Analysis of non-active power in non-sinusoidal circuits using geometric algebra

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Abstract

A new approach for the definition of non-active power in electrical systems is presented in this paper. Thanks to the use of geometric algebra, it is possible to define a new term called geometric non-active power that is applicable to both sinusoidal and non-sinusoidal systems and to linear and non-linear loads. The classic definitions of distortion and reactive power are compared and discussed with our proposal. We verify how the geometric non-active power can appear in both purely resistive and purely reactive systems. The superiority of geometric algebra is revealed through several examples of electrical circuits previously analysed in specialized literature. In addition, a new geometrical current decomposition is proposed for the first time to provide a greater physical sense to existing geometric power. The results obtained show that classic concepts based on apparent power S are based on the lack of physical meaning, which is why geometric algebra theory should be adopted instead.

Keywords: geometric algebra; clifford algebra; geometric power; non-sinusoidal circuits

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1. Introduction

Numerous studies have shown that the new electrical networks must work under many adverse conditions [1, 2, 3]. From the integration of new sources of distributed and renewable energy to the massive proliferation of non-linear loads, this issue represents a remarkable challenge that must be approached properly to avoid degradation and abnormal operation of the power grid [4].

To do this, it is essential to use valuable mathematical tools that help the engineer in daily management tasks of the power grid along with theories that allow a better understanding of the physics behind the problem. In this sense, the existing theories that describe the power flow in electrical systems have been the subject of debate and controversy over the last 100 years [5, 6, 7]. Traditionally, two major proposals

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have dominated the studies during this time: those based on time domain [8] and those based on frequency domain [6]. Some proposals have gone even further and have ventured to mix concepts of the two worlds, generating a theory halfway between the time domain and frequency domain. Without a doubt, Steinmetz's theory based on complex numbers [5] has achieved the greatest impact among the scientific community and is the basis for the definition of the apparent power concept $\boldsymbol{S}$.

$$\boldsymbol{S} = P + jQ \tag{1}$$

This concept is so ingrained in the world of electrical engineering that it is very difficult to argue against it without being criticized by the community. Despite this, numerous examples have shown that $\boldsymbol{S}$ is no longer valid under non-sinusoidal conditions, so it should give way to other alternatives with more clear physical meaning [9, 10, 11]. This implies that associated concepts such as reactive power and distortion must be re-evaluated since

$$S^2 = P^2 + N^2 = P^2 + Q^2 + D^2 = \|V\|^2\|I\|^2 \tag{2}$$

where V and I are the RMS values of voltage and current, respectively. P is the active power defined as

$$P = \frac{1}{T}\int_0^T p(t)dt = \sum_{n=0}^{\infty} P_n = \sum_{n=0}^{\infty} V_n I_n \cos\varphi_n \tag{3}$$

where V_n and I_n are the RMS voltage and current of harmonic n, respectively, and $\cos\varphi_n$ is the phase angle between V_n and I_n. In equation (2), N is the non-active power according to Fryze definition, Q is the reactive power, and D is the distortion power according to Budeanu definition.

From the definition of $\boldsymbol{S}$, it is derived that it is possible to find N, Q or D only if P and $\boldsymbol{S}$ are known. If it has been demonstrated that $\boldsymbol{S}$ is not a concept that represents a magnitude with physical meaning, then we must conclude the remaining terms don't have it either. Therefore, it is not worthwhile to continue directing efforts to justify the fitting of merely mathematical concepts, as other authors have proposed [12].

On the other hand, the development of the electric power theory based on geometric algebra (GA) has provided a new and fresh approach in the solution to the problem of power flow in electrical systems of any nature, thanks to its flexibility and ability to represent the multi-component concept of power flow in non-sinusoidal systems. The references [13, 14] demonstrate the success of GA in disciplines such as relativistic physics, electromagnetism or computer vision. Specifically, the studies of Castro-Núñez [15, 16], Montoya [17] or Castilla and Bravo [18] reveal the capabilities that GA can provide in the analysis of electrical systems. In this sense, the concept of non-active power, together with the definition of quadrature power and degraded power, acquire a certain relevance since these allow an unambiguous physical association with certain components in the time domain. Thanks to this approach, it is possible to better understand the energy balances and, even more relevant, to confirm the compliance of the principle of conservation of energy (PoCOE) or, likewise, to ensure that Tellegen's theorem is verified.

2. Traditional definitions for distorted and reactive power

In general, the different proposals throughout history have tried to justify the apparent power definition by adding quadrature terms [10, 19] to the active power P so that it satisfies

$$S^2 = P^2 + R^2 \tag{4}$$

where R is a term that justifies the observable physical evidence in many electrical systems, given by

$$S \geq P \tag{5}$$

Many authors have found it impossible to find a physical justification to (5). Apparent power is an artificial mathematical concept that does not comply with the principle of conservation of energy (PoCOE); therefore, it is not conservative. One of the main quadrature terms is the so-called reactive power Q, introduced by Budeanu [6] for sinusoidal systems and linear loads

$$Q = VI \sin\varphi \tag{6}$$

and later extended to non-linear systems with harmonic generation

$$Q = \sum_n V_n I_n \sin\varphi_n \tag{7}$$

For non-sinusoidal systems, it was necessary to add a new term D so that equation (2) remains valid, although it has no direct definition but depends on the main definition of $\boldsymbol{S}$, which a priori may seem contradictory.

$$D^2 = S^2 - P^2 - Q^2 \tag{8}$$

Other authors have made proposals in the same way [20], i.e., adding quadratic terms to justify the equation (2). Among these authors are Fryze, Kusters and Moore, and Shepard and ZakiKhani; however, a real breakthrough that manages to create a totally coherent theory of power has not been still presented. More sophisticated theories such as the currents' physical components (CPC) theory of Czarnecki [21] continue investigating the concept of apparent power through a decomposition that arises from three current components: active, scattered and reactive. These investigations have proceeded despite the deep criticism that the author himself has made of apparent power $\boldsymbol{S}$ over the years [9, 22, 23].

3. Power concepts in geometric algebra

The use of GA has recently been proved as a powerful tool for the analysis of electrical circuits specifically and engineering problems in general. Its innate ability to naturally work with multi-component systems has been used to provide the resolution of circuits with harmonic components [24, 25, 18].

GA has its origins in the study of Clifford and Grassman in the nineteenth century. Despite its advantages over Gibbs' proposals and its vector analysis, Clifford's premature death prevented its development. The research of Hestenes and others [13, 26] has rescued and promoted the use of GA again in many engineering disciplines. The lack of space does not allow an extensive introduction to GA, but the reader can refer to the classic references of Jancewicz [27], Dorst [28] or Hestenes [14].

What really makes GA an exceptional tool is its ability to mathematically accommodate vectors, complex numbers, quaternions or spinors as subspaces in GA. In addition, GA can be extended to any number of dimensions easily. One of the keys is the use of geometric multidimensional objects such as bivectors, which arise from the exterior product or Grassman product defined by

$$a \wedge b = -b \wedge a \tag{9}$$

and results in an area delimited by vectors a and b, which have a magnitude and direction (see Figure 1). The bivector is a key concept that does not exist in vector analysis and that has its own entity. Of course, like vectors, a bivector can also be expressed as a linear combination of a bivector base.

The other major pillar of GA is the geometric product. Defined mainly for vectors, it can be easily extended to multi-vectors. Consider a and b as any two vectors in the geometric space $\mathcal{G}_2$ covered by the base $\boldsymbol{\sigma_1}, \boldsymbol{\sigma_2}$

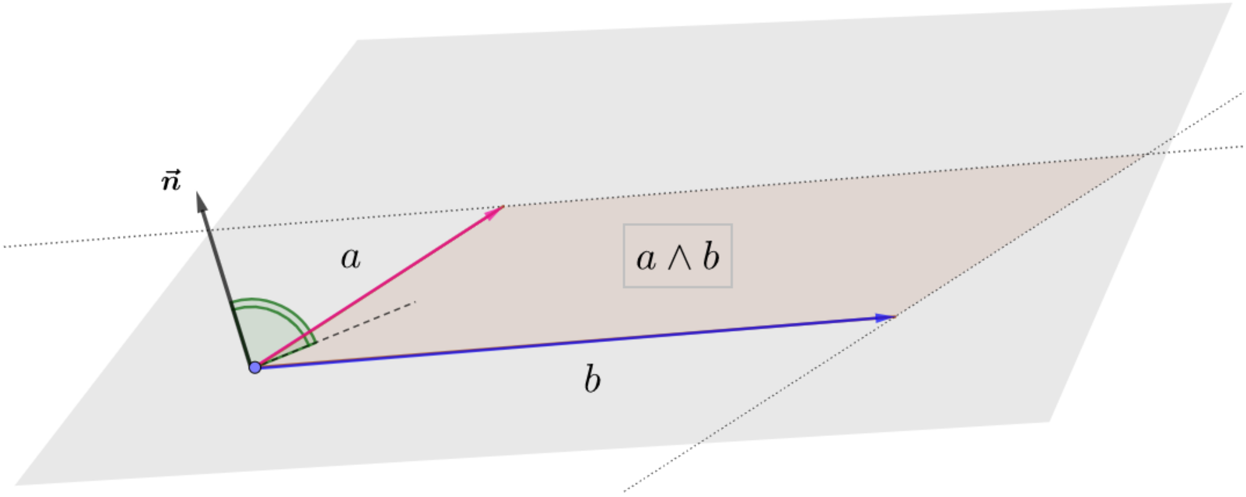

Figure 1: Bivector $\boldsymbol{a} \wedge \boldsymbol{b}$. Note that the classical vector product $\boldsymbol{a} \times \boldsymbol{b}$ is also represented as the perpendicular vector to plane formed by $\boldsymbol{a}$ and $\boldsymbol{b}$.

$$a = a_1 \boldsymbol{\sigma_1} + a_2 \boldsymbol{\sigma_2}$$

$$b = b_1 \boldsymbol{\sigma_1} + b_2 \boldsymbol{\sigma_2}$$

We then can define the geometric product as the linear combination of the scalar or internal product and the external or Grassman product. The result is a multi-vector $\boldsymbol{A}$

$$\boldsymbol{A} = ab = a \cdot b + a \wedge b = \langle \boldsymbol{A} \rangle_0 + \langle \boldsymbol{A} \rangle_2 = (a_1 a_2 + b_1 b_2) + (a_1 b_2 - b_1 a_2) \boldsymbol{\sigma_{12}} \tag{10}$$

where $\langle \boldsymbol{A} \rangle_0$ is the scalar part, and $\langle \boldsymbol{A} \rangle_2$ is the bivector.

Castro-Núñez presents the fundamental theory [16] that allows the transformation from time domain to geometric space:

$$\begin{aligned}
\varphi_{c1}(t) &= \sqrt{2}\cos\omega t &\longleftrightarrow\quad & \boldsymbol{\sigma_1} \\
\varphi_{s1}(t) &= \sqrt{2}\sin\omega t &\longleftrightarrow\quad & -\boldsymbol{\sigma_2} \\
\varphi_{c2}(t) &= \sqrt{2}\cos 2\omega t &\longleftrightarrow\quad & \boldsymbol{\sigma_2\sigma_3} \\
\varphi_{s2}(t) &= \sqrt{2}\sin 2\omega t &\longleftrightarrow\quad & \boldsymbol{\sigma_1\sigma_3} \\
& \vdots & & \\
\varphi_{cn}(t) &= \sqrt{2}\cos n\omega t &\longleftrightarrow\quad & \bigwedge_{\boldsymbol{i=2}}^{\boldsymbol{n+1}} \boldsymbol{\sigma_i} \\
\varphi_{sn}(t) &= \sqrt{2}\sin n\omega t &\longleftrightarrow\quad & \bigwedge_{\substack{\boldsymbol{i=1}\\ \boldsymbol{i\neq 2}}}^{\boldsymbol{n+1}} \boldsymbol{\sigma_i}
\end{aligned} \tag{11}$$

where $\bigwedge_n \boldsymbol{\sigma_i}$ is the product of n vectors $\boldsymbol{\sigma_i}$. As an example, consider the voltage $v(t)$

$$v(t) = \sqrt{2}\Big[(230\cos(\omega t - 30) + 20\sin(4\omega t + 45)\Big] \tag{12}$$

that can be expressed as

$$\begin{aligned}
v(t) &= \sqrt{2}\Big[230(\cos\omega t\cos 30 + \sin\omega t\sin 30) + 20(\sin 4\omega t\cos 45 + \cos 4\omega t\sin 45)\Big] \\
&= \sqrt{2}\Big[230(\frac{\sqrt{3}}{2}\cos\omega t + \frac{1}{2}\sin\omega t) + 20(\frac{\sqrt{2}}{2}\sin 4\omega t + \frac{\sqrt{2}}{2}\cos 4\omega t)\Big]
\end{aligned}$$

following the transformation proposed in (11), the transformed voltage is obtained

$$\boldsymbol{u} = \underbrace{199.18\boldsymbol{\sigma_1} - 115\boldsymbol{\sigma_2}}_{\langle \boldsymbol{u}\rangle_1} + \underbrace{14.14\boldsymbol{\sigma_{1345}} + 14.14\boldsymbol{\sigma_{2345}}}_{\langle \boldsymbol{u}\rangle_4} \tag{13}$$

Likewise, it is possible to apply the transformation (11) to find the value of the impedance and admittance of any load. In general terms, as Castro-Núñez shows in [29], impedance is defined as

$$\boldsymbol{Z} = \boldsymbol{Y}^{-1} = R + X\boldsymbol{\sigma_{12}} \tag{14}$$

It should be noted that the reactance $X = 1/\omega C$ is positive if the load is capacitive, and $X = -L\omega$ is negative if it is inductive.

In this way, any non-sinusoidal voltage can be expressed as

$$v(t)=\sum_{i=1}^{n} v_i(t) = D_1\cos(\omega t)+E_1\sin(\omega t)+\sum_{h=2}^{d} D_h\cos(h\omega t)+\sum_{h=2}^{k} E_h\sin(h\omega t) \tag{15}$$

it should be noted that (15) can be also generalized to include interharmonics and subharmonics [30]. The geometric voltage is

$$\boldsymbol{v}=D_1\boldsymbol{\sigma_1}-E_1\boldsymbol{\sigma_2}+\sum_{h=2}^{d}\left[D_h\bigwedge_{i=2}^{h+1}\boldsymbol{\sigma_i}\right]+\sum_{h=2}^{k}\left[E_h\bigwedge_{i=1,i\neq 2}^{h+1}\boldsymbol{\sigma_i}\right] \tag{16}$$

Similarly, the current can be calculated by applying Ohm's law for each of the harmonic components

$$\boldsymbol{i}=\sum_{h=1}^{n}\boldsymbol{i_h} \tag{17}$$

such that the result is

$$\boldsymbol{i}=\boldsymbol{i}_{||}+\boldsymbol{i}_{\perp}=\boldsymbol{i_g}+\boldsymbol{i_b} \tag{18}$$

with

$$\begin{aligned}\boldsymbol{i_g}=&\,G_1D_1\boldsymbol{\sigma_1}-G_1E_1\boldsymbol{\sigma_2}+\sum_{h=2}^{d}\left[G_hD_h\bigwedge_{i=2}^{h+1}\boldsymbol{\sigma_i}\right]+\\&+\sum_{h=2}^{k}\left[G_hE_h\bigwedge_{i=1,i\neq 2}^{h+1}\boldsymbol{\sigma_i}\right]\end{aligned} \tag{19}$$

$$\begin{aligned}\boldsymbol{i_b}=&-B_1E_1\boldsymbol{\sigma_1}-B_1D_1\boldsymbol{\sigma_2}+\sum_{h=2}^{d}\left[B_hD_h\bigwedge_{i=1,i\neq 2}^{h+1}\boldsymbol{\sigma_i}\right]-\\&-\sum_{h=2}^{k}\left[B_hE_h\bigwedge_{i=2}^{h+1}\boldsymbol{\sigma_i}\right]\end{aligned} \tag{20}$$

which of course can be transformed back to time domain $i(t)$ by simply performing the inverse transformation according to (11).

Finally, the geometric apparent power or net power $\boldsymbol{M}$ is defined as the product of $\boldsymbol{u}$ and $\boldsymbol{i}$:

$$\boldsymbol{M} = \boldsymbol{u}\boldsymbol{i} = \underbrace{\overbrace{\langle \boldsymbol{M_g} \rangle_0}^{P} + \overbrace{\sum_{i=1}^{n+1} \langle \boldsymbol{M_g} \rangle_i}^{\boldsymbol{CN_d}}}_{\boldsymbol{M_g}} + \underbrace{\boldsymbol{CN_{r(ps)}} + \boldsymbol{CN_{r(hi)}}}_{\boldsymbol{M_b}=\boldsymbol{CN_r}} \tag{21}$$

where

$\boldsymbol{M_g}$ is the parallel geometric apparent power

$\boldsymbol{M_b}$ is the quadrature geometric apparent power

$\boldsymbol{P}$ is the active power

$\boldsymbol{CN_d}$ is the degraded power

$\boldsymbol{CN_r}$ is the quadrature geometric power or reactive geometric power

$\boldsymbol{CN_{r(ps)}}$ is the reactive geometric power due to voltage and current phase shift of same components

$\boldsymbol{CN_{r(hi)}}$ is the reactive geometric power due to voltage and current cross products

Based on the above definitions, the net or geometric power factor can be defined as

$$pf = \frac{P}{\|\boldsymbol{M}\|} = \frac{\langle \boldsymbol{M} \rangle_0}{\sqrt{\langle \boldsymbol{M}^\dagger \boldsymbol{M} \rangle_0}} \tag{22}$$

From the expression (21), it is observed that the power $\boldsymbol{M}$ is composed of two terms: $\boldsymbol{M_g}$, which is the parallel geometric power, and $\boldsymbol{M_b}$, which is the quadrature power. Each power is obtained by multiplying the voltage $\boldsymbol{u}$ by the current $\boldsymbol{i_g}$ and $\boldsymbol{i_b}$, respectively. From the parallel geometric power $\boldsymbol{M_g}$, the active power P and the degraded power $\boldsymbol{CN_d}$ are obtained, while from the quadrature geometric power $\boldsymbol{M_b}$, the reactive geometric power of the harmonic components of the same order $\boldsymbol{CN_{r(ps)}}$ and the reactive geometric power due to cross-products of different frequency components $\boldsymbol{CN_{r(hi)}}$ are obtained. Of course, the power $\boldsymbol{M}$ can also be decomposed according to an even more basic criterion

$$\boldsymbol{M} = P + \boldsymbol{CN} \tag{23}$$

where $\boldsymbol{CN}$ includes all terms that do not contribute to the active power, called non-active geometric power.

The physical meaning of each of these terms is revealed through its definition based on the current component from which it is derived. $\boldsymbol{M_b}$ results from the current $\boldsymbol{i_b}$, i.e., the current term in quadrature with the voltage due to the susceptance B of the load. All this current $\boldsymbol{i_b}$, and therefore all the power $\boldsymbol{M_b}$, can be eliminated with a passive LC compensator, as demonstrated in [17]. Similarly, the remaining power $\boldsymbol{M_g}$, which results from the product of the voltage $\boldsymbol{u}$ and the current parallel to the voltage $\boldsymbol{i_g}$, can be decomposed into active power P and degraded power $\boldsymbol{CN_d}$. Unlike quadrature power, the purpose for compensation is not to eliminate the term $\boldsymbol{CN_d}$ (due to $\boldsymbol{i_g}$) but only a certain portion according to a new decomposition of $\boldsymbol{i_g}$, not previously published. Indeed, if we consider the proposal of Fryze, it is possible to define the active current $\boldsymbol{i_a}$ as that which contributes only to the active power P

$$\boldsymbol{i_a} = \frac{P}{V^2}\boldsymbol{u} = \frac{\langle \boldsymbol{M} \rangle_0}{\|\boldsymbol{u}\|^2}\boldsymbol{u} \tag{24}$$

where V is the RMS value of the voltage. In this way, the remaining current is

$$\boldsymbol{i_d} = \boldsymbol{i_g} - \boldsymbol{i_a} \tag{25}$$

This current $\boldsymbol{i_d}$ is called *degraded current* and coincides with the scattered current defined by the CPC theory of Czarnecki. This result is further evidence of the superiority of GA with respect to the complex number algebra and to the power theories established to date. The energy flow phenomena that occur in a circuit can be explained clearly and with physical significance using the theory defined by Castro-Núñez [31, 32] while complying with PoCOE and Tellegen's theorem.

Equation (18) is now

$$\boldsymbol{i} = \boldsymbol{i_a} + \boldsymbol{i_d} + \boldsymbol{i_b} \tag{26}$$

It is demonstrated that the three currents in (26) are orthogonal to each other and satisfy

$$\|\boldsymbol{i}\|^2 = \|\boldsymbol{i_a}\|^2 + \|\boldsymbol{i_d}\|^2 + \|\boldsymbol{i_b}\|^2 \tag{27}$$

It should be noted that, unlike the quadrature current, the current $\boldsymbol{i_d}$ can be compensated only by active elements, typically power active filters.

Figure 2: Resistor load and non-sinusoidal source

If we multiply the expression (26) by the voltage $\boldsymbol{u}$, we obtain

$$\boldsymbol{M} = \boldsymbol{ui} = \boldsymbol{ui_a} + \boldsymbol{ui_d} + \boldsymbol{ui_b} = \boldsymbol{M_a} + \boldsymbol{M_d} + \boldsymbol{M_b} \tag{28}$$

Comparing (28) with (21), it can be inferred that the power $\boldsymbol{M_g}$ is the sum of $\boldsymbol{M_a}$ and $\boldsymbol{M_d}$, although it is not always true that $\boldsymbol{M_a} = P$ or that $\boldsymbol{M_d} = \boldsymbol{CN_d}$. The above expression leads to a very interesting approach: the active geometric power includes not only the active power P but also terms derived from the cross-product of voltage and parallel current of different frequencies, which on average do not contribute to the net power flow.

4. Non active power in linear and non-linear loads

This section discusses practical examples of linear and non-linear loads under to non-sinusoidal voltages. Simple examples with resistive loads will be studied and then expanded to more complex loads.

4.1. Linear loads

4.1.1. Pure resistor

The first case study represents one of the simplest, but no less interesting, examples: a simple resistor powered by a non-sinusoidal source (Figure 2) of value

$$u(t) = 100\sqrt{2}\sin\omega t + 100\sqrt{2}\sin 2\omega t \tag{29}$$

The voltage $u(t)$ is transferred to the geometric domain by the expression (11)

$$\boldsymbol{u} = -100\boldsymbol{\sigma_2} + 100\boldsymbol{\sigma_{13}} \tag{30}$$

and, considering Ohm's law, the current is

$$\boldsymbol{i} = \frac{\boldsymbol{u}}{R} = -100\boldsymbol{\sigma_2} + 100\boldsymbol{\sigma_{13}} \tag{31}$$

The geometric apparent power $\boldsymbol{M}$ turns out to be the geometric product of the voltage and current:

$$\boldsymbol{M} = \boldsymbol{ui} = (-100\boldsymbol{\sigma_2} + 100\boldsymbol{\sigma_{13}})(-100\boldsymbol{\sigma_2} + 100\boldsymbol{\sigma_{13}}) = \underbrace{\underbrace{20,000}_{P} + \underbrace{20,000\boldsymbol{\sigma_{123}}}_{\boldsymbol{CN_d}}}_{\boldsymbol{M_a}=\boldsymbol{M_g}} \tag{32}$$

As expected, there is no quadrature power $\boldsymbol{M_b}$; there is only parallel power $\boldsymbol{M_g}$. In addition, all parallel power is active geometric power $\boldsymbol{M_a}$; that is, there is no degraded power $\boldsymbol{M_d}$ because there is no degraded current $\boldsymbol{i_d}$ since

$$\boldsymbol{i} = \boldsymbol{i_g} = \boldsymbol{i_a} \tag{33}$$

The instantaneous power in the time domain can be found easily by multiplying $u(t)$ and $i(t)$, obtaining

$$p(t) = 10,000\sin^2\omega t + 10,000\sin^2 2\omega t + 20,000\sin\omega t\sin 2\omega t \tag{34}$$

Figure 3 shows the waveform of the power $p(t)$. This wave is fluctuating but always of positive value; that is, the energy flows from the source to the load and never backwards. The average value is the active power P consumed by the resistor. It is observed that there is a clear equivalence between the expression (32) and (34).

If we analyse the circuit of Figure 2 using the classic technique of complex numbers and apply the CPC theory, the result is as follows:

$$\begin{aligned} S &= P = 20,000 \\ I &= 100\sqrt{2} \\ V &= 100\sqrt{2} \end{aligned}$$

which is in line with what is expected. Note that the CPC theory cannot capture the nuance of the undulatory term $20,000\boldsymbol{\sigma_{123}}$ present in the active geometric power $\boldsymbol{M_a}$. That is, using classical power

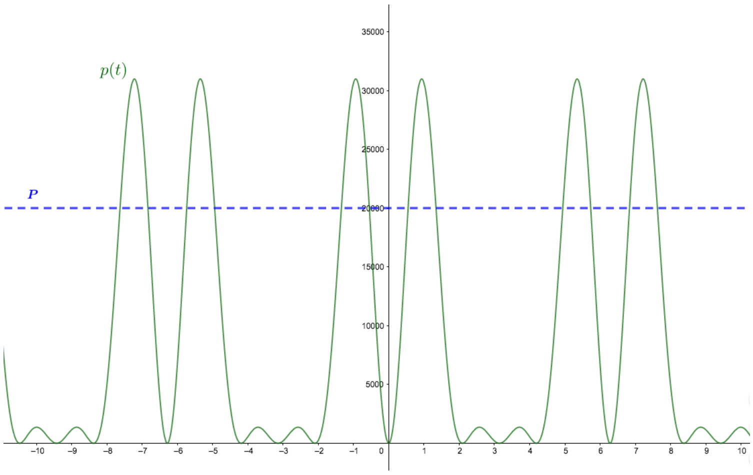

Figure 3: Instantaneous power

theories, it is impossible to capture terms due to cross-interaction of harmonics of different frequencies since it is not defined, and therefore, it is impossible to achieve the vector product of a different frequency in the complex plane. This is among the major deficiencies of the use of complex numbers in contemporary power theories.

4.1.2. Pure reactive load

The second case presents a purely reactive load composed of an inductor and a capacitor as shown in Figure 4.

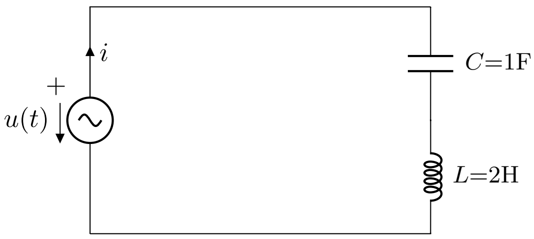

Figure 4: Pure reactive load

Following the steps described in the previous example, and considering that according to (14) the admittances are $\boldsymbol{Y_1} = \boldsymbol{\sigma_{12}}$ and $\boldsymbol{Y_2} = 0.285\boldsymbol{\sigma_{12}}$, the transferred values for voltage, current and geometric power can be obtained as

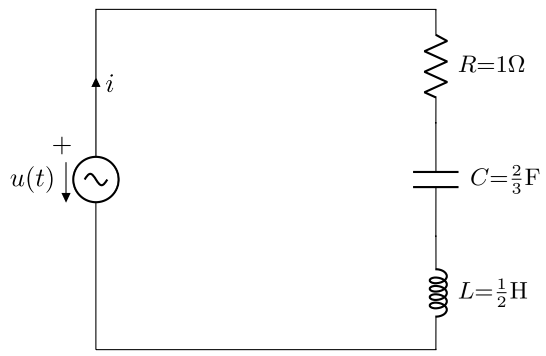

Figure 5: Arbitrary linear load

$$\boldsymbol{u} = -100\boldsymbol{\sigma_2} + 100\boldsymbol{\sigma_{13}}$$

$$\boldsymbol{i} = \boldsymbol{i_b} = \boldsymbol{Y_1}\langle \boldsymbol{u}\rangle_1 + \boldsymbol{Y_2}\langle \boldsymbol{u}\rangle_2 = -100\boldsymbol{\sigma_1} - 28.57\boldsymbol{\sigma_{23}}$$

$$\boldsymbol{M} = \boldsymbol{ui} = \underbrace{-7142.8\boldsymbol{\sigma_{12}}}_{\boldsymbol{CN_{r(ps)}}} + \underbrace{12,857.14\boldsymbol{\sigma_3}}_{\boldsymbol{CN_{r(hi)}}}$$

In this case, all current is in quadrature in this case: $\boldsymbol{i} = \boldsymbol{i_b}$. Therefore, the power in quadrature due to the interaction between harmonics of voltage and current of the same frequency is determined by $\boldsymbol{CN_{r(ps)}}$, while the cross-products are determined by $\boldsymbol{CN_{r(hi)}}$. This decomposition or detail cannot be captured by the apparent power S in any way.

The values obtained by the CPC theory are

$$S = Q = 14,708$$

$$V = 141.42$$

$$I = 104.00$$

4.1.3. Arbitrary linear load

If we apply the previous non-sinusoidal voltage to an arbitrary linear load as shown in Figure 5, we again can obtain the voltage, current and geometric power.

The impedances and admittances are as follows:

$$\boldsymbol{Z_1} = 1 + (-\frac{1}{2} + \frac{1}{\frac{2}{3}})\boldsymbol{\sigma_{12}} = 1 - \boldsymbol{\sigma_{12}} \qquad \rightarrow \boldsymbol{Y_1} = \boldsymbol{Z_1}^{-1} = 0.5 + 0.5\boldsymbol{\sigma_{12}}$$

$$\boldsymbol{Z_2} = 1 + (-2\frac{1}{2} + \frac{1}{2\frac{2}{3}})\boldsymbol{\sigma_{12}} = 1 + 0.25\boldsymbol{\sigma_{12}} \rightarrow \boldsymbol{Y_2} = \boldsymbol{Z_2}^{-1} = 0.941 - 0.235\boldsymbol{\sigma_{12}}$$

the current in this case is

$$\boldsymbol{i} = \boldsymbol{Y_1}\langle \boldsymbol{u} \rangle_1 + \boldsymbol{Y_2}\langle \boldsymbol{u} \rangle_2 = \underbrace{-50\boldsymbol{\sigma_2} + 94.11\boldsymbol{\sigma_{13}}}_{\boldsymbol{i_g}} \underbrace{-50\boldsymbol{\sigma_1} + 23.53\boldsymbol{\sigma_{23}}}_{\boldsymbol{i_b}}$$

and the geometric power is

$$\boldsymbol{M} = \underbrace{14,411.76 + 14,411.76\boldsymbol{\sigma_{123}}}_{\boldsymbol{M_g}} \underbrace{-7352,94\boldsymbol{\sigma_{12}} + 2647.05\boldsymbol{\sigma_3}}_{\boldsymbol{M_b}}$$

Once the power $\boldsymbol{M}$ is found, we can find the active current $\boldsymbol{i_a}$

$$\boldsymbol{i_a} = \frac{\langle \boldsymbol{M} \rangle_0}{\|\boldsymbol{u}\|^2}\boldsymbol{u} = \frac{14,411.76}{20,000}(-100\boldsymbol{\sigma_2} + 100\boldsymbol{\sigma_{13}}) = -72.05\boldsymbol{\sigma_2} + 72.05\boldsymbol{\sigma_{13}}$$

and the degraded current as

$$\boldsymbol{i_d} = \boldsymbol{i_g} - \boldsymbol{i_a} = 22.05\boldsymbol{\sigma_2} + 22.05\boldsymbol{\sigma_{13}}$$

It can be verified that the currents $\boldsymbol{i_a}$, $\boldsymbol{i_d}$ and $\boldsymbol{i_b}$ are mutually orthogonal:

$$\boldsymbol{i_a} \cdot \boldsymbol{i_d} = (-72.05\boldsymbol{\sigma_2} + 72.05\boldsymbol{\sigma_{13}}) \cdot (22.05\boldsymbol{\sigma_2} + 22.05\boldsymbol{\sigma_{13}}) = 1588.7 - 1588.7 = 0$$

$$\boldsymbol{i_a} \cdot \boldsymbol{i_b} = (-72.05\boldsymbol{\sigma_2} + 72.05\boldsymbol{\sigma_{13}}) \cdot (-50\boldsymbol{\sigma_1} + 23.53\boldsymbol{\sigma_{23}}) = 0$$

$$\boldsymbol{i_d} \cdot \boldsymbol{i_b} = (22.05\boldsymbol{\sigma_2} + 22.05\boldsymbol{\sigma_{13}}) \cdot (-50\boldsymbol{\sigma_1} + 23.53\boldsymbol{\sigma_{23}}) = 0$$

which satisfies

	vector		bivector		
Current	$\boldsymbol{\sigma_1}$	$\boldsymbol{\sigma_2}$	$\boldsymbol{\sigma_{13}}$	$\boldsymbol{\sigma_{23}}$	Norm
$\boldsymbol{i}$	-50.00	-50.00	94.11	23.52	120.00
$\boldsymbol{i_b}$	-50.00	-	-	23.52	55.26
$\boldsymbol{i_g}$	-	-50.00	94.11	-	106.50
$\boldsymbol{i_a}$	-	-72.05	72.05	-	101.90
$\boldsymbol{i_d}$	-	22.05	22.05	-	31.19

Table 1: Current decomposition

$$\|\boldsymbol{i}\|^2 = \|\boldsymbol{i_a}\|^2 + \|\boldsymbol{i_d}\|^2 + \|\boldsymbol{i_b}\|^2 \quad \longrightarrow \quad 14411.76 = 10384.94 + 973.18 + 3053.62$$

Table 1 shows a summary of the current decomposition for the different physical components. Unlike the CPC theory of Czarnecki, where it is possible to obtain only the magnitude of the currents, through GA it is possible to obtain a complete decomposition according to Ohm's and Kirchhoff's laws.

Figure 6 shows the waveform of each power calculated in the geometric domain but transferred to the time domain. It can be observed how the instantaneous power $p(t)$ (Figure 6.e) presents both positive and negative values, which indicates the bidirectional flow of energy. The dashed line indicates the average value or active power P. However, in Figure 6.b), it can be observed that the power is always positive, never negative. This power $\boldsymbol{M_a}$ is calculated from the active current $\boldsymbol{i_a}$.

The values obtained by the CPC theory are

$$\begin{aligned} S &= 16,970 \qquad P = 14,411.76 \qquad Q = 7814.94 \qquad D_s = 4410.93 \\ V &= 141.42 \\ I &= 120.00 \end{aligned}$$

4.2. Non linear load

The last problem has already been examined by Czarnecki in [33], establishing an unnecessary current decomposition between the source and non-linear loads due to the lack of powerful mathematical tools such as those provided by GA. Figure 7 presents the circuit to be analysed with the following characteristics

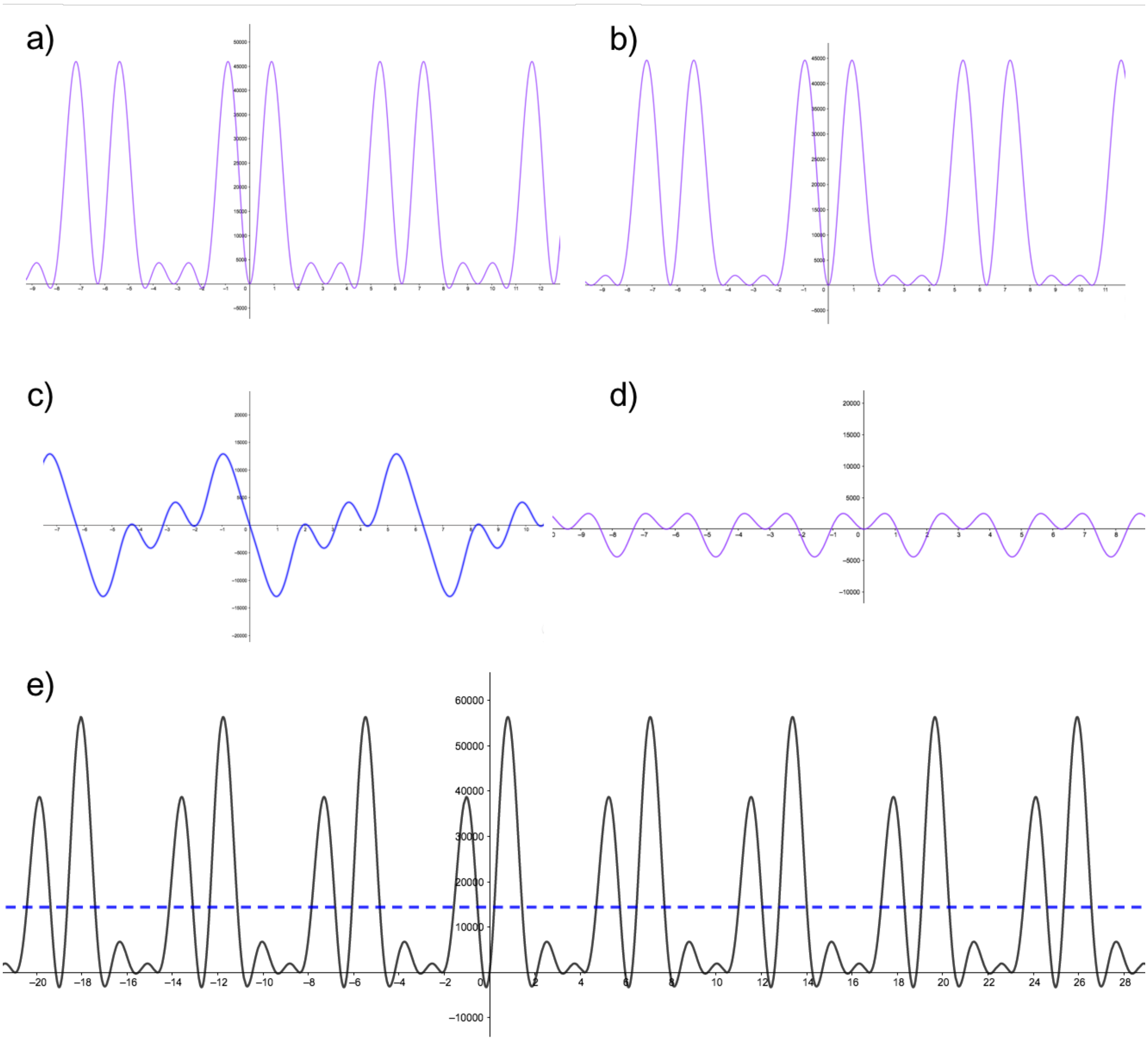

Figure 6: Instantaneous power waves: a) parallel instantaneous power $p_g(t)$, b) active instantaneous power $p_a(t)$, c) quadrature instantaneous power $p_b(t)$, d) degraded instantaneous power $p_d(t)$, e) total instantaneous power $p(t)$

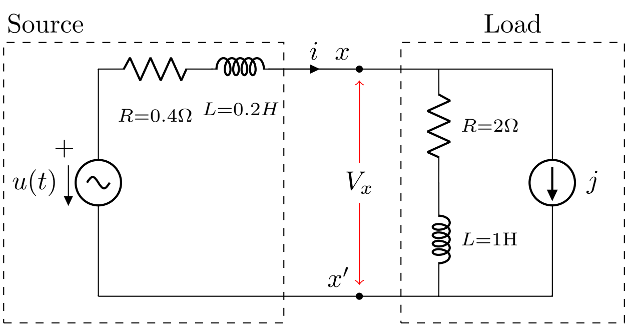

Figure 7: Non-linear load and non-sinusoidal voltage source

$$u(t) = 100\sqrt{2}\cos\omega t + 50\sqrt{2}\cos 3\omega t + 10\sqrt{2}\sin 4\omega t$$

$$j(t) = 50\sqrt{2}\cos 2\omega t + 10\sqrt{2}\sin 3\omega t + 30\sqrt{2}\cos 4\omega t$$

Following the steps of the previous examples, we proceed to carry out the transformation from the time domain to the geometric domain, so that

$$\boldsymbol{u} = 100\boldsymbol{\sigma_1} + 50\boldsymbol{\sigma_{234}} + 10\boldsymbol{\sigma_{1345}} \qquad \longrightarrow \qquad \|\boldsymbol{u}\| = 112.25 \text{ V}$$

$$\boldsymbol{j} = 50\boldsymbol{\sigma_{23}} + 10\boldsymbol{\sigma_{134}} + 30\boldsymbol{\sigma_{2345}} \qquad \longrightarrow \qquad \|\boldsymbol{j}\| = 59.16 \text{ A}$$

The impedance and admittance for every harmonic of the source and the load is

$Z_{s1} = 0.4 - 0.2\boldsymbol{\sigma_{12}}$	$Y_{s1} = 2.00 + 1.00\boldsymbol{\sigma_{12}}$
$Z_{s2} = 0.4 - 0.4\boldsymbol{\sigma_{12}}$	$Y_{s2} = 1.25 + 1.25\boldsymbol{\sigma_{12}}$
$Z_{s3} = 0.4 - 0.6\boldsymbol{\sigma_{12}}$	$Y_{s3} = 0.77 + 1.15\boldsymbol{\sigma_{12}}$
$Z_{s4} = 0.4 - 0.8\boldsymbol{\sigma_{12}}$	$Y_{s4} = 0.50 + 1.00\boldsymbol{\sigma_{12}}$
$Z_{l1} = 2.0 - 1.0\boldsymbol{\sigma_{12}}$	$Y_{l1} = 0.40 + 0.20\boldsymbol{\sigma_{12}}$
$Z_{l2} = 2.0 - 2.0\boldsymbol{\sigma_{12}}$	$Y_{l2} = 0.25 + 0.25\boldsymbol{\sigma_{12}}$
$Z_{l3} = 2.0 - 3.0\boldsymbol{\sigma_{12}}$	$Y_{l3} = 0.15 + 0.23\boldsymbol{\sigma_{12}}$
$Z_{l4} = 2.0 - 4.0\boldsymbol{\sigma_{12}}$	$Y_{l4} = 0.20 + 0.10\boldsymbol{\sigma_{12}}$

The current now can be calculated by applying Ohm's law and Kirchhoff's laws:

$$\boldsymbol{i} = \langle \boldsymbol{i} \rangle_1 + \langle \boldsymbol{i} \rangle_2 + \langle \boldsymbol{i} \rangle_3 + \langle \boldsymbol{i} \rangle_4 \tag{35}$$

$$
\begin{aligned}
\langle \boldsymbol{i} \rangle_1 &= \frac{1}{Z_{s1}+Z_{l1}} \langle \boldsymbol{u} \rangle_1 = 33.3\boldsymbol{\sigma_1} - 16.6\boldsymbol{\sigma_2} \\
\langle \boldsymbol{i} \rangle_2 &= \frac{Y_{s2}}{Y_{s2}+Y_{l2}} \langle \boldsymbol{j} \rangle_2 = 41.7\boldsymbol{\sigma_{23}} \\
\langle \boldsymbol{i} \rangle_3 &= \frac{1}{Z_{s3}+Z_{l3}} \langle \boldsymbol{u} \rangle_3 + \frac{Y_{s3}}{Y_{s3}+Y_{l3}} \langle \boldsymbol{j} \rangle_3 = 17.9\boldsymbol{\sigma_{134}} + 6.4\boldsymbol{\sigma_{234}} \\
\langle \boldsymbol{i} \rangle_4 &= \frac{1}{Z_{s4}+Z_{l4}} \langle \boldsymbol{u} \rangle_4 + \frac{Y_{s4}}{Y_{s4}+Y_{l4}} \langle \boldsymbol{j} \rangle_4 = 0.8\boldsymbol{\sigma_{1345}} + 23.3\boldsymbol{\sigma_{2345}} \\
\boldsymbol{i} &= \quad 33.3\boldsymbol{\sigma_1} - 16.6\boldsymbol{\sigma_2} + 41.7\boldsymbol{\sigma_{23}} + 17.9\boldsymbol{\sigma_{134}} + 6.4\boldsymbol{\sigma_{234}} + 0.8\boldsymbol{\sigma_{1345}} + 23.3\boldsymbol{\sigma_{2345}} \\
\|\boldsymbol{i}\| &= \; 63.5 \text{ A}
\end{aligned}
\tag{36}
$$

Once the current is found, the voltage drop in the impedance of the source can be obtained as

$$
\begin{aligned}
\boldsymbol{V_1} =& \boldsymbol{Z_{s1}} \langle \boldsymbol{i} \rangle_1 + \boldsymbol{Z_{s2}} \langle \boldsymbol{i} \rangle_2 + \boldsymbol{Z_{s3}} \langle \boldsymbol{i} \rangle_3 + \boldsymbol{Z_{s4}} \langle \boldsymbol{i} \rangle_4 \\
=& 16.7\boldsymbol{\sigma_1} - 16.7\boldsymbol{\sigma_{13}} + 16.7\boldsymbol{\sigma_{23}} + 3.3\boldsymbol{\sigma_{134}} + 13.3\boldsymbol{\sigma_{234}} - 18.3\boldsymbol{\sigma_{1345}} + 10\boldsymbol{\sigma_{2345}}
\end{aligned}
$$

Therefore, the voltage in the load V_x is

$$
\boldsymbol{V_x} = \boldsymbol{V} - \boldsymbol{V_1} = 83.3\boldsymbol{\sigma_1} + 16.7\boldsymbol{\sigma_{13}} - 16.7\boldsymbol{\sigma_{23}} - 3.3\boldsymbol{\sigma_{134}} + 36.6\boldsymbol{\sigma_{234}} + 28.3\boldsymbol{\sigma_{1345}} - 10\boldsymbol{\sigma_{2345}}
$$

$$
\|\boldsymbol{V_x}\| = 98.78 \text{ V}
$$

Once the voltage and current are found, the power $\boldsymbol{M}$ can be calculated:

		k-vector													
	Norm	**e0**	**e3**	**e4**	**e5**	**e12**	**e34**	**e45**	**e123**	**e124**	**e125**	**e345**	**e1234**	**e1245**	**e12345**
$\boldsymbol{M}$	6696.9	3662.2		-2083.3	-987.2	-535.9	961.5		4166.7		105.8	-250.0	-1025.6	-416.7	2500.0
$\boldsymbol{M_J}$	5932.5	1166.7		2000.0	816.7	-383.3	-833.3	-1000.0	-4166.7	0.0	-200.0			1916.7	-2500.0
Gen	5346.8	4828.8		-83.3	-170.5	-919.2	128.2	-1000.0	0.0	0.0	-94.2	-250.0	-1025.6	1500.0	0.0
$\boldsymbol{M_1}$	2816.2	1613.4	833.3	-363.2	-578.8	198.5	188.0	-791.7	972.2	267.1	-363.7	791.7	-393.2	1166.7	416.7
$\boldsymbol{M_l}$	3928.6	3215.5	-833.3	279.9	408.3	-1117.7	-59.8	-208.3	-972.2	-267.1	269.4	-1041.7	-632.5	333.3	-416.7
Demd	5346.8	4828.8		-83.3	-170.5	-919.2	128.2	-1000.0	0.0	0.0	-94.2	-250.0	-1025.6	1500.0	0.0
$\boldsymbol{M_x}$	5348.9	2048.8	-833.3	-1720.1	-408.3	-734.4	773.5	791.7	3194.4	-267.1	469.4	-1041.7	-632.5	-1583.3	2083.3

Table 2: Power decomposition for circuit shown in Figure 7

$$\boldsymbol{M} = \boldsymbol{ui} = 2048.8 \tag{37}$$

$$- 833.3\boldsymbol{e_3} - 1720.0\boldsymbol{e_4} - 408.33\boldsymbol{e_5} \tag{38}$$

$$- 734.4\boldsymbol{e_{12}} + 773.5\boldsymbol{e_{34}} + 791.6\boldsymbol{e_{45}} \tag{39}$$

$$+ 3194.4\boldsymbol{e_{123}} - 267.1\boldsymbol{e_{124}} + 469.4\boldsymbol{e_{125}} - 1041.6\boldsymbol{e_{345}} \tag{40}$$

$$- 632.4\boldsymbol{e_{1234}} - 1583.3\boldsymbol{e_{1245}} \tag{41}$$

$$+ 2083.3\boldsymbol{e_{12345}} \tag{42}$$

From the expression (42), the following values are highlighted

$$\begin{array}{lcr} P & = & 2048.8 \\ CN_{r(ps)} & = & 734.4 \\ CN_{r(hi)} & = & 4064.2 \\ CN_d & = & 3431.8 \\ \hline \|\boldsymbol{M}\| & = & 5348.9 \end{array}$$

Once the current and power are obtained, their decomposition can be carried out to verify that Kirchhoff's laws and PoCOE are satisfied, respectively. Tables 2 and 3 detail this decomposition. It can be observed how the power demanded by the resistors, of both the source and the load, is equal to that generated by the sources. Likewise, it is possible to perform a detailed decomposition of each specific term of active, reactive and degraded power. Moreover, the current decomposition also complies with Kirchhoff's first law for all k-vectors of which it is composed. The current I_a is the minimum current (according to Fryze) that generates the active power P necessary for the operation of the load. The strategies of compensation are several, depending on the objectives established, but thanks to the geometric power theory, these can be

Current	Norm	k-vector							
		e1	**e2**	**e13**	**e23**	**e134**	**e234**	**e1345**	**e2345**
$\boldsymbol{I}$	63.51	33.33	-16.67		41.67	17.95	6.41	0.83	23.33
$\boldsymbol{I_b}$	44.53		-16.67	20.83	20.83	18.38	1.67	7.42	21.01
$\boldsymbol{I_g}$	45.28	33.33		-20.83	20.83	-0.43	4.74	-6.58	2.32
$\boldsymbol{I_a}$	20.74	17.50		3.50	-3.50	-0.70	7.70	5.95	-2.10
$\boldsymbol{I_d}$	40.25	15.84		-24.33	24.33	0.27	-2.96	-12.53	4.42

Table 3: Current decomposition for circuit in Figure 7

accomplished without any major inconvenience.

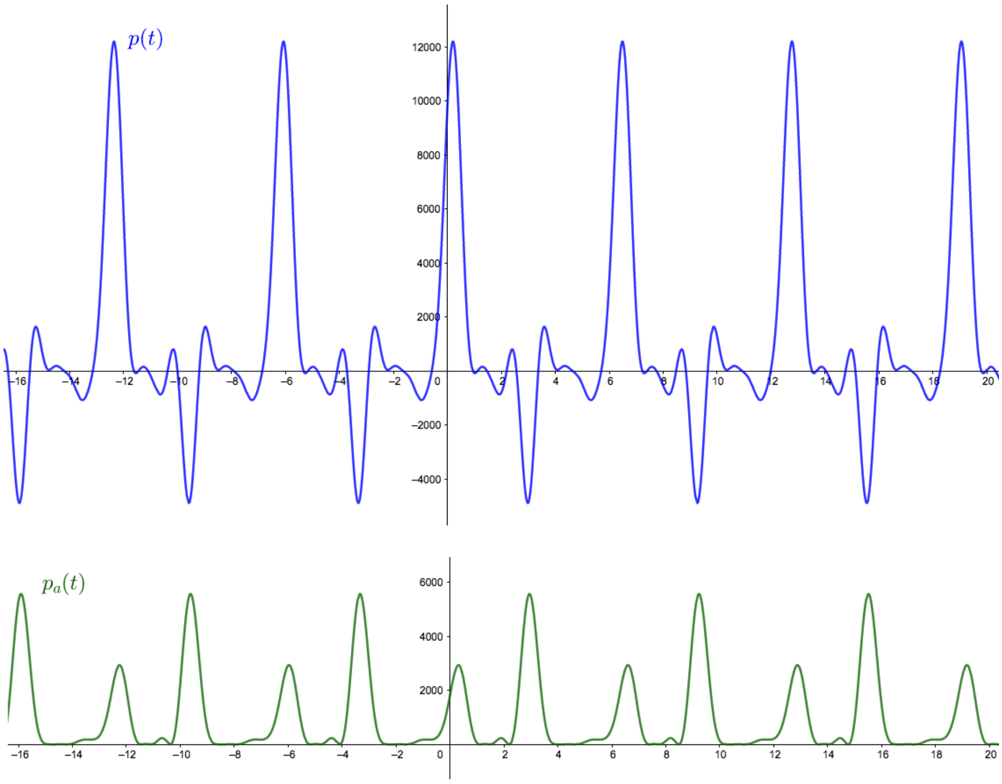

Figure 8: Instantaneous power for non linear load: top) total instantaneous power $p(t)$, bottom) active instantaneous power $p_a(t)$

Figure 8 shows the values obtained for the total instantaneous power and the active instantaneous power by applying the inverse transformation of the geometric domain to the time domain. It is shown how the power $p_a(t)$ presents only positive values.

Applying any of the existing power theories to this circuit in order to obtain the apparent power S is completely unfeasible as there is no way to collect the contributions of the harmonics' cross-terms. In addition, there is no way that the principle of energy conservation can be met, nor is there any chance that Kirchoff's first law can be met for currents in different branches of the circuit based on the proposed decomposition.

5. Conclusions

In this paper a detailed analysis of the use Geometric Algebra applied to electric power systems is conducted. Thanks to its flexibility, it is possible to correctly define new terms that comply with the principle of conservation of energy, not satisfied with the traditional definition of apparent power S. This superiority has been proven repeatedly in the scientific literature. The non-active term net geometric power is analyzed in non-sinusoidal systems and compared with traditional theories, revealing the ability of geometric algebra to verify net power flows along with their sense and magnitude. In addition, this work also proposes, for the first time, a new decomposition of currents beyond the proposed by Castro-Núñez that links up with the proposal of Fryze and Czarnecki, so it is possible to obtain an active geometrical current that minimizes the current to be supplied for the required active power of both a linear and non-linear loads. Once again, it is demonstrated how traditional concepts based on apparent power S should be abandoned due to their lack of physical meaning and the artifice of their definition.

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