CLASSIFICATION OF THE RELATIVE POSITIONS BETWEEN A SMALL ELLIPSOID AND AN ELLIPTIC PARABOLOID

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Abstract. We classify all the relative positions between an ellipsoid and an elliptic paraboloid when the ellipsoid is small in comparison with the paraboloid (*small* meaning that the two surfaces cannot be tangent at two points simultaneously when one is moved with respect to the other). This provides an easy way to detect contact between the two surfaces by a direct analysis of the coefficients of a fourth degree polynomial.

1. Introduction

As the simplest curved surfaces, quadrics have been extensively used in CAD/CAM and industrial design. This is justified by the fact that they can be manipulated through a simple algebraic expression, which makes their geometry more tractable for computational purposes. Another advantage is that quadric surfaces can be used to approximate other more complicated surfaces locally up to order two or to build a variety of shapes piecewise [14, 17, 21, 26].

The problem of contact detection is essential in fields such as robotics, computer graphics, computer animation, CAD/CAM, etc. There exists an extensive literature about contact detection between quadric surfaces (see [15] for a classical reference and references therein such as [23]), but one of the main tools relies on the use of a characteristic polynomial associated to the pencil of the quadrics to treat the collision detection problem. Indeed that was done in previous works considering conic curves (see [1, 5, 8, 16]) and other quadric surfaces, especially ellipsoids, in Euclidean or Projective spaces (see [6, 11, 24, 25]).

Although the literature dealing with the relative positions of two ellipsoids is large (see [4, 22, 24] among others), there is a lack of information on the same problem associated to other quadrics that would fit different geometric features better than an ellipsoid when considered in practical contexts. With this motivation, we address the problem of finding the relative position between an ellipsoid and an elliptic paraboloid.

A direct approach to the problem of looking for the intersection between an ellipsoid and an elliptic paraboloid involves to solve a system of two polynomial equations of degree two. In order to avoid this task, the proposed method provides a quick answer in terms of the coefficients of a fourth degree polynomial, which gives rise to algorithms of low computational cost. Additionally, the resulting algorithms do not only discern whether there is contact or not but also give information on the relative position between the quadrics. Thus, this work sets the theoretical foundations of a method to detect the relative position between the two quadrics for the follow-up applications in specific practical contexts.

We give a complete classification of the relative positions when the size of the ellipsoid is small in comparison with the size of the paraboloid. This assumption in the size is natural from a geometric point of view and is further explained in Section 3. The remaining of the paper is organized as follows. The characterization of relative positions in terms of the roots of the characteristic polynomial associated to the pencil of the quadrics is given in Section 2, which includes the main results of the paper. In Section 4 we analyze the characteristic polynomial and reduce the problem to study a paraboloid and a sphere. The particular case where the center of this sphere is in

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the OZ-axis is studied in Section 5, then extended to the whole space in Section 6, to prove the main results in Section 7. Applications of the method, implementation and examples are treated in Section 8. Conclusions are summarized in Section 9 and, finally, a technical proof of an intermediate step is left to the Appendix.

2. Classification of the relative positions

2.1. **Quadrics and characteristic polynomial.** We consider a general elliptic paraboloid $\mathcal{P}$ with associated matrix P and a general ellipsoid $\mathcal{E}$ with associated matrix E, in such a way that in homogeneous coordinates $X = (x, y, z, 1)^t$ the corresponding equations are, respectively, $X^t P X = 0$ and $X^t E X = 0$.

We consider the pencil of the quadrics $\lambda P + E$ in order to define the *characteristic polynomial* of $\mathcal{P}$ and $\mathcal{E}$ as

$$f(\lambda) = \det(\lambda P + E). \tag{1}$$

We will refer to the roots of the polynomial f as the *characteristic roots* of $\mathcal{P}$ and $\mathcal{E}$.

2.2. **The "smallness" condition.** All along this work we will assume that the ellipsoid is *small* in comparison with the elliptic paraboloid. More precisely, we will assume the following condition, which is intrinsic to the geometry of the two surfaces:

"Smallness" hypothesis: the ellipsoid and the elliptic paraboloid cannot be tangent at two points simultaneously.

This hypothesis has a very specific geometric meaning and is naturally motivated by the classification of the relative position between quadrics. This smallness condition is intrinsic to the geometry of the two quadrics and depends on their shape, but not on a particular position. Thus, it is meant that the two surfaces cannot be tangent at two points when one allows rigid moves to any of them. Therefore, the condition is equivalent to the fact that if the ellipsoid and the paraboloid intersect, then they do it just in one point or in a curve with only one connected component. Consequently, other more complicated relative positions that include multiple tangent points or intersections in curves with two connected components are excluded by this hypothesis. If the reader is interested just in the intersection curves of all the possible cases, we refer to [12, 15, 19, 20]. The "smallness" hypothesis can be phrased in terms of principal curvatures of the surfaces: the smallest principal curvature in the ellipsoid is greater than the largest principal curvature of the elliptic paraboloid (which is attained at the vertex point). In Section 3 we analyze this condition in detail and show how to check it for general ellipsoid and elliptic paraboloid.

2.3. **Relative positions between the surfaces.** To analyze the relative positions between the two quadrics we establish the following definition.

Definition 1. *We say that:*

1. *$\mathcal{P}$ and $\mathcal{E}$ are in* contact *if there exists X such that $X^t P X = X^t E X = 0$;*
2. *$\mathcal{P}$ and $\mathcal{E}$ are* tangent *at a point X if $X^t P X = X^t E X = 0$ and $Y^t P X = 0$ if and only if $Y^t E X = 0$, this is, $\mathcal{P}$ and $\mathcal{E}$ are in contact at X and have the same tangent plane at X;*
3. *a point X is* interior *to $\mathcal{P}$ if $X^t P X < 0$ and* exterior *to $\mathcal{P}$ if $X^t P X > 0$. By extension, we also say that $\mathcal{E}$ is interior (or exterior) to $\mathcal{P}$ if every point in $\mathcal{E}$ is interior (or exterior) to $\mathcal{P}$.*

Theorem 2. *Let $\mathcal{E}$ be an ellipsoid and $\mathcal{P}$ an elliptic paraboloid satisfying the "smallness" condition. The possible relative positions between $\mathcal{E}$ and $\mathcal{P}$ are those given in Table 1 and they are characterized by the corresponding configuration of the characteristic roots.*

Remark 3. Types I and TI are both detected by four negative roots. Type TI always has a double root while type I generically has four different roots with just the exception of two particular cases with a double root. The two possible double roots of type I are very specific and are determined in terms of the parameters of the paraboloid $\mathcal{P}$. For a sphere and an elliptic paraboloid with parameters a and b (see Equation (7)), these specific roots are $\hat{\lambda} = -a^2$ and $\tilde{\lambda} = -b^2$ (see Remark 15 and Theorem 16). Thus, the problem of distinguishing these two relative positions comes from the

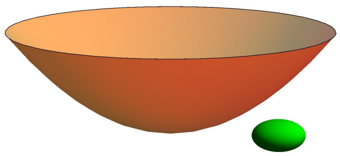
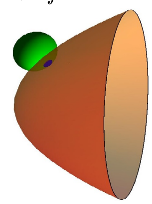
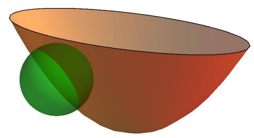
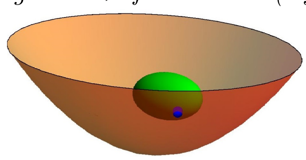
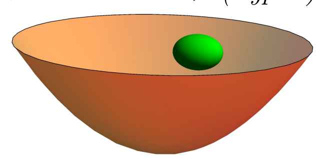
<table>
<tr><td colspan="2">Relative positions between an elliptic paraboloid $\mathcal{P}$ and a "small" ellipsoid $\mathcal{E}$
There are always two roots (λ_1 and λ_2) that satisfy: $\lambda_1 \leq \lambda_2 < 0$, the configuration of the other two roots (λ_3 and λ_4) determines the relative position between the ellipsoid and the elliptic paraboloid.</td></tr>
<tr><td>$\mathcal{E}$ is exterior to $\mathcal{P}$ (Type E)

Roots: $0 < \lambda_3 < \lambda_4$</td><td>$\mathcal{E}$ is tangent to $\mathcal{P}$ from outside (Type TE)

Roots: $0 < \lambda_3 = \lambda_4$</td></tr>
<tr><td colspan="2">Contact between $\mathcal{E}$ and $\mathcal{P}$ (Type C)

Roots: $\lambda_3 = \bar{\lambda}_4 \notin \mathbb{R}$</td></tr>
<tr><td>$\mathcal{E}$ is tangent to $\mathcal{P}$ from inside (Type TI)

Roots: $\lambda_3 = \lambda_4 < 0$</td><td>$\mathcal{E}$ is interior to $\mathcal{P}$ (Type I)

Roots: $\lambda_3 \leq \lambda_4 < 0^*$</td></tr>
<tr><td colspan="2">*(see Remark 3 to distinguish a double root in types I and TI)</td></tr>
</table>

TABLE 1. Characterization of the relative positions between a small ellipsoid and an elliptic paraboloid in terms of the characteristic roots.

fact that there are two roots that can be double roots in the Type I position. Therefore, one can distinguish between the two relative positions once it is known that $\hat{\lambda}$ and $\tilde{\lambda}$ are not characteristic roots. Hence, one has that:

(i) 4 different real negative roots imply Type I (see Example 20 in Section 8);
(ii) 3 different real negative roots, one of which has multiplicity two and is different from $\hat{\lambda}$ and $\tilde{\lambda}$, imply Type TI.

Based on Theorem 2, one can detect the relative position of $\mathcal{E}$ and $\mathcal{P}$ by a direct analysis of the coefficients of the characteristic polynomial f as follows. The first step will be to compute the characteristic polynomial f:

$$f(\lambda) = c_4\lambda^4 + c_3\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0.$$

The discriminant Δ of the fourth degree polynomial $f(\lambda)$ is given by the expression (see [7, 27]):

$$\begin{aligned}\Delta = {} & 256c_0^3c_4^3 - 192c_0^2c_1c_3c_4^2 - 128c_0^2c_2^2c_4^2 + 144c_0^2c_2c_3^2c_4 - 27c_0^2c_3^4 + 144c_0c_1^2c_2c_4^2 - 6c_0c_1^2c_3^2c_4 - 4c_1^3c_3^3 \\ & -80c_0c_1c_2^2c_3c_4 + 18c_0c_1c_2c_3^3 + 16c_0c_2^4c_4 - 4c_0c_2^3c_3^2 - 27c_1^4c_4^2 + 18c_1^3c_2c_3c_4 - 4c_1^2c_2^3c_4 + c_1^2c_2^2c_3^2.\end{aligned}$$

In view of Theorem 2, there are always two real negative roots. Therefore, there are neither two pairs of complex conjugate roots nor a double root and a pair of complex conjugate roots. The non-tangent contact position, which is characterized by two complex conjugate roots (see Table 1), is also characterized by $\Delta < 0$ (see [7, 10, 27]). The case $\Delta = 0$ corresponds to multiple roots, so it characterizes tangent contact with the exception of the double roots that may appear in the type I position (see Section 6.1 for details). $\Delta > 0$ implies that there are 4 different real roots, which results in non-contact relative positions. In the later case, the Descartes' rule of signs allows to distinguish the interior and exterior cases by checking the sign of the coefficients as shown in

Conditions to classify the relative positions between $\mathcal{E}$ and $\mathcal{P}$	
Type	*Conditions on the coefficients of f*
E	$\Delta > 0$ *and* $c_i > 0$ *for some* $i = 1, 2, 3$
TE	$\Delta = 0$ *and* $c_i > 0$ *for some* $i = 1, 2, 3$
C	$\Delta < 0$
I	$\Delta > 0$ *and* $c_i \leq 0$ *for all* $i = 1, 2, 3$
I *or* TI	$\Delta = 0$ *and* $c_i \leq 0$ *for all* $i = 1, 2, 3$ *(see Remark 3 to distinguish types I and TI)*

TABLE 2. Characterization of the relative positions between a small ellipsoid and an elliptic paraboloid in terms of the coefficients of the characteristic polynomial.

Table 2 (see also [7, 27]). Note that c_4 and c_0 are always negative, so one simply checks c_i for $i = 1, 2, 3$. This is summarized in the following result.

Corollary 4. *Let $\mathcal{E}$ be an ellipsoid and $\mathcal{P}$ an elliptic paraboloid satisfying the smallness condition. The relative positions between $\mathcal{E}$ and $\mathcal{P}$ are detected in terms of the coefficients of f as shown in Table 2.*

Although the smallness assumption is essential in Theorem 2 and Corollary 4 (see Example 8), it does not really affect the exterior case. Therefore, we obtain the following characterization of the exterior position for arbitrary ellipsoid and elliptic paraboloid (the proof is given in Section 7).

Corollary 5. *Let $\mathcal{E}$ be an ellipsoid and $\mathcal{P}$ an elliptic paraboloid. Then $\mathcal{E}$ is exterior to $\mathcal{P}$ if and only if f has two positive characteristic roots. Moreover, $\mathcal{E}$ is tangent to $\mathcal{P}$ from outside if there is one positive double root.*

Note that, following [24], the exterior position between the ellipsoid and the elliptic paraboloid corresponds to the existence of a separating plane between the quadrics. Hence Corollary 5 can be rephrased as follows: $\mathcal{E}$ and $\mathcal{P}$ are separated if and only if f has two positive roots and they touch in a point if there exists a positive double root. This result is the analogous of the seminal result given in [24] for the separation of two ellipsoids.

Possible algorithms to detect the relative position follow from Table 2 and the analysis in the following sections. We consider these aspects in Section 8.

3. The "smallness" condition

In order to understand the role played by the smallness condition and its geometric meaning, in this section we analyze it in detail. First we show how to check it by an easy computation in terms of the parameters of the ellipsoid and the elliptic paraboloid, what makes it tractable for real world applications. Then we show how to transform the space so that the ellipsoid becomes a sphere and the smallness condition is preserved. This will be useful for subsequent sections where we will work directly with a sphere. Finally, we show up to what extend the smallness condition is indeed necessary in Theorem 2 by providing a counterexample to the theorem if we remove this hypothesis.

3.1. **How to check the smallness condition.** In real world applications one must deal with ellipsoids and paraboloids which are not given in standard form. Thus, in practice, if we wish to apply the results of Theorem 2 and Corollary 4, we must first check that the smallness condition is satisfied for general ellipsoid and paraboloid. We shall emphasize that, for applications where the two surfaces move, the smallness condition only needs to be checked once at the beginning, as it only depends on the geometry and not on the position of the quadrics.

The smallness hypothesis can be interpreted in terms of the principal curvatures of the surfaces: the fact that the maximum principal curvature of the paraboloid is less than or equal to the minimum curvature of the ellipsoid guarantees the smallness condition (see Figure 1(a) and (b)). In order to do this, when considering arbitrary ellipsoids and paraboloids, we shall compare the

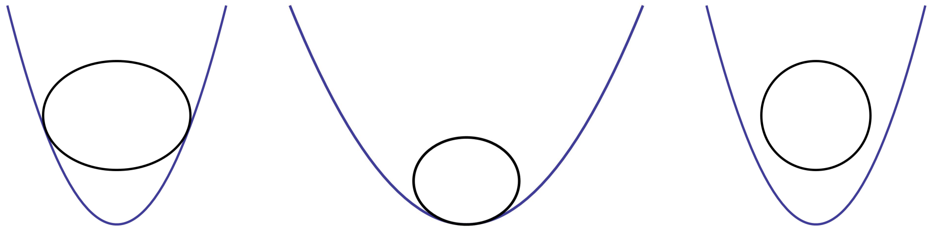


FIGURE 1. The maximum curvature of the parabola is attained at the vertex point. The condition for the ellipse not to be tangent at two points is that the curvature of the parabola at the vertex is less than or equal to the minimum curvature of the ellipse.

minimal principal curvature of the ellipsoid with the maximal principal curvature of the paraboloid. Thus, we consider the general equation of a quadric in Euclidean coordinates x_1, x_2, x_3:

$$\sum_{i,j=1}^{3} a_{ij}x_ix_j + \sum_{i=1}^{3} 2b_ix_i + c = 0. \tag{2}$$

The eigenvalues of (a_{ij}) together with the determinant of the matrix associated to the quadric suffices to determine the curvature of the surface as follows. Computing the eigenvalues of the matrix (a_{ij}) one gets for the ellipsoid values $\frac{1}{\gamma_1^2}$, $\frac{1}{\gamma_2^2}$ and $\frac{1}{\gamma_3^2}$ and for the paraboloid values $\frac{1}{\alpha^2}$, $\frac{1}{\beta^2}$ and 0, so that, in appropriate coordinates $\{x, y, z\}$ for each of the quadrics, the equations become, respectively (see, for example, [18])

$$\frac{x^2}{\gamma_1^2}+\frac{y^2}{\gamma_2^2}+\frac{z^2}{\gamma_3^2} = \gamma^2 \text{ with } \gamma^2 = -\gamma_1^2\gamma_2^2\gamma_3^2 \det(E), \text{ and } \frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2} = 2pz \text{ with } p^2 = -\alpha^2\beta^2 \det(P). \tag{3}$$

Thus, for the paraboloid, we intersect $\mathcal{P}$ with the vertical plane $y = 0$ to obtain the parabola $z = \frac{x^2}{2p\alpha^2}$. Since we are assuming $\alpha \le \beta$, this parabola is the one with greatest curvature. We parametrize the parabola by $c(t) = (t, \frac{t^2}{2p\alpha^2})$ and compute the curvature using the expression $\kappa(t) = \frac{\|c'(t)\times c''(t)\|}{\|c'(t)\|^3}$ (see, for example, [3]) to obtain

$$\kappa(t) = \frac{\alpha^4p^2}{\left(\sqrt{\alpha^4p^2+t^2}\right)^3}.$$

The curvature attains its maximum at $t = 0$, this is, the vertex of the parabola, so the maximal principal curvature is $\kappa(0) = \frac{1}{p\alpha^2}$. Assuming, without loss of generality, that $\gamma_1 \le \gamma_2 \le \gamma_3$, an analogous computation shows that the minimum principal curvature for the ellipsoid is $\frac{\gamma_1}{\gamma\gamma_3^2}$, which is attained at the co-vertices of the ellipse at the xz-plane. Now, one can check directly that $\frac{\gamma_1}{\gamma\gamma_3^2} \ge \frac{1}{p\alpha^2}$ to see that no principal curvature in the ellipsoid is smaller than any principal curvature of the paraboloid and thus check if the smallness condition is satisfied as in Table 3. When implementing this condition, since all involved numbers are positive, one may prefer to avoid using square roots for computational reasons. Thus the smallness condition can be checked simply by verifying that

$$\gamma_1^2p^2\alpha^4 \ge \gamma^2\gamma_3^4.$$

Note that γ^2 can be computed more directly as $\gamma^2 = -\frac{\det(E)}{\det(a_{ij})}$, where (a_{ij}) is the matrix of quadratic terms of the ellipsoid (see (2)).

Example 6. *We consider the elliptic paraboloid of equation*

$$\mathcal{P} : x^2 + y^2 + z^2 + 2xz - 18\sqrt{2}x + 2y + 18\sqrt{2}z + 1 = 0$$

Ellipsoid with associated matrix E	Elliptic paraboloid with associated matrix P
Eigenvalues: $\frac{1}{\gamma_1^2}, \frac{1}{\gamma_2^2}, \frac{1}{\gamma_3^2}$ with $\gamma_1 \leq \gamma_2 \leq \gamma_3$. $\gamma^2 = -\frac{\det(E)}{\det(a_{ij})}$	Eigenvalues: $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, 0$ with $\alpha \leq \beta$. $p^2 = -\alpha^2\beta^2 \det(P)$
Smallness condition: $\boldsymbol{\gamma_1^2 p^2 \alpha^4 \geq \gamma^2 \gamma_3^4}$	

Table 3. The smallness condition for general ellipsoid and elliptic paraboloid.

and the ellipsoid of equation

$$\begin{aligned}\mathcal{E} : 3.98712x^2 + 3.12189y^2 + 4.14287z^2 + 4.00064xy + 1.78427xz \\ -4.02684x + 4.2121yz - 5.99848y - 3.76504z = 0\end{aligned}$$

(see Figure 2). We compute the eigenvalues:

$$\frac{1}{\gamma_1^2} = 7.08762, \frac{1}{\gamma_2^2} = 3.17003, \frac{1}{\gamma_3^2} = 0.994229 \textit{ and } \frac{1}{\alpha^2} = 2.0, \frac{1}{\beta^2} = 1.0,$$

together with the values $\gamma^2 = 2.89045$ *and* $p^2 = 324.0$. *Now, we check that the smallness condition is satisfied by simply verifying that*

$$\gamma_1^2 p^2 \alpha^4 = 11.4284 \geq 2.95815 = \gamma^2 \gamma_3^4.$$

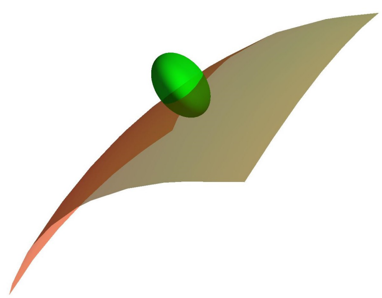


Figure 2. Quadrics in general form that satisfy the smallness condition.

3.2. **Preservation of the smallness condition.** The set of roots of the characteristic polynomial given in (1) is invariant under the action of the affine group. Indeed, for any nonsingular transformation T one has that $\det(\lambda T^t PT + T^t ET) = \det(T)^2 \det(\lambda P + E)$. The smallness condition is preserved by affine transformations that preserve the relation between the shape of the two surfaces, like rigid moves and homotheties (this can be checked directly using the condition in Table 3), however it is not preserved in general by non-uniform scalings or shears. Nevertheless, the following result shows that there exists a transformation of the space so that the ellipsoid becomes a sphere, the relative position of the surfaces is not altered and the smallness condition is also preserved. This will allow a great simplification of the problem of identifying the relative position between the quadrics in the following section.

Theorem 7. *Let $\mathcal{E}$ and $\mathcal{P}$ be an ellipsoid and an elliptic paraboloid satisfying the smallness condition. Then, there exists an affine transformation so that the ellipsoid becomes a sphere, the relative position does not change and the smallness condition is also preserved.*

Proof. Suppose given general ellipsoid $\mathcal{E}$ and elliptic paraboloid $\mathcal{P}$ that satisfy the smallness condition. Since the smallness condition is preserved by rigid moves, we can assume without loss of generality that the ellipsoid is in standard form $\frac{x^2}{\gamma_1^2} + \frac{y^2}{\gamma_2^2} + \frac{z^2}{\gamma_3^2} = \gamma^2$. We apply different scalings to each axis so that the largest principal axes become of the same length as the shortest one, thus the ellipsoid converts into a sphere $\mathcal{S}$ and the elliptic paraboloid remains as an elliptic paraboloid $\bar{\mathcal{P}}$ (see Figure 3). Assuming $\gamma_1 \leq \gamma_2 \leq \gamma_3$, this transformation applies a scaling factor of $\frac{\gamma_2}{\gamma_1}$ in the

OY axis and of $\frac{\gamma_3}{\gamma_1}$ in the OZ axes, so the transformation matrix is $\mathrm{diag}(1, \frac{\gamma_2}{\gamma_1}, \frac{\gamma_3}{\gamma_1})$. We must check that the new surfaces still satisfy the smallness condition. For that purpose we adopt the notation of Equation (3) and Table 3 and we use a bar to denote the parameters of the sphere $\mathcal{S}$ and the converted elliptic paraboloid $\bar{\mathcal{P}}$. Note that the new parameters $\bar{\gamma}_1$, $\bar{\gamma}_2$, $\bar{\gamma}_3$ and $\bar{\gamma}$ of the sphere $\mathcal{S}$ are related with the parameters of $\mathcal{E}$

$$\bar{\gamma}_1 = \bar{\gamma}_2 = \bar{\gamma}_3 = \gamma_1, \quad \bar{\gamma}^2 = -\bar{\gamma}_1^2\bar{\gamma}_2^2\bar{\gamma}_3^2 \det S = -\gamma_1^6 \frac{\gamma_2^2\gamma_3^2}{\gamma_1^4} \det E = -\gamma_1^2\gamma_2^2\gamma_3^2 \det E = \gamma^2. \tag{4}$$

On the other hand, the new parameters $\bar{\alpha}$, $\bar{\beta}$ and $\bar{p}$ of the elliptic paraboloid $\bar{\mathcal{P}}$ are related with the parameters of $\mathcal{P}$ as follows

$$\begin{gathered} \tfrac{1}{\bar{\alpha}^2} \le \tfrac{\gamma_3^2}{\gamma_1^2}\tfrac{1}{\alpha^2} \Rightarrow \bar{\alpha}^2 \ge \tfrac{\gamma_1^2}{\gamma_3^2}\alpha^2, \quad \tfrac{1}{\bar{\alpha}^2}\tfrac{1}{\bar{\beta}^2} \le \tfrac{\gamma_2^2\gamma_3^2}{\gamma_1^4}\tfrac{1}{\alpha^2}\tfrac{1}{\beta^2} \Rightarrow \bar{\alpha}^2\bar{\beta}^2 \ge \tfrac{\gamma_1^4}{\gamma_2^2\gamma_3^2}\alpha^2\beta^2, \\ \bar{p}^2 = -\bar{\alpha}^2\bar{\beta}^2 \det \bar{P} = -\bar{\alpha}^2\bar{\beta}^2 \tfrac{\gamma_2^2\gamma_3^2}{\gamma_1^4} \det P \ge -\tfrac{\gamma_1^4}{\gamma_2^2\gamma_3^2}\alpha^2\beta^2 \tfrac{\gamma_2^2\gamma_3^2}{\gamma_1^4} \det P = -\alpha^2\beta^2 \det P = p^2. \end{gathered} \tag{5}$$

Using the relations of equations (4) and (5) together with the smallness condition for $\mathcal{E}$ and $\mathcal{P}$ $(\gamma_1^2p^2\alpha^4 \ge \gamma^2\gamma_3^4)$ we see that the smallness condition is satisfied for $\mathcal{S}$ and $\bar{\mathcal{P}}$:

$$\bar{\gamma}_1^2\bar{p}^2\bar{\alpha}^4 \ge \gamma_1^2p^2\left(\frac{\gamma_1}{\gamma_3}\alpha\right)^4 \ge \frac{\gamma_1^4}{\gamma_3^4}\gamma^2\gamma_3^4 = \gamma^2\gamma_1^4 = \bar{\gamma}^2\bar{\gamma}_3^4.$$

□

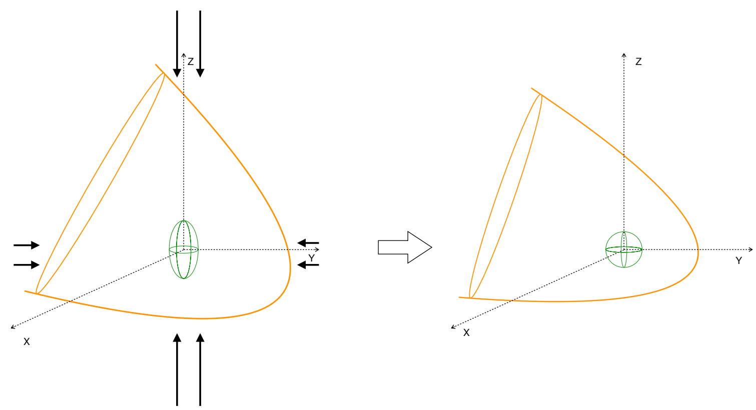


FIGURE 3. An appropriate re-scaling transforms the ellipsoid into a sphere preserving the smallness condition.

3.3. **The necessity of the smallness condition.** The smallness condition is crucial as a hypothesis for Theorem 2. Without this condition, several other relative positions arise (see Figure 4 for an illustrative example) and, moreover, the classification of Theorem 2 and the methods of detection derived from Corollary 4 fail. In particular, contact is not detected by complex roots, as the following example shows.

Example 8. *We consider the elliptic paraboloid $x^2 + \frac{y^2}{16} = z$ and the ellipsoid $\frac{x^2}{4} + y^2 + 4(z-1)^2 = 1$. The minimum principal curvature of the ellipsoid is $\frac{1}{8}$ and the maximum principal curvature of the paraboloid is 2, so the smallness condition is not satisfied. Moreover, the paraboloid and the ellipsoid intersect in two connected components as shown in Figure 4. Applying a translation rising up the ellipsoid with respect to the paraboloid would produce two tangent points between the surfaces.*

A straightforward calculation shows that the characteristic polynomial is given by

$$f(\lambda) = -\frac{1}{256}(\lambda+16)(4\lambda+1)\left(\lambda^2+16\lambda+16\right),$$

so there are 4 different real roots: $\lambda_1 = -16$, $\lambda_2 = -\frac{1}{4}$, $\lambda_3 = -4(2+\sqrt{3})$, $\lambda_4 = 4(-2+\sqrt{3})$. Thus, this example evidences that Theorem 2 fails if the smallness condition is not satisfied.

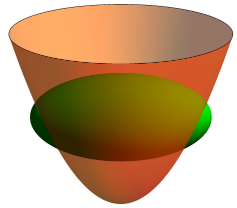


FIGURE 4. The smallness condition is not satisfied for the paraboloid $x^2 + \frac{y^2}{16} = z$ and the ellipsoid $\frac{x^2}{4} + y^2 + 4(z-1)^2 = 1$, which intersect in two connected components.

4. The characteristic polynomial

Since the roots of the characteristic polynomial given in (1) are invariant by affine transformations, we can apply Theorem 7 to simplify the analysis of the relative positions. Thus, there exists an affine map that transforms the ellipsoid into a sphere without changing the set of roots of f, the relative position between the quadrics and the validity of the smallness condition. Therefore, in what follows, instead of working with a general ellipsoid, we work with a sphere $\mathcal{S}$ of radius $r > 0$ and center at (x_c, y_c, z_c):

$$\mathcal{S} : (x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2 = r^2 \,. \tag{6}$$

In homogeneous coordinates $X = (x, y, z, 1)^t$, $\mathcal{S}$ is given by the equation $X^t S X = 0$ where

$$S = \begin{pmatrix} 1 & 0 & 0 & -x_c \\ 0 & 1 & 0 & -y_c \\ 0 & 0 & 1 & -z_c \\ -x_c & -y_c & -z_c & -r^2 + x_c^2 + y_c^2 + z_c^2 \end{pmatrix}.$$

Note that it would be possible to transform the sphere into one of radius 1 too, but it does not carry a strong simplification and we prefer to work with a general radius r to emphasize the role played by the radius along the subsequent arguments. After converting the ellipsoid $\mathcal{E}$ into a sphere $\mathcal{S}$ by a transformation that also affects the paraboloid $\mathcal{P}$, we can apply an appropriate rotation and a translation to the new elliptic paraboloid so that it is given in standard form (see [18]):

$$\mathcal{P} : \frac{x^2}{a^2} + \frac{y^2}{b^2} - z = 0 \;\text{ for } 0 < a \leq b. \tag{7}$$

The 4×4 matrix associated to $\mathcal{P}$ has the form

$$P = \begin{pmatrix} a^{-2} & 0 & 0 & 0 \\ 0 & b^{-2} & 0 & 0 \\ 0 & 0 & 0 & -1/2 \\ 0 & 0 & -1/2 & 0 \end{pmatrix}.$$

We emphasize that this process does not carry a loss of generality since the relative position between the two quadrics and the characteristic roots remain unchanged. Therefore, in what follows, we work with a sphere $\mathcal{S}$ of radius r and center (x_c, y_c, z_c) as in (6) and an elliptic paraboloid $\mathcal{P}$ given in standard form as in (7).

For $\mathcal{S}$ and $\mathcal{P}$ we compute the characteristic polynomial explicitly to obtain:

$$\begin{aligned} f(\lambda) \;\; = \;\; & -\tfrac{1}{4a^2b^2}\left\{\lambda^4 + (4z_c + a^2 + b^2)\lambda^3 \right. \\ & +(4z_c(a^2+b^2) - 4(x_c^2 + y_c^2 - r^2) + a^2b^2)\lambda^2 \\ & \left. +4(z_c a^2 b^2 - y_c^2 a^2 - x_c^2 b^2 + r^2(a^2+b^2))\lambda\right\} - r^2. \end{aligned} \tag{8}$$

The following is a general remark which will be crucial in the subsequent analysis.

Lemma 9. *The characteristic roots of f satisfy:*

(1) 0 *is not a root.*
(2) *The product of all roots is* $4a^2b^2r^2 > 0$.

Proof. Substituting in the expression of the characteristic polynomial, we see that $f(0) = -r^2 \neq 0$, so 0 is not a root of f. From expression (8), we transform f to a monic polynomial multiplying by $-4a^2b^2$ to see that the independent coefficient, which equals the product of the roots, is $4a^2b^2r^2 > 0$. □

Remark 10. For a sphere $\mathcal{S}$ of radius r as in (6) and an elliptic paraboloid as in (7), considering all possible sections, the smallness condition means that the circle cannot be tangent at two points to the parabola (see Figure 1(c)). Equivalently, the curvature of the circle is greater than or equal to the curvature of the parabola at any point. Since the curvature of the circle at any point is $\frac{1}{r}$ and the maximum curvature of the parabola is $\frac{2}{a^2}$, the smallness condition reduces to: $2r \leq a^2$.

5. Relative positions when the center of $\mathcal{S}$ is located at the OZ-axis.

Along this section we analyze the case $a \neq b$, as the case $a = b$ will be obtained as a consequence of this one. We adopt the notation established in Section 4 and assume the smallness hypothesis.

As a first step in our approach to classify the relative positions of $\mathcal{S}$ and $\mathcal{P}$ in relation with the roots of f, we are going to consider the particular case in which the center (x_c, y_c, z_c) of $\mathcal{S}$ is located in the OZ-axis. This location can be detected in terms of the roots of f as follows.

Lemma 11. *Assume $a \neq b$. Then $x_c = y_c = 0$ if and only if $-a^2$ and $-b^2$ are roots of f.*

Proof. For $a \neq b$, substitute using expression (8): $f(-a^2) = x_c^2(-b^2 + a^2)/b^2$ and $f(-b^2) = -y_c^2(-b^2 + a^2)/a^2$ and the result follows. □

In virtue of Lemma 11, the center of $\mathcal{S}$ is at the OZ-axis implies that $\lambda_1 = -a^2$ and $\lambda_2 = -b^2$ are characteristic roots. In this case, $f(\lambda) = -4(a^{-2}\lambda + 1)(b^{-2}\lambda + 1)h(\lambda)$ with $h(\lambda) = \lambda^2 + 4z_c\lambda + 4r^2$. Then, denote by $\lambda_3 \leq \lambda_4$ the other two roots and observe that

$$\lambda_3 = -2(z_c + \sqrt{z_c^2 - r^2}) \text{ and } \lambda_4 = -2(z_c - \sqrt{z_c^2 - r^2}). \tag{9}$$

Note that for $i = 3, 4$

$$\frac{1}{2}\frac{d\lambda_i}{dz_c} = -1 + (-1)^i \frac{z_c}{\sqrt{z_c^2 - r^2}}.$$

Since $|z_c/\sqrt{z_c^2 - r^2}| > 1$, the value of λ_3 decreases and λ_4 increases (both strictly) as the center of $\mathcal{S}$ ascends through the OZ-axis for $z_c > r$. Whereas λ_3 increases and λ_4 decreases as the center of $\mathcal{S}$ ascends through the OZ-axis when $z_c < -r$.

The relative positions between $\mathcal{S}$ and $\mathcal{P}$ are summarized in the following lemma.

Lemma 12. *Assume $a \neq b$. Let $\mathcal{S}$ be a sphere centered at $(0, 0, z_c)$ and $\mathcal{P}$ a standard elliptic paraboloid, then $\lambda_1 = -a^2$, $\lambda_2 = -b^2$ and all its relative positions can be characterized as follows:*

(i) *$\mathcal{S}$ is interior to $\mathcal{P}$ (Type I) if and only if $\lambda_3 < \lambda_4 < 0$ ($\lambda_4 \neq -a^2, -b^2$).*
(ii) *$\mathcal{S}$ is tangent from inside to $\mathcal{P}$ (Type TI) if and only if $-a^2 \leq \lambda_3 = -2r = \lambda_4 < 0$.*
(iii) *$\mathcal{S}$ and $\mathcal{P}$ are in non-tangent contact (Type C) if and only if $\overline{\lambda}_3 = \lambda_4 \in \mathbb{C} - \mathbb{R}$.*
(iv) *$\mathcal{S}$ is tangent from outside to $\mathcal{P}$ (Type TE) if and only if $0 < \lambda_3 = 2r = \lambda_4 \leq a^2$.*
(v) *$\mathcal{S}$ is exterior to $\mathcal{P}$ (Type E) if and only if $0 < \lambda_3 < 2r < \lambda_4$.*

Proof. Non tangent contact between the surfaces is characterized by the condition $|z_c| < r$ and this is just the case when the roots λ_3 and λ_4 are complex numbers, $\overline{\lambda}_3 = \lambda_4$ as a direct consequence of (9). This shows (iii).

Due to the smallness hypothesis, the sole possible point of tangency is $(0, 0, 0)$ so $\mathcal{S}$ and $\mathcal{P}$ are tangent if and only if $|z_c| = r$. In this cases the discriminant $\sqrt{z_c^2 - r^2}$ in (9) vanishes and thus $\lambda_3 = \lambda_4$. Furthermore, this root is $\lambda_3 = \lambda_4 = -2r$ if $z_c = r$ and $\lambda_3 = \lambda_4 = 2r$ if $z_c = -r$. This shows (ii) and (iv).

As the center of the sphere is at the OZ-axis, $\mathcal{S}$ is interior to $\mathcal{P}$ if and only if $r < z_c$ whereas $\mathcal{S}$ is exterior to $\mathcal{P}$ if and only if $z_c < -r$. In both cases the term $\sqrt{z_c^2 - r^2}$ is positive, so $\lambda_3 \neq \lambda_4$. Moreover, if $\mathcal{S}$ is interior to $\mathcal{P}$ then from (9) we have $\lambda_3 < \lambda_4 < 0$. Similarly, if $\mathcal{S}$ is exterior to $\mathcal{P}$ then $0 < \lambda_3 < \lambda_4$.

To show that $\lambda_4 \neq -a^2, -b^2$ in the interior case, note that if $z_c = r$, then $-b^2 < -a^2 \leq -2r = \lambda_4$, but λ_4 is an strictly increasing function on z_c for $z_c > r$. So $-b^2 < -a^2 \leq -2r < \lambda_4$ for $z_c > r$, this is, when $\mathcal{S}$ is interior to $\mathcal{P}$. □

Note that the case $2r = -a^2$ is a special one, where interior tangency is characterized by a triple root $-a^2$. This is precisely the case in which the maximum curvature of the vertical parabola in the plane $y = 0$ equals the curvature of a maximum circle of the sphere. A double root $-a^2$ does not correspond to a tangent position, but to the interior case (Type I) in this instance.

6. Characterization of the relative positions

In this section we explore the relation between relative positions and characteristic roots when $a \neq b$. Again, we assume the smallness hypothesis and use previous notation. We begin by relating the tangent situation with multiple roots, then we distinguish the interior and exterior cases in terms of the sign of real roots and, finally, we associate non-tangent contact with complex roots.

6.1. **Tangency points and multiple roots.** A key point in our global analysis is the relation between tangency and multiplicity of roots. Generally speaking, tangency is detected by multiple roots of the characteristic polynomial. However, there exist two multiple roots, namely $-a^2$ and $-b^2$, that can be double and are not associated to a tangent position between the surfaces. The following results express this fact. In this section we are going to carefully analyze the relation between the multiple roots and the existence of a tangent point between $\mathcal{S}$ and $\mathcal{P}$. First note that if the quadrics are tangent then there exists a multiple root. The proof of the following two results is analogous to those given in [2] (see Lemma 26 and Lemma 25, respectively), so we omit them in the interest of brevity.

Lemma 13. *If $\mathcal{P}$ and $\mathcal{S}$ are tangent, then there exists a multiple real root of the characteristic polynomial.*

A partial converse of Lemma 13 is the following.

Lemma 14. *Let $\lambda \notin \{-a^2, -b^2\}$ be a real root of $f(\lambda)$. If the multiplicity of λ is $m \geq 2$, then there exists at least one point where $\mathcal{P}$ and $\mathcal{S}$ are tangent.*

Remark 15. The roots $-a^2$ and $-b^2$ are indeed special in Lemma 14. The following example shows how $-a^2$ can be a double root and there is no tangency between $\mathcal{S}$ and $\mathcal{P}$ (see Figure 5):

$$\mathcal{P} : \tfrac{x^2}{1.2^2} + \tfrac{y^2}{1.5^2} - z = 0, \qquad \mathcal{S} : x^2 + (y - 0.5)^2 + (z - 0.712045)^2 = 0.25^2.$$

$$\textbf{Roots}: \quad \lambda_1 = -3.54808, \quad \lambda_2 = \lambda_3 = -1.44, \quad \lambda_4 = -0.110095.$$

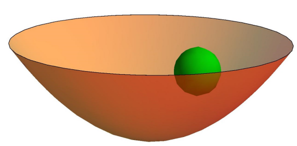


FIGURE 5. A double root $-a^2$ does not imply that $\mathcal{S}$ and $\mathcal{P}$ are tangent.

The following result summarizes the relation between tangency and multiple roots. We leave the proof for the Appendix so that we do not break the flow of the global argument.

Theorem 16. Let $a \neq b$. Then $\mathcal{P}$ and $\mathcal{S}$ are tangent if and only if one of the following possibilities holds:

(1) $-a^2$ is a triple root.
(2) $\lambda \in \mathbb{R}\backslash\{-a^2, -b^2\}$ is a multiple root.

6.2. **The interior and exterior cases.** We have already proved the characterization of relative positions when the center of $\mathcal{S}$ is in the OZ-axis (in the case $a \neq b$). To extend it to the remaining space we are going to move the center of the sphere along a path from any point to an appropriate point in the OZ-axis. Denote this path by $\alpha(t)$, with $t \in [t_0, t_1]$. Define $f_t(\lambda) = \det(\lambda P + S(t))$ as the characteristic polynomial f for $\mathcal{S}$ centered at $\alpha(t)$. We follow an analogous argument to that in [24] to prove the following result.

Lemma 17. *Let $p(t) = \alpha_n(t)\lambda^n + \alpha_{n-1}(t)\lambda^{n-1} + \cdots + \alpha_1(t)\lambda + \alpha_0(t)$ be a polynomial of degree n whose coefficients depend continuously on a parameter t. Assume $p(t_0)$ has n distinct real roots and that $p(t_1)$ has some complex (non real) root. Then there exists $t_d \in (t_0, t_1)$ so that $p(t_d)$ has a multiple real root.*

Proof. Let $t_d = \mathrm{Inf}\{t \in [t_0, t_1]/p(t)$ has a non real root$\}$. Note that $t_d \in (t_0, t_1)$. There exist $\varepsilon > 0$ small enough so that $p(t_d - \varepsilon)$ has real roots and $p(t_d + \varepsilon)$ has some non real root. For ε small enough, $p(t)$ factorizes as

$$p(t) = (\alpha_2(t)\lambda^2 + \alpha_1(t)\lambda + \alpha_0(t))s(t) \text{ in } [t_d - \varepsilon, t_d + \varepsilon],$$

where the second degree polynomial $\alpha_2(t)\lambda^2 + \alpha_1(t)\lambda + \alpha_0(t)$ has complex conjugate roots for $t = t_d + \varepsilon$. The functions α_i are continuous functions for $i = 0, 1, 2$, so the discriminant $d(t) = \alpha_1(t)^2 - 4\alpha_0(t)\alpha_2(t)$ is also a continuous function. Since $d(t_d - \varepsilon) > 0$ and $d(t_d + \varepsilon) < 0$, it follows that $d(t_d) = 0$, so there is a double root for $\alpha_2(t)\lambda^2 + \alpha_1(t)\lambda + \alpha_0(t)$ and a root of multiplicity at least two for $p(t_d)$. □

Lemma 18. *Assume $a \neq b$. Suppose that there is no contact between $\mathcal{P}$ and $\mathcal{S}$. Then f has four real roots, all of them with multiplicity 1, except $-a^2$ and $-b^2$ that could possibly have multiplicity 2. Moreover:*

(1) *If $\mathcal{S}$ is interior to $\mathcal{P}$, then all the roots are negative.*
(2) *If $\mathcal{S}$ is exterior to $\mathcal{P}$, two roots are positive and two are negative and different.*

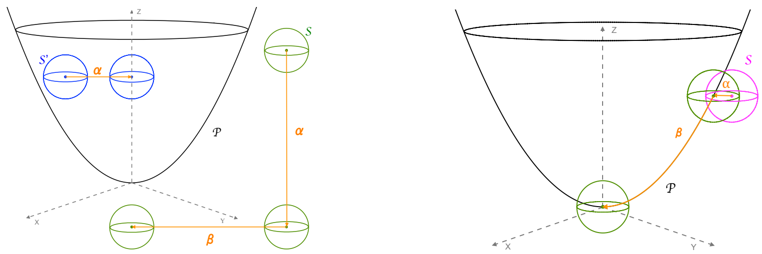
Proof. First, let us consider a sphere $\mathcal{S}$ which is interior to the paraboloid. If the center of $\mathcal{S}$ is at the OZ-axis, because of Lemma 12, the roots are $\lambda_1 = -a^2$, $\lambda_2 = -b^2$ and $\lambda_3 < \lambda_4 < 0$. If $x_c \neq 0$ and $y_c \neq 0$, we can construct the path $\alpha(t) = ((1-t)x_c, (1-t)y_c, z_c)$ with $t \in [0, 1]$ (see Figure 6 (a)). This path does not intersect the planes $x = 0$ or $y = 0$, so $-a^2$ and $-b^2$ are not roots of $f_t(\lambda)$ for $t \in [0, 1)$ (see Appendix). Since $\mathcal{S}$ is not tangent to $\mathcal{P}$ at any point of the path, there are no multiple roots in virtue of Lemma 14. Moreover, since the roots in $\alpha(1)$ are all real and negative, since 0 is not a root (see Lemma 9), and since Lemma 17 does not allow complex roots, the roots of $f_0(\lambda)$ are also real and negative.

Assume $x_c = 0$ and $y_c \neq 0$. Then $-a^2$ is a root. Consider the path $\alpha(t) = (0, (1-t)y_c, z_c)$ and note that α is contained in the plane $x = 0$, so $-a^2$ is always a root when the center of the sphere moves along α. Therefore, the characteristic polynomial decomposes as $f(\lambda) = (\lambda + a^2)g(\lambda)$. We use Lemma 9, Lemma 12 and Lemma 17 as before, together with Theorem 16 to conclude that all the roots are negative in $\alpha(1)$. Note that $g(\lambda)$ does not have any double root, but $-a^2$ could be a root of $g(\lambda)$ ($-a^2$ and $-b^2$ are the only possible double roots without an associated tangency, see Theorem 16). In that case $\mathcal{S}$ is interior to $\mathcal{P}$ and $-a^2$ is a double root of f. The argument is similar if $y_c = 0$ and $x_c \neq 0$. Thus assertion (1) follows.

To prove assertion (2) we construct a path so that the sphere moves without intersecting the paraboloid. For example, for a general $\mathcal{S}$ and $\mathcal{P}$, the sum of the paths $\alpha(t) = (x_c, y_c, z_c - t(|z_c| + 2r))$, $t \in [0, 1]$, and $\beta(t) = ((1-t)x_c, (1-t)y_c, z_c - |z_c| - 2r)$, $t \in [0, 1]$, ensures that there is not contact between the surfaces and the center at the end of the path is located in the negative part of the OZ-axis (see Figure 6 (a)). Now, a similar argument to that given in assertion (1) applies to prove assertion (2). □

6.3. **The non-tangent contact case.**

Lemma 19. *Assume $a \neq b$. If $\mathcal{S}$ and $\mathcal{P}$ are in non-tangent contact, then f has a pair of complex conjugate roots and two different real roots.*



(a) (b)

FIGURE 6. Paths translating the center of the sphere from the initial position to the OZ-axis with the same relative position all along the path.

Proof. Assume $\mathcal{S}$ and $\mathcal{P}$ are in non-tangent contact. Then we can build a path from (x_c, y_c, z_c) to $(0,0,0)$ to move the center of $\mathcal{S}$ along it in such a way that all along the path $\mathcal{S}$ and $\mathcal{P}$ are in non-tangent contact. We do it in two steps as follows. Let (x_1, y_1, z_c) be the intersection point between the horizontal half-ray starting at the OZ-axis trough (x_c, y_c, z_c). We consider the horizontal path $\alpha(t) = ((1-t)x_c + tx_1, (1-t)y_c + ty_1, z_c)$, $t \in [0,1]$, which joins (x_c, y_c, z_c) and (x_1, y_1, z_c). Now consider the arch of parabola from (x_1, y_1, z_c) to $(0,0,0)$ obtained by intersecting $\mathcal{P}$ with the vertical plane which contains these two given points and parametrized it as the curve β. The sum of α and β provides a path from (x_c, y_c, z_c) to $(0,0,0)$ (see Figure 6 (b)).

By Lemma 12, the characteristic polynomial has a pair of complex conjugate roots when the center of $\mathcal{S}$ is at $(0,0,0)$. Assume $x_c \neq 0 \neq y_c$, so $-a^2$ and $-b^2$ are not roots. Since there are no tangent points along the path we have built, there are no roots with multiplicity greater than 1 (see Lemma 14), and hence there are a pair of complex roots when the center of $\mathcal{S}$ is at (x_c, y_c, z_c) as a consequence of Lemma 17. If $x_c = 0$, then $-a^2$ is a root (see Lemma 21) and the images of α and β belong to the plane $x_c = 0$. Hence we decompose $f(\lambda) = (\lambda + a^2)h(\lambda)$ and argue as before with the polynomial h to conclude the result. □

7. THE PROOFS OF THEOREM 2 AND COROLLARY 5

Proof of Theorem 2. This section is devoted to prove the classification result of Table 1. We analyze the cases $a \neq b$ and $a = b$ separately given their different behavior.

7.1. The strictly elliptic case: $a \neq b$.

We begin with a general ellipsoid and an elliptic paraboloid in a certain relative position. Since the characteristic roots of $\mathcal{E}$ and $\mathcal{P}$ are invariant under affine transformations, we apply the corresponding homotheties and rigid moves to the space so that the ellipsoid becomes a sphere and the elliptic paraboloid is in standard form (see Section 4). We assume first $a \neq b$ and use previous results to analyze the different relative positions as follows.

- *I* *$\mathcal{S}$ is interior to $\mathcal{P}$:* as a direct consequence of Lemma 18, if $\mathcal{S}$ is interior to $\mathcal{P}$ then there are four real negative roots and either all of them are different or some of $-a^2$ and $-b^2$ are double roots (see Appendix for the $-a^2$ and $-b^2$ exceptional cases).
- *E* *$\mathcal{S}$ is exterior to $\mathcal{P}$:* if $\mathcal{S}$ is exterior to $\mathcal{P}$ then, by Lemma 18, there are two real positive and two real negative simple roots.
- *C* *Non-tangent contact between $\mathcal{S}$ and $\mathcal{P}$:* in this case Lemma 19 shows that there are two complex conjugates roots. Note that the argument in Lemma 19 also shows that the other two roots are real.
- *TI* *$\mathcal{S}$ is tangent to $\mathcal{P}$ from inside:* the tangent cases are associated with a multiple root by Theorem 16. Moreover, tangent relative positions can be obtained by moving a sphere continuously from an interior or an exterior position as in Figure 7. Since the roots are continuous functions on the coefficients of the characteristic polynomial, the interior tangency results in four real negative roots, at least one of which has multiplicity strictly greater than 1.

TE *$\mathcal{S}$ is tangent to $\mathcal{P}$ from outside:* the same argument of the interior tangency applies. Moving the sphere continuously from an exterior position to the tangent position as in Figure 7, exterior tangency results in two negative roots and one double root which is positive.

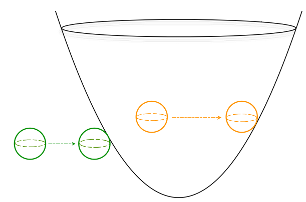


FIGURE 7. Tangencies are obtained as the limit of the exterior and interior positions.

7.2. **The circular paraboloid case: $a = b$.** To finish the proof of Theorem 2 it remains to consider the case $a = b$. If $a = b$ the paraboloid is circular and is an exceptional case in the analysis developed in Sections 4 and 6. Situations like this, where the quadric is a surface of revolution, were considered previously in the literature (see [2]). When we work with a circular paraboloid and a sphere, $-a^2$ is always a root of the characteristic polynomial. Furthermore, due to the rotational symmetry, the problem can be reduced in one dimension by considering the intersection of the plane that contains the axis of the paraboloid and the center of the sphere. Hence, $f(\lambda) = (\lambda + a^2)h(\lambda)$, so one needs to study the third degree polynomial h and the relative positions between a circle and a parabola.

However, the circular case is obtained using continuity of the roots of the characteristic polynomial from a continuous deformation of an elliptic paraboloid with $a \neq b$. Since we already have the classification of the relative position if $a \neq b$, we opt for a direct approach based on the continuity of the roots of f (see [13] for a complete reference). Thus, consider an ϵ-elliptic paraboloid of the form

$$\frac{x^2}{a^2} + \frac{y^2}{(a+\epsilon)^2} - z = 0, \text{ with } \epsilon > 0.$$

For a given relative position of the circular paraboloid, it can be obtained as a limit of an ϵ-elliptic paraboloid when $\epsilon \to 0$.

I and TI *$\mathcal{S}$ is interior or tangent from inside to $\mathcal{P}$:* since 0 is never a root by Lemma 9, a position where $\mathcal{S}$ is interior to $\mathcal{P}$ and $\mathcal{S}$ is tangent to $\mathcal{P}$ from inside has roots $\lambda_3 \leq \lambda_4 < 0$ for any ϵ-elliptic paraboloid, so in the limit, when $\epsilon \to 0$, also has $\lambda_3 \leq \lambda_4 < 0$.

E *$\mathcal{S}$ is exterior to $\mathcal{P}$:* if $\mathcal{S}$ is exterior to $\mathcal{P}$, then $0 < \lambda_3 < \lambda_4$ for any ϵ-elliptic paraboloid, so in the limit we have $0 < \lambda_3 \leq \lambda_4$ because 0 is not a root.

C *Non-tangent contact between $\mathcal{S}$ and $\mathcal{P}$:* The fact that a position of contact between $\mathcal{S}$ and $\mathcal{P}$ provides complex roots for f is obtained by redoing the argument of Lemma 19 for the circular paraboloid, taking into account that $-a^2$ is always a root and working with a third degree polynomial (see also [2] for an analogous situation with a circular hyperboloid).

TE *$\mathcal{S}$ is tangent to $\mathcal{P}$ from outside:* for any ϵ-elliptic paraboloid, that $\mathcal{S}$ is tangent from outside to $\mathcal{P}$ is characterized by a positive double root $\lambda_3 = \lambda_4 > 0$, so when $\epsilon \to 0$ this will happen too, because 0 is not a root. Now, Lemma 14 excludes the case $\lambda_3 = \lambda_4$ which corresponds to tangent contact also if $a = b$.

In summary, the classification given in Theorem 2 is also valid when $a = b$. This completes the proof of Theorem 2. □

Proof of Corollary 5. Note that the smallness assumption does not play a role when the ellipsoid is exterior to the elliptic paraboloid, since the surfaces cannot be tangent at two points simultaneously. By Theorem 16, the tangent position when the ellipsoid approaches from outside is characterized by a multiple root $\lambda \in \mathbb{R}\backslash\{-a^2, -b^2\}$, since multiple roots $-a^2$, $-b^2$ correspond to interior positions (see the proof of Theorem 16). Hence the same argument given to prove Lemma 18(2), where we build an exterior path as in Figure 6(a), works through and the exterior position is characterized by two positive roots that become a double root in case of tangency.

8. Implementation and examples

Based on the results of Section 2, the data needed to analyze the positions is small enough for the implementation of efficient algorithms to be used in real-time continuous moving frameworks. For example, as an immediate application of Corollaries 4 and 5, an algorithm to detect the relative position of an ellipsoid that moves continuously from the exterior of the elliptic paraboloid can be based only in the discriminant computation as follows:

- If $\Delta < 0$ then non-tangent contact (Type C),
  - else if $\Delta > 0$ then exterior (Type E),
    else tangent contact (Type TE).

Analysis in the development of algorithms should also take care of the efficiency in the implementation and can follow the line of techniques already used to detect the positional relationship between conics or ellipsoids, we refer to [1, 8] and references therein.

Following the scheme in Table 1, the proposed approach to characterize the relative position between an elliptic paraboloid and a small ellipsoid consists of the analysis of the real character and the sign of real roots of the characteristic polynomial of a 4×4 matrix. This direct approach requires computing and solving a fourth degree polynomial and, in computational terms, is similar to the computation of the eigenvalues of such a matrix.

Instead, we have presented an alternative method based on a simple algebraic condition over the coefficients of the fourth degree polynomial (see Table 2), as in the previous algorithm. The computational cost in this case is extremely low, requiring only a few floating point basic operations. Here, the most costly operation is the computation of the discriminant: the number of arithmetic operations (additions, subtractions, multiplications and divisions) to obtain Δ is 78. To check the smallness condition we use Table 3, which requires the explicit computation of γ and p. That means 35 and 34 additional arithmetic operations, respectively, since one needs to compute the determinants of the associated matrices E and P. In summary, the method given in Table 2 is fast, easy and has a very low computational cost, what makes it suitable for practical real world applications, especially for continuous collision detection problems, where one only needs to verify the smallness condition once at the beginning and then just check whether there is collision between the two bodies. Moreover, if the ellipsoid is outside the elliptic paraboloid, there is no need to check the smallness condition as shown in Corollary 5. The following examples illustrate the preceding analysis.

Example 20. *We consider the quadrics given in Example 6. We let the paraboloid $\mathcal{P}$ be static and translate the ellipsoid $\mathcal{E}$ by the vector $(t, 0, 0)$ with t varying in the interval $[-6, 6]$ (see Figure 8). The smallness condition was already checked in Example 6, so it does not need to be checked again. We use Theorem 2 to detect the relative position in terms of the characteristic roots: there are two positive real roots in the exterior case (E), complex roots in the contact case (C) and 4 negative roots in the interior case (I). Alternatively, one can use Corollary 4 to obtain the same conclusion with a much lower computational cost using the discriminant of the characteristic polynomial Δ (see [7, 27]) and the sign of the coefficients c_i for $i = 1, 2, 3$: Δ is negative in the contact case, Δ is positive and f has some positive coefficient in the exterior case and all negative coefficients in the interior case. We work with a step of 0.01 to estimate the intervals corresponding to each of the positions. Thus, we denote by Roots(t) and $\Delta(t)$ the set of roots and the discriminant of f_t, respectively, for the value t.*

We first see that the $\mathcal{E}$ is interior to $\mathcal{P}$ for $t \in [-6, -1.53]$:

$f_{-6}(\lambda) = -648.00\lambda^4 - 4575.8\lambda^3 - 6827.27\lambda^2 - 2670.51\lambda - 64.5678,$

$\mathit{Roots}(-6) = \{-5.1805, -1.26882, -0.586239, -0.0258579\}, \quad \Delta(-6) = 1.43617 \times 10^{20}.$

$f_{-1.53}(\lambda) = -648.00\lambda^4 - 3871.46\lambda^3 - 3547.47\lambda^2 - 876.287\lambda - 64.5678,$

$\mathit{Roots}(-1.53) = \{-4.91598, -0.722149, -0.182814, -0.153531\}, \quad \Delta(-1.53) = 5.33623 \times 10^{16}.$

We check that $\mathcal{E}$ and $\mathcal{P}$ are in contact for $t \in [-1.52, 1.24]$:

$f_{-1.52}(\lambda) = -648.00\lambda^4 - 3869.88\lambda^3 - 3539.43\lambda^2 - 871.272\lambda - 64.5678$

$\mathit{Roots}(-1.52) = \{-4.9157, -0.722174, -0.167087 \pm 0.012256i\}, \quad \Delta(-1.52) = -3.78015 \times 10^{16}.$

$f_{1.24}(\lambda) = -648.00\lambda^4 - 3434.99\lambda^3 - 1199.72\lambda^2 + 683.562\lambda - 64.5678,$

$\mathit{Roots}(1.24) = \{-4.87598, -0.753568, 0.164319 \pm 0.0108246i\}, \quad \Delta(1.24) = -2.70235 \times 10^{17}.$

And, finally, $\mathcal{E}$ is exterior to $\mathcal{P}$ for $t \in [1.25, 6]$:
$f_{1.25}(\lambda) = -648.00\lambda^4 - 3433.41\lambda^3 - 1190.8\lambda^2 + 689.814\lambda - 64.5678,$
$Roots(1.25) = \{-4.87596, -0.753711, 0.147999, 0.183196\}, \quad \Delta(1.25) = 7.1864 \times 10^{17}.$
$f_6(\lambda) = -648.00\lambda^4 - 2684.95\lambda^3 + 3398.22\lambda^2 + 4164.69\lambda - 64.5678,$
$Roots(6) = \{-4.94075, -0.821338, 0.0153146, 1.60333\}, \quad \Delta(6) = 1.37148 \times 10^{22}.$
Note that the previous calculations provide a estimation of the interior tangent position, which is attained for $t \in [-1.53, -1.52]$, and for the exterior tangent position, which is attained for $t \in [1.24, 1.25]$.

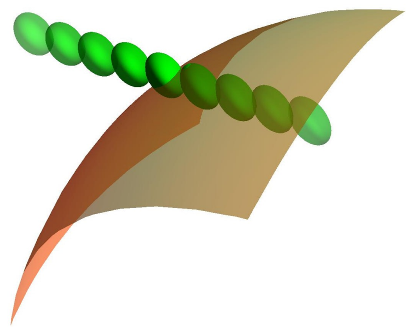


FIGURE 8. An ellipsoid moves changing its relative position with respect to an elliptic paraboloid.

9. Conclusions

We have shown that the roots of a characteristic polynomial of degree four are suitable to detect the relative position between an ellipsoid and an elliptic paraboloid if the ellipsoid is small in comparison with the elliptic paraboloid. Table 1 summarizes the five possible relative positions in terms of the roots. This classification sets the theoretical framework to develop efficient algorithms based on the analysis of the coefficients of the characteristic polynomial as in Table 2. Thus, contact detection between the two quadrics is simple enough to be applied in a continuous time-varying positional context.

The exterior case is especially interesting, as the relative positions are directly detected as follows: the surfaces are not in contact if there are two different positive roots, they are tangent if there is a double positive root and they are in non-tangent contact if there are complex conjugate roots. This classification agrees with that given for two ellipsoids or a sphere and a circular hyperboloid (see [2, 24]). The interior case is a bit more subtle from a theoretical point of view, since a double root is not necessarily associated to a tangent position. However, the computational implementation will lead to analogous results as the the previous case (see Remark 15).

10. Appendix

We complete the mathematical proofs of the results of the paper with the proof of Theorem 16. We have seen in Section 6 that the roots $-a^2$ and $-b^2$ are exceptional. We analyze their possible multiplicities as follows.

Lemma 21. *Assume $a \neq b$. Then:*

(i) *$-a^2$ is a root if and only if $x_c = 0$.*

(ii) *If $-a^2$ is a root, then it has multiplicity at least 2 if and only if* $r^2 = a^2 z_c - \dfrac{a^2}{b^2 - a^2} y_c^2 - \dfrac{a^4}{4}$.

(iii) *$-a^2$ has multiplicity 3 if and only if $x_c = y_c = 0$ and $z_c = r = \frac{a^2}{2}$.*

Proof. First, substitute using expression (8): $f(-a^2) = x_c^2(-b^2 + a^2)/b^2$ to check that $f(-a^2) = 0$ if and only if $x_c = 0$. This proves assertion (i).

Assume $x_c = 0$. Then f decomposes as $f(\lambda) = (a^{-2}\lambda + 1)g(\lambda)$ where $g(\lambda) = -(b^{-2}\lambda + 1)\left(r^2 + \lambda^2/4 + z_c\lambda\right) + b^{-2}\lambda y_c^2$. Thus $g(-a^2) = 0$ if and only if $r^2 = a^2 z_c - a^4/4 - a^2 y_c^2/(b^2 - a^2)$, so assertion (ii) follows.

Assume $x_c = 0$ and $r^2 = a^2 z_c - a^2/(b^2 - a^2)y_c^2 - a^4/4$. Then f decomposes as $f(\lambda) = (a^{-2}\lambda + 1)^2 h(\lambda)$ where

$$h(\lambda) = \lambda^2 + (4z_c + b^2 - a^2)\lambda + 4b^2 z_c - a^2 b^2 - \frac{4y_c^2 b^2}{b^2 - a^2}.$$

We have that, $h(-a^2) = 0$ if and only if $z_c = a^2/2 + y_c^2 b^2/(b^2 - a^2)^2$. Substituting z_c in $r^2 = a^2 z_c - a^2/(b^2 - a^2)y_c^2 - a^4/4$ we obtain $r^2 = a^4\left(y_c^2/(b^2 - a^2)^2 + 1/4\right)$. Since $2r \leq a^2$ by the smallness condition, we conclude that $y_c^2/(b^2 - a^2)^2 \leq 0$, so necessarily $y_c = 0$. Hence $h(-a^2) = 0$ implies that $y_c = 0$ and $2r = a^2 = z_c$ as in assertion (iii). The converse is immediate. $\square$

Note that the condition for $-a^2$ to be a double root can be rewritten as $z_c = \dfrac{y_c^2}{b^2 - a^2} + \dfrac{r^2}{a^2} + \dfrac{a^2}{4}$. This expression evidences the fact that if $-a^2$ is a double root, then the center of $\mathcal{S}$, $(0, y_c, z_c)$, is interior to $\mathcal{P}$.

Analogous assertions to (i) and (ii) in Lemma 21 follow for the root $-b^2$. However, $-b^2$ cannot be a triple root if $a \neq b$, since for that to happen it is necessary that $r = b^2/2$ and that contradicts the smallness assumption.

Proof of Theorem 16. We need to show that a root $-a^2$ with multiplicity equal to 2 is not associated with tangency between $\mathcal{S}$ and $\mathcal{P}$. Then Theorem 16 will follow from Lemma 13, Lemma 14 and Lemma 21 taking into account that $-b^2$ is never a triple root.

If $-a^2$ is a root, we have a tangent point associated to it at X_0 if and only if X_0 is an eigenvector (meaning it is a solution of $(-a^2P + S)X = 0$) and verify either $X_0^t \mathcal{S} X_0 = 0$ (these imply $X_0^t \mathcal{P} X_0 = 0$).

Assume $-a^2$ is a root of multiplicity 2. Then $x_c = 0$ and $r^2 = a^2 z_c - a^2/(b^2 - a^2)y_c^2 - a^4/4$. For $X = (x, y, z, 1)^t$, we solve $(-a^2P + S)X = 0$ where

$$-a^2P + S = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -a^2b^{-2} + 1 & 0 & -y_c \\ 0 & 0 & 1 & -(z_c - a^2/2) \\ 0 & -y_c & -(z_c - a^2/2) & -r^2 + y_c^2 + z_c^2 \end{pmatrix},$$

to obtain

$$(0, \frac{(b^2 - a^2)y}{b^2} - y_c, -z_c + z + \frac{a^2}{2}, z_c^2 + z(\frac{a^2}{2} - z_c) + y_c^2 - yy_c - r^2) = 0.$$

These equations have a solution of the form

$$x \in \mathbb{R},\ y = \frac{b^2 y_c}{b^2 - a^2},\ z = z_c - \frac{a^2}{2} \tag{10}$$

where $z_c^2 + (z_c - a^2/2)(a^2/2 - z_c) + y_c^2 - (b^2 y_c/(b^2 - a^2))y_c - r^2 = 0$. Substitute y and z in the later condition to reduce it to $z_c a^2 - y_c^2 a^2/(b^2 - a^2) - a^4/4 - r^2 = 0$, which is satisfied because $-a^2$ is a double root, so this later condition impose no new restrictions.

Now, since (x, y, z) is a tangent point we have that $r^2 = x^2 + (y - y_c)^2 + (z - z_c)^2$. We use $z = z_c - a^2/2$ from (10) to see that

$$r^2 = x^2 + (y - y_c)^2 + \frac{a^4}{4} \geq \frac{a^4}{4}$$

and due to the smallness assumption ($2r \leq a^2$) we have that $r = a^2/2$. Moreover, from (10) we conclude $x = 0$ and $y = y_c = 0$, so $-a^2$ is indeed a triple root by Lemma 21. A similar argument exclude the possibility of having a tangent point associated to the root $-b^2$ with multiplicity 2. $\square$

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