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Control Based on Saturated Time-Delay Systems Theory of Mach Number in Wind Tunnels

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Abstract A proposal for the regulation of the Mach number in wind tunnels using static state feedback for saturated systems with delays is presented here. As these systems can be precisely represented by a time-delay model with saturating inputs, a general solution for discrete delayed systems with saturating input is first derived. This general solution is based on modeling the saturation using a Lyapunov functional, using free weighting matrices and maximizing the set of admissible initial conditions. The application of this solution to the control of the Mach number in a wind tunnel is then presented, illustrating the design procedures.

Keywords Mach number · Wind tunnel · State feedback · Discrete delayed systems · Saturating input

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1 Introduction

The Mach number dynamics in a wind tunnel were studied in [15,16]. These dynamics are approximated with reasonable precision by a certain set of differential equations that include a delay in one of the state variables, representing the transportation time between the guide vanes of the fan and the test section of the tunnel. Control of this Mach number has been studied in the literature [12]: a closed-circuit fan-driven pressure tunnel, in which the Mach number is regulated by a fan motor speed regulation and by controlling small changes in the guide vane angle. Digital computers and microprocessors can easily be used to control the process, which has prompted us to develop the discrete-time controllers in this work. For example, previously in [12], a linear quadratic Gaussian (LQG) controller was proposed, albeit assuming the exact knowledge of the delay, which might be difficult in practice. Other controllers were developed in [13,14,18,19]. However, they did not take into account the saturation of the control input, which always exists in this control problem, because of the existence of technological and safety constraints. Input saturation is known to be a source of performance degradation, also generating limit cycles, multiple equilibrium points and even instability (see, for example, [4] and references therein). The presence in the same system of time delays and input saturations then makes the closed-loop stabilization a difficult problem. Therefore, in the present work, the effect of the saturation on the input is explicitly taken into account. It is emphasized that to develop a practical controller no assumption of online measurement of the delay is required.

As the model of the Mach number dynamics in a wind tunnel ([12–16] and references therein) corresponds to a set of differential equations with delays and saturated input, controller design can be facilitated by using techniques developed for feedback stabilization of time-delay systems [4–11,20–23]. From those approaches, our proposal is directly with the one discussed in [11], where six free matrices $P_{1,\ldots,6}$ were used to characterize the controller, with only one of them restricted to be positive definite, as an extension to the methodology introduced in [5]. It must be pointed out that in [10], an approach was presented to treat the actuator saturation by transforming them into a convex combination of polytopes: some ideas of that work are also used here (Lyapunov functional, free weighting matrices technique), but a simpler methodology is presented to avoid the complexity of using the convex combination of polytopes for real system. The proposed approach is then inspired by these previous results, but the system discretization inherent to periodic sampling of practical control problem is explicitly taken into account here. Thus, in this paper, a solution to general time-delay systems with saturating input is first developed, which is then particularized for the wind tunnel problem. More precisely, a static state feedback controller for the Mach number is developed to achieve good transient responses: This makes it possible to reduce the operating cost by reducing liquid nitrogen losses. The performance and effectiveness of the proposed control law are then validated through detailed simulations.

Notation. $sat(u)$ is the vector valued saturation function described by $sat(u, u_0) = [sat(u_1) \ldots sat(u_m)]^T$, where $sat(u_l) = sign(u_l) min\{u_{0_l}, u_l\}$, $l = 1, \ldots, m$. H_l denotes the l-th row of H. $\overline{\lambda}(P)$ denotes the maximal eigenvalue of matrix P.

2 Feedback Control of Mach Number in a Wind Tunnel

2.1 Dynamic Modeling

As has been shown in the literature [15], the dynamic deviations δM of the Mach number to small deviations in the guide vane angle actuator $\delta\theta_a$ in a driving fan are precisely described at a given operating point (determined by the fan speed, liquid nitrogen injection rate, and gaseous-nitrogen vent rate) by the following dynamic model [15]:

$$\begin{aligned}&\frac{1}{a}\delta\dot{M}(t) + \delta M(t) = k\,\delta\theta(t-\tau(t))\\&\delta\ddot{\theta}(t) + 2\xi w\delta\dot{\theta}(t) + w^2\delta\theta(t) = w^2\delta\theta_a(t)\end{aligned} \tag{1}$$

where $\delta\theta$ is the guide vane angle, a, k, ξ, w are parameters which, at each working point, can be assumed constant if the deviations δM, $\delta\theta$, $\delta\theta_a$ are small. The delay $\tau(t)$ represents the time required by the movement of air between the fan and the test section.

The equations above do not represent the effect of saturation in the control signal, which is always present in the guide vane angle actuator. Thus, the following model is proposed here to represent the dynamics of the real system more precisely:

$$\begin{aligned}&\frac{1}{a}\delta\dot{M}(t) + \delta M(t) = k\,\delta\theta(t-\tau(t))\\&\delta\ddot{\theta}(t) + 2\xi w\delta\dot{\theta}(t) + w^2\delta\theta(t) = w^2 sat(\delta\theta_a(t), u_0)\end{aligned} \tag{2}$$

It must be pointed out that, to simplify the numerical notation, the saturation is assumed to be symmetric and bounded by u_0; however, non-symmetric saturations that would appear in practice [1,2,17] could be easily incorporated following the ideas presented in [1].

State-space tools will be used to develop the controller, so (2) is rewritten in state-space form as follows:

$$\begin{aligned}\dot{x}(t) &= A_c x(t) + A_{\tau_c} x(t-\tau(t)) + B_c sat(u(t), u_0) + B_{w_c} w(t)\\ z(t) &= C_{z_c} x(t)\end{aligned} \tag{3}$$

where

$$x = \begin{bmatrix}\delta M\\ \delta\theta\\ \delta\dot{\theta}\end{bmatrix},\ A_c = \begin{bmatrix}-a & 0 & 0\\ 0 & 0 & 1\\ 0 & -w^2 & -2\xi w\end{bmatrix},\ A_{\tau_c} = \begin{bmatrix}0 & ak & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},$$
$$B_c = \begin{bmatrix}0\\ 0\\ w^2\end{bmatrix},\ B_{w_c} = \begin{bmatrix}0\\ 0\\ 10\end{bmatrix},\ z = \begin{bmatrix}\delta M\\ \delta\theta\end{bmatrix},\ C_{z_c} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}$$

In practice the Mach number is controlled by using microprocessor-based systems, so a digital controller will be designed for an analog time-delay system [22]. Thus, the following section discusses the discrete-time closed-loop model that will be used to derive the main results.

2.2 Problem Statement

From the previous discussion, it follows that our aim is to design a microprocessor-based controller for the class of systems whose dynamic is assumed to be described by (3), where matrices A_c, A_{τ_c}, B_c, B_{w_c}, and C_{z_c} are known. Assuming periodic sampling, it follows that our controller design will be based on the following discrete-time system with time-varying delay:

$$\begin{aligned} x(k+1) &= Ax(k) + A_d x(k-d(k)) + Bsat(u(k), u_0) + B_w w(k) \\ z(k) &= C_z x(k) \end{aligned} \tag{4}$$

where $A = e^{A_c T}$, $A_d = \int_0^T e^{A_c s} A_{\tau_c} ds$, $B = \int_0^T e^{A_c s} B_c ds$, $B_w = \int_0^T e^{A_c s} B_{w_c} ds$, and $C_z = C_{z_c}$ and $d(k)$ is a time-dependent positive integer representing the time delay, satisfying $0 \leq d(k) \leq d_M$ where d_M is a known positive and finite integer.

The disturbance vector $w(k)$ is assumed to be limited in energy, that is, $w(k) \in \mathcal{L}_2$. Hence, for some scalar ω, the disturbance $w(k)$ is bounded as follows:

$$\|w(k)\|_2^2 = \sum_{k=0}^{\infty} w^T(k) w(k) \leq \omega^{-1} < \infty \tag{5}$$

We suppose that the input vector $u(k)$ is subject to amplitude limitations $|u_l(k)| \leq u_{0_l}$, $u_{0_l} > 0$. Due to these control bounds, the effective control signal to be applied to the system (4) is $sat(u(k), u_0) = sat(Kx(k), u_0)$, where K is the vector that contains the three control gains to be determined in order to ensure closed-loop stability for the largest possible set of deviations from the working point. This set, called here estimated domain of attraction, is denoted $\Xi \subset \Phi$, and defined to be:

$$\Xi = \left\{ \phi_l(k), -d_M \leq k \leq 0 : \max \|\phi_l(k)\| \leq \delta \right\}$$

with initial condition $x_0 = \phi_l(k)$. The domain of attraction of the origin is:

$$\Phi = \left\{ \phi_l(k), -d_M \leq k \leq 0 : \lim_{k \to \infty} \phi_l(k, x_0) = 0 \right\}$$

Hence, with the proposed controller, the feedback system is:

$$\begin{aligned} x(k+1) &= Ax(k) + A_d x(k-d(k)) + Bsat(Kx(k), u_0) + B_w w(k) \\ z(k) &= C_z x(k) \end{aligned} \tag{6}$$

Let Λ be the set of all diagonal matrices in $\Re^{m\times m}$ with diagonal elements that are either 1 or 0. Then, there are 2^m elements D_j in Λ, $j = 1, \ldots, 2^m$, and denote $D_j^- = I_m - D_j$, which are elements of Λ.

The controller design goal will be mathematically transformed to embed $sat(Kx(k), u_0)$ within a convex hull of a group of linear feedbacks (to avoid saturation). Given two gain matrices K and H, the matrix set $\{D_jK + D_j^-H\}$ is formed by choosing some rows of K and the rest from H. For this, the following lemma will be used:

Lemma 2.1 [4] *Given K and H, then, $sat(Kx(k), u_0) \in \mathcal{C}o\{D_jKx(k)+D_j^-Hx(k)\}$ for all $x(k) \in \Re^n$ that satisfy $|H_lx(k)| \le u_{0_l}$.*

An ellipsoid D_e that approximates the domain of attraction is defined as follows, for a positive scalar β and positive definite symmetric matrix P:

$$D_e = \left\{x(k) \in \Re^n; \quad x^T(k)Px(k) \le \beta^{-1}\right\}$$

The polyhedral set Θ is constructed as follows:

$$\Theta = \left\{x(k) \in \Re^n; \quad |H_lx(k)| \le u_{0_l}\right\}$$

Then, the system (6) becomes:

$$\begin{aligned} x(k+1) &= \sum_{j=1}^{2^m} \lambda_j A_j x(k) + A_d x(k-d(k)) + B_w w(k) \\ z(k) &= C_z x(k) \end{aligned} \tag{7}$$

where $A_j = A + B(D_jK + D_j^-H)$, $\lambda_j \ge 0$, and $\sum_{j=1}^{2m} \lambda_j = 1$.

Our objective is to design a controller such that the closed-loop system is stable and the H_∞ performance constraint is satisfied. In other words, we aim to design a controller such that the closed-loop system satisfies the following requirement:

$$\frac{\|z(k)\|_2^2}{\|w(k)\|_2^2} = \frac{\sum_{k=0}^{\infty} z^T(k)z(k)}{\sum_{k=0}^{\infty} w^T(k)w(k)} < \gamma \tag{8}$$

where the ratio between the norm of the controlled output and that of the disturbance is less than a specified scalar γ.

3 General Results

This section provides some general results based on LMI techniques for the stabilization of any system that can be described by (7). As has been mentioned, this provides a control theory approach for addressing the Mach number control problem of a wind turbine under degradation and instability.

3.1 Stability Results

Theorem 3.1 *For given scalars* ε_1, ε_2, *if there exist positive definite symmetric matrices* X_1, $\overline{Q}$, $\overline{R}$, *and appropriately sized matrices* U, G, X_2, X_3 *satisfying:*

$$\begin{bmatrix} \Sigma_{11} & * & * & * & * & * & * & * \\ \Sigma_{21} & \Sigma_{22} & * & * & * & * & * & * \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} & * & * & * & * & * \\ \Sigma_{41} & \Sigma_{42} & 0 & \Sigma_{44} & * & * & * & * \\ 0 & \Sigma_{52} & 0 & 0 & \Sigma_{55} & * & * & * \\ \Sigma_{61} & 0 & 0 & 0 & 0 & \Sigma_{66} & * & * \\ \Sigma_{71} & \Sigma_{72} & 0 & 0 & 0 & 0 & \Sigma_{77} & * \\ \Sigma_{81} & 0 & 0 & 0 & 0 & 0 & 0 & \Sigma_{88} \end{bmatrix} < 0, \tag{9}$$

$$\begin{bmatrix} X_1 & * \\ G_l & \beta u_{0_l}^2 \end{bmatrix} \geq 0 \tag{10}$$

$$\beta - \omega \leq 0 \tag{11}$$

where

$$\begin{aligned}
&\Sigma_{11} = X_2 + X_2^T + \varepsilon_1(A_d X_1 + X_1 A_d^T),\ \Sigma_{22} = -X_3 - X_3^T \\
&\Sigma_{21} = X_3^T - X_2 - X_1 + (A + \varepsilon_2 A_d)X_1 + B(D_j U + D_j^- G),\ \Sigma_{31} = -\varepsilon_1 \overline{Q} A_d^T \\
&\Sigma_{32} = (1 - \varepsilon_2)\overline{Q} A_d^T,\ \Sigma_{33} = -\overline{Q},\ \Sigma_{41} = -d_M \varepsilon_1 \overline{R} A_d^T,\ \Sigma_{42} = -d_M \varepsilon_2 \overline{R} A_d^T \\
&\Sigma_{44} = -d_M \overline{R},\ \Sigma_{52} = B_w^T,\ \Sigma_{55} = -I,\ \Sigma_{61} = X_1,\ \Sigma_{66} = -\overline{Q},\ \Sigma_{71} = d_M X_2 \\
&\Sigma_{72} = d_M X_3,\ \Sigma_{77} = -d_M \overline{R},\ \Omega_{81} = C_z X_1,\ \Omega_{88} = -\gamma I
\end{aligned}$$

Then, the closed-loop system (7) is stabilized by the feedback gain $K = U X_1^{-1}$ where the estimated domain of attraction is given by:

$$\delta^2 \left\{ \overline{\lambda}(X_1^{-1}) + d_M \overline{\lambda}(\overline{Q}^{-1}) + 4\overline{\lambda}(\overline{R}^{-1}) \right\} \leq \beta^{-1} - \omega^{-1},\ \delta = \max \|\phi\| \tag{12}$$

Proof To prove this theorem, let us consider the following Lyapunov functional:

$$\begin{aligned}
V(k) &= V_1(k) + V_2(k) + V_3(k) \\
&= x^T(k) P_1 x(k) + \sum_{i=k-d(k)}^{k-1} x^T(i) Q x(i) + \sum_{\theta=-d(k)}^{-1} \sum_{i=k+\theta}^{k-1} y^T(i) R y(i)
\end{aligned}$$

and let us compute the difference of the Lyapunov functional:

$$\begin{aligned}\Delta V_1(k) &= \Big(x(k+1)-x(k)\Big)^T P_1 \Big(x(k+1)-x(k)\Big) \\ &\quad + 2x^T(k)P_1\Big(x(k+1)-x(k)\Big) \\ &= y^T(k)P_1 y(k) + 2x^T(k)P_1 y(k)\end{aligned} \tag{13}$$

Thus, we obtain:

$$\begin{aligned}2x^T(k)P_1 y(k) &= 2\begin{bmatrix} x(k) \\ y(k) \\ x(k-d(k)) \end{bmatrix}^T \begin{bmatrix} P_1^T & P_2^T & P_4^T \\ 0 & P_3^T & P_5^T \\ 0 & 0 & P_6^T \end{bmatrix} \begin{bmatrix} y(k) \\ 0 \\ 0 \end{bmatrix} \\ &= 2\sum_{j=1}^{2^m} \lambda_j \begin{bmatrix} x(k) \\ y(k) \\ x(k-d(k)) \end{bmatrix}^T \begin{bmatrix} P_1^T & P_2^T & P_4^T \\ 0 & P_3^T & P_5^T \\ 0 & 0 & P_6^T \end{bmatrix} \\ &\quad \times \begin{bmatrix} y(k) \\ -y(k)+(A_j-I)x(k)+A_d x(k-d(k))+B_w w(k) \\ x(k)-x(k-d(k))-\sum_{i=k-d(k)}^{k-1} y(i) \end{bmatrix} \\ &= 2\sum_{j=1}^{2^m} \lambda_j \begin{bmatrix} x(k) \\ y(k) \\ x(k-d(k)) \end{bmatrix}^T \begin{bmatrix} P_1^T & P_2^T & P_4^T \\ 0 & P_3^T & P_5^T \\ 0 & 0 & P_6^T \end{bmatrix} \left(\begin{bmatrix} 0 & I & 0 \\ A_j - I & -I & A_d \\ I & 0 & -I \end{bmatrix} \right. \\ &\quad \left. \times \begin{bmatrix} x(k) \\ y(k) \\ x(k-d(k)) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -I \end{bmatrix} \sum_{i=k-d(k)}^{k-1} y(i) + \begin{bmatrix} 0 \\ B_w \\ 0 \end{bmatrix} w(k) \right) \\ &= \sum_{j=1}^{2^m} \lambda_j \eta^T(k) \left(\Xi_j \eta(k) + 2\begin{bmatrix} -P_4^T \\ -P_5^T \\ -P_6^T \end{bmatrix} \sum_{i=k-d(k)}^{k-1} y(i) \right. \\ &\quad \left. + 2\begin{bmatrix} P_2^T B_w \\ P_3^T B_w \\ 0 \end{bmatrix} w(k) \right)\end{aligned}$$

where

$$\sum_{i=k-d(k)}^{k-1} y(i) = x(k) - x(k-d(k)), \quad \eta(k) = \begin{bmatrix} x^T(k) & y^T(k) & x^T(k-d(k)) \end{bmatrix}^T,$$

$$\Xi_j = \begin{bmatrix} (A_j - I)^T P_2 + P_2^T (A_j - I) + P_4 + P_4^T & * & * \\ P_3^T (A_j - I) + P_5^T + P_1 - P_2 & -P_3 - P_3^T & * \\ A_d^T P_2 - P_4 + P_6^T & A_d^T P_3 - P_5 & -P_6 - P_6^T \end{bmatrix}$$

On other hand, we have:

$$\Delta V_2(k) = x^T(k)Qx(k) - x^T(k-d(k))Qx(k-d(k)) \tag{14}$$

Also, by Cauchy–Schwartz inequality we obtain:

$$\begin{aligned}\Delta V_3(k) &= d(k)y^T(k)Ry(k) - \sum_{i=k-d(k)}^{k-1} y^T(i)Ry(i)\\ &\leq d_M y^T(k)Ry(k) - \left(\sum_{i=k-d(k)}^{k-1} y(i)\right)^T \frac{R}{d_M}\left(\sum_{i=k-d(k)}^{k-1} y(i)\right)\end{aligned} \tag{15}$$

From (13)–(15), it follows that:

$$\begin{aligned}\Delta V(k) + \frac{1}{\gamma}z^T(k)z(k) - w^T(k)w(k) \leq \sum_{j=1}^{2^m}\lambda_j \begin{bmatrix} \eta(k) \\ \sum_{i=k-d(k)}^{k-1} y(i) \\ w(k)\end{bmatrix}^T \Pi_j\\ \times \begin{bmatrix} \eta(k) \\ \sum_{i=k-d(k)}^{k-1} y(i) \\ w(k)\end{bmatrix}\end{aligned} \tag{16}$$

where

$$\Pi_j = \begin{bmatrix} (A_j - I)^T P_2 + P_2^T(A_j - I) + P_4 + P_4^T + Q + \frac{1}{\gamma}C_z^T C_z & * & * & * & * \\ P_3^T(A_j - I) + P_5^T + P_1 - P_2 & -P_3 - P_3^T + d_M R + P_1 & * & * & * \\ A_d^T P_2 - P_4 + P_6^T & A_d^T P_3 - P_5 & -P_6 - P_6^T - Q & * & * \\ -P_4 & -P_5 & -P_6 & -\frac{R}{d_M} & * \\ B_w^T P_2 & B_w^T P_3 & 0 & 0 & -I \end{bmatrix} \tag{17}$$

It is clear that if $\Pi_j < 0$, then:

$$\Delta V(k) + \frac{1}{\gamma}z^T(k)z(k) - w^T(k)w(k) < 0 \tag{18}$$

From (9), it can be easily seen that X_3 is non-singular, and since $P_1 = P_1^T > 0$, let:

$$X = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}^{-1} = \begin{bmatrix} X_1 & 0 \\ X_2 & X_3 \end{bmatrix}$$

Multiplying both sides of Π_j by Δ^T and Δ, respectively, where $\Delta = diag\{X, I, I, I\}$. Then, let:

$$P_1^{-1} = X_1,\ Q^{-1} = \overline{Q},\ R^{-1} = \overline{R},\ N_1 = (X_1 P_4^T + X_2^T P_5^T)X_1,\ N_2 = X_3^T P_5^T X_1.$$

Thus, some conditions are obtained that are bilinear due to cross products of P_6 with P_1, P_2, and P_3. To avoid such terms, first we select $P_6 = 0$, which leads to:

$$\begin{aligned}\begin{bmatrix} \Omega_{11} & * & * & * & * \\ \Omega_{21} & \Sigma_{22} & * & * & * \\ \Omega_{31} & \Omega_{32} & \Omega_{33} & * & * \\ \Omega_{41} & \Omega_{42} & 0 & \Omega_{44} & * \\ 0 & \Sigma_{52} & 0 & 0 & \Sigma_{55} \end{bmatrix} &+ \begin{bmatrix} X_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \overline{Q}^{-1} \begin{bmatrix} X_1 & 0 & 0 & 0 & 0 \end{bmatrix} \\ &+ \begin{bmatrix} X_2^T \\ X_3^T \\ 0 \\ 0 \\ 0 \end{bmatrix} d_M \overline{R}^{-1} \begin{bmatrix} X_2 & X_3 & 0 & 0 & 0 \end{bmatrix} \\ &+ \begin{bmatrix} X_1 C_z^T \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \frac{1}{\gamma} I \begin{bmatrix} C_z X_1 & 0 & 0 & 0 & 0 \end{bmatrix} < 0 \end{aligned} \tag{19}$$

where Σ_{22}, Σ_{52}, and Σ_{55} are defined previously and

$$\begin{aligned} &\Omega_{11} = X_2 + X_2^T + N_1 + N_1^T,\ \Omega_{21} = X_3^T - X_2 + N_2 + (A_j - I)X_1 \\ &\Omega_{31} = -X_1^{-1} N_1^T,\ \Omega_{32} = A_d^T - X_1^{-1} N_2^T,\ \Omega_{33} = -\overline{Q}^{-1} \\ &\Omega_{41} = -X_1^{-1} N_1^T,\ \Omega_{42} = -X_1^{-1} N_2^T,\ \Omega_{44} = \frac{-\overline{R}^{-1}}{d_M} \end{aligned}$$

The presence of some nonlinearities such as X_1^{-1}, N_1 and N_2 in (19) does not allow for the condition to be solved directly: the variables then have to be tuned. For this reason, we choose $N_1 = \varepsilon_1 A_d X_1$, $N_2 = \varepsilon_2 A_d X_1$. On the other hand, results derived from Lyapunov techniques would lead to nonlinear constraints that are not practical for numerical optimization, as it is difficult to transform them into LMIs. To solve this issue, the Schur complement approach is used here as it is the simplest method to transform them into LMIs [3]. Then, by replacing A_j by $A + B(D_j K + D_j^- H)$ and using the Schur complement, we obtain LMI (9) where $U = KX_1$ and $G = HX_1$.

Since (9) holds, the condition (18) is satisfied. Now, summing (18) from 0 to ∞ with respect to k yields

$$V(\infty) < V(0) + \sum_{k=0}^{\infty}\left(w^T(k)w(k) - \frac{1}{\gamma}z^T(k)z(k)\right) \tag{20}$$

Under the zero initial condition $V(0) = 0$ and by noting that $V(\infty) \geq 0$, we have (8) which implies that system (7) has its restricted $\mathcal{L}_2-$gain from $w(k)$ to $z(k)$ less than γ. Now taking $w(k) = 0$, it is easy to see that $\Delta V(k) < 0$.

Moreover, the satisfaction of LMI (10) guarantees that $|H_l x(k)| \leq u_{0_l}, \forall x(k) \in D_e$. This can be proven following similar reasonings as those in [3].

Furthermore, from $\Delta V(k) < 0$ it follows that $V(k) \leq V(0)$, and thus, we obtain:

$$V(0) \leq V_1(0) + V_2(0) + V_3(0) \leq \left(\overline{\lambda}(P) + d_M\overline{\lambda}(Q) + 4\overline{\lambda}(R)\right)\|\phi\|^2 = \varpi$$

Therefore, we have:

$$x^T(k)P_1x(k) \leq V(k) \leq V(0) + \|w(k)\|_2^2 \leq \varpi + \omega^{-1} \leq \beta^{-1}$$

Finally, we obtain (11) and (12). Then, the inequality (12) guarantees that the trajectories of $x(k)$ remain within D_e for all initial functions $\phi(k) \in \Xi$, this completes the proof. □

Remark 3.1 It must be pointed out that when deriving Theorem 3.1 we have taken P_1 that contains free matrices P_2, P_3, P_4, P_5, and P_6. Thus, our results are more general than the one of [5] since it provides more degree of freedom. In fact, it improves existing delay-dependent analysis and synthesis results that require all the symmetric matrices in a chosen Lyapunov functional to be positive definite. This constraint is relaxed in this paper.

Remark 3.2 The result of Theorem 3.1 is derived by using (17), where P_6 is fixed to be zero, in order to simplify the numerical solution: Although this makes the solution only slightly more conservative, it reduces significantly the computational cost (see [11] for a previous use of this approach).

Remark 3.3 For given ε_1 and ε_2, inequality (9) is linear, so the problem can easily be solved using off-the-shelf software like Yalmip and SeDumi. Then, to find the optimal values of ε_1 and ε_2, a numerical optimization algorithm can be used: Conservatism is reduced if we take scalars ε_1 and ε_2 as parameters of adjustment.

3.2 Controller Design by Optimization

The proposed conditions in Theorem 3.1 are in LMI form, so they can be easily considered in convex optimization problems. Below we present three problems of interest for practical controller design.

3.2.1 Maximization of the Disturbance Tolerance

The idea is to maximize the $\mathcal{L}_2-$norm bound on the disturbance so as to ensure that the system trajectories remain bounded. Considering that the initial condition is null, this can be accomplished by the following convex optimization problem:

$$\begin{aligned} &\min \beta \\ &\text{subject to (9)–(11)} \end{aligned} \tag{21}$$

3.2.2 Maximization of the Disturbance Attenuation

For a non-null positive bound on the $\mathcal{L}_2-$norm of the admissible disturbances (given by $\beta^{-1} = \omega^{-1}$), the idea is to minimize the upper bound of the $\mathcal{L}_2-$gain of $w(t)$ on $z(t)$. Considering that the initial condition is null, this can be obtained from the solution of the following convex optimization problem:

$$\begin{aligned} &\min \gamma \\ &\text{subject to (9)–(11)} \end{aligned} \tag{22}$$

3.2.3 Maximization of the Region of Admissible Initial Conditions

Now, we consider the free-disturbance case ($w(k) = 0$). As it would be very difficult to come up with a simple solution to maximize the domain of initial conditions, due to the nonlinearity of (12), our proposal is based on developing a methodology to estimate the largest possible domain of initial conditions for which it can be ensured that the closed-loop system trajectories remain bounded. As in [10,11], we impose $\sigma_1 I \geq \overline{\lambda}(X_1^{-1})$, $\sigma_2 I \geq \overline{\lambda}(\overline{Q}^{-1})$, and $\sigma_3 I \geq \overline{\lambda}(\overline{R}^{-1})$, where σ_1, σ_2, and σ_3 are weighting parameters. Consequently, by the Schur complement, the following LMIs are obtained:

$$\begin{bmatrix} \sigma_1 I & I \\ I & X_1 \end{bmatrix} \geq 0, \begin{bmatrix} \sigma_2 I & I \\ I & \overline{Q} \end{bmatrix} \geq 0, \begin{bmatrix} \sigma_3 I & I \\ I & \overline{R} \end{bmatrix} \geq 0 \tag{23}$$

It follows that condition (12) is satisfied if the following LMI holds:

$$\delta^2 \Big\{ \sigma_1 + d_M \sigma_2 + 4\sigma_3 \Big\} \leq \beta^{-1} \tag{24}$$

Combining the facts derived above, we can construct an optimization problem as follows:

$$\begin{aligned} &\text{Minimize} \quad \vartheta = \sigma_1 + d_M \sigma_2 + 4\sigma_3 \\ &\text{subject to} \quad (9), (10), (23), (24) \end{aligned} \tag{25}$$

This optimization problem can then be easily solved using off-the-shelf numerical tools, providing a methodology to design controllers that ensure stability and simultaneously maximize the operating region, as is illustrated below for the problem that

Table 1 Comparison of maximum allowable delay d_M

	[15]	[13]	Theorem 3.1
d_M	0.33	0.97	12.0

motivated these developments. It must be pointed out that the computational complexity of design conditions plays an important role for practical design and implementation of controllers. In the next section, we evaluate this complexity between different design methods.

4 Application to Mach Number Control

In this section, a controller is designed based on the approach presented in the previous section, and some simulation results are presented to show the adequate closed-loop performance.

We consider a wind tunnel with the following parameters, borrowed from the literature: $\frac{1}{a} = 1.964s$, $k = -0.0117deg^{-1}$, $\xi = 0.8$, and $w = 6rad/s$. The controller design methodology presented in the previous section can easily be applied to this problem.

First, consider a time-delay system for which control values are saturated at ± 1 and described as follows:

$$x(k+1) = Ax(k) + A_d x(k-d(k)) + Bsat(u(k), u_0)$$

where $x = \left[\, \delta M \;\; \delta\theta \;\; \delta\dot{\theta} \,\right]$.

Solving the optimization problem (25) with $T = 1$, $\beta = 1$, $\varepsilon_1 = 0.1$, and $\varepsilon_2 = 0.1$ gives the following state feedback gain, where $\delta = 949.0$:

$$K = \begin{bmatrix} 0.0001 & -0.0012 & 0.0236 \end{bmatrix}$$

The proposed solution is compared in Table 1 with previous approaches in the literature [13, 15]: It can be clearly seen that much larger delays are tolerated with the proposed approach.

Some simulation results are presented in Figs. 1 and 2, where the evolution from the constant initial conditions $x_0 = [-5 \quad 5 \quad -5]^T$ is presented. Note that, even though the control input is initially saturated ($u(0) > u_0$), the states, in due course, are driven to the working point, showing the adequate stability and performance of the proposed controller.

Thus, it is illustrated how, by using the proposed approach, a control law can be obtained which is simple to implement in a microprocessor-based control system using state variables $x(k)$, which provides stability in a wide range of initial conditions, with good transient responses and feasible control signals.

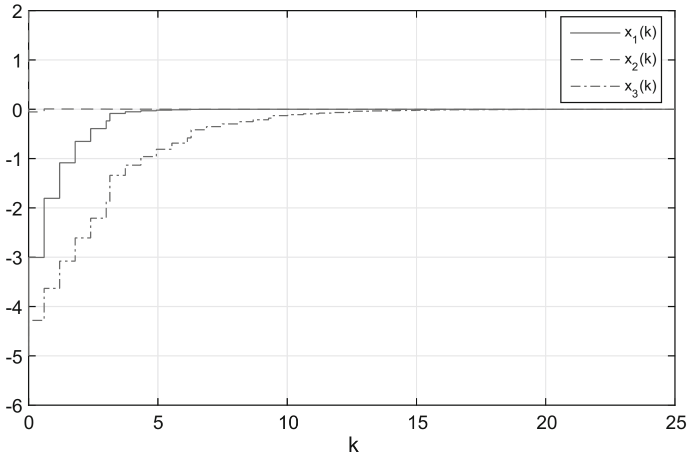

Fig. 1 Evolution with the proposed controller of the states (deviations from the desired values of M, θ and $\dot{\theta}$)

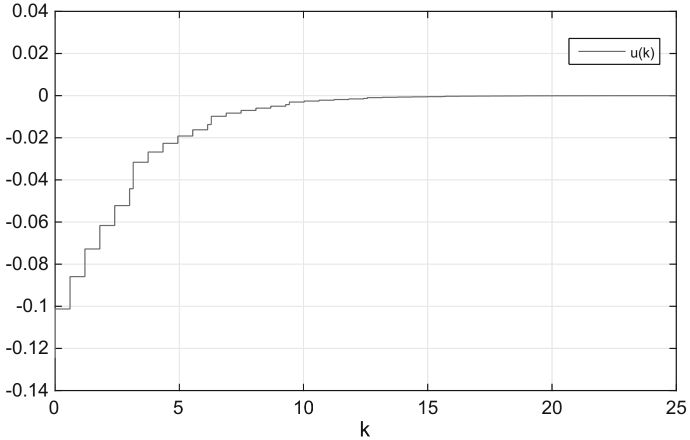

Fig. 2 Evolution with the proposed controller of the control input (deviation from the nominal value of the actuator angle θ_a)

Then, in order to take into account the effect of disturbances in the model, we consider the following time-delay system with an actuator that saturates at ± 1:

$$\begin{aligned} x(k+1) &= Ax(k) + A_d x(k - d(k)) + Bsat(u(k), u_0) + B_w w(k) \\ z(k) &= C_z x(k) \end{aligned}$$

Taking $T = 1$, $\varepsilon_1 = 0.5$, and $\varepsilon_2 = 0.5$, the algorithm proposed in (21), based on Theorem 3.1, successfully finds feedback stabilizing gains K and optimal values of β according to values of the upper bound of the delay d_M, as given in Table 2.

Table 2 Computational results for several values of the maximum delay using the algorithm in (21)

d_M	β	K
3	4.56×10^{-6}	$\begin{bmatrix} -0.0002 & -1.0310 & 1.8681 \end{bmatrix} \times 10^{-3}$
6	4.98×10^{-6}	$\begin{bmatrix} -0.0001 & -1.0447 & 15.1677 \end{bmatrix} \times 10^{-3}$
9	5.52×10^{-6}	$\begin{bmatrix} -0.0001 & -1.0485 & 19.6508 \end{bmatrix} \times 10^{-3}$

Table 3 Computational results for several values of the maximum delay using the algorithm in (22)

d_M	γ	K
2	1.94×10^{-4}	$\begin{bmatrix} -0.2099 & 8.9911 & -0.0624 \end{bmatrix}$
4	3.30×10^{-4}	$\begin{bmatrix} -0.1311 & 5.6643 & -0.0152 \end{bmatrix}$
6	5.74×10^{-4}	$\begin{bmatrix} -0.0933 & 3.5048 & -0.0010 \end{bmatrix}$

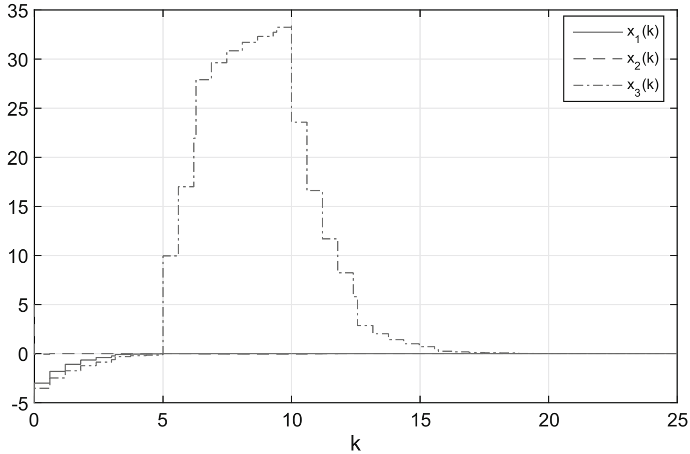

Fig. 3 Evolution of the states with the controller in Table 2, $d_M = 9$ (deviations from the desired values of M, θ, and $\dot{\theta}$)

Now, we use the method given in Theorem 3.1 and the algorithm proposed in (22) to compute the minimal value of the H_∞ performance γ for the closed-loop system. Then, Table 3 gives the values of K and γ according to the values of d_M, where $T = 1$, $\beta = 1$, $\varepsilon_1 = 0.2$, and $\varepsilon_2 = 0.2$.

Some simulation results are now presented in Figs. 3, 4 (for the controller corresponding to $d_M = 6$ in Table 2) and in Figs. 5, 6 (for the controller corresponding to $d_M = 6$ in Table 3), when the disturbance is:

$$w(k) = \begin{cases} 1, & 5 \le k \le 10 \\ 0, & k \ge 10 \end{cases}$$

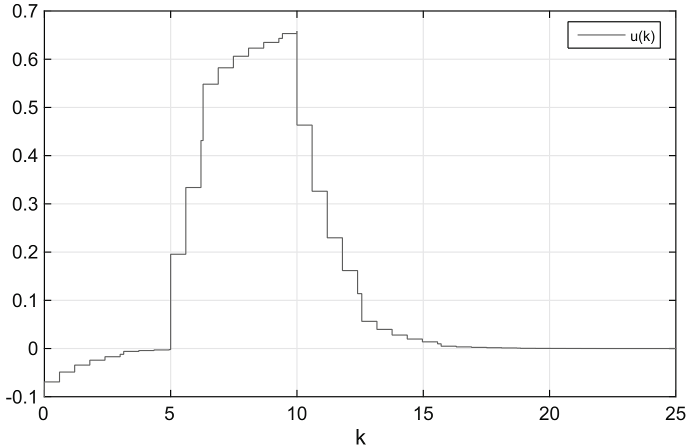

Fig. 4 Evolution of the control signal using the controller in Table 2, $d_M = 9$ (deviation from the nominal value of the actuator angle θ_a)

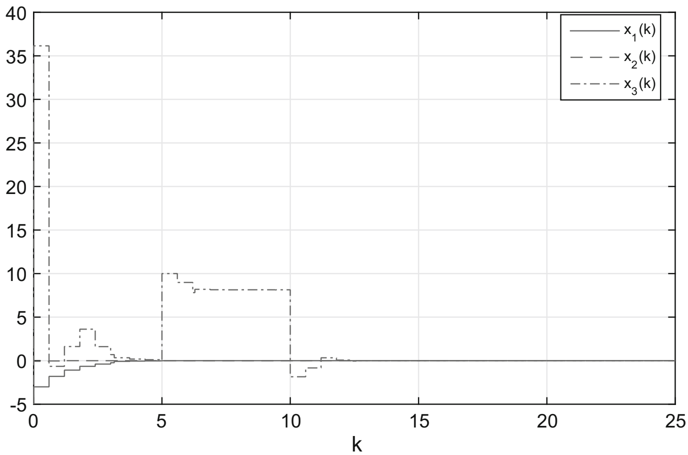

Fig. 5 Evolution of the states with the controller in Table 3, $d_M = 6$ (deviations from the desired values of M, θ, and $\dot{\theta}$)

It can be seen that the disturbance does not significantly affect neither the Mach number nor the vane angle. Finally, the Gaussian noise presented in Fig. 7 is used as disturbance to check the effect of random disturbances using the controller corresponding to $d_M = 6$ in Table 3). A stable operating condition is also reached, rejecting adequately these disturbances, as shown in Fig. 8.

In summary, the simulation results presented show that the proposed controller stabilizes the controlled system in spite of significant disturbances, combined with good transient performance.

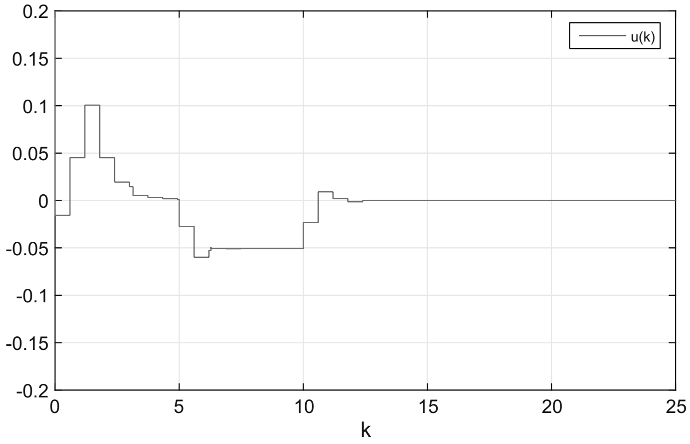

Fig. 6 Evolution of control signal using the controller in Table 3, $d_M = 6$ (deviation from the nominal value of the actuator angle θ_a)

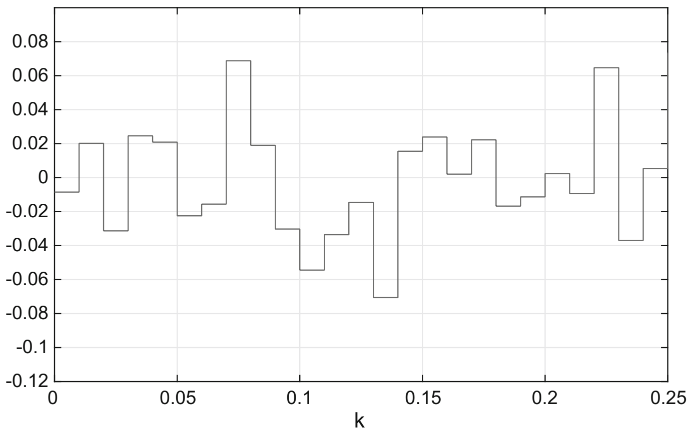

Fig. 7 Random disturbance

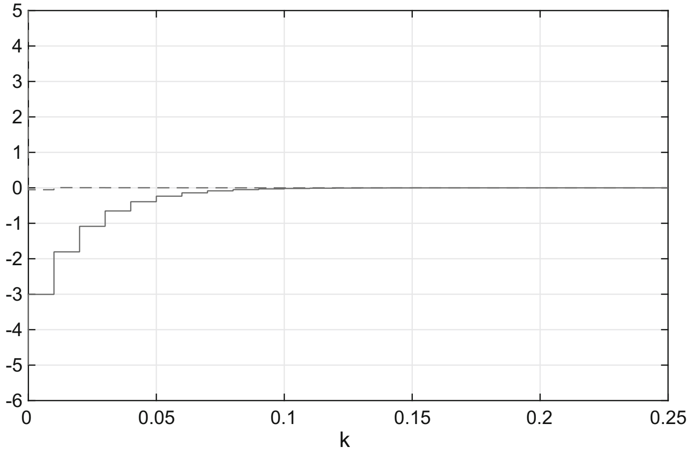

Fig. 8 Evolution with the proposed controller of the Mach number and the vane angle (deviations from the desired values of M and θ)

5 Conclusion

This paper deals with the control of the Mach number in wind tunnels. For this, a general solution is developed for a class of discrete-time systems with time-varying delay and saturating actuators with the aim of maximizing the set of admissible initial conditions. The application of the problem at hand to get improvements in the feedback control of the Mach number in a wind tunnel is proposed. This is obtained by solving an LMI optimization problem to design the state feedback gains, maximizing this estimate of the domain of attraction. Simulation results confirm the performance in terms of stability, tracking behavior, and disturbance rejection.

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