ALGEBRAIC COMBINATORICS

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Volume 6, issue 6 (2023), p. 1557-1591.
https://doi.org/10.5802/alco.323

Algebraic Combinatorics is published by The Combinatorics Consortium
and is a member of the Centre Mersenne for Open Scientific Publishing
www.tccpublishing.org www.centre-mersenne.org
e-ISSN: 2589-5486

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Abstract A now-classical cyclic extension of the descent set of a permutation has been introduced by Klyachko and Cellini. Following a recent axiomatic approach to this notion, it is natural to ask which sets of permutations admit such a (not necessarily classical) extension.

The main result of this paper is a complete answer in the case of conjugacy classes of permutations. It is shown that the conjugacy class of cycle type λ has such an extension if and only if λ is not of the form (r^s) for some square-free r. The proof involves a detailed study of hook constituents in higher Lie characters.

1. Introduction

1.1. Background and main result. The study of descent sets for permutations may be traced back to Euler. A cyclic extension of this classical concept was introduced in the study of Lie algebras [20] and descent algebras [8]. Surprising connections of the cyclic descent notion to a variety of mathematical areas were found later.

The *descent set* of a permutation $\pi = [\pi_1, \ldots, \pi_n]$ in the symmetric group S_n on n letters is

$$\mathrm{Des}(\pi) := \{1 \leqslant i \leqslant n-1 \,:\, \pi_i > \pi_{i+1}\} \quad \subseteq [n-1],$$

where $[m] := \{1, 2, \ldots, m\}$. Cellini [8] introduced a natural notion of *cyclic descent set*:

$$\mathrm{CDes}(\pi) := \{1 \leqslant i \leqslant n \,:\, \pi_i > \pi_{i+1}\} \quad \subseteq [n],$$

with the convention $\pi_{n+1} := \pi_1$. The more restricted notion of cyclic descent number had been used previously by Klyachko [20]. This cyclic descent set was further studied by Dilks, Petersen and Stembridge [10] and others.

There exists a well-established notion of descent set for standard Young tableaux (SYT), but it has no obvious cyclic analogue. In a breakthrough work, Rhoades [27] defined a notion of cyclic descent set for standard Young tableaux of rectangular shape. The properties common to Cellini's definition (for permutations) and Rhoades'

Manuscript received 13th November 2019, revised 10th February 2023 and 29th June 2023, accepted 30th June 2023.

Keywords. Cyclic descent, conjugacy class, symmetric group, higher Lie character, hook constituent.

Acknowledgements. PH was partially supported by Hungarian National Research, Development and Innovation Office (NKFIH) Grant No. K138596 and by Bar-Ilan University visiting grant. The project leading to this application has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme, Grant agreement No. 741420. RMA and YR were partially supported by the Israel Science Foundation, grant no. 1970/18 and by an MIT-Israel MISTI grant.

construction (for SYT) appeared in other combinatorial settings as well [24, 23, 11, 1]. This led to an abstract definition, as follows.

Definition 1.1 ([3]). *Let $\mathcal{T}$ be a finite set, equipped with a set valued map (called* descent map*)* Des : $\mathcal{T} \longrightarrow 2^{[n-1]}$. *Let* shift : $2^{[n]} \longrightarrow 2^{[n]}$ *be the mapping on subsets of $[n]$ induced by the cyclic shift $i \mapsto i+1 \pmod n$ of elements $i \in [n]$. A* cyclic extension *of* Des *is a pair* (cDes, p)*, where* cDes : $\mathcal{T} \longrightarrow 2^{[n]}$ *is a map and* $p : \mathcal{T} \longrightarrow \mathcal{T}$ *is a bijection, satisfying the following axioms: for all T in $\mathcal{T}$,*

$$\begin{aligned} &\textit{(extension)}\ \mathrm{cDes}(T) \cap [n-1] = \mathrm{Des}(T),\\ &\textit{(equivariance)}\ \mathrm{cDes}(p(T)) = \mathrm{shift}(\mathrm{cDes}(T)),\\ &\textit{(non-Escher)}\ \varnothing \subsetneq \mathrm{cDes}(T) \subsetneq [n]. \end{aligned}$$

The term "non-Escher" refers to M. C. Escher's drawing "Ascending and Descending", which illustrates the impossibility of the cases $\mathrm{cDes}(\pi) = \varnothing$ and $\mathrm{cDes}(\pi) = [n]$ for permutations $\pi \in S_n$.

A *ribbon* is a skew shape which contains no 2×2 square.

Theorem 1.2 ([3, Theorem 1.1]). *Let λ/μ be a skew shape with n cells. The descent map* Des *on* SYT(λ/μ) *has a cyclic extension* (cDes, p) *if and only if λ/μ is not a connected ribbon.*

The original proof used Postnikov's toric symmetric functions. A constructive proof was recently given by Huang [17].

For connections of cyclic descents to Kazhdan–Lusztig theory see [27]; for topological aspects and connections to the Steinberg torus see [10]; for twisted Schützenberger promotion see [27, 17]; for cyclic quasisymmetric functions and Schur-positivity see [2]; for Postnikov's toric Schur functions see [3]. The goal of this paper is to determine which conjugacy classes of the symmetric group carry a cyclic descent extension.

Observation 1.3. *Let* Des *and* CDes *denote the classical descent set and Cellini's cyclic descent set on permutations, respectively. Let $p : S_n \to S_n$ be the rotation*

$$[\pi_1, \pi_2, \ldots, \pi_{n-1}, \pi_n] \xmapsto{p} [\pi_n, \pi_1, \pi_2, \ldots, \pi_{n-1}].$$

Then the pair (CDes, p) *is a cyclic descent extension of* Des *on S_n in the sense of Definition* 1.1*.*

Cellini's definition provides a cyclic extension of Des on the full symmetric group, but not on some of its subsets – for example, on many conjugacy classes; see Section 7.2 below.

Example 1.4. Consider the conjugacy class of 4-cycles in S_4,

$$\mathcal{C}_{(4)} = \{2341, 2413, 3142, 3421, 4123, 4312\}.$$

Cellini's cyclic descent sets are

$$\{3\}, \{2,4\}, \{1,3\}, \{2,3\}, \{1\}, \{1,2\},$$

respectively; this family is not closed under cyclic rotation. On the other hand, redefining the cyclic descent sets to be

$$\mathrm{cDes}(2341) = \{3,4\},\ \mathrm{cDes}(2413) = \{2,4\},\ \mathrm{cDes}(3142) = \{1,3\},$$
$$\mathrm{cDes}(3421) = \{2,3\},\ \mathrm{cDes}(4123) = \{1,4\},\ \mathrm{cDes}(4312) = \{1,2\}$$

and defining the map p by

$$2341 \to 4123 \to 4312 \to 3421 \to 2341$$

and

$$3142 \to 2413 \to 3142,$$

the pair (cDes, p) does determine a cyclic extension of Des for this conjugacy class.

The goal of this paper is to show that most conjugacy classes in S_n carry a cyclic descent extension. In fact, we obtain a full characterization.

Recall that an integer is *square-free* if no prime square divides it; in particular, 1 is square-free. Our main result is the following.

Theorem 1.5. *Let λ be a partition of n, and let $\mathcal{C}_\lambda \subseteq S_n$ be the corresponding conjugacy class. The descent map* Des *on $\mathcal{C}_\lambda$ has a cyclic extension* (cDes, p) *if and only if λ is not of the form (r^s) for some square-free r.*

1.2. Proof method. The proof of Theorem 1.5 is non-constructive and involves a detailed study of the hook constituents in higher Lie characters. Here is an overview of the main ingredients.

1.2.1. *Higher Lie characters.*

Definition 1.6. *For a partition λ of n, let $\mathcal{C}_\lambda$ be the conjugacy class consisting of all the permutations in S_n of cycle type λ, and let χ^λ denote the irreducible S_n-character corresponding to λ. Let Z_λ be the centralizer of a permutation in $\mathcal{C}_\lambda$ (defined up to conjugacy). If k_i denotes the number of parts of λ equal to i, then Z_λ is isomorphic to the direct product $\times_{i=1}^n \mathbb{Z}_i \wr S_{k_i}$. Here and in the rest of the paper $\mathbb{Z}_i$ denotes the cyclic group of order i, using additive notation (integers modulo i).*

For each i, let ω_i be the linear character on $\mathbb{Z}_i \wr S_{k_i}$ indexed by the i-tuple of partitions $(\varnothing, (k_i), \varnothing, \ldots, \varnothing)$. In other words, let ζ_i be a primitive linear character on the cyclic group $\mathbb{Z}_i$, and extend it to the wreath product $\mathbb{Z}_i \wr S_{k_i}$ so that it is $\zeta_i^{k_i}$ on the base subgroup $\mathbb{Z}_i^{k_i}$ and trivial on the wreathing subgroup S_{k_i}. Denote this extension by ω_i. Now let

$$\omega^\lambda := \bigotimes_{i=1}^{n} \omega_i,$$

a linear character on Z_λ. Define the corresponding higher Lie character *to be the induced character*

$$\psi^\lambda := \omega^\lambda \uparrow_{Z_\lambda}^{S_n} .$$

The study of higher Lie characters can be traced back to Schur [30]. An old problem of Thrall [39] is to provide an explicit combinatorial interpretation of the multiplicities of the irreducible characters in the higher Lie character, see also [33, Exercise 7.89(i)]. Only partial results are known: the case $\lambda = (n)$ was solved by Kraśkiewicz and Weyman [21]; Désarménien and Wachs [9] resolved a coarser version of Thrall's problem for the sum of higher Lie characters over all derangements, see also [26]. The best result so far is Schocker's expansion [29, Theorem 3.1], which however involves signs and rational coefficients. For recent discussions see, e.g., [25, 5, 36].

A remarkable theorem of Gessel and Reutenauer [15, Theorem 2.1] applies higher Lie characters to describe the fiber sizes of the descent set map on conjugacy classes. Their proof applies an interpretation of higher Lie character ψ^λ in terms of quasisymmetric functions (Theorem 2.5 below). It follows that higher Lie characters can be used to prove the existence of cyclic descent extensions as explained below.

1.2.2. *Hook multiplicities and cyclic descent extensions.* Recall the standard notation s_λ for the Schur function indexed by a partition λ, as well as $\mathcal{F}_{n,D}$ for the fundamental quasisymmetric function indexed by a subset $D \subseteq [n-1]$; see Definition 2.1. A subset $\mathcal{A} \subseteq S_n$ is *Schur-positive* if the associated quasisymmetric function

$$\mathcal{Q}(\mathcal{A}) := \sum_{a \in \mathcal{A}} \mathcal{F}_{n,\mathrm{Des}(a)}$$

is symmetric and Schur-positive.

For an integer $0 \leqslant k < n$ and a Schur-positive subset $\mathcal{A} \subseteq S_n$ denote

$$m_{k,\mathcal{A}} := \langle \mathcal{Q}(\mathcal{A}), s_{(n-k,1^k)} \rangle,$$

where $s_{(n-k,1^k)}$ is the Schur function indexed by the hook partition $(n-k, 1^k)$.

First we prove the following key lemma, which provides an algebraic criterion for the existence of a cyclic descent extension.

Lemma 1.7. *A Schur-positive set $\mathcal{A} \subseteq S_n$ has a cyclic descent extension if and only if the following two conditions hold:*

(divisibility) *the polynomial $\sum_{k=0}^{n-1} m_{k,\mathcal{A}} x^k$ is divisible by $1+x$;*
(non-negativity) *the quotient has nonnegative coefficients.*

See Lemma 3.2 below.

1.2.3. *Divisibility.* By the Gessel–Reutenauer theorem, for every conjugacy class $\mathcal{C}_\lambda$ the quasisymmetric function $\mathcal{Q}(\mathcal{C}_\lambda)$ is the Frobenius image of the higher Lie character ψ^λ, thus $\mathcal{C}_\lambda$ is Schur-positive; see Theorem 2.5 below.

For a partition $\lambda \vdash n$ denote

$$m_{k,\lambda} := m_{k,\mathcal{C}_\lambda} = \langle \mathcal{Q}(\mathcal{C}_\lambda), s_{(n-k,1^k)} \rangle = \langle \psi^\lambda, \chi^{(n-k,1^k)} \rangle. \tag{1}$$

Proposition 1.8. *The hook-multiplicity generating function of the higher Lie character ψ^λ*

$$M_\lambda(x) := \sum_{k=0}^{n-1} m_{k,\lambda} x^k$$

is divisible by $1+x$ if and only if λ is not of the form (r^s) for a square-free integer r.

This divisibility condition is proved using an explicit evaluation of the higher Lie character on n-cycles; see Section 3.3 below.

1.2.4. *Non-negativity.* In order to prove Theorem 1.5, it remains to show that the coefficients of the quotient $M_\lambda(x)/(1+x)$ are nonnegative, whenever λ is not of the form (r^s) for a square-free r. It turns out that partitions λ which have at least two different parts, namely not of the form (r^s) for any r, are the easiest to handle. In that case, a factorization of the associated higher Lie character ψ^λ is applied to prove the following.

Lemma 1.9. *Let $\lambda = \mu \sqcup \nu$ be a disjoint union of nonempty partitions with no common part. Then*

$$\frac{M_\lambda(x)}{1+x} = M_\mu(x) M_\nu(x), \tag{2}$$

and its coefficients are thus non-negative.

The core of the proof of Theorem 1.5 is the case of $\lambda = (r^s)$. For a fixed positive integer r, consider the formal power series

$$M_r(x,y) := \sum_{\substack{i \geqslant 0 \\ s \geqslant 1}} m_{i,(r^s)} x^i y^s,$$

where $m_{i,(r^s)}$ is the multiplicity of the hook character $\chi^{(rs-i,1^i)}$ in the higher Lie character $\psi^{(r^s)}$. The following theorem completes the proof of Theorem 1.5.

Theorem 1.10. *If r is not square-free then the formal power series*

$$\frac{M_r(x,y)}{1+x}$$

has non-negative integer coefficients.

Proof of Theorem 1.5. Combine Lemma 1.7, Proposition 1.8, Lemma 1.9 and Theorem 1.10. □

1.3. Outline. The rest of the paper is organized as follows.

Necessary background, including a cyclic analogue of the Gessel–Reutenauer Theorem, is given in Section 2.

The role of hooks in the study of cyclic descent extensions is explained in Section 3. In particular, a necessary and sufficient criterion for Schur-positive sets to carry a cyclic descent extension (Lemma 1.7) is proved in Subsection 3.2. Using this criterion, the proof of Theorem 1.5 is reduced to Proposition 1.8 (divisibility) and Theorem 1.10 (non-negativity). Proposition 1.8 is proved in Subsection 3.3.

The proof of Theorem 1.10, stating the non-negativity of the coefficients of $M_\lambda(x)/(1+x)$ whenever this quotient is a polynomial, spans Sections 4, 5 and 6: the case of more than one cycle length is considered in Section 4; the case of a single cycle is considered in Section 5; and the case of cycle type (r^s) with $s>1$ is considered in Section 6. In the case of more than one cycle length, non-negativity is proved using a factorization of the associated higher Lie character (Lemma 1.9). In the case of a single cycle, combining a combinatorial formula for inner products with a variant of the Witt transform proves unimodality of the sequence of hook-multiplicities (Proposition 5.2). This, in turn, implies the desired non-negativity of the quotient.

In Section 6 we lift the single-cycle result to the case of cycle type (r^s) with $s>1$. In Subsection 6.1 we provide an explicit expression for the coefficients of $(1+x)M_r(x,y)$, see Theorem 6.3. This expression is used to obtain a product formula for the bivariate polynomial $1+(1+x)M_r(x,y)$ in Subsection 6.2, and to deduce Theorem 1.10 in Subsection 6.3.

Additional results are presented in Section 7. In Subsection 7.1 it is shown that Lemma 3.5 and Theorem 6.3 imply well-known combinatorial identities. In Subsection 7.2 it is shown that the natural approach does not provide a cyclic descent extension for conjugacy classes in S_n. Palindromicity of the hook-multiplicity generating function $M_{(r^s)}(x)$ is studied in Subsection 7.3.

Section 8 concludes the paper with final remarks and open problems.

2. Preliminaries

The role of quasisymmetric functions in the study of the distribution of descent sets is discussed in Subsection 2.1. The results presented here are used in Section 3 to establish the reduction of existence of cyclic descent extension on conjugacy classes to the study of hook-multiplicities in higher Lie characters. In Subsection 2.2 we present cyclic analogues of a few classical results. These analogues are used for certain enumerative applications; the reader may skip this subsection.

2.1. Quasisymmetric functions and descents. A symmetric function is called *Schur-positive* if all the coefficients in its expansion in the basis of Schur functions are non-negative. Recall the notation $s_{\lambda/\mu}$ for the Schur function indexed by a skew shape λ/μ.

Definition 2.1. *For each subset $D \subseteq [n-1]$ define the* fundamental quasisymmetric function

$$\mathcal{F}_{n,D}(\mathbf{x}) := \sum_{\substack{i_1 \leqslant i_2 \leqslant \ldots \leqslant i_n \\ i_j < i_{j+1} \text{ if } j \in D}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

Denote the set of standard Young tableaux of skew shape λ/μ by $\mathrm{SYT}(\lambda/\mu)$. There is an established notion of descent set for $\mathrm{SYT}(\lambda/\mu)$

(3) $\qquad \mathrm{Des}(T) := \{1 \leqslant i \leqslant n-1 \,:\, i+1 \text{ appears in a lower row of } T \text{ than } i\}.$

Theorem 2.2. (Gessel, see [33, Theorem 7.19.7]) *For every skew shape λ/μ,*

$$\sum_{T \in \mathrm{SYT}(\lambda/\mu)} \mathcal{F}_{n,\mathrm{Des}(T)} = s_{\lambda/\mu}.$$

Given any subset $\mathcal{A} \subseteq S_n$, define the quasisymmetric function

$$\mathcal{Q}(\mathcal{A}) := \sum_{a \in \mathcal{A}} \mathcal{F}_{n,\mathrm{Des}(a)}.$$

Finding subsets of permutations $\mathcal{A} \subseteq S_n$, for which $\mathcal{Q}(\mathcal{A})$ is symmetric (Schur-positive), is a long-standing problem, see [15].

We write $\lambda \vdash n$ to denote that λ is a partition of a positive integer n. For $D \subseteq [n-1]$ let $\mathbf{x}^D := \prod_{i \in D} x_i$.

Lemma 2.3. *For every subset $\mathcal{A} \subseteq S_n$ and a family $\{c_\lambda\}_{\lambda \vdash n}$ of coefficients, the equality*

$$\mathcal{Q}(\mathcal{A}) = \sum_{\lambda \vdash n} c_\lambda s_\lambda \tag{4}$$

is equivalent to the equality

$$\sum_{a \in \mathcal{A}} \mathbf{x}^{\mathrm{Des}(a)} = \sum_{\lambda \vdash n} c_\lambda \sum_{T \in \mathrm{SYT}(\lambda)} \mathbf{x}^{\mathrm{Des}(T)}. \tag{5}$$

Proof. By Theorem 2.2, Equation (4) is equivalent to

$$\sum_{a \in \mathcal{A}} \mathcal{F}_{n,\mathrm{Des}(a)} = \sum_{\lambda \vdash n} c_\lambda \sum_{T \in \mathrm{SYT}(\lambda)} \mathcal{F}_{n,\mathrm{Des}(T)}.$$

Next recall from [33, Ch. 7] that the fundamental quasisymmetric functions in $x_1, \ldots, x_n$ form a basis of the vector space QSym_n of quasisymmetric functions in n variables. Finally, apply the vector space isomorphism from QSym_n to the space of square-free polynomials in $x_1, \ldots, x_n$, which maps $\mathcal{F}_{n,D}$ to $\mathbf{x}^D$. $\square$

Corollary 2.4. *For every finite family S of skew shapes of size n and every subset $\mathcal{A} \subseteq S_n$ $\mathcal{Q}(\mathcal{A}) = \sum_{\lambda/\mu \in S} c_{\lambda/\mu} s_{\lambda/\mu}$ if and only if*

$$\sum_{a \in \mathcal{A}} \mathbf{x}^{\mathrm{Des}(a)} = \sum_{\lambda/\mu \in S} c_{\lambda/\mu} \sum_{T \in \mathrm{SYT}(\lambda/\mu)} \mathbf{x}^{\mathrm{Des}(T)}.$$

Corollary 2.4 will be combined with Theorem 1.2 to provide criteria for the existence of cyclic descent extensions for Schur-positive sets; see the proof of Lemma 3.2 and Remark 4.2 below.

Let $\lambda \vdash n$ be a partition of n and let ψ^λ be *the higher Lie character* indexed by λ (see Definition 1.6). The following result was proved by Gessel and Reutenauer.

THEOREM 2.5 ([15, Proof of Theorem 2.1]). *For every partition λ of $n \geqslant 1$,*

$$\mathcal{Q}(\mathcal{C}_\lambda) = \mathrm{ch}(\psi^\lambda),$$

where ch *is the Frobenius characteristic map. Equivalently,*

$$\mathcal{Q}(\mathcal{C}_\lambda) = \sum_{\mu \vdash n} \langle \psi^\lambda, \chi^\mu \rangle s_\mu.$$

In particular, $\mathcal{Q}(\mathcal{C}_\lambda)$ is Schur-positive.

2.2. CYCLIC ANALOGUES. Recall the complete homogeneous symmetric functions

$$h_n := \sum_{i_1 \leqslant \cdots \leqslant i_n} x_{i_1} \cdots x_{i_n} \qquad (n \geqslant 1).$$

For a composition $\alpha = (\alpha_1, \ldots, \alpha_t)$, define

$$h_\alpha := h_{\alpha_1} h_{\alpha_2} \cdots h_{\alpha_t}.$$

For any subset $J = \{j_1 < \ldots < j_t\} \subseteq [n-1]$, define

$$\alpha(J,n) := (j_1, j_2 - j_1, j_3 - j_2, \ldots, j_t - j_{t-1}, n - j_t).$$

This is a composition of n, with a corresponding *connected ribbon* having the entries of $\alpha(J,n)$ as row lengths, from bottom to top. The associated *ribbon Schur function* is

$$s_{\alpha(J,n)} := \sum_{I \subseteq J} (-1)^{\#(J \setminus I)} h_{\alpha(I,n)}.$$

THEOREM 2.6 (Gessel, an immediate consequence of [14, Theorem 3]). *Let $\mathcal{A}$ be a finite set, equipped with a descent map* $\mathrm{Des} : \mathcal{A} \longrightarrow 2^{[n-1]}$. *If*

$$\mathcal{Q}(\mathcal{A}) := \sum_{a \in \mathcal{A}} \mathcal{F}_{n,\mathrm{Des}(a)}$$

is symmetric then

$$|\{a \in \mathcal{A} \,:\, \mathrm{Des}(a) = J\}| = \langle \mathcal{Q}(A), s_{\alpha(J,n)} \rangle \qquad (\forall J \subseteq [n-1]).$$

For a subset $\varnothing \neq J = \{j_1 < j_2 < \ldots < j_t\} \subseteq [n]$ define the corresponding *cyclic composition* of n as

$$\alpha^{\mathrm{cyc}\,(J,n)} := (j_2 - j_1, \ldots, j_t - j_{t-1}, j_1 + n - j_t),$$

with $\alpha^{\mathrm{cyc}\,(J,n)} := (n)$ when $J = \{j_1\}$; note that $\alpha^{\mathrm{cyc}\,(\varnothing,n)}$ is not defined. The corresponding *affine (cyclic) ribbon Schur function* was defined in [3] as

$$\tilde{s}_{\alpha^{\mathrm{cyc}\,(J,n)}} := \sum_{\varnothing \neq I \subseteq J} (-1)^{\#(J \setminus I)} h_{\alpha^{\mathrm{cyc}\,(I,n)}}.$$

THEOREM 2.7 ([2, Cor. 4.13]). *Let $\mathcal{A}$ be a finite set, equipped with a descent map* $\mathrm{Des} : \mathcal{A} \longrightarrow 2^{[n-1]}$ *which has a cyclic extension. If*

$$\mathcal{Q}(\mathcal{A}) := \sum_{a \in \mathcal{A}} \mathcal{F}_{n,\mathrm{Des}(T)}$$

is symmetric then the fiber sizes of (any) cyclic descent map satisfy

$$|\{a \in \mathcal{A} \,:\, \mathrm{cDes}(a) = J\}| = \langle \mathcal{Q}(\mathcal{A}), \tilde{s}_{\alpha^{\mathrm{cyc}\,(J,n)}} \rangle \qquad (\forall \varnothing \subsetneq J \subsetneq [n]).$$

PROPOSITION 2.8 ([3, Lemma 2.2]). *If $A \subseteq S_n$ carries a cyclic descent extension, then the cyclic descent set generating function is uniquely determined.*

Corollary 2.9. *Let $\mathcal{A} \subseteq S_n$ be a symmetric set which carries a cyclic descent extension and S be a finite set of skew shapes of size n which are not connected ribbons. Then for every cyclic descent extension the following equations are equivalent:*

$$\mathcal{Q}(\mathcal{A}) = \sum_{\lambda/\mu \in S} c_{\lambda/\mu} s_{\lambda/\mu}, \tag{6}$$

and

$$\sum_{a \in \mathcal{A}} \mathbf{x}^{\mathrm{cDes}(a)} = \sum_{\lambda/\mu \in S} c_{\lambda/\mu} \sum_{T \in \mathrm{SYT}(\lambda/\mu)} \mathbf{x}^{\mathrm{cDes}(T)}. \tag{7}$$

Proof. By Theorem 2.7 and Theorem 2.2, if Equation (6) holds then, for every $\varnothing \subsetneq J \subsetneq [n]$,

$$\begin{aligned}|\{a \in \mathcal{A} \,:\, \mathrm{cDes}(a) = J\}| &= \langle \mathcal{Q}(\mathcal{A}), \tilde{s}_{\alpha^{\mathrm{cyc}}(J,n)} \rangle = \langle \sum_{\lambda/\mu \vdash n} c_{\lambda/\mu} s_{\lambda/\mu}, \tilde{s}_{\alpha^{\mathrm{cyc}}(J,n)} \rangle \\ &= \sum_{\lambda/\mu \vdash n} c_{\lambda/\mu} \langle \mathcal{Q}(\mathrm{SYT}(\lambda/\mu)), \tilde{s}_{\alpha^{\mathrm{cyc}}(J,n)} \rangle \\ &= \sum_{\lambda/\mu \vdash n} c_{\lambda/\mu} |\{T \in \mathrm{SYT}(\lambda/\mu) \,:\, \mathrm{cDes}(T) = J\}|.\end{aligned}$$

Thus Equation (7) holds.

For the opposite direction, let $x_n = 1$ in Equation (7) and apply Corollary 2.4 to deduce Equation (6). □

Theorem 2.5 and Theorem 2.6 imply the following.

Theorem 2.10 ([15, Theorem 2.1]). *For every conjugacy class $\mathcal{C}_\lambda$ of cycle type $\lambda \vdash n$ the descent set map* Des *has fiber sizes given by*

$$|\{\pi \in \mathcal{C}_\lambda : \ \mathrm{Des}(\pi) = J\}| = \langle \mathcal{Q}(\mathcal{C}_\lambda), s_{\alpha(J,n)} \rangle \qquad (\forall J \subseteq [n-1]).$$

The following cyclic analogue of Theorem 2.10 results from Theorem 2.5 and Theorem 2.7.

Theorem 2.11. *For every conjugacy class $\mathcal{C}_\lambda$, which carries a cyclic descent set extension, all cyclic extensions of the descent set map* Des *have fiber sizes given by*

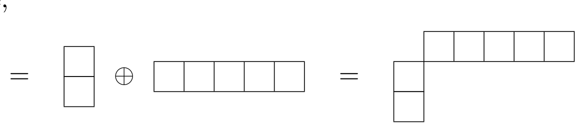
$$|\{\pi \in \mathcal{C}_\lambda : \ \mathrm{cDes}(\pi) = J\}| = \langle \mathcal{Q}(\mathcal{C}_\lambda), \tilde{s}_{\alpha^{\mathrm{cyc}}(J,n)} \rangle \qquad (\forall \varnothing \subsetneq J \subsetneq [n]).$$

3. The role of hooks

3.1. Hooks and near-hooks. It turns out that hooks and near-hooks play a crucial role in the study of cyclic descent extensions.

A *hook* is a partition with at most one part larger than 1. Explicitly, it has the form $(n-k, 1^k)$ for some $0 \leqslant k \leqslant n-1$. A *near-hook* of size n is a hook of size $n+1$ with its (northwestern) corner cell removed; see [1] for a somewhat more inclusive definition of this notion. Equivalently, recall the *direct sum* operation on shapes (partitions), denoted $\lambda \oplus \mu$, yielding a skew shape having the diagram of λ strictly southwest of the diagram of μ, with no rows or columns in common. A near-hook of size n is the direct sum of a one-column partition (1^k) and a one-row partition $(n-k)$, for some $0 \leqslant k \leqslant n$. For example,

$(1^2) \oplus (5)$ =



Given an S_n-character ϕ, let

(8) $m_{k,\phi} := \langle \phi, \chi^{(n-k,1^k)} \rangle \quad (0 \leqslant k \leqslant n-1), \quad e_{k,\phi} := \langle \phi, \chi^{(1^k)\oplus(n-k)} \rangle \quad (0 \leqslant k \leqslant n).$

When ϕ is understood from the context, we use the abbreviated notations $m_k := m_{k,\phi}$ and $e_k := e_{k,\phi}$.

By Pieri's rule [33, Theorem 7.5.17] combined with the (inverse) Frobenius characteristic map,

$$\begin{aligned}\chi^{(1^k)\oplus(n-k)} = \chi^{(1^k)}\chi^{(n-k)} = \mathrm{ch}^{-1}(s_{(1^k)}s_{(n-k)}) &= \mathrm{ch}^{-1}(s_{(n-k+1,1^{k-1})} + s_{(n-k,1^k)}) \\ &= \chi^{(n-k+1,1^{k-1})} + \chi^{(n-k,1^k)} \qquad (0 < k < n).\end{aligned}$$

Equivalently,

$$\chi^{(n-k,1^k)} = \sum_{i=0}^{k} (-1)^{k-i} \chi^{(1^i)\oplus(n-i)} \qquad (0 \leqslant k \leqslant n-1).$$

Thus the sequences $\{m_k\}_{k=0}^{n-1}$ and $\{e_k\}_{k=0}^{n}$ determine each other via the relations

(9) $$e_k = m_k + m_{k-1} \quad \text{and} \quad m_k = \sum_{i=0}^{k} (-1)^{k-i} e_i \qquad (0 \leqslant k \leqslant n),$$

where $m_k := 0$ for $k = -1$ and $k = n$. Note that, in particular,

$$\sum_{i=0}^{n} (-1)^{n-i} e_i = 0.$$

3.2. Cyclic descent extension and hook-multiplicities. Let $\mathcal{A} \subseteq S_n$ be Schur-positive with $\mathcal{Q}(\mathcal{A}) = \mathrm{ch}(\phi)$. Denote

$$m_\lambda := \langle \phi, \chi^\lambda \rangle = \langle \mathcal{Q}(\mathcal{A}), s_\lambda \rangle \qquad (\forall \lambda \vdash n).$$

For $\lambda = (n-k, 1^k)$ we use the abbreviation

$$m_k := m_{(n-k,1^k)} \qquad (0 \leqslant k < n).$$

The *hook-multiplicity generating function* is defined as

$$M_{\mathcal{A}}(x) := \sum_{k=0}^{n-1} m_k x^k.$$

The function $M_{\mathcal{A}}(x)$, where $\mathcal{A}$ is a conjugacy class, was studied and applied to the enumeration of unimodal permutations with a given cycle type by Thibon [38].

Observation 3.1. *For every $0 \leqslant k < n$*

$$m_k = |\{a \in \mathcal{A} : \ \mathrm{Des}(a) = [k]\}| .$$

Proof. For every $0 \leqslant k < n$ there exists a unique standard Young tableau T of size n with $\mathrm{Des}(T) = [k]$ (where $[0] := \varnothing$). The shape of T is $(n-k, 1^k)$. Comparing the coefficients of $\mathbf{x}^{[k]}$ on both sides of Equation (5) completes the proof. $\square$

We now restate and prove Lemma 1.7.

Lemma 3.2. *A Schur-positive set $\mathcal{A} \subseteq S_n$ carries a cyclic descent extension if and only if the hook-multiplicity generating function $M_{\mathcal{A}}(x)$ is divisible by $1 + x$ and the quotient $M_{\mathcal{A}}(x)/(1+x)$ has non-negative coefficients; equivalently, if and only if there exist non-negative integers d_k $(0 \leqslant k \leqslant n-2)$ such that*

$$m_k = d_k + d_{k-1} \qquad (0 \leqslant k \leqslant n-1),$$

where $d_k := 0$ for $k = -1$ and $k = n-1$.

Proof. If $\mathcal{A}$ carries a cyclic descent extension then, by Observation 3.1 and the equivariance of cDes, for every $0 \leqslant k \leqslant n-1$:

$$\begin{aligned} m_k &= |\{a \in \mathcal{A} :\ \mathrm{Des}(a) = [k]\}| \\ &= |\{a \in \mathcal{A} :\ \mathrm{cDes}(a) = [k]\}| + |\{a \in \mathcal{A} :\ \mathrm{cDes}(a) = [k] \sqcup \{n\}\}| \\ &= |\{a \in \mathcal{A} :\ \mathrm{cDes}(a) = [k]\}| + |\{a \in \mathcal{A} :\ \mathrm{cDes}(a) = [k+1]\}| \,. \end{aligned}$$

The numbers

$$d_k := |\{a \in \mathcal{A} :\ \mathrm{cDes}(a) = [k+1]\}| \qquad (-1 \leqslant k \leqslant n-1)$$

satisfy the required conditions, which imply the corresponding properties of $M_{\mathcal{A}}(x)$.

For the opposite direction, assume that there exist non-negative integers d_k (with $d_{-1} = d_{n-1} = 0$) such that $m_k = d_{k-1} + d_k$ for all $0 \leqslant k \leqslant n-1$. By Pieri's rule [33, Theorem 7.15.7],

$$s_{(1^k)\oplus(n-k)} = s_{(1^k)} s_{(n-k)} = s_{(n-k+1,1^{k-1})} + s_{(n-k,1^k)} \qquad (1 \leqslant k \leqslant n-1).$$

Hence

$$\sum_{k=1}^{n-1} d_{k-1} s_{(1^k)\oplus(n-k)} = \sum_{k=1}^{n-1} d_{k-1} \big(s_{(n-k+1,1^{k-1})} + s_{(n-k,1^k)}\big) = \sum_{k=0}^{n-1} m_k s_{(n-k,1^k)}.$$

Since

$$\mathcal{Q}(\mathcal{A}) = \sum_{\lambda \vdash n} \langle \mathcal{Q}(\mathcal{A}), s_\lambda \rangle s_\lambda = \sum_{\substack{\lambda \vdash n \\ \lambda \text{ non-hook}}} m_\lambda s_\lambda + \sum_{k=0}^{n-1} m_k s_{(n-k,1^k)}$$

we obtain

$$\mathcal{Q}(\mathcal{A}) = \sum_{\substack{\lambda \vdash n \\ \lambda \text{ non-hook}}} m_\lambda s_\lambda + \sum_{k=1}^{n-1} d_{k-1} s_{(1^k)\oplus(n-k)}. \tag{10}$$

By Corollary 2.4, this is equivalent to

$$\sum_{a \in \mathcal{A}} \mathbf{x}^{\mathrm{Des}(a)} = \sum_{\substack{\lambda \vdash n \\ \lambda \text{ non-hook}}} m_\lambda \sum_{T \in \mathrm{SYT}(\lambda)} \mathbf{x}^{\mathrm{Des}(T)} + \sum_{k=1}^{n-1} d_{k-1} \sum_{T \in \mathrm{SYT}((1^k)\oplus(n-k))} \mathbf{x}^{\mathrm{Des}(T)}.$$

By Theorem 1.2, the set $\mathrm{SYT}(\lambda)$ carries a cyclic descent extension if and only if $\lambda \vdash n$ is not a hook; and each of the sets $\mathrm{SYT}((1^k) \oplus (n-k))$ $(1 \leqslant k \leqslant n-1)$ carries a cyclic descent extension. Hence $\mathcal{A}$ also carries a cyclic descent extension, completing the proof. □

Corollary 3.3. *If a Schur-positive set $\mathcal{A}$ carries a cyclic descent extension then*

$$\sum_{a \in \mathcal{A}} \mathbf{x}^{\mathrm{cDes}(a)} = \sum_{\substack{\lambda \vdash n \\ \lambda \text{ non-hook}}} m_\lambda \sum_{T \in \mathrm{SYT}(\lambda)} \mathbf{x}^{\mathrm{cDes}(T)} + \sum_{k=1}^{n-1} d_{k-1} \sum_{T \in \mathrm{SYT}((1^k)\oplus(n-k))} \mathbf{x}^{\mathrm{cDes}(T)},$$

where m_λ and d_k are the non-negative integers defined above.

Proof. By Corollary 2.9 together with Equation (10), the generating function for the corresponding cyclic descent set is uniquely determined and satisfies the claimed equality. □

3.3. Divisibility of the hook-multiplicity generating function. Recall the notation

$$m_{k,\phi} := \langle \phi, \chi^{(n-k,1^k)} \rangle \qquad (0 \leqslant k < n)$$

for an S_n-character ϕ.

Lemma 3.4. *For every S_n-character ϕ, the hook-multiplicity generating function*

$$M_\phi(x) := \sum_{k=0}^{n-1} m_{k,\phi} x^k$$

is divisible by $1+x$ if and only if the value of ϕ on an n-cycle is zero, i.e., $\phi_{(n)} = 0$.

Proof. By [28, Lemma 4.10.3], for every partition $\lambda \vdash n$

$$\chi^\lambda_{(n)} = \begin{cases} (-1)^k, & \text{if } \lambda = (n-k, 1^k) \text{ for some } 0 \leqslant k < n; \\ 0, & \text{otherwise.} \end{cases}$$

Thus

$$\phi_{(n)} = \sum_{\lambda \vdash n} \langle \phi, \chi^\lambda \rangle \chi^\lambda_{(n)} = \sum_{k=0}^{n-1} \langle \phi, \chi^{(n-k,1^k)} \rangle \chi^{(n-k,1^k)}_{(n)} = \sum_{k=0}^{n-1} m_{k,\phi} \cdot (-1)^k = M_\phi(-1),$$

which equals zero if and only if $1+x$ divides $M_\phi(x)$, completing the proof. $\square$

Letting $\phi = \psi^\lambda$, the higher Lie character indexed by a partition λ, reduces Proposition 1.8 to the following character evaluation.

Recall the Möbius function $\mu(n)$, the sum of all primitive (complex) n-th roots of 1. If n has a prime square divisor then $\mu(n) = 0$; otherwise, n is a product of k distinct primes and $\mu(n) = (-1)^k$. The following lemma is equivalent to a combinatorial identity due to Garsia, as shown in Proposition 7.2 below. We give here an independent direct algebraic proof.

Lemma 3.5. *For $\lambda \vdash n$*

$$\psi^\lambda_{(n)} = \begin{cases} \mu(r), & \text{if } \lambda = (r^s); \\ 0, & \text{otherwise,} \end{cases}$$

where μ is the Möbius function.

Proof. Let c be an n-cycle in S_n, and let $Z_\lambda = Z_{S_n}(g)$ be the centralizer in S_n of a specific element $g \in \mathcal{C}_\lambda$. An explicit formula for the induced character [19, (5.1)] is

$$\psi^\lambda(c) = \omega^\lambda \uparrow_{Z_\lambda}^{S_n} (c) = \frac{1}{|Z_\lambda|} \sum_{\substack{x \in S_n \\ x^{-1}cx \in Z_\lambda}} \omega^\lambda(x^{-1}cx).$$

An n-cycle commutes only with its own powers. Thus, if λ is *not* of the form (r^s) for some r and s, then there is no n-cycle in Z_λ; equivalently, $x^{-1}cx \notin Z_\lambda$ for every $x \in S_n$. It follows that, for such partitions $\lambda \vdash n$, $\psi^\lambda_{(n)} = 0$.

Assume now that $\lambda = (r^s)$, and let let $g = g_1 g_2 \cdots g_s \in \mathcal{C}_\lambda$ be a fixed product of s disjoint r-cycles. The order of the centralizer $Z_\lambda = Z_{S_n}(g)$ is $s! r^s$. If $u \in Z_\lambda$ is an n-cycle ($n = rs$) then $g = u^k$ for some integer k with $\gcd(k,n) = s$; equivalently, $u^s = g^j$ for some $0 < j < r$ with $\gcd(j,r) = 1$. Conversely, if $u \in S_n$ is an n-cycle satisfying $u^s = g^j$ for some $0 < j < r$ with $\gcd(j,r) = 1$, then g is a power of u and therefore $u \in Z_\lambda$. Thus the number of n-cycles in Z_λ, namely the number of elements of $Z_\lambda \cap \mathcal{C}_{(n)}$, is $\varphi(r)(s-1)! r^{s-1}$, where φ is Euler's totient function.

Viewing Z_λ as the group of $s \times s$ monomial ("generalized permutation") matrices whose nonzero entries are complex r-th roots of unity, an element $u \in Z_\lambda \cap \mathcal{C}_{(n)}$ corresponds to a matrix whose underlying permutation is a full s-cycle and the product of its nonzero entries is a primitive r-th root of unity. This product is equal to $\omega^\lambda(u)$, so it is a primitive r-th root of unity. For $u, v \in Z_\lambda \cap \mathcal{C}_{(n)}$, write $u \sim v$ if $v = u^i$ for some integer i (necessarily coprime to n). This clearly defines an equivalence relation on $Z_\lambda \cap \mathcal{C}_{(n)}$. On each equivalence class, all primitive r-th roots of unity appear with the same frequency as values of ω^λ. This property thus holds for all of $Z_\lambda \cap \mathcal{C}_{(n)}$, where this frequency is $(s-1)!r^{s-1}$. Denoting by ξ any specific primitive r-th root of unity, the sum of all values of ω^λ on $Z_\lambda \cap \mathcal{C}_{(n)}$ is therefore

$$\sum_{u \in Z_\lambda \cap \mathcal{C}_{(n)}} \omega^\lambda(u) = (s-1)!r^{s-1} \sum_{j:(j,r)=1} \xi^j = (s-1)!r^{s-1}\mu(r).$$

Given any $c, u \in \mathcal{C}_{(n)}$, there are exactly $n = rs$ permutations $x \in S_n$ which satisfy $u = x^{-1}cx$. Thus

$$\begin{aligned}\psi^\lambda(c) = \omega^\lambda \uparrow_{Z_\lambda}^{S_n} (c) &= \frac{1}{|Z_\lambda|} \sum_{\substack{x \in S_n \\ x^{-1}cx \in Z_\lambda}} \omega^\lambda(x^{-1}cx) = \frac{n}{|Z_\lambda|} \sum_{u \in Z_\lambda \cap \mathcal{C}_{(n)}} \omega^\lambda(u) \\ &= \frac{n}{s!r^s}(s-1)!r^{s-1}\mu(r) = \mu(r),\end{aligned}$$

as claimed. □

Proof of Proposition 1.8. By Lemma 3.4, $1+x$ divides the hook-multiplicity generating function of the higher Lie character ψ^λ if and only if $\psi^\lambda_{(n)} = 0$. Lemma 3.5 completes the proof. □

Corollary 3.6. *Let $\lambda \vdash n$.*

1. *If $\lambda = (r^s)$ for some square-free integer r and positive integer s, then $1+x$ does not divide the hook-multiplicity generating function $M_\lambda(x)$, and the descent set map on the conjugacy class $\mathcal{C}_\lambda$ does not have a cyclic extension.*
2. *If λ is not equal to (r^s) for any square-free r, then $1+x$ divides $M_\lambda(x)$. In this case, the descent set map on $\mathcal{C}_\lambda$ has a cyclic extension if and only if the quotient $M_\lambda(x)/(1+x)$ has non-negative coefficients.*

Proof. By the Gessel–Reutenauer theorem (Theorem 2.5), for every $\lambda \vdash n$ the conjugacy class $\mathcal{C}_\lambda$ is Schur-positive, with $\mathcal{Q}(\mathcal{C}_\lambda) = \mathrm{ch}(\psi^\lambda)$. Combining this with Lemma 3.2 and Proposition 1.8 completes the proof of both parts. □

In the following sections we will prove the non-negativity of the coefficients of the quotient $M_\lambda(x)/(1+x)$ for partitions (cycle types) which are not equal to (r^s) for a square-free r: cycle types with more than one cycle length will be considered in Section 4, single cycles will be considered in Section 5, and cycle types $\lambda = (r^s)$ with non square-free r and $s > 1$ will be considered in Section 6 (this is the most difficult case).

4. Non-negativity: the case of more than one cycle length

Consider, first, the case of a conjugacy class with more than one cycle length. This is the easiest case to handle.

Proof of Lemma 1.9. The centralizer Z_λ of a permutation in $\mathcal{C}_\lambda$ is isomorphic, in this case, to the direct product $Z_\mu \times Z_\nu$. By Definition 1.6, $\omega^\lambda := \omega^\mu \otimes \omega^\nu$ and

$$\psi^\lambda = \omega^\lambda \uparrow_{Z_\lambda}^{S_n} = \left(\omega^\mu \uparrow_{Z_\mu}^{S_{|\mu|}} \otimes \omega^\nu \uparrow_{Z_\nu}^{S_{|\nu|}}\right) \uparrow_{S_{|\mu|} \times S_{|\nu|}}^{S_n} . \tag{11}$$

By the Littlewood-Richardson rule [33, Theorem A1.3.3], the outer product of two irreducible characters $\left(\chi^{\alpha}\otimes\chi^{\beta}\right)\uparrow_{S_m\times S_{n-m}}^{S_n}$ contains irreducible representations indexed by hooks if and only if both α and β are hooks; in the latter case,

$$\langle\left(\chi^{(m-i,1^i)}\otimes\chi^{(n-m-j,1^j)}\right)\uparrow_{S_m\times S_{n-m}}^{S_n},\chi^{(n-k,1^k)}\rangle=\begin{cases}1, & k\in\{i+j,i+j+1\};\\ 0, & \text{otherwise.}\end{cases}$$

Therefore

$$M_\lambda(x)=(1+x)M_\mu(x)M_\nu(x),$$

as claimed. □

Corollary 4.1. *If λ is a partition with at least two different parts, namely not of the form (r^s) for any r, then $M_\lambda(x)$ is divisible by $1+x$ and the quotient has non-negative coefficients.*

Remark 4.2. In this case, the existence of a cyclic descent extension may be proved directly as follows. By Equation (11), ψ^λ is a sum of characters indexed by disconnected shapes. Thus, by Corollary 2.4, the distribution of the descent set over $\mathcal{C}_\lambda$ is equal to a sum of distributions over the sets of SYT of various disconnected shapes. By Theorem 1.2, each of these sets carries a cyclic descent extension, hence so does $\mathcal{C}_\lambda$.

5. Non-negativity: the single cycle case

Consider now the case of a conjugacy class with a single cycle. By Corollary 3.6, if r is not square-free then $1+x$ divides $M_{(r)}(x)$. The main result of this section is the following.

Proposition 5.1. *If r is not square-free then the coefficients of $M_{(r)}(x)/(1+x)$ are non-negative*

It follows from Lemma 5.12 below that, in order to prove Proposition 5.1, it suffices to show the unimodality (to be defined) of $M_{(r)}(x)$. This is the content of the following statement.

Proposition 5.2. *For any positive integer r, the sequence $m_{0,(r)},m_{1,(r)},\dots,m_{r-1,(r)}$ is unimodal. The largest element is one of the middle ones, namely $m_{i,(r)}$ for $i=(r-1)/2$ if r is odd, and either $i=(r-2)/2$ or $i=r/2$, or both, if r is even.*

In Subsection 5.1 we use a variant of the Witt transform to produce explicit formulas for the coefficients $m_{j,(r)}$ (Lemma 5.11). Then, in Subsection 5.2, we prove their unimodality.

5.1. A variant of the Witt transform.

In this subsection we present a variant of the Witt transform, which will be used to prove non-negativity in Sections 5.2 and 6.

Denote by (i,j) the greatest common divisor of two integers i,j. Recall the arithmetical Möbius function μ.

Definition 5.3. *For a positive integer r define*

$$f_j(r):=\frac{1}{r}\sum_{d|(r,j)}\mu(d)(-1)^{(d+1)j/d}\binom{r/d}{j/d}\qquad(0\leqslant j\leqslant r).$$

Observation 5.4. *By definition,*

$$f_1(r)=f_{r-1}(r)=1\qquad(r\geqslant 1).$$

Also, the fundamental property

$$\sum_{d|r} \mu(d) = \begin{cases} 1, & \textit{if } r = 1; \\ 0, & \textit{if } r > 1, \end{cases}$$

and some case analysis (r odd, or $r \equiv 0 \pmod 4$, or $r \equiv 2 \pmod 4$) imply that

$$f_0(r) = \begin{cases} 1, & \textit{if } r = 1; \\ 0, & \textit{if } r > 1 \end{cases} \qquad \textit{and} \qquad f_r(r) = \begin{cases} 1, & \textit{if } r = 1, 2; \\ 0, & \textit{if } r > 2. \end{cases}$$

For the higher Lie character $\psi^{(r)}$, simplify slightly the notations in Equation (8):

$$m_{j,(r)} := \langle \psi^{(r)}, \chi^{(r-j,1^j)} \rangle \quad (0 \leqslant j \leqslant r-1), \quad e_{j,(r)} := \langle \psi^{(r)}, \chi^{(1^j) \oplus (r-j)} \rangle \quad (0 \leqslant j \leqslant r).$$

Proposition 5.5. *For every $0 \leqslant j \leqslant r$*

$$e_{j,(r)} = f_j(r).$$

In particular, $f_j(r)$ is a non-negative integer.

Remark 5.6. Proposition 5.5 will not be proved here, since it is the special case $s = 1$ of Theorem 6.3 below. It also follows from a well known result of Kraśkiewicz and Weyman [21] (Lemma 7.5 below). A symmetric functions proof which applies [35, Lemma 2.7] was presented by Sheila Sundaram [37]. Another proof follows from [12, Theorem 3.1]. See Subsection 7.1 below for a discussion.

Definition 5.7. *For a fixed positive integer r, collect the multiplicities $f_j(r)$ into a polynomial*

$$F_r(x) := f_0(r) + f_1(r)x + f_2(r)x^2 + \ldots + f_r(r)x^r.$$

Equation (9) and Proposition 5.5 imply the following.

Corollary 5.8.

$$F_r(x) = (1 + x)M_{(r)}(x).$$

Observation 5.9.

$$F_r(x) = \frac{1}{r} \sum_{d|r} \mu(d)(1 - (-x)^d)^{r/d}.$$

Proof. Use Definition 5.3, and write $j = kd$ if $d|(r, j)$. Then

$$\begin{aligned} F_r(x) &= \sum_{j=0}^{r} x^j \sum_{d|(r,j)} \frac{\mu(d)(-1)^{(d+1)j/d}}{r} \binom{r/d}{j/d} = \sum_{d|r} \frac{\mu(d)}{r} \sum_{k=0}^{r/d} (-1)^{(d+1)k} \binom{r/d}{k} x^{kd} \\ &= \sum_{d|r} \frac{\mu(d)}{r} (1 - (-x)^d)^{r/d}. \end{aligned}$$

□

Remark 5.10. Recall from [22] that the r-th Witt transform of a polynomial $p(x)$ is defined by

$$\mathcal{W}_p^{(r)}(x) = \frac{1}{r} \sum_{d|r} \mu(d) p(x^d)^{r/d}.$$

In our case, put $p(x) = 1 - x$ to get $F_r(x) = \mathcal{W}_p^{(r)}(-x)$. The proof of Theorem 4 and Lemma 1 in [22] could have been used to prove that the coefficients of $F_r(x)$ are non-negative integers. This non-obvious property of the numbers $f_j(r)$ also follows, of course, from their interpretation in Proposition 5.5 as inner products of two characters. What we really need, in Proposition 5.1, is the nonnegativity of the coefficients of $F_r(x)/(1 + x)^2$.

We now produce an explicit formula for each coefficient $m_{j,(r)}$. For a combinatorial interpretation of these numbers, see Lemma 7.5 below.

Lemma 5.11. *For a positive integer r,*

$$m_{j,(r)} = \frac{1}{r}\sum_{d|r}\mu(d)\binom{r/d-1}{\lfloor j/d\rfloor}(-1)^{j+\lfloor j/d\rfloor} \qquad (0 \leqslant j \leqslant r-1).$$

Proof. By Corollary 5.8,

$$(1+x)\sum_{j=0}^{r-1} m_{j,(r)}x^j = F_r(x).$$

Using Definition 5.3 and Observation 5.9, we can write

$$\sum_{j=0}^{r-1} m_{j,(r)}x^j = \frac{1}{r(1+x)}\sum_{d|r}\mu(d)(1-(-x)^d)^{r/d}.$$

Using

$$\frac{(1-(-x)^d)^{r/d}}{1+x} = (1-(-x)^d)^{r/d-1}(1-x+x^2-\ldots+(-x)^{d-1})$$

and comparing coefficients of x^j, where $j = dk+\ell$ with $0 \leqslant \ell \leqslant d-1$, we get

$$rm_{j,(r)} = \sum_{d|r}\mu(d)\binom{r/d-1}{k}(-1)^{(d+1)k+\ell} = \sum_{d|r}\mu(d)\binom{r/d-1}{\lfloor j/d\rfloor}(-1)^{j+\lfloor j/d\rfloor}. \qquad \square$$

5.2. Unimodality. A sequence $(a_0,\ldots,a_n)$ of real numbers is called *unimodal* if there exists an index $0 \leqslant i_0 \leqslant n$ such that the sequence is weakly increasing up to position i_0 and weakly decreasing afterwards: $a_0 \leqslant a_1 \leqslant \ldots \leqslant a_{i_0} \geqslant \ldots a_{n-1} \geqslant a_n$.

Lemma 5.12. *Let $a(x) = a_0 + a_1x + \ldots + a_nx^n$ be a polynomial with real, nonnegative and unimodal coefficients. Assume that $1+x$ divides $a(x)$, and let $b(x) := a(x)/(1+x)$. Then the coefficients of $b(x)$ are nonnegative.*

Proof. Let $b(x) = b_0 + \ldots + b_{n-1}x^{n-1}$. Then $a_0 = b_0$, $a_n = b_{n-1}$, and

$$a_i = b_{i-1} + b_i \qquad (1 \leqslant i \leqslant n-1) \tag{12}$$

Of course, divisibility of $a(x)$ by $1+x$ implies that

$$\sum_{i=0}^{n}(-1)^i a_i = a(-1) = 0.$$

Inverting (12) we get

$$b_i = \sum_{j=0}^{i}(-1)^{i-j}a_j \qquad (0 \leqslant i \leqslant n-1) \tag{13}$$

and, similarly,

$$b_i = \sum_{j=i+1}^{n}(-1)^{j-i-1}a_j \qquad (0 \leqslant i \leqslant n-1). \tag{14}$$

By assumption, the sequence $(a_0,\ldots,a_n)$ is nonnegative and unimodal, namely: there exists an index $0 \leqslant i_0 \leqslant n$ such that

$$0 \leqslant a_0 \leqslant \ldots \leqslant a_{i_0} \geqslant \ldots \geqslant a_n \geqslant 0.$$

It follows from (13) that, for odd indices $0 \leqslant 2i+1 \leqslant i_0$,

$$b_{2i+1} = (a_{2i+1} - a_{2i}) + \ldots + (a_1 - a_0) \geqslant 0$$

and, for even indices $0 \leqslant 2i \leqslant i_0$,

$$b_{2i} = (a_{2i} - a_{2i-1}) + \ldots + (a_2 - a_1) + a_0 \geqslant 0.$$

By (14), a similar argument holds for indices greater or equal to i_0, and the proof is complete. □

Lemma 5.12 shows that non-negativity of a sequence can be proved by showing unimodality of a related sequence. In particular, Proposition 5.1 would follow once we show the unimodality of the polynomial $M_{(n)}(x)$. In order to do that, we need the following technical lemma.

Lemma 5.13. *Assume that $r > 7$ and $1 < j < r/2$; if $j = (r-1)/2$ assume also that $r > 11$. Let $d > 1$ be a divisor of r, and denote*

$$A_{r,j,d} := \frac{\binom{r/d-1}{\lfloor j/d \rfloor}}{\binom{r-1}{j}}.$$

Then

$$\frac{(r-1)(r-j)}{r-2j} A_{r,j,d} \leqslant \begin{cases} 1, & \text{if } d > 2; \\ 3/2, & \text{if } d = 2. \end{cases}$$

Proof. Write

$$\binom{r-1}{j} = \frac{(r-1)(r-2)\cdots(r-j)}{j!} = \prod_{1 \leqslant i \leqslant j} \frac{r-i}{i}.$$

Let $\ell := \lfloor j/d \rfloor$. Then ℓd is the largest multiple of d not exceeding j, hence

$$\binom{r/d-1}{\lfloor j/d \rfloor} = \binom{r/d-1}{\ell} = \frac{(r-d)(r-2d)\cdots(r-\ell d)}{d \cdot 2d \cdots \ell d} = \prod_{1 \leqslant i \leqslant j,\, d \mid i} \frac{r-i}{i},$$

The quotient $A_{r,j,d}$ can therefore be written in the form

$$A_{r,j,d} = \prod_{1 \leqslant i \leqslant j,\, d \nmid i} \frac{i}{r-i}.$$

By assumption $j < r/2$, thus $i/(r-i) < 1$ for all $1 \leqslant i \leqslant j$. It follows that $A_{r,j,d}$ is a decreasing function of j, with $A_{r,1,d} = 1/(r-1)$.

For $d > 2$ and $2 \leqslant j \leqslant (r-2)/2$,

$$A_{r,j,d} \leqslant A_{r,2,d} = \frac{2}{(r-1)(r-2)} \leqslant \frac{r-2j}{(r-1)(r-j)},$$

where the last inequality, equivalent to $2(r-j) \leqslant (r-2j)(r-2)$, follows from $2 \leqslant r-2j$ and $r-j \leqslant r-2$.

Similarly, for $d = 2$ and $3 \leqslant j \leqslant (r-2)/2$,

$$A_{r,j,2} \leqslant A_{r,3,2} = \frac{3}{(r-1)(r-3)} \leqslant \frac{3(r-2j)}{2(r-1)(r-j)},$$

where the last inequality, equivalent to $2(r-j) \leqslant (r-2j)(r-3)$, follows from $2 \leqslant r-2j$ and $r-j \leqslant r-3$.

For $d = 2$ and $j = 2$,

$$A_{r,2,2} = \frac{1}{r-1} \leqslant \frac{3(r-4)}{2(r-1)(r-2)},$$

where the inequality, equivalent to $2(r-2) \leqslant 3(r-4)$, follows from $r \geqslant 8$.

It remains to consider the case $j = (r-1)/2$, namely $r = 2j+1$, for $d \geqslant 2$. Note that in this case we assumed that $r > 11$, namely $j > 5$.

Assume first that $d > 2$. Then

$$A_{r,j,d} \leqslant A_{r,6,d} \leqslant A_{r,4,d} = \prod_{1\leqslant i\leqslant 4,\, d\nmid i} \frac{i}{r-i} = \frac{a_d}{\binom{r-1}{4}},$$

where

$$a_d = \begin{cases} 1, & \text{if } d > 4; \\ (r-d)/d, & \text{if } d = 3, 4. \end{cases}$$

Clearly

$$1 < \frac{r-4}{4} < \frac{r-3}{3}$$

and therefore

$$A_{r,j,d} \leqslant A_{r,4,d} \leqslant A_{r,4,3} = \frac{8}{(r-1)(r-2)(r-4)} \leqslant \frac{r-2j}{(r-1)(r-j)},$$

where the last inequality, equivalent (since $r = 2j+1$) to $8(j+1) \leqslant (2j-1)(2j-3)$ and to $4j^2 - 16j \geqslant 5$, holds for $j \geqslant 5$.

Finally, assume that $r = 2j+1$ and $d = 2$. Then

$$A_{r,j,2} \leqslant A_{r,6,2} = A_{r,5,2} = \frac{15}{(r-1)(r-3)(r-5)} \leqslant \frac{3(r-2j)}{2(r-1)(r-j)},$$

where the last inequality, equivalent (since $r = 2j+1$) to $5(j+1) \leqslant 2(j-1)(j-2)$ and to $2j^2 - 11j \geqslant 1$, holds for $j \geqslant 6$. This completes the proof. $\square$

Proof of Proposition 5.2. We need to show that $m_{0,(r)} \leqslant m_{1,(r)} \leqslant \ldots \leqslant m_{\lfloor (r-1)/2 \rfloor}$ and $m_{r-1,(r)} \leqslant m_{r-2,(r)} \leqslant \ldots \leqslant m_{\lceil (r-1)/2 \rceil}$ for any positive integer r.

For $1 \leqslant r \leqslant 7$, computing the polynomial $M_r(x) := F_r(x)/(1+x)$ explicitly, using Observation 5.9, gives

$$\begin{aligned} &M_1(x) = 1; \qquad M_2(x) = M_3(x) = x; \qquad M_4(x) = x + x^2; \\ &M_5(x) = x + x^2 + x^3; \qquad M_6(x) = x + 2x^2 + x^3 + x^4; \\ &M_7(x) = x + 2x^2 + 3x^3 + 2x^4 + x^5. \end{aligned}$$

The claim clearly holds in these cases. Assume from now on that $r > 7$.

Informally, the explicit formula for $m_{j,(r)}$ in Lemma 5.11 has a dominant term corresponding to $d = 1$, i.e., $rm_{j,(r)}$ is approximately equal to $\binom{r-1}{j}$. We will show that this approximation is good enough to make the sequence $m_{0,(r)}, \ldots, m_{r-1,(r)}$ unimodal, like the sequence of binomial coefficients. Note that, unlike the binomial coefficients, this sequence is not always palindromic; see Proposition 7.13 below.

We first show that $m_{j-1,(r)} \leqslant m_{j,(r)}$ for $1 \leqslant j < r/2$. Recall that we assume $r > 7$. For $j = 1$, Lemma 5.11 shows that, for $r > 1$, $m_{0,(r)} = 0 < 1 = m_{1,(r)}$.

Assume now that $1 < j < r/2$. Clearly, for these values of j and any divisor d of r,

$$\binom{r/d-1}{\lfloor j/d \rfloor} \geqslant \binom{r/d-1}{\lfloor (j-1)/d \rfloor}.$$

We conclude, by Lemma 5.11, that

$$rm_{j,(r)} - rm_{j-1,(r)} \geqslant \left[\binom{r-1}{j} - \binom{r-1}{j-1}\right] - 2\sum_{d\mid r,\, d>1} \binom{r/d-1}{\lfloor j/d \rfloor}.$$

Since

$$\binom{r-1}{j} - \binom{r-1}{j-1} = \binom{r-1}{j}\left(1 - \frac{j}{r-j}\right) = \binom{r-1}{j}\frac{r-2j}{r-j},$$

using the notation of Lemma 5.13 we get

$$\frac{rm_{j,(r)} - rm_{j-1,(r)}}{\binom{r-1}{j}\frac{r-2j}{r-j}} \geqslant 1 - 2\frac{r-j}{r-2j} \sum_{d|r,\, d>1} A_{r,j,d}.$$

Let $d(r)$ denote the number of divisors of r. For odd $r > 7$ (unless $j = (r-1)/2$ and $r \in \{9, 11\}$), Lemma 5.13 implies that

$$\frac{rm_{j,(r)} - rm_{j-1,(r)}}{\binom{r-1}{j}\frac{r-2j}{r-j}} \geqslant 1 - \sum_{d|r,\, d>2} \frac{2}{r-1} = 1 - \frac{2d(r)-2}{r-1}.$$

For even $r > 7$, Lemma 5.13 implies that

$$\frac{rm_{j,(r)} - rm_{j-1,(r)}}{\binom{r-1}{j}\frac{r-2j}{r-j}} \geqslant 1 - \frac{3}{r-1} - \sum_{d|r,\, d>2} \frac{2}{r-1} = 1 - \frac{2d(r)-1}{r-1}.$$

We clearly have $2d(r) \leqslant r$ for $r > 7$, and therefore $m_{j-1,(r)} \leqslant m_{j,(r)}$ in both cases.

In the remaining cases, namely $j = (r-1)/2$ and $r \in \{9, 11\}$, we can compute directly using Lemma 5.11. For $r = 9$ and $j = 4$ the divisors are $d = 1, 3, 9$, but $\mu(9) = 0$. Thus

$$9m_{4,(9)} - 9m_{3,(9)} = \left[\binom{8}{4} + \binom{2}{1}\right] - \left[\binom{8}{3} - \binom{2}{1}\right] = 72 - 54 > 0.$$

For $r = 11$ and $j = 5$ the divisors are $d = 1, 11$. Thus

$$11m_{5,(11)} - 11m_{4,(11)} = \left[\binom{10}{5} + \binom{0}{0}\right] - \left[\binom{10}{4} - \binom{0}{0}\right] = 253 - 209 > 0.$$

So far we have proven that $m_{0,(r)} \leqslant m_{1,(r)} \leqslant \ldots \leqslant m_{\lfloor (r-1)/2 \rfloor,(r)}$ for $r > 7$.

The remaining inequalities, $m_{r-1,(r)} \leqslant m_{r-2,(r)} \leqslant \ldots \leqslant m_{\lceil (r-1)/2 \rceil,(r)}$, can be written as $m_{r-1-(j-1),(r)} \leqslant m_{r-1-j,(r)}$ for $1 \leqslant j < r/2$. By Lemma 5.11,

$$m_{r-1-j,(r)} = \frac{1}{r} \sum_{d|r} \mu(d) \binom{r/d - 1}{\lfloor (r-1-j)/d \rfloor} (-1)^{r-1-j+\lfloor (r-1-j)/d \rfloor} \qquad (0 \leqslant j \leqslant r-1).$$

For a divisor d of r, if $j = kd + \ell$ with $0 \leqslant \ell \leqslant d-1$ then $r-1-j = (r/d-k-1)d + (d-1-\ell)$ with $0 \leqslant d-1-\ell \leqslant d-1$, so that

$$\lfloor j/d \rfloor + \lfloor (r-1-j)/d \rfloor = k + (r/d - k - 1)) = r/d - 1.$$

It follows that

$$\begin{aligned} m_{r-1-j,(r)} &= \frac{1}{r} \sum_{d|r} \mu(d) \binom{r/d-1}{\lfloor j/d \rfloor} (-1)^{r-1-j+r/d-1-\lfloor j/d \rfloor} \\ &= \frac{1}{r} \sum_{d|r} \mu(d) \binom{r/d-1}{\lfloor j/d \rfloor} (-1)^{r+r/d-j-\lfloor j/d \rfloor} \qquad (0 \leqslant j \leqslant r-1). \end{aligned}$$

This is exactly the formula for $m_{j,(r)}$ except for the signs of the summands, which differ (for each $d|r$) by the factor $(-1)^{r+r/d}$. These signs do not play any role in the proof above that $m_{j-1,(r)} \leqslant m_{j,(r)}$ for $1 \leqslant j < r/2$, which therefore also shows, mutatis mutandis, that $m_{r-1-(j-1),(r)} \leqslant m_{r-1-j,(r)}$ for $1 \leqslant j < r/2$ — except possibly the explicit confirmation when $j = (r-1)/2$ and $r \in \{9, 11\}$. In these cases r is odd, and therefore $(-1)^{r+r/d} = 1$ for any divisor d of r. This implies that indeed

$$m_{4,(9)} - m_{5,(9)} = m_{4,(9)} - m_{3,(9)} > 0$$

and

$$m_{5,(11)} - m_{6,(11)} = m_{5,(11)} - m_{4,(11)} > 0,$$

completing the proof. □

Remark 5.14. We conjecture that Proposition 5.2 remains true for an arbitrary partition $\mu \vdash n$, in particular for every partition $(r^s) \vdash n$ with any $s \geqslant 1$; see Conjecture 8.1.

Proof of Proposition 5.1. Combine Proposition 5.2 with Lemma 5.12. □

We conclude

Corollary 5.15. *The conjugacy class of n-cycles $\mathcal{C}_{(n)}$ carries a cyclic descent extension if and only if n is not square-free.*

Proof. Combine Lemma 3.2 with Corollary 3.6(2) and Proposition 5.1. □

6. Non-negativity: the case of one cycle length

In this section we consider the case $\lambda = (r^s)$. We fix r, while s and hence $n = rs$ vary. The arguments below also work for the trivial case $r = 1$.

As in the previous section, instead of the hook multiplicities

$$m_{i,(r^s)} = \langle \psi^{(r^s)}, \chi^{(n-i,1^i)} \rangle$$

we prefer to work with their consecutive sums,

$$e_{i,(r^s)} := m_{i,(r^s)} + m_{i-1,(r^s)}.$$

Here is the structure of the current section. In Subsection 6.1 we obtain an explicit description of $e_{i,(r^s)}$ (Theorem 6.3). The proof involves a detailed computation of character values and inner products of characters. In Subsection 6.2 we transform this description into a product formula (Corollary 6.9) for the formal power series

$$E_r(x,y) := \sum_{i,s \geqslant 0} e_{i,(r^s)} x^i y^s = 1 + (1+x) M_r(x,y),$$

where

$$M_r(x,y) := \sum_{\substack{i \geqslant 0 \\ s \geqslant 1}} m_{i,(r^s)} x^i y^s.$$

The product formula is a substantial merit of working with $e_{i,(r^s)}$, and it facilitates the extension of the case $s = 1$ to $s > 1$. This is done in Subsection 6.3, where the result for $s = 1$ is used to obtain the general case.

6.1. Formulas for inner products.

In this subsection we obtain explicit formulas for the inner products

$$e_{k,(r^s)} := \langle \psi^{(r^s)}, \chi^{(1^k) \oplus (n-k)} \rangle \quad (0 \leqslant k \leqslant n).$$

Recall that, by Equation (9), the sequences $\{m_k\}_{k=0}^{n-1}$ and $\{e_k\}_{k=0}^{n}$ determine each other, via the relations

$$e_k = m_k + m_{k-1} \text{ and } m_k = \sum_{i=0}^{k} (-1)^{k-i} e_i \qquad (0 \leqslant k \leqslant n),$$

where $m_k := 0$ for $k = -1$ and $k = n$. Nota bene, these multiplicities depend on r and s but this dependence is suppressed in the notation.

Definition 6.1. *For given non-negative integers i, r and s, let*

$$P_{r,s}(i) := \left\{ \gamma = (\gamma_1, \ldots, \gamma_s) : \sum_{\ell=1}^{s} \gamma_\ell = i,\ r \geqslant \gamma_1 \geqslant \gamma_2 \geqslant \ldots \geqslant \gamma_s \geqslant 0 \right\}$$

denote the set of all partitions of i into at most s parts, each of size at most r. Denote the multiplicity of j in $\gamma \in P_{r,s}(i)$ by $k_j(\gamma) := |\{1 \leqslant \ell \leqslant s \mid \gamma_\ell = j\}|$.

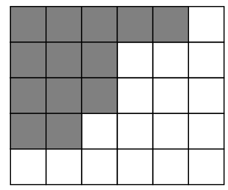


Figure 1. The partition $\gamma = (5, 3, 3, 2, 0) \in P_{6,5}(13)$

Example 6.2. Let $r = 6$, $s = 5$ and $i = 13$. Then $\gamma = (5, 3, 3, 2, 0) \in P_{6,5}(13)$ is a partition of 13 with at most 5 parts, each of size at most 6; see Figure 1. The multiplicities of the parts are $k_0(\gamma) = 1$, $k_1(\gamma) = 0$, $k_2(\gamma) = 1$, $k_3(\gamma) = 2$, $k_4(\gamma) = 0$, $k_5(\gamma) = 1$, and $k_6(\gamma) = 0$.

Recall $f_j(r)$ from Definition 5.3. The main result of this subsection is the following formula.

Theorem 6.3. *For every $s \geqslant 1$ and $i \geqslant 0$ we have*

$$e_{i,(r^s)} = \langle \psi^{(r^s)}, \chi^{(1^i)\oplus(rs-i)} \rangle = \sum_{\gamma \in P_{r,s}(i)} \prod_{j \geqslant 0} (-1)^{(j+1)k_j(\gamma)} \binom{(-1)^{j+1} f_j(r)}{k_j(\gamma)}$$

$$= \sum_{\substack{k_0,\dots,k_r \geqslant 0 \\ \sum_j k_j = s \\ \sum_j jk_j = i}} \prod_{j=0}^{r} (-1)^{(j+1)k_j} \binom{(-1)^{j+1} f_j(r)}{k_j}.$$

In particular, for $s = 1$ we have $e_{i,(r)} = f_i(r)$.

Remark 6.4. The special case $s = 1$ was stated (but not proved) in Proposition 5.5. The result in that case is not new, as noted in Remark 5.6. This case shows that $f_i(r) = e_{i,(r)}$ is an inner product of two characters, and is therefore always a non-negative integer. The factor

$$(-1)^{(j+1)k_j} \binom{(-1)^{j+1} f_j}{k_j} = \begin{cases} \binom{f_j}{k_j}, & \text{if } j \text{ is odd;} \\ \binom{f_j + k_j - 1}{k_j}, & \text{if } j \text{ is even} \end{cases}$$

is therefore also a non-negative integer, and is zero if and only if either j is odd and $k_j > f_j$, or j is even and $k_j > 0 = f_j$. If $k_j = 0$, this factor is equal to 1 and may be ignored.

In order to prove Theorem 6.3 we need a formula for a certain inner product of characters (Lemma 6.6).

First recall some notations from Definition 1.6: the centralizer $Z_{(r^s)} \cong \mathbb{Z}_r \wr S_s$ of an element of cycle type (r^s), the linear character $\omega^{(r^s)}$ on $Z_{(r^s)}$, and the higher Lie character $\psi^{(r^s)} := \omega^{(r^s)} \uparrow_{Z_{(r^s)}}^{S_n}$.

Embed $Z_{(r^s)} \cong \mathbb{Z}_r \wr S_s$ into $K_{r,s} \cong S_r \wr S_s \leqslant S_n$, where $\mathbb{Z}_r \leqslant S_r$ is generated by a full cycle. Denote

$$\phi_{r,s} := \omega^{(r^s)} \uparrow_{Z_{(r^s)}}^{K_{r,s}},$$

so that $\psi^{(r^s)} = \phi_{r,s} \uparrow_{K_{r,s}}^{S_n}$.

Observation 6.5. *If $s = s_1 + s_2$ then $Z_{(r^{s_1})} \times Z_{(r^{s_2})} \leqslant Z_{(r^s)}$, $K_{r,s_1} \times K_{r,s_2} \leqslant K_{r,s}$ and also for the characters*

$$\omega^{(r^s)} = \omega^{(r^{s_1})} \otimes \omega^{(r^{s_2})} \text{ and } \phi_{r,s} \downarrow_{K_{r,s_1} \times K_{r,s_2}}^{K_{r,s}} = \phi_{r,s_1} \otimes \phi_{r,s_2}.$$

In the lemma below we express the multiplicity of a certain linear character in a restriction of $\phi_{r,s}$. This expression will be used, in the proof of Theorem 6.3, to compute $e_{i,(r^s)}$.

For every $0 \leqslant j \leqslant r$, let $R_{r,j} := S_j \times S_{r-j} \leqslant S_r$, in the natural embedding. Then

$$R_{r,j} \wr S_s = K_{r,s} \cap \left(S_{js} \times S_{(r-j)s}\right).$$

Denote by 1_{S_n} the trivial character and by ε_{S_n} the sign character of S_n, so that

$$\chi^{(1^k)\oplus(n-k)} = (\varepsilon_{S_k} \times 1_{S_{n-k}}) \uparrow_{S_k \times S_{n-k}}^{S_n}.$$

Define

$$\nu_{r,j,s} := (\varepsilon_{S_{js}} \times 1_{S_{(r-j)s}}) \downarrow_{R_{r,j} \wr S_s}^{S_{js} \times S_{(r-j)s}}.$$

This is a linear character on $R_{r,j} \wr S_s$.

Lemma 6.6. *For every $0 \leqslant j \leqslant r$, $f_j(r)$ is a non-negative integer and*

$$\langle \phi_{r,s} \downarrow_{R_{r,j} \wr S_s}^{K_{r,s}}, \nu_{r,j,s} \rangle = (-1)^{(j+1)s} \binom{(-1)^{j+1} f_j(r)}{s} = \begin{cases} \binom{f_j(r)}{s}, & \text{if } j \text{ is odd;} \\ \binom{f_j(r)+s-1}{s}, & \text{if } j \text{ is even.} \end{cases}$$

Remark 6.7. As a byproduct, Lemma 6.6 provides a new proof of the non-negativity of $f_j(r)$. Indeed, if $s = 1$ then $\langle \phi_{r,1} \downarrow_{R_{r,j}}^{K_{r,1}}, \nu_{r,j,1} \rangle = f_j(r)$ is clearly a non-negative integer.

The rest of this subsection consists of the proofs of Lemma 6.6 and Theorem 6.3. In these proofs r, s and j are fixed, unless specified otherwise. For convenience, we omit the indices and write $Z := Z_{(r^s)}$, $\omega := \omega^{(r^s)}$, $\psi := \psi^{(r^s)}$, $K := K_{r,s}$, $\phi := \phi_{r,s}$, $R := R_{r,j}$, and $\nu := \nu_{r,j,s}$,

Proof of Lemma 6.6. The proof consists of two parts. First we determine the character values of the induced character $\phi = \omega \uparrow_Z^K$ on the wreath product $K = S_r \wr S_s$; the resulting formula is Equation (16). In the second part we apply this formula to compute the inner product.

Let $\zeta : \mathbb{Z}_r \to \mathbb{C}$ be the primitive linear character used to define ω; see Definition 1.6. Recall the explicit formula for an induced character [19, (5.1)]: For a subgroup $H \leqslant G$ and a character χ of H, define $\chi^0 : G \to \mathbb{C}$ by $\chi^0(g) = \chi(g)$ if $g \in H$ and $\chi^0(g) = 0$ otherwise. Then

$$\chi \uparrow_H^G (y) = \frac{1}{|H|} \sum_{x \in G} \chi^0(x^{-1}yx) = \sum_{t \in T} \chi^0(t^{-1}yt), \tag{15}$$

where T is a full set of right coset representatives of H in G.

An element of $K = S_r \wr S_s$ can be represented by an s-tuple of elements of S_r and a wreathing permutation from S_s, so $K = \{(x_1, \ldots, x_s; \sigma) \mid x_1, \ldots, x_s \in S_r,\ \sigma \in S_s\}$ with the product $(x_1, \ldots, x_s; \sigma)(y_1, \ldots, y_s; \tau) = (x_1 y_{\sigma^{-1}(1)}, \ldots, x_s y_{\sigma^{-1}(s)}; \sigma\tau)$. A full set of right coset representatives of $\mathbb{Z}_r$ in S_r is S_{r-1} (in the natural embedding). Hence, a full set of right coset representatives of $Z = \mathbb{Z}_r \wr S_s$ in $K = S_r \wr S_s$ is $T = \{(x_1, \ldots, x_s; 1) \mid (\forall i)\, x_i \in S_{r-1}\}$. For any $z_1, \ldots, z_s \in \mathbb{Z}_r$, the shifted set $(z_1, \ldots, z_s; 1)T$ is also a full set of right coset representatives. Instead of taking the sum over T in (15), we will take it over the union of all the shifted sets, namely $\{(x_1, \ldots, x_s; 1) | \, (\forall i)\, x_i \in S_r\}$, and divide by r^s. Since

$$(x_1, \ldots, x_s; 1)^{-1}(y_1, \ldots, y_s; \sigma)(x_1, \ldots, x_s; 1) = (x_1^{-1} y_1 x_{\sigma^{-1}(1)}, \ldots, x_s^{-1} y_s x_{\sigma^{-1}(s)}; \sigma),$$

we conclude that, for any $y = (y_1, \ldots, y_s; \sigma) \in K$,

$$\begin{aligned}\phi(y) = \omega \uparrow_Z^K (y_1, \ldots, y_s; \sigma) &= \frac{1}{r^s} \sum_{x_1,\ldots,x_s \in S_r} \omega^0(x_1^{-1} y_1 x_{\sigma^{-1}(1)}, \ldots, x_s^{-1} y_s x_{\sigma^{-1}(s)}; \sigma) \\ &= \frac{1}{r^s} \sum_{\substack{x_1,\ldots,x_s \in S_r \\ (\forall i)\, x_i^{-1} y_i x_{\sigma^{-1}(i)} \in \mathbb{Z}_r}} \omega(x_1^{-1} y_1 x_{\sigma^{-1}(1)}, \ldots, x_s^{-1} y_s x_{\sigma^{-1}(s)}; \sigma) \\ &= \frac{1}{r^s} \sum_{\substack{x_1,\ldots,x_s \in S_r \\ (\forall i)\, x_i^{-1} y_i x_{\sigma^{-1}(i)} \in \mathbb{Z}_r}} \prod_{i=1}^{s} \zeta(x_i^{-1} y_i x_{\sigma^{-1}(i)}).\end{aligned}$$

The factors in the product over i are complex numbers, thus commute. We can therefore rearrange them in an order fitting the decomposition of $\sigma^{-1} \in S_s$ into disjoint cycles: if

$$\sigma^{-1} = C_1 \cdots C_t$$

is a product of t disjoint cycles, choose an element a_k in each cycle C_k. Then

$$C_k = (a_k, \sigma^{-1}(a_k), \sigma^{-2}(a_k), \ldots) \qquad (1 \leqslant k \leqslant t).$$

Since ζ is a linear character, cancellation gives

$$\begin{aligned}\prod_{i \in C_k} \zeta(x_i^{-1} y_i x_{\sigma^{-1}(i)}) &= \zeta(x_{a_k}^{-1} y_{a_k} x_{\sigma^{-1}(a_k)}) \zeta(x_{\sigma^{-1}(a_k)}^{-1} y_{\sigma^{-1}(a_k)} x_{\sigma^{-2}(a_k)}) \cdots \\ &= \zeta(x_{a_k}^{-1} c_k x_{a_k}),\end{aligned}$$

where

$$c_k := y_{a_k} y_{\sigma^{-1}(a_k)} \cdots \in S_r \qquad (1 \leqslant k \leqslant t).$$

The condition $x_i^{-1} y_i x_{\sigma^{-1}(i)} \in \mathbb{Z}_r$ $(\forall i)$ implies that the products $x_{a_k}^{-1} c_k x_{a_k} \in \mathbb{Z}_r$ $(\forall k)$. Hence if $\phi(y) \neq 0$ then, necessarily, each cycle-product $c_k \in S_r$ must be conjugate to an element of $\mathbb{Z}_r$. Since $\mathbb{Z}_r \leqslant S_r$ is generated by a full cycle, a necessary and sufficient condition for c_k to have a conjugate in $\mathbb{Z}_r$ is that it is a product of disjoint cycles of the same length.

For any divisor d of r, if $c_k \in S_r$ is a product of disjoint d-cycles and $x_{a_k} \in S_r$ is such that $x_{a_k}^{-1} c_k x_{a_k} \in \mathbb{Z}_r$, then the value of $\zeta(x_{a_k}^{-1} c_k x_{a_k})$ is a primitive d-th root of unity. By varying the conjugating element x_{a_k}, each element of $\mathbb{Z}_r$ of order d is obtained with the same multiplicity $|Z_{(d^{r/d})}| = (r/d)! d^{r/d}$. The other x_i's, for $i \in C_k \setminus \{a_k\}$, are arbitrary, as long as $x_i^{-1} y_i x_{\sigma^{-1}(i)} \in \mathbb{Z}_r$ $(\forall i)$. There are $r^{\ell_k - 1}$ such choices, where ℓ_k is the length of the cycle C_k. We conclude that, for any $y \in K$ for which $c_k \in S_r$ is a product of disjoint d_k-cycles $(1 \leqslant k \leqslant t)$,

$$\phi(y) = \frac{1}{r^s} \prod_{k=1}^{t} (r/d_k)! d_k^{r/d_k} r^{\ell_k - 1} \sum_{z \in \mathbb{Z}_r \,:\, o(z) = d_k} \zeta(z).$$

If g is a generator of $\mathbb{Z}_r$, then $o(g^m) = d$ if and only if $m = jr/d$ for some integer j coprime to d. It follows that

$$\sum_{z \in \mathbb{Z}_r \,:\, o(z) = d} \zeta(z) = \sum_{0 \leqslant j < d \,:\, (j,d) = 1} \zeta(g^{jr/d}) = \mu(d).$$

Since $\sum_{k=1}^{t} (\ell_k - 1) = s - t$, we now have an explicit formula for the values of ϕ:

$$\phi(y) = \begin{cases} \displaystyle\prod_{k=1}^{t} \frac{\mu(d_k) d_k^{r/d_k} (r/d_k)!}{r}, & \text{if } c_k \text{ is a product of disjoint } d_k\text{-cycles } (\forall k); \\ 0, & \text{otherwise.} \end{cases} \tag{16}$$

To determine the inner product $\langle \phi \downarrow^K_{R \wr S_s}, \nu\rangle$ we evaluate the linear character ν on $R \wr S_s$. Let $y = (v_1 z_1, \ldots, v_s z_s; \sigma) \in R \wr S_s$, where $v_i \in S_j, z_i \in S_{r-j}$ $(1 \leqslant i \leqslant s)$ and $\sigma \in S_s$. Then, by the definition of ν,

$$\nu(y) = \nu(v_1 z_1, \ldots, v_s z_s; \sigma) = \operatorname{sgn}(\sigma)^j \prod_{i=1}^{s} \varepsilon(v_i)$$

where $\operatorname{sgn}(\sigma)$ denotes the sign of $\sigma \in S_s$. We obtain

$$\begin{aligned}\langle \phi \downarrow^K_{R \wr S_s}, \nu\rangle &= \frac{1}{|R \wr S_s|} \sum_{(v_1 z_1, \ldots, v_s z_s; \sigma) \in R \wr S_s} \phi(v_1 z_1, \ldots, v_s z_s; \sigma) \nu(v_1 z_1, \ldots, v_s z_s; \sigma) \\ &= \frac{1}{s!} \sum_{\sigma \in S_s} \frac{\operatorname{sgn}(\sigma)^j}{(j!(r-j)!)^s} \sum_{\substack{v_1, \ldots, v_s \in S_j \\ z_1, \ldots, z_s \in S_{r-j}}} \phi(v_1 z_1, \ldots, v_s z_s; \sigma) \prod_{i=1}^{s} \varepsilon(v_i).\end{aligned}$$

By Equation (16), for each nonzero summand and each cycle C_k of σ^{-1} $(1 \leqslant k \leqslant t)$, the cycle-product $c_k \in R \leqslant S_r$ of y has cycle type d_k^{r/d_k}, and its restrictions to S_j and S_{r-j} have cycle types d_k^{j/d_k} and $d_k^{(r-j)/d_k}$, respectively. In particular, $d_k | (r, j)$. It follows that the sign

$$\prod_{i=1}^{s} \varepsilon(v_i) = \prod_{k=1}^{t} (-1)^{(d_k+1)j/d_k}.$$

For any $d | (r, j)$, let n_d denote the number of elements of R which are products of disjoint d-cycles. If C_k has length ℓ_k, then the cycle-product c_k can be a product of disjoint d_k-cycles in exactly $(j!(r-j)!)^{\ell_k - 1} n_{d_k}$ ways. The choices of d_k for different cycles C_k are independent; the only restriction is $d_k | (r, j)$. Using again the equality $\sum_{k=1}^{t} (\ell_k - 1) = s - t$ and Equation (16), we obtain

$$\begin{aligned}\langle \phi \downarrow^K_{R \wr S_s}, \nu\rangle &= \frac{1}{s!} \sum_{\sigma \in S_s} \frac{\operatorname{sgn}(\sigma)^j}{(j!(r-j)!)^s} \sum_{\substack{v_1, \ldots, v_s \in S_j \\ z_1, \ldots, z_s \in S_{r-j}}} \phi(v_1 z_1, \ldots, v_s z_s; \sigma) \prod_{i=1}^{s} \varepsilon(v_i) \\ &= \frac{1}{s!} \sum_{\sigma \in S_s} \frac{\operatorname{sgn}(\sigma)^j}{(j!(r-j)!)^s} \sum_{d_1, \ldots, d_t | (r,j)} \\ &\qquad \prod_{k=1}^{t} \frac{(j!(r-j)!)^{\ell_k - 1} n_{d_k} \mu(d_k) d_k^{r/d_k} (r/d_k)! (-1)^{(d_k+1)j/d_k}}{r} \\ &= \frac{1}{s!} \sum_{\sigma \in S_s} \frac{\operatorname{sgn}(\sigma)^j}{(j!(r-j)!)^t} \sum_{d_1, \ldots, d_t | (r,j)} \prod_{k=1}^{t} \frac{n_{d_k} \mu(d_k) d_k^{r/d_k} (r/d_k)! (-1)^{(d_k+1)j/d_k}}{r} \\ &= \frac{1}{s!} \sum_{\sigma \in S_s} \frac{\operatorname{sgn}(\sigma)^j}{(j!(r-j)!)^t} \left(\sum_{d | (r,j)} \frac{n_d \mu(d) d^{r/d} (r/d)! (-1)^{(d+1)j/d}}{r} \right)^t.\end{aligned}$$

Of course, the number of elements of $R = S_j \times S_{r-j}$ which are products of disjoint d-cycles is

$$n_d = \frac{j!}{d^{j/d} (j/d)!} \cdot \frac{(r-j)!}{d^{(r-j)/d} ((r-j)/d)!} = \frac{j!(r-j)!}{d^{r/d} (j/d)! ((r-j)/d)!}.$$

Putting everything together and recalling that $t = \mathrm{cyc}(\sigma)$ is the number of cycles of σ, we obtain by Definition 5.3

$$\begin{aligned}\langle \phi\downarrow^K_{R\wr S_s}, \nu\rangle &= \frac{1}{s!}\sum_{\sigma\in S_s}\mathrm{sgn}(\sigma)^j\left(\sum_{d\mid(r,j)}\frac{\mu(d)(-1)^{(d+1)j/d}}{r}\binom{r/d}{j/d}\right)^{\mathrm{cyc}(\sigma)}\\ &= \frac{1}{s!}\sum_{\sigma\in S_s}\mathrm{sgn}(\sigma)^j f_j(r)^{\mathrm{cyc}(\sigma)}.\end{aligned}$$

In particular, if $s = 1$ then $f = \langle \phi\downarrow^K_R, \nu\rangle$ is a non-negative integer.

Finally, by [32, Proposition 1.3.4], for any $s \geqslant 0$ and indeterminate x,

$$\frac{1}{s!}\sum_{\sigma\in S_s} x^{\mathrm{cyc}(\sigma)} = \binom{x+s-1}{s}.$$

Substituting $-x$ for x and noting that $\mathrm{sgn}(\sigma) = (-1)^{s-\mathrm{cyc}(\sigma)}$, we get

$$\frac{1}{s!}\sum_{\sigma\in S_s}\mathrm{sgn}(\sigma)x^{\mathrm{cyc}(\sigma)} = \binom{x}{s}.$$

This yields the desired formula, depending on the parity of j, for $\langle \phi\downarrow^K_{R\wr S_s}, \nu\rangle$. □

To prove Theorem 6.3 we need a final ingredient, a combinatorial parametrization of $(S_r \wr S_s, S_i \times S_{rs-i})$ double cosets of S_{rs} by partitions.

Recall Definition 6.1. The idea and definition of $P_{r,s}(i)$ actually appear already in the work of Giannelli [16], in a similar context but without explicit reference to double cosets or Mackey's formula; see Definitions 2.8–2.10 and Proposition 2.11 there.

An example of a partition of 13 representing a certain $(S_6 \wr S_5, S_{13} \times S_{17})$ double coset appears in Figure 1.

Lemma 6.8. *Let $n = rs$, $K = K_{r,s} \cong S_r \wr S_s \leqslant S_n$. There is a bijection between the $(K, S_i \times S_{n-i})$ double cosets of S_n and $P_{r,s}(i)$, the set of partitions of i into at most s parts, all of size at most r.*

Proof. To describe the bijection from $K\backslash S_n/(S_i \times S_{n-i})$ to $P_{r,s}(i)$ explicitly, first fix the underlying decomposition $\{1,\dots,n\} = \{1,\dots,i\} \cup \{i+1,\dots,n\}$ for the action of $S_i \times S_{n-i}$. The left cosets in $S_n/(S_i \times S_{n-i})$ are clearly in bijection with the subsets of size i in $\{1,\dots,n\}$:

$$g(S_i \times S_{n-i}) \longleftrightarrow g(\{1,\dots,i\}).$$

Now fix a decomposition for the action of $K \cong S_r \wr S_s$: $\{1,\dots,n\} = B_1 \cup \ldots \cup B_s$, where $B_j := \{(j-1)r+1, (j-1)r+2, \dots, (j-1)r+r\}$ $(j = 1,\dots,s)$. The elements of S_s permute these blocks, and each of the s copies of S_r acts on one of the blocks. Given $g \in S_n$, we map the double coset $Kg(S_i \times S_{n-i})$ to the partition γ which is the non-increasing rearrangement of the sequence

$$(|B_1 \cap g(\{1,\dots,i\})|, \dots, |B_s \cap g(\{1,\dots,i\})|).$$

This sequence consists of s non-negative integers, each at most r, which sum up to i. Thus $\gamma \in P_{r,s}(i)$. We will show that this map is a bijection.

For arbitrary $x \in K$ and $y \in S_i \times S_{n-i}$ we have, for each $1 \leqslant j \leqslant s$,

$$|B_j \cap xgy(\{1,\dots,i\})| = |B_j \cap xg(\{1,\dots,i\})| = \left|x^{-1}(B_j) \cap g(\{1,\dots,i\})\right|.$$

The element $x^{-1} \in K = S_r \wr S_s$ permutes the blocks and permutes the elements of each block. This shows that the mapping $Kg(S_i \times S_{n-i}) \mapsto \gamma$ is well defined.

The mapping from $K\backslash S_n/(S_i \times S_{n-i})$ to $P_{r,s}(i)$ is clearly onto, since for each partition $\gamma = (a_1, \dots, a_s) \in P_{r,s}(i)$ there exists a permutation $g \in S_n$ such that $|B_j \cap g(\{1, \dots, i\})| = a_j$ $(1 \leqslant j \leqslant s)$.

Finally, if $Kg_1(S_i \times S_{n-i})$ and $Kg_2(S_i \times S_{n-i})$ are mapped to the same partition γ, then there exists a permutation $\pi \in S_s$ satisfying

$$|B_j \cap g_1(\{1, \dots, i\})| = \left|B_{\pi(j)} \cap g_2(\{1, \dots, i\})\right| \qquad (1 \leqslant j \leqslant s).$$

Therefore there exist an element $x \in K = S_r \wr S_s$ such that

$$B_j \cap g_1(\{1, \dots, i\}) = B_j \cap xg_2(\{1, \dots, i\}) \qquad (1 \leqslant j \leqslant s).$$

It follows that

$$g_1(\{1, \dots, i\}) = xg_2(\{1, \dots, i\}),$$

and therefore $g_1 = xg_2y$ for a suitable permutation $y \in S_i \times S_{n-i}$. □

We are now ready to prove Theorem 6.3. The proof applies Lemma 6.8 and the explicit bijection described in its proof, combined with Lemma 6.6.

Proof of Theorem 6.3. Recall that $n = rs$, $K = K_{r,s} \cong S_r \wr S_s \leqslant S_n$ and $\psi = \phi \uparrow_K^{S_n}$. By Frobenius reciprocity (twice) and Mackey's formula [19, (5.2), Problem (5.6)],

$$\begin{aligned} e_i &= \langle \psi, \chi^{(1^i) \oplus (n-i)} \rangle = \langle \phi \uparrow_K^{S_n}, (\varepsilon_{S_i} \times 1_{S_{n-i}}) \uparrow_{S_i \times S_{n-i}}^{S_n} \rangle \\ &= \langle \phi \uparrow_K^{S_n} \downarrow_{S_i \times S_{n-i}}^{S_n}, \varepsilon_{S_i} \times 1_{S_{n-i}} \rangle \\ &= \sum_{[g] \in K\backslash S_n/(S_i \times S_{n-i})} \langle \phi^g \downarrow_{K^g \cap (S_i \times S_{n-i})}^{K^g} \uparrow_{K^g \cap (S_i \times S_{n-i})}^{S_i \times S_{n-i}}, \varepsilon_{S_i} \times 1_{S_{n-i}} \rangle \\ &= \sum_{[g] \in K\backslash S_n/(S_i \times S_{n-i})} \langle \phi^g \downarrow_{K^g \cap (S_i \times S_{n-i})}^{K^g}, (\varepsilon_{S_i} \times 1_{S_{n-i}}) \downarrow_{K^g \cap (S_i \times S_{n-i})}^{S_i \times S_{n-i}} \rangle. \end{aligned} \tag{17}$$

The above sums are indexed by the $(K, S_i \times S_{n-i})$ double cosets in S_n. For each representative g of a double coset, $K^g := g^{-1}Kg$ is the corresponding conjugate of K and ϕ^g is the character on K^g defined by $\phi^g(g^{-1}kg) := \phi(k)$ for all $k \in K$.

By Lemma 6.8, these double cosets are parametrized by the partitions in $P_{r,s}(i)$. Let us determine the summand of (17) corresponding to a partition $\gamma \in P_{r,s}(i)$ in which part j occurs with multiplicity $k_j = k_j(\gamma)$ $(0 \leqslant j \leqslant r)$. By the bijection described in the proof of Lemma 6.8,

$$k_j = |\{t : |B_t \cap g(\{1, \dots, i\})| = j\}| \qquad (0 \leqslant j \leqslant r),$$

where $B_t := \{(t-1)r + 1, \dots, (t-1)r + r\}$ $(1 \leqslant t \leqslant s)$. Thus

$$K^g \cap (S_i \times S_{n-i}) \cong (R_{r,0} \wr S_{k_0}) \times (R_{r,1} \wr S_{k_1}) \times \cdots \times (R_{r,r} \wr S_{k_r}),$$

where $R_{r,j} = S_j \times S_{r-j}$, as above. In particular,

$$(\varepsilon_{S_i} \times 1_{S_{n-i}}) \downarrow_{K^g \cap (S_i \times S_{n-i})}^{S_i \times S_{n-i}} = \bigotimes_j (\varepsilon_{S_{jk_j}} \times 1_{S_{(r-j)k_j}}) \downarrow_{R_{r,j} \wr S_{k_j}}^{S_{jk_j} \times S_{(r-j)k_j}} = \bigotimes_j \nu_{r,j,k_j}.$$

Note that, by Observation 6.5, $\phi_{r,s}$ factors similarly. Therefore the corresponding summand in (17) is

$$\langle \phi^g \downarrow_{K^g \cap (S_i \times S_{n-i})}^{K^g}, (\varepsilon_{S_i} \times 1_{S_{n-i}}) \downarrow_{K^g \cap (S_i \times S_{n-i})}^{S_i \times S_{n-i}} \rangle = \prod_{j=0}^{r} \langle \phi_{r,k_j} \downarrow_{R_{r,j} \wr S_{k_j}}^{S_r \wr S_{k_j}}, \nu_{r,j,k_j} \rangle. \tag{18}$$

We have already computed these inner products in Lemma 6.6. By (17), (18), Lemma 6.6 and Definition 5.3 we obtain

$$e_{i,(r^s)} = \langle \phi \uparrow_K^{S_n}, (\varepsilon_{S_i} \times 1_{S_{n-i}}) \uparrow_{S_i \times S_{n-i}}^{S_n} \rangle = \sum_{\gamma \in P_{r,s}(i)} \prod_{j=0}^{r} (-1)^{(j+1)k_j(\gamma)} \binom{(-1)^{j+1} f_j(r)}{k_j(\gamma)},$$

as claimed. If $s = 1$ then $P_{r,1}(i)$ contains (for $0 \leqslant i \leqslant r$) a unique partition $\gamma = (i)$, for which $k_i(\gamma) = 1$ is the unique nonzero multiplicity. Thus, in this case,

$$e_{i,(r)} = (-1)^{i+1}\binom{(-1)^{i+1}f_i(r)}{1} = f_i(r). \qquad \square$$

6.2. A product formula. Now we derive a restatement of Theorem 6.3 as a product formula for formal power series.

Corollary 6.9. *For a positive integer r, define the formal power series*

$$E_r(x,y) := \sum_{i,s\geqslant 0} e_{i,(r^s)} x^i y^s.$$

Then

$$E_r(x,y) = \prod_{j=0}^{r} (1-(-x)^j y)^{(-1)^{j+1} f_j(r)}.$$

Proof. Recall the following formal power series expansion, valid for any integer f:

$$(1+t)^f = \sum_{n=0}^{\infty} \binom{f}{n} t^n.$$

By Theorem 6.3, with the obvious extension for $s = 0$,

$$\begin{aligned} E_r(x,y) &= \sum_{i,s\geqslant 0} \left(\sum_{\substack{k_0,\ldots,k_r \geqslant 0 \\ \sum_j k_j = s \\ \sum_j jk_j = i}} \prod_{j=0}^{r} (-1)^{(j+1)k_j} \binom{(-1)^{j+1} f_j(r)}{k_j} \right) x^i y^s \\ &= \prod_{j=0}^{r} \sum_{k_j=0}^{\infty} (-1)^{(j+1)k_j} \binom{(-1)^{j+1} f_j(r)}{k_j} x^{jk_j} y^{k_j} \\ &= \prod_{j=0}^{r} (1+(-1)^{j+1} x^j y)^{(-1)^{j+1} f_j(r)}, \end{aligned}$$

as required. $\square$

Remark 6.10. For small r and arbitrary s, Corollary 6.9 enables us to determine explicitly the hook-multiplicities $m_{i,(r^s)}$. This is done by recalling Equation (9) and the fact that, by definition, $e_{i,(r^s)}$ is the coefficient of $x^i y^s$ in $E_r(x,y)$. For example, by Corollary 6.9 and Observation 5.4, $E_2(x,y) = E_3(x,y) = (1+xy)(1-x^2y)^{-1}$. Thus, for $r \in \{2,3\}$ and any $s \geqslant 1$, the value of $e_{i,(r^s)}$ is 1 for $i \in \{2s-1, 2s\}$ and zero otherwise. Combining this with Equation (9), it follows that, for $r \in \{2,3\}$ and $s \geqslant 1$, the hook multiplicity $m_{i,(r^s)}$ is 1 for $i = 2s-1$ and zero otherwise.

6.3. Non-negativity. Now we are ready to prove Theorem 1.10.

Proof of Theorem 1.10. Assume that r is not square-free. Recall, from Definition 5.7, the notation $F_r(x) := \sum_{j=0}^{r} f_j(r) x^j$. By Proposition 5.1 and Corollary 5.8 we may write $F_r(x) = (1+x)^2 G_r(x)$, where $G_r(x) = \sum_{j=0}^{r-2} g_j(r) x^j$ is a polynomial with non-negative integer coefficients. Let $g_j(r) := 0$ for $j < 0$ or $j > r-2$. Then

$$f_j(r) = g_j(r) + 2g_{j-1}(r) + g_{j-2}(r) \qquad (\forall j).$$

Therefore, by Corollary 6.9,

$$\begin{aligned}E_r(x,y) &= \prod_{j\geqslant 0}(1-(-x)^j y)^{(-1)^{j+1}f_j(r)}\\ &= \prod_{j\geqslant 0}(1-(-x)^j y)^{(-1)^{j+1}(g_j(r)+2g_{j-1}(r)+g_{j-2}(r))}\\ &= \prod_{j\geqslant 0}\left(\frac{(1-(-x)^j y)(1-(-x)^{j+2}y)}{(1-(-x)^{j+1}y)^2}\right)^{(-1)^{j+1}g_j(r)}.\end{aligned}$$

We claim that each factor in this product has the form $1+(1+x)^2 p_j(x,y)$, with $p_j(x,y)$ a formal power series with non-negative integer coefficients. This implies that $(E_r(x,y)-1)/(1+x)^2$ is itself a formal power series with non-negative integer coefficients, completing the proof of Theorem 1.10.

Indeed, if j is odd then the corresponding factor is

$$\left(\frac{(1+x^j y)(1+x^{j+2}y)}{(1-x^{j+1}y)^2}\right)^{g_j(r)} = \left(1+\frac{(x+1)^2 x^j y}{(1-x^{j+1}y)^2}\right)^{g_j(r)},$$

where

$$\frac{x^j y}{(1-x^{j+1}y)^2} = x^j y \cdot \sum_{i\geqslant 0}(i+1)(x^{j+1}y)^i$$

is a formal power series with non-negative integer coefficients.

Finally, if j is even then the corresponding factor is

$$\left(\frac{(1+x^{j+1}y)^2}{(1-x^j y)(1-x^{j+2}y)}\right)^{g_j(r)} = \left(1+\frac{(x+1)^2 x^j y}{(1-x^j y)(1-x^{j+2}y)}\right)^{g_j(r)},$$

where

$$\frac{x^j y}{(1-x^j y)(1-x^{j+2}y)} = x^j y \cdot \sum_{i\geqslant 0}(x^j y)^i \cdot \sum_{k\geqslant 0}(x^{j+2}y)^k$$

is a formal power series with non-negative integer coefficients. □

7. Additional results

7.1. Combinatorial identities. In this subsection it will be shown that Lemma 3.5 and Theorem 6.3 imply well-known combinatorial identities.

Recall the *major index* of a permutation $\pi \in S_n$,

$$\mathrm{maj}(\pi) := \sum_{i\in \mathrm{Des}(\pi)} i.$$

The following identity is due to Garsia [13]. A purely combinatorial proof was given by Wachs [40].

Proposition 7.1 ([13, Equation 5.8]). *For every partition $\lambda \vdash n$,*

$$\sum_{\pi\in\mathcal{C}_\lambda}\zeta^{\mathrm{maj}(\pi)} = \begin{cases}\mu(r), & \text{if } \lambda=(r^s);\\ 0, & \text{otherwise},\end{cases}$$

where ζ is a primitive n-th root of unity and μ is the Möbius function.

The following lemma follows from the work of Stembridge [34].

Lemma 7.2. *For every Schur-positive set $\mathcal{A} \subseteq S_n$ with associated S_n-character $\phi := \mathrm{ch}^{-1}(\mathcal{Q}(\mathcal{A}))$, the value of ϕ at an n-cycle $c \in S_n$ is*

$$\phi(c) = \sum_{\pi \in \mathcal{A}} \zeta^{\mathrm{maj}(\pi)},$$

where ζ is a primitive n-th root of unity.

Proof. First, recall the definition of the descent set of standard Young tableaux (SYT) from Equation (3). By [34, Lemma 3.4], for every partition $\nu \vdash n$ the value of the irreducible S_n-character χ^ν at an n-cycle $c \in S_n$ is

$$\chi^\nu(c) = \sum_{T \in \mathrm{SYT}(\nu)} \zeta^{\mathrm{maj}(T)}. \tag{19}$$

Let $\mathcal{A} \subseteq S_n$ be Schur-positive with associated S_n-character ϕ, i.e., $\mathcal{Q}(A) = \mathrm{ch}(\phi)$. By Lemma 2.3 together with Equation (19),

$$\sum_{\pi \in \mathcal{A}} \zeta^{\mathrm{maj}(\pi)} = \sum_{\nu \vdash n} \langle \mathcal{Q}(\mathcal{A}), s_\nu \rangle \sum_{T \in \mathrm{SYT}(\nu)} \zeta^{\mathrm{maj}(T)} = \sum_{\nu \vdash n} \langle \phi, \chi^\nu \rangle \chi^\nu(c) = \phi(c),$$

completing the proof. □

In light of Lemma 7.2 we deduce the following.

Corollary 7.3. *Lemma 3.5 is equivalent to Garsia's identity (Proposition 7.1).*

Proof. By the Gessel–Reutenauer Theorem (Theorem 2.10), for every $\lambda \vdash n$, the conjugacy class of cycle type λ is Schur-positive with $\mathrm{ch}^{-1}(\mathcal{Q}(\mathcal{C}_\lambda)) = \psi^\lambda$. Letting $\mathcal{A} = \mathcal{C}_\lambda$ in Lemma 7.2, Proposition 7.1 implies Lemma 3.5 and vice versa. □

There is a combinatorial description, due to Schocker, of the multiplicity of an arbitrary irreducible character of S_n in the higher Lie character. In its full generality it is too complicated to be presented here, see [29] for the details. The special case of a full cycle, $\lambda = (n)$ (for which $\psi^{(n)} = \omega^{(n)} \uparrow_{\mathbb{Z}_n}^{S_n}$ is the Lie character), is due to Kraśkiewicz and Weyman [21].

Theorem 7.4 (Kraśkiewicz–Weyman, see [13, Theorem 8.4]). *For every partition $\nu \vdash n$, the multiplicity $m_{\nu,(n)} := \langle \psi^{(n)}, \chi^\nu \rangle$ is equal to the cardinality of the set*

$$\{T \in \mathrm{SYT}(\nu) : \ \mathrm{maj}(T) \equiv 1 \pmod n\}.$$

Corollary 7.5. *For every $0 \leqslant k \leqslant n$, the multiplicity $m_{k,(n)} := \langle \psi^{(n)}, \chi^{(n-k,1^k)} \rangle$ is equal to the cardinality of the set*

$$\left\{ 1 \leqslant a_1 < \cdots < a_k \leqslant n-1 : \sum_{i=1}^k a_i \equiv 1 \ (\mathrm{mod}\ n) \right\}.$$

Proof. The map $\mathrm{Des} : \mathrm{SYT}(n-k, 1^k) \to \binom{[n-1]}{k}$ is a bijection, where $\binom{[n-1]}{k}$ denotes the set of all k-subsets of $[n-1]$. The major index of a tableau is the sum of the elements of its descent set. □

Consider the following combinatorial identity.

Proposition 7.6. *For every $0 \leqslant k \leqslant n$,*

$$\left| \left\{ 1 \leqslant a_1 < \cdots < a_k \leqslant n : \sum_{i=1}^k a_i \equiv 1 \ (\mathrm{mod}\ n) \right\} \right| = \sum_{d | (n,k)} \frac{\mu(d)(-1)^{(d+1)k/d}}{n} \binom{n/d}{k/d}.$$

Proof. Let $H(x,q) := \prod_{i=1}^{n}(1+xq^i)$. Writing

$$H(x,q) \equiv \sum_{k=0}^{n}\sum_{t=0}^{n-1} c_{k,t}x^k q^t \mod (q^n-1),$$

the cardinality we are interested in is the coefficient $c_{k,1}$ of $x^k q$. For any $d|n$ and η a primitive d-th root of unity (we write $o(\eta)=d$),

$$H(x,\eta) = \prod_{i=1}^{n}(1+x\eta^i) = \left(\prod_{i=1}^{d}(1+x\eta^i)\right)^{n/d} = (1-(-x)^d)^{n/d},$$

where the last equality holds since both sides are polynomials of the same degree, with exactly the same roots and the same constant term.

Let ω be a primitive n-th root of unity. Then

$$H(x,\omega^j) = \sum_{t=0}^{n-1} h_t(x)\omega^{jt} \qquad (0 \leqslant j \leqslant n-1),$$

where $h_t(x) = \sum_{k=0}^{n} c_{k,t}x^k$. Fourier inversion gives

$$\begin{aligned}\sum_{k=0}^{n} c_{k,1}x^k = h_1(x) &= \frac{1}{n}\sum_{j=0}^{n-1} H(x,\omega^j)\omega^{-j} \\ &= \frac{1}{n}\sum_{d|n}(1-(-x)^d)^{n/d} \sum_{j\,:\,o(\omega^j)=d} \omega^{-j} \\ &= \frac{1}{n}\sum_{d|n}(1-(-x)^d)^{n/d}\mu(d) \\ &= \frac{1}{n}\sum_{k=0}^{n}\sum_{d|(n,k)}\binom{n/d}{k/d}(-1)^{(d+1)k/d}x^k\mu(d),\end{aligned}$$

as required. □

Observation 7.7. *Proposition* 5.5 *is equivalent to Proposition* 7.6*.*

Proof. By considering separately the cases $a_k < n$ and $a_k = n$, Corollary 7.5 yields

$$\left|\left\{1 \leqslant a_1 < \cdots < a_k \leqslant n : \sum_{i=1}^{k} a_i \equiv 1 \ (\mathrm{mod}\ n)\right\}\right| = m_{k,(n)} + m_{k-1,(n)} = e_{k,(n)}.$$

By Proposition 5.5 and Definition 5.3,

$$e_{k,(n)} = f_k(n) = \sum_{d|(n,k)} \frac{\mu(d)(-1)^{(d+1)k/d}}{n}\binom{n/d}{k/d},$$

proving Proposition 7.6. The opposite direction is similar. □

Remark 7.8. Noting that Proposition 5.5 is the special case $s=1$ of Theorem 6.3, one concludes that Proposition 7.6 is a consequence of the latter.

It remains a challenge to find such a direct link between Schocker's general description of the multiplicity and our version in Theorem 6.3.

Remark 7.9. Proposition 5.5 is not new. For example, by the Gessel–Reutenauer Theorem (Theorem 2.10) together with Observation 3.1, Theorem 6.3 at $s = 1$ is equivalent to the equation

$$|\{\pi \in \mathcal{C}_{(n)} :\ \mathrm{Des}(\pi) = [j]\}| = \frac{1}{n} \sum_{d|n} \mu(d)(-1)^{j-\lfloor j/d \rfloor} \binom{n-1}{j-1} \qquad (0 \leqslant j < n),$$

which is an immediate consequence of a recent result of Elizalde and Troyka [12, Theorem 3.1]. An older proof was presented to us by Sheila Sundaram [37], deducing Proposition 5.5 from [35, Lemma 2.7]. The reader is referred to [12] for further discussion and relations to the enumeration of Lyndon words.

7.2. Cellini's cyclic descents. In this subsection it is shown that the apparently natural approach does not provide a cyclic descent extension for many conjugacy classes in S_n.

Recall the original notion of cyclic descent set defined by Cellini [8],

$$\mathrm{CDes}(\pi) := \{1 \leqslant i \leqslant n \,:\, \pi_i > \pi_{i+1}\} \qquad (\forall \pi \in S_n),$$

with the convention $\pi_{n+1} := \pi_1$.

Elizalde and Roichman [6] presented several subsets of S_n on which the image of Cellini's cyclic descent map is closed under cyclic rotation, thus leading to a cyclic extension of Des. However, as we shall see, many conjugacy classes do not have this property. In fact, we conjecture that only two conjugacy classes have this property. Here are some partial results.

Proposition 7.10. *For $n = rs > 1$, the image of Cellini's cyclic descent map on any conjugacy class of cycle type (r^s) is not closed under cyclic rotation.*

Proof. Recall the notation $\mathcal{C}_\lambda$ from Section 2. For $r = 1$ and $n = s > 1$, $\mathcal{C}_{(1^n)}$ consists of the identity permutation only. Cellini's $\mathrm{CDes}(id)$ is the singleton $\{n\}$ and, for $n > 1$, the set $\{\{n\}\}$ is not closed under cyclic rotation. For $r > 1$ (and $s \geqslant 1$), let $\sigma = [s+1, s+2, \ldots, n, 1, 2, \ldots, s]$. In other words, σ is the permutation in S_n defined by

$$\sigma(i) = i + s \pmod n \qquad (\forall i \in [n]).$$

Then $\sigma \in \mathcal{C}_{(r^s)}$, and Cellini's $\mathrm{CDes}(\sigma)$ is the singleton $\{n-s\}$. If the image of CDes on $\mathcal{C}_{(r^s)}$ is invariant under cyclic rotation then there is a permutation $\pi \in \mathcal{C}_{(r^s)}$ with $\mathrm{CDes}(\pi) = \{n\}$. The only permutation in S_n with this property is the identity permutation, which is not in $\mathcal{C}_{(r^s)}$. This is a contradiction. ☐

Proposition 7.11. *For $n > 1$, the image of Cellini's cyclic descent map on any conjugacy class of k-cycles in S_n, except 2-cycles in S_3 and 3-cycles in S_4, is not closed under cyclic rotation.*

Proof. Letting $r = n$ in Proposition 7.10, statement holds on n-cycles. For $k < n$ let $\sigma \in \mathcal{C}_{(k,1^{n-k})}$ be the permutation $[k, 1, 2, \ldots, k-1, k+1, k+2, \ldots, n]$. Then $\mathrm{CDes}(\sigma) = \{1, n\}$. By the equivariance property, there must be a k-cycle π with cyclic descent set $\{1, 2\}$. Then $\pi(3)$ is the minimal value thus it is equal to 1, and $\pi(1)$ is the maximal value thus it is equal to n. Let $\pi(2) = x$, thus

$$\pi = [n, x, 1, 2, \ldots, x-1, x+1, \ldots, n-1],$$

namely, for every $3 < i \leqslant x+1$, $\pi(i) = i-2$ and for every $x+1 < i \leqslant n$, $\pi(i) = i-1$. Thus π has no fixed points unless $x = 2$, in this case π has cycle type $(n-1, 1)$. One deduces that statement holds for $k < n-1$.

For the $(n-1)$-cycles an argument similar to the above works. For $n > 4$ the permutation $\sigma = (1, n-1, n-2, \ldots, 5, 4, 2, 3)(n)$, i.e., $[n-1, 3, 1, 2, 4, 5, \ldots, n-2, n] \in$

$\mathcal{C}_{(n-1,1)}$ has cyclic descent set $\mathrm{CDes}(\sigma) = \{1, 2, n\}$. By the equivariance property, there is a permutation $\pi \in \mathcal{C}_{(n-1,1)}$ with cyclic descent set $\{1, n-1, n\}$. Then $\pi(2)$ is the minimal value thus it is equal to 1, and $\pi(n-1)$ is the maximal value thus it is equal to n. If $\pi(n) = l$ and $\pi(1) = k$ then $1 < k < l < n$ and $\pi = [k, 1, \ldots, k-1, k+1, ..., l-1, l+1, \ldots, n, l]$, so the cycle $(1, k, k-1, \ldots, 2)$ of π has length $2 \leqslant k < n-1$, contradicting the equivariance property. □

Conjecture 7.12. *For $n > 1$, the image of Cellini's cyclic descent map on a conjugacy class $\mathcal{C}$ is invariant under cyclic rotation if and only if $n \in \{3, 4\}$ and $\mathcal{C}$ is the conjugacy class of $(n-1)$-cycles.*

7.3. Palindromicity of hook multiplicities. A sequence $a_0, \ldots, a_n$ (equivalently, the polynomial $a_0 + a_1 x + \cdots + a_n x^n$) is *palindromic* (or *symmetric*) if $a_i = a_{n-i}$ for all $0 \leqslant i \leqslant n$.

For a partition $\lambda \vdash n$ recall the notation

$$m_{k,\lambda} := \langle \psi^\lambda, \chi^{(n-k,1^k)} \rangle \qquad (0 \leqslant k < n).$$

In this subsection we prove the following.

Proposition 7.13. *Consider the partition $\lambda = (r^s)$ for positive integers r and s.*

1. *If $s = 1$ then the hook-multiplicity sequence $m_{0,(r)}, m_{1,(r)}, \ldots, m_{r-1,(r)}$ is palindromic if and only if either r is odd or $r \equiv 0 \pmod 4$.*
2. *If $s > 1$ then the hook-multiplicity sequence $m_{0,(r^s)}, m_{1,(r^s)}, \ldots, m_{rs-1,(r^s)}$ is palindromic if and only if $r \equiv 0 \pmod 4$.*

Proof. **1.** Assume first that $s = 1$. Recall the notations $M_{(r)}(x) := \sum_{j=0}^{r-1} m_{j,(r)} x^j$ and $F_r(x) := \sum_{j=0}^{r} f_j(r) x^j$. By Corollary 5.8, $F_r(x) = (1+x) M_{(r)}(x)$. Hence $M_{(r)}(x)$ is palindromic if and only if $F_r(x)$ is palindromic.

If r is odd then, for every j, every divisor $d|(r,j)$ is odd, hence $(d+1)j/d$ is even. Thus, by Definition 5.3,

$$f_j(r) = \sum_{d|(r,j)} \frac{\mu(d)}{r} \binom{r/d}{j/d} = \sum_{d|(r,r-j)} \frac{\mu(d)}{r} \binom{r/d}{(r-j)/d} = f_{r-j}(r) \qquad (0 \leqslant j \leqslant r),$$

and $F_r(x)$ is palindromic.

Next consider the case $r \equiv 0 \pmod 4$. Since $\mu(d) = 0$ for $d \equiv 0 \pmod 4$, Definition 5.3 implies that

$$\begin{aligned} f_j(r) &= \sum_{d|(r,j)} \frac{\mu(d)(-1)^{(d+1)j/d}}{r} \binom{r/d}{j/d} \\ &= \sum_{\substack{d|(r,j) \\ d \text{ odd}}} \frac{\mu(d)(-1)^{(d+1)j/d}}{r} \binom{r/d}{j/d} + \sum_{\substack{d|(r,j) \\ d \equiv 2 \pmod 4}} \frac{\mu(d)(-1)^{(d+1)j/d}}{r} \binom{r/d}{j/d}. \end{aligned}$$

Again, if $d|(r,j)$ is odd then $(d+1)j/d$ is even. If $d \equiv 2 \pmod 4$ then

$$(-1)^{(d+1)j/d} (-1)^{(d+1)(r-j)/d} = (-1)^{(d+1)r/d} = 1,$$

since r/d is even. It follows that

$$(-1)^{(d+1)j/d} = (-1)^{(d+1)(r-j)/d}$$

in this case. We deduce that if $r \equiv 0 \pmod 4$ then, for every $0 \leqslant j \leqslant r$,

$$\begin{aligned} f_j(r) &= \sum_{\substack{d|(r,j) \\ d \text{ odd}}} \frac{\mu(d)}{r}\binom{r/d}{j/d} + \sum_{\substack{d|(r,j) \\ d\equiv 2 \pmod 4}} \frac{\mu(d)(-1)^{(d+1)j/d}}{r}\binom{r/d}{j/d} \\ &= \sum_{\substack{d|(r,r-j) \\ d \text{ odd}}} \frac{\mu(d)}{r}\binom{r/d}{(r-j)/d} + \sum_{\substack{d|(r,r-j) \\ d\equiv 2 \pmod 4}} \frac{\mu(d)(-1)^{(d+1)(r-j)/d}}{r}\binom{r/d}{(r-j)/d} \\ &= f_{r-j}(r), \end{aligned}$$

hence $F_r(x)$ is palindromic.

On the other hand, if $r \equiv 2 \pmod 4$ then, letting $j = 2$,

$$f_2(r) = \frac{1}{r}\left[\binom{r}{2} + \binom{r/2}{1}\right] = \frac{r}{2} \neq \frac{r-2}{2} = \frac{1}{r}\left[\binom{r}{r-2} - \binom{r/2}{(r-2)/2}\right] = f_{r-2}(r),$$

thus $F_r(x)$ is not palindromic in this case.

2. Assume now that $s > 1$. By Equation (9),

$$\sum_{j=0}^{rs} e_{j,(r^s)}x^j = (1+x)\sum_{j=0}^{rs-1} m_{j,(r^s)}x^j.$$

Thus the sequence $m_{0,(r^s)}, \dots m_{rs-1,(r^s)}$ is palindromic if and only if the sequence $e_{0,(r^s)}, \dots, e_{rs,(r^s)}$ is. We will show that this happens if and only if $r \equiv 0 \pmod 4$.

Assume first that $r \equiv 0 \pmod 4$. Following Definition 6.1, for each partition $\gamma = (\gamma_1, \dots, \gamma_s) \in P_{r,s}(i)$ consider the complementary partition $\bar\gamma = (\bar\gamma_1, \dots, \bar\gamma_s) \in P_{r,s}(rs-i)$, defined by $\bar\gamma_\ell := r - \gamma_{s+1-\ell}$ $(1 \leqslant \ell \leqslant s)$, or equivalently by $k_j(\bar\gamma) = k_{r-j}(\gamma)$ $(0 \leqslant j \leqslant r)$. Since $r \equiv 0 \pmod 4$, $f_j(r) = f_{r-j}(r)$ $(0 \leqslant j \leqslant r)$, by Part 1 of the current proof. In addition, $j+1$ and $r-j+1$ have the same parity when r is even and j is arbitrary. Using Proposition 6.3, it follows that for $r \equiv 0 \pmod 4$ and any $0 \leqslant i \leqslant rs$,

$$\begin{aligned} e_{i,(r^s)} &= \sum_{\gamma \in P_{r,s}(i)} \prod_{j=0}^{r} (-1)^{(j+1)k_j(\gamma)}\binom{(-1)^{j+1}f_j(r)}{k_j(\gamma)} \\ &= \sum_{\gamma \in P_{r,s}(rs-i)} \prod_{j=0}^{r} (-1)^{(r-j+1)k_{r-j}(\gamma)}\binom{(-1)^{r-j+1}f_{r-j}(r)}{k_{r-j}(\gamma)} = e_{rs-i,(r^s)}, \end{aligned}$$

proving palindromicity in this case. For the converse, consider again the explicit formula for $e_{i,(r^s)}$ (from Proposition 6.3) written above. Each summand corresponds to a partition $\gamma \in P_{r,s}(i)$. According to Remark 6.4, the summand is zero if and only if either $k_j(\gamma) > f_j(r)$ for some odd j, or $k_j(\gamma) > 0 = f_j(r)$ for some even j.

We have $f_0(r) = 0$ for $r > 1$, $f_r(r) = 0$ for $r > 2$, and $f_1(r) = f_{r-1}(r) = 1$ for $r > 0$ thanks to Observation 5.4. It follows that for $\gamma \in P_{r,s}(i)$ to contribute a nonzero summand, it is necessary that $k_0(\gamma) = 0$ (for $r > 1$), $k_r(\gamma) = 0$ (for $r > 2$), $k_1(\gamma) \in \{0,1\}$ (for $r > 0$) and $k_{r-1}(\gamma) \in \{0,1\}$ (for $r-1$ odd).

Assume first that $r > 1$ is odd. The restrictions on $k_0(\gamma)$ and $k_1(\gamma)$ imply that for $i = s$ there is no relevant $\gamma \in P_{r,s}(s)$ (since $s > 1$). The restriction on $k_r(\gamma)$ implies that for $i = rs - s$ there is only one relevant $\gamma \in P_{r,s}(rs-s)$, with $k_{r-1}(\gamma) = s$ (and $k_j(\gamma) = 0$ for all other values of j). Thus

$$e_{rs-s,(r^s)} = \binom{s}{s} = 1 > 0 = e_{s,(r^s)},$$

and there is no palindromicity.

If $r = 1$ then $f_0(1) = f_1(1) = 1$ and the unique partition $\gamma \in P_{1,s}(i)$ has $k_1(\gamma) = i$ and $k_0(\gamma) = s - i$ ($0 \leqslant i \leqslant s$). It follows that

$$e_{i,(1^s)} = \binom{s-i}{s-i}\binom{1}{i} = \begin{cases} 1, & \text{if } i \in \{0, 1\}; \\ 0, & \text{otherwise.} \end{cases}$$

For $s > 1$ this sequence is not palindromic.

Assume now that $2 < r \equiv 2 \pmod 4$. The restrictions on $k_0(\gamma)$ and $k_1(\gamma)$ imply (actually, for any $r > 1$) that $e_{i,(r^s)} = 0$ for $0 \leqslant i < 2s-1$ and $e_{2s-1,(r^s)} = \binom{f_2(r)+s-2}{s-1}$. Similarly, the restrictions on $k_r(\gamma)$ and $k_{r-1}(\gamma)$ imply (for even $r > 2$) that $e_{rs-i,(r^s)} = 0$ for $0 \leqslant i < 2s-1$ and $e_{rs-2s+1,(r^s)} = \binom{f_{r-2}(r)+s-2}{s-1}$. Recall, from Part 1 of the current proof, that for $r \equiv 2 \pmod 4$

$$f_2(r) = \frac{r}{2} > \frac{r-2}{2} = f_{r-2}(r).$$

Therefore

$$e_{2s-1,(r^s)} = \binom{f_2(r)+s-2}{s-1} > \binom{f_{r-2}(r)+s-2}{s-1} = e_{rs-2s+1,(r^s)}$$

and the sequence is not palindromic.

Finally, if $r = 2$ then, by Remark 6.10,

$$e_{i,(2^s)} = \begin{cases} 1, & \text{if } i \in \{2s-1, 2s\}; \\ 0, & \text{otherwise} \end{cases}$$

and this sequence is not palindromic. This completes the proof. $\square$

8. Final remarks and open problems

Recall the notation $m_{k,\lambda} := \langle \psi^\lambda, \chi^{(n-k,1^k)} \rangle$. By Proposition 5.2, the *hook-multiplicity sequence* $m_{0,(n)}, m_{1,(n)}, \ldots, m_{n-1,(n)}$ is unimodal; We conjecture that it is unimodal for all partitions $\lambda \vdash n$.

Conjecture 8.1. *For every partition $\lambda \vdash n$, the sequence*

$$m_{0,\lambda}, m_{1,\lambda}, \ldots, m_{n-1,\lambda}$$

is unimodal.

The conjecture has been verified for all partitions of size $n \leqslant 15$ and for all partitions of rectangular shape (r^s) with $r \leqslant 40$ and $s \leqslant 5$.

Note that, by Lemma 5.12, Conjecture 8.1 would provide an alternative proof of Theorem 1.10.

A sequence $a_0, \ldots, a_n$ of real numbers is *log-concave* if $a_{i-1}a_{i+1} \leqslant a_i^2$ for all $0 < i < n$. It is not hard to show that a log-concave sequence with no internal zeros is unimodal, see e.g. [7, 31]. Since log-concavity implies unimodality, it is tempting to ask whether the hook-multiplicity sequence is log-concave.

Conjecture 8.2. *For every partition $\lambda = (r^s)$ with even $r \neq 6$, the hook-multiplicity sequence $m_{0,\lambda}, m_{1,\lambda}, \ldots, m_{n-1,\lambda}$ is log-concave.*

Conjecture 8.2 was verified for all $r \leqslant 40$ and $s \leqslant 5$.

Finally, recall that our proof of Theorem 1.5 is not constructive. We conclude the paper with the following challenging problem.

Problem 8.3. *Find an explicit combinatorial description of a cyclic descent extension on the conjugacy class of each cycle type not equal to (r^s) for a square-free r.*

A solution of this problem for the conjugacy classes of involutions is presented in [4]. The analogous problem for standard Young tableaux of fixed non-ribbon shape was solved by Huang [17].

Acknowledgements. The second author is grateful for the hospitality of Bar-Ilan University during his Sabbatical leave. Special thanks are due to Jan Saxl, whose personality influenced the development of this paper in many ways; in particular, the paper is inspired by the ingenious use of a higher Lie character in [18]. Thanks also to Eli Bagno, Jonathan Bloom, Sergi Elizalde, Darij Grinberg, Vic Reiner, Richard Stanley, Sheila Sundaram and Josh Swanson for their comments.

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