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On the Persson-Strang's Identity for the Legendre Polynomials

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Abstract:

We show an alternative proof of an identity given by Persson-Strang for the well known Legendre polynomials.

Keywords: *Zudilin's identity, Legendre polynomials, Persson-Strang's identity.*

Introduction

The Legendre's polynomials (Legendre, 1785) $P_n(x), -1 \leq x \leq 1,$ can be defined via the following recurrence relation (Lanczos, 1972; Chihara, 1978; Oldham, & Spanier, 1987):

$$(n+1)\, P_{n+1} = (2n+1)x\, P_n - n\, P_{n-1}, P_0 = 1, \quad P_1 = x, \quad n = 1, 2, \dots, \tag{1}$$

hence:

$$P_2 = \frac{1}{2}(3x^2 - 1), P_3 = \frac{1}{2}(5x^3 - 3x), P_4 = \frac{1}{8}(35x^4 - 30x^2 + 3), \dots \tag{2}$$

These polynomials also are determined univocally through the conditions (Sommerfeld, 1964; Broman, 1989):

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = 0, \quad m \neq n, P_n(1) = 1, \quad \forall\, n, \tag{3}$$

therefore:

$$\int_{-1}^{1} x^m\, P_n(x)\, dx = 0, \qquad m < n\,, \tag{4}$$

and the Laplace's integral formula (Chihara, 1978; Oldham, & Spanier, 1987; Sommerfeld, 1964; López-Bonilla, López-Vázquez, & Torres-Silva, 2015; Doman, 2016; Foupouagnigni, & Koepf, 2020) gives an alternative way to generate the expressions (2):

$$P_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (x + \sqrt{x^2 - 1}\, cos\, \beta)^n\, d\beta, \qquad n = 0, 1, 2 \dots \tag{5}$$

or equivalently:

$$P_n(x) = \frac{1}{2^n}\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^k \binom{n}{k}\binom{2n-2k}{n} x^{n-2k}. \tag{6}$$

Persson-Strang (2003), Amdeberhan et al. (2021), Bulnes, López-Bonilla, & Prajapati (2023) obtained the interesting identity:

$$\int_{-1}^{1} [\frac{P_{2n+1}(x)}{x}]^2\, dx = 2\,, \qquad n = 0, 1, 2, \ldots \tag{7}$$

then in Sec. 2 we use an identity of Zudilin (2014) to generate an alternative proof of (7).

Persson-Strang's identity

Zudilin (2014) and López-Bonilla, & Prajapati (2023) deduced the following property:

$$[\, P_m(x)\,]^2 = \sum_{k=0}^{m} \binom{m}{k}\binom{m+k}{m}\binom{2k}{k}(-\frac{1-x^2}{4})^k\,, \quad m = 0, 1, 2, \ldots \tag{8}$$

therefore:

$$Q(m) \equiv \int_{-1}^{1}[\frac{P_m(x)}{x}]^2\, dx = \sum_{k=0}^{m} \binom{m}{k}\binom{m+k}{m}\binom{2k}{k}\frac{(-1)^k}{4^k}\int_{-1}^{1}\frac{(1-x^2)^k}{x^2}\, dx, \tag{9}$$

however, it easy to obtain the expression:

$$\int_{-1}^{1}\frac{(1-x^2)^k}{x^2}\, dx = -\frac{2\,(4^k)}{\binom{2k}{k}}, \tag{10}$$

then from (9) and (10):

$$Q(m) = -2\, \sum_{k=0}^{m}(-1)^k \binom{m}{k}\binom{m+k}{m} = -2\ {}_2F_1(-m, m+1; 1; 1). \tag{11}$$

On the other hand, we know the relation (Quaintance & Gould, 2016):

$$\sum_{k=0}^{m}(-1)^k \binom{m}{k}\binom{z+ky}{m} = (-y)^m\,, \qquad y \neq 0, \tag{12}$$

which for $z = m$ and $y = 1$ implies the result:

$$\sum_{k=0}^{m}(-1)^k \binom{m}{k}\binom{m+k}{m} = (-1)^m = P_m(-1) = {}_2F_1(-m, m+1; 1; 1), \tag{13}$$

hence from (9), (11) and (13):

$$\int_{-1}^{1} [\frac{P_m(x)}{x}]^2\, dx = 2(-1)^{m-1}\,, \qquad m = 0, 1, 2, \ldots \tag{14}$$

which reproduces the Persson-Strang's identity (7) when *m* is odd.

Remark.-In Bulnes, López-Bonilla, & Prajapati (2023) it was proved the property:

$$\int_{-1}^{1}\frac{1}{x}P_{2n+1}(x)\, dx = \frac{2\,(-4)^n (n!)^2}{(2n+1)!}. \tag{15}$$

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