Dualities for modal N4-lattices

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Abstract

We introduce a new Priestley-style topological duality for N4-lattices, which are the algebraic counterpart of paraconsistent Nelson logic. Our duality differs from the existing one, due to S. Odintsov, in that we only rely on Esakia duality for Heyting algebras and not on the duality for De Morgan algebras of Cornish and Fowler. A major advantage of our approach is that we obtain a simple description for our topological structures, which allows us to extend the duality to other algebraic structures such as N4-lattices with monotonic modal operators, and also to provide a neighbourhood semantics for the non-normal modal logic corresponding to these algebras.

Keywords: N4-lattice, paraconsistent Nelson logic, Priestley duality, twist-structure, Esakia duality.

1 Introduction

Paraconsistent Nelson logic, which was introduced in [1] as an inconsistency-tolerant counterpart of the better-known logic of Nelson [16, 23], combines interesting features of intuitionistic, classical and many-valued logics (e.g. Belnap–Dunn four-valued logic); recent work has shown that it can also be seen as one member of the wide family of substructural logics [25].

The work we present in this article is a contribution towards a better topological understanding of the algebraic counterpart of paraconsistent Nelson logic, namely a variety of involutive lattices called *N4-lattices* in [17]. We present a Priestly-style duality for those lattices and we develop a topological duality for N4-lattices expanded with a monotone modal operator.

A Priestley-style duality for N4-lattices was already introduced by Odintsov [19], generalizing the duality developed by Cignoli [5] for a subclass of N4-lattices called N3-lattices. The main differences between the Cignoli–Odintsov approach and ours are the following:

- we only rely on Esakia duality for Heyting algebras [12], whereas [5, 19] use both Esakia duality and the duality for De Morgan algebras [8, 9]: as a consequence, the dual spaces that we obtain are, in our opinion, easier to understand than those considered in [5, 19];
- [5, 19] only deal with *bounded* N4-lattices, whereas we cover the non-bounded case as well.

From our perspective, our duality has the further advantage that it can be easily extended to obtain topological counterparts of N4-lattices with modal operators such as those introduced in [20, 24], and the resulting duality can be used to provide a state-based semantics for the paraconsistent modal logic introduced in [24]. The duality we present for non-modal N4-latices has already been introduced in [15], to which we will refer in the sequel.

This article is organized as follows.

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Section 2 introduces the abstract algebraic definition of N4-lattices and we state a fundamental result of Odintsov [18], namely that every N4-lattice can be represented through a concrete construction called *twist-structure*. We show that this can be extended to a categorial equivalence, which will allow us to work, for our duality, with a category of twist-structures instead of the category of N4-lattices as defined in [19].

Section 3 contains the details of our duality. At the level of objects, on the algebraic side we have *twist-structures*, that is, tuples $\langle \mathbf{A}, \nabla, \Delta \rangle$ where $\mathbf{A}$ is a Brouwerian lattice (i.e. the 0-free subreduct of a Heyting algebra) and ∇, Δ are, respectively, a filter and an ideal of $\mathbf{A}$. On the topological side we have tuples $\langle X, C, O \rangle$ such that X is the Priestley space corresponding to a Heyting algebra (an *Esakia space*) and $C, O \subseteq X$ are, respectively, a closed and an open set of the Priestley topology on X. We prove that the two resulting categories are dually equivalent via the usual functors involved in Priestley (and Esakia) duality.

Section 4 recalls the algebraic definitions of *monotonic N4-lattices* [24] and *BK-lattices* [20], which are both classes of N4-lattices augmented with monotone modal operators. We see that twist-structure representations are available for these algebras as well and, as in the non-modal case, we extend them to categorial equivalences that employ the same functors.

Section 5 extends the duality of Section 3 to twist-structures corresponding to N4-lattices with modal operators. At the level of objects, on the algebraic side we have as before tuples $\langle \mathbf{A}, \nabla, \Delta \rangle$, but where $\mathbf{A}$ is now a Brouwerian lattice augmented with modal operators. On the topological side, these operators are represented by *neighbourhood functions* on the corresponding spaces [14]. We show that the usual Priestley functors establish dualities between twist-structures augmented with modal operators and the spaces thus obtained.

Finally, Section 6 shows how the duality of Section 5 can be used to provide a state-based semantics which is complete with respect to the paraconsistent modal logic introduced in [24], thus solving one of the open problems posed in [24, Section 5].

2 Equivalence between N4-lattices and twist-structures

In this section we prove a result which is implicitly contained in [18], namely that N4-lattices, viewed as a category, are equivalent to a category of twist-structures over (i.e. special second powers of) Brouwerian lattices. This restricts to an equivalence between bounded N4-lattices and twist-structures over bounded Brouwerian lattices (i.e. Heyting algebras). In the next section we will develop a duality for the latter category based on Esakia duality for Heyting algebras, which will allow us to obtain a dual equivalence between the topological spaces thus introduced and the category of (bounded) N4-lattices.

Let us start by introducing N4-lattices, which are our main objects of interest [18, Definition 2.3].

Definition 2.1
An *N4-lattice* is an algebra $\mathbf{B} = \langle B, \wedge, \vee, \rightarrow, \sim \rangle$ such that:

1. the reduct[1] $\langle B, \wedge, \vee, \sim \rangle$ is a De Morgan lattice, i.e. a distributive lattice equipped with a unary operation $\sim : B \rightarrow B$ (usually called *negation*) such that $\sim\sim a = a$ and $\sim(a \vee b) = \sim a \wedge \sim b$ for all $a, b \in B$,
2. the relation $\preceq$ defined, for all $a, b \in B$, as $a \preceq b$ iff $a \rightarrow b = (a \rightarrow b) \rightarrow (a \rightarrow b)$, is a pre-ordering (i.e. reflexive and transitive),

[1] By a *reduct* of **B** we mean an algebra with the same carrier set, in which some of the algebraic operations of **B** have been suppressed.

3. the relation $\equiv$ defined, for all $a,b\in B$, as $a\equiv b$ iff $a\preceq b$ and $b\preceq a$, is a congruence relation w.r.t. $\wedge,\vee,\to$ and the quotient algebra $\mathbf{B}_{\bowtie}=\langle B,\wedge,\vee,\to\rangle/\equiv$ is a Brouwerian lattice (see below),
4. for all $a,b\in B$, $\sim(a\to b)\equiv a\wedge\sim b$,
5. for all $a,b\in B$, $a\leq b$ iff ($a\preceq b$ and $\sim b\preceq\sim a$), where $\leq$ is the lattice order of $\mathbf{B}$.

B is said to be *bounded* if its lattice reduct is bounded, in which case we include the bounds as constants in the algebraic signature.

We remind the reader that a *Brouwerian lattice* is a lattice $\langle L,\wedge,\vee\rangle$ equipped with a binary operation $\to$ that satisfies the following residuation condition: for all $a,b,c\in L$, $a\wedge b\leq c$ if and only if $b\leq a\to c$. Brouwerian lattices are precisely the 0-free subreducts (i.e. subalgebras of reducts) of Heyting algebras.[2]

Although it is not apparent in the above definition, it is known [17] that N4-lattices form a variety (therefore, the class of bounded N4-lattices is also a varietiy).

Condition (3) of the definition provides a way to associate a Brouwerian lattice to any given N4-lattice. Conversely, we are now going to describe a method introduced in [18, Definition 2.1] that allows us to construct an N4-lattice as a special power of a Brouwerian lattice, and we shall see that each N4-lattice is isomorphic to one obtained in this way.[3]

Let $\mathbf{A}=\langle A,\wedge,\vee,\to,1\rangle$ be a Brouwerian lattice. Consider the algebra $\mathbf{A}^{\bowtie}=\langle A\times A,\wedge,\vee,\to,\sim\rangle$ with operations defined as follows:

- $\langle a,b\rangle\wedge\langle c,d\rangle:=\langle a\wedge c,b\vee d\rangle$
- $\langle a,b\rangle\vee\langle c,d\rangle:=\langle a\vee c,b\wedge d\rangle$
- $\langle a,b\rangle\to\langle c,d\rangle:=\langle a\to c,a\wedge d\rangle$
- $\sim\langle a,b\rangle:=\langle b,a\rangle$.

It is not difficult to check that $\mathbf{A}^{\bowtie}$ satisfies all conditions of Definition 2.1, i.e. $\mathbf{A}^{\bowtie}$ is an N4-lattice. If **A** has a minimum element 0, then by defining $\top:=\langle 1,0\rangle$ and $\bot:=\langle 0,1\rangle$ we obtain a bounded N4-lattice. Notice that the operations $\wedge,\vee,\to$ of $\mathbf{A}^{\bowtie}$ are defined component-wise just as in a direct product in the first component, while they are somehow 'twisted' in the second one. This explains the name *twist-structure over* **A** for the algebra $\mathbf{A}^{\bowtie}$ used for instance in [18].

Although the construction described above indeed produces an N4-lattice, not all N4-lattices are isomorphic to one constructed in this way. In order to obtain all of them, we need to consider all $\{\wedge,\vee,\to,\sim\}$-subalgebras of $\mathbf{A}^{\bowtie}$. The following construction, due to Odintsov, provides a way of producing all such subalgebras.

Given our Brouwerian lattice **A**, denote by $D(\mathbf{A})$ the set of *dense elements* of **A**, defined as

$$D(\mathbf{A}):=\{a\vee(a\to b):a,b\in A\}.$$

$D(\mathbf{A})$ is always a lattice filter of **A**, so we may also call it *the filter of dense elements of* **A**. If **A** has a bottom element 0, (i.e. if **A** is in fact a Heyting algebra, except for the fact that no symbol for 0 is included as a constant in the algebraic language), then the dense elements can also be obtained as follows:

$$D(\mathbf{A})=\{a\vee\neg a:a\in A\}=\{a\in A:\neg\neg a=1\}$$

where $\neg$ is the Heyting negation of **A**, i.e. $\neg a:=a\to 0$.

[2] Brouwerian lattices are also known in the literature as *generalized Heyting algebras* [6], *Brouwerian algebras* [10], implicative lattices [17] or *relatively pseudo-complemented lattices* [23]. Note also that some authors call 'Brouwerian lattices' structures that are (lattice-theoretic) dual to ours.

[3] The origins of this construction can be traced back to [26] and also, independently, [13].

Now consider a lattice filter $\nabla \subseteq A$ such that $D(\mathbf{A}) \subseteq \nabla$ and let $\Delta \subseteq A$ be a lattice ideal. Then the set

$$B := \{\langle a,b\rangle \in A\times A :\ a\vee b\in\nabla,\ a\wedge b\in\Delta\}$$

is closed under the operations $\wedge, \vee, \rightarrow, \sim$ of $\mathbf{A}^{\bowtie}$. Therefore $\langle B, \wedge, \vee, \rightarrow, \sim\rangle$ is an N4-lattice. Following [18], we denote this algebra by $Tw(\mathbf{A}, \nabla, \Delta)$. Notice also that, for every $a \in A$, there is $b \in A$ such that $\langle a, b\rangle \in Tw(\mathbf{A}, \nabla, \Delta)$. To see this, take $a' \in \Delta$. Then $a \vee (a \rightarrow a') \in D(\mathbf{A}) \subseteq \nabla$ and since $a \wedge (a \rightarrow a') = a \wedge a' \in \Delta$, we have $\langle a, a \rightarrow a'\rangle \in B$. Thus, letting $b := a \rightarrow a'$, we obtain the desired result.

In order to show that any N4-lattice can be obtained as $Tw(\mathbf{A}, \nabla, \Delta)$ for a suitable choice of $(\mathbf{A}, \nabla, \Delta)$, we define, for an arbitrary N4-lattice $\mathbf{B}$,

$$\nabla(\mathbf{B}) := \{[a\vee\sim a] : a\in B\}$$

where $[b]$ denotes the equivalence class of $b \in B$ modulo the relation $\equiv$ introduced in Definition 2.1. Similarly, we let

$$\Delta(\mathbf{B}) := \{[a\wedge\sim a] : a\in B\}.$$

It is not difficult to check that $\nabla(\mathbf{B})$ is a lattice filter of the Brouwerian lattice $\mathbf{B}_{\bowtie} = \langle B, \wedge, \vee, \rightarrow\rangle/\equiv$ which contains the dense elements of $\mathbf{B}_{\bowtie}$, and that $\Delta(\mathbf{B})$ is an ideal of $\mathbf{B}_{\bowtie}$. Thus, we can construct the N4-lattice $Tw(\mathbf{B}_{\bowtie}, \nabla(\mathbf{B}), \Delta(\mathbf{B}))$, which turns out to be isomorphic to $\mathbf{B}$, as shown by the following result [18, Corollary 3.2].

Proposition 2.2 (cf. [15], Prop. 2.2)
Every N4-lattice (bounded N4-lattice) $\mathbf{B}$ is isomorphic to the algebra

$$Tw(\mathbf{B}_{\bowtie}, \nabla(\mathbf{B}), \Delta(\mathbf{B}))$$

where $\mathbf{B}_{\bowtie}$ is a Brouwerian lattice (Heyting algebra), through the map $j_{\mathbf{B}} : B \rightarrow B/\equiv \times B/\equiv$ defined, for all $a \in B$, as

$$j_{\mathbf{B}}(a) := \langle [a], [\sim a]\rangle.$$

Thus, any (bounded) N4-lattice can be associated to a triple of the form $(\mathbf{A}, \nabla, \Delta)$ with $\mathbf{A}$ a (bounded) Brouwerian lattice and ∇, Δ, respectively, a filter and an ideal of $\mathbf{A}$. We are going to see that $j_{\mathbf{B}}$ is in fact the unit of a categorical equivalence between two naturally associated categories.

We denote by $\mathsf{N4}$ the category whose objects are N4-lattices and whose morphisms are the algebraic N4-lattice homomorphisms. The category of bounded N4-lattices (denoted $\mathsf{N4}^{\perp}$) is defined analogously, the corresponding objects being bounded N4-lattices and the morphisms being the algebraic N4-lattice homomorphisms that preserve the bounds.

On the other side of our equivalence, we define a category Twist whose objects are triples $\mathcal{A} = (\mathbf{A}, \nabla, \Delta)$ such that:

- $\mathbf{A}$ is a Brouwerian lattice,
- ∇ is a lattice filter of $\mathbf{A}$ containing the dense elements $D(\mathbf{A})$,
- Δ is a lattice ideal of $\mathbf{A}$.

We will refer to objects in this category as *twist-structures*, but notice that we view them just as triples $(\mathbf{A}, \nabla, \Delta)$ rather than as the product algebra $Tw(\mathbf{A}, \nabla, \Delta)$ defined above.

A *morphism between two twist-structures* $\mathcal{A}_1 = (\mathbf{A}_1, \nabla_1, \Delta_1)$ *and* $\mathcal{A}_2 = (\mathbf{A}_2, \nabla_2, \Delta_2)$ is defined as a Brouwerian lattice homomorphism $h\colon \mathbf{A}_1 \to \mathbf{A}_2$ such that

- $h[\nabla_1] \subseteq \nabla_2$
- $h[\Delta_1] \subseteq \Delta_2$.

It is easy to check that the set-theoretic composition of morphisms gives a morphism and that the identity morphism of a twist-structure is the identity homomorphism of the underlying Brouwerian lattice. We define the category $\mathsf{Twist}^{\perp}$ by restricting the objects to twist-structures over bounded Brouwerian lattices (i.e. Heyting algebras) and by requiring that morphisms preserve the bounds. Note that $\mathsf{Twist}^{\perp}$ is a subcategory of Twist which is not full because of the requirement that morphisms preserve the bounds.

We proceed to define functors $T\colon \mathsf{N4} \to \mathsf{Twist}$ and $N\colon \mathsf{Twist} \to \mathsf{N4}$ that will allow us to prove the equivalence between the two categories.

Given an N4-lattice $\mathbf{B}$, we let

$$T(\mathbf{B}) := (\mathbf{B}_{\bowtie}, \nabla(\mathbf{B}), \Delta(\mathbf{B})).$$

If $f\colon \mathbf{B_1} \to \mathbf{B_2}$ is an N4-lattice homomorphism, we define $T(f)\colon (\mathbf{B_1})_{\bowtie} \to (\mathbf{B_2})_{\bowtie}$ as

$$T(f)([a]_{\equiv_1}) := [f(a)]_{\equiv_2}$$

where $[a]_{\equiv_1}$ is the equivalence class of $a \in B_1$ modulo the relation introduced in Definition 2.1 and likewise $[b]_{\equiv_2} \in B_2/{\equiv_2}$ for all $b \in B_2$. The definition is sound because $a \equiv_1 a'$ implies that $f(a) \equiv_2 f(a')$. The map $T(f)$ is a morphism from $(\mathbf{B_1})_{\bowtie}$ to $(\mathbf{B_2})_{\bowtie}$ satisfying that $T(f) \circ \pi_1 = \pi_2 \circ f$, where $\pi_i\colon \mathbf{B}_i \to \mathbf{B}_i/{\equiv}$ is defined by $\pi_i(b) := [b]_{\equiv_i}$ for all $b \in B_i$.

It is straightforward to check that T is indeed a functor from $\mathsf{N4}$ to Twist. Note that if $\mathbf{B_1}, \mathbf{B_2}$ are bounded and f preserves the bounds, then $T(f)$ also preserves the bounds. Thus T also gives a functor from $\mathsf{N4}^{\perp}$ to $\mathsf{Twist}^{\perp}$.

Conversely, given a twist-structure $\mathcal{A} = (\mathbf{A}, \nabla, \Delta)$, we let

$$N(\mathcal{A}) := Tw(\mathbf{A}, \nabla, \Delta).$$

We know from Proposition 2.2 that $Tw(\mathbf{A}, \nabla, \Delta)$ is an N4-lattice. For a morphism $h\colon \mathcal{A}_1 \to \mathcal{A}_2$ between twist-structures $\mathcal{A}_1 = (\mathbf{A}_1, \nabla_1, \Delta_1)$ and $\mathcal{A}_2 = (\mathbf{A}_2, \nabla_2, \Delta_2)$, we define the map $N(h)\colon N(\mathcal{A}_1) \to N(\mathcal{A}_2)$, for all $a, b \in A_1$, as

$$N(h)\langle a, b\rangle := \langle h(a), h(b)\rangle.$$

It is easy to see that this map is well defined, that is, if $\langle a, b\rangle \in N(\mathcal{A}_1)$, then $\langle h(a), h(b)\rangle \in N(\mathcal{A}_2)$, and that it is a homomorphism. As with T, it is straightforward to see that N is a functor from Twist to $\mathsf{N4}$. Moreover, if $\mathcal{A}_1, \mathcal{A}_2$ are twist-structures over bounded Brouwerian lattices and $h\colon \mathcal{A}_1 \to \mathcal{A}_2$ preserves the bounds, then $N(h)\colon N(\mathcal{A}_1) \to N(\mathcal{A}_2)$ is a bounded N4-lattice homomorphism. Therefore N gives a functor from $\mathsf{Twist}^{\perp}$ to $\mathsf{N4}^{\perp}$.

Now, given an N4-lattice $\mathbf{B}$, by Proposition 2.2 we have an algebraic isomorphism

$$j_{\mathbf{B}}\colon \mathbf{B} \cong N(T(\mathbf{B})).$$

It is easy to see that this implies that $j_{\mathbf{B}}$ is an isomorphism in the category $\mathsf{N4}$.

Conversely, given a twist-structure $\mathcal{A}$, we define a function $\eta_{\mathcal{A}}\colon \mathcal{A} \to T(N(\mathcal{A}))$ as follows: for all $a \in A$,

$$\eta_{\mathcal{A}}(a) := [\langle a, a' \rangle] \tag{2.1}$$

where $a' \in A$ is an element we choose such that $\langle a, a' \rangle \in N(\mathcal{A})$ and $[\langle a, a' \rangle]$ is the equivalence class of $\langle a, a' \rangle$ modulo the equivalence relation on $T(N(\mathcal{A}))$ introduced in Definition 2.1. In order to see that this definition is sound, notice first that such an element $a' \in A$ always exists because $\pi_1[N(\mathcal{A})] = A$, and secondly notice that for all $a, b, a', b' \in A$, it holds that $[\langle a, a' \rangle] = [\langle b, b' \rangle]$ if and only if $a = b$.

PROPOSITION 2.3 (cf. [15], Prop. 2.3)
For any twist-structure $\mathcal{A}$, the map $\eta_{\mathcal{A}}\colon \mathcal{A} \to T(N(\mathcal{A}))$ defined in (2.1) is an isomorphism in the category Twist.

PROOF. (a). The map $\eta_{\mathcal{A}}$ is one-to-one. Let $a, b, a', b' \in A$ and suppose that $[\langle a, a' \rangle] = [\langle b, b' \rangle]$. Then, as noted above, $a = b$.
(b). $\eta_{\mathcal{A}}$ is onto. Let $\langle a, b \rangle \in N(\mathcal{A})$. As observed above, $[\langle a, b \rangle] = [\langle a, c \rangle]$ for any $c \in A$ such that $\langle a, c \rangle \in N(\mathcal{A})$. Hence, $[\langle a, b \rangle] = [\langle a, a' \rangle] = \eta_{\mathcal{A}}(a)$.
(c). $\eta_{\mathcal{A}}$ is a homomorphism. Let $a, b \in A$. Then $\eta_{\mathcal{A}}(a) \wedge \eta_{\mathcal{A}}(b) = [\langle a, a' \rangle] \wedge [\langle b, b' \rangle] = [\langle a, a' \rangle \wedge \langle b, b' \rangle] = [\langle a \wedge b, a' \vee b' \rangle]$. But $[\langle a \wedge b, a' \vee b' \rangle] = [\langle a \wedge b, c \rangle]$ for any $c \in A$ such that $\langle a \wedge b, c \rangle \in N(\mathcal{A})$, so in particular we have that $[\langle a \wedge b, a' \vee b' \rangle] = \eta_{\mathcal{A}}(a \wedge b)$. A similar reasoning establishes the cases of the other operations.
(d). $\eta_{\mathcal{A}}[\nabla] = \nabla(N(\mathcal{A}))$. It is sufficent to observe that

$$\begin{aligned}
\nabla(N(\mathcal{A})) &= \{[\langle a, b \rangle \vee {\sim} \langle a, b \rangle] : \langle a, b \rangle \in N(\mathcal{A})\} \\
&= \{[\langle a \vee b, a \wedge b \rangle] : a \vee b \in \nabla,\ a \wedge b \in \Delta\} \\
&= \{[\langle c, d \rangle] : c \in \nabla,\ d \in \Delta,\ d \leq c\} \\
&= \{[\langle c, d \rangle] : c \in \nabla\} \\
&= \eta_{\mathcal{A}}[\nabla].
\end{aligned}$$

(e). $\eta_{\mathcal{A}}[\Delta] = \Delta(N(\mathcal{A}))$. Similar to the proof of the previous item. ∎

PROPOSITION 2.4 (cf. [15], Prop. 2.4)
Let $f\colon \mathbf{B_1} \to \mathbf{B_2}$ be a morphism of N4-lattices. Then $N(T(f)) \circ j_{\mathbf{B_1}} = j_{\mathbf{B_2}} \circ f$.

PROOF. For any $a \in B_1$, we have

$$\begin{aligned}
N(T(f)) \circ j_{\mathbf{B_1}}(a) &= N(T(f))(\langle [a]_{\equiv_1}, [{\sim} a]_{\equiv_1} \rangle) \\
&= \langle T(f)([a]_{\equiv_1}), T(f)([{\sim} a]_{\equiv_1}) \rangle \\
&= \langle [f(a)]_{\equiv_2}, [f({\sim} a)]_{\equiv_2} \rangle \\
&= \langle [f(a)]_{\equiv_2}, [{\sim} f(a)]_{\equiv_2} \rangle \\
&= j_{\mathbf{B_2}} \circ f(a).
\end{aligned}$$

∎

PROPOSITION 2.5 (cf. [15], Prop. 2.5)
Let $h\colon \mathcal{A}_1 \to \mathcal{A}_2$ be a morphism of twist-structures. Then $T(N(h)) \circ \eta_{\mathcal{A}_1} = \eta_{\mathcal{A}_2} \circ h$.

$$\begin{tikzcd} \mathsf{N4}\ (\mathsf{N4}^{\perp}) \arrow[rr, bend left, "T"] & & \mathsf{Twist}\ (\mathsf{Twist}^{\perp}) \arrow[ll, bend left, "N"] \end{tikzcd}$$

FIG. 1. Equivalence between (bounded) N4-lattices and twist-structures over (bounded) Brouwerian lattices.

PROOF. For any $a \in A_1$, we have

$$\begin{aligned} T(N(h)) \circ \eta_{\mathcal{A}_1}(a) &= T(N(h))([\langle a, a' \rangle]_{\equiv_1}) \\ &= [N(h)(\langle a, a' \rangle)]_{\equiv_2} \\ &= [\langle h(a), h(a') \rangle]_{\equiv_2} \\ &= [\langle h(a), (h(a))']_{\equiv_2} \\ &= \eta_{\mathcal{A}_2} \circ h(a). \end{aligned}$$

Let us remind the reader that the equality $[\langle h(a), h(a') \rangle]_{\equiv_2} = [\langle h(a), (h(a))']_{\equiv_2}$ holds because $[\langle h(a), h(a') \rangle]_{\equiv_2} = [\langle h(a), b \rangle]_{\equiv_2}$ for any $b \in A_2$ as long as $\langle h(a), b \rangle \in N(\mathcal{A}_2)$. ∎

Propositions 2.4 and 2.5 imply the announced equivalence result (Figure 1).

THEOREM 2.6 (cf. [15], Thm. 2.6)
Functors $T: \mathsf{N4} \to \mathsf{Twist}$ and $N: \mathsf{Twist} \to \mathsf{N4}$ establish a natural equivalence between the category $\mathsf{N4}$ of (bounded) N4-lattices and the category Twist of twist-structures over Brouwerian lattices (Heyting algebras).

3 Topological duality for twist-structures

In this section, we introduce a category of topological structures that will be proven to be equivalent to the twist-structures considered in the previous section. As we will build on Esakia duality for Heyting algebras, we begin by recalling essential definitions and results on Esakia duality [12], which is itself based on Priestley duality for distributive lattices [11].

3.1 Esakia duality

Recall that a *Priestley space* is a compact topological ordered space $\mathcal{X} = \langle X, \tau, \leq \rangle$ that satisfies the following separation condition: for every $x, y \in X$ such that $x \not\leq y$ there exists a clopen up-set U with $x \in U$ and $y \notin U$. A Priestley space is an *Esakia space* if in addition it satisfies that for every clopen set $U \subseteq X$, the down-set $\downarrow U$ is clopen.

If $\mathbf{A}$ is a Heyting algebra, then $\langle X(\mathbf{A}), \tau, \subseteq \rangle$ is an Esakia space, where $X(\mathbf{A})$ is the set of the prime filters of $\mathbf{A}$ and τ is the topology generated by the sub-basis

$$\{\sigma_{\mathbf{A}}(a) : a \in A\} \cup \{X(\mathbf{A}) - \sigma_{\mathbf{A}}(a) : a \in A\}.$$

with

$$\sigma_{\mathbf{A}}(a) := \{P \in X(\mathbf{A}) : a \in P\} \tag{3.1}$$

Conversely, if $\mathcal{X}=\langle X,\tau,\leq\rangle$ is an Esakia space, then the distributive lattice of its clopen up-sets forms a Heyting algebra when endowed with the following implication operation. For clopen up-sets $U,V\subseteq X$, we let

$$U\rightarrow_X V:=\{x\in X:\uparrow x\cap U\subseteq V\},$$

which also is a clopen up-set. We denote this Heyting algebra by $A(\mathcal{X})$. Notice that implication can be equivalently defined as

$$U\rightarrow_X V:=(\downarrow(U-V))^c.$$

The correspondence between Heyting algebras and Esakia spaces given by the maps $X(.)$ and $A(.)$ can be turned into a dual equivalence between the category of Heyting algebras and the category of Esakia spaces by extending those maps to contravariant functors between the two categories.

The category of Heyting algebras has as objects these algebras and as morphisms the algebraic homomorphisms between them. The objects of the category of Esakia spaces are these spaces and the morphisms are Esakia functions, defined as follows. Let $\mathcal{X},\mathcal{Y}$ be Esakia spaces. A map $f:X\rightarrow Y$ is an *Esakia function* if it is continuous, order-preserving and satisfies that $\uparrow_Y f(x)\subseteq f[\uparrow_X x]$ for every $x\in X$.

If $h:\mathbf{A}_1\rightarrow\mathbf{A}_2$ is a homomorphism of Heyting algebras, then the map $X(h):X(\mathbf{A}_2)\rightarrow X(\mathbf{A}_1)$ between the corresponding Esakia spaces defined by $X(h)(P)=h^{-1}[P]$ for every $P\in X(\mathbf{A}_2)$ is an Esakia function. Conversely, if $f:X_1\rightarrow X_2$ is an Esakia function, then the map $A(f):A(X_2)\rightarrow A(X_1)$ defined by $A(f)(U)=f^{-1}[U]$ for every clopen up-set of X_2 is a Heyting algebra homomorphism.

The map $X(.)$ so obtained is a contravariant functor from the category of Heyting algebras to the category of Esakia spaces and the map $A(.)$ is a contravariant functor in the other direction. The two functors establish a dual equivalence between the two categories. The natural transformations are given by the following families of morphisms. For a Heyting algebra $\mathbf{A}$, the map $\sigma_{\mathbf{A}}:\mathbf{A}\rightarrow A(X(\mathbf{A}))$ defined in (3.1) is an isomorphism. If $\mathcal{X}$ is an Esakia space, the map $\epsilon_{\mathcal{X}}:X\rightarrow X(A(\mathcal{X}))$ defined by $\epsilon_{\mathcal{X}}(x)=\{U\in A(\mathcal{X}):x\in U\}$ for every $x\in X$ is a homeomorphism and an order isomorphism.

Esakia duality can be adapted to obtain a topological duality for Brouwerian lattices as these can be seen as Heyting algebras which possibly lack the bottom element. Our strategy is the following. To a Brouwerian lattice $\mathbf{A}$ we add a new bottom element, thus obtaining a Heyting algebra $\mathbf{A}^*$, and then we consider its dual Esakia space $X(\mathbf{A}^*)$. Our original $\mathbf{A}$ can then be recovered as the algebra of the *non-empty* clopen up-sets of $X(\mathbf{A}^*)$. Let us expound the details.

Let $\mathbf{A}=\langle A,\wedge,\vee,\rightarrow,1\rangle$ be a Brouwerian lattice. Regardless of whether $\mathbf{A}$ already has a bottom element, we add a new one 0^* and we set $0^*\leq a$ for all $a\in A\cup\{0^*\}$. This uniquely determines the behaviour of the Heyting implication, because on the one hand it must hold that $0^*\rightarrow a=1$ for all $a\in A\cup\{0^*\}$, and on the other hand residuation implies that

$$a\rightarrow 0^*=\bigvee\{b\in A\cup\{0^*\}:a\wedge b\leq 0^*\}$$

which means that for $a\neq 0^*$ the only possible choice is $b=0^*$. Hence we are led to the following definition:

$$a\rightarrow 0^*:=\begin{cases}0^* & \text{if } a\in A\\ 1 & \text{otherwise (i.e. if } a=0^*).\end{cases}$$

Extending in this way $\to$ to the new universe $A \cup \{0^*\}$ we obtain a Heyting algebra, which we denote by $\mathbf{A}^*$. Note that $X(\mathbf{A}^*) = X(\mathbf{A}) \cup A$ and the prime filter A of $\mathbf{A}^*$ contains all prime filters of $\mathbf{A}^*$.

Concerning the dense elements, we observe that $D(\mathbf{A}^*) = A$ because for every $a \in A$, $a \vee (a \to 0^*) = a \vee 0^* = a$. Moreover, a Heyting algebra $\mathbf{B}$ is isomorphic to $\mathbf{A}^*$ for some Brouwerian lattice $\mathbf{A}$ if and only if, for every $b \in B - \{0\}$, $b \to 0 = 0$. Indeed, if $\mathbf{B}$ satisfies this last condition, then it is isomorphic to $(\mathbf{B}_*)^*$ where $\mathbf{B}_*$ is the Brouwerian lattice we obtain by deleting 0 from $\mathbf{B}$.

Notice that if $\mathbf{A}_1, \mathbf{A}_2$ are Brouwerian lattices, a map $h\colon \mathbf{A}_1^* \to \mathbf{A}_2^*$ is a Heyting algebra homomorphism if and only if the restriction $h{\restriction}_{A_1}\colon \mathbf{A}_1 \to \mathbf{A}_2$ is a Brouwerian lattice homomorphism. This implies that Brouwerian lattices, viewed as a category, are equivalent to a full subcategory of Heyting algebras. Moreover the objects of this subcategory are the Heyting algebras that satisfy the quasiequation: $x \wedge y \approx \bot \Rightarrow x \approx \bot$.

Let $\mathbf{A}$ be a Brouwerian lattice. If we look at $X(\mathbf{A}^*)$, the Esakia space corresponding to $\mathbf{A}^*$, we have that $X(\mathbf{A}^*)$ has a greatest element, namely A, and it holds that $A \in \sigma_{\mathbf{A}^*}(a)$ for every $a \in A$. Moreover, the map $\sigma_{\mathbf{A}^*}$ restricted to A establishes an isomorphism between $\mathbf{A}$ and the algebra of non-empty clopen up-sets of $X(\mathbf{A}^*)$. This makes it possible to recover the Brouwerian lattice $\mathbf{A}$ as the lattice of non-empty clopen up-sets of the Esakia dual of $\mathbf{A}^*$.

We say that an Esakia space $\langle X, \leq, \tau \rangle$ is a *pointed Esakia space* if the poset $\langle X, \leq \rangle$ has a greatest element 1_X. It immediately follows that the set of non-empty clopen up-sets of a pointed Esakia space is closed under finite intersections, and therefore it is a Brouwerian lattice (it may or may not have a least element, depending on whether the subspace given by the elements different from 1_X is an Esakia space or not).

Let $\mathcal{X} = \langle X, \leq, \tau \rangle$ be a pointed Esakia space. Note that for every non-empty clopen up-set $U \subseteq X$, we have $U \to_X \emptyset = \{x \in X : \uparrow x \cap U \subseteq \emptyset\} = \emptyset$. Denoting by $A_*(\mathcal{X})$ the Brouwerian lattice of non-empty clopen up-sets of $\mathcal{X}$, we have that the algebra of all clopen up-sets $A(\mathcal{X})$ is isomorphic to $(A_*(\mathcal{X}))^*$. We can thus, without loss of generality, identify $A(\mathcal{X})$ and $(A_*(\mathcal{X}))^*$.

For a Brouwerian lattice $\mathbf{A}$, we set $X^*(\mathbf{A}) := X(\mathbf{A}^*)$. This is clearly a pointed Esakia space and, as observed above, $\sigma_{\mathbf{A}^*}$ restricted to $\mathbf{A}$ establishes an isomorphism between $\mathbf{A}$ and $A_*(X^*(\mathbf{A}))$.

Now let $h\colon \mathbf{A}_1 \to \mathbf{A}_2$ be a homomorphism between Brouwerian lattices $\mathbf{A}_1, \mathbf{A}_2$. Then h extends to a unique Heyting algebra homomorphism $h^*\colon \mathbf{A}_1^* \to \mathbf{A}_2^*$ that maps the new element $0_1^* \in \mathbf{A}_1^*$ to the new element $0_2^* \in \mathbf{A}_2^*$. So the dual Esakia function $X(h^*)\colon X(\mathbf{A}_2^*) \to X(\mathbf{A}_1^*)$ maps the top element of $X(\mathbf{A}_2^*)$ (namely, A_2) to the top element of $X(\mathbf{A}_1^*)$ (namely, A_1). We denote the map $X(h^*)$ by $X^*(h)$.

If $\mathcal{X}_1, \mathcal{X}_2$ are pointed Esakia spaces and $f\colon X_1 \to X_2$ is an Esakia function, then the dual $A(f)\colon A(\mathcal{X}_2) \to A(\mathcal{X}_1)$ restricts to a Brouwerian lattice homomorphism $A_*(f)\colon A_*(\mathcal{X}_2) \to A_*(\mathcal{X}_1)$ when, for every non-empty clopen up-set $U \subseteq X_2$, $f^{-1}[U]$ is non-empty. This holds if and only if $f(1_{X_1}) = 1_{X_2}$. In fact, $f(1_{X_1}) = 1_{X_2}$ obviously implies that $f^{-1}[U]$ is non-empty for every non-empty clopen up-set $U \subseteq X_2$. On the other hand, suppose that, for every non-empty clopen up-set $U \subseteq X_2$, we had that $f^{-1}[U]$ is non-empty and $f(1_{X_1}) \neq 1_{X_2}$. Then, since $1_{X_2} \not\leq f(1_{X_1})$, there is a clopen up-set $U \subseteq X_2$ such that $1_{X_2} \in U$ and $f(1_{X_1}) \notin U$. This would imply $1_{X_1} \notin f^{-1}[U]$, which is not possible because $f^{-1}[U]$ is a non-empty up-set.

Accordingly, we say that an Esakia function $f\colon X_1 \to X_2$ between pointed Esakia spaces $\mathcal{X}_1, \mathcal{X}_2$ is a *pointed Esakia function* (or morphism) if $f(1_{X_1}) = 1_{X_2}$.

Of course if $\mathcal{X}$ is a pointed Esakia space, then $X(A(\mathcal{X})) = X^*(A_*(\mathcal{X}))$ and the homeomorphism and order isomorphism $\epsilon_{\mathcal{X}}\colon X \to X(A(\mathcal{X}))$ is a pointed Esakia function. Therefore Esakia duality easily implies that $X^*(.)$ and $A_*(.)$ are contravariant functors that establish a dual equivalence between the category of Brouwerian lattices with their homomorphism and the category of pointed Esakia spaces with pointed Esakia functions (Figure 2).

$$\begin{tikzcd} \mathsf{BrLat} \arrow[r, bend left, "X"] & \mathsf{pEsSp} \arrow[l, bend left, "A"] \end{tikzcd}$$

FIG. 2. Equivalence between Brouwerian lattices and pointed Esakia spaces.

3.2 Duality for twist-structures

The following property is going to be useful for the description of our topological spaces.

LEMMA 3.1 (cf. [15], Lemma 4.1)
Let $P \subseteq A$ be a prime filter of a Brouwerian lattice $\mathbf{A}$. Then $D(\mathbf{A}) \subseteq P$ if and only if P is a maximal element of the poset of prime filters of $\mathbf{A}$.

PROOF. We will prove that, if $P \subsetneq Q$ for some prime filter Q, then $Q = A$, so Q is not prime. Assume that $P \subseteq Q$ and there is $a \in Q$ such that $a \notin P$. We claim that, for an arbitrary element $b \in A$, it holds that $b \in Q$. By assumption we have $a \vee (a \to b) \in D(\mathbf{A}) \subseteq P$. Since P is prime and $a \notin P$, we conclude that $a \to b \in P \subseteq Q$. Now $a, a \to b \in Q$ imply that $a \wedge (a \to b) = a \wedge b \in Q$. This means that $b \in Q$ as we claimed.
Suppose now that P is a maximal element of the poset of prime filters of $\mathbf{A}$. Let $a, b \in A$ and assume that $a \vee (a \to b) \notin P$. Consider the filter F generated by $P \cup \{a\}$. Then $a \to b \notin F$. On the contrary there would be $c \in P$ such that $c \wedge a \leq a \to b$. Then $c \leq a \to (a \to b) = a \to b$. It would thus follow that $a \to b \in P$, against our assumption. So there is a prime filter Q such that $P \subsetneq F \subseteq Q$ and $a \to b \notin Q$. Therefore P is not maximal: a contradiction. Hence $D(\mathbf{A}) \subseteq P$. ∎

COROLLARY 3.2
Let $P \subseteq A$ be a prime filter of a Brouwerian lattice $\mathbf{A}$ such that $D(\mathbf{A}) \subseteq P$. Then P is a maximal element of the poset $X(\mathbf{A}^*) - \{A\}$.

In the rest of the section we first present a duality for twist-structures over Heyting algebras, and later we extend it to obtain a duality for all twist-structures.

3.3 Duality for twist-structures over Heyting algebras

In this subsection, unless otherwise specified, we consider only twist-structures $(\mathbf{A}, \nabla, \Delta) \in \mathsf{Twist}^{\perp}$, that is twist-structures over Heyting algebras.

Let $\mathcal{A} = (\mathbf{A}, \nabla, \Delta) \in \mathsf{Twist}^{\perp}$ and let $\langle X(\mathbf{A}), \tau, \subseteq \rangle$ be the dual Esakia space of $\mathbf{A}$. Thanks to the isomorphism $\sigma_{\mathbf{A}}$ between $\mathbf{A}$ and the algebra of clopen up-sets of $X(\mathbf{A})$, the sets $\nabla, \Delta \subseteq A$ can be represented as follows. We let

$$C_{\mathcal{A}} := \bigcap \{\sigma_{\mathbf{A}}(a) : a \in \nabla\}$$

which is obviously a closed up-set, and

$$O_{\mathcal{A}} := \bigcup \{\sigma_{\mathbf{A}}(a) : a \in \Delta\}$$

which is an open up-set. It is easy to check that

$$C_{\mathcal{A}} = \{P \in X(\mathbf{A}) : \nabla \subseteq P\} \qquad O_{\mathcal{A}} = \{P \in X(\mathbf{A}) : P \cap \Delta \neq \emptyset\}.$$

In the rest of the article we will thus be using whichever of the above definitions is more convenient. Let us also notice that $C_{\mathcal{A}}$ is included in the set $\max(X(\mathbf{A}))$ of maximal elements of our Esakia space (which also implies, trivially, that $C_{\mathcal{A}}$ is an up-set). This follows from Lemma 3.1, because $P \in C_{\mathcal{A}}$ implies that $D(\mathbf{A}) \subseteq \nabla \subseteq P$. We use this insight to introduce formally the spaces we will deal with.

Definition 3.3 (cf. [15], Definition 4.2)
An *NE-space* is a structure $\mathcal{X} = \langle X, \leq, \tau, C, O\rangle$ such that

1. $\langle X, \leq, \tau\rangle$ is an Esakia space,
2. C is a closed set such that $C \subseteq \max(X)$,
3. O is an open up-set.

In order to view NE-spaces as a category, we need to introduce a notion of NE-morphism. We propose the following definition.

Definition 3.4 (cf. [15], Definition 4.3)
Let $\mathcal{X}_1 = \langle X_1, \leq_1, \tau_1, C_1, O_1\rangle$ and $\mathcal{X}_2 = \langle X_2, \leq_2, \tau_2, C_2, O_2\rangle$ be two NE-spaces. A *morphism* is a map $f: X_1 \to X_2$ such that

1. f is an Esakia function, i.e. f is monotone, continuous and for every $x \in X_1$, $\uparrow f(x) \subseteq f[\uparrow x]$,
2. $f[C_1] \subseteq C_2$,
3. $f^{-1}[O_2] \subseteq O_1$.

Given NE-spaces $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$ and NE-morphisms $f: \mathcal{X}_1 \to \mathcal{X}_2$, $g: \mathcal{X}_2 \to \mathcal{X}_3$, it is easy to see that $g \circ f: \mathcal{X}_1 \to \mathcal{X}_3$ is also a morphism. Moreover, the identity map on an NE-space is a morphism. So we indeed have a category $\mathsf{NE\text{-}Sp}$ of NE-spaces.

We are going to see that $\mathsf{NE\text{-}Sp}$ is dually equivalent to the category $\mathsf{Twist}^{\perp}$ of twist-structures over Heyting algebras. It will thus follow that $\mathsf{NE\text{-}Sp}$ is also dually equivalent to the category $\mathsf{N4}^{\perp}$ of bounded N4-lattices.

The definition immediately implies that, for any twist-structure $\mathcal{A} = (\mathbf{A}, \nabla, \Delta)$,

$$X(\mathcal{A}) := \langle X(\mathbf{A}), \tau, \subseteq, C_{\mathcal{A}}, O_{\mathcal{A}}\rangle$$

is an NE-space. Given a morphism of twist-structures $h: \mathcal{A}_1 \to \mathcal{A}_2$, we define the map $X(h): X(\mathcal{A}_2) \to X(\mathcal{A}_1)$ as in Esakia duality, i.e. we let $X(h)(P) := h^{-1}[P]$ for any $P \in X(\mathbf{A}_2)$.

It is obvious that $X(h)$ is an Esakia function. Let us check that the other requirements of Definition 3.4 are also met.

Lemma 3.5 (cf. [15], Lemma 4.4)
Let $h: \mathcal{A}_1 \to \mathcal{A}_2$ be a morphism between twist-structures $\mathcal{A}_1 = \langle \mathbf{A}_1, \nabla_1, \Delta_1\rangle$ and $\mathcal{A}_2 = \langle \mathbf{A}_2, \nabla_2, \Delta_2\rangle$. Then $X(h): X(\mathcal{A}_2) \to X(\mathcal{A}_1)$ is a morphism between the corresponding NE-spaces.

Proof. In order to see that $X(h)[C_{\mathcal{A}_2}] \subseteq C_{\mathcal{A}_1}$, assume $Q \in X(h)[C_{\mathcal{A}_2}]$, i.e. $Q \in h^{-1}[C_{\mathcal{A}_2}]$. This means that there is $Q' \in X(\mathbf{A}_2)$ such that $\nabla_2 \subseteq Q'$ and $Q = h^{-1}[Q']$. Since h is a morphism of twist-structures, we have that $h[\nabla_1] \subseteq \nabla_2$. This implies that $\nabla_1 \subseteq h^{-1}[\nabla_2] \subseteq h^{-1}[Q'] = Q$. We conclude that $\nabla_1 \subseteq Q$, which means that $Q \in C_{\mathcal{A}_1}$ as desired.
Assume now that $P \in X(h)^{-1}[O_{\mathcal{A}_1}]$. This means that $X(h)(P) = h^{-1}[P] \in O_{\mathcal{A}_1}$. Then $h^{-1}[P] \cap \Delta_1 \neq \emptyset$. Let $a \in A_1$ be such that $a \in h^{-1}[P] \cap \Delta_1$. We then have $h(a) \in P \cap h(\Delta_1)$. From the assumptions we have $P \cap h(\Delta_1) \subseteq P \cap \Delta_2$, so we obtain $h(a) \in P \cap \Delta_2 \neq \emptyset$, which implies $P \in O_{\mathcal{A}_2}$ as required. Thus, $X(h)$ is indeed a morphism of NE-spaces. ∎

It follows from Esakia duality that the map X preserves composition and identity maps. So we actually have a functor $X\colon \mathsf{Twist}^{\perp} \to \mathsf{NE\text{-}Sp}$. We are now going to define a functor $A\colon \mathsf{NE\text{-}Sp} \to \mathsf{Twist}^{\perp}$ in the opposite direction.

To each NE-space $\mathcal{X} = \langle X, \leq, \tau, C, O\rangle$ we associate a twist-structure in the following way. Let $A(\mathcal{X})$ be the Heyting algebra of clopen up-sets of $\mathcal{X}$. To the closed set C we associate the following filter of $A(\mathcal{X})$:

$$\nabla_C := \{U \in A(\mathcal{X}) : C \subseteq U\}.$$

Likewise, to the open up-set O we associate the following ideal of $A(\mathcal{X})$:

$$\Delta_O := \{U \in A(\mathcal{X}) : U \subseteq O\}.$$

We need to ensure that ∇_C does indeed contain all dense elements of $A(\mathcal{X})$. For this we notice that, since every dense element has the form $U \cup ({\downarrow} U)^c$ for some clopen up-set $U \in A(\mathcal{X})$, condition (2) of Definition 3.3 is equivalent to the following property: $C \subseteq U \cup ({\downarrow} U)^c$ for all $U \in A(\mathcal{X})$. In fact, we have

$$\max(X) = \bigcap\{U \cup ({\downarrow} U)^c : U \in A(\mathcal{X})\}.$$

To see this, assume $x \in \max(X)$. Then, for every clopen up-set U, we have $x \notin ({\downarrow} U)^c$ iff $x \in {\downarrow} U$ iff there is $y \in U$ such that $x \leq y$. By maximality of x, this means that $x = y$, so $x \in U$. Hence, $x \in U \cup ({\downarrow} U)^c$ for every $U \in A(\mathcal{X})$. Conversely, suppose $x \notin \max(X)$, i.e. there is $y \in X$ such that $x < y$. Since X is a Priestley space, we know that there is a clopen up-set V such that $x \notin V$ and $y \in V$. Moreover, $x \in {\downarrow} V$, i.e. $x \notin ({\downarrow} V)^c$. This means that $x \notin V \cup ({\downarrow} V)^c$, so $x \notin \bigcap\{U \cup ({\downarrow} U)^c : U \in A(\mathcal{X})\}$.

The above reasoning immediately implies that $\langle A(\mathcal{X}), \nabla_C, \Delta_O\rangle$ is a twist-structure over a Heyting algebra. Thus, for every object $\mathcal{X} \in \mathsf{NE\text{-}Sp}$, we have $A(\mathcal{X}) \in \mathsf{Twist}^{\perp}$. Let us now look at morphisms.

Let $\mathcal{X}_1 = \langle X_1, \leq_1, \tau_1, C_1, O_1\rangle$ and $\mathcal{X}_2 = \langle X_2, \leq_2, \tau_2, C_2, O_2\rangle$ be NE-spaces, and let $f\colon X_1 \to X_2$ be an NE-morphism. Consider the dual map $A(f)\colon A(\mathcal{X}_2) \to A(\mathcal{X}_1)$ between the Heyting algebras of clopen up-sets of the two spaces. We know from Esakia duality that $A(f)$ is a Heyting algebra homomorphism. Let us check that it is in fact a twist-structure morphism (as defined in Section 2) from $\langle A(\mathcal{X}_2), \nabla_{C_2}, \Delta_{O_2}\rangle$ to $\langle A(\mathcal{X}_1), \nabla_{C_1}, \Delta_{O_1}\rangle$.

LEMMA 3.6 (cf. [15], Lemma 4.5)
Let $f\colon \mathcal{X}_1 \to \mathcal{X}_2$ be a morphism of NE-spaces. Then $A(f)\colon A(\mathcal{X}_2) \to A(\mathcal{X}_1)$ is a twist-structure morphism.

PROOF. We need to show that $A(f)[\nabla_{C_2}] \subseteq \nabla_{C_1}$ and $A(f)[\Delta_{O_2}] \subseteq \Delta_{O_1}$. Let $U \in A(f)[\nabla_{C_2}]$ and $V \in \nabla_{C_2}$ be such that $U = A(f)(V) = f^{-1}[V]$. Since $V \in \nabla_{C_2}$ we have $C_2 \subseteq V$. So $f^{-1}[C_2] \subseteq f^{-1}[V]$. Then, since $f[C_1] \subseteq C_2$, we have $C_1 \subseteq f^{-1}[C_2]$. Therefore, $C_1 \subseteq f^{-1}[V]$. Hence, $f^{-1}[V] \in \nabla_{C_1}$. Now let $U \in A(f)[\Delta_{O_2}]$ and assume that $V \in \Delta_{O_2}$ is such that $A(f)(V) = U$, so that $f^{-1}[V] = U$. Since $V \in \Delta_{O_2}$, we have $V \subseteq O_2$. Therefore, $U = f^{-1}[V] \subseteq f^{-1}[O_2] \subseteq O_1$. Hence, $U \in \Delta_{O_1}$. ∎

We thus have a functor $A\colon \mathsf{NE\text{-}Sp} \to \mathsf{Twist}^{\perp}$ from the category NE-spaces to the category of twist-structures over Heyting algebras. We are now going to see that, for any twist-structure $\mathcal{A}$ and any NE-space $\mathcal{X}$, there are natural isomorphisms $\sigma_{\mathbf{A}}\colon \mathcal{A} \cong A(X(\mathcal{A}))$ and $\epsilon_{\mathcal{X}}\colon \mathcal{X} \cong X(A(\mathcal{X}))$.

Given a twist-structure $\mathcal{A} = (\mathbf{A}, \nabla, \Delta)$, consider the twist-structure associated with the dual space of $\mathcal{A}$, that is, $\langle A(X(\mathcal{A})), \nabla_{C_{\mathcal{A}}}, \Delta_{O_{\mathcal{A}}}\rangle$. We know by Esakia duality that the map $\sigma_{\mathbf{A}}\colon A \to A(X(\mathcal{A}))$ defined by

$$\sigma_{\mathbf{A}}(a) = \{P \in X(\mathbf{A}) : a \in P\}$$

is a Heyting algebra isomorphism. Thus, we only need to check that $\sigma_{\mathbf{A}}$ is a twist-structure morphism too. This follows from next lemma (we will omit the subscript of $\sigma_{\mathbf{A}}$ when there is no ambiguity).

LEMMA 3.7 (cf. [15], Lemma 4.6)
For any twist-structure $\mathcal{A} = \langle \mathbf{A}, \nabla, \Delta \rangle$, the map $\sigma_{\mathbf{A}} : A \to A(X(\mathcal{A}))$ satisfies:

1. $\sigma_{\mathbf{A}}[\nabla] = \nabla_{C_{\mathcal{A}}}$,
2. $\sigma_{\mathbf{A}}[\Delta] = \Delta_{O_{\mathcal{A}}}$.

PROOF. (i) Let $a \in \nabla$. Then $C_{\mathcal{A}} \subseteq \sigma(a)$, so $\sigma(a) \in \nabla_{C_{\mathcal{A}}}$. Let now $\sigma(a) \in \nabla_{C_{\mathcal{A}}}$. Then $C_{\mathcal{A}} \subseteq \sigma(a)$. Suppose that $a \notin \nabla$. Let P be a prime filter such that $\nabla \subseteq P$ and $a \notin P$. Since $P \in C_{\mathcal{A}}$, we have $C_{\mathcal{A}} \nsubseteq \sigma(a)$, i.e. $\sigma(a) \notin \nabla_{C_{\mathcal{A}}}$, a contradiction.
(ii) Let $a \in \Delta$. Then $\sigma(a) \subseteq O_{\mathcal{A}}$. Therefore, $\sigma(a) \in \Delta_{O_{\mathcal{A}}}$. Suppose now $\sigma(a) \in \Delta_{O_{\mathcal{A}}}$, i.e. $\sigma(a) \subseteq O_{\mathcal{A}}$. Suppose $a \notin \Delta$. Let P be a prime filter such that $a \in P$ and $P \cap \Delta = \emptyset$. Then $P \in \sigma(a)$ and $P \notin O_{\mathcal{A}}$, i.e. $\sigma(a) \nsubseteq O_{\mathcal{A}}$, a contradiction. ∎

Conversely, consider the NE-space corresponding to the twist-structure $A(\mathcal{X})$:

$$\langle X(A(\mathcal{X})), \subseteq, \tau_{\mathcal{A}}, C_{A(\mathcal{X})}, O_{A(\mathcal{X})} \rangle.$$

Recall that the map $\epsilon_{\mathcal{X}} : X \to X(A(\mathcal{X}))$ defined, for all $x \in X$, by

$$\epsilon_{\mathcal{X}}(x) = \{U \in A(\mathcal{X}) : x \in U\}$$

is an Esakia-homeomorphism between $\langle X, \leq, \tau \rangle$ and $\langle X(A(\mathcal{X})), \subseteq, \tau_{\mathcal{A}} \rangle$. We check that $\epsilon_{\mathcal{X}}$ is in fact an NE-morphism as well.

LEMMA 3.8 (cf. [15], Lemma 4.7)
For any NE-space $\mathcal{X} = \langle X, \leq, \tau, C, O \rangle$, the map $\epsilon_{\mathcal{X}} : X \to X(A(\mathcal{X}))$ satisfies:

1. $\epsilon_{\mathcal{X}}[C] = C_{A(\mathcal{X})}$,
2. $\epsilon_{\mathcal{X}}[O] = O_{A(\mathcal{X})}$.

PROOF. (i) Recall that in a Priestley space any closed up-set is the intersection of all the clopen up-sets containing it, and similarly any open up-set is the union of all the clopen up-sets it contains (see, e.g. [7, Proposition A.1]). Given $x \in C$, we have to see that

$$\epsilon_{\mathcal{X}}(x) \in \bigcap \{\sigma(U) : U \in \nabla_C\},$$

that is, that for every $U \in A(\mathcal{X})$ such that $C \subseteq U$, it holds that $U \in \epsilon_{\mathcal{X}}(x)$. Assume then $C \subseteq U \in A(\mathcal{X})$. Then $x \in U$, so $U \in \epsilon_{\mathcal{X}}(x)$. Conversely, assume $x \in U$ for every clopen up-set $U \supseteq C$. Since C is a closed up-set, $C = \bigcap \{U \in A(\mathcal{X}) : C \subseteq U\}$. Therefore, $x \in C$ and $\epsilon_{\mathcal{X}}(x) \in \epsilon_{\mathcal{X}}[C]$.
(ii) For $x \in O$, we have to see that

$$\epsilon_{\mathcal{X}}(x) \in \bigcup \{\sigma(U) : U \in \Delta_O\}.$$

That is, that $U \in \epsilon_{\mathcal{X}}(x)$ for some $U \in A(\mathcal{X})$ with $U \subseteq O$. Suppose the contrary. Then, for every $U \in A(\mathcal{X})$ with $U \subseteq O$, it holds that $x \notin U$. Since O is an open up-set, $O = \bigcup \{U \in A(\mathcal{X}) : U \subseteq O\}$. It follows that $x \notin O$, a contradiction. Hence, $\epsilon_{\mathcal{X}}(x) \in \bigcup \{\sigma(U) : U \in \Delta_O\}$. Assume now $\epsilon_{\mathcal{X}}(x) \in \bigcup \{\sigma(U) : U \in \Delta_O\}$. Then there is a clopen up-set $U \subseteq O$ such that $x \in U$. Therefore, $x \in O$ and $\epsilon_{\mathcal{X}}(x) \in \epsilon_{\mathcal{X}}[O]$. ∎

$$\begin{tikzcd} \mathsf{N4}^{\perp} \arrow[r, bend left, "T"] & \mathsf{Twist}^{\perp} \arrow[l, bend left, "N"] \arrow[r, bend left, "X"] & \mathsf{NE\text{-}Sp} \arrow[l, bend left, "A"] \end{tikzcd}$$

FIG. 3. Equivalence between bounded N4-lattices and NE-spaces.

The fact that $\sigma_{\mathbf{A}}$ and $\epsilon_{\mathcal{X}}$ are natural follows immediately from Esakia duality. We highlight these facts in the following lemmas.

LEMMA 3.9
Let $h:\mathcal{A}_1\to\mathcal{A}_2$ be a morphism of twist-structures. Then $\sigma_{\mathcal{A}_2}\circ h=A(X(h))\circ\sigma_{\mathcal{A}_1}$.

LEMMA 3.10
Let $f:\mathcal{X}_1\to\mathcal{X}_2$ be a morphism of NE-spaces. Then $\epsilon_{\mathcal{X}_2}\circ f=X(A(f))\circ\epsilon_{\mathcal{X}_1}$.

Joining the previous results, we obtain the announced dual equivalences (Figure 3).

THEOREM 3.11 (cf. [15], Thm. 4.8)
The functors $X:\mathsf{Twist}^{\perp}\to\mathsf{NE\text{-}Sp}$ and $A:\mathsf{NE\text{-}Sp}\to\mathsf{Twist}^{\perp}$ establish a dual equivalence between the category $\mathsf{Twist}^{\perp}$ of twist-structures over Heyting algebras and the category $\mathsf{NE\text{-}Sp}$ of NE-spaces.

COROLLARY 3.12 (cf. [15], Cor. 4.9)
The category $\mathsf{N4}^{\perp}$ of bounded N4-lattices and the category $\mathsf{NE\text{-}Sp}$ of NE-spaces are dually equivalent via functors $X\circ T:\mathsf{N4}^{\perp}\to\mathsf{NE\text{-}Sp}$ and $N\circ A:\mathsf{NE\text{-}Sp}\to\mathsf{N4}^{\perp}$.

3.4 Duality for Twist

The above duality for $\mathsf{Twist}^{\perp}$ can be adapted to obtain a topological duality for Twist, the category of twist-structures over Brouwerian lattices. In this subsection, unless otherwise specified, by twist-structure we mean a twist-structure over a Brouwerian lattice.

Let $\mathcal{A}=(\mathbf{A},\nabla,\Delta)\in\mathsf{Twist}$. We consider the Heyting algebra $\mathbf{A}^*$ and its dual Esakia space $X(\mathbf{A}^*)$, which is a pointed Esakia space in our terminology. We define $C_{\mathcal{A}}$ and $O_{\mathcal{A}}$ similarly as before:

$$C_{\mathcal{A}}:=\bigcap\{\sigma_{\mathbf{A}^*}(a):a\in\nabla\}\qquad O_{\mathcal{A}}:=\bigcup\{\sigma_{\mathbf{A}^*}(a):a\in\Delta\}.$$

Then,

$$C_{\mathcal{A}}=\{P\in X(\mathbf{A}):\nabla\subseteq P\}\cup\{A\}\qquad\text{and}\qquad O_{\mathcal{A}}=\{P\in X(\mathbf{A}):P\cap\Delta\neq\emptyset\}\cup\{A\}.$$

So these sets are respectively a non-empty closed up-set and a non-empty open up-set. Moreover, the elements of $C_{\mathcal{A}}-\{A\}$ are maximal among the points of $X(\mathbf{A}^*)-\{A\}$. The objects of the category which we will prove to be dual to Twist are topological structures defined as follows.

DEFINITION 3.13
A *pointed NE-space* is a structure $\mathcal{X}=\langle X,\leq,\tau,C,O\rangle$ such that

1. $\langle X,\leq,\tau\rangle$ is a pointed Esakia space,
2. C is a non-empty closed up-set such that the elements of $C-\{1_X\}$ are maximal in $X-\{1_X\}$,
3. O is a non-empty open up-set.

It follows from the above considerations that, if $\mathcal{A}=(\mathbf{A},\nabla,\Delta)$ is a twist-structure, then $X^*(\mathcal{A}):=\langle X(\mathbf{A}^*),C_{\mathcal{A}},O_{\mathcal{A}}\rangle$ is a pointed NE-space which we take as our candidate for the dual of $\mathcal{A}$.

We observe that condition (2) of Definition 3.13 is equivalent to the following: for all clopen up-sets $U,V\in A(\mathcal{X})$, if $V\neq\emptyset$, then $C\subseteq U\cup(U\to_X V)$. To see this, let us prove that

$$\max(X-\{1_X\})=\bigcap\{U\cup(U\to_X V):U,V \text{ clopen up-sets and } V\neq\emptyset\}-\{1_X\}.$$

Suppose $x\in\max(X-\{1_X\})$. Let U,V be clopen up-sets with $V\neq\emptyset$. Suppose that $x\notin U\to_X V$. Then $\uparrow x\cap U\nsubseteq V$. By maximality of X, $\uparrow x=\{x,1_X\}$. Therefore if $x\notin U$, then $\uparrow x\cap U=\{1_X\}\subseteq V$, a contradiction. Thus $x\in U$. Hence, $x\in\bigcap\{U\cup(U\to_X V):U,V$ clopen up-sets and $V\neq\emptyset\}$. Conversely, suppose $x\notin\max(X-\{1_X\})$ and $x\neq 1_X$, i.e. there is $y\in X$ such that $x<y<1_X$. Since X is a Priestley space, we know that there are clopen up-sets U,V such that $x\notin U$, $y\in U$, $y\notin V$ and $1_X\in V$. Then $x\notin U\to_X V$, because $y\in\uparrow x\cap U$ and $y\notin V$. Therefore, $x\notin U\cup(U\to_X V)$. So $x\notin\bigcap\{U\cup(U\to_X V):U,V$ clopen up-sets and $V\neq\emptyset\}$.

Let $\mathcal{X}=\langle X,\leq,\tau,C,O\rangle$ be a pointed NE-space. In the Brouwerian lattice $A_*(\mathcal{X})=A(\mathcal{X})-\emptyset$ of non-empty clopen up-sets, we define the following filter ∇_C and ideal Δ_O:

$$\nabla_C:=\{U\in A_*(X):C\subseteq U\}\qquad \Delta_O:=\{U\in A_*(\mathcal{X}):U\subseteq O\}.$$

From the discussion above it immediately follows that $(A_*(X),\nabla_C,\Delta_O)$ is a twist-structure, which we will take as our candidate for the dual of $\mathcal{X}$.

We denote by $\mathsf{pNE\text{-}Sp}$ the category having as objects pointed NE-spaces and whose morphisms are defined as follows.

Definition 3.14
Let $\mathcal{X}_1=\langle X_1,\leq_1,\tau_1,C_1,O_1\rangle$ and $\mathcal{X}_2=\langle X_2,\leq_2,\tau_2,C_2,O_2\rangle$ be pointed NE-spaces. A *pNE-morphism* is a map $f:X_1\to X_2$ such that

1. f is a pointed Esakia function,
2. $f[C_1]\subseteq C_2$,
3. $f^{-1}[O_2]\subseteq O_1$.

Let $h:\mathcal{A}_1\to\mathcal{A}_1$ be a morphism between twist-structures $\mathcal{A}_1=(\mathbf{A}_1,\nabla_1,\Delta_1)$ and $\mathcal{A}_2=(\mathbf{A}_2,\nabla_2,\Delta_2)$. Let us consider the dual $X(h^*):X(\mathbf{A}_2^*)\to X(\mathbf{A}_1^*)$ of the extension $h^*:\mathbf{A}_1^*\to\mathbf{A}_2^*$ of h. This map is a pointed Esakia function which satisfies that $X(h^*)[C_{\mathcal{A}_2}]\subseteq C_{\mathcal{A}_1}$ and $X(h^*)^{-1}[O_{\mathcal{A}_1}]\subseteq O_{\mathcal{A}_2}$. The proof of this is analogous to that of Lemma 3.5. Therefore $X(h^*)$ is a pNE-morphism from $X^*(\mathcal{A}_2)$ to $X^*(\mathcal{A}_1)$. In this context we denote $X(h^*)$ by $X^*(h)$.

Let now $\mathcal{X}_1=\langle X_1,\leq_1,\tau_1,C_1,O_1\rangle$ and $\mathcal{X}_2=\langle X_2,\leq_2,\tau_2,C_2,O_2\rangle$ be pointed NE-spaces and $f:\mathcal{X}_1\to\mathcal{X}_2$ a pNE-morphism. Since f is a pointed Esakia function, the dual $A(f):A(\mathcal{X}_2)\to A(\mathcal{X}_1)$ of f as a pointed Esakia function when restricted to $A_*(\mathcal{X}_2)$ is a Brouwerian lattice homomorphism from $A_*(\mathcal{X}_2)$ to $A_*(\mathcal{X}_1)$. We denote this restriction by $A_*(f)$. By a proof similar to that of Lemma 3.6, and taking into account that $f(t_1)=t_2$, we obtain that $A_*(f)[\nabla_{C_2}]\subseteq\nabla_{C_1}$ and $A_*(f)[\Delta_{O_2}]\subseteq\Delta_{O_1}$. Therefore $A_*(f)$ is a twist-structure morphism from $A_*(\mathcal{X}_2)$ to $A_*(\mathcal{X}_1)$. We take the map $A_*(f)$ as the dual of f in the category Twist.

If $\mathcal{A}=(\mathbf{A},\nabla,\Delta)$ is a twist-structure, it can be shown with a proof similar to that of Lemma 3.7 that the map $\sigma_{\mathbf{A}^*}$ restricted to A is an isomorphism between $\mathcal{A}$ and $A_*(X^*(\mathcal{A}))$.

Let $\mathcal{X}=\langle X,\leq,\tau,C,O\rangle$ be a pointed NE-space and consider the space $X^*(A_*(\mathcal{X}))$ corresponding to its dual twist-structure $A_*(\mathcal{X})$. We have that $\mathcal{X}^*(A_*(\mathcal{X}))$ is the pointed Esakia space dual to the Brouwerian lattice of the twist-structure $A_*(\mathcal{X})$, so $X^*(A_*(\mathcal{X}))$ is the dual Esakia space of the

$$\begin{tikzcd} \mathsf{N4} \arrow[r, bend left, "T"] & \mathsf{Twist} \arrow[l, bend left, "N"] \arrow[r, bend left, "X"] & \mathsf{pNE\text{-}Sp} \arrow[l, bend left, "A"] \end{tikzcd}$$

FIG. 4. Equivalence between N4-lattices and pointed NE-spaces.

Heyting algebra $A(\mathcal{X})$ of clopen up-sets of $\mathcal{X}$. Hence, the map $\epsilon_{\mathcal{X}}: X \to X(A(\mathcal{X})) = X^*(A_*(X)$ is a homeomorphism and an order isomorphism. Moreover, $\epsilon_{\mathcal{X}}[C] = C_{A_*(\mathcal{X})}$ and $\epsilon_{\mathcal{X}}[O] = O_{A_*(\mathcal{X})}$. The proof is analogous to the proof of Lemma 3.8. Therefore $\epsilon_{\mathcal{X}}$ is an isomorphism in the category $\mathsf{pNE\text{-}Sp}$.

From the above considerations the next theorem easily follows (Figure 4).

THEOREM 3.15
$X^*: \mathsf{Twist} \to \mathsf{pNE\text{-}Sp}$ and $A_*: \mathsf{pNE\text{-}Sp} \to \mathsf{Twist}$ are contravariant functors that establish a dual equivalence between the category Twist of twist-structures and the category $\mathsf{pNE\text{-}Sp}$ of pointed NE-spaces.

COROLLARY 3.16
The category $\mathsf{N4}$ of N4-lattices and the category $\mathsf{pNE\text{-}Sp}$ of pointed NE-spaces are dually equivalent via functors $X^* \circ T: \mathsf{N4} \to \mathsf{pNE\text{-}Sp}$ and $N \circ A_*: \mathsf{pNE\text{-}Sp} \to \mathsf{N4}$.

4 Modal N4-lattices

We are now going to extend the topological duality introduced in the previous section to N4-lattices with unary modal operations.

DEFINITION 4.1 ([24])
A *monotone modal N4-lattice (MN4-lattice)* is an algebra $\mathbf{B} = \langle B, \wedge, \vee, \to, \sim, \Box \rangle$ such that the reduct $\langle B, \wedge, \vee, \to, \sim \rangle$ is an N4-lattice and, for all $a, b \in B$,

(Q1) if $a \preceq b$, then $\Box a \preceq \Box b$,
(Q2) if $\sim a \preceq \sim b$, then $\sim \Box a \preceq \sim \Box b$,

where $\preceq$ is the pre-order introduced in Definition 2.1. An *MN4$^{\perp}$-lattice* (or a *bounded MN4-lattice*) is an MN4-lattice whose lattice reduct is bounded.

It is easy to check that by defining $\Diamond := \sim \Box \sim$ we obtain another unary operation satisfying (Q1) and (Q2), i.e. for all $a, b \in B$,

if $a \preceq b$, then $\Diamond a \preceq \Diamond b$,
if $\sim a \preceq \sim b$, then $\sim \Diamond a \preceq \sim \Diamond b$.

MN4-lattices obviously form a quasivariety; at present we do not know whether this class is in fact a variety or not. Let us mention one subvariety of MN4-lattices that is already known in the literature.

DEFINITION 4.2
A *BK-lattice* is an algebra $\mathbf{B} = \langle B, \wedge, \vee, \to, \sim, \Box, \perp \rangle$ such that the reduct $\langle B, \wedge, \vee, \to, \sim, \Box \rangle$ is a bounded MN4-lattice (with bottom element $\perp$) and, for all $a, b \in B$,

(E1) $(a \to b) \to a \preceq a$,
(E2) $\Box a \wedge \Box b \preceq \Box(a \wedge b)$,

(E3) $\quad \Box(a \to a) = \Box(a \to a) \to \Box(a \to a)$,
(E4) $\quad -\Box a \equiv \Diamond - a$,
(E5) $\quad -\Diamond a \equiv \Box - a$.

where $a \equiv b$ abbreviates the two equalities $a \preceq b$ and $b \preceq a$, while $-a$ abbreviates $a \to \bot$.

BK-lattices were introduced in [20] as an algebraic counterpart of the modal expansion of the Belnap–Dunn logic of [21], although the definition presented above is adapted from [24, Definition 3.5]. It is worth pointing out that, since BK-lattices are particular examples of (bounded) MN4-lattices, all the results that we will prove about the latter apply to BK-lattices as well.

Conditions (Q1) and (Q2) ensure that in any MN4-lattice operations $\Box$ and $\Diamond$ are compatible with the relation $\equiv$ introduced in Definition 2.1. So we can define operations $\Box$, $\Diamond$ on the Brouwerian lattice $\mathbf{B}_{\bowtie} = \langle B, \wedge, \vee, \to \rangle / \equiv$ as follows:

$$\Box[a] = [\Box a] \qquad \text{and} \qquad \Diamond[a] = [\sim\Box\sim a].$$

These operations satisfy the following monotonicity properties (here $\leq$ denotes the lattice order of $\mathbf{B}_{\bowtie}$):

if $[a] \leq [b]$, then $\Box[a] \leq \Box[b]$,
if $[a] \leq [b]$, then $\Diamond[a] \leq \Diamond[b]$.

These properties explain the choice of the terminology 'monotone modal N4-lattice'.

Moreover, the filter $\nabla(\mathbf{B}) \subseteq B/\equiv$ and the ideal $\Delta(\mathbf{B}) \subseteq B/\equiv$ satisfy:

- if $a \vee b \in \nabla(\mathbf{B})$ and $a \wedge b \in \Delta(\mathbf{B})$, then $\Box a \vee \Diamond b \in \nabla(\mathbf{B})$ and $\Box a \wedge \Diamond b \in \Delta(\mathbf{B})$.

These observations suggest a way to represent MN4-lattices as twist-structure products. We will need the following definitions.

Definition 4.3
A *monotone bimodal Brouwerian lattice* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \to \Box, \Diamond \rangle$ such that the reduct $\langle A, \wedge, \vee, \to \rangle$ is a Brouwerian lattice and $\Box, \Diamond \colon A \to A$ are monotone maps. A *monotone bimodal Heyting algebra* is a bounded monotone bimodal Brouwerian lattice.

Given a monotone bimodal Brouwerian lattice $\mathbf{A}$, we can construct an MN4-lattice $\mathbf{A}^{\bowtie}$ in the following way:

- the N4-lattice reduct of $\mathbf{A}^{\bowtie}$ is the twist-structure $\langle A, \wedge, \vee, \to \rangle^{\bowtie}$ introduced above,
- $\Box\langle a, b \rangle = \langle \Box a, \Diamond b \rangle$ for any $\langle a, b \rangle \in A \times A$.

Routine checking shows that the algebra $\mathbf{A}^{\bowtie}$ is in fact an MN4-lattice [24, Proposition 4.3]. However, as in the case of N4-lattices, not all MN4-lattices arise in this way as we need to consider all subalgebras of $\mathbf{A}^{\bowtie}$. These can be characterized in the following way.

Let $\mathbf{A} = \langle A, \wedge, \vee, \to \Box, \Diamond \rangle$ be a bimodal Brouwerian lattice, $\nabla \subseteq A$ a filter containing the dense elements of $\mathbf{A}$ and $\Delta \subseteq A$ an ideal satisfying:

- if $a \vee b \in \nabla$ and $a \wedge b \in \Delta$, then $\Box a \vee \Diamond b \in \nabla$ and $\Box a \wedge \Diamond b \in \Delta$.

It is easy to check that the set

$$B := \{\langle a, b \rangle \in A \times A \colon\ a \vee b \in \nabla,\ a \wedge b \in \Delta\}$$

is a subalgebra of $\mathbf{A}^{\bowtie}$. Let us denote by $Tw(\mathbf{A},\nabla,\Delta)$ the MN4-lattice obtained through this construction. The following result shows that all MN4-lattices arise in this way.

Proposition 4.4
Every MN4-lattice (bounded MN4-lattice) $\mathbf{B}$ is isomorphic to the twist-structure $Tw(\mathbf{B}_{\bowtie},\nabla(\mathbf{B}),\Delta(\mathbf{B}))$, where $\mathbf{B}_{\bowtie}$ is a monotone bimodal Brouwerian lattice (Heyting algebra), through the map $j_{\mathbf{B}}: B \to B/{\equiv} \times B/{\equiv}$ defined, for all $a \in B$, as

$$j_{\mathbf{B}}(a) := \langle [a], [\sim a] \rangle.$$

Proof. We know by Proposition 2.2 that the map $j_{\mathbf{B}}$ is an isomorphism between the N4-lattice reducts of $\mathbf{B}$ and $Tw(\mathbf{B}_{\bowtie},\nabla(\mathbf{B}),\Delta(\mathbf{B}))$. By [24, Theorem 4.5], we have that $j_{\mathbf{B}}$ preserves the modal operator, which concludes our proof. ∎

Using the above result, we are going to extend the categorial equivalence between N4-lattices and twist-structures to MN4-lattices and modal twist-structures, defined as follows.

Definition 4.5
A *monotone modal twist-structure* is a triple $\mathcal{A} = \langle \mathbf{A},\nabla,\Delta \rangle$ where

(i) $\mathbf{A}$ is a monotone bimodal Brouwerian lattice,
(ii) ∇ is a filter that includes the dense elements of $\mathbf{A}$,
(iii) Δ is an ideal of $\mathbf{A}$,
(iv) for every $a,b \in A$, if $a \vee b \in \nabla$ and $a \wedge b \in \Delta$, then $\Box a \vee \Diamond b \in \nabla$ and $\Box a \wedge \Diamond b \in \Delta$.

Definition 4.6
Let $\mathcal{A}_1 = \langle \mathbf{A}_1,\nabla_1,\Delta_1 \rangle$ and $\mathcal{A}_2 = \langle \mathbf{A}_2,\nabla_2,\Delta_2 \rangle$ be monotone modal twist-structures. A *morphism* from $\mathcal{A}_1$ to $\mathcal{A}_2$ is an homomorphism of monotone bimodal Brouwerian lattices $h: \mathbf{A}_1 \to \mathbf{A}_2$ such that

(i) $h[\nabla_1] \subseteq \nabla_2$,
(ii) $h[\Delta_1] \subseteq \Delta_2$.

The category of monotone modal twist-structures, denoted MTwist, has as objects monotone modal twist-structures and as morphisms the above-defined maps between them. We define the category $\mathsf{MTwist}^{\perp}$ by restricting the objects to bounded Brouwerian lattices (i.e. Heyting algebras) and by requiring that the morphisms preserve the bounds. We are going to prove that MTwist ($\mathsf{MTwist}^{\perp}$) is equivalent to the category $\mathsf{MN4}$ ($\mathsf{MN4}^{\perp}$) having as objects (bounded) MN4-lattices and as morphisms algebraic (bounded) MN4-lattice homomorphisms.

We define functors T: $\mathsf{MN4} \to \mathsf{MTwist}$ and N: $\mathsf{MTwist} \to \mathsf{MN4}$ in the same way as in the non-modal case, and likewise for the functions $j_{\mathbf{B}}$: $\mathbf{B} \to N(H(\mathbf{B}))$ and $\eta_{\mathcal{A}}$: $\mathcal{A} \to T(N(\mathcal{A}))$. We proceed to check that these definitions work in the modal case as well.

Lemma 4.7
For any monotone modal twist-structure $\mathcal{A}$, the map $\eta_{\mathcal{A}}$: $\mathcal{A} \to T(N(\mathcal{A}))$ defined in (2.1) is an isomorphism in the category MTwist.

PROOF. We need only to show that $\eta_{\mathcal{A}}(\Box a) = \Box \eta_{\mathcal{A}}(a)$ and $\eta_{\mathcal{A}}(\Diamond a) = \Diamond \eta_{\mathcal{A}}(a)$ for all $a \in A$. We have

$$\begin{aligned}\eta_{\mathcal{A}}(\Box a) &= [\langle \Box a, (\Box a)' \rangle]_{\equiv} \\ &= [\langle \Box a, \Diamond(a') \rangle]_{\equiv} \\ &= [\Box \langle a, a' \rangle]_{\equiv} \\ &= \Box [\langle a, a' \rangle]_{\equiv} \\ &= \Box \eta_{\mathcal{A}}(a).\end{aligned}$$

We recall once more that the equality $[\langle \Box a, (\Box a)' \rangle]_{\equiv} = [\langle \Box a, \Diamond(a') \rangle]_{\equiv}$ holds because it only depends on the first component of each pair. A similar argument allows us to prove that $\eta_{\mathcal{A}}(\Diamond a) = \Diamond \eta_{\mathcal{A}}(a)$. ∎

LEMMA 4.8
Let $h: \mathcal{A}_1 \to \mathcal{A}_1$ be a morphism of monotone modal twist-structures. Then the map $N(h): N(\mathcal{A}_1) \to N(\mathcal{A}_2)$ is such that $N(h)(\Box \langle a,b \rangle) = \Box N(h)(\langle a,b \rangle)$ for all $\langle a,b \rangle \in N(\mathcal{A}_1)$. Therefore $N(h)$ is an MN4-lattice morphism.

PROOF. It is sufficient to observe that $N(h)(\Box \langle a,b \rangle) = N(h)(\langle \Box a, \Diamond b \rangle = \langle h(\Box a), h(\Diamond b) \rangle = \langle \Box h(a), \Diamond h(b) \rangle = \Box \langle h(a), h(b) \rangle = \Box N(h)(\langle a,b \rangle)$. ∎

LEMMA 4.9
Let $f: \mathbf{B}_1 \to \mathbf{B}_2$ be an MN4-lattice homomorphism. Then:

$$T(f)(\Box [a]_{\equiv}) = \Box T(f)([a]_{\equiv}) \qquad \text{and} \qquad T(f)(\Diamond [a]_{\equiv}) = \Diamond T(f)([a]_{\equiv}).$$

Therefore $T(f): T(\mathbf{B}_1) \to T(\mathbf{B}_2)$ is a monotone modal twist-structure morphism.

PROOF. It is sufficient to observe that $T(f)(\Box [a]_{\equiv}) = T(f)([\Box a]_{\equiv}) = [f(\Box a)]_{\equiv} = [\Box f(a)]_{\equiv} = \Box [f(a)]_{\equiv} = \Box T(f)([a]_{\equiv})$. Similarly, $T(f)(\Diamond [a]_{\equiv}) = T(f)([\sim \Box \sim a]_{\equiv}) = [f(\sim \Box \sim a)]_{\equiv} = [\sim \Box \sim f(a)]_{\equiv} = \Diamond [f(a)]_{\equiv} = \Diamond T(f)([a]_{\equiv})$ ∎

The previous lemmas (together with Proposition 4.4) immediately imply the announced equivalence result.

THEOREM 4.10
Functors $T: \mathsf{MN4} \to \mathsf{MTwist}$ and $N: \mathsf{MTwist} \to \mathsf{MN4}$ establish a natural equivalence between the category $\mathsf{MN4}$ of (bounded) MN4-lattices and the category MTwist of twist-structures over monotone bimodal Brouwerian lattices (Heyting algebras).

As mentioned above, BK-lattices are particular examples of bounded MN4-lattices. To be more precise, we can rephrase the representation result proved in [20] in our terms saying that BK-lattices correspond exactly to modal twist-structures $\mathcal{A} = \langle \mathbf{A}, \nabla, \Delta \rangle$ such that:

- $\mathbf{A}$ is a modal Boolean algebra
- $a \in \nabla$ implies $\Box a \in \nabla$
- $a \in \Delta$ implies $\Diamond a \in \Delta$.

We remind the reader that a *modal Boolean algebra* or simply a *modal algebra* [4] is an algebra $\langle A, \wedge, \vee, \to, \Box, \Diamond, 0, 1 \rangle$ such that the reduct $\langle A, \wedge, \vee, \to, 0, 1 \rangle$ is a Boolean algebra and the modal operations satisfy, for all $a, b \in A$: $\Box(a \wedge b) = \Box a \wedge \Box b$, $\Box 1 = 1$ and $\Box a = \neg \Diamond \neg a$, where $\neg a$ denotes the Boolean complement of a.

It is easy to prove that the equivalence stated in Theorem 4.10 restricts to an equivalence between full sub-categories corresponding to BK-lattices and to twist-structures over modal Boolean algebras (Figure 5).

$$\begin{tikzcd} \mathsf{MN4}\ (\mathsf{MN4}^{\perp}) \arrow[r, bend left, "T"] & \mathsf{MTwist}\ (\mathsf{MTwist}^{\perp}) \arrow[l, bend left, "N"] \end{tikzcd}$$

$$\begin{tikzcd} \mathsf{BKlat} \arrow[r, bend left, "T"] & \mathsf{ModBA} \arrow[l, bend left, "N"] \end{tikzcd}$$

FIG. 5. Equivalence between modal N4-lattices and monotone bimodal Brouwerian lattices.

5 Duality for modal twist-structures

In this section we extend our topological duality for twist-structures to modal twist-structures. As in the non-modal case we relied on Esakia duality for Heyting algebras, we will now build on the duality for distributive lattices with monotone operators of [14].

Let $\mathbf{A}=\langle A,\wedge,\vee,\rightarrow\Box,\Diamond\rangle$ be a monotone bimodal Heyting algebra and denote by $\langle X(\mathbf{A}),\tau_{\mathbf{A}},\subseteq\rangle$ the corresponding Esakia space. We denote by $\mathcal{P}^{\uparrow}(X(\mathbf{A}))$ the collection of upward subsets of $X(\mathbf{A})$ and by $\mathrm{Fi}(\mathbf{A})$ the set of all lattice filters of $\mathbf{A}$. For each operation $\bullet\in\{\Box,\Diamond\}$ we define a *neighbourhood function* $\nu_{\bullet}: X(\mathbf{A})\rightarrow\mathcal{P}(\mathcal{P}^{\uparrow}(X(\mathbf{A})))$ as follows: for every prime filter $P\in X(\mathbf{A})$,

$$\nu_{\bullet}(P)=\{U\in A(X(\mathbf{A})):\exists F\in\mathrm{Fi}(\mathbf{A})\text{ s.t. }\bullet[F]\subseteq P\text{ and }\{Q\in X(\mathbf{A}):F\subseteq Q\}\subseteq U\}.$$

Notice that $\nu_{\bullet}(P)$ is an up-set of $\langle\mathcal{P}^{\uparrow}(X(\mathbf{A})),\subseteq\rangle$. It is also obvious that $\nu_{\bullet}$ is monotone with respect to the Esakia order of $X(\mathbf{A})$. The structure $\langle X(\mathbf{A}),\tau_{\mathbf{A}},\subseteq,\nu_{\Box},\nu_{\Diamond}\rangle$ will be called the *monotone modal Esakia space* of $\mathbf{A}$.

Using the neighbourhood function $\nu_{\bullet}$ we can represent the algebraic operation $\bullet$ in the Heyting algebra of clopen up-sets of $X(\mathbf{A})$ through the following definition: for any $U\in A(X(\mathbf{A}))$,

$$\bullet_{\nu_{\bullet}}(U):=\{P\in X(\mathbf{A}):U\in\nu_{\bullet}(P)\}.$$

The following proposition shows that the above definitions make sense and that, using them, we obtain that the isomorphism $\sigma_{\mathbf{A}}: A\rightarrow A(X(\mathbf{A}))$ preserves the monotone modal operators as well.

PROPOSITION 5.1
For every $a\in A$, $\sigma_{\mathbf{A}}(\bullet a)=\bullet_{\nu_{\bullet}}\sigma_{\mathbf{A}}(a)$.

PROOF. Let $P\in\sigma_{\mathbf{A}}(\bullet a)$. Then $\bullet a\in P$. Since $\bullet$ is monotone in $\mathbf{A}$, it follows that $\bullet[\uparrow a]\subseteq P$. Moreover, if $Q\in X(\mathbf{A})$ is such that $\uparrow a\subseteq Q$, then $Q\in\sigma_{\mathbf{A}}(a)$. From the fact that $\uparrow a$ is a filter, it then follows that $P\in\bullet_{\nu_{\bullet}}\sigma_{\mathbf{A}}(a)$. Suppose now that $P\in\bullet_{\nu_{\bullet}}\sigma_{\mathbf{A}}(a)$. Let F be a filter of $\mathbf{A}$ such that $\bullet[F]\subseteq P$ and $\{Q\in X(\mathbf{A}):F\subseteq Q\}\subseteq\sigma_{\mathbf{A}}(a)$. We have to show that $\bullet a\in P$. Suppose the contrary. Then $a\notin F$. So there is a prime filter Q such that $F\subseteq Q$ and $a\notin Q$. This contradicts the fact that $\{Q\in X(\mathbf{A}):F\subseteq Q\}\subseteq\sigma_{\mathbf{A}}(a)$. Hence we conclude that $\bullet a\in P$. ∎

By Proposition 5.1 we already know that $\sigma_{\mathbf{A}}$ is an isomorphism between the monotone bimodal Heyting algebra $\mathbf{A}=\langle A,\wedge,\vee,\rightarrow,0,1,\Box,\Diamond\rangle$ and $\langle A(X(\mathbf{A})),\cap,\cup,\rightarrow,\emptyset,\ X(\mathbf{A}),\Box_{\nu_{\Box}},\Diamond_{\nu_{\Diamond}}\rangle$. Thus, in order to extend this to a monotone modal twist-structure $\langle\mathbf{A},\nabla,\Delta\rangle$, we only need to take care of representing ∇ and Δ. Let us look at how properties (ii)-(iv) of Definition 4.5 are reflected on the NE-space corresponding to a monotone modal twist-structure.

PROPOSITION 5.2
For any monotone modal twist-structure $\mathcal{A}=\langle \mathbf{A},\nabla,\Delta\rangle$ over a monotone bimodal Heyting algebra, the maps $\nu_\Box,\nu_\Diamond: X(\mathbf{A})\to\mathcal{P}(\mathcal{P}^\uparrow(X(\mathbf{A})))$ satisfy the following condition: for all clopen up-sets $U,V\in X(\mathbf{A})$,

$$\text{if } C_{\mathcal{A}}\subseteq U\cup V \text{ and } U\cap V\subseteq O_{\mathcal{A}}, \text{ then } C_{\mathcal{A}}\subseteq \Box_{\nu_\Box}U\cup\Diamond_{\nu_\Diamond}V \text{ and } \Box_{\nu_\Box}U\cap\Diamond_{\nu_\Diamond}V\subseteq O_{\mathcal{A}}.$$

PROOF. Let $a,b\in A$ be such that $\sigma_{\mathbf{A}}(a)=U$ and $\sigma_{\mathbf{A}}(b)=V$. Assume $C_{\mathcal{A}}\subseteq U\cup V$ and $U\cap V\subseteq O_{\mathcal{A}}$. From the first assumption we have $C_{\mathcal{A}}\subseteq\sigma_{\mathbf{A}}(a)\cup\sigma_{\mathbf{A}}(b)=\sigma_{\mathbf{A}}(a\vee b)$, from the second $\sigma_{\mathbf{A}}(a)\cap\sigma_{\mathbf{A}}(b)=\sigma_{\mathbf{A}}(a\wedge b)\subseteq O_{\mathcal{A}}$. Let us check that $a\vee b\in\nabla$ and $a\wedge b\in\Delta$. The first assumption means that, for every prime filter $P\supseteq\nabla$, one has $a\vee b\in P$. Now, if $a\vee b\notin\nabla$, there would be a prime filter $P\supseteq\nabla$ such that $a\vee b\notin P$, a contradiction. Similarly, $a\wedge b\notin\Delta$ implies that there is a prime filter P such that $a\wedge b\in P$ and $\Delta\cap P=\emptyset$. Then $P\in\sigma_{\mathbf{A}}(a\wedge b)$ and therefore $P\in O_{\mathcal{A}}$. This means $P\cap\Delta\neq\emptyset$, a contradiction. We conclude that $a\vee b\in\nabla$ and $a\wedge b\in\Delta$. Applying Definition 4.5 (iv), we have $\Box a\vee\Diamond b\in\nabla$ and $\Box a\wedge\Diamond b\in\Delta$. Therefore, $C_{\mathcal{A}}\subseteq\sigma_{\mathbf{A}}(\Box a\vee\Diamond b)=\sigma_{\mathbf{A}}(\Box a)\cup\sigma_{\mathbf{A}}(\Diamond b)=\Box_{\nu_\Box}\sigma_{\mathbf{A}}(a)\cup\Diamond_{\nu_\Diamond}\sigma_{\mathbf{A}}(b)=\Box_{\nu_\Box}U\cup\Diamond_{\nu_\Diamond}V$. Similarly we obtain $\sigma_{\mathbf{A}}(\Box a\wedge\Diamond b)=\sigma_{\mathbf{A}}(\Box a)\cap\sigma_{\mathbf{A}}(\Diamond b)=\Box_{\nu_\Box}U\cap\Diamond_{\nu_\Diamond}V\subseteq O_{\mathcal{A}}$. ∎

We put the above observations together to introduce our formal definition of spaces corresponding to monotone modal twist-structures over Heyting algebras.

Let $\langle X,\leq,\tau,C,O\rangle$ be an NE-space and let ν be a neighborhood function. Recall that the corresponding modal operation on the set of clopen up-sets is defined by

$$\bullet_\nu(U)=\{x\in X: U\in\nu(x)\}.$$

DEFINITION 5.3
A *monotone modal NE-space (MNE-space)* is a structure $\mathcal{X}=\langle X,\leq,\ \tau,\nu_1,\nu_2,C,O\rangle$ such that $\langle X,\leq,\tau,C,O\rangle$ is an NE-space and $\nu_i: X\to\mathcal{P}(\mathcal{P}^\uparrow(X))$ are neighbourhood functions satisfying the following properties: for all $x,y\in X$ and all clopen up-sets $U,V\in A(\mathcal{X})$,

(i) $x\leq y$ implies $\nu_i(x)\subseteq\nu_i(y)$ with $i\in\{1,2\}$,
(ii) $\Box_{\nu_1}U,\ \Diamond_{\nu_2}U\in A(\mathcal{X})$,
(iii) if $C\subseteq U\cup V$ and $U\cap V\subseteq O$, then $C\subseteq\Box_{\nu_1}U\cup\Diamond_{\nu_2}V$ and $\Box_{\nu_1}U\cap\Diamond_{\nu_2}V\subseteq O$,
(iv) $\nu_i(x)$ is an up-set for each $i\in\{1,2\}$ and for all $x\in X$,

where $\Box_{\nu_1},\Diamond_{\nu_2}$ are, respectively, the modal operations corresponding to the neighborhood functions ν_1 and ν_2.

The above observations immediately imply that the space corresponding to a monotone modal twist-structure satisfies the properties of Definition 5.3.

PROPOSITION 5.4
Let $\mathcal{A}=\langle\mathbf{A},\nabla,\Delta\rangle$ be a monotone modal twist-structure over a Heyting algebra. Then

$$X(\mathcal{A})=\langle X(\mathbf{A}),\subseteq,\tau_{\mathbf{A}},\nu_\Box,\nu_\Diamond,C_{\mathcal{A}},O_{\mathcal{A}}\rangle$$

is an MNE-space.

Conversely, let us check that the Heyting algebra of clopen up-sets of an MNE-space is the algebraic reduct of a monotone modal twist-structure.

PROPOSITION 5.5
Let $\mathcal{X} = \langle X, \leq, \tau, \nu_1, \nu_2, C, O\rangle$ be an MNE-space. Then the twist-structure $\mathcal{A}(\mathcal{X}) = \langle A(\mathcal{X}), \nabla_C, \Delta_O\rangle$ is a monotone modal twist-structure over a Heyting algebra, when we endow $A(\mathcal{X})$ with the operations $\Box_{\nu_1}$ and $\Diamond_{\nu_2}$.

PROOF. Let $U, V \in A(\mathcal{X})$. If $U \subseteq V$, then $\Box_{\nu_1} U \subseteq \Box_{\nu_1} V$ and $\Diamond_{\nu_2} U \subseteq \Diamond_{\nu_2} V$ because $\nu_1(x), \nu_2(x) \in \mathcal{P}^{\uparrow}(X)$ for all $x \in X$. Hence, $\langle A(\mathcal{X}), \Box_{\nu_1}, \Diamond_{\nu_2}\rangle$ is a monotone bimodal Heyting algebra. It remains to check that ∇_C and Δ_O satisfy property (iv) of Definition 4.5. Suppose that $U \cup V \in \nabla_C$ and $U \cap V \in \Delta_O$. The former means that $C \subseteq U \cup V$, the latter that $U \cap V \subseteq O$. Then, by Definition 5.3 (iii), we have $C \subseteq \Box_{\nu_1} U \cup \Diamond_{\nu_2} V$ and $\Box_{\nu_1} U \cap \Diamond_{\nu_2} V \subseteq O$. Hence we obtain $\Box_{\nu_1} U \cup \Diamond_{\nu_2} V \in \nabla_C$ and $\Box_{\nu_1} U \cap \Diamond_{\nu_2} V \in \Delta_O$. ∎

In order to view MNE-spaces as a category, we need to specify the morphisms. This is done through the following definition.

DEFINITION 5.6
A map $f: X \to X'$ between two MNE-spaces $\mathcal{X}$ and $\mathcal{X}'$ is an *MNE-morphism* if f is an NE-morphism which additionally satisfies that, for every $x \in X$ and every clopen up-set $U \in A(X')$,

(i) $U \in \nu_1'(f(x))$ if and only if $f^{-1}[U] \in \nu_1(x)$,
(ii) $U \in \nu_2'(f(x))$ if and only if $f^{-1}[U] \in \nu_2(x)$.

It is easy to see that the composition of MNE-morphisms is an MNE-morphism and that the identity map of an MNE-space is a morphism. We can thus define a category $\mathsf{MNE\text{-}Sp}$ having as objects MNE-spaces and as morphisms MNE-morphisms. We proceed to introduce functors $X: \mathsf{MTwist}^{\perp} \to \mathsf{MNE\text{-}Sp}$ and $A: \mathsf{MNE\text{-}Sp} \to \mathsf{MTwist}^{\perp}$ adopting the same definitions as for (non-modal) twist-structures and NE-spaces.

Let us check that Definition 5.6 is actually capturing the essential properties of morphisms between spaces that are dual to modal twist-structures.

LEMMA 5.7
Let $h: \mathcal{A} \to \mathcal{A}'$ be a morphism between monotone modal twist-structures $\mathcal{A} = \langle \mathbf{A}, \nabla, \Delta\rangle$ and $\mathcal{A}' = \langle \mathbf{A}', \nabla', \Delta'\rangle$ over a Heyting algebra. Then $X(h): \mathcal{X}(\mathcal{A}') \to \mathcal{X}(\mathcal{A})$ is an MNE-morphism between the corresponding spaces.

PROOF. By Lemma 3.5, we just need to check that conditions (i) and (ii) of Definition 5.6 are satisfied. We only prove (i) as the proof of (ii) is analogous. Let $P \in X(\mathbf{A}')$ and $U \in A(X(\mathbf{A}))$. We can assume that $U = \sigma_{\mathbf{A}}(a)$ for some $a \in A$. Suppose $U \in \nu_1(X(h)(P))$. Let $F \subseteq A$ be a filter such that $\Box[F] \subseteq X(h)(P) = h^{-1}[P]$ and such that $\{Q \in X(\mathbf{A}) : F \subseteq Q\} \subseteq \sigma_{\mathbf{A}}(a)$. This implies that $a \in F$. Thus, $\Box a \in h^{-1}[P]$, i.e. $h(\Box a) = \Box' h(a) \in P$. From the monotonicity of $\Box'$ it follows that $\Box'[\uparrow h(a)] \subseteq P$. Moreover, if $Q \in X(\mathbf{A})$ is such that $\uparrow h(a) \subseteq Q$, then $Q \in \sigma(h(a))$. Therefore, since $\uparrow h(a)$ is a filter of $\mathbf{A}'$, we conclude that $\sigma_{\mathbf{A}'}(h(a)) \in \nu_1'(P)$. Note that $Q \in X(h)^{-1}[\sigma_{\mathbf{A}}(a)]$ iff $h^{-1}[Q] \in \sigma(a)$ iff $h(a) \in Q$ iff $Q \in \sigma_{\mathbf{A}'}(h(a))$. Thus, $X(h)^{-1}[\sigma_{\mathbf{A}}(a)] = \sigma_{\mathbf{A}'}(h(a)) \in \nu_1'(P)$.
To prove the other implication of (i), suppose $X(h)^{-1}[\sigma(a)] \in \nu_1'(P)$. Then, $\sigma_{\mathbf{A}'}(h(a)) \in \nu_1'(P)$. Let $G \subseteq A'$ be a filter such that $\Box'[G] \subseteq P$ and $\{Q \in X(\mathbf{A}') : G \subseteq Q\} \subseteq \sigma_{\mathbf{A}'}(h(a))$. Then $h^{-1}[G] \subseteq h^{-1}[\Box'^{-1}[P]] \subseteq \Box[h^{-1}[P]] = \Box^{-1}[X(h)(P)]$. Suppose $Q' \in X(\mathbf{A})$ is such that $h^{-1}[G] \subseteq Q'$ but $a \notin Q'$. Then $h(a) \notin G$. So there is $Q \in X(\mathbf{A}')$ such that $G \subseteq Q$ and $h(a) \notin Q$, a contradiction. Thus $\{Q' \in X(\mathbf{A}) : h^{-1}[G] \subseteq Q'\} \subseteq \sigma_{\mathbf{A}}(a)$. We conclude that $U = \sigma_{\mathbf{A}}(a) \in \nu_1(X(h)(P))$. ∎

Next we check that the NE-space isomorphism $\epsilon_{\mathcal{X}}: \mathcal{X} \cong X(A(\mathcal{X}))$ is an MNE-morphism, and therefore an isomorphism in the category $\mathsf{MNE\text{-}Sp}$.

LEMMA 5.8
Let $\mathcal{X}=\langle X,\leq,\tau,C,O,\nu_1,\nu_2\rangle$ be an MNE-space. Then the map $\epsilon_{\mathcal{X}}:X\to X(A(\mathcal{X}))$ satisfies:

(i) $\nu_{\Box_{\nu_1}}(\epsilon_{\mathcal{X}}(x))=\{\epsilon_{\mathcal{X}}[U]:U\in\nu_1(x)\}$,
(ii) $\nu_{\Diamond_{\nu_2}}(\epsilon_{\mathcal{X}}(x))=\{\epsilon_{\mathcal{X}}[U]:U\in\nu_2(x)\}$.

PROOF. We only prove (i) as the proof of (ii) is analogous. Suppose $V\in\nu_{\Box_{\nu_1}}(\epsilon_{\mathcal{X}}(x))$. Let D be a closed up-set of $X(A(\mathcal{X}))$ such that $D\subseteq V$ and with the property that for every clopen up-set W of $X(A(\mathcal{X}))$ with $D\subseteq W$ it holds that $W\in\nu_{\Box_{\nu_1}}(\epsilon_{\mathcal{X}}(x))$. Then $\epsilon_{\mathcal{X}}^{-1}[D]$ is a closed set. We consider $U=\epsilon_{\mathcal{X}}^{-1}[V]$, which is a clopen up-set of X. So, $\epsilon_{\mathcal{X}}^{-1}[D]\subseteq U$. Suppose now that W is a clopen up-set of X such that $\epsilon_{\mathcal{X}}^{-1}[D]\subseteq W$. Then $D\subseteq\epsilon_{\mathcal{X}}[W]$ and $\epsilon_{\mathcal{X}}[W]$ is a clopen up-set of $X(A(\mathcal{X}))$. So, $\epsilon_{\mathcal{X}}[W]\in\nu_{\Box_{\nu_1}}(\epsilon_{\mathcal{X}}(x))$. Note that, for every clopen up-set V of X, it holds that $\sigma_{X^+}(V)=\epsilon_{\mathcal{X}}[V]$. So from Proposition 5.1 we have $\Box_{\nu_{\Box_{\nu_1}}}\sigma_{A(\mathcal{X})}(W)=\sigma_{A(\mathcal{X})}(\Box_{\nu_1}W)$. Thus, $\Box_{\nu_{\Box_{\nu_1}}}\epsilon_{\mathcal{X}}[W]=\epsilon_{\mathcal{X}}[\Box_{\nu_1}W]$. Since $\epsilon_{\mathcal{X}}[W]\in\nu_{\Box_{\nu_1}}(\epsilon_{\mathcal{X}}(x))$, we have $\epsilon_{\mathcal{X}}(x)\in\Box_{\nu_{\Box_{\nu_1}}}\epsilon_{\mathcal{X}}[W]$ and so $\epsilon_{\mathcal{X}}(x)\in\epsilon_{\mathcal{X}}[\Box_{\nu_1}W]$. Thus, $x\in\Box_{\nu_1}W$ and so $W\in\nu_1(x)$. It follows that $U\in\nu_1(x)$. Therefore $\epsilon_{\mathcal{X}}[U]=\epsilon_{\mathcal{X}}[\epsilon_{\mathcal{X}}^{-1}[V]]=V\in\{\epsilon_{\mathcal{X}}[U]:U\in\nu_1(x)\}$.
Suppose now that $U\in\nu_1(x)$. Then $x\in\Box_{\nu_1}U$. So $\epsilon_{\mathcal{X}}(x)\in\epsilon_{\mathcal{X}}[\Box_{\nu_1}U]=\sigma_{A(\mathcal{X})}(\Box_{\nu_1}U)=\Box_{\nu_{\Box_{\nu_1}}}\sigma_{A(\mathcal{X})}(U)=\Box_{\nu_{\Box_{\nu_1}}}\epsilon_{\mathcal{X}}[U]$. So, $\epsilon_{\mathcal{X}}[U]\in\nu_{\Box_{\nu_1}}(\epsilon_{\mathcal{X}}(x))$. ∎

Finally, let us check that MNE-morphisms give rise to monotone modal twist-structure morphisms between the corresponding monotone modal twist-structures.

LEMMA 5.9
Let $f:\mathcal{X}\to\mathcal{X}'$ be a morphism of MNE-spaces. Then $A(f)\colon\mathcal{A}(\mathcal{X}')\to\mathcal{A}(\mathcal{X})$ is a monotone modal twist-structure morphism.

PROOF. Recalling Lemma 3.6, we only need to prove that for every clopen up-set $U\in A(\mathcal{X}')$, it holds that $A(f)(\Box_{\nu_1'}U)=\Box_{\nu_1}A(f)(U)$ and $A(f)(\Box_{\nu_2'}U)=\Box_{\nu_2}A(f)(U)$. We only prove the first equality as the proof of the second one is similar. Let $x\in X'$. Then $x\in A(f)(\Box_{\nu_1'}U)$ if and only if $f(x)\in\Box_{\nu_1'}U$ if and only if $U\in\nu_1'(f(x))$ if and only if $A(f)(U)=f^{-1}[U]\in\nu_1(x)$ if and only if $x\in\Box_{\nu_1}A(f)(U)$. ∎

Joining the previous results, one immediately sees that functors defined in the same way as for (non-modal) twist-structures and NE-spaces yield an equivalence in the modal case.

THEOREM 5.10
The functors $X\colon\mathsf{MTwist}^{\perp}\to\mathsf{MNE\text{-}Sp}$ and $A\colon\mathsf{MNE\text{-}Sp}\to\mathsf{MTwist}^{\perp}$ establish a dual equivalence between the category $\mathsf{MTwist}^{\perp}$ of modal twist-structures over bimodal Heyting algebras and the category $\mathsf{MNE\text{-}Sp}$ of MNE-spaces.

COROLLARY 5.11
The category $\mathsf{MN4}^{\perp}$ of bounded MN4-lattices and the category $\mathsf{MNE\text{-}Sp}$ of MNE-spaces are dually equivalent via the functors $X\circ T\colon\mathsf{MN4}^{\perp}\to\mathsf{MNE\text{-}Sp}$ and $N\circ A\colon\mathsf{MNE\text{-}Sp}\to\mathsf{MN4}^{\perp}$.

The above result can be used to obtain a topological duality for BK-lattices by restricting the objects of MTwist to modal twist-structures $\mathcal{A}=\langle\mathbf{A},\nabla,\Delta\rangle$ corresponding to BK-lattices, in which case we know that $\mathbf{A}$ is a modal Boolean algebra. It is then easy to see that the objects of the dual category are structures $\mathcal{X}=\langle X,\leq,\tau,C,O,\nu_1,\nu_2\rangle$ such that $\langle X,\tau\rangle$ is a Stone space and, for every $U\in A(\mathcal{X})$,

- if $C\subseteq U$, then $C\subseteq\Box_{\nu_1}U$,
- if $U\subseteq O$, then $\Diamond_{\nu_2}V\subseteq O$.

$$\begin{tikzcd} \mathsf{MN4}^{\perp} \arrow[r, bend left, "T"] & \mathsf{MTwist}^{\perp} \arrow[l, bend left, "N"] \arrow[r, bend left, "X"] & \mathsf{MNE\text{-}Sp} \arrow[l, bend left, "A"] \end{tikzcd}$$

FIG. 6. Equivalence between bounded MN4-lattices and MNE-spaces.

Moreover, since $\Box$ and $\Diamond$ are modal operators in the classical sense (i.e. they preserve, respectively, finite meets and finite joins and are dual of one another), we can replace the neigbourhood functions ν_1, ν_2 by a relation R and follow the duality theory for Boolean algebras with operators [3, Chapter 5].

The equivalences obtained so far are displayed in Figure 6.

We are now going to expound how to extend the above duality between $\mathsf{MNE\text{-}Sp}$ and $\mathsf{MTwist}^{\perp}$ to a duality between the category of $\mathsf{MTwist}^{\perp}$ and a category of monotone modal NE-spaces. For this purpose we are interested in monotone bimodal Heyting algebras $\mathbf{A} = \langle A, \wedge, \vee, \to \Box, \Diamond\rangle$ satisfying the properties that $\Box 0 = 0$, $\Diamond 0 = 0$ and that, for all $a \in A - \{0\}$, $\Box a, \Diamond a \neq 0$.

PROPOSITION 5.12
Let $\mathbf{A} = \langle A, \wedge, \vee, \to \Box, \Diamond\rangle$ be a monotone bimodal Heyting algebra.

(i) if $\Box 0 = 0$, $\Diamond 0 = 0$, then for every neighbourhood function $\nu_\bullet$ with $\bullet \in \{\Box, \Diamond\}$ and every prime filter P, $\emptyset \notin \nu_\bullet(P)$, and therefore $\bullet_{\nu_\bullet}(\emptyset) = \emptyset$,

(ii) if for all $a \in A - \{0\}$, $\Box a, \Diamond a \neq 0$, then for every neighborhood function $\nu_\bullet$ with $\bullet \in \{\Box, \Diamond\}$ and every $a \in A - \{0\}$, $\bullet_{\nu_\bullet}(\sigma_{\mathbf{A}}(a)) \neq \emptyset$.

PROOF. (i). Suppose $\emptyset \in \nu_\bullet(P)$. Thus there is a filter F of $\mathbf{A}$ such that $\bullet[F] \subseteq P$ and $\{Q \in X(\mathbf{A}) : F \subseteq Q\} \subseteq \emptyset$. Since every proper filter is included in some prime filter, it follows that $F = A$. Therefore $\{\bullet a : a \in A\} \subseteq P$. Now since $\bullet 0 = 0$, $0 \in P$, a contradiction. Therefore, $\emptyset \notin \nu_\bullet(P)$. Hence, $\bullet_{\nu_\bullet}(\emptyset) = \emptyset$.
(ii) Let $a \in A - \{0\}$. Then $\Box a \neq 0$. Let P be a prime filter such that $\Box a \in P$. Then $\Box[\uparrow a] \subseteq P$. Moreover, $\{Q \in X(\mathbf{A}) : \uparrow a \subseteq Q\} \subseteq \sigma_{\mathbf{A}}(a)$. Therefore, $P \in \Box_{\nu_\Box}(\sigma_{\mathbf{A}}(a))$. In a similar way we obtain that $\Diamond_{\nu_\Diamond}(\sigma_{\mathbf{A}}(a)) \neq \emptyset$. ∎

Now we are in a position to introduce the dual space of a monotone modal twist-structure $\mathcal{A} = \langle \mathbf{A}, \nabla, \Delta\rangle$. Let us consider the dual pointed NE-space $X^*(\mathcal{A}) = \langle X(\mathbf{A}^*), \tau_{\mathbf{A}^*}, \subseteq, C_{\mathcal{A}}, O_{\mathcal{A}}\rangle$ of the twist-structure $\mathcal{A}$ (disregarding the monotone modal operations). Recall that the Heyting algebra $\mathbf{A}^*$ is obtained from $\mathbf{A}$ by adding a new bottom element 0^*. We expand the operations $\Box$ and $\Diamond$ to $A^* = A \cup \{0^*\}$ by setting $\Box 0^* = 0^*$ and $\Diamond 0^* = 0^*$. Then we consider the dual monotone modal Esakia space $X(\mathcal{A}^*) = \langle X(\mathbf{A}^*), \subseteq, \tau_{\mathbf{A}^*}, \nu_\Box, \nu_\Diamond\rangle$ of $\langle \mathbf{A}^*, \Box, \Diamond\rangle$. Recall that A is a prime filter of $\mathbf{A}^*$. Also note that $\langle X(\mathbf{A}^*), \subseteq, \tau_{\mathbf{A}^*}\rangle$ is a pointed Esakia space. The structure $X^*(\mathcal{A}) = \langle X(\mathbf{A}^*), \subseteq, \tau_{\mathbf{A}^*}, \nu_\Box, \nu_\Diamond, C_{\mathcal{A}}, O_{\mathcal{A}}\rangle$ will be the dual of $\mathcal{A}$.

For a monotone modal twist-structure $\mathcal{A} = \langle \mathbf{A}, \nabla, \Delta\rangle$, we already know that $\sigma_{\mathbf{A}^*} : \mathcal{A} \cong A_*(X^*(\mathcal{A}))$ is an isomorphism of twist-structures between $\mathcal{A}$ and the twist-structure $A_*(X^*(\mathcal{A}))$, when we disregard the modal part of $\mathbf{A}$. In order to see that $\sigma_{\mathbf{A}^*}$ establishes an isomorphism of monotone modal twist-structures between $\mathcal{A}$ and $A_*(X^*(\mathcal{A}))$, using Proposition 5.1 it only remains to see that for every $a \in A$, $\Box_{\nu_\Box}\sigma_{\mathbf{A}^*}(a)$ and $\Diamond_{\nu_\Diamond}\sigma_{\mathbf{A}^*}(a)$ are non-empty. This follows from Proposition 5.12.

In order to characterize abstractly the duals of monotone modal twist-structures, we are now going to study some properties of the structure $X^*(\mathcal{A})$ that is dual to $\mathcal{A}$.

PROPOSITION 5.13
Let $\mathcal{A} = \langle \mathbf{A}, \nabla, \Delta\rangle$ be a monotone modal twist-structure. Then in the dual structure $\langle X(\mathbf{A}^*), \subseteq, \tau_{\mathbf{A}^*}, \nu_\Box, \nu_\Diamond\rangle$ we have $\nu_\Box(A) = \{\sigma_{\mathbf{A}^*}(a) : a \in A\}$ and $\nu_\Diamond(A) = \{\sigma_{\mathbf{A}^*}(a) : a \in A\}$.

PROOF. Let $a \in A$. Consider the filter $\uparrow a$. Then $\{Q \in X(\mathbf{A}^*) : \uparrow a \subseteq Q\} = \sigma_{\mathbf{A}^*}(a)$. Moreover, it is obvious that $\Box[\uparrow a] \subseteq A$. Thus, $\sigma_{\mathbf{A}^*}(a) \in \nu_\Box(A)$. Since $\emptyset \notin \nu_\Box(A)$, $\sigma_{\mathbf{A}^*}(0^*) = \emptyset \notin \nu_\Box(A)$. Hence, $\nu_\Box(A) = \{\sigma_{\mathbf{A}^*}(a) : a \in A\}$. In a similar way we obtain that $\nu_\Diamond(A) = \{\sigma_{\mathbf{A}^*}(a) : a \in A\}$. ∎

Let $\mathcal{A} = \langle \mathbf{A}, \nabla, \Delta \rangle$ be a monotone modal twist-structure. By a proof similar to that of Proposition 5.2 we obtain that conditions (i)-(iv) in Definition 5.3 hold in $\langle X(\mathbf{A}^*), \subseteq, \tau_{\mathbf{A}^*}, \nu_\Box, \nu_\Diamond \rangle$. This and Proposition 5.13 lead us to the next definition.

DEFINITION 5.14
A *pointed monotone modal NE-space (pointed MNE-space)* is a structure $\mathcal{X} = \langle X, \leq, \tau, C, O, \nu_1, \nu_2 \rangle$ such that $\langle X, \leq, \tau, C, O \rangle$ is a pointed NE-space and $\nu_i : X \to \mathcal{P}(\mathcal{P}^\uparrow(X))$ are neighbourhood functions satisfying conditions (i)-(iv) in Definition 5.3 as well as the following ones:

(i) $\emptyset \notin \nu_1(x)$ and $\emptyset \notin \nu_2(x)$, for every $x \in X$,
(ii) $\nu_1(1_X) = \nu_2(1_X) = A_*(X)$, where 1_X is the greatest element of $\langle X, \leq \rangle$.

From Propositions 5.12 and 5.13 it follows that if $\mathcal{A} = \langle \mathbf{A}, \nabla, \Delta \rangle$ is a monotone modal twist-structure, then $X(\mathcal{A}^*)$ is a pointed monotone modal NE-space. In this context we denote it by $X^*(\mathcal{A})$.
We are now going to obtain the monotone modal twist-structure dual of a pointed monotone modal NE-space.

LEMMA 5.15
Let $\mathcal{X} = \langle X, \leq, \tau, C, O, \nu_1, \nu_2 \rangle$ be a pointed monotone modal NE-space. Then for every non-empty clopen up-set U, $\Box_{\nu_1}(U)$ and $\Diamond_{\nu_2}(U)$ are non-empty.

PROOF. Let U be a non-empty clopen up-set and let 1_X be the greatest element of $\langle X, \leq \rangle$. Then $U \in \nu_1(1_X)$. Therefore $1_X \in \Box_{\nu_1}(U)$ and hence $\Box_{\nu_1}(U) \neq \emptyset$. In a similar way we obtain that $\Diamond_{\nu_2}(U) \neq \emptyset$. ∎

The above lemma implies that the restrictions of $\Box_{\nu_1}$ and $\Diamond_{\nu_2}$ to non-empty clopen up-sets are monotone operations on the algebra of non-empty clopen up-sets of $\mathcal{X}$. Therefore, we take as dual of $\mathcal{X}$ the algebra $\mathcal{A}^-(\mathcal{X}) = (A_*(\mathcal{X}), \nabla_C, \Delta_O)$ corresponding to the pointed NE-space $\langle X, \leq, \tau, C, O \rangle$ with $A_*(\mathcal{X})$ endowed with the operations $\Box_{\nu_1}$ and $\Diamond_{\nu_2}$ restricted to the universe of $A_*(\mathcal{X})$. It is then easy to see that $\mathcal{A}^-(\mathcal{X}) = (A_*(\mathcal{X}), \nabla_C, \Delta_O)$ is a monotone modal twist-structure.

Let $h : \mathcal{A}_1 \to \mathcal{A}_2$ be a morphism of monotone modal twist-structures $\mathcal{A}_1 = \langle \mathbf{A}_1, \nabla_1, \Delta_1 \rangle$ and $\mathcal{A}_2 = \langle \mathbf{A}_2, \nabla_2, \Delta_2 \rangle$. Then the extension of h to the homomorphism $h^* : \mathbf{A}_1^* \to \mathbf{A}_2^*$ of Heyting algebras that maps 0_1^* to 0_2^* is also a homomorphism from the monotone modal Heyting algebra $\mathbf{A}_1^*$ to the monotone modal Heyting algebra $\mathbf{A}_2^*$, because $h^*(\Box_1 0_1^*) = h^*(0_1^*) = 0_2^* = \Box_2 0^* = \Box_2 h^*(0_1^*)$, and similarly $h^*(\Diamond_1 0_1^*) = \Diamond_2 h^*(0_1^*)$. We already know that $X(h^*)$ is a pNE-morphism from $\mathcal{X}^*(\mathcal{A}_2)$ to $\mathcal{X}^*(\mathcal{A}_1)$, if we disregard the modal part. With a proof similar to that of Lemma 5.7 we obtain that, for every $P \in X(\mathbf{A}_2^*)$ and every clopen up-set U of $\mathcal{X}^*(\mathcal{A}_1)$,

- $U \in \nu_{\Box_1}(X(h^*)(P))$ if and only if $X(h^*)^{-1}[U] \in \nu_{\Box_2}(P)$,
- $U \in \nu_{\Diamond_1}(X(h^*)(P))$ if and only if $X(h^*)^{-1}[U] \in \nu_{\Diamond_2}(P)$.

Thus we define morphisms between pointed monotone modal NE-spaces as follows.

DEFINITION 5.16
A map $f : X \to X'$ between two pointed monotone modal NE-spaces $\mathcal{X}$ and $\mathcal{X}'$ is an *pMNE-morphism* if f is a pNE-morphism and for every $x \in X$ and every clopen up-set $U \in A^*(X')$,

(i) $U \in \nu_1'(f(x))$ if and only if $f^{-1}[U] \in \nu_1(x)$,
(ii) $U \in \nu_2'(f(x))$ if and only if $f^{-1}[U] \in \nu_2(x)$.

TABLE 1. Summary of (dual) equivalences.

	twist-structures over	topological structures
N4-lattices	Brouwerian lattices	pointed NE-spaces
bounded N4-lattices	Heyting algebras	NE-spaces
monotone modal N4-lattices	monotone bimodal Brouwerian lattices	pointed MNE-spaces
bounded monotone modal N4-lattices	monotone bimodal Heyting algebras	MNE-spaces
BK-lattices	modal Boolean algebras	BK-spaces

The definition implies that if $h : \mathcal{A}_1 \to \mathcal{A}_2$ is a morphism of monotone modal twist-structures, then $X(h^*) : \mathcal{X}^*(\mathcal{A}_2) \to \mathcal{X}^*(\mathcal{A}_1)$ is an pMNE-morphism, which we denote by $X^*(h)$.

Let $f : X \to X'$ be an pMNE-morphism from a pointed monotone modal NE-space $\mathcal{X}$ to a pointed monotone modal NE-space $\mathcal{X}'$. The dual $A_*(f) : \mathcal{A}^-(X') \to \mathcal{A}^-(X)$ of f as a pNE-morphism preserves also the modal operations (the proof is similar to that of Lemma 5.9). Therefore, it is a monotone modal twist-structure homomorphism.

If $\mathcal{X}$ is a pointed MNE-space, then we know that the map $\epsilon_{\mathcal{X}} : X \to \mathcal{X}^*(\mathcal{A}^-(\mathcal{X}))$ is an pNE-isomorphism (if we disregard the neighborhood maps). A proof analogous to that of Lemma 5.8 allows us to establish that $\epsilon_{\mathcal{X}}$ is an pMNE-isomorphism.

Let $\mathsf{pMNE\text{-}Sp}$ be the category of pointed MNE-spaces with pMNE-morphisms. Using the duality between $\mathsf{MTwist}^{\perp}$ and $\mathsf{MNE\text{-}Sp}$ together with the considerations above, it is not difficult to prove the following theorem. Let $\mathcal{X}^*(.)$ and $A_*(.)$ be the maps we obtain from the above definitions.

THEOREM 5.17
The maps $\mathcal{X}^* : \mathsf{MTwist} \to \mathsf{pMNE\text{-}Sp}$ and $A_* : \mathsf{pMNE\text{-}Sp} \to \mathsf{MTwist}$ are contravariant functors which establish a dual equivalence between the category MTwist of modal twist-structures and the category $\mathsf{pMNE\text{-}Sp}$ of pointed MNE-spaces.

COROLLARY 5.18
The category $\mathsf{MN4}$ of MN4-lattices and the category $\mathsf{pMNE\text{-}Sp}$ of pointed MNE-spaces are dually equivalent via the functors $\mathcal{X}^* \circ T : \mathsf{MN4} \to \mathsf{pMNE\text{-}Sp}$ and $N \circ A_* : \mathsf{pMNE\text{-}Sp} \to \mathsf{MN4}$.

The equivalences established by the above results are displayed below:

$$\begin{tikzcd} \mathsf{MN4} \arrow[r, bend left, "T"] & \mathsf{MTwist} \arrow[l, bend left, "N"] \arrow[r, bend left, "X"] & \mathsf{pMNE\text{-}Sp} \arrow[l, bend left, "A"] \end{tikzcd}$$

Table 1 below summarizes all the equivalence results established in this article (we have called *BK-spaces* the topological structures corresponding to BK-lattices, which can be easily obtained by restricting Definition 5.3 to *modal spaces* [2, Definition 3.1]).

6 A semantics for paraconsistent modal logic $\mathcal{MN}4$

Pointed monotone modal NE-spaces can be used to provide a semantics for the paraconsistent modal logic $\mathcal{MN}4$ introduced in [24]. This is a logic in the language $\langle \wedge, \vee, \rightarrow, \sim, \Box \rangle$ that can be syntactically defined by adding to any complete calculus for paraconsistent Nelson logic (see, e.g. [24, Definition 2.1]) the following rules [24, Definition 3.1]:

$$(\Box 1) \qquad \frac{p \rightarrow q}{\Box p \rightarrow \Box q} \qquad\qquad (\Box 2) \qquad \frac{\neg p \rightarrow \neg q}{\neg \Box p \rightarrow \neg \Box q}$$

This calculus, the consequence thereof we denote by $\vdash_{\mathcal{MN}4}$, is complete with respect to the algebraic semantics given by MN4-lattices as follows. Define the relation $\models_{MN4}$ by

$\Gamma \models_{MN4} \varphi$ if and only if for every MN4-lattice $\mathbf{B}$ and every homomorphism $h : Fm \rightarrow \mathbf{B}$, if for every $\psi \in \Gamma$, $h(\psi) = h(\psi \rightarrow \psi)$, then $h(\varphi) = h(\varphi \rightarrow \varphi)$.

Then [24, Theorem 3.6] implies the following:

Theorem 6.1
For every set of formulas Γ and every formula φ,

$$\Gamma \vdash_{\mathcal{MN}4} \varphi \ \text{ iff } \ \Gamma \models_{MN4} \varphi.$$

We are going to show that $\vdash_{\mathcal{MN}4}$ is also complete with respect to a semantics provided by pointed MNE-spaces.

Let $\mathcal{X} = \langle X, \leq, \tau, C, O, \nu_1, \nu_2 \rangle$ be a pointed MNE-space. A valuation on $\mathcal{X}$ is a map $V : Var \rightarrow A_*(\mathcal{X}) \times A_*(\mathcal{X})$ such that for every propositional variable $p \in Var$,

(i) $C \subseteq \pi_1(V(p)) \cup \pi_2(V(p))$,
(ii) $\pi_1(V(p)) \cap \pi_2(V(p)) \subseteq O$.

Let V be a valuation on a pointed NE-space $\mathcal{X} = \langle X, \leq, \tau, C, O, \nu_1, \nu_2 \rangle$. We extend it to a map $V : For \rightarrow A_*(\mathcal{X}) \times A_*(\mathcal{X})$ by setting

- $V(\varphi \wedge \psi) = \langle \pi_1(V(\varphi)) \cap \pi_1(V(\psi)), \pi_2(V(\pi) \cup \pi_2(V(\psi))) \rangle$,
- $V(\varphi \vee \psi) = \langle \pi_1(V(\varphi)) \cup \pi_1(V(\psi)), \pi_2(V(\varphi) \cap \pi_2(V(\psi))) \rangle$,
- $V(\varphi \rightarrow \psi) = \langle \pi_1(V(\varphi)) \rightarrow \pi_1(V(\psi)), \pi_1(V(\varphi)) \cap \pi_2(V(\psi)) \rangle$,
- $V(\sim \varphi) = \langle \pi_2(V(\varphi)), \pi_1(V(\varphi)) \rangle$,
- $V(\Box \varphi) = \langle \Box_{\nu_1} V(\varphi), \Diamond_{\nu_2} V(\varphi) \rangle$.

Let V be a valuation on a pointed NE-space $\mathcal{X} = \langle X, \leq, \tau, C, O, \nu_1, \nu_2 \rangle$. For every φ we let $V_1(\varphi) := \pi_1(V(\varphi))$ and $V_2(\varphi) := \pi_2(V(\varphi))$.

Lemma 6.2
Let V be a valuation on a pointed NE-space $\mathcal{X} = \langle X, \leq, \tau, C, O, \nu_1, \nu_2 \rangle$. Then for every formula φ, $C \subseteq V_1(\varphi) \cup V_2(\varphi)$ and $V_1(\varphi) \cap V_2(\varphi) \subseteq O$.

Proof. By definition the valuation of the statement holds for every propositional variable. So V is a map from the set of variables to the domain of the MN4-lattice $Tw(A_*(\mathcal{X}), \nabla_C, \Delta_O)$. Therefore, if the conditions corresponding to (i) and (ii) of the definition of valuation hold for φ and ψ, then the corresponding conditions hold for $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$ and $\sim \varphi$. Moreover, from the definition of pointed NE-space it immediately follows that the corresponding condition holds for $\Box \varphi$. ∎

Corollary 6.3
Let V be a valuation on a pointed NE-space $\mathcal{X}=\langle X,\leq,\ \tau,C,O,\nu_1,\nu_2\rangle$. Then the extension of V to the algebra of formulas is a homomorphism from this algebra to $Tw(A_*(\mathcal{X}),\nabla_C,\Delta_O)$. Moreover, for every formula φ,

$$V_1(\varphi)=X \text{ if and only if } V(\varphi)=V(\varphi\to\varphi).$$

Proof. The rightward implication follows from the lemma. Suppose that $V_1(\varphi)=X$. Since $V(\varphi\to\varphi)=\langle V_1(\varphi)\to V_1(\varphi),V_1(\varphi)\cap V_2(\varphi)\rangle$ we have $V(\varphi\to\varphi)=\langle X,V_2(\varphi)\rangle=\langle V_1(\varphi),V_2(\varphi)\rangle=V(\varphi)$. Assume now that $V(\varphi)=V(\varphi\to\varphi)$. Then $V_1(\varphi)=V_1(\varphi)\to V_1(\varphi)=X$. ∎

We are now in a position to define a consequence relation.

Definition 6.4
For every set of formulas Γ and every formula φ, let

$\Gamma\models_{pMNE}\varphi$ if and only if for every pointed MNE-space $\mathcal{X}$ and every valuation V on $\mathcal{X}$, if for every $\psi\in\Gamma$, $V_1(\psi)=X$, then $V_1(\varphi)=X$.

In order to show that the paraconsistent modal logic $\mathcal{MN}4$ is complete with respect to the semantics provided by pointed MNE-spaces, we need to make some observations. Let $\mathbf{B}$ be an MN4-lattice and h a homomorphism from the algebra of formulas to $\mathbf{B}$. We consider the monotone modal twist-structure $(\mathbf{B}_{\bowtie},\nabla(\mathbf{B}),\Delta(\mathbf{B}))$ and its dual pointed NE-space $\langle X(\mathbf{B}_{\bowtie}),\subseteq,\ \tau_{\mathbf{B}_{\bowtie}},C_{\mathbf{B}_{\bowtie}},O_{\mathbf{B}_{\bowtie}},\nu_\Box,\nu_\Diamond\rangle$. Recall that $\mathbf{B}$ is isomorphic to $Tw(\mathbf{B}_{\bowtie},\nabla(\mathbf{B}),\Delta(\mathbf{B}))$ via the map $j_{\mathbf{B}}$ defined by $j_{\mathbf{B}}(a)=\langle[a],[\sim a]\rangle$ for every $a\in B$ and that $(\mathbf{B}_{\bowtie},\nabla(\mathbf{B}),\Delta(\mathbf{B}))$ is isomorphic to $(A_*(X^*(\mathbf{B}_{\bowtie})),\nabla_{C_{\mathbf{B}_{\bowtie}}},\Delta_{O_{\mathbf{B}_{\bowtie}}})$ via the map $\sigma_{\mathbf{B}_{\bowtie}}$. Thus, $Tw(\mathbf{B}_{\bowtie},\nabla(\mathbf{B}),\Delta(\mathbf{B}))$ is isomorphic to $Tw(A_*(X^*(\mathbf{B}_{\bowtie})),\nabla_{C_{\mathbf{B}_{\bowtie}}},\Delta_{O_{\mathbf{B}_{\bowtie}}})$ via the map k defined by

$$k(\langle[a],[\sim a]\rangle)=\langle\sigma_{\mathbf{B}_{\bowtie}}([a]),\sigma_{\mathbf{B}_{\bowtie}}([\sim a])\rangle.$$

Therefore, $\mathbf{B}$ is isomorphic to $Tw(A_*(X^*(\mathbf{B}_{\bowtie})),\nabla_{C_{\mathbf{B}_{\bowtie}}},\Delta_{O_{\mathbf{B}_{\bowtie}}})$ through the map $k\circ j_{\mathbf{B}}$. Then, the map $k\circ j_{\mathbf{B}}\circ h$ is a valuation on $X^*(\mathbf{B}_{\bowtie})$. Recall that the set of points of $X^*(\mathbf{B}_{\bowtie})$ is the set $X((\mathbf{B}_{\bowtie})^*)$ of prime filters of $(\mathbf{B}_{\bowtie})^*$, which is the set of prime filters of $\mathbf{B}_{\bowtie}$ together with $B_{\bowtie}$.

Lemma 6.5
Let $\mathbf{B}$ be an MN4-lattice and h a homomorphism from the algebra of formulas to $\mathbf{B}$. The valuation $k\circ j_{\mathbf{B}}\circ h$ on $X^*(\mathbf{B}_{\bowtie})$ satisfies, for every formula ψ,

$$h(\psi)=h(\psi\to\psi) \text{ if and only if } (k\circ j_{\mathbf{B}}\circ h)_1(\psi)=X((\mathbf{B}_{\bowtie})^*).$$

Proof. For every formula ψ,

$$(k\circ j_{\mathbf{B}}\circ h)(\psi)=k(j_{\mathbf{B}}(h(\psi)))=\langle\sigma_{\mathbf{B}_{\bowtie}}([h(\psi)]],\sigma_{\mathbf{B}_{\bowtie}}([h(\sim\psi)])\rangle.$$

Suppose $h(\psi)=h(\psi\to\psi)$. Then $[h(\psi)]=[h(\psi\to\psi)]=[h(\psi)]\to[h(\psi)]$. Therefore,

$$(k\circ j_{\mathbf{B}}\circ h)_1(\psi)=\sigma_{\mathbf{B}_{\bowtie}}([h(\psi)])=\sigma_{\mathbf{B}_{\bowtie}}([h(\psi)]\to[h(\psi)])=X((\mathbf{B}_{\bowtie})^*).$$

Suppose now $(k\circ j_{\mathbf{B}}\circ h)_1(\psi)=X((\mathbf{B}_{\bowtie})^*)$. That is, $\sigma_{\mathbf{B}_{\bowtie}}([h(\psi)]])=X((\mathbf{B}_{\bowtie})^*)$. Since $\sigma_{\mathbf{B}_{\bowtie}}([h(\psi)]\to[h(\psi)])=X((\mathbf{B}_{\bowtie})^*)$, injectivity of $\sigma_{\mathbf{B}_{\bowtie}}$ implies that $[h(\psi)]\to[h(\psi)]=[h(\psi)]$. But then $[h(\psi)]$ is the top element of the Heyting algebra $(\mathbf{B}_{\bowtie})^*$ which is also the top element of $\mathbf{B}_{\bowtie}$. Therefore, $h(\psi)=h(\psi)\to h(\psi)$. ∎

PROPOSITION 6.6
For every set of formulas Γ and every formula φ,

$$\Gamma \models_{pMNE} \varphi \text{ if and only if } \Gamma \models_{MN4} \varphi.$$

PROOF. Suppose $\Gamma \models_{MN4} \varphi$ and $\Gamma \not\models_{pMNE} \varphi$. Then, let $\mathcal{X} = \langle X, \leq, \tau, C, O, \nu_1, \nu_2 \rangle$ be an pointed NE-space and V a valuation on $\mathcal{X}$ such that for every $\psi \in \Gamma$, $V_1(\psi) = X$ and $V_1(\varphi) \neq X$. We consider the MN4-lattice $Tw(A_*(\mathcal{X}), \nabla_C, \Delta_0)$. The valuation V gives a homomorphism from the algebra of formulas to $Tw(A_*(\mathcal{X}), \nabla_C, \Delta_0)$. It holds that for every $\psi \in \Gamma$, $V(\psi) = V(\psi \to \psi)$. Therefore, since $\Gamma \models_{MN4} \varphi$, $V(\varphi) = V(\varphi \to \varphi)$, and so $V_1(\varphi) = X$, a contradiction.
To prove the converse suppose $\Gamma \models_{pMNE} \varphi$. Let $\mathbf{B}$ be an MN4-lattice and h a homomorphism from the algebra of formulas to $\mathbf{B}$ such that for every $\psi \in \Gamma$, $h(\psi) = h(\psi \to \psi)$. Consider the space $\mathcal{X}^*(\mathbf{B}_{\bowtie})$ and the valuation $k \circ j_{\mathbf{B}} \circ h$. Then for every $\psi \in \Gamma$, $(k \circ j_{\mathbf{B}} \circ h)_1(\psi) = X(\mathbf{B}_{\bowtie})$. Therefore, $(k \circ j_{\mathbf{B}} \circ h)_1(\varphi) = X((\mathbf{B}_{\bowtie})^*)$, and so $h(\varphi) = h(\varphi \to \varphi)$. We conclude that $\Gamma \models_{MN4} \varphi$. ■

As a corollary we obtain the announced completeness result.

THEOREM 6.7
For every set of formulas Γ and every formula φ,

$$\Gamma \vdash_{\mathcal{MN}4} \varphi \ \text{ iff } \ \Gamma \models_{pMNE} \varphi.$$

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