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# On The Optimization of DC-Pension Fund Investment by Lie Symmetry Analysis

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## ABSTRACT

**We applied the works inspired by Kumei and Bluman in the determination and evaluation of time-dependent financial investments especially in the DC-Pension fund optimization. We established the applicability of Lie symmetry analysis and the reductions techniques of the Lie-symmetry to transform (2+1) dimensional nonlinear pde into a system of (1+1) linear equation with its solutions and obtained the optimal strategy.**

## KEYWORDS

**DC pension, Lie symmetry, CRRA utility function maximization, optimal portfolio strategy.**

**INTRODUCTION:**

There are two different methods to implement pension funds policies: Defined-Benefit plan (DB) and Defined-Contribution plan (DC). In DB, the benefits are fixed. The benefits accruable are defined in advanced and the contributions are systematically adjusted to guarantee that the fund balance is maintained in line with government policies. The participant of this type of scheme is responsible for all the associated financial risks. The defined-contribution scheme is designed so that the contributions are defined in advance and benefits on the return on the assets of the fund, all the associated financial risks are borne by the beneficiary.

The demographic advancement that affects the sustainability of retirement income and the subsequent evolution of equity market has increased adoption of the subject. The benefit payments depend on the fund portfolio and the efficiency of the investment strategy. In the classical work of Merton dynamics portfolio selection model, return rates and volatilities of risk assets are all assumed to be deterministic. As the pension management and the plan members are given more flexibilities to select without restrictions appropriate benefit outgo, the optimal benefit outgo is described as control variable in some recent literatures.

The benefit outgo is dynamically chosen by the PFM to achieve the plan member's objectives. The benefit payment policy has not gained popularity in a DC pension fund study. For the benefit payment policy, the retirements depends on traditional DC scheme as demonstrated in actuarial principles and concepts. In a defined contribution (DC) pension plan, the financial risk is borne by the member: contributions are fixed in advance, and the benefits provided by the plan depend on the investment performance experienced during the active membership and on the price of the annuity at retirement, in the case that the benefits are given in the form of an annuity. Therefore, the financial risk can be split into two parts: investment risk, during the accumulation phase, and annuity risk, focused at retirement. Recently, due to the demographic evolution and the development of the equity market, DC schemes have become popular in global pension market. A successful DC scheme will deliver good annuity at retirement, so the investment strategy for the accumulation phase in DC schemes is very critical.
The research in the use of symmetry reductions for the solution of partial differential equations in the of optimization strategies of investment is now expanding as a result of its numerous applications. The pioneer work of Gazizov and Ibragimov (see[1]) on Black-Scholes equations (see[2,3,4]), where the evolution equation is symmetry. The recent work of Myeni and Leach (see[5,6]) extended the complete symmetry groups in its applications to ODE (see [7,8,9,10]) similar to the type of systems commonly found in modelling financial derivatives.

In this work, we are interested in the optimization of pension fund portfolio which corresponds to a nonlinear evolution partial differential equation in 2+1 variables, and used symmetry analysis of Lie group to evaluate its applicability. It's easily verifiable that the peculiarity of Hamilton-Jacobi-Bellman equations that the amount of symmetry in Lie analysis is very appropriate to the solution. Benth and Karlsen showed in their work that *Ansatz* provided the structure for the solution of the problem which coincides with methodology of the symmetry analysis. An interesting structures and properties of the equation described in their paper showed the robustness and elegance application of the evolution equation of nonlinear structure (see [11]).

We noted that Lie symmetries provide tools for solving linear and nonlinear differential equations with their exact solutions (see [12]).The invariance properties of the multiple-term fractional equation of Kolmogorov – Petrovskii – Piskunov (KPP) as stated in their work (see[13]), where Lie symmetry analysis method was applied for the exact solution of the system. The works of Gazizov and Ibragimov (see [1]), Loand Hui (see[15,16,17]), equally presented elegant techniques for the determination and evaluation of time-dependent financial assets.

The direction of our work is the extension of Leach, et al. (see[18]) where he showed the applicability of Lie theory in the solution of a model on the optimization of risky and riskless investments.

However, Sinkaka et al in (see [19]) formulated closed-form solutions using Cox-Ingersoll-Ross (CIR) interest rate model of the PDE of some derivatives. A major advancement is the derivation of Lie symmetry and optimal system of parabolic equation from the optimal investments.

Our novel in this work is to extend the applicability of Lie group in solving the terminal value problem of HJB equation derived from the power utility function in the defined contribution (DC) pension scheme and the explicit solution by the CRRA utility function using Lie symmetry reduction method and also ensure that the invariance of the terminal is consistent with the solutions, provided that the symmetry can be determined, and subsequently the similarity reductions and solutions be evaluated.

The challenges in handling the problem of HJB equation by the method of Lie symmetry method are to ensure among others consistency in the determination of combinations of symmetries admitted and find the compatibilities with other terminal conditions, then the similarity reductions and solutions may be evaluated. The objective of the Pension Fund Manager (PFM) is to maximize the expected terminal utility of wealth in a complete market setting under the constant relative risk aversion (CRRA) that will to an extent guarantee the investor's lifestyle before the exit from the scheme.

In real life, we consider stochastic income with inflation risk and formulate the DC-pension scheme dynamics using the dynamic programming principle (DPP). The explicit representation of the solution of the HJB equation consistent with the portfolio optimization for an investor that is interested in maximizing the expected power utility of the terminal wealth was derived by the application of Lie symmetry method to the nonlinear PDE.

## MATHEMATICAL MODEL

In this section, we consider the market structure and define the stochastic dynamics of the asset values and the contributions. We consider a complete and frictionless financial market which is continuously open over the fixed time interval $[0,T], where\ T>0$ denotes the retirement time. We also consider the uncertainty involved in the financial market is defined and shaped by a complete filtered probability space $(\Omega,\mathscr{F},\mathcal{P})\ and\ \mathscr{F}=f(t)_{t\geq 0}$, represents the information available before time $t$ in the market, and adapted from $\mathscr{F}(t)_{t\geq 0}$.

A: **The Financial Market**:

We suppose that the market is composed of three kinds of financial assets namely: fixed income securities (bond/ bank account), equities (stocks) and real estate securities (property). These assets operate independently with their associated risks depending on market environments.

A1: **Fixed income securities**: The first asset in a financial market is the risk-free asset (bond/ bank account) denoted by $\mathscr{B}(t)$ and the price of this asset at time $t$ evolves according to the dynamics of differential equations:

$$d\mathscr{B}(t)=\mathscr{B}(t)t(t)dt\ \ ,\ \mathscr{B}(0)=\mathscr{B}_0, r(t)>0. \tag{1}$$

where $\mathscr{B}_0$ is the initial price of the risk-free asset, and $r(t)$ is the instantaneous rate of interest follows Hull-White model (1990). On the historical probability measure $\mathbb{P}$, the dynamics of $r(t)$ is given by the mean-reverting stochastic differential equation (SDE) as

$$dr(t) = (\theta(t) - \beta r(t))dt + \sigma dW(t), r(0) = r_0. \quad (2)$$

Where $\sigma > 0,\ \theta(t),\ and\ \beta$ denotes the interest rate volatility, the mean-reversion which is time dependent, and the reversion rate. $W(t)$ is a standard Brownian motion.

A2: **The Stock**: The stock is denoted by $S(t)$ whose dynamic follows SDE governed by

$$dS(t) = S(t)\theta_s dt + \sigma_s dW_s(t); \quad S(0) = S_0 \quad (3)$$

Where $S_0, \theta_s(t),\ and\ \sigma_s$ denote; the initial stock price, the expected rate of return, and the stock volatility rate respectively.

A3: **The real estate security**: The real estate security exhibits a significant investment asset and a potential contributor to pension fund wealth. We denote property by $R(t)$ with the dynamics governed by SDE

$$dR(t) = R(t)\theta_R(t)dt + \sigma_R dW_R(t),\ R(0) = R_0 \quad (4)$$

Where $R(0) > 0, \theta_R(t)\ , and\ \sigma_s > 0$ the initial returns are expected rate of return, and the real estate price volatility respectively.

B: **Contributions to the Funds**:

In the defined contribution (DC) management, the members will be continuously contributing the part/ proportion of their salaries to the retirement time $T$. The contributions process at time $t$ is given by the SDE.

$$dC(t) = C(t)\theta_c(t)dt + \sigma_C W_C(t), C(0) = C_0. \quad (5)$$

Where $\theta_C > 0, \sigma_C > 0,\ and\ C_0 > 0$ are respectively the rates of contribution, contributions volatility, and the initial contribution.

C: **Pension Wealth Process**:

We consider a complete probability space $(\Omega, \mathscr{F}, \mathcal{P})$ with a filtration$\{\mathscr{F}_t\}_{t\geq 0}$ , $t \geq 0$. The filtration $\{\mathscr{F}_t\}_{t\geq 0}$ is generated by the trajectories of a 1-dimensional standard Brownian motion$B(t)$ , $t \geq 0$. Given the financial market composed of two types assets: riskless and risky assets respectively. The riskless asset $S_0(t), t \geq 0$; evolves according to the dynamics. $dS_0(t) = rS_0(t)dt, S_0(0) = 1$ (6)

Where $r \geq 0$ is the instantaneous spot rate of return. The price of the risky asset $S_0(t), t \geq 0$ follows the Itô's process evolves and satisfies the SDE:

$$dS_1(t) = rS_1(t)dt + \sigma S_1(t)dB(t), \quad S_1(0) = S_0 \tag{7}$$

$r$ is the rate of expected return and $\sigma > 0$ is the instantaneous rate of volatility.

We assume that the financial market consists of two assets namely: risky and risk-free. Risk-free is usually at time $t$ evolves as ODE

$$d\mathscr{B}(t) = \mathscr{B}(t)r(t)dt \ , \ r(0) = r_0 > 0 \tag{8}$$

The price of the risk-free asset is controlled by the ODE as stated in (8). The dynamics of a risky asset price evolve according to the models of SDEs as sated as:

$$\left.\begin{aligned} dS(t) &= S(t)\left\{r(t)dt + (v(t)^{\frac{1}{2}}dW_a(t)\right\}, S(0) > 0, \\ dV(t) &= \kappa\big(\psi - v(t)\big)dt + w(v(t)^{\frac{1}{2}})dW_b(t), \ V(0) > 0 \end{aligned}\right\} \tag{9}$$

The parameters in (9) are described as follows:

(a) $S(t)$ is the price of asset evolution
(b) $V(t)$ is the volatility and evolves as CIR are the Wiener process caused by risky assets, volatility and interest rate processes.

From (9) $r > 0$ a constant interest rate, correlation $dW_a(t)dW_b(t) = \sigma_{a,b}(t)d(t)$ and $|\sigma_{a,b}| < 1$. The variance process, $(t)$ , of the risky asset $S(t)$ is a mean-reverting square root process, in which $\kappa > 0$ determines the speed of adjustment of the volatility towards its theoretical mean $\psi > 0$ and $w > 0$ is the second-order volatility (i.e. the validity of the volatility)

## OPTIMIZATION FRAMEWORK AND TRANSFORMATION

We recall that an investor starts at initialization time $(t)$ of investment where $t \leq T \ \ at\ wealth\ \Lambda_t = \lambda$ and $price\ at\ \mathrm{P}_t = p$. The market structure of DC-Pension scheme consists of two assets which operates independently with their associated risk-depending on the investment. The DC-pension fund manager(DC-PFM) allocates a fraction, $\Pi_u, u \in [t, T], \Pi_u$ is a progressively measurable process of the wealth denoted by $\Lambda_u$ in the risky asset.

The assumptions by the DC-PFM has risk preference governed by power utility function

$$U(w) = [\gamma^{-1}w^{\gamma}]\ 0 < \gamma < 1. \tag{10}$$

The returns on investment depends on the maximization of the utility function using a defined and robust strategy $\Pi^*$ at the final time $T$ of exit.

The value function

$$H(s,t,w) = \sup_{\pi \in A_t} E[U(W_T)|S_t = s, W_t = w] \quad , t \in (0,T) \tag{11}$$

The corresponding HJB equation $Z$ of (11) is given by

$$H_t + rwH_w + \alpha(\mu - \log s)sH_s + \frac{1}{2}\sigma^2 s^2 H_{ss} - \frac{1}{2}\sigma^2 s^2 \frac{H_{ws}^2}{H_{ww}} - \frac{[\alpha(\mu-\log s)-r]^2}{2\sigma^2}\frac{H_w^2}{H_{ww}} - [\alpha(\mu - \log s) - r]s\frac{H_w H_{ws}}{H_{ww}} = 0 \quad (12)$$

subject to the boundary conditions:

$$H(s,T,w) = \gamma^{-1}w^{\gamma} \ \ and\ H(s,T,0) = 0 \quad (13)$$

By the work of Hopf-Cole (see[20]) and by the solution of Merton (see[21]).

The solution of (12) can be reformulated as

$$H(s,T,w) = \gamma^{-1}w^{\gamma}\mathcal{G}(s,t)^{1-\gamma} \quad (14)$$

Where $\mathcal{G}(s,t)$ will be determined by the evolution equation:

$$\mathcal{G}_t + \frac{1}{1-\gamma}[\alpha(\mu - \log s) - \gamma r]s\mathcal{G}_s + \frac{1}{2}\sigma^2 s^2 \mathcal{G}_{ss} + \left\{\frac{r\gamma}{1-\gamma} + \frac{r[\alpha(\mu-\log s)-r]^2}{2\sigma^2(1-\gamma)^2}\right\}\mathcal{G} \quad (15)$$

subject to the terminal condition $\mathcal{G}(s,T) = 1$ (16)

**SYMMETRY ANALYSIS OF EQUATION (12) AND ITS SIMILARITY**

The HJB stated by equation (12) has the structure corresponding to the one parameter transformation Group, using the symbolic Lie program for Lie analysis of differential equations (see [22, 23, 24, and 25]), and we obtain the following structure symmetries.

$$\psi_1 = \partial_t \quad (17)$$

$$\psi_2 = w\partial_w \quad (18)$$

$$\psi_3 = H\partial_H \quad (19)$$

$$\psi_4 = F(t,y)\partial_H \quad , \quad y = \mu - \log s \quad (20)$$

$$\psi_5 = \mathcal{G}(t,y)\partial_w \quad (21)$$

$F(t,y)$ *and* $\mathcal{G}(t,y)$ are the solutions of the linear parabolic equations stated in (22)

$$F_t + \frac{1}{2}\sigma^2 F_{tt} + \left(\frac{1}{2}\sigma^2 - \alpha(\mu - \log s)F_y\right) = 0 \quad (22)$$

*where* $F_t$ *and* $F_y$ are the first partial derivatives and $F_{tt}$ is the second partial derivative.

Similarly, $\mathcal{G}_t + \frac{1}{2}\sigma^2\mathcal{G}_{yy} + \left(\frac{1}{2}\sigma^2 - r\right)\mathcal{G}_y - r\mathcal{G} = 0$ (23)

Note by the effective code of Nucci (see[26,27]) yields the same symmetries.

By the combination of the symmetries define by:

$$\psi = \sum_{k=1}^{5} \lambda_k \psi_k \quad (24)$$

With the same terminal condition as in (13). From the symmetries structure of (22) which has the same structure of heat equation. However, are interested in the existence symmetry in the partial differential equation. From (22) and (23), the first-order maximum condition for the optimal strategy

$\Pi^*$ will be derived using the HJB reconstructed/ derived and obtain the explicit solution using the power utility function as in (11).

$$H(s,t,w) = \underset{\pi_t}{Sup} E[U(W_T)|S_t = S, W_t = w], t \in (0,T) \qquad (25)$$

The corresponding HJB to (25) is given by:

$$H_t + \mu s H_s + (rw+\lambda)H_w + \frac{1}{2}\xi^2 s^{2(\beta+1)} H_{ss} +$$
$$\underset{\pi_t}{Sup}\left[\frac{1}{2}\pi^2\xi^2 w^2 s^{2\beta} + \pi\left[(\mu - r)wH_w + \xi^2 w s^{2\beta+1} H_{ws}\right]\right] = 0 \qquad (26)$$

$$H' = \frac{\partial H}{\partial t}dt + \frac{\partial H}{\partial s}ds + \frac{\partial H}{\partial w}ds + \frac{\partial H}{\partial w}dw + \frac{1}{2}\frac{\partial^2 H}{\partial S^2}\left[\xi^2 s^{2(\beta+1)}\right]dt$$
$$+\frac{1}{2}\frac{\partial^2 H}{\partial w^2}\left[\pi^2\xi^2 w^2 s^{2\beta}\right]dt + \frac{\partial^2 H}{\partial w \partial s}dwds \qquad (27)$$

Then the first order maximum condition for the strategy is:

$$\pi_t^* = \frac{(\mu - r)\frac{\partial H}{\partial w} + \xi^2 s^{2\beta}\frac{\partial^2 H}{\partial w \partial s}}{w\xi^2 s^{2\beta}\frac{\partial^2 H}{\partial w^2}} \qquad (28)$$

The PDE for value function follows: $\frac{\partial H}{\partial t} + \mu s\frac{\partial H}{\partial s} + (rw+\lambda)\frac{\partial H}{\partial w} + \frac{1}{2}\xi^2 s^{2(\beta+1)}\frac{\partial^2 H}{\partial s^2} - \frac{\left[(\mu - r)\frac{\partial H}{\partial w} + \xi^2 s^{2\beta+1}\frac{\partial^2 H}{\partial w \partial s}\right]}{2\xi^2 s^{2\beta}\frac{\partial^2 H}{\partial w^2}}$ $, H(s,T,w) = U(w)$

(29)

With the given terminal condition: $H(s,T,w) = U(w)$ as in (29)

Using the power transformation with the new variable, we obtain

$$H(s,t,w) = \ell^*(s,t)[w - a(t)]^p \qquad (30)$$

$$\ell^*(s,T) = 1 \quad at\ a(T) = 0 \qquad (31)$$

$$\ell^*(s,t) = \left[\ell(y,t)\ \right]^{1-p},\ y = s^{-2\beta},\ \ \ell(y,T) = 1 \qquad (32)$$

$$\ell^*(s,t) = X(t)e^{Y(t)l},\ X(T) = 1,\ Y(T) = 0 \qquad (33)$$

(30,31,32 and 33) is to allow for the implementation of symmetry and subsequently reduced the HJB to a first order ODE. A symmetry group transforms any solutions of equation into solution of the same equation. Consider the symmetry structure obtained by the code of Nocci (see [26 and 27]),

then from (24), the variance

$$\lambda_1 = 0\ \ and\ (\lambda_3 - \lambda_2)\frac{w^p}{p} + \lambda_4 F(s,T) - \lambda_5 w^{p-1}\mathcal{G}(s,T) = 0 \qquad (34)$$

For the derivation of one-parameter generator using Lie symmetry analysis (see[28,29 and 30]), with the equivalent variables : $\tilde{\psi} = w\partial_w + pH\partial_H$ (35)

Where (35) results in a similarity transformation

$H(s,t,w) = \ell^* \frac{(s,t)w^p}{p}$ such that $\widetilde{\ell}^*_t - \frac{1}{2}\frac{s^{2\beta+2}p\xi^2\tilde{\ell}_s^{\,2}}{(p-1)\tilde{\ell}} - \frac{s(\mu-pr)}{p-1}\tilde{\ell}_s + \frac{1}{2}\xi^2 s^{2(\beta+1)}\tilde{\ell}^*_{ss} + \left[r - \frac{(\mu-r)^2}{2\xi^2(p-1)s^{2\beta}}\right]p\ell^* + \frac{\lambda p}{w}\ell^* = 0$ (36)

Using equations (17-21), by conjecture , the similarity transformation as :

$$H(s,w,t) = \frac{\ell^*(s,t)[w-\mathcal{G}(t)]^p}{p} \quad (37)$$

generated by: $\psi^* = w\partial_w + \mathcal{G}(t)\partial_w + pH\partial_H \;\; and \;\; \mathcal{G}_t - r\mathcal{G} + \lambda \;,\; \mathcal{G}(T) = 0$ (38)

with solution given as : $\mathcal{G}(t) = \frac{\lambda\left[1-e^{-\gamma(T-t)}\right]}{\gamma}$ (39)

Then, the transformed function of

$\ell^*(s,t) = \ell^*{}_t - \frac{1}{2}\frac{s^{2(\beta+1)}p\xi^2\ell^{*2}_s}{(p-1)\ell^*} - \frac{s(\mu-pr)}{p-1}\ell^*{}_s + \frac{1}{2}\xi^2 s^{2(\beta+1)}\ell^*{}_{ss} + \left[r - \frac{(\mu-r)^2}{2\xi^2(p-1)s^{2\beta}}\right]p\ell^* =$
$0 at \;\; \ell^*(s,T) = 1$ (40)

The resolution of (40) , which is nonlinear is achieved by a point transformations of

$$\sum = l(s,t)\ell^{*\frac{p}{p-1}}\partial_{\ell^*} \quad (41)$$

$where \;\; l(s,t)$ is a solution of the linear system:

$$l_t - \frac{s(\mu-pr)}{(p-1)}l_s + \frac{1}{2}\xi^2 s^{2(\beta+1)}l_{ss} + \left[r - \frac{(\mu-r)^2}{2\xi^2(p-1)s^{2\beta}}\right]\frac{p}{p-1}l = 0 \quad (42)$$

The work of Kumei and Bluman stated the necessary and sufficient conditions for a given nonlinear system of PDEs can be transformed into a linear system of PDEs by invertible mapping. Using the same algorithm presented in ([31] and [32]), (40) maps into (42) as a linear equation and further reduced (42) into $\ell^*(s,t) = [l(s,t)]^{1-p}$ , $l(s,T) = 1$.

Recall that from (11), the group classification generated is given by:

$\hat{P}(s,t)u_t + \hat{Q}(s,t)u_s + \hat{R}(s,t)u_{ss} + \hat{S}(s,t)u = 0 \;,\; \hat{P} \neq 0 \; and \; \hat{Q} \neq 0$ (43)

And the principal lie symmetry structure by (43) using $u\partial_u, and\, w(s,t)\partial_u$ as span index with solution of (43) finally reduced to: $V_\tau = V_{yy} + Z(\tau,y)V$ (44)

by a Lie's equivalence: $y = \alpha(t,\varphi), \tau = \vartheta(t), v = \gamma(t,\varphi), \alpha_\varphi \neq 0, \vartheta_t \neq 0$ (45)

(see pg12 of draft for ref.)

Consequently, (42) is transformed into :$l(s,t) = \phi(s,t)u(s,t)$ (46)

$l(s,t) = \phi(s,t)u(s,t)$ (46)

$and \; l(s,t) = \phi(s,t)e^{at+bs^{-2\beta}}$ (47)

$such \; that \; a = \frac{\mu+pr+2(\mu-pr)\beta}{2(p-1)}$ (48)

$$b = \frac{\mu - pr}{2\xi^2\beta(1-p)} \tag{49}$$

From (42), the linear system reduces to:

$$\bar{C}_k + \frac{1}{2}s^{2\beta+2}\xi^2\phi_{ss} + \frac{1}{2}\frac{\mu^2 - pr^2}{\xi^2(p-1)s^{2\beta}}\bar{C} = 0 \tag{50}$$

$$such\ that \quad \bar{C}(s,T) = e^{-aT - bs^{-2\beta}} \tag{51}$$

$Let\ \theta^2 = \frac{\mu^2 - pr^2}{1-p}$ , $\theta > 0$. It follows that $\bar{C}_t + \frac{1}{2}s^{2\beta+2}\xi^2\phi_{ss} - \frac{1}{2}\frac{\theta^2}{\xi^2 s^{2\beta}}\bar{C} = 0$ (52)

Further transformation of (52) is reduced to heat equation with potential term.

$$\psi_\tau - \psi_{yy} - \frac{(\beta^2 - 1)\tau\psi}{4\beta^2 y^2} = 0 \tag{53}$$

Symmetrically (50) is similar to (52) (see [33])

If $\beta^2 - 1 \Longrightarrow \beta = 1\ \ or\ \beta = -1$. We will consider only the case of $\beta = -1$.

$$Let\ \bar{C}(s,t) = \rho(t)e^{\bar{C}(t)s^{-2\beta}} \tag{54}$$

Which forms a Riccati ODE. $\quad \varphi(t) = -2\xi^2\beta^2\varphi^2(t) + \frac{\theta^2}{2k^2}\ , at\ \ \theta(T) = -b$ (55)

With the associated first-order ODE:

$$\rho^{'}(t) = -\xi^2\beta(2\beta + 1)\rho(t)\theta(t)\ ,\ and\ \ \rho(T) = e^{-aT} \tag{56}$$

By subsequent computation, we arrived at the required optimal strategy:

$$\pi^* = \frac{\upsilon + \mathcal{G}(t)}{\upsilon} \cdot \frac{\mu - r}{(1-p)(\xi s^\beta)^2} \cdot \left\{1 - \frac{2\xi^2\beta(1-p)[b + \theta(t)]}{\mu - r}\right\} \tag{57}$$

## Conclusion

We obtained the optimal strategy for the PFM using the power utility function and also examined the model proposed by Benth and Karlsen for the optimization of financial investment in the DC-Pension scheme. The derived model which is a nonlinear evolution partial differential equation in 2+1 dimensional variables subjected to the Lie symmetry analysis.

However, a linear 1+1 dimensional evolution equation transformed to a classical heat equation and the HJB equations revealed by the Lie symmetry analysis is elegant and helpful for the solution of the equation that established the explicit optimal portfolio strategy for the DC-Pension investor within the context of the problem.

The difficulties to the solution of partial differential equations by Lie symmetry is the requirement to transform the boundary conditions. We proposed a further work in area of finding the rogue wave solutions necessary in describing some important phenomenon in financial derivatives.

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