Research of Dynamic RQ System M/M/1 with Unreliable Server

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Abstract. The paper considers a single-line retrial queueing (RQ) system with an unreliable server controlled by a dynamic random multiple access protocol. A study of the prelimit probability distribution of the number of applications in orbit has been carried out. To study this system, the method of generating functions is used.

Keywords: Retrial queue · Dynamic random multiple access protocol · Unreliable server

1 Introduction

When designing or upgrading a data transmission network, it often becomes necessary to quantify network characteristics, such as the intensity of data flows over network communication lines, delays that occur at various stages of processing and transmitting packets. At the moment, such a network research tool is a protocol analyzer. But it does not give the opportunity to obtain probabilistic-temporal characteristics. Therefore, to study such systems, the apparatus of the theory of queuing is used, which makes it possible to build mathematical models of the data transmission network and find the main characteristics of the system.

A large number of works are devoted to the study of models of data transmission networks with various access protocols [1–8]. Various modifications of access protocols are proposed to solve the problems of repetitive applications. In [9–14], the authors investigate models with adaptive access protocols. The papers [15–23] consider the study of queuing systems with a dynamic access protocol. In this paper, we study a single-channel RQ system with an unreliable server controlled by a dynamic access protocol. The server is considered unreliable if it fails from time to time and requires restoration (repair). Only after that, the server resumes servicing new requests.

2 Description of the Mathematical Model

A prerequisite in data transmission networks over a communication line is the availability of a common resource. Any subscriber station, having generated requests, sends them to a common resource (server).

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If the server is free, then the customer is serviced. If the server fails during the service of the customer, then it is sent for repair, and the customer goes into orbit. To study such systems, consider a single-line RQ system with an unreliable server controlled by a dynamic access protocol.

Let's consider a single-server retrial queueing system with an unreliable server and the stationary Poisson flow of customers with parameter λ. A customer is serviced during random time distributed exponentially with parameter μ_1. We assume that the server is unreliable. An unreliable server may be in the following states: idle, busy or under repair. If the server is idle, and customer arrives, then the servicing immediately begins. If the server is busy at an arrival moment, then the customer goes into the orbit and waits for the opportunity to occupy the server at the next attempt. After a random time interval, a customer with intensity σ/i again tries to occupy the server for service, where i is the number of customers in orbit at time t (see Fig. 1). The working time is distributed exponentially with parameter γ_1, if server is idle and with parameter γ_2, if the server is busy. As soon as a breakdown occurs, the server is sent to repair and the servicing customer goes into the orbit. During repairing, all incoming customers go into the orbit. The recovery time is distributed exponentially with parameter μ_2. The goal of the research is to study such a system, as well as to determine its main characteristics and to find a stationary probability distribution of the number of customers in the orbit.

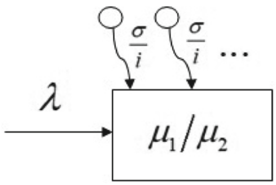

Fig. 1. Model of dynamic retrial queueing system M/M/1 with unreliable server

Let $i(t)$ be the number of customers in the orbit at time t and $k\,(t)$ determine the state of the server as follows:

$$k(t) = \begin{cases} 0, \text{if the server is idle,} \\ 1, \text{if the server is busy,} \\ 2, \text{if the server is under repair.} \end{cases}$$

3 Method of Generating Functions

Denote $P\,\{i(t) = i, k(t) = k\} = P(k, i, t)$ the probability that at time t the server is in state k and there are i customers in the orbit.

The probability distribution $P(k,i,t)$ satisfies the following system of equations:

$$\begin{cases} P\left(0,i,t+\varDelta t\right)=(1-\lambda\varDelta t)(1-\sigma\varDelta t)(1-\gamma_1\varDelta t)P\left(0,i,t\right)+ \\ +\mu_1\varDelta tP\left(1,i,t\right)+\mu_2\varDelta tP\left(2,i,t\right)+o(\varDelta t), \\ P\left(1,i,t+\varDelta t\right)=(1-\lambda\varDelta t)(1-\mu_1\varDelta t)(1-\gamma_2\varDelta t)P\left(1,i,t\right)+ \\ +\lambda\varDelta tP\left(0,i,t\right)+\sigma\varDelta tP\left(0,i+1,t\right)+\lambda\varDelta tP\left(1,i-1,t\right)+o(\varDelta t), \\ P\left(2,i,t+\varDelta t\right)=(1-\lambda\varDelta t)(1-\mu_2\varDelta t)P\left(2,i,t\right)+\gamma_1\varDelta tP\left(0,i,t\right)+ \\ +\gamma_2\varDelta tP\left(1,i-1,t\right)+\lambda\varDelta tP\left(2,i-1,t\right)+o(\varDelta t). \end{cases}$$

Let us compose a system of Kolmogorov differential equations for $i\geq 1$:

$$\begin{cases} \dfrac{\partial P\left(0,i,\ t\right)}{\partial t}=-\left(\lambda+\sigma+\gamma_1\right)P\left(0,i,\ t\right)+\mu_1P\left(1,i,\ t\right)+ \\ +\mu_2P\left(2,i,t\right), \\ \dfrac{\partial P\left(1,i,\ t\right)}{\partial t}=-\left(\lambda+\mu_1+\gamma_2\right)P\left(1,i,\ t\right)+\lambda P\left(0,i,\ t\right)+ \\ +\sigma P\left(0,i+1,t\right)+\lambda P\left(1,i-1,t\right), \\ \dfrac{\partial P\left(2,i,\ t\right)}{\partial t}=-\left(\lambda+\mu_2\right)P\left(2,i,\ t\right)+\gamma_1P\left(0,i,\ t\right)+ \\ +\gamma_2P\left(1,i-1,t\right)+\lambda P\left(2,i-1,t\right). \end{cases} \tag{1}$$

We assume that the system operates in the steady-state regime, i.e.

$$P(k,i,t)\equiv P(k,t).$$

Then we can rewrite System in the following form

$$\begin{cases} -\left(\lambda+\gamma_1\right)P\left(0,0\right)+\mu_1P\left(1,0\right)+\mu_2P\left(2,0\right)=0, i=0, \\ -\left(\lambda+\mu_1+\gamma_2\right)P\left(1,0\right)+\lambda P\left(0,0\right)+\sigma P\left(0,1\right)=0, i=0, \\ -\left(\lambda+\mu_2\right)P\left(2,0\right)+\gamma_1P\left(0,0\right)=0, i=0, \\ -\left(\lambda+\sigma+\gamma_1\right)P\left(0,i\right)+\mu_1P\left(1,i\right)+\mu_2P\left(2,i\right)=0, i\ \geq\ 1, \\ -\left(\lambda+\mu_1+\gamma_2\right)P\left(1,i\right)+\lambda P\left(0,i\right)+\sigma P\left(0,i+1\right)+ \\ +\lambda P\left(0,i-1\right)=0, i\ \geq\ 1, \\ -\left(\lambda+\mu_2\right)P\left(2,i\right)+\gamma_1P\left(0,i\right)+\gamma_2P\left(1,i-1\right)+ \\ +\lambda P\left(2,i-1\right)=0, i\ \geq\ 1. \end{cases} \tag{2}$$

To find a solution of System (2), it is necessary to define the generating functions:

$$G\left(k,x\right)=\sum_{i=0}^{\infty}x^iP\left(k,i\right).$$

Partial generating function

$$G(k,x) = \sum_{i=0}^{\infty} x^i P(k,i) \neq Mx^i,$$

but

$$G(k,x) = Mx^i,$$

therefore

$$G(x) = \{G(0,x),\ G(1,x),\ G(2,x)\}$$

is a generating function, and we introduce its components $G(k,x)$ as partial generating functions.

Then we get the following system of equations

$$\begin{cases} -(\lambda+\sigma+\gamma_1)\,G(0,x) + \mu_1 G(1,x) + \mu_2 G(2,x) = -\sigma P(0,0)\,, \\ \left(\lambda+\dfrac{\sigma}{x}\right) G(0,x) + (\lambda x - \lambda - \mu_1 - \gamma_2)\,G(1,x) = \dfrac{\sigma}{x} P(0,0)\,, \\ \gamma_1 G(0,x) + \gamma_2 x G(1,x) + (\lambda x - \lambda - \mu_2)\,G(2,x) = 0. \end{cases} \quad (3)$$

Let us multiply the second equation of System (3) by x.

Then we have the following system of equation

$$\begin{cases} -(\lambda+\sigma+\gamma_1)\,G(0,x) + \mu_1 G(1,x) + \mu_2 G(2,x) = -\sigma P(0,0)\,, \\ (\lambda x+\sigma)\,G(0,x) + \left(\lambda x^2 - \lambda x - \mu_1 x - \gamma_2 x\right) G(1,x) = \sigma P(0,0)\,, \\ \gamma_1 G(0,x) + \gamma_2 x G(1,x) + (\lambda x - \lambda - \mu_2)\,G(2,x) = 0. \end{cases} \quad (4)$$

We will find a solution of System (4) by denoting

$$G(x) = G(0,x) + G(1,x) + G(2,x)\,.$$

Then we have the following system of equation

$$\begin{aligned} &G(x) = P(0,0)\sigma((x-1)\lambda-\gamma_1-\mu_2)((x-1)\lambda-\gamma_2-\mu_1)/((x^2-x)* \\ &*\lambda^3 + x((\sigma+\gamma_1)x-\sigma-\gamma_1-\gamma_2-\mu_2)\lambda^2 + (((-\gamma_2-\mu_1-\mu_2)\sigma- \\ &-\gamma_1(\gamma_2+\mu_1))x+\sigma\mu_1)\lambda+\sigma\mu_2(\gamma_2+\mu_1)). \end{aligned} \quad (5)$$

Taking into account the normalization condition, where

$$G(1) = 1,$$

we obtain an expression for $P(0,0)$:

$$P(0,0) = \frac{(-\gamma_2-\mu_2)\lambda^2 - (\sigma\mu_2 + (\sigma+\gamma_1)\gamma_2 + \gamma_1\mu_1)\lambda + \sigma\mu_2(\gamma_2+\mu_1)}{\sigma(\gamma_1+\mu_2)(\gamma_2+\mu_1)}.$$

By substituting $P(0,0)$ into the Eq. (5), we obtain the expression for the generating function:

$$G(x) = ((-\gamma_2 - \mu_2)\lambda^2 - ((\gamma_2 + \mu_2)\sigma + \gamma_1(\gamma_2 + \mu_1))\lambda + \sigma\mu_2(\gamma_2 + \mu_1))*$$
$$*((x-1)\lambda - \gamma_1 - \mu_2)((x-1)\lambda - \gamma_2 - \mu_1)/((\gamma_1 + \mu_2)((x^2 - x)\lambda^3 +$$
$$+ x((\sigma + \gamma_1)x - \sigma - \gamma_1 - \gamma_2 - \mu_2)\lambda^2 + (((-\gamma_2 - \mu_1 - \mu_2)\sigma -$$
$$-\gamma_1(\gamma_2 + \mu_1))x + \sigma\mu_1)\lambda + \sigma\mu_2(\gamma_2 + \mu_1))(\gamma_2 + \mu_1)).$$

The values of the stationary distribution of server states R_k will look like this:

$$R_0 = G(0,1) = \frac{(-\lambda + \gamma_2 + \mu_1)\mu_2 - \lambda\gamma_2}{(\gamma_2 + \mu_1)(\gamma_1 + \mu_2)},$$

$$R_1 = G(1,1) = \frac{\lambda}{\gamma_2 + \mu_1},$$

$$R_2 = G(2,1) = \frac{(-\lambda + \gamma_2 + \mu_1)\gamma_1 + \lambda\gamma_2}{(\gamma_2 + \mu_1)(\gamma_1 + \mu_2)}.$$

In case of the probabilities $P(0,0)$ must be positive, the following inequalities must be true:

$$\frac{\lambda}{\mu_1} \leq \frac{\sigma\mu_2(\gamma_2 + \mu_1)}{(\lambda + \sigma)(\gamma_2\mu_1 + \mu_2\mu_1) + \gamma_1\mu_1(\gamma_2 + \mu_1)} = S, \quad (6)$$

where S is the throughput of the system under consideration.

Definition. Throughput is the upper limit of those load values $\rho = \frac{\lambda}{\mu_1}$, for which there is the steady-state regime.

The inequality (6) determines the condition for the existence of a steady-state regime for the considered dynamical system.

4 Method of Characteristic Functions

Introducing the partial characteristic function

$$H(k,u) = \sum_{i=0}^{\infty} e^{jui} P(k,i),$$

where $j = \sqrt{-1}$, System (2) can be rewritten as

$$\begin{cases} -(\lambda + \sigma + \gamma_1) H(0,u) + \mu_1 H(1,u) + \mu_2 H(2,u) = -\sigma P(0,0), \\ \left(\lambda + \frac{\sigma}{e^{ju}}\right) H(0,u) + \left(\lambda e^{ju} - \lambda - \mu_1 - \gamma_2\right) H(1,u) = \frac{\sigma}{e^{ju}} P(0,0), \\ \gamma_1 H(0,u) + \gamma_2 e^{ju} H(1,u) + \left(\lambda e^{ju} - \lambda - \mu_2\right) H(2,u) = 0. \end{cases} \quad (7)$$

We will find a solution of System (7) by denoting

$$H(u) = H(0,u) + H(1,u) + H(2,u).$$

Then we have the following system of equation

$$H(u) = P(0,0)\sigma((e^{ju}-1)\lambda-\gamma_1-\mu_2)((e^{ju}-1)\lambda-\gamma_2-\mu_1)/((e^{ju^2}- e^{ju})*\lambda^3+e^{ju}((\sigma+\gamma_1)e^{ju}-\sigma-\gamma_1-\gamma_2-\mu_2)\lambda^2+(((-\gamma_2-\mu_1-\mu_2)\sigma- -\gamma_1(\gamma_2+\mu_1))e^{ju}+\sigma\mu_1)\lambda+\sigma\mu_2(\gamma_2+\mu_1)). \tag{8}$$

By substituting $P(0,0)$ into the Eq. (8), we obtain

$$H(u) = ((-\gamma_2-\mu_2)\lambda^2-((\gamma_2+\mu_2)\sigma+\gamma_1(\gamma_2+\mu_1))\lambda+\sigma\mu_2(\gamma_2+\mu_1))* *((e^{ju}-1)\lambda-\gamma_1-\mu_2)((e^{ju}-1)\lambda-\gamma_2-\mu_1)/((\gamma_1+\mu_2)((e^{ju^2}-e^{ju})\lambda^3+ +e^{ju}((\sigma+\gamma_1)e^{ju}-\sigma-\gamma_1-\gamma_2-\mu_2)\lambda^2+(((-\gamma_2-\mu_1-\mu_2)\sigma- -\gamma_1(\gamma_2+\mu_1))e^{ju}+\sigma\mu_1)\lambda+\sigma\mu_2(\gamma_2+\mu_1))(\gamma_2+\mu_1)).$$

5 Asymptotic Analysis

Let us denoting

$$H(k,u) = \sum_{i=0}^{\infty} e^{jui} P(k,i),$$

where $j=\sqrt{-1}$ is the imaginary unit.

Thus System (2) can be rewritten as

$$\begin{cases} -(\lambda+\sigma+\gamma_1)\,H(0,u)+\mu_1 H(1,u)+\mu_2 H(2,u) = -\sigma P(0,0), \\ \left(\lambda+\dfrac{\sigma}{e^{ju}}\right)H(0,u)+\left(\lambda e^{ju}-\lambda-\mu_1-\gamma_2\right)H(1,u) = \dfrac{\sigma}{e^{ju}}P(0,0), \\ \gamma_1 H(0,u)+\gamma_2 e^{ju}H(1,u)+\left(\lambda e^{ju}-\lambda-\mu_2\right)H(2,u)=0. \end{cases} \tag{9}$$

We introduce a parameter

$$\rho = \frac{\lambda}{\mu_1},$$

that characterizes the system load.

Dividing all equations of System (9) by μ_1, we obtain the following system of equations

$$\begin{cases} -\left(\rho+\dfrac{\sigma}{\mu_1}+\dfrac{\gamma_1}{\mu_1}\right)H(0,u)+H(1,u)+\dfrac{\mu_2}{\mu_1}H(2,u)=\dfrac{\sigma}{\mu_1}P(0,0), \\ \left(\rho+\dfrac{\sigma}{\mu_1}e^{-ju}\right)H(0,u)+\left(\rho e^{ju}-\rho-1-\dfrac{\gamma_2}{\mu_1}\right)H(1,u)= \\ \qquad\qquad = \dfrac{\sigma}{\mu_1}e^{-ju}P(0,0), \\ \dfrac{\gamma_1}{\mu_1}H(0,u)+\dfrac{\gamma_2}{\mu_1}e^{ju}H(1,u)+\left(\rho e^{ju}-\rho-\dfrac{\mu_2}{\mu_1}\right)H(2,u)=0, \end{cases} \tag{10}$$

Denoting three-dimensional vectors

$$H(u) = \{H(0,u),\ H(1,u),\ H(2,u)\},$$

$$P(0) = \{P(0,0),\ P(1,0),\ P(2,0)\}$$

and matrices

$$A(ju,\rho) = \begin{bmatrix} -\left(\rho + \frac{\sigma}{\mu_1} + \frac{\gamma_1}{\mu_1}\right) & \left(\rho + \frac{\sigma}{\mu_1}e^{-ju}\right) & \frac{\gamma_1}{\mu_1} \\ 1 & \left(\rho e^{ju} - \rho - 1 - \frac{\gamma_2}{\mu_1}\right) & \frac{\gamma_2}{\mu_1}e^{ju} \\ \frac{\mu_2}{\mu_1} & 0 & \left(\rho e^{ju} - \rho - \frac{\mu_2}{\mu_1}\right) \end{bmatrix},$$

$$B(ju) = \begin{bmatrix} -\frac{\sigma}{\mu_1} & \frac{\sigma}{\mu_1}e^{-ju} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

we rewrite the System (10) in the form

$$H(u)\ A(ju,\rho) + P(0)\ B(ju) = 0. \tag{11}$$

To find the value of the throughput S, we will solve the Eq. (11) by the method of asymptotic analysis in the condition of a large load, denoting $\varepsilon = S - \rho$ and setting that $\varepsilon \to 0$.

Let us introduce the following substitutions

$$\rho = S - \varepsilon,\ u = \varepsilon w,\ H(u) = F(w,\varepsilon),\ P(0) = \varepsilon \Pi, \tag{12}$$

we get

$$F(w,\varepsilon)A(j\varepsilon w,\ S-\varepsilon) + \varepsilon \Pi B(j\varepsilon w) = 0, \tag{13}$$

where

$$A(j\varepsilon w,\ S-\varepsilon) = \sum_{n=0}^{\infty} \frac{(j\varepsilon w)^n}{n!} A_n(S) - \varepsilon \sum_{n=0}^{\infty} \frac{(j\varepsilon w)^n}{n!} A'_n(S). \tag{14}$$

Theorem 1. *The value S of throughput is equal to the value of the root of the equation*

$$R(S)A_1(S)E = 0, \tag{15}$$

where the vector R is determined by the equation $R(S)A_0(S) = 0$ and the normalization condition $RE = 1$, and equality $A_0(S) = A(0,S)$.

The characteristic function of the normalized number of customers in the server has the form

$$\Phi(w) = \lim_{\varepsilon\to 0} \{F(w,\varepsilon)E\} = \frac{\kappa}{\kappa - jw}, \tag{16}$$

where

$$\kappa = \frac{R(S)A'_1(S)E + f_2 A_1(S)E}{f_1 A_1(S)E + \frac{1}{2}R(S)A_2(S)E}. \tag{17}$$

Proof. Let us denote

$$\lim_{\varepsilon\to 0} A(j\varepsilon w,\ S-\varepsilon) = A(0,S) = A_0(S),$$

where $A_0(S)$ is determined from the Eq. (14). Then by performing the limit transition in Eq. (13), we obtain

$$F(w)A_0(S) = 0. \tag{18}$$

It follows from the form of the matrix $A_0(S)$ that its properties are similar to those of the matrix of infinitesimal characteristics, so the solution of the homogeneous System (13) can be written as

$$F(w) = \varPhi(w)R(S),$$

where $R(S)$ is the probability distribution of chain values determined by the equation

$$R(S)A_0(S) = 0$$

and the normalization condition $RE = 1$.

Using matrix decomposition

$$A(j\varepsilon w,\ S-\varepsilon) = A_0(S) + j\varepsilon w A_1(S) - \varepsilon A'_0(S) + O(\varepsilon^2),$$

$$B(j\varepsilon w) = B_0 + O(\varepsilon^2),$$

we write the Eq. (13) in the following form

$$F(w,\varepsilon)(A_0(S) + j\varepsilon w A_1(S) - \varepsilon A'_0(S)) + \varepsilon \varPi B_0. \tag{19}$$

The solution of this equation is written in the form of decomposition

$$F(w,\varepsilon) = \varPhi(w)R(S) + \varepsilon f(w) + O(\varepsilon^2), \tag{20}$$

Substitute the decomposition in the Eq. (19), we get

$$(\varPhi(w)R(S) + \varepsilon f(w))(A_0(S) + j\varepsilon w A_1(S) - \varepsilon A'_0(S)) + \varepsilon \varPi B_0 =$$
$$= \varPhi(w)R(S)(A_0(S) + j\varepsilon w A_1(S) - \varepsilon A'_0(S)) + \varepsilon f(w)A_0(S) + \varepsilon \varPi B_0 = O(\varepsilon^2).$$

Considering that $R(S)A_0(S) = 0$, then for the function $f(w)$ at $\varepsilon \to 0$ we can write the equation

$$f(w)A_0(S) + jw\varPhi(w)R(S)A_1(S) - \varPhi(w)R(S)A'_0(S) + \varPi B_0,$$

This equation is an inhomogeneous system of linear algebraic equations, so the solution of $f(w)$ can be written as

$$f(w) = \varPhi(w)(jwf_1 - f_2) + f_3, \tag{21}$$

where vectors f_1, f_2, f_3 are solutions of systems:

$$f_1 A_0(S) + R(S) A_1(S) = 0, \tag{22}$$

$$f_2 A_0(S) + R(S) A'_0(S) = 0, \tag{23}$$

$$f_3 A_0(S) + \Pi B_0 = 0.$$

The solution of the System (23) has the form

$$f_2 = R'(S).$$

To find the solution of System (22), the Eq. (15) must be satisfied. Thus, the Eq. (21) can be written as

$$f(w) = \Phi(w)(jwf_1 - R'(S)) + f_3, \tag{24}$$

So the Eq. (20) can be written as follows:

$$F(w,\varepsilon) = \Phi(w)R(S) + \varepsilon\Phi(w)(jwf_1 - R'(S)) + \varepsilon f_3 + O(\varepsilon^2). \tag{25}$$

To find the function $\Phi(w)$, we sum over k all the equations of System (13), then we obtain

$$F(w,\varepsilon)A(j\varepsilon w,\ S-\varepsilon)E + \varepsilon\Pi B(j\varepsilon w)E = 0. \tag{26}$$

For matrices A and B from this equation we write the decompositions

$$A(j\varepsilon w,\ S-\varepsilon) = A_0(S) + j\varepsilon w A_1(S) + \frac{(j\varepsilon w)^2}{2}A_2(S) - \varepsilon A'_0(S) +$$
$$+\frac{\varepsilon^2}{2}A''_0(S) - j\varepsilon^2 w A'_1(S) + O(\varepsilon^2),$$

$$B(j\varepsilon w) = B_0 + j\varepsilon w B_1 + O(\varepsilon^2).$$

It follows from the form of matrices $A(ju)$ and $B(ju)$ that

$$A''_0(S) = 0,\ A_0(S)E = 0,\ A'_0(S)E = 0,\ B_0 E = 0,$$

therefore

$$A(j\varepsilon w,\ S-\varepsilon)E = j\varepsilon w A_1(S)E + \frac{(j\varepsilon w)^2}{2}A_2(S)E - j\varepsilon^2 w A'_1(S) + O(\varepsilon^2),$$

$$B(j\varepsilon w)E = j\varepsilon w B_1 + O(\varepsilon^2),$$

Then the Eq. (26) will take the form

$$F(w,\varepsilon)(A_1(S) + \frac{jw\varepsilon}{2}A_2(S) - \varepsilon A'_1(S))E + \varepsilon\Pi B_1 E = O(\varepsilon^2).$$

Substituting the Eq. (25), we get the following form

$$\begin{aligned}
&(\Phi(w)R(S) + \varepsilon\Phi(w)(jwf_1 - R'(S)) + \varepsilon f_3)\times \\
&\times (A_1(S)E + \frac{jw\varepsilon}{2}A_2(S)E - \varepsilon A'_1(S)E) + \varepsilon \Pi B_1 E + O(\varepsilon^2) = \\
&= \Phi(w)R(S)\left[jw\varepsilon\ A_1(S)E + \frac{(jw\varepsilon)^2}{2}A_2(S)E - j\varepsilon^2 wA'_1(S)E\right] + \\
&+ \varepsilon(\Phi(w)(jwf_1 - R'(S)) + \varepsilon f_3)jw\varepsilon A_1(S)E + \varepsilon \Pi j\varepsilon w B_1 + O(\varepsilon^2).
\end{aligned}$$

Since the condition $R(S)A_1(S)E = 0$ is satisfied then at $\varepsilon \to 0$, we obtain

$$\begin{aligned}
&\Phi(w)\left[jw\ (f_1A_1(S)E + \frac{1}{2}R(S)A'_2(S)E) - (R(S)A'_1(S)E + \right.\\
&\left. +\ f_2A_1(S)E)\right] + f_3A_1(S)E + \Pi B_1 E = 0.
\end{aligned}$$

Then

$$\Phi(w) = \frac{f_3A_1(S)E + \Pi B_1 E}{R(S)A'_1(S)E + f_2A_1(S)E - jw(f_1A_1(S)E + \frac{1}{2}R(S)A'_2(S)E)}.$$

Taking into account that $\Phi(0) = 1$, we get

$$\kappa = \frac{R(S)A'_1(S)E + f_2A_1(S)E}{f_1A_1(S)E + \frac{1}{2}R(S)A_2(S)E}.$$

So Theorem 1 is proved.

6 Numerical Example

In order to find the probability distribution $P(i)$, it suffices to apply the inverse Fourier transform to the characteristic function.

$$P(i) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-jui}H(u)du,.$$

where $H(u) = G(e^{ju}),\ j = \sqrt{-1}$.

In a numerical example, we take

$$\mu_1 = 5,\quad \mu_2 = 2,\quad \gamma_1 = 0.03,\quad \gamma_2 = 0.03,\quad \lambda = 1,\quad \sigma = 1.$$

The value of throughput for given parameters of a given RQ system $S = 0,34$.

Table 1 and Fig. 2 show the distribution of the number of customers in orbit for this system.

Table 1. The probability distribution of the number of customers in orbit, i=0, 1, 2,...

i	0	1	2	3	4	5
$P(i)$	0,48213	0,19421	0,12012	0,07523	0,04734	0,02985
i	6	7	8	9	10	11
$P(i)$	0,01883	0,01188	0,00750	0,00473	0,00299	0,00189
i	12	13	14	15	16	...
$P(i)$	0,00119	0,00075	0,00047	0,00030	0,00019	

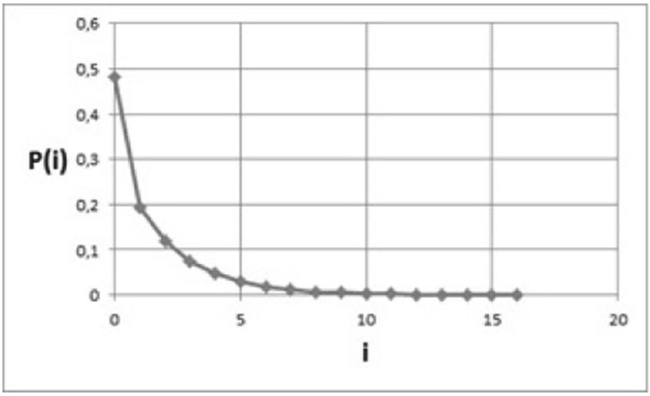

Fig. 2. The probability distribution of the number of customers

The values of the stationary distribution of server states R_k are:

$$R_0 = \frac{(-\lambda + \gamma_2 + \mu_1)\mu_2 - \lambda\gamma_2}{(\gamma_2 + \mu_1)(\gamma_1 + \mu_2)} = 0,68,$$

$$R_1 = \frac{\lambda}{\gamma_2 + \mu_1} = 0,19,$$

$$R_2 = \frac{(-\lambda + \gamma_2 + \mu_1)\gamma_1 + \lambda\gamma_2}{(\gamma_2 + \mu_1)(\gamma_1 + \mu_2)} = 0,13.$$

Table 2. Variation of S at different values σ

σ	1	5	10	50	100	500
S	0,342	0,689	0,788	0,891	0,906	0,919

Table 2 shows as the σ increases, the throughput S increases.

7 Conclusion

In this paper, we study the dynamic RQ-system M/M/1 with an unreliable server. As a result of the study, the generating and characteristic functions for the probability distribution of the number of applications in orbit are obtained. Further, the study was carried out by the method of asymptotic analysis under the condition of a large system load. The main characteristics of the system, the stationary distribution of server states, and the throughput of the system under consideration are found.

References

1. Lyubina T.V., Nazarov A.A.: Research of dynamic and adaptive RQ-systems with an incoming MMPP flow of applications. Bull. Tomsk State Univ. Manag. Comput. Technol. Inform. **3**(24), 104–112 (2013)
2. Lyubina T., Bublik Y.: Analysis of dynamic and adaptive retrial queue system with the incoming MAP-flow of requests. In: International Conference on Information Technologies and Mathematical Modelling, pp. 384–392 (2015)
3. Nazarov, A.A., Tsoy, S.A.: A general approach to the analysis of Markov models of data communication networks controlled by static random multiple access protocols. Autom. Control. Comput. Sci. **38**(4), 64–75 (2004)
4. Apostolopoulos, T.K., Protonotarios, E.N.: Queueing analysis of buffered slotted multiple access protocols. Comput. Commun. **8**(1), 9–21 (1985)
5. Ephremides A.: Analysis of protocols of multiple access. In: Baccelli, F., Fayolle, G. (eds.) Modelling and Performance Evaluation Methodology, vol. 60, pp. 563–575. Springer, Heidelberg (1984). https://doi.org/10.1007/BFb0005192
6. Nazarov, A.A., Nikitina, M.A.: Application of Markov chain ergodicity conditions to the study of the existence of stationary regimes in communication networks. Autom. Control. Comput. Sci. **37**(1), 50–55 (2003)
7. Gao, J., Rubin, I.: Analysis of a random-access protocol under long-range-dependent traffic. IEEE Trans. Veh. Technol. **52**(3), 693–700 (2003)
8. Nazarov, A.A., Sudyko, E.A.: Method of asymptotic semiinvariants for studying a mathematical model of a random access network. Probl. Inf. Transm. **46**(1), 86–102 (2010)
9. Nazarov A.A., Kuznetsov D.Yu.: Adaptive random access networks. Tomsk: TPU, p. 256 (2002)
10. Lyubina T.V., Nazarov A.A.: Study of an adaptive RQ-system with an incoming MMPP flow of applications using asymptotic analysis. In: Information technologies and mathematical modeling (ITMM-2012): materials of the XI All-Russian. Scientific-Practical. Conf. with International Participation (Anzhero-Sudzhensk, 23–24 November 2012). Kemerovo: Practice, Part 2, pp. 94–99 (2013)
11. Kuznetsov, D.Y., Nazarov, A.A.: Analysis of a communication network governed by an adaptive random multiple access protocol in critical load. Probl. Inf. Transm. **40**(3), 243–253 (2004)
12. Kuznetsov, D.Y., Nazarov, A.A.: Investigation of communication networks with adaptive random multiple access protocol under heavy load conditions for an infinite number of nodes. Autom. Control. Comput. Sci. **37**(3), 47–55 (2003)

13. Nazarov, A.A., Odyshev, Y.D.: Analysis of a communication network with the adaptive ALOHA protocol for a finite number of stations under overload. Problemy Peredachi Informatsii **36**(3), 83–93 (2000)
14. Anyugu Francis Lin, B., Ye, X., Hao, S.: Adaptive protocol for full-duplex two-way systems with the buffer-aided relaying. IET Commun. **13**(1), 54–58 (2019)
15. Shokhor, S.L.: Distribution of the number of messages in a communication network with channel reservation and a dynamic access protocol. Bull. Tomsk State Univ. **271**, 67–69 (2000)
16. Lyubina T.V., Nazarov A.A.: Investigation of a Markovian dynamical RQ-system with conflicts of requests. Bull. Tomsk State Univ. Manag. Comput. Technol. Inform. **3**(12), 73–84 (2010)
17. Nazarov, A.A., Shokhor, S.L.: Comparison of an asymptotic and prelimit model of a communication network with a dynamic protocol of random multiple access. In: Alexandrova, I.A., et al. (eds.) Mathematical Modeling and Probability Theory, Tomsk: Peleng, pp. 233–241 (1988)
18. Klimenok, V.I.: Optimization of dynamic management of the operating mode of data systems with repeat calls. Autom. Control. Comput. Sci. **24**(1), 23–28 (1990)
19. Gupur, G., Li, X.Z., Zhu, G.T.: The application of C 0 -semigroup theory to dynamic queueing systems. Semigroup Forum **62**, 205–216 (2001). https://doi.org/10.1007/s002330010030
20. Ouyang, H., Nelson, B.L.: Simulation-based predictive analytics for dynamic queueing systems. In: 2017 Winter Simulation Conference (WSC), pp. 1716–1727. IEEE (2017)
21. Mounce, R.: Existence of equilibrium in a continuous dynamic queueing model for traffic networks with responsive signal control. In: Lam, W., Wong, S., Lo, H. (eds.) Transportation and Traffic Theory 2009: Golden Jubilee, pp. 327–344. Springer, Boston (2009). https://doi.org/10.1007/978-1-4419-0820-9_16
22. Filipiak, J.: Dynamic routing in a queueing system with a multiple service facility. Oper. Res. **32**(5), 1163–1180 (1984)
23. Nazarov, A.A., Odyshev, Y.D.: Investigation of communications networks with dynamic synchronous ALOHA protocol under heavy load conditions. Avtomat. Vychisl. Tekhn. **35**(1), 77–84 (2001)