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Sign-symmetry and frustration index in signed graphs

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A graph in which every edge is labeled positive or negative is called a signed graph. We determine the number of ways to sign the edges of the McGee graph with exactly two negative edges up to switching isomorphism. We characterize signed graphs that are both sign-symmetric and have a frustration index of 1. We prove some results about which signed graphs on complete multipartite graphs have frustration indices 2 and 3. In the final part, we derive the relationship between the frustration index and the number of parts in a sign-symmetric signed graph on complete multipartite graphs.

Key words: signed graph, balance, frustration index, switching, switching isomorphism and sign-symmetric

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LIST OF SYMBOLS, ABBREVIATIONS, AND NOMENCLATURE

$G :$ A graph (V, E).

$\Sigma :$ A signed graph.

$-\Sigma :$ A negation of the signed graph.

$\Sigma_i \cong \Sigma_j :$ Denote that Σ_i is switching isomorphism to Σ_j.

$K_{n_1,\ldots,n_k} :$ Denote to complete k-partite graph.

$\tau(k_{n_1,\ldots,n_k}) :$ Denotes the number of triangles in a complete k-partite graph.

$\ell(\Sigma) :$ The frustration index of signed graph.

$\ell_0(\Sigma) :$ The frustration number of signed graph.

$C_n^+ :$ Denote the number of positive cycles of length n.

$C_n^- :$ Denote the number of negative cycles of length n.

$t^+ :$ Denote the number of positive triangles.

$t^- :$ Denote the number of negative triangles.

CHAPTER I

INTRODUCTION

1.1 History of Graph Theory and Signed Graphs

It is possible to pinpoint the beginning of graph theory to 1736 when mathematician Leonhard Euler found an answer to the Königsberg bridge puzzle [8]. The Königsberg Bridge Problem was an old conundrum that involved trying to find a way over each of the seven bridges that span a branched river that flows by an island without using them more than once. Such a way does not exist, according to Euler. His proof proved the first theorem in graph theory; however, it only made passing mention of the actual configuration of the bridges. Additionally, since the 1950s, signed graphs have been explored. Harary initially mentioned them in his structural balance theory, a generalization of Heider's thesis (Heider, 1946) from sociology [1].

1.2 Graphs

One of the most vibrant areas of combinatorics is graph theory. It is a trustworthy source for graph theory[[11] and [14]]. A graph can accurately represent any set of items having a binary relationship, and the theory of graphs directs one's analysis of the problem. Officially, a graph G is a triple consisting of a vertex set V(G), an edge set E(G), and a relation that associates with each edge two vertices called its endpoints [29]. Although a graph appears to have a simple idea, complicated analysis methods have been developed for them. Numerous studies have been done on hundreds, if not thousands, of different graph invariants. According to a compilation, there are various ”nice” graphs. Graphs have been examined from a variety of perspectives and orientations. When the value of an invariant is known, structural queries are often used to probe the existence of substructures.

1.2.1 Some Types of Graph

There are many types of graphs, and in this section, we will discuss some of them:

- Null Graph: Graphs containing zero or more vertices but no edges are called null graphs.
- Trivial Graph: The trivial graph has just one vertex and no edges.
- Undirected Graph: Undirected graphs contain edges without a direction.
- Directed Graph: In a directed graph, each edge has a direction assigned to it, and edges connect the vertices.
- Connected Graph: If a path connects any two of a graph's vertices, the graph is said to be connected.
- Disconnected Graph: If there is no path connecting at least two of the graph's vertices, the graph is said to be disconnected.
- Simple graph: A simple graph is a graph having no loops or no multiple edges. In a simple graph with n vertices, every vertex's degree is at most n -1.

1.3 Signed graphs

In many aspects of our lives now, signed graphs are crucial. For instance, one can describe and examine the geometry of subsets of the classical root systems in mathematics [34]. Both topological graph theory and group theory contain them. Signed graphs have been employed in social psychology [16]. The signed graph is also helpful today in network science and other domains. Thus, a signed graph is a graph with each edge receiving either a positive or negative sign [32]. Let us give a quick and essential example of a signed graph for friendship. This can be represented by a signed graph, where vertices are the people, a positive edge signifies that the two corresponding people like each other, and a negative edge hates each other.

1.4 Background material

This section provides the essential definitions that this dissertation requires. Additionally, if a definition is needed within any chapter of this dissertation, it will be included in that same chapter. The majority of the definition is standard.

Definition 1.4.1. *A cycle is a path that begins and ends at the same vertex, and the length of a cycle is the number of edges it contains and is denoted by C_n. Moreover, a cycle in Σ is said to be positive if the product of signs on its edges is +1 and negative otherwise. C_n^+ or C_n^- denotes it based on the product of signs on its edges.*

Definition 1.4.2. *A graph that can be obtained from G by deleting some of its vertices is called an induced subgraph of G [27].*

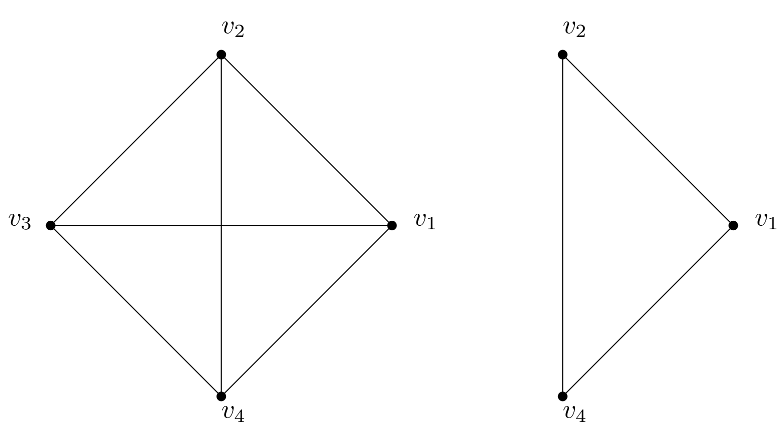

Figure 1.1

A graph and an induced subgraph

Figure 1.1 shows the graph G with four vertices and its induced subgraph by deleting vertex v_3

Definition 1.4.3. *We denote the signed graph by* $\Sigma = (G, \sigma)$ *where* G *is the underlying graph and* σ *is the signature of* Σ.

Definition 1.4.4. *The negation of signed graph* Σ *is the same underlying graph with all signs reversed. We denote it by* $-\Sigma = (G, -\sigma)$

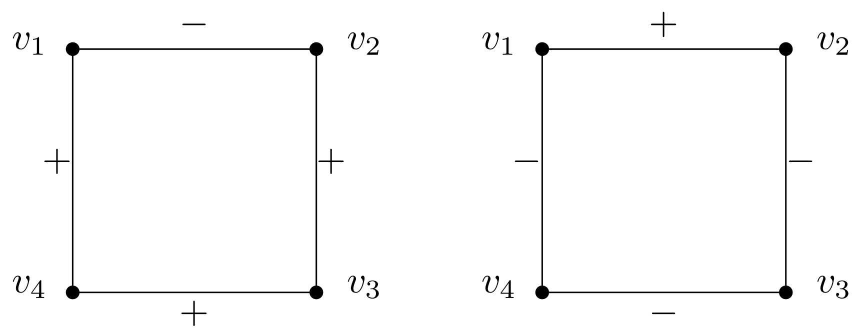

Figure 1.2

A signed graph Σ and its negation $-\Sigma$

Figure 1.2 shows a signed graph Σ and its negation $-\Sigma$ by taking reversed of Σ.

Definition 1.4.5. *A signed graph is balanced if all its cycles have a positive sign product.*

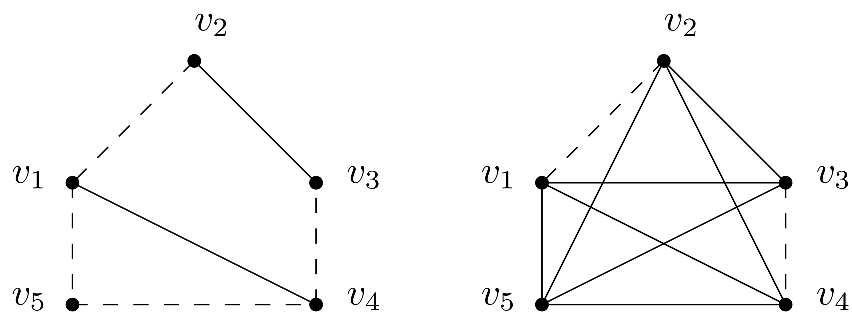

Figure 1.3

A balanced and unbalanced signed graph

Figure 1.3 shows the first signed graph is balanced because it contains positive cycles, and the second signed graph is not balanced.

Theorem 1.4.1. *([15] Harary) A signed graph* Σ *is balanced if and only if there is a bipartition of its vertex set,* $V = X \cup Y$*, such that every positive edge is induced by X or Y while every negative edge has one endpoint in X and one in Y. Also, if and only if for any two vertices v,w, every path between them has the same sign.*

Definition 1.4.6. *The frustration index is the minimum number of edges to be deleted in a signed graph to get a balanced signed graph. We denote the frustration index of a signed graph by* $\ell(\Sigma)$.

Definition 1.4.7. *The frustration number is the minimum number of vertices to be deleted in a signed graph to get a balanced signed graph. We denote the frustration number of a signed graph by* $\ell_0(\Sigma)$.

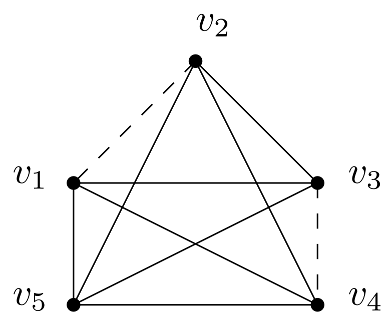

Figure 1.4

Unbalanced signed graph

It is clear to us the above-signed graph is not balanced, so we need to delete two negative edges because we have two negative disjoint triangles. Hence $\ell(\Sigma) = 2$. Similarly, we have two negative disjoint triangles for the frustration number, so $\ell_0(\Sigma) = 2$. However, it is unnecessary to be $\ell(\Sigma) = \ell_0(\Sigma)$.

Theorem 1.4.2. *[26] For every signed subcubic graph* Σ*,* $\ell_0(\Sigma) = \ell(\Sigma)$.

Lemma 1.4.1. *[5] The frustration index of a signed graph is invariant under switching.*

Lemma 1.4.2. $min|E^-(\Sigma_i)| = \ell(\Sigma)$ *where* $i \in \{1, ..., k\}$.

Proof. $(\Rightarrow)$

Let $\ell(\Sigma) \leq min|E^-(\Sigma_i)| \implies \ell(\Sigma) \leq |E^-(\Sigma_i)|$. Σ_i has $|E^-(\Sigma_i)|$ negative edges. Deleting those edges gives a balanced signed graph. Hence $\ell(\Sigma_i) \leq |E^-(\Sigma_i)|$. But $\ell(\Sigma) = \ell(\Sigma_i)$

$(\Leftarrow)$

$\min|E^-(\Sigma_i)| \leq \ell(\Sigma) = \ell$. The frustration index of Σ is ℓ. This means we can find edges $\{e_1, ..., e_\ell\}$ so that $\Sigma - \{e_1, ..., e_\ell\}$ is balanced. Now we can switch $\Sigma - \{e_1, ..., e_\ell\}$ to be all positive. Now put back edges $\{e_1, ..., e_\ell\}$. They all have to be negative. Otherwise, the frustration index of Σ will be less than ℓ. We know for some i, $|E^-(\Sigma_i)| = \ell$. ■

Definition 1.4.8. *Switching at a vertex v reverses the sign of edges incident on v.*

Definition 1.4.9. *Two signed graphs are called switching isomorphic if one is isomorphic to a switching equivalent of the other. We denote it by $\Sigma_1 \cong \Sigma_2$.*

Theorem 1.4.3. *[35] Let Σ and Σ' be two signed graphs with the same underlying graph Γ. Then $C^+(\Sigma) = C^+(\Sigma')$ if and only if Σ' is obtained by switching Σ. In particular, Σ is balanced if and only if it switches to the all-positive signed graph $+\Gamma$.*

Lemma 1.4.3. *[36] Switching does not change the sign of any circle.*

Definition 1.4.10. *A signed graph is said to be sign-symmetric if it is switching isomorphic to its negation. We denote it by $\Sigma \cong -\Sigma$.*

Lemma 1.4.4. *Every signed graph is an induced subgraph of a sign-symmetric signed graph.*

Proof. Let $\Sigma^{'}$ be a signed graph. Let $\Sigma^{'} = \Sigma \cup -\Sigma$. Now, $\Sigma^{'}$ is sign-symmetric and Σ is an induced subgraph of $\Sigma^{'}$. ■

1.5 Organization of the dissertation

Chapter two demonstrates how many different ways to sign the McGee graph with exactly two negative edges. We did this by counting the odd negative cycles. We also identified the McGee signed graph with two negative edges sign-symmetric. In Chapter Three, we examine different kinds of signed graphs with $\ell = 1$, including Wheel signed graphs, Heawood signed graphs, and others, to identify which are sign-symmetric. Additionally, we obtain a theorem for the signed graph of the Broken Wheel. In Chapter Four, we proved four theorems for the complete k-partite signed graphs with $\ell = 1$ and $\ell = 2$, which are sing-symmetric. Additionally, we obtain a theorem for $k \geq 5$ with $\ell = 1$ *and* $\ell = 2$. In Chapter Five, we study the complete k-partite signed graphs with $\ell = 3$, where 3 $\leq$k$\leq$ 6, and determine the sign-symmetric. In Chapter Six, we study the relationship between k-partite and the frustration index for negating the complete k-partite signed graphs, and we get a theorem for the given lower bound. Also, we study the relationship between k parts, n vertices, and frustration index ℓ. Moreover, we also propose several conjectures.

CHAPTER II

MCGEE SIGNED GRAPH

2.1 Introduction

In this chapter, we study how signs can be assigned to the precisely two negative edges of the McGee graph and analyze the resulting signed graphs. Along the way, we will determine whether McGee-signed graphs are sign-symmetric. The McGee graph shown in Figure 2.1 has many incarnations. It is a symmetric graph and illustrates several aspects of signed graph theory. The McGee graph is a famous cubic symmetric graph on 24 vertices and 36 edges and is a (3,7) cage. This means it is the smallest 3-regular in which the shortest cycle has length 7 [17].

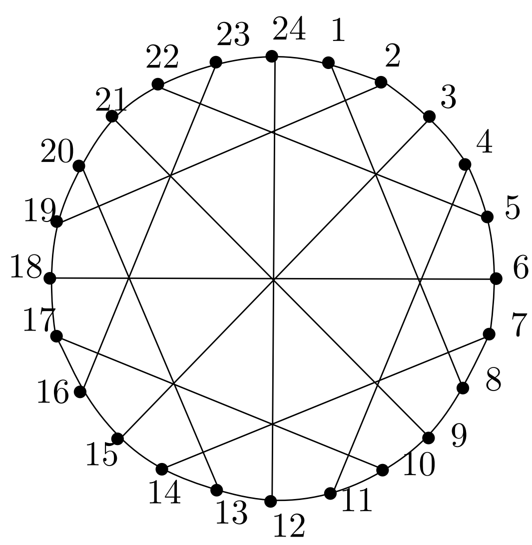

Figure 2.1

The McGee graph

2.2 Signings on the McGee graph with two negative edges

Signed graphs are frequently seen in mathematics, biology, chemistry, social networks, and In several other fields([19], [20], [21] and [12]). Modeling social interaction with the help of this tool is quite helpful. It is a great source of a signed graph([31] and [9]). Zaslavsky, Vaidyanathan, Deepak Sehrawat, Bikash Bhattacharjya recently conducted a thorough analysis of the different signed graphs, respectively Petersen signed graphs, Heawood signed graph, and Signed Complete Graphs on Six Vertices([33], [24] and [22]). In this section, On the McGee graph, how many signings have exactly two negative edges? To put it another way, the question is how many different ways to sign the two negative edges of the McGee graph, where two signatures are distinct if they are not switching isomorphic.

Theorem 2.2.1. *[33] There are exactly six signed Petersen graphs up to switching isomorphism.*

Theorem 2.2.2. *[24] There are exactly seven signed Heawood graphs up to switching isomorphism. They are* $+H$, H_1, $H_{2,1}$, $H_{2,2}$, $H_{3,1}$, $H_{3,2}$ *and* H_4.

Theorem 2.2.3. *[22] There are exactly 16 different signatures on* K_6 *up to switching isomorphism.*

Theorem 2.2.4. *There are fifteen ways to sign the McGee graph up to switching isomorphism with exactly 2-negative edges. They are* Σ_1, Σ_2, Σ_3, Σ_4, Σ_5, Σ_6, Σ_7, Σ_8, Σ_9, Σ_{10}, Σ_{11}, Σ_{23}, Σ_{24}, Σ_{25} *and* Σ_{26}.

The fifteen signed graphs are shown in Figure 2.2. Black lines represent positive edges; dashed lines represent negative edges.

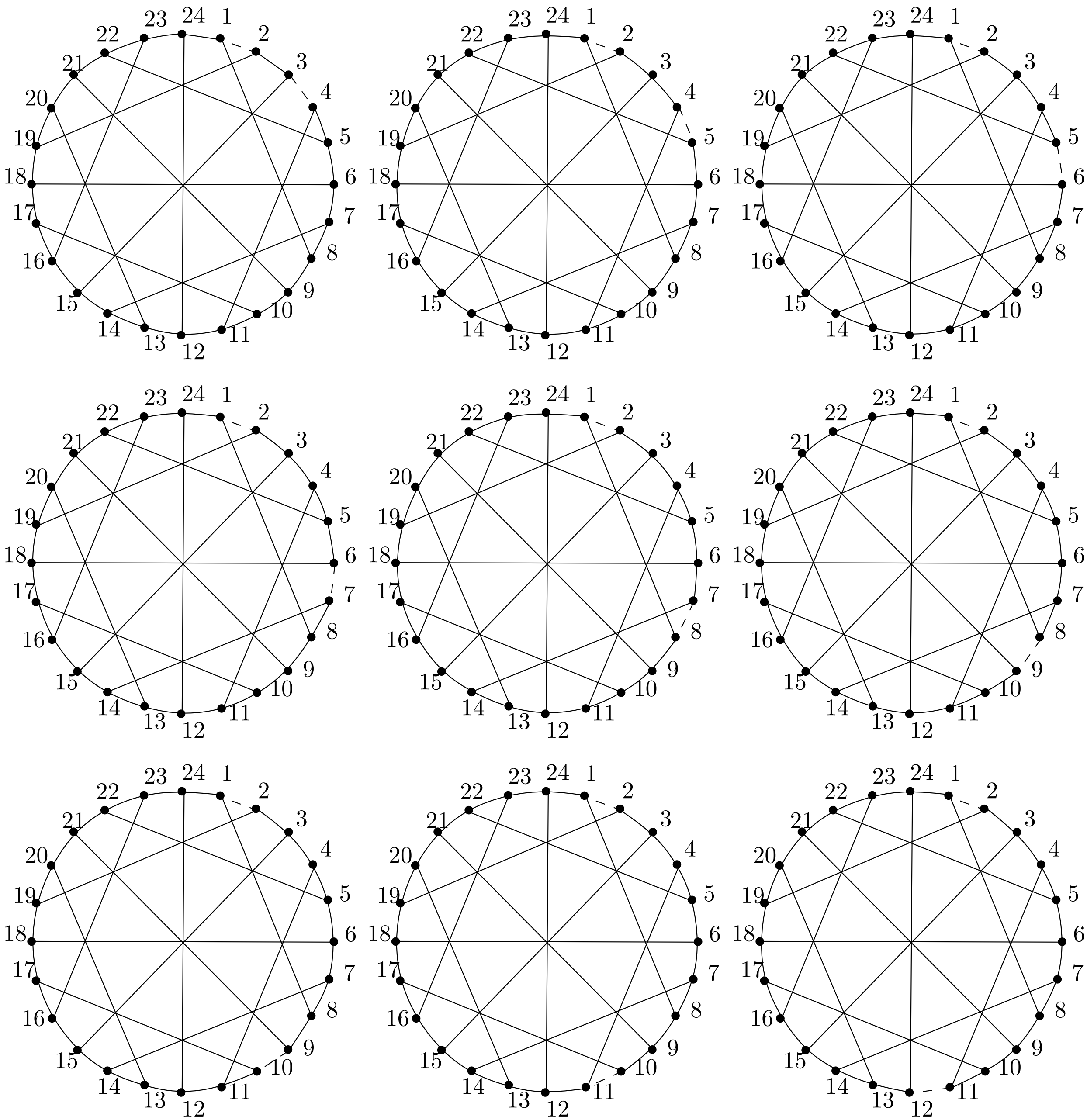

Figure 2.2

The fifteen switching isomorphism types of signed McGee graphs with $\ell = 2$

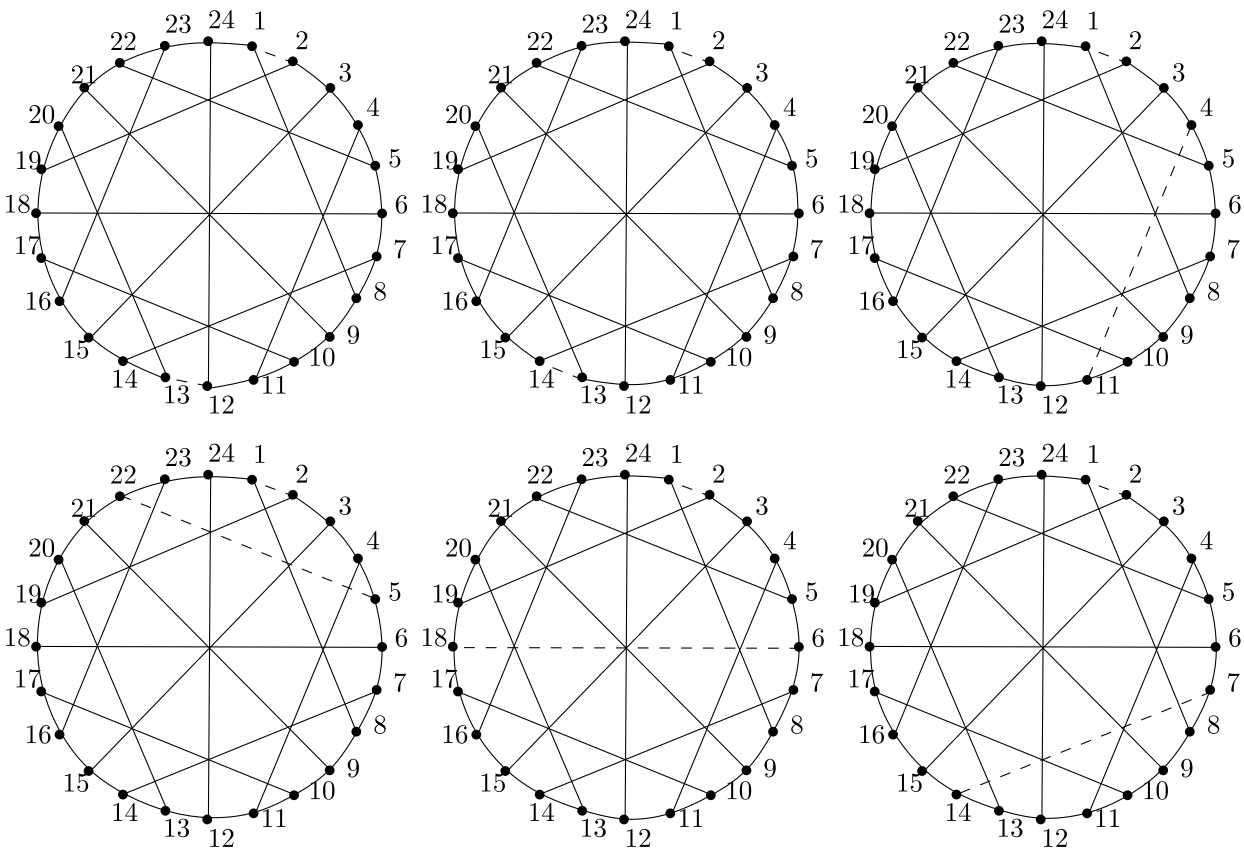

Figure 2.2 (continued)

2.3 Proof of the Main Results

Proof. Let edge 1-2 be a negative edge and 1-24 and 1-8 to be positive edges. Now, take another edge to be negative, and any two edges connected to this one be a positive edge. Then, we get 31 cases with two negative edges. Now, Assume the number of negative m-cycles of a signed graph Σ is denoted by $|C_n^-|$. Now, the table shows the number of negative 7-cycles and 8-cycles in 31 cases. Also, we put groups based on the total number of negative 7-cycles and 8-cycles.

Table 2.1

Number of negative 7-cycles and 8-cycles from Σ_1 *to* Σ_{31}

$C^- \backslash \Sigma$	Σ_1	Σ_2	Σ_3	Σ_4	Σ_5	Σ_6	Σ_7	Σ_8	Σ_9	Σ_{10}	Σ_{11}	Σ_{12}	Σ_{13}	Σ_{14}	Σ_{15}	Σ_{16}
$C^-_{7\ (1-2)}$	5	6	6	5	4	5	6	6	5	5	6	5	5	6	6	5
C^-_7	5	6	6	5	4	5	6	6	5	5	6	5	5	6	6	5
Total C^-_7	10	12	12	10	8	10	12	12	10	10	12	10	10	12	12	10
$C^-_{8\ (1-2)}$	5	6	7	7	6	6	6	7	8	7	6	7	8	7	6	6
C^-_8	5	6	7	7	6	6	6	7	8	7	6	7	8	7	6	6
Total C^-_8	10	12	14	14	12	12	12	14	16	14	12	14	16	14	12	12
Groups	1	4	6	5	2	3	4	6	7	5	4	5	7	6	4	3

$C^- \backslash \Sigma$	Σ_{17}	Σ_{18}	Σ_{19}	Σ_{20}	Σ_{21}	Σ_{22}	Σ_{23}	Σ_{24}	Σ_{25}	Σ_{26}	Σ_{27}	Σ_{28}	Σ_{29}	Σ_{30}	Σ_{31}
$C^-_{7\ (1-2)}$	4	5	6	6	5	4	5	6	5	5	5	6	4	5	5
C^-_7	4	5	6	6	5	6	5	6	7	5	7	6	6	5	5
Total C^-_7	8	10	12	12	10	10	10	12	12	10	12	12	10	10	10
$C^-_{8\ (1-2)}$	6	7	7	6	5	7	7	7	8	7	8	7	7	7	7
C^-_8	6	7	7	6	5	3	7	7	4	7	4	7	3	7	7
Total C^-_8	12	14	14	12	10	10	14	14	12	14	12	14	10	14	14
Groups	2	5	6	4	1	1	5	6	4	5	4	6	1	5	5

Now, we take group 1 with Σ_1, Σ_{21}, Σ_{22}, and Σ_{29}. We get:

- In Σ_1 by switching and relabeling, we get Σ_1 is switching isomorphic to Σ_{29}.
- In Σ_{21} by switching and relabeling, we get Σ_{21} is switching isomorphic to Σ_1.
- In Σ_{22} by switching and relabeling, we get Σ_{22} is switching isomorphic to Σ_{29}.

Thus Σ_1 is switcing isomorphic to Σ_{21},Σ_{22} and Σ_{29}. Now, we take group 2, which has Σ_5 and Σ_{17}. We get:

- In Σ_5 by switching and relabeling, we get Σ_5 is switching isomorphic to Σ_{17}.

Thus Σ_5 is switching isomorphic to Σ_{17}. Now, we take group 3 with Σ_6 and Σ_{16}. We get:

- In Σ_6 by switching and relabeling, we get Σ_6 is switching isomorphic to Σ_{16}.

Thus Σ_6 is switching isomorphic to Σ_{16}. Now, we take group 4 which have Σ_2, Σ_7 Σ_{11} , Σ_{15}, Σ_{20}, Σ_{25} and Σ_{27}. We get:

- In Σ_2 by switching and relabeling, we get Σ_2 is switching isomorphic to Σ_{20}.
- In Σ_7 by switching and relabeling, we get Σ_7 is switching isomorphic to Σ_{15}.
- In Σ_{11} by switching and relabeling, we get Σ_{11} is not switching isomorphic to any Σ.
- In Σ_{25} by switching and relabeling, we get Σ_{25} is switching isomorphic to Σ_{27}.

Thus Σ_2 is switching isomorphic to Σ_{20},Σ_7 is switching isomorphic to Σ_{15} and Σ_{25} is switching isomorphic to Σ_{27}.Now, we take group 5 which have Σ_4, Σ_{10} Σ_{12} ,Σ_{18}, Σ_{23}, Σ_{26}, Σ_{30} and Σ_{31}. We get:

- In Σ_4 by switching and relabeling, we get Σ_4 is switching isomorphic to Σ_{18}.
- In Σ_{10} by switching and relabeling, we get Σ_{10} is switching isomorphic to Σ_{12}.
- In Σ_{23} by switching and relabeling, we get Σ_{23} is switching isomorphic to Σ_{31}.
- In Σ_{26} by switching and relabeling, we get Σ_{26} is switching isomorphic to Σ_{30}.

Thus Σ_4 is switching isomorphic to Σ_{18},Σ_{10} is switching isomorphic to Σ_{12}, Σ_{23} is switching isomorphic to Σ_{31} and Σ_{26} is switching isomorphic to Σ_{30} .Now, we take group 6 which has Σ_3, Σ_8 Σ_{14} ,Σ_{19}, Σ_{24} and Σ_{28}. We get:

- In Σ_3 by switching and relabeling, we get Σ_3 is switching isomorphic to Σ_{19}.
- In Σ_8 by switching and relabeling, we get Σ_8 is switching isomorphic to Σ_{14}.
- In Σ_{24} by switching and relabeling, we get Σ_{24} is switching isomorphic to Σ_{28}.

Thus Σ_3 is switching isomorphic to Σ_{19},Σ_8 is switching isomorphic to Σ_{14} and Σ_{24} is switching isomorphic to Σ_{28}. Now, we take group 7, which has Σ_9 and Σ_{13}. We get:

- In Σ_9 by switching and relabeling, we get Σ_9 is switching isomorphic to Σ_{13}.

Thus Σ_9 is switching isomorphic to Σ_{13}. Now, we need to find the number of negative 9-cycles and 10-cycles for groups 4,5 and 6.

Table 2.2

Number of negative 9-cycles and 10-cycles for groups 4, 5, and 6

$C^- \setminus \Sigma$	$\Sigma_2 \cong \Sigma_{20}$	$\Sigma_7 \cong \Sigma_{15}$	Σ_{11}	$\Sigma_{25} \cong \Sigma_{27}$	$\Sigma_4 \cong \Sigma_{18}$	$\Sigma_{10} \cong \Sigma_{12}$
$C^-_{9\ (1-2)}$	3	3	4	3	4	4
C^-_9	3	3	4	2	4	4
Total C^-_9	6	6	8	5	8	8
$C^-_{10\ (1-2)}$	13	12	10	10	11	11
C^-_{10}	13	10	10	12	9	9
Total C^-_{10}	26	22	20	22	20	20
Groups	4	4	4	4	5	5

$C^- \setminus \Sigma$	$\Sigma_{23} \cong \Sigma_{31}$	$\Sigma_{26} \cong \Sigma_{30}$	$\Sigma_3 \cong \Sigma_{19}$	$\Sigma_8 \cong \Sigma_{14}$	$\Sigma_{24} \cong \Sigma_{28}$
$C^-_{9\ (1-2)}$	4	4	3	3	2
C^-_9	4	4	3	3	2
Total C^-_9	8	8	6	6	4
$C^-_{10\ (1-2)}$	11	11	11	10	12
C^-_{10}	11	11	9	10	12
Total C^-_{10}	22	22	20	20	24
Groups	5	5	6	6	6

In group 4, no one is switching isomorphic to another. Therefore, there are four ways to sign McGee's graph up to switching isomorphism. Similarly, in group 6, no one is switching isomorphic to another. Therefore, there are three ways to sign McGee's graph up to switching isomorphism. Now, we need to find the number of negative 11-cycles for group 5.

Table 2.3

Number of negative 11-cycles for group 5

$C^- \backslash \Sigma$	$\Sigma_4 \cong \Sigma_{18}$	$\Sigma_{10} \cong \Sigma_{12}$	$\Sigma_{23} \cong \Sigma_{31}$	$\Sigma_{26} \cong \Sigma_{30}$
$C^-_{11\ (1-2)}$	13	13	13	15
C^-_{11}	17	17	13	15
Total C^-_{11}	20	20	26	30
Groups	5	5	5	5

Table 2.3 shows two ways to sign the McGee graph up to switching isomorphism, which is $\Sigma_{23} \cong \Sigma_{31}$ and $\Sigma_{26} \cong \Sigma_{30}$. Now, we need to find the number of negative 12-cycles for $\Sigma_4 \cong \Sigma_{18}$ and $\Sigma_{10} \cong \Sigma_{12}$.

Table 2.4

Number of negative 12-cycles for $\Sigma_4 \cong \Sigma_{18}$ and $\Sigma_{10} \cong \Sigma_{12}$

$C^- \backslash \Sigma$	$\Sigma_4 \cong \Sigma_{18}$	$\Sigma_{10} \cong \Sigma_{12}$
$C^-_{12\ (1-2)}$	38	37
C^-_{12}	42	40
Total C^-_{12}	80	77
Groups	5	5

From the table 2.4, we can see $\Sigma_4 \cong \Sigma_{18}$ and $\Sigma_{10} \cong \Sigma_{12}$ are not switching isomorphism. Therefore, there are two ways to sign McGee's graph up to switching isomorphism. The number of negative cycles leads us to conclude that the fifteen signed graphs shown in Figure 2.2 are pairwise non-switching-isomorphic. The theorem's proof is complete at this point. ■

2.4 The McGee Signed Graphs that are Sign-Symmetric with Frustration index = 2

Theorem 2.4.1. *There is no McGee signed graph that is sign-symmetric with $\ell = 2$.*

Proof. It is easily to see $\ell(\Sigma_i) = 2$, where i=1,2,...,31. Now, by taking the negation of $\ell(\Sigma_i)$. We get that:

The following are three edge-disjoint negative circles in $-\Sigma_i$.

- In $-\Sigma_1$: 1-24-12-13-14-7-8 , 2-3-15-16-17-18-19 and 22-5-4-11-10-9-21.
- In $-\Sigma_2$: 1-24-12-13-14-7-8 , 10-17-16-15-3-4-11 and 5-6-18-19-20-21-22.
- In $-\Sigma_3$: 1-24-12-13-14-7-8 , 10-17-18-19-20-21-9 and 16-23-22-5-4-3-15.
- In $-\Sigma_4$: 1-24-12-13-14-7-8 , 10-17-16-15-3-4-11 and 5-6-18-19-20-21-22.
- In $-\Sigma_5$: 1-24-12-11-10-9-8 , 16-23-22-5-4-3-15 and 18-6-7-14-13-20-19.
- In $-\Sigma_6$: 1-24-12-13-14-7-8 , 16-23-22-5-4-3-15 and 9-21-20-19-18-17-10.
- In $-\Sigma_7$: 1-24-12-13-14-7-8 , 10-17-16-15-3-4-11 and 5-6-18-19-20-21-22.
- In $-\Sigma_8$: 1-24-12-13-14-7-8 , 9-21-20-19-18-17-10 and 16-23-22-5-4-3-15.
- In $-\Sigma_9$: 1-24-12-13-14-7-8 , 16-23-22-5-4-3-15 and 9-21-20-19-18-17-10.
- In $-\Sigma_{10}$: 24-23-16-17-10-11-12 , 7-14-13-20-21-9-8 and 2-9-18-6-5-4-3.
- In $-\Sigma_{23}$: 1-24-12-13-14-7-8 , 2-3-4-5-6-18-19 and 10-17-16-23-22-21-9.
- In $-\Sigma_{24}$: 1-24-12-13-14-7-8 , 2-19-18-6-5-4-3 and 16-23-22-21-9-10-17.
- In $-\Sigma_{25}$: 1-24-12-13-14-7-8 , 9-10-17-18-19-20-21 and 23-16-15-3-4-5-22.
- In $-\Sigma_{26}$: 24-23-16-17-10-11-12 , 15-14-13-20-19-2-3 and 2-19-18-6-5-4-3.

The following are two edge-disjoint negative circles in $-\Sigma_{11}$.

- In $-\Sigma_{11}$: 1-24-12-11-10-9-8 and 6-7-14-15-16-17-18.

Since $\ell(\Sigma_{11}) = \ell(-\Sigma_{11}) = 2$, we switching $(-\Sigma_{11})$ to check if it is switching isomorphism to (Σ_{11}). After switching, we find two disjoint negative C_7^-, 3-4-5-6-7-14-15 and 21-22-23-24-1-8-9. Now we need to delete one edge form C_7^- to check if the signed graph is balanced or not, and we have 49 cases:

1. If we delete edges 3-4 and 21-22, then the signed graph is not balanced because it contains C_7^-, which is 1-24-12-11-10-9-8.
2. If we delete edges 3-4 and 22-23, then the signed graph is not balanced because it contains C_7^-, which is 1-24-12-11-10-9-8.
3. If we delete edges 3-4 and 23-24, then the signed graph is not balanced because it contains C_7^-, which is 1-24-12-11-10-9-8.
4. If we delete edges 3-4 and 24-1, then the signed graph is not balanced because it contains C_7^-, which is 6-18-17-10-9-8-7.
5. If we delete edges 3-4 and 1-8, then the signed graph is not balanced because it contains C_7^-, which is 6-18-17-10-9-8-7.
6. If we delete edges 3-4 and 8-9, then the signed graph is not balanced because it contains C_7^-, which is 6-5-22-21-20-19-18.
7. If we delete edges 3-4 and 9-21, then the signed graph is not balanced because it contains C_7^-, which is 6-18-17-10-9-8-7.
8. If we delete edges 4-5 and 21-22, then the signed graph is not balanced because it contains C_7^-, which is 6-18-17-16-15-14-7.
9. If we delete edges 4-5 and 22-23, then the signed graph is not balanced because it contains C_7^-, which is 6-18-17-16-15-14-7.
10. If we delete edges 4-5 and 23-24, then the signed graph is not balanced because it contains C_7^-, which is 6-18-17-16-15-14-7.
11. If we delete edges 4-5 and 24-1, then the signed graph is not balanced because it contains C_7^-, which is 21-20-13-12-11-10-9.
12. If we delete edges 4-5 and 1-8, then the signed graph is not balanced because it contains C_7^-, which is 21-20-13-12-11-10-9.
13. If we delete edges 4-5 and 8-9, then the signed graph is not balanced because it contains C_7^-, which is 6-18-17-16-15-14-7.
14. If we delete edges 4-5 and 9-21, then the signed graph is not balanced because it contains C_7^-, which is 5-18-19-20-21-22.
15. If we delete edges 5-6 and 21-22, then the signed graph is not balanced because it contains C_7^-, which is 2-3-15-16-17-18-19.
16. If we delete edges 5-6 and 22-23, then the signed graph is not balanced because it contains C_7^-, which is 6-18-17-16-15-14-7.

17. If we delete edges 5-6 and 23-24, then the signed graph is not balanced because it contains C_7^-, which is 2-3-15-16-17-18-19.
18. If we delete edges 5-6 and 24-1, then the signed graph is not balanced because it contains C_7^-, which is 6-18-17-16-15-14-7.
19. If we delete edges 5-6 and 1-8, then the signed graph is not balanced because it contains C_7^-, which is 2-3-15-16-17-18-19.
20. If we delete edges 5-6 and 8-9, then the signed graph is not balanced because it contains C_7^-, which is 6-18-17-16-15-14-7.
21. If we delete edges 5-6 and 9-21, then the signed graph is not balanced because it contains C_7^-, which is 2-3-15-16-17-18-19.
22. If we delete edges 6-7 and 21-22, then the signed graph is not balanced because it contains C_7^-, which is 1-24-12-11-10-9-8.
23. If we delete edges 6-7 and 22-23, then the signed graph is not balanced because it contains C_7^-, which is 2-3-15-16-17-18-19.
24. If we delete edges 6-7 and 23-24, then the signed graph is not balanced because it contains C_7^-, which is 1-24-12-11-10-9-8.
25. If we delete edges 6-7 and 24-1, then the signed graph is not balanced because it contains C_7^-, which is 2-3-15-16-17-18-19.
26. If we delete edges 6-7 and 1-8, then the signed graph is not balanced because it contains C_7^-, which is 5-6-18-19-20-21-22.
27. If we delete edges 6-7 and 8-9, then the signed graph is not balanced because it contains C_7^-, which is 2-3-15-16-17-18-19.
28. If we delete edges 6-7 and 9-21, then the signed graph is not balanced because it contains C_7^-, which is 6-18-19-2-3-4-5.
29. If we delete edges 7-14 and 21-22, then the signed graph is not balanced because it contains C_7^-, which is 1-24-12-11-10-9-8.
30. If we delete edges 7-14 and 22-23, then the signed graph is not balanced because it contains C_7^-, which is 1-24-12-11-10-9-8.
31. If we delete edges 7-14 and 23-24, then the signed graph is not balanced because it contains C_7^-, which is 1-24-12-11-10-9-8.
32. If we delete edges 7-14 and 24-1, then the signed graph is not balanced because it contains C_7^-, which is 2-3-15-16-17-18-19.

33. If we delete edges 7-14 and 1-8, then the signed graph is not balanced because it contains C_7^-, which is 2-3-15-16-17-18-19.

34. If we delete edges 7-14 and 8-9, then the signed graph is not balanced because it contains C_7^-, which is 2-3-15-16-17-18-19.

35. If we delete edges 7-14 and 9-21, then the signed graph is not balanced because it contains C_7^-, which is 2-3-15-16-17-18-19.

36. If we delete edges 14-15 and 21-22, then the signed graph is not balanced because it contains C_7^-, which is 1-24-12-11-10-9-8.

37. If we delete edges 14-15 and 22-23, then the signed graph is not balanced because it contains C_7^-, which is 1-24-12-11-10-9-8.

38. If we delete edges 14-15 and 23-24, then the signed graph is not balanced because it contains C_7^-, which is 1-24-12-11-10-9-8.

39. If we delete edges 14-15 and 24-1, then the signed graph is not balanced because it contains C_7^-, which is 2-3-15-16-17-18-19.

40. If we delete edges 14-15 and 1-8, then the signed graph is not balanced because it contains C_7^-, which is 2-3-15-16-17-18-19.

41. If we delete edges 14-15 and 8-9, then the signed graph is not balanced because it contains C_7^-, which is 2-3-15-16-17-18-19.

42. If we delete edges 14-15 and 9-21, then the signed graph is not balanced because it contains C_7^-, which is 2-3-15-16-17-18-19.

43. If we delete edges 15-3 and 21-22, then the signed graph is not balanced because it contains C_7^-, which is 1-24-12-11-10-9-8.

44. If we delete edges 15-3 and 22-23, then the signed graph is not balanced because it contains C_7^-, which is 1-24-12-11-10-9-8.

45. If we delete edges 15-3 and 23-24, then the signed graph is not balanced because it contains C_7^-, which is 1-24-12-11-10-9-8.

46. If we delete edges 15-3 and 24-1, then the signed graph is not balanced because it contains C_7^-, which is 6-7-14-15-16-17.

47. If we delete edges 15-3 and 1-8, then the signed graph is not balanced because it contains C_7^-, which is 2-3-4-5-6-18-19.

48. If we delete edges 15-3 and 8-9, then the signed graph is not balanced because it contains C_7^-, which is 2-3-4-5-6-18-19.

49. If we delete edges 15-3 and 9-21, then the signed graph is not balanced because it contains C_7^-, which is 2-3-4-5-6-18-19.

Hence, the frustration index of the signed graph that is obtained by switching the negation of $\Sigma_{11} \geq 3$, we get that $\Sigma_{11} \ncong -\Sigma_{11}$.

Table 2.5

Frustration index of Σ_i and $-\Sigma_i$

$\ell \setminus \Sigma$	Σ_1	Σ_2	Σ_3	Σ_4	Σ_5	Σ_6	Σ_7	Σ_8
$\ell(\Sigma)$	2	2	2	2	2	2	2	2
$\ell(-\Sigma)$	≥ 3	≥ 3	≥ 3	≥ 3	≥ 3	≥ 3	≥ 3	≥ 3

$\ell \setminus \Sigma$	Σ_9	Σ_{10}	Σ_{11}	Σ_{23}	Σ_{24}	Σ_{25}	Σ_{26}
$\ell(\Sigma)$	2	2	2	2	2	2	2
$\ell(-\Sigma)$	≥ 3	≥ 3	≥ 3	≥ 3	≥ 3	≥ 3	≥ 3

■

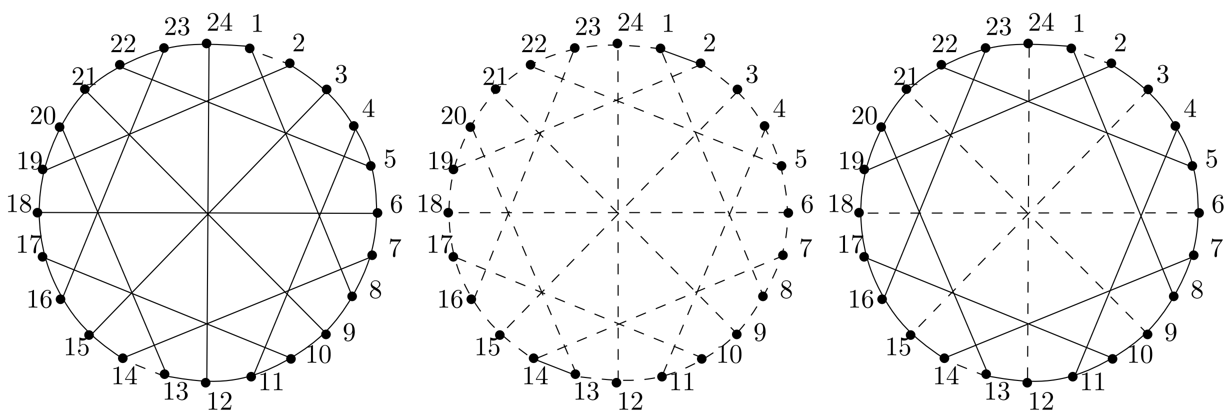

Figure 2.3

Σ_{11} signed graph, a negation of Σ_{11} signed graph, and the signed graph obtains by switching the negation of Σ_{11}

CHAPTER III

WHICH SIGNED GRAPHS ARE SIGN-SYMMETRIC WITH $\ell = 1$?

In this chapter, we will study signed graphs that are sign-symmetric with frustration index=1 ($\ell = 1$).

3.1 Introduction

A particular type of signed graph known as a sign-symmetric signed graph displays symmetry in the sign of the edges. Numerous studies have been conducted in various domains on this characteristic, which has significant consequences for the construction and behavior of these graphs [4]. In [13], gives new constructions of non-bipartite sign-symmetric signed graphs, and we will study different types of signed graphs and obtain a theorem for the signed graph of the Broken Wheel [30], which is a sign-symmetric signed graph.

Corollary 3.1.1. *A signed graph containing an odd number of triangles cannot be sign-symmetric [7].*

Lemma 3.1.1. *A necessary condition for a signed graph Σ to be sign-symmetric is*

$$C_3^-(\Sigma) = C_3^+(\Sigma).$$

3.2 Wheel Signed Graphs that are Sign-Symmetric

Definition 3.2.1. *A wheel graph is a graph formed by connecting a single universal vertex to all vertices of a cycle.*

Observation 1. *A signed graph on W_4 with ℓ=1 is sign-symmetric.*

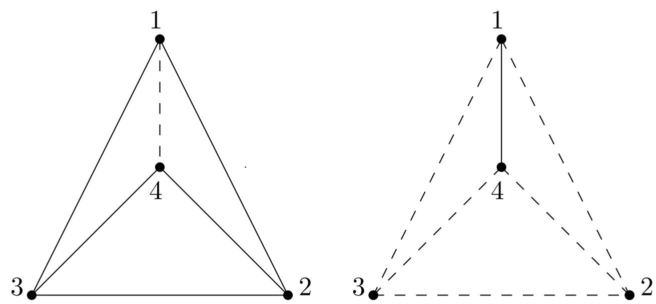

Figure 3.1

A signed graph on W_4 and its negation with $\ell = 1$

Lemma 3.2.1. *A signed graph on W_{2n+1} where $n = 2$ with $\ell = 1$ and all external edges are positive, and it contains a hub connected respectively by one negative edge with an odd vertex, one positive edge with an even vertex and one positive edge with an odd vertex. Then, a signed graph on W_{2n+1} where $n = 2$ is sign-symmetric with $\ell = 1$.*

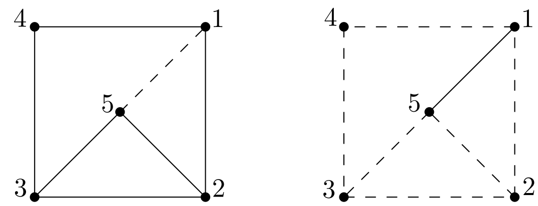

Figure 3.2

A signed graph on W_{2n+1} where $n = 2$ with three adjacent respectively internal edges, one negative and two positives, and its negation with $\ell = 1$

Theorem 3.2.1. *A signed graph on W_{2n+1} where $n \geq 2$ with $\ell = 1$ and all external edges are positive, and it contains a hub connected respectively by one negative edge with an odd vertex, one positive edge with an even vertex and one positive edge with an odd vertex. Then, a signed graph on W_{2n+1} where $n \geq 2$ is sign-symmetric with $\ell = 1$.*

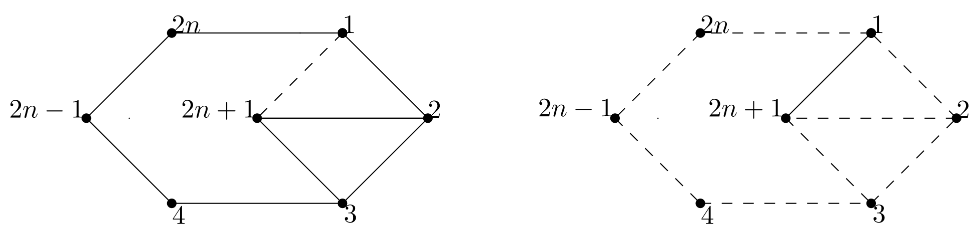

Figure 3.3

A signed graph on W_{2n+1} where $n \geq 2$ with three adjacent respectively internal edges, one negative and two positives, and its negation with $\ell = 1$

Proof. Let V = $\{1, 2, \ldots, 2n, 2n + 1\}$ be vertices of a signed graph. Assume all external edges 2n are positive, with three adjacent respectively internal edges, one negative with an odd vertex and two positive edges, one with an even vertex and one with an odd vertex. Taking the negation of the signed graph, the external edges 2n are negative, and there are three adjacent respectively internal edges, one positive with an odd vertex and two negative edges, one with an even vertex and one with an odd vertex. We get the originally signed graph by switching to an even external vertex and relabeling all the vertices. ■

Observation 2. *A signed graph on W_5 with $\ell = 1$ is sign-symmetric.*

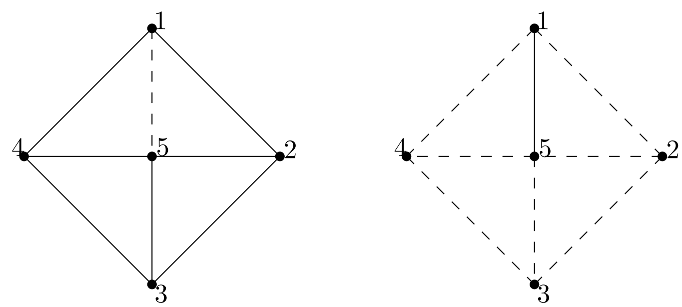

Figure 3.4

A signed graph on W_5 and its negation with $\ell = 1$

Lemma 3.2.2. *A signed graph on W_{2n+1} where $n = 3$ with $\ell = 1$ and all external edges are positive, and it contains a hub connected respectively by a positive edge with even vertex, one negative edge and one positive edge with odd vertices, and by one positive edge with even vertex. Then, a signed graph on W_{2n+1} where $n = 3$ is sign-symmetric with $\ell = 1$.*

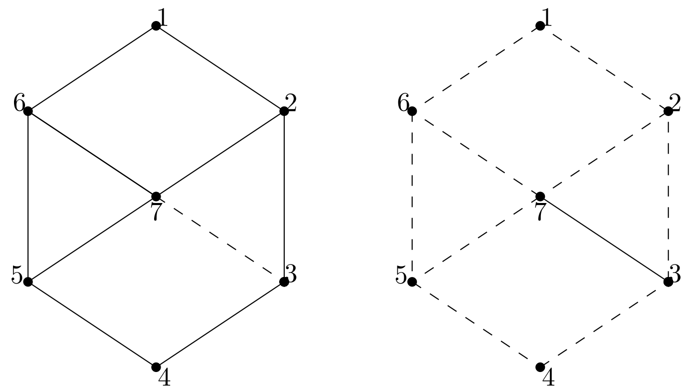

Figure 3.5

A signed graph on W_{2n+1} where $n = 3$ with a hub connected respectively by a positive edge with even vertex, one negative edge and one positive edge with odd vertices, and by one positive edge with even vertex and its negation with $\ell = 1$

Theorem 3.2.2. *A signed graph on W_{2n+1} where $n \geq 3$ with $\ell = 1$ and all external edges are positive, and it contains a hub connected respectively by a positive edge with even vertex, one negative edge and one positive edge with odd vertices, and by one positive edge with even vertex. Then, a signed graph on W_{2n+1} where $n \geq 3$ is sign-symmetric with $\ell = 1$.*

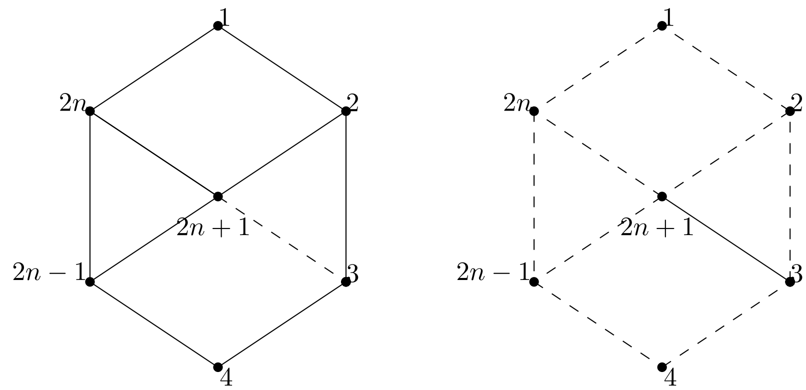

Figure 3.6

A signed graph on W_{2n+1} where $n \geq 3$ with a hub connected respectively by a positive edge with even vertex, one negative edge and one positive edge with odd vertices, and by one positive edge with even vertex and its negation with $\ell = 1$

Proof. Let V = $\{1, 2, \ldots, 2n, 2n + 1\}$ be vertices of a signed graph , $n \geq 3$. Assume all external edges 2n are positive, and the hub is connected respectively by a positive edge with an even vertex, one negative edge and one positive edge with odd vertices, and by one positive edge with an even vertex. Taking the negation of the signed graph, the outer edges 2n are negative. Also, three internal edges are negative, two with an even vertex and one with an odd vertex, and one positive internal edge with an odd vertex. Switching to the even external vertex and relabeling all the vertices results in the originally signed graph. ■

Definition 3.2.2. *A broken wheel is a wheel that obtains by deleting some internal edges.*

Lemma 3.2.3. *A signed graph on W_{2n+1} where $n = 3$, with $\ell = 1$ and all external edges are positive, and it contains a hub connected by a positive edge with all even vertices and with only two respectively odd vertices by one negative edge and one positive edge. Then, a signed graph on W_{2n+1} with $\ell = 1$ where $n = 3$ is sign-symmetric.*

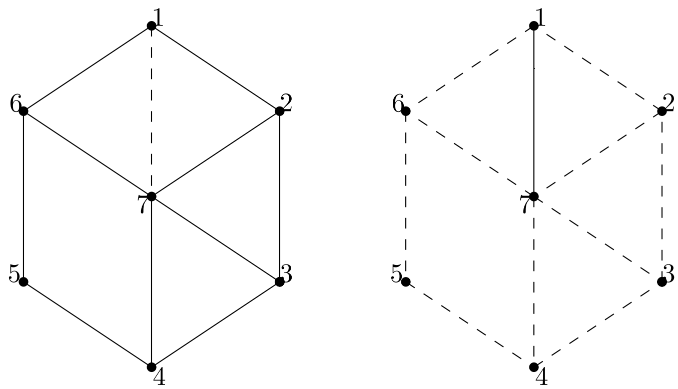

Figure 3.7

A signed graph on a broken wheel W_7 and its negation with $\ell = 1$

Theorem 3.2.3. *A signed graph on W_{2n+1} where $n \geq 3$ with $\ell = 1$ and all external edges are positive, and it contains a hub connected by a positive edge with all even vertices and with only two respectively odd vertices by one negative and one positive edge. Then, a signed graph on W_{2n+1} with $\ell = 1$ where $n \geq 3$ is sign-symmetric.*

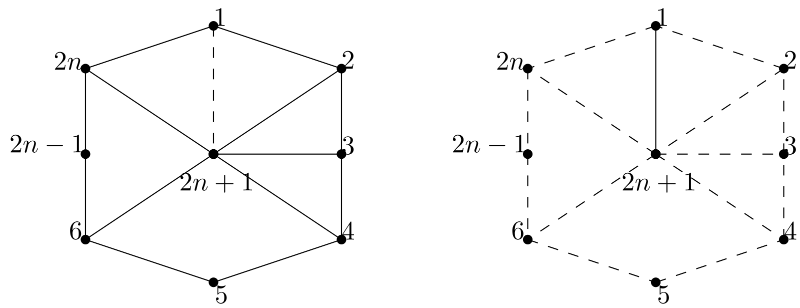

Figure 3.8

A signed graph on a Broken wheel and its negation with $\ell = 1$

Proof. Let V = $\{1, 2, \ldots, 2n, 2n + 1\}$ be vertices of a signed graph , $n \geq 3$. Assume all external edges 2n are positive, and the hub connects by positive edges with the even vertices and with only two respectively odd vertices by one negative and one positive edge. The negation of the signed graph results in the outer edges 2n beginning negative. Also, their internal edges are negative with an even vertex, one with an odd vertex, and one positive with an odd vertex. Switching to an even external vertex and relabeling results in the originally signed graph. ■

3.3 Cycle Graphs that are Sign-Symmetric

Observation 3. *A signed graph on C_{2n} where $n \geq 2$ with $\ell = 1$ is sign-symmetric.*

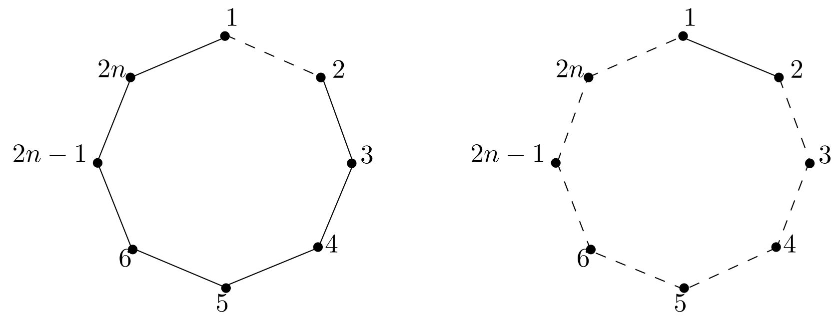

Figure 3.9

A signed graph on C_{2n} and its negation with $\ell = 1$

Lemma 3.3.1. *A signed graph on C_{2n} where $n = 3$, with $\ell = 1$ and all external edges are positive, and one vertex connects to all non-adjacent vertices respectively by two positive edges with an even vertex and one with an odd vertex and one negative edge with odd vertex. Then, a signed graph on C_{2n} where $n = 3$ is sign-symmetric with $\ell = 1$.*

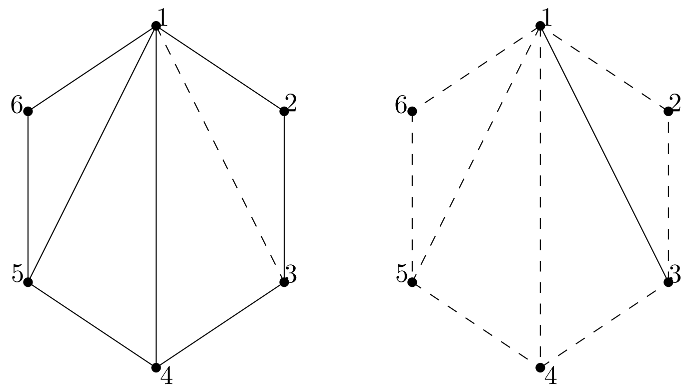

Figure 3.10

A signed graph on C_{2n} where $n = 3$ with one vertex connects to all non-adjacent vertices and its negation with $\ell = 1$

Theorem 3.3.1. *A signed graph on C_{2n} where $n \geq 3$ with $\ell = 1$ and all external edges are positive, and one vertex connects to all non-adjacent by a negative edge with an odd vertex, positive edges with even vertices, and by a positive edge with vertex 2n-1. Then, a signed graph on C_{2n} where $n \geq 3$ is sign-symmetric with $\ell = 1$.*

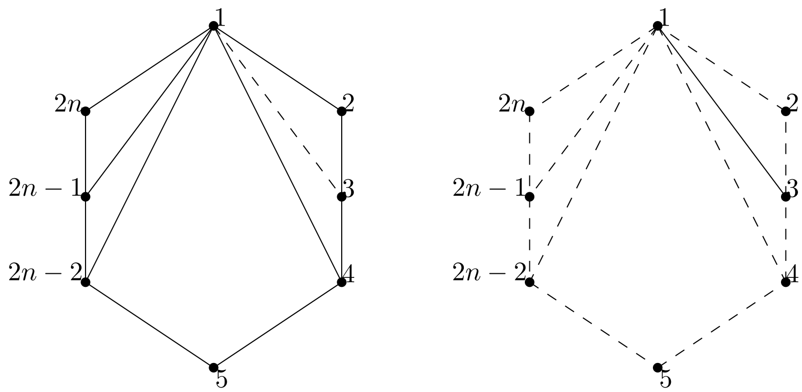

Figure 3.11

A signed graph on C_{2n} where $n \geq 3$ with one vertex connects to all non-adjacent vertices and its negation with $\ell = 1$

Proof. Let Σ be a signed graph on C_{2n} where $n \geq 3$ with$\ell(\Sigma) = 1$. Taking the negation of Σ, results in $\ell(-\Sigma) = 1$. Switching all even vertices results in all edges becoming positive except one negative edge incident to an odd vertex. Relabeling the vertices produces a signed graph C_{2n} where $n \geq 3$, with $\ell = 1$, is a sign-symmetric. ■

3.4 Complete Signed Graphs that are Sign-Symmetric

Definition 3.4.1. *A complete graph is a graph in which an edge connects each pair of graph vertices.*

Observation 4. *A signed graph on K_4 with $\ell = 1$ is sign-symmetric.*

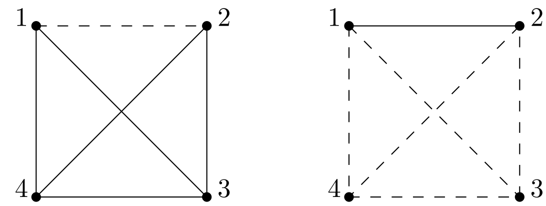

Figure 3.12

K_4 signed graph and its negation with $\ell = 1$

3.5 Famous Signed Graphs that are Sign-Symmetric

This section discusses some famously signed graphs that are sign-symmetric with $\ell = 1$. According to [25], there is one way to put one negative edge in a Heawood graph, and according to [6], there is one way to put one negative edge in the Petersen graph.

Lemma 3.5.1. *A Heawood signed graph with* $\ell = 1$ *is sign-symmetric.*

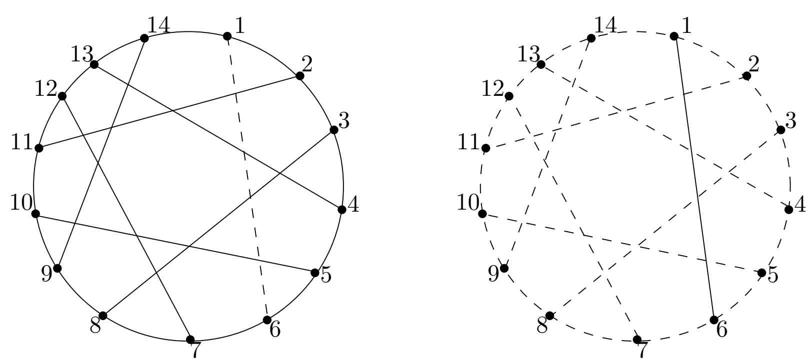

Figure 3.13

A Heawood signed graph and its negation with $\ell = 1$

Proof. Let Σ be a signed graph on a Heawood graph with $\ell(\Sigma) = 1$. Taking the negation of Σ results in all the edges becoming negative except one edge from vertex 1 to vertex 6, which becomes positive. Since the Heawood graph does not contain any odd cycles (bipartite), we now get the originally signed graph by switching all even vertices or odd vertices. Therefore, a Heawood signed graph with $\ell = 1$ is a sign-symmetric. ■

Lemma 3.5.2. *A Petersen signed graph with $\ell = 1$ is not sign-symmetric.*

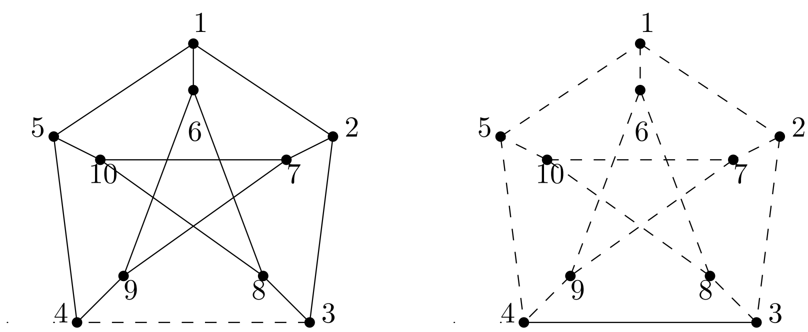

Figure 3.14

The Petersen signed graph and its negation with $\ell = 1$

Proof. Let Σ be a signed graph on a Petersen graph with $\ell(\Sigma) = 1$. By taking the negation of Σ, we get $-\Sigma$, which is not a balanced signed graph with at least two negative disjoint circles$\{1, 2, 3, 8, 6\}$ and $\{5, 10, 7, 9, 4\}$. Hence $\ell(-\Sigma) \geq 2$ and since $\ell(\Sigma) \neq \ell(-\Sigma)$. Therefore, a Petersen signed graph is not sign-symmetric. ■

Lemma 3.5.3. *A McGee signed graph with $\ell = 1$ is not sign-symmetric.*

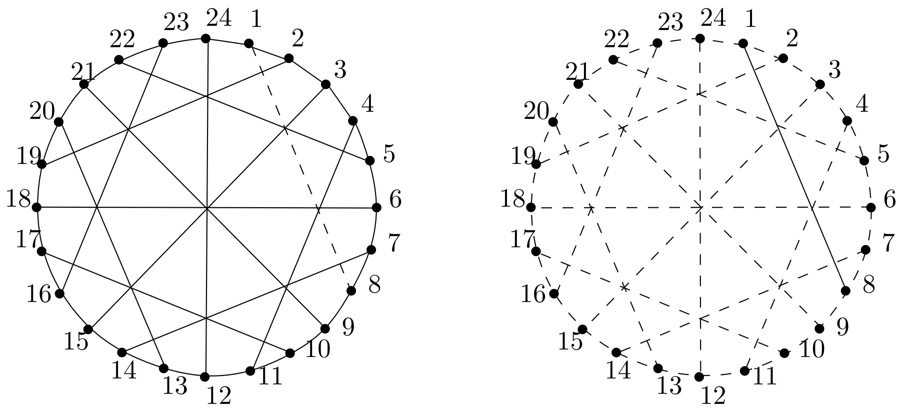

Figure 3.15

McGee signed graph and its negation with $\ell = 1$

Proof. Let Σ be a signed graph on a McGee graph with $\ell(\Sigma) = 1$. Now, by taking the negation of Σ we get $\ell(-\Sigma) \geq 2$ and since $\ell(\Sigma) \neq \ell(-\Sigma)$. Therefore, a McGee signed graph is not sign-symmetric with $\ell = 1$. ■

Lemma 3.5.4. *A prism signed graph with* $\ell = 1$ *is sign-symmetric if and only if n is even.*

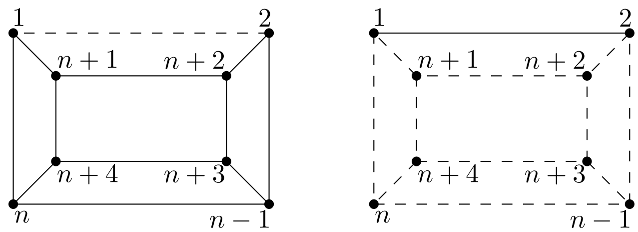

Figure 3.16

A Prism signed graph and its negation with $\ell = 1$

Proof. (⇒)

Let Σ be a signed graph on a prism graph with $\ell(\Sigma) = 1$. Taking the negation of Σ results in all the edges becoming negative except one edge that becomes positive. Switching the even vertices in the outer and switching the odd vertices in the internal produces the originally signed graph. Therefore, A prism-signed graph with $\ell = 1$ is a sign-symmetric where n is even.

(⇐)

Assume Σ is sign-symmetric since $\ell(\Sigma) = \ell(-\Sigma) = 1$. We claim n is even. Now, let n is odd. Then, we get $\ell(\Sigma) = 1$ but $\ell(-\Sigma)$ 1, which contradicts n is even. Therefore, A prism signed graph with $\ell = 1$ is a sign-symmetric where n is even. ■

Table 3.1

Table of Sign-Symmetric and Not Sign-Symmetric of Signed Graphs with $\ell = 1$

List of which Signed Graphs that are Sign-Symmetric with $\ell = 1$	
Sign-Symmetric	Not Sign-Symmetric
C_{2n}	C_{2n+1}
K_4	K_3 , K_5, ...
W_4, W_5	W_6, W_7, ...
W_{2n+1} with one negative edge and one positive from respectively odd vertices connect with the hub	W_{2n} with one negative edge and one positive from respectively odd vertices connect with the hub
C_{2n} with one vertex connected to all non-adjacent even vertices by positive edges and only two odd vertices by one negative edge and one positive	C_{2n+1} with one vertex connected to all non-adjacent even vertices by positive edges and only two odd vertices by one negative edge and one positive
Heawood signed graph	Petersen signed graph
Prism signed graph	McGee signed graph

CHAPTER IV

COMPLETE k-PARTITE SIGNED GRAPHS WITH $\ell = 1$ AND $\ell = 2$

4.1 Introduction

A complete k-partite sign-symmetric signed graph is a special case of a sign-symmetric signed graph. It has several interesting properties, including a high degree of symmetry and a well-defined spectral structure. The complete k-partite graph has been studied in various mathematics and computer science areas, including coding theory, graph theory, and optimization [10]. In [13], it gives new constructions of non-bipartite sign-symmetric signed graphs. If k=2, it is called a complete bipartite signed graph and a trivially sign-symmetric signed graph. In this chapter, we will study the complete k-partite signed graphs and characterize the complete k-partite signed graphs where $k \geq 3$ that are sign-symmetric with ℓ=1 and ℓ=2, and the results from this chapter have been submitted for publication [2].

4.2 Complete k-Partite Signed Graphs with $\ell = 1$

Definition 4.2.1. *A k-partite graph in which every two vertices from different partition classes are adjacent is called complete; the complete k-partite graphs for all k together are the complete multipartite graphs. The complete k-partite graph is denoted by $K_{n_1,\ldots,n_k}$; if $n_1 = \ldots = n_k = s$, we abbreviate this to K_s^k. Thus, K_s^k is the complete k-partite graph in which every partition class contains exactly s vertices [11].*

Lemma 4.2.1. *A necessary condition for a signed graph* Σ *to be sign-symmetric is* $C_3^-(\Sigma) = \frac{1}{2}\, C_3^+\,(\Sigma)$.

Lemma 4.2.2. *A signed graph on* $K_{1,1,1}$ *is not sign-symmetric with* ℓ *=1.*

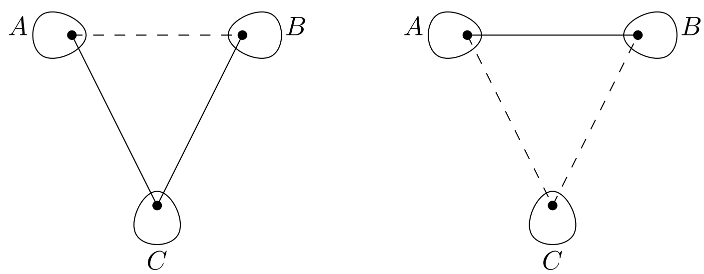

Figure 4.1

A signed graph on $K_{1,1,1}$ and its negation with $\ell = 1$

Proof. We see Σ is not a balanced signed graph and $\ell(\Sigma) = 1$. Now, by taking the negation of Σ, we get $-\Sigma$, a balanced signed graph, and $\ell(\Sigma) = 0$. Since $\ell(\Sigma) \neq \ell(-\Sigma)$, a signed graph on $K_{1,1,1}$ is not sign-symmetric with ℓ =1. ■

Lemma 4.2.3. *A signed graph on* $K_{1,n_2=\ell,n_3}$ *with* ℓ *is not sign-symmetric if negative edges between parts A and B.*

Proof. We see Σ is not a balanced signed graph and $\ell(\Sigma) = 1$. Now, by taking the negation of Σ we get $-\Sigma$, which is a balanced signed graph, and $\ell(-\Sigma) = 0$ since $\ell(\Sigma) \neq \ell(-\Sigma)$. Therefore, a signed graph on $K_{1,n_2=\ell,n_3}$ is not sign-symmetric with ℓ. ■

Lemma 4.2.4. *A signed graph on* $K_{1,3,3}$ *with* $\ell = 1$ *is not sign-symmetric.*

Proof. We see Σ is not a balanced signed graph and $\ell(\Sigma) = 1$. Now, by counting the number of triangles $[C_3]$ in the underlying graph for $K_{1,3,3}$ we get $\tau(K_{3,3,3})$ =9. Since the number of $C_3^+ \neq C_3^-$, a signed graph on $K_{1,3,3}$ with one negative edge is not sign-symmetric. ■

Lemma 4.2.5. *A signed graph on $K_{2,3,3}$ with ℓ=1 is not sign-symmetric.*

Proof. We see Σ is not a balanced signed graph and $\ell(\Sigma) = 1$. Now, by taking the negation of Σ we get $-\Sigma$, which is not a balanced signed graph, and $\ell(-\Sigma) > 2$ since $\ell(\Sigma) \neq \ell(-\Sigma)$. Therefore, a signed graph on $K_{2,3,3}$ is not sign-symmetric with ℓ=1. ■

Lemma 4.2.6. *A signed graph on $K_{3,3,3}$ with ℓ=1 is not sign-symmetric.*

Proof. We see Σ is not a balanced signed graph and $\ell(\Sigma) = 1$. Now, by counting the number of triangles $[C_3]$ in the underlying graph for $K_{3,3,3}$ we get $\tau(K_{3,3,3})$ =27. Since the number of $C_3^+ \neq C_3^-$, a signed graph on $K_{3,3,3}$ with one negative edge is not sign-symmetric. ■

Lemma 4.2.7. *A signed graph on $K_{4,4,4}$ with ℓ=1 is not sign-symmetric.*

Proof. Let Σ be a signed graph on $K_{4,4,4}$ with $\ell(\Sigma) = 1$. It is easy to see that $-\Sigma$ has at least three vertex-disjoint negative circles. Hence $\ell(-\Sigma) \geq 3$. We conclude that Σ is not sign-symmetric. ■

Lemma 4.2.8. *If $k = 3$ and $n_1 \geq 2$, then a signed graph on K_{n_1,n_2,n_3} with ℓ=1 is not sign-symmetric.*

Proof. Assume we have k = 3 parts. Parts A, B, and C contain at least two vertices. We connect parts A to B with one negative and three positive edges. Also, we connect parts C to A and B by positive edges. Now, by taking the negation of the signed graph, all edges become negative except one edge from A to B, which becomes positive. Now, we get $\ell(\Sigma) = 1$ but $\ell(-\Sigma) > 2$. Therefore, the signed graph on K_{n_1,n_2,n_3} where $n_1 \geq 2$ is not sign-symmetric with ℓ=1. ■

Lemma 4.2.9. *If $k = 3$ and $n_2 \geq 3$, then a signed graph on K_{1,n_2,n_3} with ℓ=1 is not sign-symmetric.*

Proof. Assume we have k = 3 parts. Part A contains one vertex, and B and C contain at least three vertices. We connect parts A to B with one negative and two positive edges. Also, we connect parts C to A and B by positive edges. Now, by taking the negation of the signed graph, all edges become negative except one edge from A to B, which becomes positive. Now, we get $\ell(\Sigma) = 1$ but $\ell(-\Sigma) > 2$. Therefore, the signed graph on K_{1,n_2,n_3} where $n_2 \geq 3$ is not sign-symmetric. ■

Theorem 4.2.1. *A signed graph on K_{n_1,n_2,n_3} with $\ell = 1$, is sign-symmetric if and only if*

1. *$n_1 = 1$ and $n_2 = 2$.*

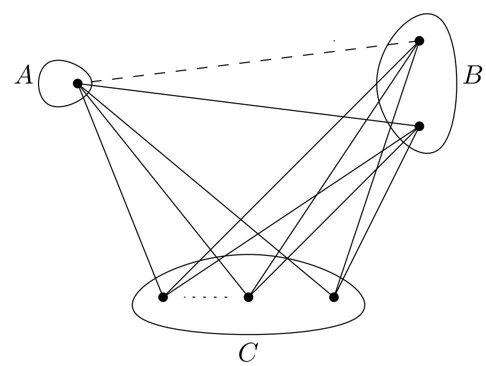

Figure 4.2

A sign-symmetric signed graph on $K_{1,2,n_{3\geq 2}}$ with $\ell = 1$

Proof. (⇒)

Assume we have three parts. A contains one vertex, B contains two vertices, and C contains at least two vertices. First, connect parts A to B with one negative and one positive edge. Also, connect parts C to A and B by positive edges. Next, by taking the negation of the signed graph, all edges become negative except one edge from A to B, which becomes positive. Then, switching all vertices on C, the signed graph becomes sign-symmetric.

($\Leftarrow$)

Assume Σ is a sign-symmetric signed graph. Since $\ell(\Sigma) = \ell(-\Sigma) = 1$, we claim $n_2 = 2$ and suppose $n_2 > 2$. Then, n_2 *and* $n_3 \geq 3$ so $\{a, b_2, c_2\}$ *and* $\{a, b_3, c_3\}$ are negative triangles. Therefore, $\ell(-\Sigma) > 1$, which contradicts $n_2 = 2$. We can see the theorem is verified.

■

Lemma 4.2.10. *If* $k = 4$ *and* $n_4 > 1$*, then* K_{n_1,n_2,n_3,n_4} *with* ℓ*=1 is not sign-symmetric.*

Proof. Assume Σ is a sign-symmetric signed graph. Since $\ell(\Sigma) = \ell(-\Sigma) = 1$, we claim $n_4 > 1$ and suppose $n_4 \geq 2$. Now, by counting the number of triangles $[C_3]$ in the underlying graph for $K_{1,1,1,2}$ we get $\tau(K_{1,1,1,2})$ =7. Since the number of $C_3^+ \neq C_3^-$, a signed graph on $K_{1,1,1,2}$ with one negative edge is not sign-symmetric. Now, take $n_4 = 3$. Then, $n_4 \geq 3$ so $\{b, c, d_1\}$ *and* $\{a, c, d_2\}$ are negative triangles. Therefore, we get $\ell(-\Sigma) = 2$, a signed graph on $K_{n_1,n_2,n_3,n_{4>1}}$ with ℓ=1 is not sign-symmetric. ■

Theorem 4.2.2. *A signed graph on* K_{n_1,n_2,n_3,n_4} *with* ℓ*=1 is sign-symmetric if and only if*

1. $n_1 = n_2 = n_3 = n_4 = 1.$

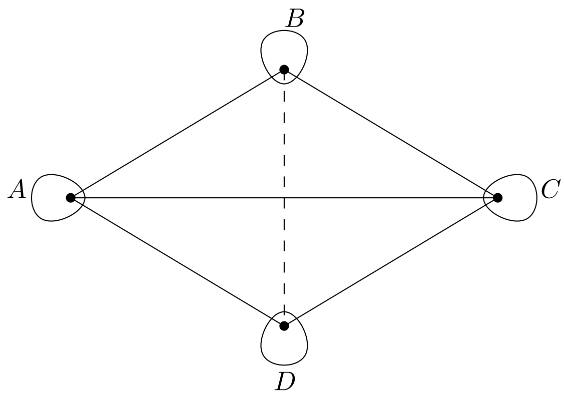

Figure 4.3

A sign-symmetric signed graph on $K_{1,1,1,1}$ with $\ell = 1$

Proof. $(\Rightarrow)$

Assume we have four parts, A, B, C, and D containing one vertex. Now, connect any two parts by one negative edge and the other by positive edges. Now, by taking the negation of the signed graph, all edges become negative except one edge between two parts is positive. Switching all vertices containing negative edges shows that all edges are positive except one. Now, by relabeling, the signed graph becomes sign-symmetric.

$(\Leftarrow)$

Assume Σ is a sign-symmetric signed graph and since $\ell(\Sigma) = \ell(-\Sigma) = 1$. We claim $n_4 = 1$ and suppose $n_4 > 1$. Then, $n_4 \geq 2$ so $\{a, b, d_2\}$ *and* $\{a, c, d_1\}$ are negative triangles. Therefore, we get $\ell(-\Sigma) > 1$, which contradicts $n_4 = 1$. ■

Lemma 4.2.11. *A signed graph on $K_{1,1,1,1,1}$ with ℓ=1 is not sign-symmetric.*

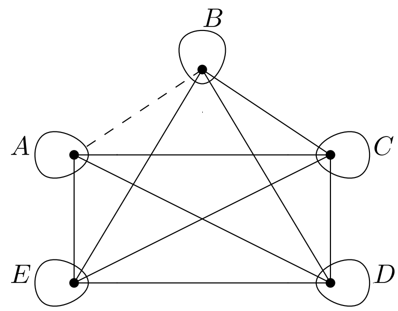

Figure 4.4

A signed graph on $K_{1,1,1,1,1}$ with ℓ=1

Proof. Assume Σ is a sign-symmetric signed graph and since $\ell(\Sigma) = \ell(-\Sigma) = 1$. We claim $n_5 = 1$ and suppose $n_5 > 1$. Then, $n_5 \geq 2$ so $\{a, d, e_1\}$ *and* $\{b, c, e_2\}$ are negative triangles. Therefore, we get $\ell(-\Sigma) > 1$, which contradicts $n_5 = 1$. Now, if $n_5 = 1$, then $\{b, c, e\}$ *and* $\{c, d, a\}$ are negative triangles. Hence $\ell(-\Sigma) > 1$. Therefore, we conclude that a signed graph on $K_{1,1,1,1,1}$ is not sign-symmetric with ℓ=1. ■

4.3 Complete k-Partite Signed Graphs with $\ell = 2$

Lemma 4.3.1. *A signed graph on $K_{1,2,2}$ with $\ell = 2$ is not sign-symmetric if two negative edges are between vertices from part C to B and A or from part B to A and C.*

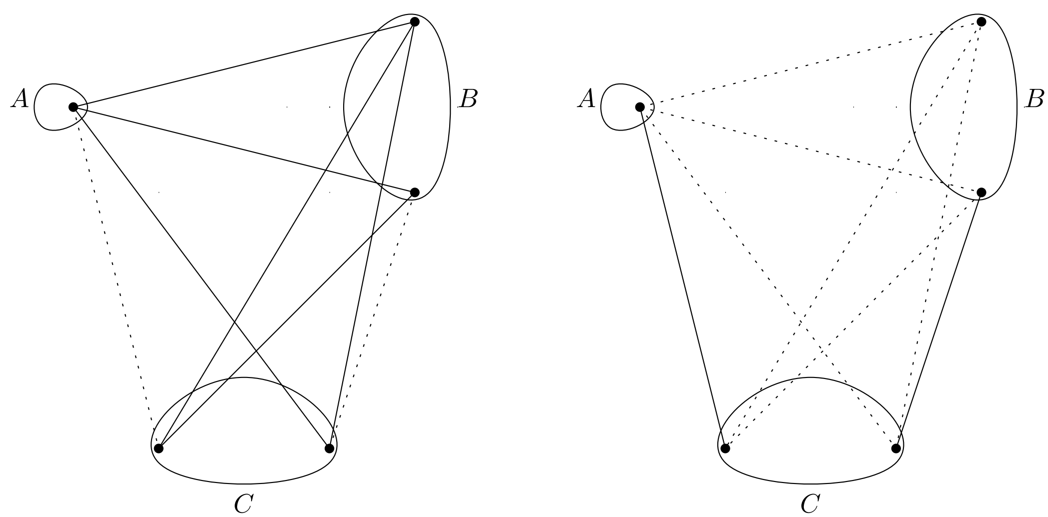

Figure 4.5

A signed graph on $K_{1,2,2}$ and its negation with $\ell = 2$

Proof. It is clear from the figure 4.5 the number of disjoint C_3^- of $K_{1,2,2}$ equals 2. Now by counting the number of disjoint C_3^- for the negation of $K_{1,2,2}$, equal 1. Therefore, a signed graph on $K_{1,2,2}$ with $\ell = 2$ is not sign-symmetric. ■

Theorem 4.3.1. *A signed graph on K_{n_1,n_2,n_3} is sign-symmetric with ℓ=2 if and only if :*

1. $n_1 = 1,\ n_2 = 2$ *and* $n_3 \geq 2$
2. $n_1 = 1,\ n_2 = 3$ *and* $n_3 = 4$
3. $n_1 = 1,\ n_2 = 4$ *and* $n_3 \geq 4$
4. $n_1 = n_2 = 2$ *and* $n_3 \geq 2$

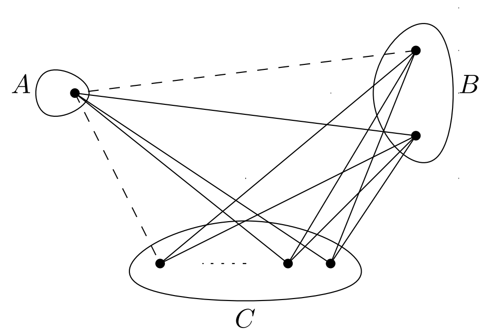

Figure 4.6

A sign-symmetric signed graph on $K_{1,2,n_3\geq 2}$ with $\ell = 2$

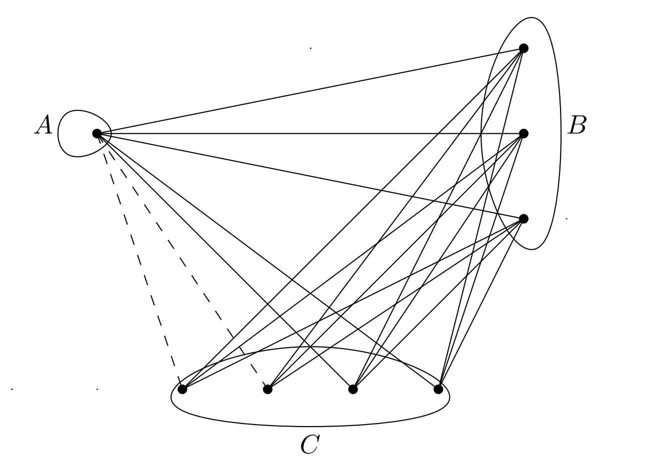

Figure 4.7

A sign-symmetric signed graph on $K_{1,3,4}$ with $\ell = 2$

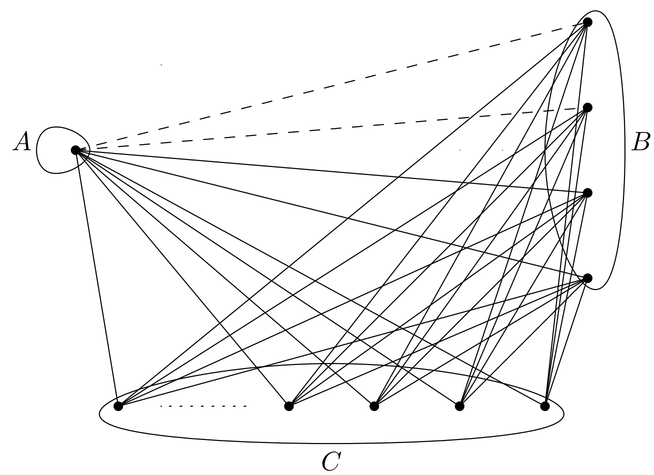

Figure 4.8

A sign-symmetric signed graph on $K_{1,4,n_3\geq 4}$ with $\ell = 2$

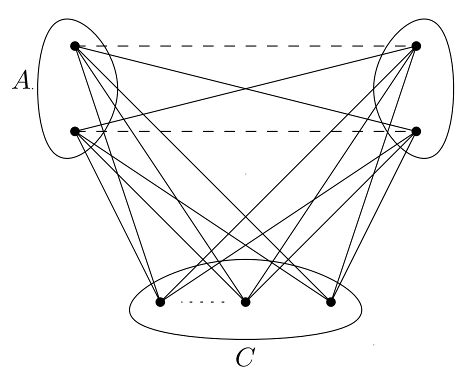

Figure 4.9

A sign-symmetric signed graph on $K_{2,2,n_3\geq 2}$ with $\ell = 2$

Proof. (⇒)

It is simple to verify that all signed graphs displayed above in Figures 4.6 to 4.9 are sign-symmetric.

(⇐)

Assume Σ is a sign-symmetric signed graph since $\ell(\Sigma) = \ell(-\Sigma) = 2$. We claim $n_1 \leq 2$ and suppose $n_1 > 2$. Then, $n_1,\ n_2$ *and* $n_3 \geq 3$ so $\{a_1, b_2, c_1\},\ \{a_2, b_3, c_2\}$ *and* $\{a_3, b_1, c_3\}$ are negative triangles. Therefore, we get $\ell(-\Sigma) > 2$, which contradicts $n_1 \leq 2$.

Now we have two cases:

Case (1) if $n_1 = 1$. We claim $2 \leq n_2 \leq 4$. Now let $n_2 \geq 4$. Then, n_2 *and* $n_3 \geq 5$, so $\{a, b_3, c_1\},\ \{a, b_4, c_2\}$ *and* $\{a, b_5, c_3\}$ are negative triangles. Therefore, we get $\ell(-\Sigma) > 2$, which contradicts $2 \leq n_2 \leq 4$. We can see 1 and 3 verified.

If $n_1 = 1$ *and* $n_2 = 3$. We claim $n_3 = 4$ and assume that $n_3 > 4$. Then, $n_3 \geq 5$, so we have two cases:

(1) If n_3 is an odd number. Then, by counting the number of triangles $[C_3]$ in the underlying graph for $K_{1,3,n_3}$ we get $\tau(K_{1,3,n_3})$ = odd number. Since the number of $C_3^+ \neq C_3^-$, a signed graph on $K_{1,3,n_3}$ with one negative edge is not sign-symmetric.

(2) If n_3 an even number and $n_3 > 4$. Then, $n_3 \geq 6$, we have two cases:

Case 1:

If 2 negative edges between parts A and B, then we get $\ell(\Sigma) = 2$ but $\ell(-\Sigma) = 1$.

If 2 negative edges between parts A and C, then we get $\ell(\Sigma) = 2$ but $\ell(-\Sigma) = 3$.

If 2 negative edges between parts B and C, then we get $\ell(\Sigma) = 2$ but $\ell(-\Sigma) = 3$.

Case 2:

If one negative edge between parts A and B and one negative edge between parts A and C, then we get $\ell(\Sigma) = 2$ but $\ell(-\Sigma) = 3$.

If one negative edge between parts A and B and one negative edge between parts B and C, then we get $\ell(\Sigma) = \ell(-\Sigma) = 2$, but after switching $\ell(-\Sigma)$ we get $\Sigma \ncong -\Sigma$.

If one negative edge between parts A and C and one negative edge between parts B and C, then we get $\ell(\Sigma) = 2$ but $\ell(-\Sigma) = 3$.

Therefore, a signed graph on $K_{1,3,n_3 \geq 6}$ is not sign-symmetric with $\ell = 2$. We can see 2 from the theorem is verified.

Case (2) if $n_1 = 2$. We claim $n_2 = 2$. Now assume that $n_2 > 2$. Then, n_2 *and* $n_3 \geq 3$, so $\{a_1, b_2, c_1\}$, $\{a_2, b_2, c_2\}$ *and* $\{a_1, b_3, c_3\}$ are negative triangles. Therefore, we get $\ell(-\Sigma) > 2$, which contradicts $n_2 = 2$. We proved 4 from the theorem. ■

Lemma 4.3.2. *If two negative edges incident on one vertex and contain an odd cycle, then a signed graph on* $K_{1,1,2,2}$ *is not-sign-symmetric.*

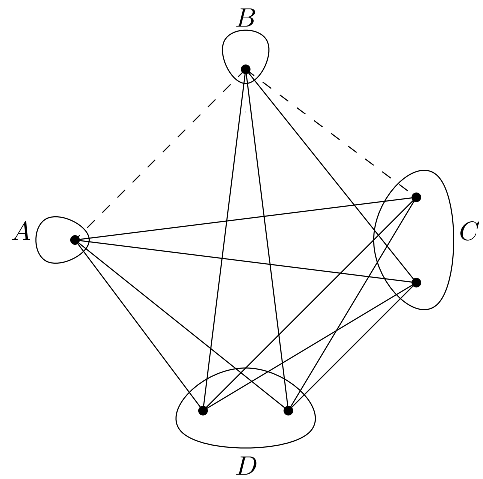

Figure 4.10

A signed graph on $K_{1,1,2,2}$ with 2 negative edges incident on one vertex adjacent to two different partitions

Proof. We can see $\ell(\Sigma) = 2$, but when we take the negation of the signed graph, we get $\ell(-\Sigma) > 2$. Therefore, a signed graph on $K_{1,1,2,2}$ with two negative edges incident on one vertex containing an odd cycle is not sign-symmetric with $\ell = 2$. ■

Lemma 4.3.3. *If two negative edges from two vertices in the same partition adjacent to two different partitions, then a signed graph on $K_{1,1,2,2}$ is not-sign-symmetric.*

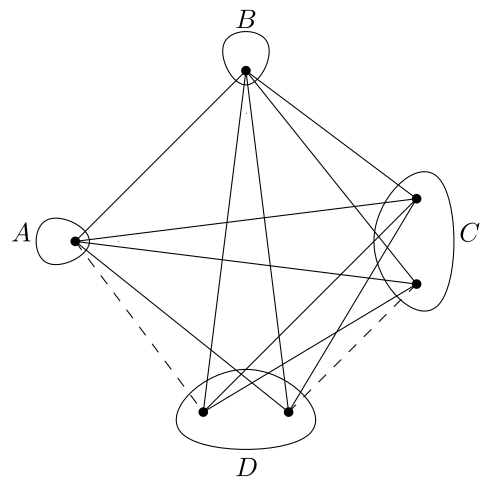

Figure 4.11

A signed graph on $K_{1,1,2,2}$ with 2 negative edges incident on two different vertices in the same partition adjacent to two different partitions

Proof. We can see $\ell(\Sigma) = 2$ but $\ell(-\Sigma) \geq 3$. Therefore, a signed graph on $K_{1,1,2,2}$ with two negative edges from two vertices in the same partition adjacent to two different partitions is not sign-symmetric with $\ell = 2$. ■

Theorem 4.3.2. *A signed graph on K_{n_1,n_2,n_3,n_4} is sign-symmetric with $\ell = 2$ if and only if $n_1 = n_2 = 1$ and $n_3 = n_4 = 2$.*

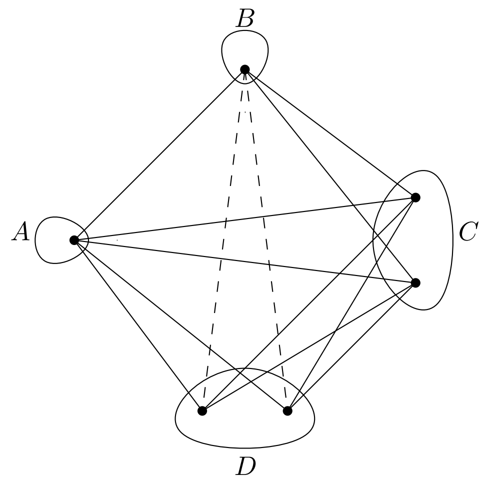

Figure 4.12

A sign-symmetric signed graph on $K_{1,1,2,2}$ with $\ell = 2$

Proof. ($\Rightarrow$)

Assume we have 4 parts. A, B, C, and D. Parts A and B contain one vertex, and parts C and D contain two vertices. Now, connect the part containing one vertex to the part containing two vertices, one of them by positive edges and the other by negative edges, or both by one positive edge and one negative edge. Also, connect all other vertices by positive edges. Now, by taking the negation of the signed graph, all edges become negative except the part that contains one vertex to the part that contains two vertices, one of them by positive edges and the other one by negative edges, or both of them by one positive edge and one negative edge. Now, by switching the vertices containing a maximum number of negative edges. Then, by relabeling, the signed graph $K_{1,1,2,2}$ becomes sign-symmetric with $\ell = 2$.

($\Leftarrow$)

Assume Σ is sign-symmetric since $\ell(\Sigma) = \ell(-\Sigma) = 2$. We claim $n_1 = 1$ and suppose $n_1 > 1$. Then, $n_1,\ n_2,\ n_3\ and\ n_4 \geq 2$ so $\{b_1, c_1, d_1\}\ \{b_2, c_2, d_2\}\ and\ \{a_2, b_1, d_2\}$ are negative triangles.

Therefore, we get $\ell(-\Sigma) > 2$, a contradiction. Hence $n_1 = 1$. Now, we claim $n_2 = 1$. Suppose $n_2 > 1$. Then, n_2, n_3 *and* $n_4 \geq 2$ so $\{a_1, b_2, d_2\}$, $\{a_1, b_2, c_1\}$ *and* $\{b_1, c_1, d_1\}$ are negative triangles. Therefore, we get $\ell(-\Sigma) > 2$, a contradiction. Hence $n_2 = 1$. Now also, we claim $n_3 = 2$. Suppose $n_3 > 2$. Then, n_3 and $n_4 \geq 3$ so $\{a, c_1, d_1\}$ $\{a, c_2, d_2\}$ *and* $\{a, b, c_2\}$ are negative triangles. Therefore, we get $\ell(-\Sigma) > 2$, a contradiction. Hence $n_3 = 2$. Now, we claim $n_4 = 2$. Suppose $n_4 > 2$. Then, $n_4 \geq 3$ so $\{a, b, c_2\}$ $\{a, b, d_2\}$ *and* $\{a, c_2, d_3\}$ are negative triangles. Therefore, we get $\ell(-\Sigma) > 2$, a contradiction. Hence $n_4 = 2$. We conclude that a signed graph on $K_{1,1,2,2}$ with $\ell = 2$ is sign-symmetric. ■

Lemma 4.3.4. *A signed graph on K_5 has two ways to be signed with two negative edges.*

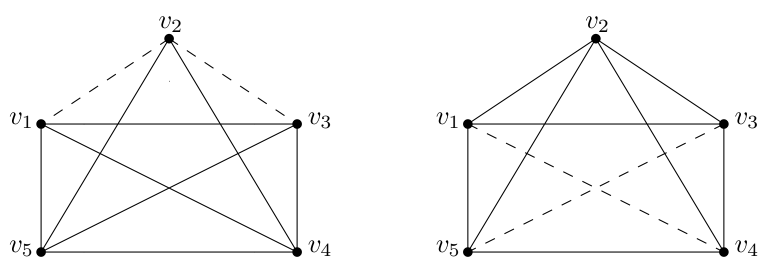

Figure 4.13

A signed graph on K_5 with two negative edges

Lemma 4.3.5. *A signed graph on $K_{1,1,1,1,1}$ is not sign-symmetric with $\ell = 2$.*

Proof. There are two cases:

Case 1): If two negative edges are adjacent on one vertex.

We see $\ell(\Sigma) = \ell(-\Sigma) = 2$, but if we switch the negation of $K_{1,1,1,1,1}$. We get two negative edges not adjacent to one vertex. Therefore, this case is not sign-symmetric.

Case 2): If two negative edges are not adjacent on one vertex.

Similar argument, we see $\ell(\Sigma) = \ell(-\Sigma) = 2$, but if we switch the negation, we get two negative edges adjacent to one vertex. Therefore, this case is not sign-symmetric.

(Other proof): if two negative edges are adjacent on one vertex.

By counting the number of triangles $[C_3]$ in the underlying graph for $K_{1,1,1,1,1}$, we get that

$$\tau(K_{1,1,1,1,1}) = n_1n_2n_3 + n_1n_2n_4 + n_1n_2n_5 + n_1n_3n_4 + n_1n_4n_5 + n_2n_3n_4 + n_2n_3n_5 + n_3n_4n_5 + n_4n_5n_2 + n_5n_3n_1 = 10$$

Next, we are counting the number of $C_3^+ = \{1,4,3\},\ \{1,2,5\},\ \{2,5,3\},\ \{2,5,4\},\ \{3,5,4\}$ *and* $\{4,2,3\} = 6$. Then, we are counting the number of $C_3^- = \{1,5,4\},\ \{1,4,2\},\ \{1,3,5\}$ *and* $\{1,2,3\} = 4$. Since the number of $C_3^+ \neq C_3^-$. Therefore, a signed graph on $K_{1,1,1,1,1}$ with two negative edges adjacent on one vertex is not sign-symmetric.

(Other proof): if two negative edges are not adjacent on one vertex.

By counting the number of triangles $[C_3]$ in the underlying graph for $K_{1,1,1,1,1}$, we get that, $\tau(K_{1,1,1,1,1}) = 10$. Next, we are counting the number of $C_3^+ = \{1,2,5\},\ \{1,2,3\},\ \{1,3,4\}$ *and* $\{1,4,5\} = 4$. Then, we are counting the number of $C_3^- = \{1,3,5\},\ \{1,2,4\},\ \{2,5,3\},\ \{2,3,4\},$ $\{2,5,4\}$ and $\{3,5,4\} = 6$. Since the number of $C_3^+ \neq C_3^-$. Therefore, a signed graph on $K_{1,1,1,1,1}$ with two negative edges not adjacent on one vertex is not sign-symmetric. ■

Observation 5. *We can see the number of* C_3^+ *in* Σ *in case 1 is equal to the number of* C_3^- *in* $-\Sigma$ *in case 2, which means* Σ *in case 1 and* $-\Sigma$ *in case 2 is switching isomorphism. Also, we can see the number of* C_3^- *in* $-\Sigma$ *in case 1 is equal to the number of* C_3^+ *in* Σ *in case 2, which means* $-\Sigma$ *in case 1 and* Σ *in case 2 is switching isomorphism for each other.*

Theorem 4.3.3. *A signed graph on* $K_{n_1,n_2,\ldots,n_k}$*, if* $k \geq 5$ *with* $\ell = 2$ *is not sign-symmetric.*

Proof. Assume Σ has at least five parts and has at least one vertex in each part. By using the previous proof of lemma 4.3.5, we get that the small piece of a signed graph is not sign-symmetric. Therefore, a signed graph on $K_{n_1,n_2,\ldots,n_k}$, if $k \geq 5$ with $\ell = 2$ is not sign-symmetric. ■

CHAPTER V

COMPLETE k-PARTITE SIGNED GRAPHS WITH $\ell = 3$

5.1 Introduction

The complete k-partite signed graphs that are sign-symmetric with $\ell = 1$ and $\ell = 2$ were described in Chapter 4, and we will continue to describe the complete k-partite signed graphs that are sign-symmetric with $\ell = 3$ in this chapter.

5.2 Complete k-Partite Signed Graphs where $k = 3$

Lemma 5.2.1. *Suppose a sign-symmetric signed graph exists on K_{n_1,n_2,n_3} with $\ell = 3$. Then,*

$$n_1 n_2 \leq 6$$

Proof. Let Σ be a sign-symmetric signed graph on K_{n_1,n_2,n_3}. By convention $n_1 \leq n_2 \leq n_3$. The number of triangles in K_{n_1,n_2,n_3} is $n_1 n_2 n_3$. Let C_3^+ and C_3^- denote the number of positive and negative triangles, respectively. We have

$$C_3^+(\Sigma) + C_3^-(\Sigma) = n_1 n_2 n_3 \tag{5.1}$$

$$C_3^+(-\Sigma) + C_3^-(-\Sigma) = n_1 n_2 n_3 \tag{5.2}$$

since Σ is sign-symmetric, $C_3^+(\Sigma) = C_3^+(-\Sigma)$ and $C_3^-(\Sigma) = C_3^-(-\Sigma)$. Also, $C_3^+(\Sigma) = C_3^-(-\Sigma)$ and $C_3^-(\Sigma) = C_3^+(-\Sigma)$.

$$C_3^-(\Sigma) \geq \frac{1}{2}\, n_1 n_2 n_3 \tag{5.3}$$

Let e_1, e_2, e_3 be three negative edges. Also, let us count the number of negative triangles containing e_1, e_2, e_3. The number of negative triangles containing e_i is at most n_3 for $i = 1, 2, 3$.

$$C_3^-(\Sigma) \leq n_3 + n_3 + n_3 \tag{5.4}$$

Then, we get:

$$\frac{1}{2}\, n_1 n_2 \leq C_3^-(\Sigma) \leq 3 \tag{5.5}$$

This completes the proof of the lemma. ∎

Theorem 5.2.1. *There exists a sign-symmetric signed graph on K_{n_1,n_2,n_3} with $\ell = 3$ if and only if:*

1. $n_1 = 1, n_2 = 3$ *and* $n_3 = 4$.
2. $n_1 = 1, n_2 = 3$ *and* $n_3 = 6$
3. $n_1 = 1, n_2 = 4$ *and* $n_3 \geq 4$.
4. $n_1 = 1,\ n_2 = 5$ *and* $n_3 = 6$.
5. $n_1 = 1\ n_2 = 6$ *and* $n_3 \geq 6$.
6. $n_1 = n_2 = 2$ *and* $n_3 \geq 2$.
7. $n_1 = 2,\ n_2 = 3$ *and* $n_3 \geq 3$.

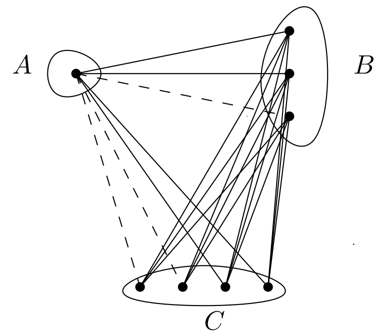

Figure 5.1

A sign-symmetric signed graph on $K_{1,3,4}$ with $\ell = 3$

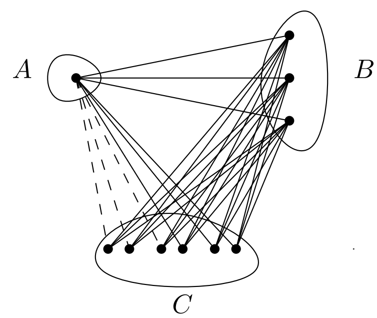

Figure 5.2

A sign-symmetric signed graph on $K_{1,3,6}$ with $\ell = 3$

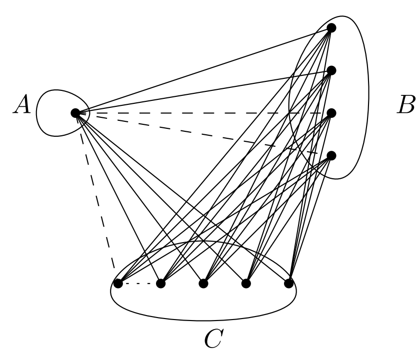

Figure 5.3

A sign-symmetric signed graph on $K_{1,4,n_3 \geq 4}$ with $\ell = 3$

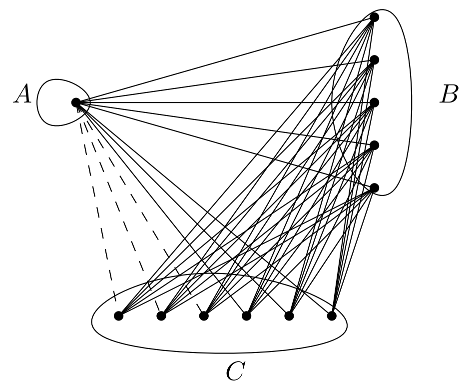

Figure 5.4

A sign-symmetric signed graph on $K_{1,5,6}$ with $\ell = 3$

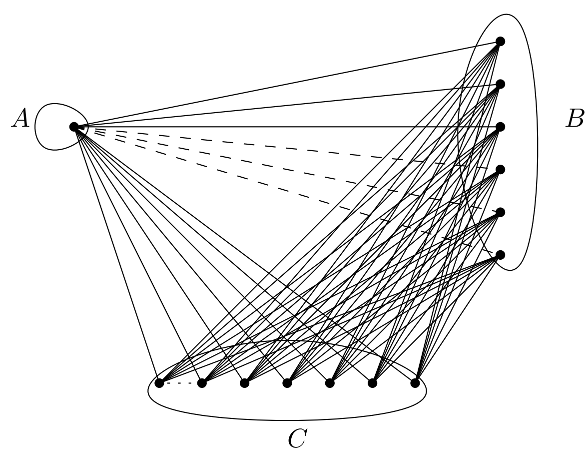

Figure 5.5

A sign-symmetric signed graph on $K_{1,6,n_3\geq 6}$ with $\ell = 3$

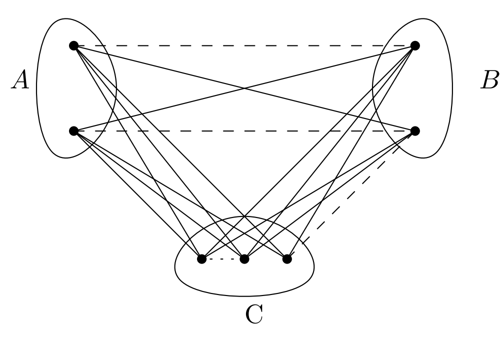

Figure 5.6

A sign-symmetric signed graph on $K_{2,2,n_3\geq 2}$ with $\ell = 3$

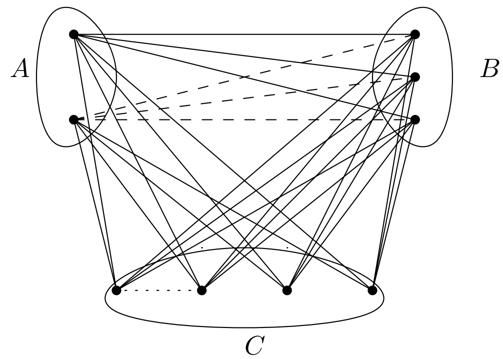

Figure 5.7

A sign-symmetric signed graph on $K_{2,3,n_3\geq 3}$ with $\ell = 3$

Proof. $(\Longrightarrow)$

It is easily checked that all signed graphs shown above in Figures (1 to 7) are sign-symmetric.

$(\Longleftarrow)$

Assume Σ is a sign-symmetric signed graph. Since $\ell(\Sigma) = \ell(-\Sigma) = 3$ we claim $1 \leq n_1 \leq 2$.

Suppose $n_1 > 2$. Then, $n_1,\ n_2\ and\ n_3 \geq 3$ so by using previous lemma

$\Longrightarrow\ 3 \geq \frac{1}{2}\, n_1 n_2$

$\Longrightarrow 3 \geq \frac{1}{2}\, 3 \cdot 3$

$\Longrightarrow\ 3 \geq 4.5$

Therefore, we get a contradiction, hence $n_1 \leq 2$.

Now we have two cases:

Case 1): Let $n_1 = 1$. We claim $n_2 \leq 6$. Then, by using the previous lemma

$\Longrightarrow\ 3 \geq \frac{1}{2}(n_1 \cdot n_2)$

we get:

$\Longrightarrow\ 3 \geq \frac{1}{2}(1 \cdot n_2)$

$\implies 6 \geq n_2$

Hence $n_2 \leq 6$. We can see 3 and 5 from the theorem are verified.

There are four cases:

1) $n_1 = 1$ *and* $n_2 = 1$. We claim $n_3 = 1$. Now assume that $n_3 > 1$. Then, $n_3 \geq 2$ so $\{a, b, c_1\}$ is a negative triangle. Therefore, we get $\ell(-\Sigma) < 3$, which contradicts $\ell(\Sigma) = \ell(-\Sigma)$=3, hence a signed graph on $K_{1,1,2}$ is not sign-symmetric. Now, if $n_3 = 1$, then a signed graph on $K_{1,1,1}$ with $\ell = 3$ does not balance, but the negation of $K_{1,1,1}$ is balance. Therefore, we get $\ell(-\Sigma) = 0$, which contradicts $\ell(\Sigma) = \ell(-\Sigma) = 3$, hence if $n_1 = n_2 = n_3 = 1$ is is not sign-symmetric with $\ell = 3$.

2) $n_1 = 1$ *and* $n_2 = 2$. We claim $n_3 = 2$. Now assume that $n_3 > 2$. Then, $n_3 \geq 3$ so $\{a, b_1, c_1\}$ *and* $\{a, b_2, c_3\}$ are negative triangles. Therefore, we get $\ell(-\Sigma) < 3$, which contradicts $\ell(\Sigma) = \ell(-\Sigma) = 3$, hence a signed graph on $K_{1,2,3}$ is not sign-symmetric with $\ell = 3$. Now, if $n_3 = 2$, so $\{a, b_1, c_1\}$ is a negative triangle. Therefore, we get $\ell(-\Sigma) = 1$, which contradicts $\ell(\Sigma) = \ell(-\Sigma) = 3$, hence if $n_1 = 1$ *and* $n_2 = n_3 = 2$ is is not sign-symmetric with $\ell = 3$.

3) $n_1 = 1$ *and* $n_2 = 3$. We have two cases:

(1) If n_3 is an odd number. Then, by counting the number of triangles $[C_3]$ in the underlying graph for $K_{1,3,n_3}$ we get $\tau(K_{1,3,n_3})$ = odd number. Since the number of $C_3^+ \neq C_3^-$, a signed graph on $K_{1,3,n_3}$ with $\ell = 3$ is not sign-symmetric.

(2) If n_3 is an even number and $n_3 > 6$. Then, $n_3 \geq 8$, we have there cases:

Case (1):

If 3 negative edges between parts A and B, then we get $\ell(\Sigma) = 3$ but $\ell(-\Sigma) = 0$.

If 3 negative edges between parts A and C, then we get $\ell(\Sigma) = \ell(-\Sigma) = 3$, but after switching $-\Sigma$, we get $C_3^- \neq C_3^+$ so $\Sigma \ncong -\Sigma$.

If 3 negative edges between parts B and C, then we get $\ell(\Sigma) = \ell(-\Sigma) = 3$, but after switching $-\Sigma$, we get $C_3^- \neq C_3^+$ so $\Sigma \ncong -\Sigma$.

Case (2):

If two negative edges between parts A and B and one negative edge between parts A and C, then we get $\ell(\Sigma) = 3$ but $\ell(-\Sigma) = 2$.

If two negative edges between parts A and B and one negative edge between parts B and C, then we get $\ell(\Sigma) = 3$ but $\ell(-\Sigma) = 1$.

If two negative edges between parts A and C and one negative edge between parts A and B, then we get $\ell(\Sigma) = \ell(-\Sigma) = 3$, but after switching $-\Sigma$, we get $C_3^- \neq C_3^+$ so $\Sigma \ncong -\Sigma$.

If two negative edges between parts A and C and one negative edge between parts B and C, then we get $\ell(\Sigma) = \ell(-\Sigma) = 3$, but after switching $-\Sigma$, we get $C_3^- \neq C_3^+$ so $\Sigma \ncong -\Sigma$.

If there are two negative edges between parts B and C and one negative edge between parts A and B, then we get $\ell(\Sigma) = 3$ but $\ell(-\Sigma) = 2$.

If there are two negative edges between parts B and C and one negative edge between parts A and C, then we get $\ell(\Sigma) = \ell(-\Sigma) = 3$, but after switching $-\Sigma$, we get $C_3^- \neq C_3^+$ so $\Sigma \ncong -\Sigma$.

Case (3):

If one negative edge between parts A and B, one negative edge between parts B and C, and one negative edge between C and A, then we get $\ell(\Sigma) = \ell(-\Sigma) = 3$ but after switching $-\Sigma$, we get $C_3^- \neq C_3^+$ so $\Sigma \ncong -\Sigma$.

Therefore, a signed graph on $K_{1,3,n_3>6}$ is not sign-symmetric with $\ell = 3$. We can see 1 and 2 from the theorem are verified.

4) If $n_1 = 1$ *and* $n_2 = 5$. We have two cases:

(1) If n_3 is an odd number. Then, by counting the number of triangles $[C_3]$ in the underlying graph for $K_{1,5,n_3}$, we get $\tau(K_{1,5,n_3})$ = odd number. Since the number of $C_3^+ \neq C_3^-$, a signed graph on $K_{1,5,n_3}$ with three negative edges is not sign-symmetric.

(2) If n_3 an even number and $n_3 > 6$. Then, $n_3 \geq 8$, we have there cases:

Case (1):

If there are 3 negative edges between parts A and B, then we get $\ell(\Sigma) = 3$ but $\ell(-\Sigma) = 2$.

If there are 3 negative edges between parts A and C, then we get $\ell(\Sigma) = 3$ but $\ell(-\Sigma) > 3$.

If there are 3 negative edges between parts B and C, then we get $\ell(\Sigma) = 3$ but $\ell(-\Sigma) = 5$.

Case (2):

If there are two negative edges between parts A and B and one negative edge between parts A and C, then we get $\ell(\Sigma) = 3$ but $\ell(-\Sigma) = 4$.

If there are two negative edges between parts A and B and one negative edge between parts B and C, then we get $\ell(\Sigma) = \ell(-\Sigma) = 3$, but after switching $-\Sigma$, we get $C_3^- \neq C_3^+$ so $\Sigma \ncong -\Sigma$.

If there are two negative edges between parts A and C and one negative edge between parts A and B, then we get $\ell(\Sigma) = 3$ but $\ell(-\Sigma) = 4$.

If there are two negative edges between parts A and C and one negative edge between parts B and C, then we get $\ell(\Sigma) = 3$ but $\ell(-\Sigma) > 3$.

If there are two negative edges between parts B and C and one negative edge between parts A and B, then we get $\ell(\Sigma) = 3$ but $\ell(-\Sigma) > 3$.

If there are two negative edges between parts B and C and one negative edge between parts A and C, then we get $\ell(\Sigma) = \ell(-\Sigma) = 3$, but after switching $-\Sigma$, we get $C_3^- \neq C_3^+$ so $\Sigma \ncong -\Sigma$.

Case (3):

If one negative edge between parts A and B, one negative edge between parts B and C, and one negative edge between C and A, we get $\ell(\Sigma) = 3$ but $\ell(-\Sigma) = 4$. Therefore, a signed graph on $K_{1,5,n_3>6}$ is not sign-symmetric with $\ell = 3$. We can see 4 from the theorem is verified.

Case 2) : Let $n_1 = 2$. We claim $n_2 \leq 3$. Then, by using the previous lemma

$\implies 3 \geq \frac{1}{2}(n_1 \cdot n_2)$

we get:

$\implies 3 \geq \frac{1}{2}(2 \cdot n_2)$

$\implies 3 \geq n_2$.

Therefore, we get $n_2 \leq 3$. We can see (6) and (7) from the theorem is verified. ■

5.3 Complete k-Partite Signed Graphs where $k = 4$

Lemma 5.3.1. *We use the following formula to count the number of triangles in the underlying graph for* K_{n_1,n_2,n_3,n_4}.

$$\tau(K_{n_1,n_2,n_3,n_4}) = n_1n_2n_3 + n_1n_2n_4 + n_1n_3n_4 + n_2n_3n_4$$

Lemma 5.3.2. *Assume there is a signed graph on* K_{n_1,n_2,n_3,n_4} *with* $\ell = 3$ *that is sign-symmetric. Then,*

$$3(n_3 + n_4) \geq \frac{1}{2}\tau(K_{n_1,n_2,n_3,n_4})$$

Theorem 5.3.1. *There exists a sign-symmetric signed graph on* $K_{n1,n2,n3,n4}$ *with* $\ell = 3$ *if and only if:*

1. $n_1 = n_2 = 1$ *and* $n_3 = n_4 = 2$
2. $n_1 = n_2 = 1$, $n_3 = 2$ *and* $n_4 = 4$

3. $n_1 = n_2 = 1$ *and* $n_3 = n_4 = 3$

4. $n_1 = 1$ *and* $n_2 = n_3 = n_4 = 2$

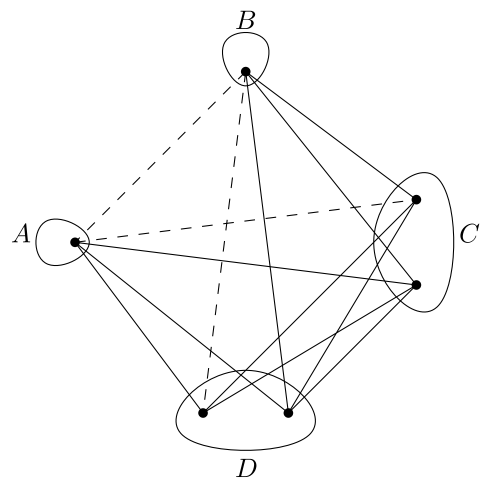

Figure 5.8

A sign-symmetric signed graph on $K_{1,1,2,2}$ with $\ell = 3$

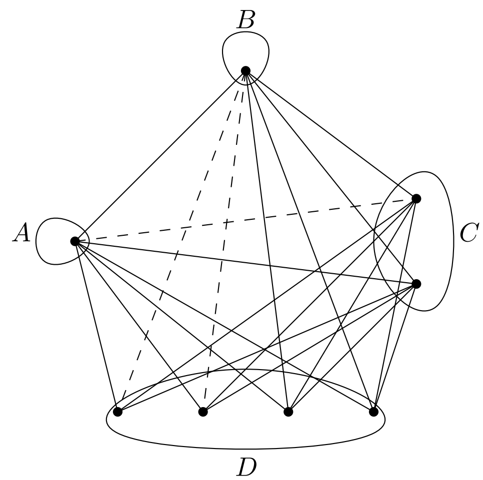

Figure 5.9

A sign-symmetric signed graph on $K_{1,1,2,4}$ with $\ell = 3$

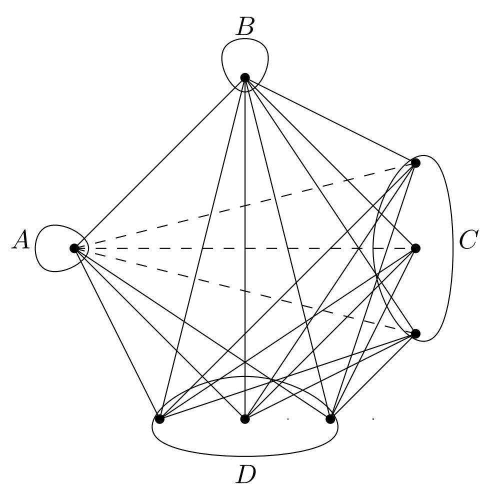

Figure 5.10

A sign-symmetric signed graph on $K_{1,1,3,3}$ with $\ell = 3$

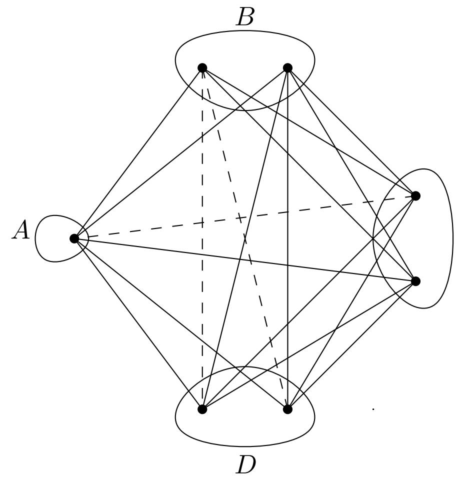

Figure 5.11

A sign-symmetric signed graph on $K_{1,2,2,2}$ with $\ell = 3$

Proof. $(\Longrightarrow)$

Verifying that every signed graph shown in Figures 8 through 11 above is a sign-symmetric signed graph is simple.

$(\Longleftarrow)$

Assume Σ is sign-symmetric signed graph since $\ell(\Sigma) = \ell(-\Sigma) = 3$. We claim $n_1 = 1$. Suppose $n_1 > 1$. Then, n_1, n_2, n_3 and $n_4 \geq 2$ so by using the previous lemma, we get

$3(n_3 + n_4) \geq \frac{1}{2}\, \tau(K_{n_1,n_2,n_3,n_4})$

$\Longrightarrow\ 3(2+2) \geq \frac{1}{2}(2\cdot2\cdot2 + 2\cdot2\cdot2 + 2\cdot2\cdot2 + 2\cdot2\cdot2)$

$\Longrightarrow\ 12 \geq \frac{1}{2}(32)$

Therefore, we get a contradiction, hence $n_1 = 1$.

Now, we claim $n_2 \leq 2$ and suppose $n_2 > 2$. Then, n_2, n_3 *and* $n_4 \geq 3$ so by using the previous lemma, we get

$3(n_3 + n_4) \geq \frac{1}{2}(n_1 \cdot n_2 \cdot n_3 + n_1 \cdot n_2 \cdot n_4 + n_1 \cdot n_3 \cdot n_4 + n_2 \cdot n_3 \cdot n_4)$

$\implies 3(3+3) \geq \frac{1}{2}(1 \cdot 3 \cdot 3 + 1 \cdot 3 \cdot 3 + 1 \cdot 3 \cdot 3 + 3 \cdot 3 \cdot 3)$

$\implies 18 \geq \frac{1}{2}(54).$

Therefore, we get a contradiction, hence $n_2 \leq 2$.

Now we have two cases:

Case 1) If $n_1 = 1$ *and* $n_2 = 1$. Now by using the formula

$$(6n_4 + n_3 + 4) \geq (n_4 + 2n_3 \cdot n_4) \tag{5.6}$$

Then, by taking $n_3 = 4$, we get that

$\implies (6n_4 + 4 + 4) \geq 9n_4$

$\implies 8 \geq 3n_4$

$\implies n_4 \leq 3$. Hence $n_3 \leq 3$.

Now If $n_1 = 1$ *and* $n_2 = 1$ we have three cases:

Case(1): $n_1 = 1$, $n_2 = 1$ *and* $n_3 = 1$ we have three cases:

(1) If n_4=1, then, by counting the number of triangles $[C_3]$ in the underlying graph for $K_{1,1,1,1}$, we get $\tau(K_{1,1,1,1}) = 4$. Since the number of $C_3^+ + C_3^- = 4$, a signed graph on $K_{1,1,1,1}$ with $\ell = 3$ is not sign-symmetric.

(2) If n_4=2 is an even number. Then, by counting the number of triangles $[C_3]$ in the underlying graph for $K_{1,1,1,2}$, we get $\tau(K_{1,1,1,2}) = 7$. Since the number of $C_3^+ \neq C_3^-$, a signed graph on $K_{1,1,1,2}$ with $\ell = 3$ is not sign-symmetric.

(3) If n_4=3 and by using the previous lemma,

$3(n_3 + n_4) \geq \frac{1}{2}(n_1 \cdot n_2 \cdot n_3 + n_1 \cdot n_2 \cdot n_4 + n_1 \cdot n_3 \cdot n_4 + n_2 \cdot n_3 \cdot n_4)$. Since

$3(1 + 3) \geq \frac{1}{2}(10)$.

Now, we have four cases:

(1) If there are two negative edges between parts D and C and one negative edge between parts D and A, then we get $\ell(\Sigma) = 3$ but $\ell(-\Sigma) = 2$.

(2) If there is one negative edge between parts D and A, one negative edge between parts D and B, and one negative edge between D and C, then we get $\ell(\Sigma) = \ell(-\Sigma) = 3$ but after switching $-\Sigma$, we get $C_3^- \neq C_3^+$ so $\Sigma \ncong -\Sigma$.

(3) If there is one negative edge between parts C and B, one negative edge between parts C and D, and one negative edge between D and A, then we get $\ell(\Sigma) = 3$, but $\ell(-\Sigma) = 2$.

(4) If there is one negative edge between parts C and A, one negative edge between parts C and B, and one negative edge between C and D, then we get $\ell(\Sigma) = 3$, but $\ell(-\Sigma) = 2$. Therefore, a signed graph on $K_{1,1,1,3}$ is not sign-symmetric with $\ell = 3$.

Case(2): $n_1 = 1$, $n_2 = 1$ *and* $n_3 = 2$ we have two cases:

(1) If n_4 is an odd number. Then, by counting the number of triangles $[C_3]$ in the underlying graph for $K_{1,1,2,n_4}$, we get $\tau(K_{1,1,2,n_4})$ = odd number. Since the number of $C_3^+ \neq C_3^-$, a signed graph on $K_{1,1,2,n_4}$ where n_4 is an odd number with $\ell = 3$ is not sign-symmetric.

(2) If n_3 is an even number and $n_4 > 4$. Then, $n_3 \geq 6$, so by using the previous lemma, we get

$3(n_3 + n_4) \geq \frac{1}{2}(n_1 \cdot n_2 \cdot n_3 + n_1 \cdot n_2 \cdot n_4 + n_1 \cdot n_3 \cdot n_4 + n_2 \cdot n_3 \cdot n_4)$. Since

$3(2 + 6) \geq \frac{1}{2}(32)$ this mean three negative edges between parts C *and* D, so we get $\ell(-\Sigma) > 3$, therefore a signed graph on $K_{1,1,2,6}$ is not sign-symmetric with $\ell = 3$. Hence a signed graph on

$K_{1,1,2,2}$ and $K_{1,1,2,4}$ are sign-symmetric with $\ell = 3$. We proved (1) and (2) from the theorem.

Case(3): $n_1 = 1$, $n_2 = 1$ *and* $n_3 = 3$ and by using equation number 5.6:

$$(6n_4 + n_3 + 4) \geq (n_4 + 2n_3 \cdot n_4)$$

and by substitute $n_3 = 3$, we get that,

$$(6n_4 + 3 + 4) \geq (n_4 + 2 \cdot 3 \cdot n_4)$$

then, after solving the equation, we obtain

$$7 \geq n_4$$

Now, we have two cases:

(1) If $n_4 = 3$ is an odd number, we claim $n_4 = 3$ and suppose $n_4 > 3$. Then, $n_4 \geq 5$, so by using the previous lemma, we get that

$3(n_3 + n_4) \geq \frac{1}{2}(n_1 \cdot n_2 \cdot n_3 + n_1 \cdot n_2 \cdot n_4 + n_1 \cdot n_3 \cdot n_4 + n_2 \cdot n_3 \cdot n_4)$. Since

$3(3 + 5) \geq \frac{1}{2}(38)$ this mean three negative edges between parts C *and* D, so we get $\ell(-\Sigma) > 3$, therefore a signed graph on $K_{1,1,3,5}$ is not sign-symmetric with $\ell = 3$.

(2) If n_4 is an even number. Then, by counting the number of triangles $[C_3]$ in the underlying graph for $K_{1,1,3,n_4}$, we get $\tau(K_{1,1,3,n_4})$ is an odd number. Since the number of $C_3^+ \neq C_3^-$, a signed graph on $K_{1,1,3,n_4}$ with $\ell = 3$ is not sign-symmetric. Hence a signed graph on $K_{1,1,3,3}$ is sign-symmetric with $\ell = 3$. We proved (3) from the theorem.

Case 2) if $n_1 = 1$ and $n_2 = 2$ we claim $n_3 = 2$ and suppose $n_3 > 2$. Now by taking $n_3 = 3$ and by using the formula

$$(n_3 + n_4) + (n_3 + n_4) + (n_4 + 2) \geq \frac{1}{2}(2 \cdot n_3 + 2 \cdot n_4 + n_3 \cdot n_4 + 2 \cdot n_3 \cdot n_4) \tag{5.7}$$

Then, by taking $n_3 = 3$ we get that,

$$12 + 6n_4 + 4 \geq 6 + n_4 + 9n_4$$

By resolving the aforementioned equation, we obtain that

$$10 \geq 4n_4 \implies 2.5 \geq n_4 \implies n_3 < 3$$

Therefore, we get a contradiction, hence $n_3 = 2$.

Now If $n_1 = 1$ *and* $n_2 = n_3 = 2$ and by using equation 5.7 to find n_4,

$$(n_3 + n_4) + (n_3 + n_4) + (n_4 + 2) \geq \frac{1}{2}(2 \cdot n_3 + 2 \cdot n_4 + n_3 \cdot n_4 + 2 \cdot n_3 \cdot n_4)$$

Next, we obtain that by substituting $n_3 = 2$,

$$(2 + n_4) + (2 + n_4) + (n_4 + 2) \geq \frac{1}{2}(2 \cdot 2 + 2 \cdot n_4 + 2 \cdot n_4 + 2 \cdot 2 \cdot n_4)$$

After solving the equation above for n_4, we can see that,

$$12 + 6n_4 \geq 4 + 8n_4 \implies 8 \geq 2n_4 \implies 4 \geq n_4$$

We claim $n_4 = 2$ and suppose $n_4 > 2$. Then, $n_4 \geq 3$ so by using the previous lemma, we get

$3(n_3 + n_4) \geq \frac{1}{2}(n_1 \cdot n_2 \cdot n_3 + n_1 \cdot n_2 \cdot n_4 + n_1 \cdot n_3 \cdot n_4 + n_2 \cdot n_3 \cdot n_4)$

$\implies 3(2 + 3) \geq \frac{1}{2}(1 \cdot 2 \cdot 2 + 1 \cdot 2 \cdot 3 + 1 \cdot 2 \cdot 3 + 2 \cdot 2 \cdot 3)$

$\implies 15 \geq \frac{1}{2}(28)$. This means three negative edges between parts $A,\ B\ and\ C$, so we have four cases:

(1) If there are two negative edges between parts A and B and one negative edge between parts A and C, then we get $\ell(\Sigma) = 3$ but $\ell(-\Sigma) \geq 4$.

(2) If there are two negative edges between parts A and B and one negative edge between parts B and C, then we get $\ell(\Sigma) = 3$ but $\ell(-\Sigma) \geq 4$.

(3) If there is one negative edge between parts A and B, one negative edge between parts B and C, and one negative edge between A and C, then we get $\ell(\Sigma) = 3$, but $\ell(-\Sigma) \geq 4$.

(4) If there is one negative edge between parts A and B and two negative edges between parts B and C, then we get $\ell(\Sigma) = 3$ but $\ell(-\Sigma) \geq 4$.

Therefore, a signed graph on $K_{1,2,2,3}$ with $\ell = 3$ is not sign-symmetric. Hence a signed graph on $K_{1,2,2,2}$ with $\ell = 3$ is sign-symmetric. We proved (4) from the theorem. ■

5.4 Complete k-Partite Signed Graphs where $k = 5$

Lemma 5.4.1. *The following formula is used to determine how many triangles are present in the underlying graph given* K_{n_1,n_2,n_3,n_4,n_5}

$$\tau(K_{n_1,n_2,n_3,n_4,n_5}) = n_1n_2n_3 + n_1n_2n_4 + n_1n_2n_5 + n_1n_3n_4 + n_1n_3n_5 + n_1n_4n_5 + n_2n_3n_4 + n_2n_3n_5 + n_2n_4n_5 + n_3n_4n_5$$

Lemma 5.4.2. *Consider a signed graph on* K_{n_1,n_2,n_3,n_4,n_5} *with* $\ell = 3$ *that is sign-symmetric. Then,*

$$3(n_3 + n_4 + n_5) \geq \frac{1}{2}\tau(K_{n_1,n_2,n_3,n_4,n_5})$$

Theorem 5.4.1. *There exists a sign-symmetric signed graph on* K_{n_1,n_2,n_3,n_4,n_5} *with* $\ell = 3$ *if and only if:*

1. $n_1 = n_2 = n_3 = n_4 = 1$ *and* $n_5 \geq 1$.

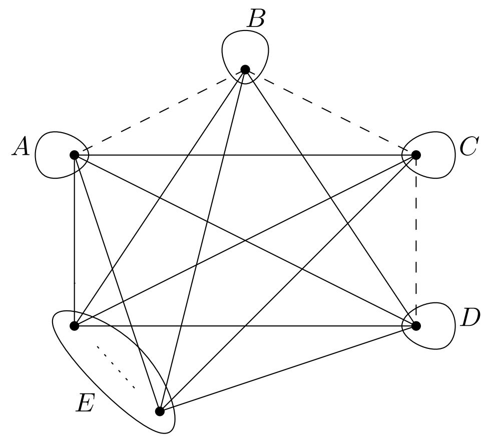

Figure 5.12

A sign-symmetric signed graph on $K_{1,1,1,1,n_5\geq 1}$ with $\ell = 3$

Proof. $(\Longrightarrow)$

It is evident that the aforementioned signed graph is sign-symmetric.

$(\Longleftarrow)$

Assume Σ is sign-symmetric signed graph since $\ell(\Sigma) = \ell(-\Sigma) = 3$. We claim $n_1 = 1$. Suppose $n_1 > 1$. Then, $n_1,\ n_2,\ n_3,\ n_4$ *and* $n_5 \geq 2$ so by using previous lemma we get,

$3(n_3 + n_4 + n_5) \geq \frac{1}{2}\ \tau(K_{n_1,n_2,n_3,n_4,n_5})$

$\implies\ 3(2+2+2) \geq \frac{1}{2}\ (80)$

$\implies\ 18 \geq 40$

Therefore, we get a contradiction, hence $n_1 = 1$.

Now we claim $n_2 = 1$. Suppose $n_2 > 1$. Then, $n_2,\ n_3,\ n_4$ *and* $n_5 \geq 2$ so by using the previous lemma, we get,

$3(n_3 + n_4 + n_5) \geq \frac{1}{2}\ \tau(K_{n_1,n_2,n_3,n_4,n_5})$

$\implies\ 3(2+2+2) \geq \frac{1}{2}\ (56)$

$\implies 18 \geq 28$

Therefore, we get a contradiction, hence $n_2 = 1$.

Now we claim $n_3 = 1$. Suppose $n_3 > 1$. Then, $n_3,\ n_4\ and\ n_5 \geq 2$ so by using the previous lemma, we get,

$3(n_3 + n_4 + n_5) \geq \frac{1}{2}\,\tau(K_{n_1,n_2,n_3,n_4,n_5})$

$\implies 3(2 + 2 + 2) \geq \frac{1}{2}\,(38)$

$\implies 18 \geq 19$

Therefore, we get a contradiction, hence $n_3 = 1$.

Now we claim $1 \leq n_4 \leq 2$. Suppose $n_4 > 2$. Then, $n_4\ and\ n_5 \geq 3$ so by using the previous lemma, we get,

$3(n_3 + n_4 + n_5) \geq \frac{1}{2}\,\tau(K_{n_1,n_2,n_3,n_4,n_5})$

$\implies 3(1 + 3 + 3) \geq \frac{1}{2}(46)$

$\implies 21 \geq 23$

Therefore, we get a contradiction, hence $1 \leq n_4 \leq 2$. We have two cases.

Case 1) if $n_4 = 1$, a signed graph on $K_{1,1,1,1,n_5}$ with $\ell = 3$ is sign-symmetric. We proved the theorem.

Case 2) if $n_4 = 2$, we claim $2 \leq n_5 \leq 3$. Suppose $n_5 > 3$. Then, $n_5 \geq 4$ so by using the previous lemma, we get,

$3(n_3 + n_4 + n_5) \geq \frac{1}{2}\,\tau(K_{n_1,n_2,n_3,n_4,n_5})$

$\implies 3(1 + 2 + 4) \geq \frac{1}{2}\,(43)$

$\implies 21 \geq 21.5$

Therefore, we get a contradiction, hence $2 \leq n_5 \leq 3$. We have two cases:

(1) Now If $n_1 = n_2 = n_3 = 1$ and $n_4 = n_5 = 2$, so by counting the number of triangles $[C_3]$ for K_{n_1,n_2,n_3,n_4,n_5}, we get $\tau(K_{1,1,1,2,2})$=25. Since the number of $C_3^+ \neq C_3^-$, a signed graph on $K_{1,1,1,2,2}$ with three negative edges is not sign-symmetric.

(2) Now If $n_1 = n_2 = n_3 = 1$, $n_4 = 2$ *and* $n_5 = 3$. Then, by using the previous lemma.

$3(n_3 + n_4 + n_5) \geq \frac{1}{2}\,\tau(K_{n_1,n_2,n_3,n_4,n_5})$

$\Longrightarrow\ 3(1 + 2 + 3) \geq \frac{1}{2}\,\tau(K_{1,1,1,2,3})$

$\Longrightarrow\ 18 \geq \frac{1}{2}(34)$. This means three negative edges between parts A, B, and C, so we get $\ell(-\Sigma) > 3$; hence $(K_{1,1,1,2,3})$ is not sign-symmetric. Therefore, a signed graph on $K_{1,1,1,2,n_5}$ with $\ell = 3$ is not sign-symmetric. ■

5.5 Complete k-Partite Signed Graphs where $k = 6$

Lemma 5.5.1. *We use the following formula to count the number of triangles in the underlying graph for $K_{n_1,n_2,n_3,n_4,n_5,n_6}$ as shown below*

$$\begin{aligned}\tau(K_{n_1,n_2,n_3,n_4,n_5,n_6}) =& n_1n_2n_3 + n_1n_2n_4 + n_1n_2n_5 + n_1n_2n_6 + n_1n_3n_4 + n_1n_3n_5 + n_1n_3n_6 +\\ & n_1n_4n_5 + n_1n_4n_6 + n_2n_3n_4 + n_2n_3n_5 + n_2n_3n_6 + n_2n_4n_5 + n_2n_4n_6 +\\ & n_2n_5n_6 + n_3n_4n_5 + n_3n_4n_6 + n_4n_5n_6 + n_5n_6n_1 + n_5n_6n_3\end{aligned}$$

Lemma 5.5.2. *Consider a signed graph on $K_{n_1,n_2,n_3,n_4,n_5,n_6}$ with $\ell = 3$ that is sign-symmetric. Then,*

$$3(n_3 + n_4 + n_5 + n_6) \geq \frac{1}{2}(\tau(K_{n_1,n_2,n_3,n_4,n_5,n_6}))$$

Theorem 5.5.1. *There exists a sign-symmetric signed graph on* $K_{n_1,n_2,n_3,n_4,n_5,n_6}$ *with* $\ell = 3$ *if and only if:*

1. $n_1 = n_2 = n_3 = n_4 = n_5 = n_6 = 1.$

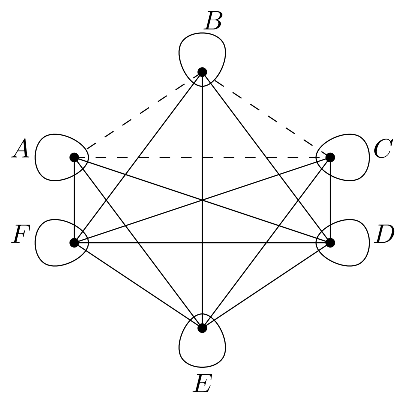

Figure 5.13

A sign-symmetric signed graph on $K_{1,1,1,1,1,1}$ with $\ell = 3$

Proof. ($\Rightarrow$)

It is easy to check the aforementioned signed graph is sing-symmetric

($\Leftarrow$)

Assume Σ is sign-symmetric signed graph since $\ell(\Sigma) = \ell(-\Sigma) = 3$. We claim $n_1 = 1$. Suppose $n_1 > 1$. Then, $n_1,\ n_2,\ n_3,\ n_4,\ n_5$ *and* $n_6 \geq 2$ so by using previous lemma we get,

$3(n_3 + n_4 + n_5 + n_6) \geq \frac{1}{2}\, \tau(K_{n_1,n_2,n_3,n_4,n_5,n_6})$

$\implies\ 3(2 + 2 + 2 + 2) \geq \frac{1}{2}\,(160)$

$\implies\ 24 \geq 80$

Therefore, we get a contradiction, hence $n_1 = 1$. Next, we claim $n_2 = 1$. Suppose $n_2 > 1$. Then,

$n_2,\ n_3,\ n_4,\ n_5$ and $n_6 \geq 2$ so by using previous lemma we get,

$3(n_3 + n_4 + n_5 + n_6) \geq \frac{1}{2}\,\tau(K_{n_1,n_2,n_3,n_4,n_5,n_6})$

$\implies 3(2+2+2+2) \geq \frac{1}{2}\,(120)$

$\implies 24 \geq 60$

The result is a contradiction, hence $n_2 = 1$. Afterward, we claim $n_3 = 1$. Suppose $n_3 > 1$. Then, $n_3,\ n_4,\ n_5\ and\ n_6 \geq 2$ so by using previous lemma we get,

$3(n_3 + n_4 + n_5 + n_6) \geq \frac{1}{2}\,\tau(K_{n_1,n_2,n_3,n_4,n_5,n_6})$

$\implies 3(2+2+2+2) \geq \frac{1}{2}\,(88)$

$\implies 24 \geq 44$

As a result, we obtain a contradiction, leading to $n_3 = 1$. Then, we claim $n_4 = 1$. Suppose $n_4 > 1$. Then, $n_4,\ n_5\ and\ n_6 \geq 2$ so by applying the previous lemma, we get

$3(n_3 + n_4 + n_5 + n_6) \geq \frac{1}{2}\,\tau(K_{n_1,n_2,n_3,n_4,n_5,n_6})$

$\implies 3(1+2+2+2) \geq \frac{1}{2}\,(63)$

$\implies 21 \geq 31.5$

A contradiction is produced as a result, which leads to $n_4 = 1$.

After that, we claim $n_5 = 1$. Suppose $n_5 > 1$. Then, $n_5\ and\ n_6 \geq 2$ hence, by using the previous lemma, we have

$3(n_3 + n_4 + n_5 + n_6) \geq \frac{1}{2}\,\tau(K_{n_1,n_2,n_3,n_4,n_5,n_6})$

$\implies 3(1+1+2+2) \geq \frac{1}{2}\,(44)$

$\implies 18 \geq 22$

As a result, there is a contradiction, which results in $n_5 = 1$. Finally, we claim $n_6 = 1$. Suppose $n_6 > 1$. Then, $n_6 \geq 2$ as a result, by applying the previous lemma, we have

$3(n_3 + n_4 + n_5 + n_6) \geq \frac{1}{2}\,\tau(K_{n_1,n_2,n_3,n_4,n_5,n_6})$

$\implies 3(1 + 1 + 1 + 2) \geq \frac{1}{2}\,(30)$

$\implies 15 \geq 15$

This means three negative edges between parts $A,\ B\ and\ C$, so we get $\ell(-\Sigma) > 3$, hence $(K_{1,1,1,1,1,2})$ is not sign-symmetric. Therefore, a signed graph on $K_{1,1,1,1,1,1}$ with $\ell = 3$ is sign-symmetric. ■

5.6 Complete k-Partite Signed Graphs where $k \geq 7$

Lemma 5.6.1. *The number of triangles in the underlying graph for $K_{n_1,n_2,n_3,n_4,n_5,n_6,n_7}$ is calculated using the following formula.*

$$\begin{aligned}\tau(K_{n_1,n_2,n_3,n_4,n_5,n_6,n_7}) =\ & n_1n_2n_3 + n_1n_2n_4 + n_1n_2n_5 + n_1n_2n_6 + n_1n_2n_7 + n_1n_3n_4 + n_1n_3n_5+\\ & n_1n_3n_6 + n_1n_3n_7 + n_1n_4n_5 + n_1n_4n_6 + n_1n_4n_7 + n_1n_5n_6 + n_1n_5n_7+\\ & n_1n_6n_7 + n_2n_3n_4 + n_2n_3n_5 + n_2n_3n_6 + n_2n_3n_7 + n_2n_4n_5 + n_2n_4n_6+\\ & n_2n_4n_7 + n_2n_5n_6 + n_2n_5n_7 + n_2n_6n_7 + n_3n_4n_5 + n_3n_4n_6 + n_3n_4n_7+\\ & n_3n_5n_6 + n_3n_5n_7 + n_3n_6n_7 + n_4n_5n_6 + n_4n_5n_7 + n_4n_6n_7 + n_5n_6n_7\end{aligned}$$

Lemma 5.6.2. *Consider a signed graph on $K_{n_1,n_2,n_3,n_4,n_5,n_6,n_7}$ with $\ell = 3$ that is sign-symmetric. Then,*

$$3(n_3 + n_4 + n_5 + n_6 + n_7) \geq \frac{1}{2}\tau(K_{n_1,n_2,n_3,n_4,n_5,n_6,n_7})$$

Theorem 5.6.1. *A signed graph on $K_{1,1,1,1,1,1,1}$ with $\ell = 3$ is not sign-symmetric.*

Proof. Assume Σ is sign-symmetric signed graph since $\ell(\Sigma) = \ell(-\Sigma) = 3$. Then, by using the previous lemma, we have

$3(n_3 + n_4 + n_5 + n_6 + n_7) \geq \frac{1}{2}\,\tau(K_{n_1,n_2,n_3,n_4,n_5,n_6,n_7})$

$\implies 3(1 + 1 + 1 + 1 + 1) \geq \frac{1}{2}\,(35)$

$\implies 15 \geq 17.5$

Therefore, we get a signed graph on $K_{1,1,1,1,1,1,1}$ with $\ell = 3$ is not sign-symmetric. ■

CHAPTER VI

NUMBER OF PARTS AND FRUSTRATION INDEX IN A SIGN-SYMMETRIC SIGNED COMPLETE MULTIPARTITE GRAPH

6.1 Introduction

In this chapter, we will study the relationship between k parts of the complete k-partite signed graphs and the frustration index of signed graphs ℓ. Also, we will study the relationship between k parts, n vertices, and frustration index ℓ. In [23], the authors mention that the maximum frustration index of numerous families of signed graphs, including the complete graph K_n with n vertices, has upper bounds provided by the authors in [28]. According to them, the frustration index of a complete signed graph is

$$\ell(K_n, \sigma) \leq \lfloor \frac{(n-1)^2}{4} \rfloor \tag{6.1}$$

and it was proved in [18], and the results from this chapter have been submitted for publication [3].

6.2 Relationship Between k Parts and Frustration Index ℓ

Theorem 6.2.1. *Let Σ be a sign-symmetric signed graph on a complete k-partite graph. Then,*

$$k - 2\sqrt{k} + 1 \leq \ell(\Sigma) \tag{6.2}$$

Table 6.1 shows the relationship between k parts and frustration index ℓ.

Table 6.1

List of Number of parts k and Frustration Index ℓ

k	$k-2\sqrt{k}+1$	ℓ_{min}
1	0	0
2	0	0
3	0.535	1
4	1	1
5	1.527	3
6	2.1	3
7	2.7	-
8	3.34	-

Proof. Let Σ be a sign-symmetric complete k-partite signed graph

$$\ell(K_n, \sigma) = \ell$$

Now by taking the negation of the complete k-partite signed graph and using 6.1 and by substituting n by k - ℓ into the equation 6.1, we get that

$$\ell(-K_n, \sigma) \leq \lfloor \frac{(k-\ell-1)^2}{4} \rfloor$$

Now, since $k \leq \ell$, then, we get

$$\frac{(k-\ell-1)^2}{4} \leq \ell$$

By taking the square root of both sides, we see that

$$\frac{(k-\ell-1)}{2} \leq \sqrt{\ell}$$

Consequently, by multiplying both sides by 2, we have that

$$(k-\ell-1) \leq 2\sqrt{\ell}$$

Then, move the k to the left side and put the constant and all the variable terms on the right side

$$k \leq 2\sqrt{\ell} + \ell + 1$$

Now, by completing the square, we obtain that

$$k \leq (1+\sqrt{\ell})^2$$

Next, taking square root on both sides

$$\sqrt{k} \leq 1 + \sqrt{\ell}$$

Then, we obtain that by subtracting -1 from both sides of the equation

$$\sqrt{k} - 1 \leq \sqrt{\ell}$$

Finally, square both sides of the equation, we get that

$$k - 2\sqrt{k} + 1 \leq \ell$$

Thus, this completes the proof of theorem 6.2.1. ■

Conjecture 6.2.2. *$k \leq \ell(\Sigma)$ if $k \geq 7$.*

6.3 Relationship Between k Parts, n Vertices and Frustration Index ℓ

Theorem 6.3.1. *Let Σ be a sign-symmetric signed graph on a complete k-partite graph ($k \geq 3$) such that all parts have the same number of vertices. Then,*

$$\ell \geq \frac{1}{54} n^2 \tag{6.3}$$

Proof. Let Σ be a sign-symmetric complete k-partite signed graph and let t^+ be a number of positive triangles and t^- be a number of negative triangles. Then, $t^- \leq \ell n$. Now, by using the formula where all parts have the same size of vertices,

$$t^+ + t^- \geq \binom{k}{3} \left(\frac{n}{k}\right)^3$$

Since Σ is a sign-symmetric complete k-partite signed graph $t^+ = t^-$. Now, since $t^- \leq \ell n$ and by substitution in the above formal, we get that

$$\ell n + \ell n \geq \binom{k}{3} \left(\frac{n}{k}\right)^3$$

By simplifying the left side, we get

$$2\ \ell n \geq \binom{k}{3}\left(\frac{n}{k}\right)^3$$

Now, by simplifying the right side

$$2\ \ell n \geq \frac{k(k-1)(k-2)}{6}\ \frac{n^3}{k^3}$$

Then, we get by deleting one k from the numerator and one from the denominator that

$$2\ \ell n \geq \frac{(k-1)(k-2)}{6}\ \frac{n^3}{k^2}$$

Now, by dividing both sides by $2n$, we obtain

$$\ell \geq \frac{(k-1)(k-2)}{12}\ \frac{n^2}{k^2}$$

Then, we get by simplifying the right side that

$$\ell \geq \frac{1}{12}\ (1-\frac{1}{k})\ (1-\frac{2}{k})\ n^2$$

Hence, $k \geq 3$ since complete k-partite signed graph

$$\ell \geq \frac{1}{54}\ n^2$$

As a result, the proof of the theorem is complete. ■

Corollary 6.3.1. *Let* Σ *be a sign-symmetric signed graph on a complete multipartite graph with all parts equal, then* $\ell(\Sigma) \geq |V(\Sigma)|$.

CHAPTER VII

FUTURE WORK

In this chapter, we mention some open problems that arise from this dissertation:

1. Give a complete characterization of signed-symmetric signed graphs with frustration number 1.
2. Give a complete characterization of sign-symmetric signed graphs with frustration index 3.
3. Give a characterization of signed graphs that are both sign-symmetric and projective-planar.
4. What is the relationship between the frustration index and frustration number in sign-symmetric signed graphs?
5. What is the complexity status of recognizing a sign-symmetric signed graph?

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