Available online at www.sciencedirect.com

ScienceDirect

Fuzzy Sets and Systems 456 (2023) 197–207

FUZZY sets and systems

www.elsevier.com/locate/fss

Short communication

A generalization of a copula-based construction of fuzzy implications

Juan Fernández-Sánchez [a,b], Anna Kolesárová [c], Radko Mesiar [d,e], José Juan Quesada-Molina [f], Manuel Úbeda-Flores [g,*]

[a] Research Group of Theory of Copulas and Applications, University of Almería, 04120 Almería, Spain
[b] Interdisciplinary Mathematics Institute, Complutense University of Madrid, 28040 Madrid, Spain
[c] Institute of Information Engineering, Automation and Mathematics, Faculty of Chemical and Food Technology, Slovak University of Technology, 812 37 Bratislava, Slovakia
[d] Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Slovak University of Technology, 810 05 Bratislava, Slovakia
[e] University of Ostrava, Institute for Research and Applications of Fuzzy Modelling, 30. Dubna 22, Ostrava, Czech Republic
[f] Department of Applied Mathematics, University of Granada, 18071 Granada, Spain
[g] Department of Mathematics, University of Almería, 04120 Almería, Spain

Received 12 March 2022; received in revised form 11 May 2022; accepted 16 May 2022
Available online 24 May 2022

Abstract

In this paper we complement and generalize some constructions of fuzzy implications based on two arbitrary copulas, obtaining new fuzzy implications. By means of (restricted) aggregation functions acting on $[0, 1]^S$, where S is a fixed finite or infinite set, and related S-systems of fuzzy implications and transforming functions, we introduce and discuss a rather general method for constructing fuzzy implications. Several examples illustrating our results are also included.

Keywords: Aggregation function; Copula; Fuzzy implication

1. Introduction

In fuzzy logic and approximate reasoning, the most usual method for managing the conditional statements "If A, then B" is done through functions $I : [0, 1]^2 \to [0, 1]$ in such a way that the result value of the conditional is functionally stated from the truth values of the fuzzy statements A and B. These functions I are the so-called fuzzy implications (or simply, implications), and they play an important role in many fields where fuzzy logic applies (see [2]).

* Corresponding author.
E-mail addresses: juanfernandez@ual.es (J. Fernández-Sánchez), anna.kolesarova@stuba.sk (A. Kolesárová), radko.mesiar@stuba.sk (R. Mesiar), jquesada@ugr.es (J.J. Quesada-Molina), mubeda@ual.es (M. Úbeda-Flores).

https://doi.org/10.1016/j.fss.2022.05.013
0165-0114/

A binary operation $I : [0,1]^2 \to [0,1]$ is called a *fuzzy implication* if it satisfies the following conditions:

(i) $I(0,0) = I(1,1) = 1$,
(ii) $I(1,0) = 0$,
(iii) $I(u_1, v) \geq I(u_2, v)$ for all $u_1, u_2, v \in [0,1]$, $u_1 \leq u_2$,
(iv) $I(u, v_1) \leq I(u, v_2)$ for all $u, v_1, v_2 \in [0,1]$, $v_1 \leq v_2$,

which means that fuzzy implications are hybrid monotone extensions of the classical Boolean implication.

Implications are usually derived from t-norms and semi-copulas, being both of them subclasses of aggregation functions with zero annihilator. See [4,10] for more details of these concepts, and [3,17–19] for some constructions of implications based on these functions.

Methods for constructing fuzzy implications based on copulas –bivariate probability distribution functions with uniform margins on [0, 1]– can be found, for example, in [9,14,20,21]. Our aim in this paper is to provide a generalization of some known copula-based constructions of fuzzy implications.

The rest of the paper is organized as follows. After some preliminaries concerning copulas (Section 2), our goal in this paper is to complement and generalize some of the results related to the new method for constructing fuzzy implications given in [20] (see also [21]), which is based on any two copulas and a fuzzy implication (Section 3). To illustrate the generality of our approach, several examples are also included. Section 4 is devoted to the conclusions indicating some possible applications of our results for constructing fuzzy implications with some particular properties, such as the neutrality principle or the ordering property, and also some possible directions for further research.

2. Preliminaries

A (bivariate) *copula* is a binary operation $C\colon [0,1]^2 \to [0,1]$ which satisfies:

(i) for every $u \in [0,1]$, $C(u,0) = C(0,u) = 0$ and $C(u,1) = C(1,u) = u$, and
(ii) $V_C([u_1,u_2] \times [v_1,v_2]) := C(u_2,v_2) - C(u_1,v_2) - C(u_2,v_1) + C(u_1,v_1) \geq 0$ for every $u_1, v_1, u_2, v_2 \in [0,1]$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$.

We refer to [10,22] for a complete study of these functions.

Let Π denote the copula for independent random variables, i.e., $\Pi(u,v) := uv$ for all $(u,v) \in [0,1]^2$. For any copula C we have $W(u,v) := \max(0, u+v-1) \leq C(u,v) \leq \min(u,v) =: M(u,v)$ for all (u,v) in $[0,1]^2$, where M and W are themselves copulas (which are also the Fréchet-Hoeffding upper and lower bounds, respectively).

The importance of copulas in probability and statistics comes from *Sklar's theorem* [24], which shows that the joint distribution H of a pair of random variables and the corresponding univariate marginal distributions F and G are linked by a copula C in the following manner: $H(x,y) = C(F(x), G(y))$ for all $(x,y) \in]-\infty, +\infty[^2$. If F and G are continuous, then the copula C is unique; otherwise, the copula is uniquely determined on Range $F \times$ Range G (see [7] for details).

Let $\mathcal{B}([0,1])$ and $\mathcal{B}([0,1]^2)$ denote the Borel σ-algebras in [0, 1] and $[0,1]^2$, respectively, and let λ denote the Lebesgue measure on [0, 1]. A measure μ on $\mathcal{B}([0,1]^2)$ is *doubly stochastic* if $\mu(B \times [0,1]) = \mu([0,1] \times B) = \lambda(B)$ for every $B \in \mathcal{B}([0,1])$. Each copula C induces a doubly stochastic measure μ_C on $\mathcal{B}([0,1]^2)$ by setting $\mu_C(R) = V_C(R)$ for every rectangle $R = [u_1,u_2] \times [v_1,v_2] \subseteq [0,1]^2$, and extending μ_C to $\mathcal{B}([0,1]^2)$ (see [22]). The *support* of a copula C is the complement of the union of all open subsets of $[0,1]^2$ with μ_C-measure zero.

We also recall the known Disintegration theorem [1], which we adapt for our purposes, and where 1_E denotes the indicator function of a set $E \subseteq [0,1]$.

Theorem 1 *(Disintegration theorem). Let μ be a doubly stochastic measure on $\mathcal{B}([0,1]^2)$. Then there exists a λ-almost everywhere uniquely determined Borel family of probability measures $\{\mu_x\}_{x\in[0,1]}$ on $\mathcal{B}([0,1])$ such that*

$$\mu(B) = \int_{[0,1]} \left(\int_{[0,1]} 1_{B_x}(y)\, \mathrm{d}\mu_x(y) \right) dx = \int_{[0,1]} \mu_x(B_x)\, \mathrm{d}x,$$

for every $B \in \mathcal{B}([0,1]^2)$, where $B_x = \{y \in [0,1] : (x,y) \in B\}$.

3. Implications based on copulas

In [20], a copula-based method for constructing fuzzy implications is provided. Specifically, let C be a copula, and for each $x \in [0,1]$, let $\nu_{C,x} \colon [0,1] \to [0,x]$ be the function defined by $\nu_{C,x}(t) = C(x,t)$, and let $g_{C,x} \colon [0,1] \to [0,1]$ be the function given by

$$g_{C,x}(t) = \begin{cases} \dfrac{\mathrm{d}\nu_{C,x}(t)}{\mathrm{d}t} & \text{if this derivative exists,} \\ 0 & \text{otherwise.} \end{cases}$$

Now, for each $x \in [0,1]$, let $f_{C,x} \colon [0,1] \to [0,1]$ be the left-continuous function defined by

$$f_{C,x}(t) = \sup\{g_{C,z}(t) : z \in [0,x]\}. \tag{1}$$

Then the authors proved in [20] that, for any two copulas C_1 and C_2, and a Borel integrable fuzzy implication I, the function $J_{C_1,C_2,I} \colon [0,1]^2 \to [0,1]$ given by

$$J_{C_1,C_2,I}(u,v) = \int\limits_{[0,1]} I\left(f_{C_1,u}(t), f_{C_2,v}(t)\right) \mathrm{d}t \tag{2}$$

is a fuzzy implication.

In [20, Remark 2.9(iv)], the authors pointed out that construction (2) can be applied to any two monotone non-decreasing systems of functions $\left(f_{1,x}\right)_{x\in[0,1]}$ and $\left(f_{2,y}\right)_{y\in[0,1]}$ such that $f_{1,0}(t) = f_{2,0}(t) = 0$ and $f_{1,1}(t) = f_{2,1}(t) = 1$ for all $t \in [0,1]$. We can even generalize this result in the next theorem, but first, following [12], we "extend" the arity of an aggregation function, where, for any interval $\mathcal{I}$ and any non-empty set S, $\mathcal{I}^S$ denotes the set of all functions from S to $\mathcal{I}$.

Definition 1. Let $\mathcal{I}$ be an interval, and let S be a non-empty set. The function $A \colon \mathcal{I}^S \longrightarrow \mathcal{I}$ is an (S-)aggregation function on $\mathcal{I}$ if:

(i) A is non-decreasing in each variable; and
(ii) A fulfills the boundary conditions

$$\inf_{\mathbf{x}\in\mathcal{I}^S} A(\mathbf{x}) = \inf \mathcal{I} \quad \textit{and} \quad \sup_{\mathbf{x}\in\mathcal{I}^S} A(\mathbf{x}) = \sup \mathcal{I}.$$

Theorem 2. *Let S be a non-empty set, let $\mathcal{G}_1 = \left(g_{1,s}\right)_{s\in S}$ and $\mathcal{G}_2 = \left(g_{2,s}\right)_{s\in S}$ be two systems of non-decreasing functions from $[0,1]$ into $[0,1]$ such that $g_{1,s}(0) = g_{2,s}(0) = 0$ and $g_{1,s}(1) = g_{2,s}(1) = 1$ for all $s \in S$. Let $\mathbf{I}_S = (I_s)_{s\in S}$ be a family of fuzzy implications, and let A be an (S-)aggregation function on $[0,1]$. Then the function $K_{\mathcal{G}_1,\mathcal{G}_2,\mathbf{I}_S,A}$ defined on $[0,1]^2$ by*

$$K_{\mathcal{G}_1,\mathcal{G}_2,\mathbf{I}_S,A}(u,v) = A\left(\left(I_s\left(g_{1,s}(u), g_{2,s}(v)\right)\right)_{s\in S}\right) \tag{3}$$

is a fuzzy implication.

Proof. First, assume $u = v = 0$. Then we have $I_s\left(g_{1,s}(u), g_{2,s}(v)\right) = I_s(0,0) = 1$ for each $s \in S$, whence

$$K_{\mathcal{G}_1,\mathcal{G}_2,\mathbf{I}_S,A}(0,0) = A\left(\left(I_s\left(g_{1,s}(0), g_{2,s}(0)\right)\right)_{s\in S}\right) = A(\mathbf{1}) = 1;$$

and similarly, $K_{\mathcal{G}_1,\mathcal{G}_2,\mathbf{I}_S,A}(1,1) = A(\mathbf{1}) = 1$ and $K_{\mathcal{G}_1,\mathcal{G}_2,\mathbf{I}_S,A}(1,0) = A(\mathbf{0}) = 0$.

Now, for any $u \in [0,1]$ and given $v_1, v_2 \in [0,1]$ such that $v_1 < v_2$, we have $I_s\left(g_{1,s}(u), g_{2,s}(v_1)\right) \le I_s\left(g_{1,s}(u), g_{2,s}(v_2)\right)$ for each $s \in S$, whence

$$K_{\mathcal{G}_1,\mathcal{G}_2,\mathbf{I}_S,A}(u,v_1) = A\left(\left(I_s\left(g_{1,s}(u), g_{2,s}(v_1)\right)\right)_{s\in S}\right) \le A\left(\left(I_s\left(g_{1,s}(u), g_{2,s}(v_2)\right)\right)_{s\in S}\right) = K_{\mathcal{G}_1,\mathcal{G}_2,\mathbf{I}_S,A}(u,v_2);$$

and similarly, for any $v \in [0,1]$ and given $u_1, u_2 \in [0,1]$ such that $u_1 < u_2$, $K_{\mathcal{G}_1,\mathcal{G}_2,\mathbf{I}_S,A}(u_1,v) \ge K_{\mathcal{G}_1,\mathcal{G}_2,\mathbf{I}_S,A}(u_2,v)$; i.e., the function given by (3) is a fuzzy implication. □

We apply Theorem 2 in the next examples.

Example 1. Let $S =]0, 1]$, let $g_{1,s}(t) = g_{2,s}(t) = t^s$ for all $t \in [0, 1]$, and, for each $s \in S$, let $I_s = I_G$ be the Goguen fuzzy implication, i.e.,

$$I_G(u, v) = \begin{cases} 1 & \text{if } u \le v, \\ \dfrac{v}{u} & \text{otherwise} \end{cases}$$

(see [2]). If A is the S-supremum aggregation function on $[0, 1]$, i.e., $A\left((u_s)_{s\in S}\right) = \sup\{u_s : s \in S\}$, then the fuzzy implication (3) is given by

$$K_{\mathcal{G}_1,\mathcal{G}_2,\mathbf{I}_S,A}(u, v) = \sup\left\{I_G\left(u^s, v^s\right) : s \in]0, 1]\right\} = \begin{cases} 0 & \text{if } v = 0 \text{ and } u \neq 0, \\ 1 & \text{otherwise} \end{cases}$$

for all $(u, v) \in [0, 1]^2$.

Example 2. Let $S = [1, +\infty[$, let $g_{1,s}(t) = g_{2,s}(t) = t^s$ for all $t \in [0, 1]$, and, for each $s \in S$, let $I_s = I_G$ be the Goguen fuzzy implication. If A is the S-infimum aggregation function on $[0, 1]$, i.e., $A\left((u_s)_{s\in S}\right) = \inf\{u_s : s \in S\}$, then the fuzzy implication $K_{\mathcal{G}_1,\mathcal{G}_2,\mathbf{I}_S,A}$ is the Rescher fuzzy implication (see [2]), that is,

$$K_{\mathcal{G}_1,\mathcal{G}_2,\mathbf{I}_S,A}(u, v) = \inf\left\{I_G\left(u^s, v^s\right) : s \in [1, +\infty[\right\} = \begin{cases} 1 & \text{if } u \le v, \\ 0 & \text{otherwise} \end{cases}$$

for all $(u, v) \in [0, 1]^2$.

Remark 1. For infinite sets S, it is often necessary to deal with some restricted domains for an aggregation function A instead of $[0, 1]^S$, i.e., a suitable subset of $[0, 1]^S$ as a domain of an aggregation function A is considered. Namely, one can deal with $A\colon \mathcal{H} \to [0, 1]$, where $\mathcal{H} \subset [0, 1]^S$ such that $1_\emptyset, 1_S \in \mathcal{H}$, and $A(1_\emptyset) = 0$, $A(1_S) = 1$ and $A(\mathbf{x}) \le A(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{H}$ satisfying $\mathbf{x} \le \mathbf{y}$. Obviously, to apply Theorem 2, one needs to ensure that $\mathbf{x}_{\mathcal{G}_1,\mathcal{G}_2,\mathbf{I}_S,u,v} \in \mathcal{H}$ for all $u, v \in [0, 1]$, where

$$\mathbf{x}_{\mathcal{G}_1,\mathcal{G}_2,\mathbf{I}_S,u,v}(s) = I_s\left(g_{1,s}(u), g_{2,s}(v)\right).$$

In such a case, (3) can be rewritten as

$$K_{\mathcal{G}_1,\mathcal{G}_2,\mathbf{I}_S,A}(u, v) = A\left(\left(\mathbf{x}_{\mathcal{G}_1,\mathcal{G}_2,\mathbf{I}_S,u,v}(s)\right)_{s\in S}\right). \tag{4}$$

As typical examples for such aggregation functions A one can consider the Lebesgue, Choquet and Sugeno integrals defined for appropriate functions from $[0, 1]^S$ only. In general, the Borel measurability of fuzzy implications I_s should be considered. As an example of a fuzzy implication which is not Borel measurable one can take a function $I_E\colon [0, 1]^2 \to [0, 1]$ given by

$$I_E(u, v) = \begin{cases} 1 & \textit{if } u > v \textit{ or } (u = v \textit{ and } u \notin E), \\ 0 & \textit{otherwise}, \end{cases}$$

where E is any subset of $]0, 1[$ which is not Borel measurable. It can be checked that I_E is a fuzzy implication, but it is not a Borel measurable function.

Example 3. Let $S =]0, 1]$, let $g_{1,s}(t) = g_{2,s}(t) = t^s$ for all $t \in [0, 1]$, and, for each $s \in S$, let $I_s = I_G$ be the Goguen fuzzy implication. Then $\mathbf{x}_{\mathcal{G}_1,\mathcal{G}_2,\mathbf{I}_S,u,v}(s) = \min\left\{1, \left(\frac{v}{u}\right)^s\right\}$ and thus $\mathbf{x}_{\mathcal{G}_1,\mathcal{G}_2,\mathbf{I}_S,u,v}$ is Borel measurable for any $u, v \in [0, 1]$.

(i) If A is the Lebesgue integral (see [15]) on $(]0, 1], \mathcal{B}(]0, 1]))$ and considering the Lebesgue measure λ, we have

$$K_{\mathcal{G}_1,\mathcal{G}_2,\mathbf{I}_S,A}(u, v) = \begin{cases} 1 & \textit{if } u \le v, \\ \dfrac{v - u}{u \ln\left(\frac{v}{u}\right)} & \textit{otherwise}. \end{cases}$$

(ii) If $A = Ch_\mu$ is the Choquet integral [5] with respect to the fuzzy measure $\mu = \lambda^2$, then we get

$$K_{\mathcal{G}_1,\mathcal{G}_2,\mathbf{I}_S,A}(u,v) = \begin{cases} 1 & \text{if } u \le v, \\ \dfrac{2u + 2v \ln \frac{v}{u} - 2v}{u \ln^2 \frac{v}{u}} & \text{otherwise.} \end{cases}$$

(iii) If $A = Su_\lambda$ is the Sugeno integral [25] with respect to the Lebesgue measure λ, then we have

$$K_{\mathcal{G}_1,\mathcal{G}_2,\mathbf{I}_S,A}(u,v) = \begin{cases} 1 & \text{if } u \le v, \\ h^{-1}\left(\ln \dfrac{v}{u}\right) & \text{otherwise,} \end{cases}$$

$h^{-1}\colon [-\infty, 0] \to [0, 1]$ being an inverse function to the function $h\colon [0, 1] \to [-\infty, 0]$ given by $h(t) = \frac{\ln t}{t}$.

As a consequence of Theorem 2 we have the following result.

Corollary 3. *Let S be a non-empty set, and let μ be a measure that can either be non-additive defined on 2^S, finitely additive defined on an algebra on S, or σ-additive defined on a σ-algebra on S. Let $\mathcal{G}_1 = (g_{1,s})_{s\in S}$ and $\mathcal{G}_2 = (g_{2,s})_{s\in S}$ be two systems of non-decreasing functions from $[0, 1]$ into $[0, 1]$ such that $g_{1,s}(0) = g_{2,s}(0) = 0$ and $g_{1,s}(1) = g_{2,s}(1) = 1$ for all $s \in S$. Let $\mathbf{I}_S = (I_s)_{s\in S}$ be a family of fuzzy implications fulfilling adequate integrability conditions for μ. Then the function $K_{\mathcal{G}_1,\mathcal{G}_2,\mathbf{I}_S}$ defined on $[0, 1]^2$ by*

$$K_{\mathcal{G}_1,\mathcal{G}_2,\mathbf{I}_S}(u,v) = \int_S I_s\left(g_{1,s}(u), g_{2,s}(v)\right) \mathrm{d}\mu(s) \tag{5}$$

is a fuzzy implication.

Remark 2. (i) Note that formula (5) has been obtained from formula (3) when an aggregation function A is an appropriate integral, $A(\cdot) = \int_S \cdot \, \mathrm{d}\mu$. In such a case, the letter A will not be explicitly written in the notation of the constructed fuzzy implication.

(ii) We want to stress that though there is a variety of integrals which could be applied in (5), we will only consider the Lebesgue integral with respect to a σ-additive measure μ (see [15]), or the Choquet and Sugeno integrals with respect to a fuzzy measure μ (see [5,8] and [25], respectively).

Remark 3. Obviously, depending on what kind of set S and measure μ are considered in Corollary 3, we obtain different implications.

1. If the set S is finite and μ is a probability measure on S, then we get a convex linear combination of fuzzy implications.
2. If $S = [0, 1]$, μ is a probability measure on $\mathcal{B}([0, 1])$ and, for any $u, v \in [0, 1]$, $I_s\left(g_{1,s}(u), g_{2,s}(v)\right)$ is a Borel measurable function on $[0, 1]$, then we can obtain new fuzzy implications, see, e.g., Example 3(i). Moreover, observe that (5) is a generalization of (2): if μ is the Lebesgue measure on $[0, 1]$, $(g_{1,s})_{s\in S} := (\mu_{C_1,s})_{s\in S}$ and $(g_{2,s})_{s\in S} := (\mu_{C_2,s})_{s\in S}$ are the disintegrations of the measures associated with the copulas C_1 and C_2, respectively, and, for any $u, v \in [0, 1]$, $I_s\left(g_{1,s}(u), g_{2,s}(v)\right)$ is a Borel measurable function on $[0, 1]$, then we obtain (2). We note that there is a difference between our construction and the fuzzy implication given by (2), since in this last case the function $f_{C,x}$ defined by (1) is left-continuous, but not right-continuous, in general.

We provide an additional example.

Example 4. Let $S = \{1, 2, 3\}$, let I_1 be the Reichenbach implication —i.e., $I_1(u, v) = 1 - u + uv$ for all $(u, v) \in [0, 1]^2$ (see [2])—, let I_2 be the Goguen implication, let I_3 be the Łukasiewicz implication —i.e., $I_3(u, v) = \min(1, 1 - u + v)$ for all $(u, v) \in [0, 1]^2$ (see [2])—, let $g_{1,s}(t) = g_{2,s}(t) = t$ for all $t \in [0, 1]$ and for every $s \in S$, and let μ be the fuzzy measure on 2^S given by

$$\begin{aligned}
\mu(\emptyset) &= 0, \\
\mu(\{1\}) = \mu(\{3\}) &= 1/3, \quad \mu(\{2\}) = 2/3, \\
\mu(\{1,3\}) &= 2/3, \quad \mu(\{2,3\}) = \mu(\{1,2\}) = 5/6, \\
\mu(\{1,2,3\}) &= 1.
\end{aligned}$$

Then, after some elementary algebra, we can conclude that the fuzzy implication defined by (5), when the Choquet integral is considered, is given by

$$K_{\mathcal{G}_1,\mathcal{G}_2,\mathbf{I}_S}(u,v) = \begin{cases} 1 - \dfrac{u(1-v)}{6} & \text{if } u \le v, \\ \dfrac{u^2(v-2) + u(v+2) + v}{3u} & \text{if } \dfrac{u}{u+1} < v < u, \\ \dfrac{u^2(v-3) + u(2v+3) + 3v}{6u} & \text{otherwise.} \end{cases}$$

Observe that all I_i, $i = 1, 2, 3$, satisfy the left neutrality principle, i.e., $I_i(1, v) = v$ for all $v \in [0, 1]$, and that $K_{\mathcal{G}_1,\mathcal{G}_2,\mathbf{I}_S}$ also has the same property.

For our next result we recall several known concepts.

Given two measure spaces $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$, a mapping $f: \Omega_1 \to \Omega_2$ is said to be a *measure-preserving transformation* if:

(i) it is measurable with respect to the σ-fields $\mathcal{F}_1$ and $\mathcal{F}_2$, in the sense that, for every set $B \in \mathcal{F}_2$, $f^{-1}(B) \in \mathcal{F}_1$; and
(ii) $\mu_1\left(f^{-1}(B)\right) = \mu_2(B)$ for every set $B \in \mathcal{F}_2$.

For the particular case $(\Omega_1, \mathcal{F}_1, \mu_1) = (\Omega_2, \mathcal{F}_2, \mu_2) = ([0, 1], \mathcal{B}([0, 1]), \lambda)$, a relationship between copulas and measure-preserving transformations is the following [23]: If f_1 and f_2 are measure-preserving transformations on the space $([0, 1], \mathcal{B}([0, 1]), \lambda)$, then the function $C_{f_1,f_2}: [0, 1]^2 \to [0, 1]$ defined by

$$C_{f_1,f_2}(u,v) := \lambda\left(f_1^{-1}([0,u]) \cap f_2^{-1}([0,v])\right) \tag{6}$$

is a copula; and, conversely, for every copula C there exist two measure-preserving transformations f_1 and f_2 such that C can be expressed in the form (6).

For any fuzzy implication I, the function $N_I: [0, 1] \longrightarrow [0, 1]$, defined by $N_I(u) = I(u, 0)$, is called the *natural negation of* I. If $N_I = N_Z$, where the function $N_Z: [0, 1] \longrightarrow [0, 1]$ is defined as $N_Z(u) = 1 - u$ for all $u \in [0, 1]$, then the natural negation of I is called the *Zadeh negation* (or the *standard negation*).

Given two copulas C_1 and C_2, consider the $*$ "product" of C_1 and C_2 (see [6]) defined in terms of the two first-order partial derivatives of these copulas in the following manner:

$$(C_1 * C_2)(r,s) = \int_0^1 \frac{\partial C_1}{\partial t}(r,t) \cdot \frac{\partial C_2}{\partial t}(t,s)\,\mathrm{d}t$$

(if some derivative does not exist, its value equals zero, by convention).

We note that $C_1 * C_2$ is always a copula and, for any copula C, we have $\Pi * C = C * \Pi = \Pi$, $M * C = C * M = C$ and $(W * C)(u, v) = v - C(1 - u, v)$ and $(C * W)(u, v) = u - C(u, 1 - v)$ for all $(u, v) \in [0, 1]^2$.

Using measure-preserving transformations and a measurable implication with an additional boundary condition —in terms of a natural negation—, we obtain an interesting expression for (5) as stated in the following result, where id will denote the identity map on $[0, 1]$.

Theorem 4. *Let f be a measure-preserving transformation, let D be any copula, and let $\mathcal{G}_1 = \left(g_{1,s}\right)_{s\in[0,1]} = \left(\mu_{D,s}\right)_{s\in[0,1]}$ and $\mathcal{G}_2 = \left(g_{2,s}\right)_{s\in[0,1]} = \left(\mu_{C_{id,f},s}\right)_{s\in[0,1]}$ be the disintegrations of the measures associated with the copulas D and $C_{id,f}$, respectively. For each $s \in [0, 1]$ and every $u \in [0, 1]$, let $N_{I_s}(u) = I_s(u, 0)$ be the natural negation of a measurable implication I_s. Then the fuzzy implication $K_{\mathcal{G}_1,\mathcal{G}_2,\mathbf{I}_{[0,1]}}$ given by (5) can be expressed as*

$$K_{\mathcal{G}_1,\mathcal{G}_2,\mathbf{I}_{[0,1]}}(u,v) = 1 - \lambda(T_v) + \int_{T_v} N_{I_s}(\mu_{D,s}(u))\,\mathrm{d}\lambda(s) \tag{7}$$

for all $(u,v) \in [0,1]^2$, *where* $T_v = \{s \in [0,1] : f(s) > v\}$. *In particular, if* $N_{I_s} = N_Z$ *for every* $s \in [0,1]$, *we have*

$$K_{\mathcal{G}_1,\mathcal{G}_2,\mathbf{I}_{[0,1]}}(u,v) = 1 - u + \left(D * C_{id,f}\right)(u,v) \tag{8}$$

for all $(u,v) \in [0,1]^2$.

Proof. First note that $T_v = \left\{s \in [0,1] : \mu_{C_{id,f},s}(v) = 0\right\}$. On the other hand, since I_s is non-increasing in the first variable, from $I_s(1,1) = 1$ we have $I_s(u,1) = 1$ for every $u \in [0,1]$. Then we obtain

$$\begin{aligned} K_{\mathcal{G}_1,\mathcal{G}_2,\mathbf{I}_{[0,1]}}(u,v) &= \int_{[0,1]} I_s\left(\mu_{D,s}(u), \mu_{C_{id,f},s}(v)\right)\mathrm{d}\lambda(s) \\ &= \int_{T_v} I_s\left(\mu_{D,s}(u), 0\right)\mathrm{d}\lambda(s) + \int_{[0,1]\setminus T_v} I_s\left(\mu_{D,s}(u), 1\right)\mathrm{d}\lambda(s) \\ &= \int_{T_v} N_{I_s}(\mu_{D,s}(u))\mathrm{d}\lambda(s) + \int_{[0,1]\setminus T_v} 1\,\mathrm{d}\lambda(s), \end{aligned}$$

whence Equation (7) follows. In particular, if, for every $s \in [0,1]$, the natural negation of I_s is the Zadeh negation, i.e., $N_{I_s} = N_Z$ for every $s \in [0,1]$, then we have

$$\begin{aligned} K_{\mathcal{G}_1,\mathcal{G}_2,\mathbf{I}_{[0,1]}}(u,v) &= 1 - \lambda(T_v) + \int_{T_v}\left(1 - \mu_{D,s}(u)\right)\mathrm{d}\lambda(s) = 1 - \int_{T_v} \mu_{D,s}(u)\,\mathrm{d}\lambda(s) \\ &= 1 - \mu_D([0,u] \times T_v) = 1 - \mu_D([0,u] \times f^{-1}([v,1])) = 1 - u + \mu_D([0,u] \times f^{-1}([0,v])) \\ &= 1 - u + \left(D * C_{id,f}\right)(u,v), \end{aligned}$$

where the last equality follows from [11,13], and this completes the proof. □

In the following examples we use the expression given by (8) in order to find new implications by choosing different copulas D or measure-preserving transformations f.

Example 5. For different choices of the copula D in (8) we obtain the following fuzzy implications:

1. If $D = \Pi$ we have

   $$K_{\mathcal{G}_1,\mathcal{G}_2,\mathbf{I}_{[0,1]}}(u,v) = 1 - u + uv$$

   for all $(u,v) \in [0,1]^2$, i.e., the Reichenbach implication, regardless of the choice of the function f.
2. If $D = M$ then

   $$K_{\mathcal{G}_1,\mathcal{G}_2,\mathbf{I}_{[0,1]}}(u,v) = 1 - u + C_{id,f}(u,v)$$

   for all $(u,v) \in [0,1]^2$.
3. If $D = W$ then

   $$K_{\mathcal{G}_1,\mathcal{G}_2,\mathbf{I}_{[0,1]}}(u,v) = 1 - u + v - C_{id,f}(1-u,v)$$

   for all $(u,v) \in [0,1]^2$.

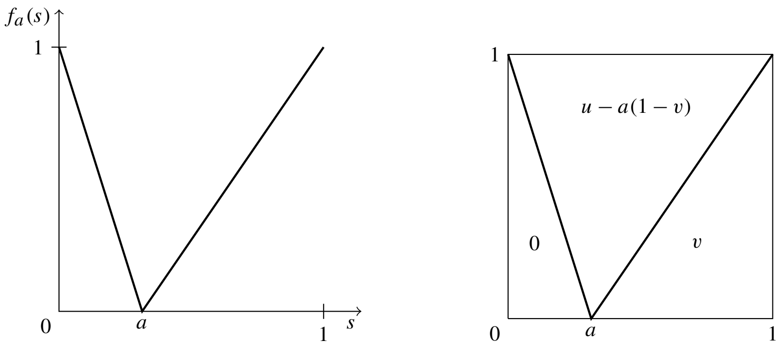

Fig. 1. The graph of the function f_a given by (9) (left) and 2D-visualization of the copula C_1 from Example 6 (right) whose support corresponds to the graph of f_a.

Example 6. Let $a \in]0, 1[$, and consider the function f_a defined by

$$f_a(s) = \begin{cases} 1 - \dfrac{s}{a} & \text{if } 0 \le s \le a, \\ \dfrac{s - a}{1 - a} & \text{if } a \le s \le 1. \end{cases} \tag{9}$$

Moreover, $f_0(s) = s$ and $f_1(s) = 1 - s$ for all $s \in [0, 1]$. Since $id^{-1}([0, u]) = [0, u]$ and $f_a^{-1}([0, v]) = [a(1 - v), v + a(1 - v)]$ then

$$C_{id,f_a}(u, v) = \begin{cases} 0 & \text{if } 0 \le u \le a(1 - v), \\ u - a(1 - v) & \text{if } a(1 - v) \le u \le v + a(1 - v), \\ v & \text{if } v + a(1 - v) \le u \le 1, \end{cases}$$

so that

$$\frac{\partial C_{id,f_a}}{\partial u}(u, v) = \begin{cases} 1 & \text{if } a(1 - v) \le u \le v + a(1 - v), \\ 0 & \text{otherwise,} \end{cases}$$

and hence, for any copula C we have

$$\left(C * C_{id,f_a}\right)(u, v) = \int_{a(1-v)}^{v+a(1-v)} \frac{\partial C}{\partial t}(u, t)\, \mathrm{d}t = C(u, v + a(1 - v)) - C(u, a(1 - v)).$$

Therefore, for each $a \in [0, 1]$, from (8) we obtain the fuzzy implication $I_{C,a}$ given in [20, Corollary 2.6], i.e.,

$$I_{C,a}(u, v) = 1 - u + C(u, v + a(1 - v)) - C(u, a(1 - v))$$

for all $(u, v) \in [0, 1]^2$. We note that $I_{C,a}(u, v) + u - 1$ is a subfamily of the biparametric family of copulas given by

$$V_C([c(1 - u), u + c(1 - u)] \times [d(1 - v), v + d(1 - v)]),$$

for any $c, d \in [0, 1]$ (see [16,22]). Observe also that if, for instance, we consider the copula $C = M$, then we have

$$I_{M,a}(u, v) = 1 - u + \min(u, v + a(1 - v)) - \min(u, a(1 - v)),$$

and hence $C_1(u, v) := u - 1 + I_{M,a}(u, v)$ is a copula whose support is related —in some sense— to the graph of the function f_a given by (9): Fig. 1 shows the graph of the function f_a and the support of the resulting copula C_1. Finally, note also that $I_{W,a} = I_{M,1-a}$ for all $a \in [0, 1]$.

Example 7. Let $a \in]0,1[$, and let f_a^* be the function given by

$$f_a^*(s) = \begin{cases} \dfrac{s}{a} & \text{if } 0 \le s \le a, \\ \dfrac{1-s}{1-a} & \text{if } a \le s \le 1. \end{cases}$$

Moreover, $f_0^*(s) = 1 - s$ and $f_1^*(s) = s$ for all $s \in [0,1]$. Then, for any copula C and each $a \in [0,1]$, from (8) we obtain the fuzzy implication

$$I_{C,a}^*(u,v) = 1 + C(u, av) - C(u, av + 1 - v)$$

for all $(u,v) \in [0,1]^2$.

Example 8. Consider a partition $0 = b_0 < a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n = 1$, and let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$. Let $f_{\mathbf{a},\mathbf{b}} \colon [0,1] \to [0,1]$ be the function defined on each subinterval $[b_{i-1}, b_i]$, for $i = 1, 2, \ldots, n$, by

$$f_{\mathbf{a},\mathbf{b}}(s) = \begin{cases} \dfrac{a_i - s}{a_i - b_{i-1}} & \text{if } b_{i-1} \le s \le a_i, \\ \dfrac{s - a_i}{b_i - a_i} & \text{if } a_i \le s \le b_i. \end{cases}$$

Then, for any copula C, we obtain from (8) the fuzzy implication

$$I_{C,(\mathbf{a},\mathbf{b})}(u,v) = 1 - u + \sum_{i=1}^{n} \left[C(u, a_i + (b_i - a_i)v) - C(u, a_i - (a_i - b_{i-1})v) \right]$$

for all $(u,v) \in [0,1]^2$. We note that, by using different notation, this expression corresponds to Equation (7) in [20].

Example 9. Consider a partition $0 = a_0 < a_1 < a_2 < \cdots < a_n = 1$, and let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$. Let $f_{\mathbf{a}} \colon [0,1] \to [0,1]$ be the function defined on each subinterval $[a_{i-1}, a_i]$, for $i = 1, 2, \ldots, n$, by

$$f_{\mathbf{a}}(s) = \frac{s - a_{i-1}}{a_i - a_{i-1}}.$$

Then, for any copula C, we obtain from (8) the fuzzy implication

$$I_{C,\mathbf{a}}(u,v) = 1 - u + \sum_{i=1}^{n} \left[C(u, a_{i-1} + (a_i - a_{i-1})v) - C(u, a_{i-1}) \right]$$

for all $(u,v) \in [0,1]^2$.

Example 10. Consider a partition $0 = b_0 < a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n = 1$, and let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$. Let $f_{\mathbf{a},\mathbf{b}}^* \colon [0,1] \to [0,1]$ be the function defined on each subinterval $[b_{i-1}, b_i]$, for $i = 1, 2, \ldots, n$, by

$$f_{\mathbf{a},\mathbf{b}}^*(s) = \begin{cases} \dfrac{s - b_{i-1}}{a_i - b_{i-1}} & \text{if } b_{i-1} \le s \le a_i, \\ \dfrac{b_i - s}{b_i - a_i} & \text{if } a_i \le s \le b_i. \end{cases}$$

Then, for any copula C, we obtain from (8) the fuzzy implication

$$I_{C,(\mathbf{a},\mathbf{b})}^*(u,v) = 1 + \sum_{i=1}^{n} \left[C(u, b_{i-1} + (a_i - b_{i-1})v) - C(u, b_i - (b_i - a_i)v) \right]$$

for all $(u,v) \in [0,1]^2$.

4. Conclusions

In this paper we have developed a rather general construction method for constructing fuzzy implication functions given in Theorem 2 —see formula (3)—, which extends several other special construction methods, such as, for example, the convex sums and generalized convex sums (usually in the form of some integral) of fuzzy implication functions. This method can also be applied for constructing fuzzy implication functions with some particular properties. For example, the neutrality principle of fuzzy implications —i.e., the property $I(1, v) = v$ for each $v \in [0, 1]$— is satisfied by any fuzzy implication constructed by means of formula (8), independently of the considered copula D and the measure-preserving transformation f. Also, if each I_s satisfies the neutrality principle, then the neutrality principle is preserved by our construction (3) whenever A is an idempotent aggregation function and $g_{2,s} = id$ for any $s \in S$. Similarly, any fuzzy implication constructed by means of formula (3) has the Zadeh —or the standard— fuzzy negation $N_Z : [0, 1] \longrightarrow [0, 1]$ given by $N_Z(x) = 1 - x$ as its natural negation, i.e., $I(u, 0) = N_Z(u)$, and this property is preserved by our construction (3) whenever A is an idempotent aggregation function and $g_{1,s} = id$ for any $s \in S$. We also recall another important property of fuzzy implications, namely the ordering property: $I(u, v) = 1$ if, and only if, $u \leq v$. This property is satisfied by any implication constructed by means of formula (3) whenever $g_{1,s} = g_{2,s}$ is strictly increasing for any $s \in S$, and A has no unit multipliers, i.e., if $A((x_s)_{s\in S}) = 1$ only if $x_s = 1$ for some $s \in S$. Note that if S is finite, then $A \leq \max$ is a sufficient condition for A not having unit multipliers. The use of the proposed construction method (3), as well as some other particular construction methods proposed in the paper (given, e.g., by (5) or (8)) for obtaining fuzzy implications with some other special properties, such as, for example, contrapositive symmetry, is a challenging topic for further research.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

The authors are grateful for the comments of two anonymous referees. The second and third authors acknowledge the support of Slovak Research and Development Agency, project APVV-18-0052. The second author was also supported by the project VEGA 1/0267/21 and the third author by the project VEGA 1/0006/19. The fourth and fifth authors acknowledge the support of the program FEDER-Andalucía (Spain) 2014-2020 under research project UAL2020-AGR-B1783.

References

[1] L. Ambrosio, N. Gigli, G. Savaré, Gradient Flows in Metric Spaces and in the Space of Probability Measures, second edition, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2008.
[2] M. Baczyński, B. Jayaram, Fuzzy Implications, Studies in Fuzziness and Soft Computing, vol. 231, Springer, Heidelberg, 2008.
[3] M. Baczyński, P. Grzegorzewski, R. Mesiar, P. Helbin, W. Niemyska, Fuzzy implications based on semicopulas, Fuzzy Sets Syst. 323 (2017) 138–151.
[4] G. Beliakov, A. Pradera, T. Calvo, Aggregation Functions: A Guide for Practitioners, Springer, New York, 2007.
[5] G. Choquet, Theory of capacities, Ann. Inst. Fourier 5 (1954) 131–295.
[6] W.F. Darsow, B. Nguyen, E.T. Olsen, Copulas and Markov processes, Ill. J. Math. 36 (1992) 600–642.
[7] E. de Amo, M. Díaz Carrillo, J. Fernández Sánchez, Characterization of all copulas associated with non-continuous random variables, Fuzzy Sets Syst. 191 (2012) 103–112.
[8] D. Denneberg, Non-additive Measure and Integral, Kluwer Academic Publishers, Dordrecht, 1994.
[9] A. Dolati, J. Fernández-Sánchez, M. Úbeda-Flores, A copula-based family of fuzzy implication operators, Fuzzy Sets Syst. 211 (2013) 55–61.
[10] F. Durante, C. Sempi, Principles of Copula Theory, Chapman & Hall/CRC, Boca Raton, FL, 2016.
[11] F. Durante, P. Sarkoci, C. Sempi, Shuffles of copulas, J. Math. Anal. Appl. 352 (2009) 914–921.
[12] M. Grabisch, J.-L. Marichal, R. Mesiar, E. Pap, Aggregation Functions, Encyclopedia of Mathematics and Its Applications, vol. 127, Cambridge University Press, Cambridge, 2009.
[13] F. Griessenberger, J. Fernández-Sánchez, W. Trutschnig, Some properties of double shuffles of bivariate copulas and (extreme) copulas invariant with respect to Lüroth double shuffles, Fuzzy Sets Syst. 428 (2022) 102–120.
[14] P. Grzegorzewski, Probabilistic implications, Fuzzy Sets Syst. 226 (2013) 53–66.

[15] P.R. Halmos, Measure Theory, Springer, New York, 1974.
[16] L'. Horanská, A. Kolesárová, On (a, b)-transformations of conjunctive functions, Fuzzy Sets Syst. 326 (2017) 89–105.
[17] H.W. Liu, Two classes of pseudo-triangular norms and fuzzy implications, Comput. Math. Appl. 61 (2011) 783–789.
[18] M. Mas, M. Monserrat, J. Torrens, E. Trillas, A survey on fuzzy implication functions, IEEE Trans. Fuzzy Syst. 15 (2007) 1107–1121.
[19] S. Massanet, J. Recasens, J. Torrens, Fuzzy implication functions based on powers of continuous t-norms, Int. J. Approx. Reason. 83 (2017) 265–279.
[20] R. Mesiar, A. Kolesárová, Copulas and fuzzy implications, Int. J. Approx. Reason. 117 (2020) 52–59.
[21] R. Mesiar, A. Kolesárová, Quasi-copulas, copulas and fuzzy implicators, Int. J. Comput. Intell. Syst. 13 (2020) 681–689.
[22] R.B. Nelsen, An Introduction to Copulas, second edition, Springer, New York, 2006.
[23] E.T. Olsen, W.F. Darsow, B. Nguyen, Copulas and Markov operators, in: L. Rüschendorf, B. Schweizer, M.D. Taylor (Eds.), Distributions with Fixed Marginals and Related Topics, in: Lecture Notes-Monograph Series, vol. 28, Institute of Mathematical Statistics, Hayward, CA, 1996, pp. 244–259.
[24] A. Sklar, Fonctions de répartition à n dimensions et leurs marges, Publ. Inst. Stat. Univ. Paris 8 (1959) 229–231.
[25] M. Sugeno, Theory of fuzzy integrals and its applications, PhD thesis, Tokyo Institute of Technology, 1974.