# Analog Transmission of Spatio-Temporal Correlated Sources over MAC with Modulo Mappings

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*Abstract*—**Modulo mappings are an appealing scheme for the analog transmission of spatially correlated discrete-time analog source symbols over Multiple Access Channels (MACs). In this work, we show that modulo mappings are also useful to exploit temporal correlation with zero-delay and without impairing encoding/decoding computational complexity. Spatio-temporal source correlation is exploited with a Kalman Filter-based receiver, coupled with a sphere decoder, that takes into account the non-linearities of the modulo mappings. We also explain how to optimize the mapping parameters.**

*Index Terms*—**Correlation, Kalman filtering, Multiaccess communication.**

## I. INTRODUCTION

THE transmission of correlated discrete-time analog information over a fading Multiple Access Channel (MAC) is a fundamental problem in wireless communications. As an example, the information sent by the sensors in a Wireless Sensor Network (WSN), like temperature or humidity measurements, is typically correlated both spatially, between different users, and temporally, between consecutive time instants.

In general, analog mappings are an appealing alternative to traditional digital schemes in scenarios with severe delay constraints. In particular, modulo mappings have been proposed to transmit spatially correlated sources over fading MACs with zero-delay, hence providing better performance than uncoded transmissions [1], [2]. In addition, low complexity receivers based on a sphere decoder were designed for scenarios with a large number of users [3].

Those works considered only spatial correlation while this work also looks at temporal correlation with the aid of a Kalman Filter (KF). KF is a widely known algorithm based on linear equations to estimate parameters observed during time [4]. In addition, popular non-linear approximations to this problem have been addressed in the literature using alternatives like the Extended Kalman Filter [5], among others.

The contributions of this letter extend [3] on the following points to exploit temporal correlations with modulo mappings:

- A receiver based on the structure of the KF that integrates a sphere decoder, providing an accurate model of the non-linearities of the modulo mapping, as well as low complexity decoding of the source symbols. This receiver simultaneously exploits temporal and spatial correlations.
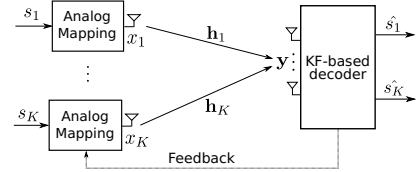
Fig. 1. Block diagram of a fading SIMO MAC with feedback.

- An optimization strategy for the modulo mappings assuming the existence of a feedback link from the centralized receiver to the users. Again, this optimization allows obtaining parameters that take into account both temporal and spatial correlations.

*Notation*: Bold lower-case and upper-case letters are used for vectors and matrices, respectively, while regular letters denote scalars. For a given matrix $A$, $A^T$ represents the transpose operation, $A^+ = (A^T A)^{-1} A^T$ is the pseudo-inverse, $|A|$ denotes the determinant, and $[A]_{i,j}$ is the element in the $i$-th row and $j$-th column. $\mathcal{N}(\mu, \Sigma)$ denotes a multivariate Gaussian distribution with mean $\mu$ and covariance $\Sigma$, $p_{\mathcal{N}}(s; \mu, \Sigma)$ its corresponding probability density function (pdf), and $\mathbb{E}[x]$ is the expectation of a random variable $x$. The operator $\mathrm{diag}(\cdot)$ constructs a diagonal matrix with the arguments in its main diagonal. $\lfloor \cdot \rfloor$ is the element-wise floor operation and $\mathrm{mod}(a, b) = a/b - \lfloor a/b \rfloor$. Finally, $\mathbf{0}$ and $\mathbf{1}$ denote the vectors of zeros and ones of the right dimension, respectively.

## II. SYSTEM MODEL

Fig. 1 shows the block diagram of the fading Single-Input and Multiple-Output (SIMO) MAC model considered in this work. A group of $K$ single-antenna users transmits discrete-time analog Gaussian source symbols to a common receiver over the fading MAC. The receiver is equipped with $N_R$ receive antennas, and a feedback link from the receiver to the users is available to send information about the encoder configuration.

Vectors of $K$ source symbols during $T$ consecutive time instants are assumed to be correlated both spatially and temporally according to the following autoregressive model

$$s_t = F s_{t-1} + g_t, \quad \forall t = 1, 2, \dots T \qquad (1)$$

where $s_t \in \mathbb{R}^K$ is the vector of source symbols at the $t$-th time instant, $F$ is a state transition matrix and $g_t \sim \mathcal{N}(0, G)$ is the noise component of the autoregressive model, with covariance $G$. We assume $s_t \sim \mathcal{N}(0, C_s)$ with $C_s$ its covariance matrix and $s_0 = \mathbf{0}$.

Source symbols are encoded on a per-user basis with a non-linear mapping $z(\cdot)$ to produce the symbols $x_t = z(s_t) =$
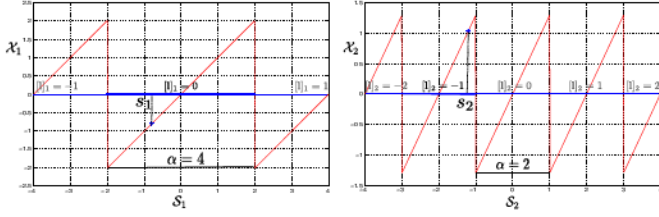
Fig. 2. Example of modulo mapping with $\alpha_1 = 4$ and $\alpha_2 = 2$. Each component of the $l$ vector represents the interval where the symbol of each user falls, in this case $l = [0, -1]^T$ (from [3]).

$[z_1(s_{1t}), \ldots, z_K(s_{Kt})]^T$ to be transmitted over the fading MAC. The received signal is

$$y_t = Hx_t + n_t, \tag{2}$$

where $H \in \mathbb{R}^{N_R \times K}$ is the fading MAC response which remains constant over the transmission of a block of $T$ symbols, and $n_t \sim \mathcal{N}(0, \sigma_n^2 I)$ is the channel Additive White Gaussian Noise (AWGN). Symbols transmitted by each user satisfy an individual power constraint such that $E\left[|z_k(s_k)|^2\right] \leq P_k, \forall k$. Eq. (1) and (2) constitute a non-linear KF setting.

We consider that the source symbols are encoded by means of modulo mappings defined as $z_k(s_k) = \Delta_k (\text{mod}(s_k + \alpha_k/2, \alpha_k) - \alpha_k/2), \forall k$. The parameters $\alpha_k$ determine the shape of the mapping while $\Delta_k$'s are gain factors to satisfy the power constraints. Two examples of this modulo mapping are shown in Fig. 2. As observed, the mappings can be alternatively defined by intervals using linear functions. Each of these intervals is indexed with an integer value, and this allows to rewrite the mapping function in vector form as

$$z(s) = D(s - Al), \tag{3}$$

where $l = \left\lfloor A^{-1}s + \frac{1}{2}1 \right\rfloor \in \mathbb{Z}^K$ stacks the $K$ integer values $l_k$ such that $s_k \in [\alpha_k(l_k - 1/2), \alpha_k(l_k + 1/2)], \forall k$. The matrix $A = \text{diag}(\alpha_1, \ldots, \alpha_K)$ contains the modulo mapping parameters for all users and the matrix $D = \text{diag}(\Delta_1, \ldots, \Delta_K)$ comprises the gain factors. Both $A$ and $D$ should be designed to minimize the distortion considering the correlation model in (1).

At the receiver, an estimate of the source symbols is obtained from the received signal $y_t$ by using a Minimum Mean Squared Error (MMSE) decoder that follows the structure of a non-linear KF and that relies on a sphere decoder to reduce the computational complexity. This filter is described in the next section.

Note that this system model assumes that the variables are real-valued. If complex signaling is considered, it can be transformed into a real-valued equivalent model, doubling the number of variables of the original system from $K$ to $2K$ by treating separately the real and imaginary parts [3].

## III. Modulo Mapping Decoder

This section describes a KF-based decoder that leverages both the spatial and the temporal correlation of the source symbols. The decoding operation consists of two steps: an observation step that estimates the source symbols using the posterior probability for the received signal, and a prediction step that generates prior information for the observation step

in the next time instant. In our system, the prediction step is computed with linear operations using $F$ and $G$ as

$$\hat{s}_{t|t-1} = F\hat{s}_{t-1} \tag{4}$$

$$P_{t|t-1} = FP_{t-1}F^T + G \tag{5}$$

but the observation step is affected by non-linearities due to the modulo mappings. We next show how to model the posterior probability of the received signal when using these mappings. In the following, the time subindex $t$ will be omitted for clarity.

For a given prediction $\bar{s} = \hat{s}_{t|t-1}$ and the corresponding covariance matrix $P = P_{t|t-1}$, the posterior probability of the modulo mapping for Gaussian sources with the form $s \sim \mathcal{N}(\bar{s}, P)$ can be expressed as

$$p_m(s|y) \propto p(y|s)p(s)$$
$$\propto \sum_{l \in \mathbb{Z}^K} \mathcal{T}_l(y, HD(s - Al), \sigma_n^2 I) p_{\mathcal{N}}(s; \bar{s}, P) \tag{6}$$
$$\propto \sum_{l \in \mathbb{Z}^K} \phi_l \mathcal{T}_l(s, \mu_l, \Sigma), \tag{7}$$

where

$$\mathcal{T}_l(s, \mu, \Sigma) = \begin{cases} p_{\mathcal{N}}(s; \mu, \Sigma) & a_l < s \leq b_l \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

represents a truncated Gaussian function with mean $\mu_l = \bar{s} + \frac{1}{\sigma_n^2} \Sigma D^T H^T (y + HDAl - HD\bar{s})$ and covariance $\Sigma = \left(\frac{1}{\sigma_n^2} D^T H^T HD + P^{-1}\right)^{-1}$. The vectors $a_l = A\left(l - \frac{1}{2}1\right)$ and $b_l = A\left(l + \frac{1}{2}1\right)$ represent, respectively, the lower and upper limits for the combination of intervals given by $l$.

From (6) to (7), we grouped the terms not depending on $s$ in $\phi_l$ after rearranging the exponents in (6) as a single quadratic form. In that case, $\phi_l = \exp\left(-\frac{1}{2\sigma_n^2} \|y + HDAl\|^2 + \frac{1}{2} \mu_l^T \Sigma^{-1} \mu_l\right)$, and it can be rewritten using an approach similar to [3] as

$$\phi_l = \exp\left(-(l + l_o)^T \Lambda (l + l_o) + m^T (l + l_o)\right), \tag{9}$$

where

$$\Lambda = \frac{1}{2} A^T D^T H^T \left(\sigma_n^2 I + HDPD^T H^T\right)^{-1} HDA, \tag{10}$$

$$m = A^T D^T H^T HD\Sigma P^{-1}\bar{s}, \tag{11}$$

and $l_o = (HDA)^+ y$. When $\bar{s} = 0$, the previous expressions agree with those obtained in [3].

Estimating the source symbols requires to compute the mean of the weighted sum of truncated Gaussian functions in (7), which is a complicated problem [6]. An approximation of (8) can be obtained with a weighted Gaussian pdf, in the form $\mathcal{T}_l(s) \approx \lambda_l p_{\mathcal{N}}(s; \hat{s}_l, \Sigma)$, with

$$\hat{s}_l = \arg\max_s \mathcal{T}_l(s, \mu_l, \Sigma), \tag{12}$$

and $\lambda_l = \sqrt{|2\pi\Sigma|} p_{\mathcal{N}}(\hat{s}_l; \mu_l, \Sigma)$. In case $\hat{s}_l = \mu_l$, the factor $\lambda_l = 1$; but if $\hat{s}_l \neq \mu_l$ it means that $\hat{s}_l$ falls outside the interval $[a_l, b_l]$, and the factor $\lambda_l < 1$ helps to reflect the importance of that tail. Note that this is a sensible approach since in most cases the Gaussian functions in (8) with significant weights $\phi_l$ are truncated just by their tails. The problem in (12) can

be efficiently approximated with a quadratic programming algorithm applied to the problem

$$\hat{s}_l = \arg\max_{s} \ \frac{1}{2} s^T \Sigma^{-1} s - v^T s, \tag{13}$$

$$s.t. \quad a_l \leq s < b_l,$$

where $v = \frac{1}{\sigma_n^2}(D^T H^T y + D^T H^T H D A l) + P^{-1}\bar{s}$. These expressions help to approximate the posterior probability in (7) by a weighted sum of Gaussian functions as

$$p_m(s|y) \approx \sum_{l \in \mathbb{Z}^K} \bar{w}_l p_{\mathcal{N}}(s; \hat{s}_l, \Sigma), \tag{14}$$

where $\bar{w}_l = w_l / \sum w_l$, and $w_l = \phi_l p_{\mathcal{N}}(\hat{s}_l; \mu_l, \Sigma)$.

The main drawback of the representations in (7) and (14) is that $l$ spans over a potentially large set but its cardinality can be significantly reduced by searching for $l$ vectors with large associated weights $\phi_l$ [3]. Therefore, we are interested in vectors $l$ such that their corresponding exponent in $\phi_l$ (see (9)) is below some threshold $R$, i.e.

$$h(l) = \left(\tilde{l} + \tilde{l}_o\right)^T \tilde{\Lambda} \left(\tilde{l} + \tilde{l}_o\right) < R, \tag{15}$$

with $R$ a parameter that sets the maximum considered weights, and where the original terms $l$, $l_o$, $m$ and $\Lambda$ are transformed into lattice form as

$$\tilde{\Lambda} = \begin{pmatrix} \frac{1}{4}m^T \Lambda^{-1} m & -\frac{1}{2}m^T \\ -\frac{1}{2}m & \Lambda \end{pmatrix}, \tag{16}$$

$\tilde{l} = [1, l^T]^T$ and $\tilde{l}_o = [0, l_o^T]^T$. Now, a sphere decoder can be applied over the lattice represented by $\tilde{\Lambda}$ to recursively obtain a set of feasible $l$ vectors as $\mathcal{L}_R = \{l | h(l) < R\}$. In particular, decomposing $\tilde{\Lambda} = L^T L$ where $L$ is a lower triangular matrix, feasible integers for the $k$-th component of $l$ can be found as

$$\underbrace{[l_o]_k - d_k - \frac{\sqrt{R - c_k}}{[L]_{k,k}}}_{I_k(l)} \leq [l]_k \leq \underbrace{[l_o]_k - d_k + \frac{\sqrt{R - c_k}}{[L]_{k,k}}}_{O_k(l)} \tag{17}$$

with

$$d_k = \sum_{i=1}^{k-1} \frac{[L]_{k,i}}{[L]_{k,k}}([l]_i + [l_o]_i),$$

$$c_k = c_{k-1} + \left(\sum_{i=1}^{k-1}[L]_{k-1,i}([l]_i + [l_o]_i)\right)^2$$

where $c_0 = 0$. Hence, $\mathcal{L}_R = \mathcal{L}_R^K$, where

$$\mathcal{L}_R^k = \{l \in \mathbb{Z}^n | $$
$$l = [n^T n]^T, I_k(n) \leq n \leq O_k(n), \forall n \in \mathcal{L}_R^{k-1}\}, \tag{18}$$

and $\mathcal{L}_R^0 = \emptyset$. Finally, thanks to this set, the expression in (14) can be accurately described with a small number of vectors $l$.

Recall that KF is based on linear equations. However, its structure can be adapted for this scenario to consider the non-linearities of the modulo mapping. In particular, the posterior probability is approximated by its mean and its covariance matrix, which can be computed at a low cost by exploiting its structure as a sum of Gaussian functions given by (14). We define a decoding algorithm with the elements defined

above, as shown in Algorithm 1. A more accurate model could consider, for instance, to use the weights $w_l$ and the vectors $\hat{s}_l$ in a Gaussian Sum Filter (GSF) [7] but at the cost of a larger computational complexity.

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**Algorithm 1** KF-based decoder for modulo mappings.

$\hat{s}_{1|0} \leftarrow 0, P_{1|0} \leftarrow C_s$
$A_1 \leftarrow \text{diag}(\alpha_1, \ldots, \alpha_K), D_1 \leftarrow \text{diag}(\Delta_1, \ldots, \Delta_K)$ for $P_{1|0}$
**for all** $t \in [1, T]$ **do**
  $\Sigma \leftarrow \left(\frac{1}{\sigma_n^2} D_t^T H^T H D_t + P_{t|t-1}^{-1}\right)^{-1}$
  $\mathcal{L}_R \leftarrow$ (18) on $y_t$ assuming a $\mathcal{N}(\hat{s}_{t|t-1}, P_{t|t-1})$
  **for all** $l \in \mathcal{L}_R$ **do**
    $\hat{s}_l \leftarrow$ Solve (13)
    $\mu_l \leftarrow \bar{s} + \frac{1}{\sigma_n^2}\Sigma D_t^T H^T(y_t + HD_t A_t l - HD_t \bar{s})$
    $w_l \leftarrow \phi_l p_{\mathcal{N}}(\hat{s}_l; \mu_l, \Sigma)$
  **end for**
  $\hat{s}_t \leftarrow \sum_{l \in \mathcal{L}_R} \bar{w}_l \hat{s}_l$
  $P_t \leftarrow \Sigma + \sum_{l \in \mathcal{L}_R} \bar{w}_l(\hat{s}_l - \hat{s}_t)(\hat{s}_l - \hat{s}_t)^T$
  $\hat{s}_{t+1|t} \leftarrow F\hat{s}_t$
  $P_{t+1|t} \leftarrow FP_t F^T + G$
  Update $A_{t+1}, D_{t+1}$ for $P_{t+1|t}$ with (19). {Feedback the $(\alpha_1, \ldots, \alpha_K)$ to the users}
**end for**

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### A. Parameter Optimization

The modulo parameters $\alpha_k$ can be optimized at the receiver assuming that the channel is known and that the source covariance is given by $P_{t+1|t}$. As shown in [3], the $\Delta_k$ gain factors increase if their corresponding $\alpha_k$'s are lower and this reduces distortion when using modulo mappings. Thus, lower values for $\alpha_k$ are desirable to lower the distortion in general, but too low values may prevent the correct decoding of the source symbols. In the latter case, several decoded symbols could be equally feasible, which mathematically translates into similar weights $\phi_l$ for different $l$ in the posterior probability in (7). Thus, the optimization of the $\alpha_k$ parameters should consider this trade-off, which will depend on the channel state and the source correlation. To address this problem we will follow an approach similar to [3].

First, we assume the decomposition $\Delta_k = p_k \delta_k$ with $p_k, \delta_k \in \mathbb{R}$. We then set $p_k = \sqrt{P_k}$ and initialize $\delta_k = 1, \forall k$. Finally, using $P_{t+1|t}$ as the source covariance matrix at each time instant, and decomposing the matrix $\Xi = A^{-1}\Lambda A^{-1}$ as $\Xi = Q^T Q$, the parameters $\alpha_k$ are obtained as

$$\alpha_k = \frac{\sqrt{S}}{[Q]_{k,k}}, \tag{19}$$

for some design parameter $S$ that will be of the same order of magnitude as the radius chosen during the decoding stage. The $\alpha_k$ parameters can be fed back to the users before they transmit their next symbol. Each $\delta_k$ is updated from the corresponding $\alpha_k$ according to [3, Eq. (7)], and the resulting values are used to compute the gain factors $\Delta_k$. Better performance could be achieved by considering optimal power allocations $p_k \in \mathbb{C}$, but this goes beyond the scope of this work. This strategy relies on an accurate description of the error covariance, and it does not require the actual transmitted symbols $s_t$.
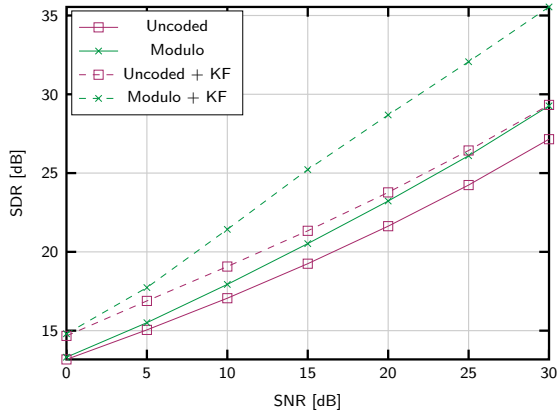
Fig. 3. SDR vs SNR for a $10 \times 10$ scenario, $\rho = 0.95$ and different decoders.



Fig. 4. SDR vs SNR for $10 \times 10$ scenario with different correlation factors using the KF-based receivers with uncoded and modulo mappings.

## B. Computational Complexity

The main computational burden of Algorithm 1 is on the generation of the $\mathcal{L}_{\tilde{R}}$ set with the sphere decoder. A general analysis of the sphere decoder algorithm is provided in [8], and it relies on estimating the number of lattice points enclosed by a sphere of a given radius. In our case, this number is proportional to the number of users and inversely proportional to the $\alpha_k$ values. In fact, a lower bound on the overall computational complexity can be defined for $\alpha_k$ parameters arbitrarily large (i.e. $\alpha_k \gg [\boldsymbol{C}_s]_{k,k}$), since it would match that of the linear KF. This would allow choosing larger $\alpha_k$ values than the optimal ones to lower the computational cost.

## IV. SIMULATION RESULTS

In this section, the performance of the proposed decoding algorithm based on the KF is evaluated by means of computer simulations. The results are obtained considering a particular correlation model where $\boldsymbol{F} = \rho \boldsymbol{I}$ and $\boldsymbol{G} = \left(1 - \rho^2\right) \boldsymbol{C}_s$, with $[\boldsymbol{C}_s]_{i,i} = 1$ and $[\boldsymbol{C}_s]_{i,j} = \rho, \forall i, j, i \neq j$, with $0 \leq \rho < 1$. The parameters of the modulo mappings are optimized following the strategy described in Section III-A. The power constraints are set to $P_k = P, \forall k$, hence the Signal-to-Noise Ratio (SNR) in the plots is defined as SNR $= P / \sigma_n^2$. Blocks of size $T = 1000$ symbols are considered, a size large enough to show that the decoder does not diverge after the successive decoding of the received samples. Also, for very low values (e.g. $T < 10$), the gain of this decoder would be negligible with respect to ignoring the spatial correlation. Finally, the fading MAC is modeled according to a Rayleigh distribution.

Performance is measured in terms of Signal-to-Distortion Ratio (SDR), computed as the inverse of the Mean Squared Error (MSE) given by $\xi = \frac{1}{KT} \sum_t \|\boldsymbol{s}_t - \hat{\boldsymbol{s}}_t\|^2$, hence SDR[dB] $= 10 \log(1/\xi)$.

Figure 3 shows the SDR obtained for the scheme with modulo mappings and the proposed KF-based decoding in a scenario with $K = 10$ users, $n_R = 10$ antennas and a correlation factor $\rho = 0.95$. This strategy is compared to other three approaches: 1) uncoded scheme with the linear MMSE decoder that only considers spatial correlation, 2) uncoded scheme with a linear KF that incorporates spatial and temporal correlation; and 3) modulo mappings with a MMSE decoder
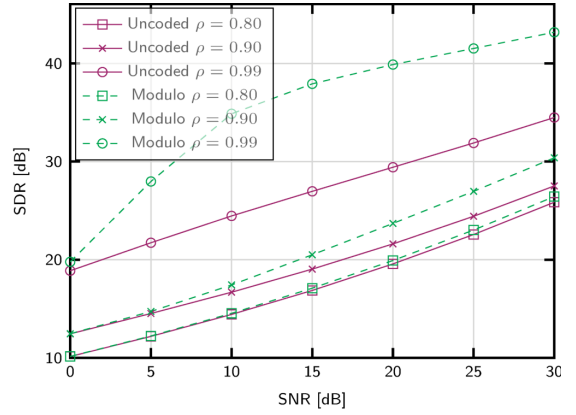
that ignores temporal correlation [3]. As observed, modulo mappings reach the uncoded scheme with KF receiver only on high SNRs, while the modulo with KF receiver provides gains along all the SNR values. Also, if this scheme is compared to uncoded transmissions with KF, gains up to 8 dB are obtained.

Figure 4 shows the SDR for uncoded and modulo mappings combined with the KF-based receiver for different correlation factors. As observed, the performance gain of modulo mappings increases with the correlation factor, especially for medium and high SNRs. This gain goes from 0.5 dB for $\rho = 0.8$ up to 10 dB for $\rho = 0.99$.

## V. CONCLUSION

We have studied the design of modulo mappings to exploit both the temporal and the spatial correlation of source symbols transmitted over a fading SIMO MAC. This task encompasses both the decoding strategy and the parameter design. Simulation results show that significant gains are obtained with respect to uncoded schemes, and also against a modulo-based transmitter that only considers spatial correlation.

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