Krasnosel'skii type compression-expansion fixed point theorem for set contractions and star convex sets

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Abstract. In this paper, we give or improve compression-expansion results for set contractions in conical domains determined by balls or star convex sets. In the compression case, we use Potter's idea of proof, while the expansion case is reduced to the compression one by means of a change of variable. Finally, to illustrate the theory, we give an application to the initial value problem for a system of implicit first order differential equations.

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1. Introduction

Krasnosel'skii compression-expansion fixed point theorem is a powerful tool to prove the existence of positive solutions to several classes of boundary value problems and also to obtain multiple solutions (see, for example [4], [5], [8], [11], [15], [16] and [17]).

Krasnosel'skii proved his theorem directly, using only arguments of fixed point theory, particularly Schauder's fixed point theorem (see [6] and [7]). However, it is well-known that this result appears as a consequence of the topological degree theory (see [3]). Nevertheless, for applications, it is more convenient to use Krasnosel'skii theorem, because it offers directly the conditions - of compression or expansion - that have to be verified.

Also for the theory, a direct approach without using degree arguments could be useful when trying to extend the results from compact mappings to more general ones. Such a possibility is shown in Chapter 10 of [10], where some compression-expansion results are established for mappings from a smaller family for which the topological degree has not been developed.

The direct approach owed to Krasnosel'skii was also followed by Potter [12], who extended the compression result from compact mappings to set contractions. Notice that both Krasnosel'skii and Potter obtained a solution localized in annular conical sets defined using the norm. An expansion type result for set contractions was proved by Các and Gatica [1], but under a more restrictive condition on the lower boundary of the annular set.

In this paper, we prove the expansive type result for set contractions without restrictive conditions on the lower boundary, using the idea of reducing the expansive case to the compressing one, as shown in [14] for compact operators. We shall do even more, proving the results on star convex conical sets. The motivation for working with star convex sets is given in the Introduction of [9]. It comes from the necessity to distinguish between two solutions in case that they have the same norm.

The difficulty when working with k-set contractions ($k \in [0, 1)$) comes from the fact that the geometric transformations which are used in the proofs can change uncontrollably the constant k. As regards the use of star convex sets, they introduce much more complicated geometric transformations connected to their retro-activity property, which have to be put in accordance with the constant k, as Lemma 3.1 shows.

The results are presented as follows. First, in Section 2, we give some preliminary notions and results about $\alpha-$Lipschitz mappings (set contractions, in particular) and star convex sets, and we recall Potter's compression result for balls.

In Section 3, we first state and prove a technical lemma improving the corresponding one in [9], which allows us to extend the compression fixed point theorem in star convex sets given in [9], to general set contractions. Next, in Subsection 3.2, we improve the expansion result of Các and Gatica by simplifying the condition on the lower boundary of the conical annular set. We conclude this section by proving the expansion fixed point theorem for set contractions and star convex sets.

Finally, in Section 4, as an application, we consider the initial value problem for a system of implicit first order differential equations, which leads to a fixed point equation with a non-compact operator. Adding a Lipschitz condition over the derivative-dependent term, this operator becomes a set contraction.

2. Preliminaries

We start by briefly giving some notations, definitions and preliminary results, for details, see [9], [12].

If X is a real vector space and A a subset of X, we denote by $\mathrm{co}(A)$ the *convex hull* of A, that is

$$\mathrm{co}(A) := \left\{ \sum_{i=1}^{n} \lambda_i x_i : \ n \in \mathbb{N}, \ x_i \in A, \ \lambda_i \in [0,1] \text{ for } i = 1, \cdots, n, \ \sum_{i=1}^{n} \lambda_i = 1 \right\}$$

and, in case that X is a topological vector space, by $\overline{\text{co}}(A)$ the closure of $\text{co}(A)$.

The following concept allows us to relax the compactness hypothesis assumed in the mentioned classical results owed to Krasnosel'skii. Some of its principal properties can be found in [12], [9].

Let (X, d) be a metric space and $A \subset X$ be a bounded set. The *Kuratowski measure of noncompactness* of A is the non negative real number

$$\alpha(A) := \inf\{\varepsilon > 0 : \text{ there exist finitely many sets of diameter at most } \varepsilon \\ \text{which cover } A\}.$$

The measure of noncompactness can be considered as a tool to determine how much a particular set differs from being compact. In this way, it is possible to define a concept close to compact mapping, known as *set contraction*.

Let X, Y be metric spaces and $D \subset X$. Assume that the mapping $T : D \subset X \longrightarrow Y$ is continuous. We say that T is *α-Lipschitz* (see [2]) if there exists a constant $k \geq 0$ such that

$$\alpha(T(A)) \leq k\alpha(A), \quad \text{for all bounded } \; A \subset D.$$

When k is important to be mentioned, we say that T is a *k-α-Lipschitz* mapping, or that T is an *α-Lipschitz mapping with constant k*. In case that $k \in [0, 1)$, we say that T is a *set contraction* or a *k-set contraction.*

Note that any continuous and compact mapping is a 0-set contraction.

Recall the following properties.

Proposition 2.1. *Let (X_i, d_i) be metric spaces for $i = 1, 2, 3$, and $(X, \|\cdot\|)$ be a Banach space.*

- *(i) If $T_1 : X_1 \to X_2$ is a k_1-α-Lipschitz mapping and $T_2 : X_2 \to X_3$ is a k_2-α-Lipschitz mapping, then $T_2 \circ T_1 : X_1 \to X_3$ is a k_1k_2-α-Lipschitz mapping.*
- *(ii) If $T : D \subset X \to X$ is a k-α-Lipschitz mapping and $\lambda : D \to \mathbb{R}_+$ is a continuous function such that $\sup_{x\in D} \lambda(x) = l < \infty$, then the mapping*

$$\widehat{T} : D \subset X \to X, \quad \widehat{T}\,(x) = \lambda\,(x)\, T\,(x)$$

is kl-α-Lipschitz.

Now, we give some definitions and results about star convex sets, which are needed in what follows. For details, see [9].

For a fixed $x_0 \in X$, we say that a set $E \subset X$ is *x_0-star convex* if

$$\lambda x_0 + (1 - \lambda)x \in E, \quad \text{for all } \; \lambda \in [0, 1] \text{ and } x \in E.$$

In case that $x_0 = 0$, E is simply called a *star convex* set. Clearly, if E is x_0-star convex, then $x_0 \in E$.

We have the following result about the projection mapping on the boundary of an x_0-star convex set.

Theorem 2.2. *Let $(X, \|\cdot\|)$ be a Banach space, $x_0 \in X$ and $E \subset X$ be a bounded closed x_0-star convex set such that its boundary F does not contain x_0, and*

$$\text{for all } x \in E \setminus \{x_0\}, \text{ there is a unique } \lambda_x > 0 \text{ with } \lambda_x x + (1-\lambda_x)x_0 \in F. \tag{2.1}$$

Then there exists a continuous mapping $\partial : E \setminus \{x_0\} \to F$ such that

$$\begin{aligned} &\partial(x) = \partial(\lambda x + (1-\lambda)x_0), \quad \text{for all} \quad x \in E \setminus \{x_0\},\ \lambda \in (0,1], \\ &\partial(x) = x, \quad \text{for all} \quad x \in F. \end{aligned} \tag{2.2}$$

Proof. As $x_0 \in E \setminus F$, there exists $\gamma > 0$ such that $B := \{x \in X : \ \|x - x_0\| \leq \gamma\} \subset E \setminus F$. Let S be the boundary of B, i.e., $S := \{x \in X : \ \|x - x_0\| = \gamma\}$. We define the mapping ∂ as the composition $\eta \circ \eta_0$, where η_0 is the radial projection

$$\eta_0 : E \setminus \{x_0\} \to S, \quad \eta_0(x) = \frac{\gamma}{\|x - x_0\|}(x - x_0),$$

and

$$\eta : S \to F, \quad \eta(x) = \lambda_x x + (1-\lambda_x)x_0.$$

From (2.1), the mapping η is well-defined. Also, it is easy to see that condition (2.2) is satisfied. Clearly, η_0 is continuous. It remains to prove the continuity of η. To this aim, it suffices to prove the continuity of the function

$$\lambda : S \longrightarrow \mathbb{R}^+, \quad \lambda(x) = \lambda_x.$$

For that purpose, let $\{y_n\}_{n\in\mathbb{N}}$ be any sequence in S converging to some $y \in S$. Since E is bounded and $\lambda(S) \subset [0, +\infty)$, we can assert that there exists $m \in \mathbb{R}^+$ such that $\lambda(S) \subset [0, m]$. Then the sequence $\{\lambda(y_n)\}_{n\in\mathbb{N}}$ is included in the compact interval $[0, m]$, so any of its limit points is finite. Let l be any limit point of $\{\lambda(y_n)\}_{n\in\mathbb{N}}$. From

$$\eta(y_n) = \lambda(y_n)y_n + (1-\lambda(y_n))x_0 \in F,$$

we find that

$$ly + (1-l)x_0 \in F.$$

This, in view of (2.1), gives $l = \lambda(y)$. Hence $\lambda(y_n) \to \lambda(y)$ as $n \to +\infty$. Therefore, λ is continuous as wished. □

In the following, we shall consider only star convex sets E which satisfy the following condition.

Condition 1. E is bounded, closed, $0 \notin F$ where F is the boundary of E in X, and for every $x \in E \setminus \{0\}$ there exists a unique $\lambda_x > 0$ with $\lambda_x x \in F$.

Let us now consider a cone C in X and its intersection with a star convex set E. Clearly $0 \in C \cap E$ and we can make the following remark.

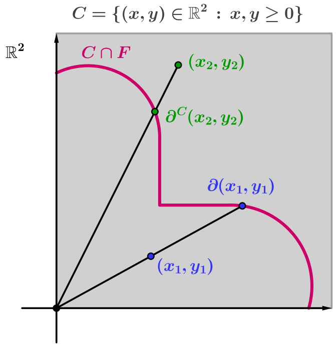

FIGURE 1. An example of mappings ∂ and ∂^C.

Remark 2.3. If E is a star convex set satisfying Condition 1, then the restriction of the mapping ∂ to $C \cap (E \setminus \{0\})$ can be continuously extended to $C \setminus \{0\}$, as follows

$$\partial^C : C \setminus \{0\} \to F, \quad \partial^C(x) = \begin{cases} \partial(x), & x \in (C \cap E) \setminus \{0\}, \\ \partial\left(\frac{d(0,F)}{\|x\|}x\right), & x \in C \setminus E. \end{cases}$$

Figure 1 illustrates the behavior of ∂^C in a particular case.

Proposition 2.4 ([9]). *Let E be a star convex set satisfying Condition 1. Then, for every $x \in (C \cap E) \setminus \{0\}$, there exists a unique number $\beta_x \in [1, +\infty)$ such that $\beta_x x \in C \cap F$. Moreover, the mapping*

$$\beta : (C \cap E) \setminus \{0\} \to [1, +\infty), \quad \beta(x) = \beta_x = \frac{d(0, \partial(x))}{d(0, x)}$$

is continuous and $\beta(x) \to +\infty$ as $x \to 0$.

Remark 2.5. By using ∂^C, the map β can be continuously extended to $C\setminus\{0\}$ as follows: $\beta^C(x) = d(0, \partial^C(x))/d(0, x)$, for $x \in C\setminus\{0\}$.

For two star convex sets E_i, $i = 1, 2$, satisfying Condition 1, we let:

- F_i be the boundary of E_i, $\mathring{E}_i$ be the interior of E_i.
- $\partial_i^C : C \setminus \{0\} \longrightarrow F_i$ be the continuous mapping associated whit E_i according to Remark 2.3.
- $\beta_i : (C \cap E_i) \setminus \{0\} \longrightarrow [1, +\infty)$ be the continuous mapping associated to E_i as shown in Proposition 2.4, and β_i^C be the corresponding continuous extension to $C \setminus \{0\}$.

Assume next that $E_1 \subset E_2$, $F_1 \cap F_2 = \emptyset$, and let $T : C \cap \left(E_2 \setminus \mathring{E}_1\right) \to C$ be a continuous mapping.

We say that T is a *compression* of the set $C \cap \left(E_2 \setminus \mathring{E}_1\right)$ (see Figure 2 (a)) if:

(C_1) $x - T(x) \notin C$, for all $x \in C \cap F_1$,
(C_2) $T(x) - (1+\varepsilon)x \notin C$, for all $\varepsilon > 0$ and $x \in C \cap F_2$.

We say that $T : C \cap \left(E_2 \setminus \mathring{E}_1\right) \to C$ is an *expansion* of the set $C \cap \left(E_2 \setminus \mathring{E}_1\right)$ (see Figure 2 (b)) if

(E_1) $T(x) - (1+\varepsilon)x \notin C$, for all $\varepsilon > 0$ and $x \in C \cap F_1$,
(E_2) $x - T(x) \notin C$, for all $x \in C \cap F_2$.

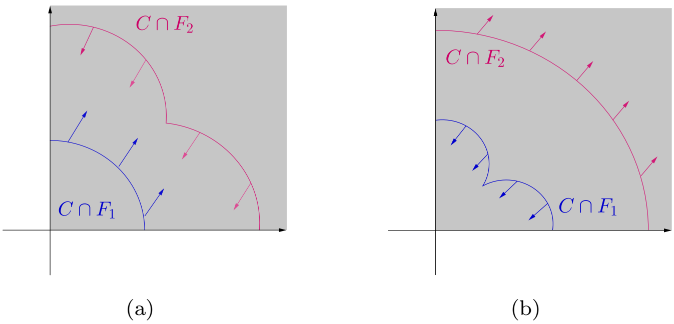

FIGURE 2. Figures (a) and (b) represent, respectively, the boundary conditions for a compression and an expansion of the set $C \cap \left(E_2 \setminus \mathring{E}_1\right)$, where $C = \{(x,y) \in \mathbb{R}^2 \ : \ x, y \geq 0\}$, and E_1, E_2 are star convex sets.

We conclude this section by recalling Potter's compression result for balls. Given two real numbers r, R with $0 < r < R$, denote

$$C_{r,R} = \{x \in C : \ r \leq \|x\| \leq R\},$$
$$C_r = \{x \in C : \ \|x\| \leq r\},$$
$$S_r = \{x \in C : \ \|x\| = r\}.$$

Theorem 2.6 ([12]). *If $T : C_{r,R} \to C$ is a k-set contraction and a compression of the set $C_{r,R}$, i.e.,*

$$x - T(x) \notin C, \quad \text{for all } x \in S_r,$$
$$T(x) - (1+\varepsilon)x \notin C, \text{ for all } \varepsilon > 0 \text{ and } x \in S_R,$$

then T has at least one fixed point in $C_{r,R}$.

3. Main results

In this section, we first give an extension of the main compression result from [9]. This extension allows k to be any number with $0 \leq k < 1$, compared to the more restrictive assumption $0 \leq k < d(0,F)/L$, where $L := sup\{d(0,x) : x \in F\}$, considered in [9]. Next, we improve the expansion type result given in [1]. This is done by reducing the expansion case to the compression one. Finally, the expansion result is extended to star convex sets.

3.1. Compression fixed point theorem for set contractions and star convex sets

In order to accomplish our first aim, we need to improve the technical Lemma 3.16 in [9].

Lemma 3.1. *Assume that $(X, \|\cdot\|)$ is a Banach space, C a cone in X and E a set satisfying Condition 1. Let $T : C\cap F \longrightarrow C$ be $\alpha-$Lipschitz with constant k and consider the mapping*

$$\begin{aligned} &\tilde{T} : C\cap E \longrightarrow C \\ &x \longmapsto \tilde{T}(x) := \begin{cases} \frac{1}{\beta(x)}T(\beta(x)x), & x\neq 0, \\ 0, & x=0. \end{cases} \end{aligned}$$

Then $\tilde{T}$ is $\alpha-$Lipschitz with constant $\tilde{k}$ for every $\tilde{k} > k$ as close to k as we wish.

Proof. We only need to adapt the part of the proof where we show the existence of $\tilde{k}$ as close to k as we wish such that

$$\alpha\left(\tilde{T}(A)\right) \leq \tilde{k}\alpha(A), \text{ for every } A\subset C\cap E,\, \alpha(A)\neq 0. \tag{3.1}$$

Thus, let $A\subset C\cap E$ be such that $\alpha(A)\neq 0$ and $k>0$ (if this is not true, we can consider $\hat{k}>0$ as close to k as we wish). As k, $\alpha(A)>0$, there exists $d>0$ such that $k\,\alpha(A)/2>d>0$. We fix d satisfying these conditions.

To accomplish the aim of proving that (3.1) is satisfied, we proceed as follows.

Step 1: Cover any set A by a finite number of subsets with the property that the restrictions of $\tilde{T}$ to each of them is $\alpha-$Lipschitz for some suitable constant. To this aim, we start by using the continuity of $\tilde{T}$ proved in [9], that ensures the existence of $\delta_d>0$ such that

$$\tilde{T}((C\cap E)\cap B(0,\delta_d)) \subset B(0,d).$$

Then, we have

$$\tilde{T}(A\cap B(0,\delta_d)) \subset B(0,d). \tag{3.2}$$

On the other hand, for each natural number $n\geq 1$, take $\varepsilon_n=\delta_d/n$ and, for all natural number $m\geq 1$, define

$$A_m^n := \{x\in A \,:\, \beta(x)\in[1+(m-1)\varepsilon_n,\, 1+m\varepsilon_n]\}.$$

For any natural number $n\geq 1$, there exists $N_n\in\mathbb{N}$, $N_n>1$, such that, if $\beta(x)>1+N_n\varepsilon_n$, then $x\in A\cap B(0,\delta_d)$. Therefore,

$$A\subset (A\cap B(0,\delta_d)) \cup \left(\bigcup_{m=1}^{N_n} A_m^n\right). \tag{3.3}$$

Step 2: Fix $n \in \mathbb{N}$, $n > 1$, arbitrarily and, for each $m \in \{1, \dots, N_n\}$, we show that $\tilde{T}_{|A_m^n}$ is $\alpha-$Lipschitz with constant $k\,(1+m\varepsilon_n)/(1+(m-1)\varepsilon_n)$. For each $m \in \{1, \dots, N_n\}$, we have $0 \notin A_m^n$, so

$$\begin{aligned} \tilde{T}_{|A_m^n} : A_m^n &\longrightarrow C \\ x &\longmapsto \tilde{T}_{|A_m^n}(x) = \frac{1}{\beta(x)} T(\beta(x)x). \end{aligned}$$

Let $m \in \{1, \dots, N_n\}$ and define the following auxiliary mappings:

$$\begin{aligned} \frac{1}{\beta_{|A_m^n}} : A_m^n \longrightarrow C, \;\; x \longmapsto \frac{1}{\beta_{|A_m^n}}(x) := \frac{1}{\beta(x)}, \\ S_m^n : A_m^n \longrightarrow C, \;\; x \longmapsto S_m^n(x) := \beta(x)x. \end{aligned}$$

Since β is continuous and its image is a subset of $[1, \infty)$, we deduce that $1/\beta_{|A_m^n}$ is a continuous function. Besides, for all $x \in A_m^n$, it is satisfied that

$$\beta(x) \in [1 + (m-1)\,\varepsilon_n, 1 + m\,\varepsilon_n] \;\Leftrightarrow\; \frac{1}{\beta(x)} \in \left[\frac{1}{1 + m\,\varepsilon_n}, \frac{1}{1 + (m-1)\,\varepsilon_n}\right],$$

hence

$$sup\left\{\frac{1}{\beta(x)} \;:\; x \in A_m^n\right\} \leq \frac{1}{1 + (m-1)\,\varepsilon_n}.$$

As T is a k-set contraction, in view of Proposition 2.1, it remains to prove that S_m^n is $\alpha-$Lipschitz with constant $(1 + m\varepsilon_n)$. Let $B \subset A_m^n$, then B is bounded and, using the definition of A_m^n, we have

$$\begin{aligned} S_m^n(B) &= \Big\{\beta(x)x \;:\; x \in B\Big\} \\ &\subset \Big\{[\lambda(1 + (m-1)\varepsilon_n) + (1-\lambda)(1 + m\varepsilon_n)]x \;:\; \lambda \in [0,1],\, x \in B\Big\} \\ &= \mathrm{co}\Big(\big\{[1 + (m-1)\varepsilon_n]B\big\} \cup \big\{[1 + m\varepsilon_n]B\big\}\Big). \end{aligned}$$

Now, by using the properties of the measure of noncompactness, we obtain

$$\alpha(S_m^n(B)) \leq \max\Big\{\alpha\big([1 + (m-1)\varepsilon_n]B\big),\, \alpha\big([1 + m\varepsilon_n]B\big)\Big\} = (1 + m\varepsilon_n)\alpha(B).$$

Also, as β is continuous, then S_m^n is continuous. Therefore S_m^n is $\alpha-$Lipschitz with constant $(1 + m\varepsilon_n)$.

As a consequence, by using the properties of $\alpha-$Lipschitz mapping in Proposition 2.1, one has that $\tilde{T}_{|A_m^n}$ is $\alpha-$Lipschitz with constant $k\,(1 + m\varepsilon_n)/(1 + (m-1)\varepsilon_n)$.

Step 3: Taking into account the two previous steps, we can conclude that $\tilde{T}$ is $\alpha-$Lipschitz with constant $\tilde{k}$ as close to k as we wish. It follows from (3.2), (3.3), and the properties of the measure of noncompactness, that

$$
\begin{aligned}
\alpha\left(\tilde{T}(A)\right) &= \alpha\left(\tilde{T}(A\cap B(0,\delta_d)) \cup \bigcup_{m=1}^{N_n} \tilde{T}(A_m^n)\right)\\
&= \alpha\left(\tilde{T}(A\cap B(0,\delta_d)) \cup \tilde{T}(A_1^n) \cup \ldots \cup \tilde{T}(A_{N_n}^n)\right)\\
&= \max\left\{\alpha\left(\tilde{T}(A\cap B(0,\delta_d))\right), \alpha\left(\tilde{T}(A_1^n)\right), \ldots, \alpha\left(\tilde{T}(A_{N_n}^n)\right)\right\}\\
&\le \max\left\{\alpha\left(B(0,d)\right), \alpha\left(\tilde{T}(A_1^n)\right), \ldots, \alpha\left(\tilde{T}(A_{N_n}^n)\right)\right\}\\
&\le \max\left\{2d, \frac{1+\varepsilon_n}{1}k\alpha(A_1^n), \ldots, \frac{1+N_n\varepsilon_n}{1+(N_n-1)\varepsilon_n}k\alpha(A_{N_n}^n)\right\}\\
&\le \max\left\{2d, \frac{1+\varepsilon_n}{1}k\alpha(A), \ldots, \frac{1+N_n\varepsilon_n}{1+(N_n-1)\varepsilon_n}k\alpha(A)\right\}\\
&\le (1+\varepsilon_n)k\alpha(A),
\end{aligned}
$$

where the last inequality holds since $2d < k\alpha(A)$ and

$$
\frac{1+m\varepsilon_n}{1+(m-1)\varepsilon_n} \le \frac{1+\varepsilon_n}{1} \text{ for all } m \in \{1,\ldots,N_n\}.
$$

Since $\varepsilon_n \to 0$ as $n \to \infty$, one has that, if n is large enough, the number $\tilde{k} := (1+\varepsilon_n)k$ is as close to k as we wish. $\square$

Now, we can state the extension of the compression result from [9], for any constant $0 \le k < 1$.

Theorem 3.2. *Let $(X, \|\cdot\|)$ be a Banach space, C be a cone in X, and E_1, E_2 be star convex sets fulfilling Condition 1. Assume that $T : C\cap\left(E_2\backslash\mathring{E}_1\right) \longrightarrow C$ is a k-set contraction and a compression of the set $C\cap\left(E_2\backslash\mathring{E}_1\right)$. Then T has at least one fixed point in $C\cap\left(E_2\backslash\mathring{E}_1\right)$.*

Proof. The proof follows the one of the corresponding theorem in [9], using this time the improved Lemma 3.1. $\square$

3.2. Expansion fixed point theorem for set contractions and balls

Our second aim is to improve the expansion result given in [1]. We shall do this by simplifying the condition on the lower boundary of the conical annular set.

Theorem 3.3. *Let $(X, \|\cdot\|)$ be a Banach space, C be a cone in X, $r, R \in \mathbb{R}$, $0 < r < R$, and $T : C_{r,R} \longrightarrow C$ be a $k-$set contraction verifying the following properties:*

$$
\begin{aligned}
Tx-(1+\varepsilon)x \notin C,&\ \textit{for all } \varepsilon>0 \textit{ and all } x\in C,\ \|x\|=r;\\
x-Tx \notin C,&\ \textit{for all } x\in C,\ \|x\|=R.
\end{aligned}
$$

Then, T has a fixed point in $C_{r,R}$.

Proof. We consider an auxiliary mapping $\tilde{T}$ which we show that satisfies the hypothesis of the mentioned fixed point theorem due to Potter. Let

$$\begin{aligned} \tilde{T} : C_{r,R} &\longrightarrow C \\ x &\longmapsto \tilde{T}(x) := \frac{1}{\theta(x)} T\left(\theta(x)x\right), \end{aligned}$$

where $\theta(x) = (r+R)/\|x\| - 1$, for every $x \in C_{r,R}$.

First of all, we have to prove that $\tilde{T}$ is well-defined. We need to show that $\theta(x)x \in C_{r,R}$ for every $x \in C_{r,R}$. Since $\theta(x) > 0$ and $x \in C$, we can assert that $\theta(x)x \in C$. Therefore, it remains to prove that $r \leq \|\theta(x)x\| \leq R$. If $x \in C_{r,R}$, then $r \leq \|x\| \leq R$, whence $r \leq r + R - \|x\| \leq R$, that is, $r \leq \theta(x)\|x\| \leq R$. Due to this, we can assert that $\theta(x)x \in C_{r,R}$ and, finally, $\tilde{T}$ is well-defined.

Secondly, it is easy to prove that, if $\|x\| = r$, then $\|\theta(x)x\| = R$ and, if $\|x\| = R$, then $\|\theta(x)x\| = r$. As a consequence, T verifying (E_1) implies that $\tilde{T}$ verifies (C_2) and T verifying (E_2) implies that $\tilde{T}$ verifies (C_1).

Thirdly, we have to prove that $\tilde{T}$ is a set contraction. It is clear that $\tilde{T}$ is continuous since T, θ are continuous and $\theta > 0$. Therefore, it remains to prove that, for every $A \subset C_{r,R}$ (A is bounded), we have

$$\alpha(\tilde{T}(A)) \leq \tilde{k}\alpha\left(A\right), \tag{3.4}$$

for some constant $0 \leq \tilde{k} < 1$, independent of A. For that purpose, we proceed in a similar way to the ones in the proof of Lemma 3.1 or Lemma 3.1 in [12]. We begin by distinguishing two cases:
(a) If $\alpha(A) = 0$, then $\overline{A}$ is compact. As $\tilde{T}$ is continuous, then $\tilde{T}(\overline{A})$ is compact and, therefore,

$$\alpha(\tilde{T}(A)) \leq \alpha(\tilde{T}(\overline{A})) = 0 = \tilde{k}\,\alpha(A), \quad \text{for any } \tilde{k} \geq 0.$$

(b) If $\alpha(A) \neq 0$, let us assume that $k \neq 0$, but, if it is not the case, we consider $0 < \hat{k} < 1$ as close to k as we wish. As $0 < r < R$, then $\delta_{r,R} = \frac{R}{r} - \frac{r}{R} > 0$ and, for each $n \in \mathbb{N}$, we can consider

$$\varepsilon^n_{r,R} := \frac{\delta_{r,R}}{n}.$$

Let $n \in \mathbb{N}$, $n \geq 1$, be arbitrarily fixed, for each integer number $m \geq 0$, we define the following sets:

$$A^n_m := \left\{ x \in A \ : \ \theta(x) \in \left[\frac{r}{R} + m\,\varepsilon^n_{r,R}, \frac{r}{R} + (m+1)\,\varepsilon^n_{r,R} \right] \right\}.$$

Since $A \subset C_{r,R}$, we can assert that $\frac{r}{R} \leq \theta(x) \leq \frac{R}{r}$. Moreover, noticing that $\frac{r}{R} + 0\,\varepsilon^n_{r,R} = \frac{r}{R}$ and $\frac{r}{R} + [(n-1)+1]\,\varepsilon^n_{r,R} = \frac{R}{r}$, we get

$$A \subset \bigcup_{m=0}^{n-1} A^n_m.$$

From this, by using properties of the measure of noncompactness, it follows that

$$\alpha\left(\tilde{T}(A)\right) = \alpha\left(\bigcup_{m=0}^{n-1} \tilde{T}(A_m^n)\right) = \max_{m\in\{0,\ldots,n-1\}} \left\{\alpha\left(\tilde{T}(A_m^n)\right)\right\}.$$

For each $m \in \{0,\ldots,n-1\}$, we deal with $\alpha\left(\tilde{T}(A_m^n)\right)$ and, for that purpose, we consider

$$\tilde{T}_{|A_m^n} : A_m^n \longrightarrow C,\ x \longmapsto \tilde{T}_{|A_m^n}(x) = \tilde{T}(x).$$

With the aim of proving that $\tilde{T}_{|A_m^n}$ is $\alpha-$Lipschitz with constant $k\,(\frac{r}{R}+(m+1)\,\varepsilon_{r,R}^n)/(\frac{r}{R}+m\,\varepsilon_{r,R}^n)$, we study some properties of the following auxiliary mappings:

$$\begin{aligned} \frac{1}{\theta_{|A_m^n}} &: A_m^n \longrightarrow C,\ x \longmapsto \frac{1}{\theta(x)}; \\ S_m^n &: A_m^n \longrightarrow C,\ x \longmapsto S_m^n(x) := \theta(x)x. \end{aligned}$$

On the one hand, for each $x \in A_m^n$, it is satisfied that $1/\theta(x) \le 1/(\frac{r}{R}+m\,\varepsilon_{r,R}^n)$, then

$$\sup\left\{\frac{1}{\theta(x)}\ :\ x \in A_m^n\right\} \le \frac{1}{\frac{r}{R}+m\,\varepsilon_{r,R}^n}.$$

On the other hand, for every $B \subset A_m^n$, we have

$$\begin{aligned} &S_m^n(B) = \{\theta(x)x\ :\ x \in B\} \\ &\subset \left\{\left[\lambda\left(\frac{r}{R}+m\,\varepsilon_{r,R}^n\right)+(1-\lambda)\left(\frac{r}{R}+(m+1)\,\varepsilon_{r,R}^n\right)\right]x\ :\ \lambda\in[0,1],\ x\in B\right\} \\ &= \operatorname{co}\left(\left\{\left[\frac{r}{R}+m\,\varepsilon_{r,R}^n\right]B\right\}\cup\left\{\left[\frac{r}{R}+(m+1)\,\varepsilon_{r,R}^n\right]B\right\}\right). \end{aligned}$$

Now, by using properties of the measure of noncompactness, we can assert

$$\begin{aligned} \alpha\left(S_m^n(B)\right) &\le \alpha\left(\left\{\left[\frac{r}{R}+m\,\varepsilon_{r,R}^n\right]B\right\}\cup\left\{\left[\frac{r}{R}+(m+1)\,\varepsilon_{r,R}^n\right]B\right\}\right) \\ &= \left(\frac{r}{R}+(m+1)\,\varepsilon_{r,R}^n\right)\alpha(B). \end{aligned}$$

Consequently, S_m^n is $\alpha-$Lipschitz with constant $r/R+(m+1)\,\varepsilon_{r,R}^n$.

As T is a $k-$set contraction and $\tilde{T}_{|A_m^n} = (T\circ S_m^n)/\theta_{|A_m^n}$, we finally get that $\tilde{T}_{|A_m^n}$ is $\alpha-$Lipschitz with constant $k\,(\frac{r}{R}+(m+1)\,\varepsilon_{r,R}^n)/(\frac{r}{R}+m\,\varepsilon_{r,R}^n)$.

Moreover, for each $m\in\{1,\ldots,n-1\}$, it follows that

$$\frac{\frac{r}{R}+(m+1)\,\varepsilon_{r,R}^n}{\frac{r}{R}+m\,\varepsilon_{r,R}^n} < \frac{\frac{r}{R}+\varepsilon_{r,R}^n}{\frac{r}{R}}.$$

Finally, taking into account the different statements which have been proved, we have

$$\begin{aligned}\alpha\left(\tilde{T}(A)\right) &= \max_{m\in\{0,\ldots,n-1\}}\left\{\alpha\left(\tilde{T}(A_m^n)\right)\right\}\\ &\le \max_{m\in\{0,\ldots,n-1\}}\left\{\frac{\frac{r}{R}+(m+1)\,\varepsilon_{r,R}^n}{\frac{r}{R}+m\,\varepsilon_{r,R}^n}\,\alpha(A_m^n)\right\}\\ &= \frac{\frac{r}{R}+\varepsilon_{r,R}^n}{\frac{r}{R}}k\,\alpha(A).\end{aligned}$$

Since $\varepsilon_{r,R}^n \to 0$ as $n\to\infty$, one has that, if n is large enough, the number

$$\tilde{k} := \frac{\frac{r}{R}+\varepsilon_{r,R}^n}{\frac{r}{R}}k$$

is as close to k as we wish. Therefore, we can guarantee that $\tilde{k}\in(0,1)$.

To finish the proof, we apply Theorem 2.6 to the operator $\tilde{T}$. Hence, $\tilde{T}$ has a fixed point in $\tilde{x}\in C_{r,R}$, then

$$\tilde{x}=\tilde{T}(\tilde{x})=\frac{1}{\theta(\tilde{x})}T(\theta(\tilde{x})\tilde{x}),$$

or, equivalently,

$$\theta(\tilde{x})\tilde{x}=T(\theta(\tilde{x})\tilde{x}).$$

This shows that the point $x:=\theta(\tilde{x})\tilde{x}$ is a fixed point of T in $C_{r,R}$. □

3.3. Expansion fixed point theorem for set contractions and star convex sets

The last goal is to extend Theorem 3.3 for star convex sets.

Theorem 3.4. *Let $(X,\|\cdot\|)$ be a Banach space, C be a cone in X, and E_1, E_2 be star convex sets fulfilling Condition* 1. *If $T: C\cap\left(E_2\backslash\mathring{E}_1\right)\longrightarrow C$ is a $k-$set contraction and an expansion of the set $C\cap\left(E_2\backslash\mathring{E}_1\right)$, then T has a fixed point in $C\cap\left(E_2\backslash\mathring{E}_1\right)$.*

Proof. The idea of the proof is the same as the one for balls, we consider the auxiliary mapping

$$\begin{aligned}\tilde{T}: C\cap\left(E_2\backslash\mathring{E}_1\right)&\longrightarrow C\\ x\longmapsto \tilde{T}(x)&:=\frac{1}{\theta(x)}T(\theta(x)x),\end{aligned}$$

where $\theta(x)=\beta_1^C(x)+\beta_2(x)-1$, for $x\in C\cap\left(E_2\backslash\mathring{E}_1\right)$.

As in Theorem 3.3, we begin by proving that $\tilde{T}$ is well-defined. We need to show that $\theta(x)x\in C\cap\left(E_2\backslash\mathring{E}_1\right)$, for every $x\in C\cap\left(E_2\backslash\mathring{E}_1\right)$. Since $\theta(x)>0$ and $x\in C$, we clearly have $\theta(x)x\in C$. To prove that $\theta(x)x\in E_2\backslash\mathring{E}_1$, let us note the equivalence between $\lambda x\in E_2\backslash\mathring{E}_1$ and the inequality $\beta_1^C(x)\le\lambda\le\beta_2(x)$. Now, if $x\in E_2\backslash\mathring{E}_1$, then $\beta_1^C(x)\le 1\le\beta_2(x)$, whence

$\beta_1^C(x) \leq \beta_1^C(x) + \beta_2(x) - 1 \leq \beta_2(x)$, that is, $\beta_1^C(x) \leq \theta(x) \leq \beta_2(x)$, which shows that $\theta(x)x \in E_2 \backslash \mathring{E}_1$. Therefore, we conclude that $\tilde{T}$ is well-defined.

Next, we show that $\tilde{T}$ is a compression of the set $C \cap \left(E_2 \backslash \mathring{E}_1\right)$. For that purpose, we prove that, if $x \in F_1$, then $\theta(x)x \in F_2$, and, if $x \in F_2$, then $\theta(x)x \in F_1$. Consequently, if T fulfills (E_1) then $\tilde{T}$ fulfills (C_2), and if T fulfills (E_2) then $\tilde{T}$ fulfills (C_1). This way, $\tilde{T}$ is a compression of the set $C \cap \left(E_2 \backslash \mathring{E}_1\right)$. Indeed, if $x \in F_1$, then $\beta_1^C(x) = 1$ and so $\theta(x) = \beta_2(x)$. Hence, according to Proposition 2.4, $\theta(x)x \in F_2$. Similarly, if $x \in F_2$, then $\beta_2(x) = 1$ and so $\theta(x) = \beta_1^C(x)$. Thus, by using the definition of β_1^C, $\theta(x)x \in F_1$.

Finally, it remains to prove that there exists $0 \leq \tilde{k} < 1$ such that $\tilde{T}$ is a $\tilde{k}-$set contraction. On the one hand, as β_1^C, β_2 and T are continuous and $\theta(x) \neq 0$, for all $x \in C \cap \left(E_2 \backslash \mathring{E}_1\right)$, one has that $\tilde{T}$ is continuous. On the other hand, for every $A \subset C \cap \left(E_2 \backslash \mathring{E}_1\right)$, we have to prove that

$$\alpha\left(\tilde{T}(A)\right) \leq \tilde{k}\alpha(A), \tag{3.5}$$

for some constant $0 \leq \tilde{k} < 1$, independent of A.

The proof of (3.5) is identical to that of formula (3.4) in the proof of Theorem 3.3, once we have shown the existence of two positive numbers r and R with $r < R$ such that

$$\frac{r}{R} \leq \theta(x) \leq \frac{R}{r}, \tag{3.6}$$

for every $x \in C \cap \left(E_2 \backslash \mathring{E}_1\right)$.

Indeed, as $0 \in E_1 \backslash F_1$, there exists $r > 0$ such that $B_r = \overline{B}(0, r) \subset \mathring{E}_1$. As $E_1 \subset E_2$ and E_2 is bounded, there exists $R > r$ such that $E_2 \subset B_R = \overline{B}(0, R)$. Therefore, $C \cap \left(E_2 \backslash \mathring{E}_1\right) \subset C \cap (B_R \backslash \mathring{B}_r)$. Hence, for any $x \in C \cap \left(E_2 \backslash \mathring{E}_1\right)$, we have

$$x,\ \theta(x)x \in C \cap (B_R \backslash \mathring{B}_r),$$

implying

$$r \leq \|x\|,\ \theta(x)\|x\| \leq R,$$

which immediately yield (3.6).

To finish the proof, we apply Theorem 3.2 to the operator $\tilde{T}$. Thus, $\tilde{T}$ has a fixed point $\tilde{x} \in C \cap \left(E_2 \backslash \mathring{E}_1\right)$. Then

$$\tilde{x} = \tilde{T}(\tilde{x}) = \frac{1}{\theta(\tilde{x})} T(\theta(\tilde{x})\tilde{x}),$$

or, equivalently,

$$\theta(\tilde{x})\tilde{x} = T(\theta(\tilde{x})\tilde{x}).$$

This shows that the point $x := \theta(x)x$ is a fixed point of T in $C \cap \left(E_2 \backslash \mathring{E}_1\right)$. $\square$

4. Application

Let us consider the following initial value problem for a first order implicit differential system,

$$\begin{cases} x'(t) = f(t, x(t)) + g(t, x'(t)), \quad t \in [0,1], \\ x(0) = 0, \end{cases} \tag{4.1}$$

where $f, g : [0,1] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$. We search for solutions $x \in \mathcal{C}^1([0,1], \mathbb{R}^n)$. For that purpose, with the substitution $y = x'$, the initial value problem (4.1) becomes

$$y(t) = f\left(t, \int_0^t y(s)ds\right) + g(t, y(t)), \quad t \in [0,1], \tag{4.2}$$

with $\int_0^t y(s)ds = \left(\int_0^t y_1(s)ds, \dots, \int_0^t y_n(s)ds\right)$. Denote by T the operator associated to the right-hand side of (4.2), namely, $T : \mathcal{C}([0,1], \mathbb{R}^n) \to \mathcal{C}([0,1], \mathbb{R}^n)$,

$$T(y)(t) = f\left(t, \int_0^t y(s)ds\right) + g(t, y(t)).$$

Thus, the solutions to (4.1) are the fixed points in $\mathcal{C}([0,1], \mathbb{R}^n)$ of the operator T. To localize the fixed points of T, we use Theorem 3.2 in the Banach space $(\mathcal{C}([0,1], \mathbb{R}^n), \| \cdot \|_\infty)$, with

$$||y||_\infty = \max\{||y_1||_\infty, \dots, ||y_n||_\infty\}, \text{ for } y = (y_1, \dots, y_n).$$

First, we state some conditions on the mappings f and g to guarantee that the operator $T : \mathcal{C}([0,1], \mathbb{R}^n) \longrightarrow \mathcal{C}([0,1], \mathbb{R}^n)$ is well-defined and a set contraction:

(H_1) $f, g : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous.

(H_2) For each $t \in [0,1]$, there exists $L(t) \in \mathbb{R}_+$ such that

$$|g_i(t, u) - g_i(t, v)| \leq L(t) \max\{|u_j - v_j|, \ j = 1, \dots, n\},$$

for all $u, v \in \mathbb{R}^n$, $i \in \{1, \dots, n\}$, and

$$\sup_{t \in [0,1]} L(t) =: k < 1. \tag{4.3}$$

Under conditions (H_1) and (H_2), the operator T is a sum of a compact operator and a k-set contraction. Indeed, $T = A + B$, where the operators $A, B : \mathcal{C}([0,1], \mathbb{R}^n) \to \mathcal{C}([0,1], \mathbb{R}^n)$ are given by

$$A(y)(t) = f\left(t, \int_0^t y(s)ds\right), \quad B(y)(t) = g(t, y(t)).$$

The operator A is compact as a composition of three continuous bounded operators J, i and N_f, such that one of them is compact. In fact, $A = N_f \circ i \circ J$, where

$$J : \mathcal{C}([0,1], \mathbb{R}^n) \to \mathcal{C}^1([0,1], \mathbb{R}^n), \quad J(y)(t) = \int_0^t y(s)\, ds,$$

$$i : \mathcal{C}^1([0,1], \mathbb{R}^n) \to \mathcal{C}([0,1], \mathbb{R}^n), \quad i(y) = y,$$

$$N_f : \mathcal{C}([0,1], \mathbb{R}^n) \to \mathcal{C}([0,1], \mathbb{R}^n), \quad N_f(y)(t) = f(t, y(t)).$$

All these operators are continuous and bounded (map bounded sets into bounded sets), while i is compact as a consequence of the Arzelà–Ascoli theorem (see, e.g., Section 1.2 in [13]).

The operator B is a contraction with the Lipschitz constant k given by (4.3). Indeed, for any y, $\overline{y} \in \mathcal{C}([0,1],\mathbb{R}^n)$, and every $t \in [0,1]$, one has

$$\begin{aligned} |B_i(y)(t) - B_i(\overline{y})(t)| &= |g_i(t,y(t)) - g_i(t,\overline{y}(t))| \\ &\le L(t) \max\{\left|y_j(t) - \overline{y}_j(t)\right|,\ j = 1,\dots,n\} \\ &\le k \left\|y - \overline{y}\right\|_\infty , \end{aligned}$$

for each $i \in \{1,\dots,n\}$, whence

$$\left\|B(y) - B(\overline{y})\right\|_\infty \le k \left\|y - \overline{y}\right\|_\infty .$$

Next, we consider, in $\mathcal{C}([0,1],\mathbb{R}^n)$, the cone of non-negative functions $C := \mathcal{C}([0,1],\mathbb{R}^n_+)$, and we add the following positivity condition in order to guarantee the invariance condition $T(C) \subset C$:

(H_3) $f([0,1] \times \mathbb{R}^n_+) \subset \mathbb{R}^n_+$, and $g([0,1] \times \mathbb{R}^n_+) \subset \mathbb{R}^n_+$.

Finally, we define two star convex sets E_1, $E_2 \subset \mathcal{C}([0,1],\mathbb{R}^n)$ satisfying Condition 1, such that $T : C \cap (E_2 \setminus \mathring{E}_1) \longrightarrow C$ is a compression of the set $C \cap (E_2 \setminus \mathring{E}_1)$. For their definition, we use the norm $\left\|\cdot\right\|_\infty$ and the functional

$$\varphi_i : \mathcal{C}([0,1],\mathbb{R}) \to \mathbb{R}_+, \quad \varphi_i(z) = a_i \min_{t\in[0,1]} |z(t)| + b_i \left\|z\right\|_\infty ,$$

where $a_i, b_i \in \mathbb{R}_+$ and $b_i \neq 0$, $i \in \{1,\dots,n\}$. Notice the relationship between the functional φ_i and the norm,

$$b_i \left\|z\right\|_\infty \le \varphi_i(z) \le (a_i + b_i) \left\|z\right\|_\infty ,$$

or, equivalently,

$$\frac{\varphi_i(z)}{a_i + b_i} \le \left\|z\right\|_\infty \le \frac{\varphi_i(z)}{b_i}. \tag{4.4}$$

Let $r, R \in (0,\infty)^n$ be two vectors with $(a_i + b_i)\, r_i < R_i$, for $i = 1,\dots,n$, and take

$$\begin{aligned} E_1 &:= \{y \in \mathcal{C}([0,1],\mathbb{R}^n) :\ \left\|y_i\right\|_\infty \le r_i,\ i = 1,\dots,n\}, \\ E_2 &:= \{y \in \mathcal{C}([0,1],\mathbb{R}^n) :\ \varphi_i(y_i) \le R_i,\ i = 1,\dots,n\}. \end{aligned}$$

Clearly, E_1 and E_2 are star convex sets satisfying Condition 1.

For each $i \in \{1,\dots,n\}$, denote

$$\begin{aligned} \underline{f}_i &:= \min\{f_i(t,y) :\ t \in [0,1],\ y_j \in [0,r_j],\ j = 1,\dots,n\}, \\ \underline{g}_i &:= \min\left\{g_i(t,y) :\ t \in [0,1],\ y_j \in \left[0, \frac{R_j}{b_j}\right],\ j = 1,\dots,n,\ j \neq i,\ y_i = r_i\right\}, \\ \overline{f}_i &:= \max\left\{f_i(t,y) :\ t \in [0,1],\ y_j \in \left[0, \frac{R_j}{b_j}\right],\ j = 1,\dots,n\right\}, \\ \overline{g}_i &:= \max\left\{g_i(t,y) :\ t \in [0,1],\ y_j \in \left[r_j, \frac{R_j}{b_j}\right],\ j = 1,\dots,n,\ j \neq i, \right. \\ &\qquad\qquad\qquad\qquad \left. y_i \in \left[\frac{R_i}{a_i + b_i}, \frac{R_i}{b_i}\right]\right\}, \end{aligned}$$

and assume that the following conditions are satisfied:

$(H_{4,C})$ For each $i \in \{1, \ldots, n\}$,

$$\underline{f}_i + \underline{g}_i > r_i, \tag{4.5}$$

$$\overline{f}_i + \overline{g}_i \leq \frac{R_i}{a_i + b_i}. \tag{4.6}$$

Under these conditions, $E_1 \subset \mathring{E}_2$, and (C_1), (C_2) hold, that is, the operator T is a compression of the set $C \cap (E_2 \setminus \mathring{E}_1)$.

First, using the inequalities $(a_i + b_i)\, r_i < R_i$, we have $E_1 \subset \mathring{E}_2$. Indeed, if $y \in E_1$, then, for every $i \in \{1, \ldots, n\}$, one has

$$\varphi_i(y_i) = a_i \min_{t \in [0,1]} |y_i(t)| + b_i \|y_i\|_\infty \leq (a_i + b_i) \|y_i\|_\infty \leq (a_i + b_i) r_i < R_i,$$

which gives $y \in \mathring{E}_2$.

Next, we show that (4.5) guarantees that condition (C_1) is fulfilled. Indeed, if we assume the contrary, then there exists $y \in C$ with $\|y_j\|_\infty \leq r_j$, for all $j \in \{1, \ldots, n\}$, and $\|y_k\|_\infty = r_k$ for some $k \in \{1, \ldots, n\}$, such that

$$y(t) \geq T(y)(t), \quad \text{for every} \;\; t \in [0, 1].$$

Let $t_0 \in [0, 1]$ be such that $y_k(t_0) = \|y_k\|_\infty = r_k$. From the previous inequality, by using the definition of T, we obtain

$$r_k = y_k(t_0) \geq T_k(y)(t_0) = f_k\left(t_0, \int_0^{t_0} y(s)ds\right) + g_k(t_0, y(t_0)) \geq \underline{f}_k + \underline{g}_k,$$

which contradicts (4.5). Hence, (C_1) holds.

Finally, we prove that (C_2) is also satisfied by using (4.6). If we assume the contrary, then there exist $\varepsilon > 0$ and $y \in C$ with $\varphi_j(y_j) \leq R_j$ for all $j \in \{1, \ldots, n\}$, and $\varphi_k(y_k) = R_k$ for some $k \in \{1, \ldots, n\}$, such that

$$T(y)(t) \geq (1 + \varepsilon) y(t), \quad \text{for any} \;\; t \in [0, 1].$$

Let $t_0 \in [0, 1]$ be such that $y_k(t_0) = \|y_k\|_\infty$. Then, using the last inequality, the expression of T, and (4.4), we obtain

$$\overline{f}_k + \overline{g}_k \geq f_k\left(t_0, \int_0^{t_0} y(s)ds\right) + g_k\left(t_0, y(t_0)\right) > y_k(t_0) \geq \frac{R_k}{a_k + b_k},$$

which contradicts (4.6). Hence, (C_2) holds.

Therefore, since all the assumptions of Theorem 3.2 are fulfilled, we have the following existence and localization result.

Theorem 4.1. *Under conditions $(H_1) - (H_{4,C})$, problem (4.1) has a nonnegative and increasing solution $x \in \mathcal{C}^1([0, 1], \mathbb{R}^n)$ such that*

$$\begin{aligned} & r_k \leq \|x_k'\|_\infty, \text{ for at least one } k \in \{1, \ldots, n\}, \text{ and} \\ & \varphi_i(x_i') \leq R_i, \text{ for all } i \in \{1, \ldots, n\}. \end{aligned} \tag{4.7}$$

In particular, if we assume that $n = 1$, and the monotonicity of f and g, more exactly:

(H_5) For each $t \in [0, 1]$, the functions $f(t, \cdot)$ and $g(t, \cdot)$ are increasing in $\mathbb{R}_+$, then conditions (4.5) and (4.6) in $(H_{4,C})$, with r_1, R_1, a_1, b_1 simply denoted by r, R, a, b, turn into

$$f(t,0) + g(t,r) > r, \quad \text{for every} \quad t \in [0,1], \tag{4.8}$$

$$f\left(t, \frac{R}{b}\right) + g\left(t, \frac{R}{b}\right) \leq \frac{R}{a+b}, \quad \text{for every} \quad t \in [0,1]. \tag{4.9}$$

Let us present two examples. The first one, which in fact is a solvable equation, is given to test the compression conditions.

Example 1. Let

$$f(s) = \lambda s \quad \text{and} \quad g(s) = \alpha s + \beta \quad (s \in \mathbb{R}),$$

where $\lambda, \alpha \geq 0$, $\beta > 0$ and $\lambda + \alpha < \frac{b}{a+b}$. If

$$r < \frac{\beta}{1-\alpha} \quad \text{and} \quad R\left(\frac{1}{a+b} - \frac{\lambda+\alpha}{b}\right) \geq \beta, \tag{4.10}$$

then the assumptions of Theorem 4.1 are fulfilled.

For example, condition (4.10) is satisfied for $a = b = 1$, $r = 1$, $R = 6$, $\beta = 1$ and $\lambda = \alpha = 1/6$. In this case, the exact solution of the problem is $x(t) = 6\left(e^{\frac{t}{5}} - 1\right)$, for $t \in [0,1]$, and $x'(t) = \frac{6}{5}e^{\frac{t}{5}}$, for $t \in [0,1]$, then

$$\|x'\|_\infty = \frac{6}{5}e^{\frac{1}{5}} \text{ and } \varphi(x') = \frac{6}{5}\left(1 + e^{\frac{1}{5}}\right),$$

and it is easy to see that conditions (4.7) hold.

The second example deals with equations that can not be explicitly solved.

Example 2. Consider the equation

$$x'(t) = \lambda x(t) + \alpha x'(t) + \beta + \gamma \sin(x'(t)), \;\; t \in [0,1]. \tag{4.11}$$

In this case,

$$f(s) = \lambda s \;\; \text{and} \;\; g(s) = \alpha s + \beta + \gamma \sin s \;\; (s \in \mathbb{R}),$$

where we assume that α, β, γ and λ are non-negative.

Now, we explain how to fulfill the conditions of Theorem 4.1. Clearly, condition (H_1) holds. Next, if $\alpha < 1-\gamma$, then $|g'(s)| = |\alpha + \gamma \cos s| \leq \alpha + \gamma < 1$. Therefore, (H_2) is satisfied for $k = \alpha + \gamma < 1$. To guarantee condition (H_3), we need $g(\mathbb{R}_+) \subset \mathbb{R}_+$, which takes place if $\beta \geq \gamma$. Furthermore, condition (H_5) is fulfilled if g is increasing in $\mathbb{R}_+$, and this happens if $\alpha \geq \gamma$. This condition, together with $\alpha < 1-\gamma$, gives $\gamma \leq \alpha < 1-\gamma$. Then, obviously, γ has to satisfy $0 \leq \gamma < \frac{1}{2}$. Finally, we have to check conditions (4.8) and (4.9). For the first, we need $r > 0$ such that $g(r) > r$, that is, $\alpha r + \beta + \gamma \sin r > r$. This clearly happens if $\alpha r + \beta - \gamma > r$, or, equivalently, $r < \frac{\beta-\gamma}{1-\alpha}$, which requires $\beta > \gamma$ since r has to be positive. Condition (4.9) reads as

$$\lambda \frac{R}{b} + \alpha \frac{R}{b} + \beta + \gamma \sin \frac{R}{b} \leq \frac{R}{a+b}. \tag{4.12}$$

We show that there exists R large enough that satisfies this inequality. Indeed, if we divide by $\frac{R}{b}$, we obtain

$$\lambda + \alpha + \frac{\beta b}{R} + \gamma \frac{sin \frac{R}{b}}{\frac{R}{b}} \leq \frac{b}{a+b}.$$

The limit of the left hand side, when R tends to ∞, being $\lambda + \alpha$ guarantees the existence of R provided that $\lambda + \alpha < \frac{b}{a+b}$ or, equivalently, $\lambda < \frac{b}{a+b} - \alpha$. In view of $\lambda \geq 0$, it requires that $\alpha < \frac{b}{a+b}$.

Therefore, the conditions of Theorem 4.1 are fulfilled if the non-negative parameters α, β, γ and λ satisfy:

$$\gamma \leq \alpha < 1 - \gamma,$$

$$\lambda + \alpha < \frac{b}{a+b},$$

$$\gamma < \min \left\{ \frac{1}{2}, \frac{b}{a+b}, \beta \right\}.$$

Under these conditions, for every

$$r < \frac{\beta - \gamma}{1 - \alpha},$$

there exists a solution $x \in \mathcal{C}^1([0,1], \mathbb{R})$ of equation (4.11) with $x(0) = 0$ that is non-negative, increasing and with $\|x'\|_\infty \geq r$.

If, in addition, a number R is chosen such that inequality (4.12) holds, then the solution x also satisfies

$$a \min_{t \in [0,1]} x'(t) + b \max_{t \in [0,1]} x'(t) \leq R.$$

Notice that, if $\gamma = 0$, Example 2 reduces to Example 1 and also our sufficient conditions on parameters α, β and λ required in Example 2 become those in Example 1.

To conclude this paper, we claim that a similar approach with expansion type conditions does not work, that is, Theorem 3.4 does not apply for the initial value problem (4.1).

Indeed, if we take

$$E_1 = \{ y \in \mathcal{C}([0,1], \mathbb{R}) : \ \varphi(y) \leq r \}, \quad E_2 = \{ y \in \mathcal{C}([0,1], \mathbb{R}) : \ \|y\|_\infty \leq R \},$$

where r, R are positive numbers with $r < bR$, and we proceed similarly to the compression case, we arrive to the following sufficient conditions of expansion:

$(H_{4,E})$

$$\max_{t \in [0,1], \ y \in \left[0, \frac{r}{b}\right]} f(t,y) + \max_{t \in [0,1], \ y \in \left[\frac{r}{a+b}, \frac{r}{b}\right]} g(t,y) \leq \frac{r}{a+b}, \tag{4.13}$$

$$\min_{t \in [0,1], \ y \in [0,R]} f(t,y) + \min_{t \in [0,1]} g(t,R) > R. \tag{4.14}$$

These conditions ensure that $E_1 \subset \overset{\circ}{E}_2$ and $(E_1), (E_2)$ are fulfilled but, unfortunately, they are not compatible with hypothesis (H_2), and thus Theorem 3.4 can not be applied. We shall prove this incompatibility in the autonomous case, that is, when f and g do not depend on t, and conditions (4.13), (4.14) are

$$\max_{y\in\left[0,\frac{r}{b}\right]} f(y) + \max_{y\in\left[\frac{r}{a+b},\frac{r}{b}\right]} g(y) \leq \frac{r}{a+b},$$

$$\min_{y\in[0,R]} f(y) + g(R) > R.$$

Subtracting the two inequalities yields

$$g\left(R\right) - \max_{y\in\left[\frac{r}{a+b},\frac{r}{b}\right]} g(y) > R - \frac{r}{a+b} + \max_{y\in\left[0,\frac{r}{b}\right]} f(y) - \min_{y\in[0,R]} f(y). \tag{4.15}$$

From $r < bR$, we have $[0, r/b] \subset [0, R]$, whence

$$\min_{y\in[0,R]} f(y) \leq \min_{y\in\left[0,\ \frac{r}{b}\right]} f\left(y\right) \leq \max_{y\in\left[0,\ \frac{r}{b}\right]} f\left(y\right).$$

Hence, the right-hand side in (4.15) is greater than or equal to $R - r/\left(a+b\right)$ and, so,

$$g\left(R\right) - \max_{y\in\left[\frac{r}{a+b},\frac{r}{b}\right]} g(y) > R - \frac{r}{a+b}. \tag{4.16}$$

On the other hand, if $\hat{y} \in \left[r/\left(a+b\right), r/b\right]$ is such that $g(\hat{y}) = \max_{y\in[r/(a+b),\ r/b]} g(y)$, then, also using (H2), we find

$$\begin{aligned} g\left(R\right) - \max_{y\in\left[\frac{r}{a+b},\frac{r}{b}\right]} g(y) &= g\left(R\right) - g\left(\hat{y}\right) \leq k\left(R - \hat{y}\right) \\ &\leq k\left(R - \frac{r}{a+b}\right) < R - \frac{r}{a+b}. \end{aligned}$$

This together with (4.16) clearly yields a contradiction, proving our claim.

Nevertheless, expansion conditions are possible for many other problems involving compact operators, as shown by lots of papers in the literature.

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