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Algebraic Geometry: Birational Classification, Derived Categories, and Moduli Spaces

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ABSTRACT. The workshop covered a number of active areas of research in algebraic geometry with a focus on derived categories, moduli spaces (of varieties and sheaves) and birational geometry (often in positive characteristic) and their interactions. Special emphasis was put on hyperkähler manifolds and singularity theory.

Mathematics Subject Classification (2010): 14E, 14F05, 14D22.

Introduction by the Organisers

We have kept the structure of the last workshop in the series and planned 21 talks each 50 minutes. This provided sufficient time for questions and discussions after the talks and for new and ongoing collaborations during the breaks and in the evenings.

Most of the talks concentrated on various topics in singularity theory (Kovacs, Schwede, Totaro, Yasuda), on families of hyperkähler manifolds (Bayer, O’Grady, Macrì, Sarti), derived categories (Kuznetsov, Kawamata, Perry) and aspects of cohomology theories (Bhatt, Moonen). Other talks touched upon recent developments in a variety of nearby areas: From Lesieutre’s talk on the existence of varieties with non-finitely generated automorphism group, Lehmann’s talk on the geometric version of the Manin conjecture to Patakfalvi’s discussion of his construction of the moduli space of surfaces in mixed characteristic. Talks on slightly more exotic topics for this group of algebraic geometers were delivered by Richard

Thomas (on higher rank virtual cycles) and Brendan Hassett (on stable rationality in families). In cases where the subject risked to be particularly technical or a further away from the mainstream, we specifically asked the speaker to start with a gentle introduction (e.g. Ben Moonen on the Tate conjecture and Mihnea Popa on Hodge modules). Thematically the workshop was more focused than the last one but still allowing for cross fertilization between nearby research directions in algebraic geometry. It certainly does not happen all that often that experts in the minimal model program and people working on homological algebraic geometry or on hyperkähler manifolds would enter an intense mathematical exchange.

Recently the theory of MMP and its applications have advanced significantly. One of the major questions: BAB conjecture concerning boundedness of Fano varieties has been announced by Birkar (who unfortunately cancelled at the very last minute). People find lots of new connections with other subjects, like Kähler–Einstein metrics on Fano varieties, deep Hodge theory machineries, and others. In positive characteristics, the corresponding theory is also moving forward quickly. In the theory of hyperkähler manifolds the interplay between Hodge theory and derived categories has been explored from various angles. Interesting new complete families of polarized hyperkähler manifolds have been found and exotic isomorphisms between different constructions shed new light on the landscape. One can expect the theory to incorporate more arithmetic features in the near future and the Kuga-Satake construction linking hyperkähler varieties to abelian varieties is likely to play a central role in it.

Many participants expressed to the organizers that this edition of the workshop profited from a particularly friendly and stimulating atmosphere. The hike and the soccer match (Italy won!) have certainly contributed to it, but the large number of younger participants and the slightly larger number (compared to our last meeting) of female participants might have had an effect as well. This was very visible during the talks, with interesting questions and lively discussions, but also during the breaks and after dinner. Talks later in the week would refer to the ones earlier and sometimes even new results obtained in discussions during the week were mentioned. We are convinced that the workshop has had an impact on research in the various branches of algebraic geometry covered by it that will be felt in the next months and years. For example, a preprint of Keiji Oguiso, one of the participants, has appeared on the arxiv only three weeks after the workshop. In it he produces new examples of the phenomenon described by Lesieutre in his talk. We expect other papers will follow where the impact of the workshop can be felt directly.

The response to the workshop was overwhelmingly positive, before and during the week. Only a few people declined the original invitation and only two or three had to cancel at a later date. (With the small drawback that there were very few slots to fill with PhD students at short notice at the end.) Although the time of the year was not optimal for US participants, many of them arranged their beginning for term teaching so that they could attend. The international mix with a strong group coming from Japan, France, US, Italy and the UK was appreciated by the

community. Once again the Oberwolfach meeting was a well received occasion to meet other algebraic geometers with similar interests again or for the first time.

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Workshop: Algebraic Geometry: Birational Classification, Derived Categories, and Moduli Spaces

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Abstracts

Stability conditions in families and families of Hyperkähler varieties

AREND BAYER

(joint work with Martí Lahoz, Emanuele Macrì, Howard Nuer, Alexander Perry,
Paolo Stellari)

In this talk, I described a new construction of families of Hyperkähler varieties associated to families of cubic fourfolds, obtained in work in progress with the co-authors listed above. Our construction is based on crucial technical progress in the theory of Bridgeland stability conditions on derived categories of algebraic varieties. More specifically, we develop a notion of a “family of stability conditions” on a family of varieties, as well as a version of that for families with *Kuznetsov subcategories* of the derived categories of the fibers; both come with a notion of *relative moduli spaces* of stable objects. Our construction allows us to prove analogues of the powerful results for moduli spaces of stable sheaves on K3 surfaces, due to Mukai, Huybrechts, O’Grady, Yoshioka and others, in the setting of cubic fourfolds.

0.1. Setting: The Kuznetsov category of a cubic fourfold, and stability conditions. Let $X \subset \mathbb{P}^5$ be a smooth cubic fourfold. By its *Kuznetsov category* we denote the triangulated subcategory

$$\mathcal{K}u(X) := \mathcal{O}_X^\perp \cap \mathcal{O}_X(1)^\perp \cap \mathcal{O}_X(2)^\perp \subset D^b(X) = D^b(\text{Coh } X)$$

of its bounded derived category of coherent sheaves. This category shares many properties with the derived category of K3 surfaces; its foundations were developed in [Kuz10, AT14, Huy17]:

- (1) $\mathcal{K}u(X)$ is a CY 2-category: $\text{Hom}(E, F) = \text{Hom}(F, E[2])^\vee$.
- (2) Topological K -theory of $\mathcal{K}u(X)$, along with the faithful functor $\mathcal{K}u(X) \rightarrow D^b(X)$ and the Hodge structure on $H^4(X)$ equips $\mathcal{K}u(X)$ with an extended *Mukai lattice*, which by some abuse of notation we will denote $\tilde{H}(\mathcal{K}u(X), \mathbb{Z})$: as a lattice, it is isomorphic to $H^*(K3)$; it carries a weight two Hodge structure with $h^{2,0} = 1$; and it admits a Mukai vector

$$v: K(\mathcal{K}u(X)) \rightarrow \tilde{H}(\mathcal{K}u(X), \mathbb{Z})$$

satisfying $(v(E), v(F)) = -\chi(E, F)$.

Often, $\mathcal{K}u(X)$ is equivalent to the derived category of a K3 surface, see Corollary 4. By $\tilde{H}_{\text{Hodge}}(\mathcal{K}u(X), \mathbb{Z})$ we will denote the sublattice of integral $(1,1)$ -classes.

The recent preprint [BLMS17] gives a construction of a component $\text{Stab}^\dagger(\mathcal{K}u(X))$ of the space of *Bridgeland stability conditions* on $\mathcal{K}u(X)$. A stability condition consists of the datum of a subcategory $\mathcal{P}(\phi)$ of semistable objects of phase ϕ for all $\phi \in \mathbb{R}$, and a central charge, i.e. group homomorphism $Z: \tilde{H}_{\text{Hodge}}(\mathcal{K}u(X)) \rightarrow \mathbb{C}$, such that there is a notion of Harder-Narasimhan filtrations for all objects in

$\mathrm{Ku}(X)$, and such that the central charge $Z(E)$ of a semistable object $E \in \mathcal{P}(\phi)$ of phase ϕ is a complex number with argument $\pi\phi$.

0.2. Main results. Let $\mathbf{v} \in \tilde{H}_{\mathrm{Hodge}}(\mathrm{Ku}(X), \mathbb{Z})$ be a primitive class, and let $\sigma \in \mathrm{Stab}^\dagger(\mathrm{Ku}(X))$ be a stability condition as constructed in [BLMS17]. The first result concerns the existence and non-emptiness of the moduli space $M_\sigma(\mathrm{Ku}(X), \mathbf{v})$ of σ -stable objects in $\mathrm{Ku}(X)$ of Mukai vector \mathbf{v} .

Theorem 1. *If σ is generic with respect to \mathbf{v} , then $M_\sigma(\mathrm{Ku}(X), \mathbf{v})$ is non-empty if and only if $\mathbf{v}^2 \geq -2$. It is a smooth projective irreducible holomorphic symplectic variety of dimension $\mathbf{v}^2 + 2$, deformation-equivalent to a Hilbert scheme of points on a K3 surface.*

Here *generic* means that σ is not on a wall, so that stability and semistability coincide for objects of Mukai vector \mathbf{v} . We can also describe $H^2(M_\sigma(\mathbf{v}))$ in terms of the Hodge structure on $\tilde{H}(\mathrm{Ku}(X))$, and thus on $H^4(X)$, analogous to the corresponding result by Yoshioka for moduli of sheaves on K3s

Theorem 1 is proved by deformation to the case where $\mathrm{Ku}(X)$ is known to be equivalent to the derived category of a K3 surface. Such deformation arguments rely on the existence of *relative moduli spaces* given by Theorem 2 below.

Consider a family $\mathcal{X} \rightarrow S$ of smooth cubic fourfolds. Let \mathbf{v} be a primitive section of the local system given by the Mukai lattices $\tilde{H}(\mathrm{Ku}(\mathcal{X}_s), \mathbb{Z})$ of the fibers over $s \in S$, such that \mathbf{v} is algebraic on all fibers. Assume that for $s \in S$ very general, there exists a stability condition $\sigma_s \in \mathrm{Stab}^\dagger(\mathrm{Ku}(\mathcal{X}_s))$ that is generic with respect to \mathbf{v} , and such that the associated central charge $Z: \tilde{H}_{\mathrm{Hodge}}(\mathrm{Ku}(\mathcal{X})) \rightarrow \mathbb{C}$ is monodromy-invariant. (This is, for example, automatic when S is the moduli space of all cubic fourfolds.)

Theorem 2. (1) *There exists a finite cover $\tilde{S} \rightarrow S$, an algebraic space $\tilde{M}(\mathbf{v})$, and a proper morphism $\tilde{M} \rightarrow \tilde{S}$ that makes \tilde{M} a relative moduli space over \tilde{S} : the fibers over $s \in \tilde{S}$ are a moduli space $M_{\sigma_s}(\mathrm{Ku}(\mathcal{X}_s), \mathbf{v})$ of stable objects in the Kuznetsov category of the corresponding cubic.*

(2) *There exists an open subset $S^0 \subset S$, a projective variety $M^0(\mathbf{v})$, and a projective morphism $M^0(\mathbf{v}) \rightarrow S^0$ that makes $M^0(\mathbf{v})$ a relative moduli space over S^0 .*

Note that every fiber of the morphism $\tilde{M}(\mathbf{v}) \rightarrow \tilde{S}$ is projective, but the morphism itself might not be.

Example 3. Let S be the moduli space of cubic fourfolds. For a very general cubic fourfold, $\tilde{H}_{\mathrm{Hodge}}(\mathrm{Ku}(X), \mathbb{Z})$ is isomorphic to the A_2 -lattice, generated by two roots λ_1, λ_2 with $(\lambda_1, \lambda_2) = -1$. If we choose $\mathbf{v} = \lambda_1$ in Theorem 2, then $S^0 = \tilde{S} = S$, and $M(\mathbf{v})$ is the Fano variety of lines. For $\mathbf{v} = \lambda_1 + 2\lambda_2$, we have $S^0 \subset S$ the complement of cubics containing a plane, $\tilde{S} = S$, and $M^0(\mathbf{v})$ is the family of Hyperkähler eight-folds constructed in recent work [LLSvS15] of Lehn, Lehn, Sorger and van Straten. In particular, the algebraic space $\tilde{M}(\mathbf{v})$ partially

compactifies their family, at the cost of losing projectivity, over cubics containing a plane; here our moduli spaces agree with those considered by Ouchi in [Ouc17]. Finally, for $\mathbf{v} = 2\lambda_1$, we get an algebraic construction of a 20-dimensional family of singular 10-dimensional O’Grady spaces.

0.3. Applications. Recall from Hassett’s work on cubic fourfolds, [Has00], that there is a countable union of divisors of special cubics for which one can Hodge-theoretically associate a K3 surface to its primitive cohomology in $H^4(X)$. In our notation, a cubic is contained in one of Hassett’s special divisors if and only if $\tilde{H}_{\text{Hodge}}(\mathcal{K}u(X), \mathbb{Z})$ contains a hyperbolic plane.

Corollary 4. *Let X be a cubic fourfold. Then X has a Hodge-theoretically associated K3 if and only if there exists a smooth projective K3 surface S and an equivalence $\mathcal{K}u(X) \cong \mathbb{D}^b(S)$.*

This (literally) completes a result by Addington and Thomas, [AT14], who proved that every divisor described by Hassett contains an open subset of cubics admitting a derived equivalence as above. A version of the Corollary also holds for K3s with a Brauer twist; the corresponding Hodge-theoretic condition is the existence of a square-zero class in $\tilde{H}_{\text{Hodge}}(\mathcal{K}u(X))$.

As pointed out to us by Voisin, the non-emptiness of moduli spaces also produces enough algebraic cohomology classes to reprove her result on the integral Hodge conjecture for cubic fourfolds:

Corollary 5 ([Voi07, Theorem 18]). *The integral Hodge conjecture holds for X .*

Our results also provide the full machinery of [BM14], describing the birational geometry of $M_\sigma(\mathcal{K}u(X), \mathbf{v})$ in terms of wall-crossing.

0.4. Stability conditions in families. As hinted at in the introductory paragraph, the notion of relative moduli spaces depends on developing a notion and construction of stability conditions for a family of varieties $\pi: \mathcal{Y} \rightarrow S$.

Here we only sketch the underlying definition, in the simplest possible case where $S = \mathbb{C}$ is a smooth curve over \mathbb{C} . Both the definitions and the technical setup borrows results and ideas from the work [AP06, Pol07] by Abramovich and Polishchuk on sheaves of t-structures over a base.

The first ingredient of a stability condition on $\mathbb{D}^b(\mathcal{Y})$ over C is again a *slicing*, i.e. a list $\mathcal{P}(\phi)$ of semistable objects of phase ϕ satisfying Harder-Narasimhan filtration; we require that each $\mathcal{P}(\phi)$ is invariant under tensoring with pull-backs of line bundles from C . Second, our central charge $Z: \mathbb{D}^b(\mathcal{Y})_{C\text{-tor}} \rightarrow \mathbb{C}$ is defined for C -torsion objects, i.e. objects whose pull-back to the generic fiber over C vanishes. We require Z is constant in families, in the sense that for any object $F \in \mathbb{D}^b(\mathcal{Y})$, the complex number $Z(F|_{\pi^{-1}(c)})$ is independent of the closed point $c \in C$. Finally, we require that stability is an open property in families.

We show the existence of such stability conditions over C in the same generality that the existence of stability conditions on the fibers can be proved in the framework of [BMT14]. It comes with proper relative moduli spaces of semistable objects, generalizing work by Piyaratne and Toda, [PT15]. Finally, it extends to

the notion of stability on Kuznetsov categories, as for cubic fourfolds in the setup above.

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Canonical deformations of de Rham cohomology

BHARGAV BHATT

(joint work with Matthew Morrow, Peter Scholze)

The goal of this talk was to explain two instances (one classical, one recent) in arithmetic geometry where the de Rham cohomology of smooth varieties admits a canonical deformation. More precisely, we explained the following:

- (1) (Crystalline cohomology) Given a perfect field k of characteristic p and a smooth k -scheme X , its de Rham cohomology $R\Gamma_{dR}(X/k)$ admits a canonical deformation $R\Gamma_{crys}(X/W(k))$ to the ring $W(k)$ of Witt vectors of k given by Grothendieck’s theory of crystalline cohomology.

The assignment $X \mapsto H_{crys}^*(X/W(k))[\frac{1}{p}]$ gives a Weil cohomology theory on proper varieties, so each $H_{crys}^i(X/W(k))[\frac{1}{p}]$ has the same rank as the corresponding ℓ -adic cohomology group. In particular, the existence of this deformation can be used to meaningfully reformulate what it means to have “pathologies” in the de Rham cohomology of X : it means there must be p -torsion in crystalline cohomology. For instance, calculations of Lang and Illusie imply that the de Rham cohomology of Enriques surfaces in characteristic 2 is “pathological”, and thus there must be non-trivial 2-torsion in their crystalline cohomology.

Just like $R\Gamma_{dR}(X/k)$ is computed as the hypercohomology of an explicit complex $\Omega_{X/k}^*$ (and not merely an object of the derived category), the crystalline cohomology $R\Gamma_{crys}(X/W(k))$ is also calculated as the hypercohomology of an explicit (but rather intricate) complex $W\Omega_X^*$ known as the de Rham-Witt complex.

- (2) (A_{inf} -cohomology) Given a complete and algebraically closed extension C/\mathbf{Q}_p with ring of integers \mathcal{O}_C and a smooth formal \mathcal{O}_C -scheme X , its de Rham cohomology $R\Gamma_{dR}(X/\mathcal{O}_C)$ admits a canonical deformation $R\Gamma_A(X)$ to Fontaine’s ring $A_{\text{inf}} := A_{\text{inf}}(\mathcal{O}_C)$ (see [1]).

For proper X , the theory $R\Gamma_A(X)[\frac{1}{q-1}]$ (for a specific unit $q \in A_{\text{inf}}$) is essentially given by the étale cohomology $R\Gamma(X_C, \mathbf{Z}_p)$ of the generic fibre X_C of X . In particular, the existence of this deformation implies (by semicontinuity) that the mod p Betti numbers of X_C are a lower bound for the de Rham Betti numbers of the special fibre X_0 . For instance, this gives a topological explanation of the calculations of Lang and Illusie mentioned in (1): the de Rham cohomology of an Enriques surface in characteristic 2 must be “pathological” (i.e., larger than the corresponding groups in other characteristics) because the étale (or singular) cohomology of an Enriques surface in characteristic 0 has non-trivial 2-torsion. (Here we implicitly use Ekedahl’s result that such surfaces always lift to characteristic 0.)

- (3) ($A\Omega$ -complexes) The theory mentioned in (2) is constructed locally. More precisely, in [1], we construct a complex $A\Omega_X$ of A_{inf} -modules on the formal scheme X and deduce the relevant comparisons with de Rham and étale cohomology by local arguments.

A representative example is the following: if X is the formal affine line defined by $R = \widehat{\mathcal{O}_C[t]}$, then the global sections $A\Omega_R$ of $A\Omega_X$ are computed by “ q -deformations of the de Rham complex”, i.e., we have

$$A\Omega_R \simeq \bigoplus_{i \geq 0} \left(A_{\text{inf}} \cdot t^i \xrightarrow{[i]_q = \frac{q^i - 1}{q - 1}} A_{\text{inf}} \cdot t^{i-1} \cdot dt \right).$$

When we specialize to \mathcal{O}_C , we have $q = 1$, so the above complex reduces to the usual de Rham complex for R/\mathcal{O}_C . In general, however, the deformation is quite non-trivial, and not computed by the de Rham complex of lift. Note that the right hand side of the equation above is extremely sensitive to the choice of the co-ordinate t ! Thus, the well-definedness of $A\Omega_R$ (which is clear from its functorial definition) captures a surprising quasi-isomorphism invariance property of q -de Rham complexes in this setting. Precise conjectures for more liberal notions of q -de Rham complexes were formulated by Scholze.

Unlike the de Rham-Witt complex, the $A\Omega$ -complexes are only well-defined up to quasi-isomorphism: we do not have canonical functorial representatives for calculating $A\Omega_R$. In fact, one can show that such representatives cannot exist in a fashion compatible with the natural E_∞ -algebra structure on $A\Omega_R$.

- (4) (Topological Hochschild homology) Finally, we explained the homotopy-theoretic motivation for guessing the existence of the complex $A\Omega_R$ from (3). For this, recall that one defines the *topological Hochschild homology* spectrum $THH(R)$ for a ring R by mimicing the construction of the Hochschild complex $HH(R)$, but by replacing the base ring \mathbf{Z} by the sphere spectrum \mathbf{S} . By construction, this spectrum carries an S^1 -action, whose cohomological shadow is Connes' differential. Decades ago, Waldhausen suggested that working over \mathbf{S} instead of \mathbf{Z} produces better behaved and denominator free answers, at least in homotopy theory. Hesselholt's [3] vindicates this philosophy: setting $TC^-(R) := THH(R)^{hS^1}$ to be the homotopy S^1 -invariants, he calculated that $\pi_0(TC^-(\mathcal{O}_C))$ is simply Fontaine's period ring A_{inf} , thus giving a very conceptual and thoroughly topological interpretation of the latter. As $TC^-(R)$ was known to be roughly related to the étale cohomology of R_C for smooth \mathcal{O}_C -algebras R , this suggested the following guess: for such R 's, the graded pieces of a natural filtration on $TC^-(R)$ would form an A_{inf} -valued cohomology theory (i.e., would give $A\Omega_R$). In fact, Hesselholt had himself arrived at a similar guess in the characteristic p setting. Whilst this is not how $A\Omega_R$ was defined in [1], this picture has now been realized [2].

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The blow-up of \mathbb{P}^4 at 8 points and its Fano model, via vector bundles on a degree 1 del Pezzo surface

CINZIA CASAGRANDE

(joint work with Giulio Codogni, Andrea Fanelli)

The classical notion of association, or Gale duality, gives a bijection between sets of 8 general points in \mathbb{P}^2 and sets of 8 general points in \mathbb{P}^4 , up to projective equivalence. This correspondence has been used by Mukai to establish a beautiful relation among the blow-up X of \mathbb{P}^4 at 8 general points, and a degree one del Pezzo surface S , via moduli of sheaves on S , as follows.

Theorem 1 ([Muk05]). *Let $\{q_1, \dots, q_8\} \subset \mathbb{P}^2$ and $\{p_1, \dots, p_8\} \subset \mathbb{P}^4$ be associated sets of points, and set $S = \text{Bl}_{q_1, \dots, q_8} \mathbb{P}^2$ and $X = \text{Bl}_{p_1, \dots, p_8} \mathbb{P}^4$. Then X is isomorphic to the moduli space of rank 2 torsion free sheaves F on S , with $c_1(F) = -K_S$ and $c_2(F) = 2$, Gieseker semistable with respect to $-K_S + 2h$, where $h \in \text{Pic}(S)$ is the pull-back of $\mathcal{O}_{\mathbb{P}^2}(1)$ under the blow-up map $S \rightarrow \mathbb{P}^2$.*

This result gives a very useful tool to describe the birational geometry of X ; let us explain how.

Moduli of vector bundles on a degree 1 del Pezzo surface. Let S be a smooth del Pezzo surface of degree one, and $L \in \text{Pic}(S)$ ample. Let $M_{S,L}$ be the moduli space of rank 2 torsion free sheaves F on S , with $c_1(F) = -K_S$ and $c_2(F) = 2$, Gieseker semistable with respect to L .

To describe the moduli spaces $M_{S,L}$, we introduce two convex rational polyhedral cones

$$\Pi \subset \mathcal{E} \subset \text{Nef}(S) \subset H^2(S, \mathbb{R})$$

(see [CCF17] for the explicit definitions). Let us first state some general properties of $M_{S,L}$.

Proposition 2 ([Muk05, CCF17]). *The moduli space $M_{S,L}$ is non-empty if and only if $L \in \mathcal{E}$, and in this case $M_{S,L}$ is a smooth, projective, irreducible, rational 4-fold. Every sheaf parametrized by $M_{S,L}$ is locally free and stable.*

Let us consider now the birational geometry of the moduli spaces $M_{S,L}$. The relation between variation of polarization via wall-crossings, and birational geometry of moduli spaces of sheaves on surfaces, is classical and has been intensively studied. In the case of $M_{S,L}$, this relation can be made completely explicit. There are finitely many walls for slope-semistability, that are explicitly described; these walls determine the stability fan in $H^2(S, \mathbb{R})$, supported on the cone \mathcal{E} . When the polarization L varies in the interior of a cone of maximal dimension of the stability fan, the stability condition determined by L is constant, and so is $M_{S,L}$. When L moves to a different cone of the stability fan, the moduli space $M_{S,L}$ undergoes a simple birational transformation. This study of the birational geometry of $M_{S,L}$, via the variation of the stability condition, is due to Mukai [Muk05].

The determinant map. The moduli spaces $M_{S,L}$ are Mori dream spaces, hence the stability fan in $H^2(S, \mathbb{R})$ has a counterpart in $H^2(M_{S,L}, \mathbb{R})$, the fan given by the Mori chamber decomposition, defined via birational geometry. To relate these two combinatorial structures, we use the classical construction of determinant line bundles on the moduli space $M_{S,L}$. This yields a group homomorphism

$$\rho: \text{Pic}(S) \longrightarrow \text{Pic}(M_{S,L}),$$

and we show the following result, which relies on the classical positivity properties of the determinant line bundle, and on Theorem 1.

Theorem 3. *Let $L \in \text{Pic}(S)$ ample, $L \in \Pi$. The map $\rho: H^2(S, \mathbb{R}) \rightarrow H^2(M_{S,L}, \mathbb{R})$ is an isomorphism, and yields an isomorphism between the stability fan in $H^2(S, \mathbb{R})$, and the Mori chamber decomposition in $H^2(M_{S,L}, \mathbb{R})$. In particular, $\rho(\mathcal{E})$ is the cone of effective divisors, and $\rho(\Pi)$ is the cone of movable divisors.*

Let us describe two applications of this study to the moduli spaces $M_{S,L}$. We recall that a pseudo-isomorphism is a birational map which is an isomorphism in codimension one, and similarly we define a pseudo-automorphism. When the polarization L varies in the cone Π , we get finitely many pseudo-isomorphic moduli spaces $M_{S,L}$, related by sequences of flips. We show that in fact, when $L \in \Pi$, the moduli space $M_{S,L}$ determines the surface S .

Theorem 4. *Let S_1 and S_2 be del Pezzo surfaces of degree one, and $L_i \in \text{Pic}(S_i)$ ample line bundles with $L_i \in \Pi_i \subset H^2(S_i, \mathbb{R})$, for $i = 1, 2$. Then $S_1 \cong S_2$ if and only if M_{S_1, L_1} and M_{S_2, L_2} are pseudo-isomorphic.*

We also describe the group of pseudo-automorphisms of the moduli space $M_{S,L}$, when the polarization L is in the cone Π .

Theorem 5. *Let $L \in \text{Pic}(S)$ ample, $L \in \Pi$. Then the group of pseudo-automorphisms of $M_{S,L}$ is isomorphic to the automorphism group $\text{Aut}(S)$ of S , where $f \in \text{Aut}(S)$ acts on $M_{S,L}$ as $[F] \mapsto [(f^{-1})^*F]$.*

Geometry of the Fano model Y . The anticanonical class $-K_S$ in $H^2(S, \mathbb{R})$ belongs to the cone Π , and lies in the interior of a cone of the stability fan. It follows again from the classical properties of the determinant line bundle that for the polarization $L = -K_S$, the moduli space $M_{S,L}$ is Fano. More precisely, we have the following.

Proposition 6. *The moduli space $Y := M_{S, -K_S}$ is a smooth, rational Fano 4-fold with $b_2(Y) = 9$, $(-K_Y)^4 = 13$, and $h^0(Y, -K_Y) = 6$.*

Let us notice that, except products of del Pezzo surfaces, there are very few known examples of Fano 4-folds with $b_2 \geq 7$. In particular, to the authors' knowledge, the family of Fano 4-folds Y is the only known example of Fano 4-fold with $b_2 \geq 9$ which is not a product of surfaces. It is a very interesting family, whose construction and study was one of the motivations for this work.

The following explains the relation among X and Y .

Proposition 7. *Let X be a blow-up of \mathbb{P}^4 at 8 general points. Then there is a pseudo-isomorphism $\xi: X \dashrightarrow Y$ where Y is a smooth Fano 4-fold, and ξ is a sequence of 36 antitlips. The flipping curves in X are the transforms of the 28 lines through two blown-up points, and of the 8 rational normal quartics through 7 blown-up points.*

By Theorem 3, the determinant map gives isomorphisms $H^2(S, \mathbb{R}) \rightarrow H^2(Y, \mathbb{R})$ and $H^2(S, \mathbb{R}) \rightarrow H^2(X, \mathbb{R})$, and these yield a completely explicit description of the relevant cones of effective, movable, and nef divisors of Y and X , and more generally of the Mori chamber decompositions. We refer to [CCF17] for the explicit results.

Finally, motivated by the low values of $h^0(Y, -K_Y)$ and $(-K_Y)^4$, and also by the analogy with degree one del Pezzo surfaces, we study the base loci of the anticanonical and bianticanonical linear system of Y , and prove the following.

Theorem 8. *The linear system $| -K_Y |$ has a base locus of positive dimension, while the linear system $| -2K_Y |$ is base point free.*

This result is proved using the birational map $X \dashrightarrow Y$ and studying the corresponding linear systems in X . We show that the base locus of $| -K_X |$ contains the transform R of a smooth rational quintic curve in \mathbb{P}^4 through the 8 blown-up points. We have $-K_X \cdot R = 1$, and R is not contained in the stable base locus of $| -K_X |$; the transform of R in Y is contained in the base locus of $| -K_Y |$.

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Relative semi-ampleness in positive characteristic

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(joint work with Hiromu Tanaka)

1. INTRODUCTION

It is a crucial problem to understand under what condition a line bundle L on a projective variety X is semi-ample, i.e. there exists a positive integer m such that the natural map

$$H^0(X, L^{\otimes m}) \otimes \mathcal{O}_X \rightarrow L^{\otimes m}$$

is surjective.

Some classical results in this direction include:

- (Zariski) If X is normal, L is big and nef then L is semi-ample if and only if the associated ring of section $R(X, L)$ is finitely generated.
- (Fujita) If X is normal, and Z is the stable base locus of L , i.e.

$$Z = \bigcap_{m \in \mathbb{N}} Bs|L^{\otimes m}|,$$

then $L|_Z$ is not ample.

On the other hand, note that over $\overline{\mathbb{F}}_p$, it is hard to find line bundles which are nef and not semi-ample (see [4]) and, in particular, we do not know any line bundle on a surface X which has positive degree on every curve and it is not ample.

Over any algebraically closed field of positive characteristic, Keel [3] showed that a line bundle L is semi-ample if and only if the restriction $L|_{\mathbb{E}(L)}$ is semi-ample. Recall that

$$\mathbb{E}(L) = \bigcup_{L|_V \text{ is not big}} V$$

is the exceptional locus of L .

The same result does not hold in characteristic zero: if $X = C \times C$ where C is a curve of genus greater than 1 and $L = p_1^* \omega_C \otimes \mathcal{O}_X(\Delta)$. Then L is big and nef and $\mathbb{E}(L) = \Delta$. Moreover $L|_\Delta = \mathcal{O}_\Delta$, and in particular $L|_{\mathbb{E}(L)}$ is semi-ample. But, over \mathbb{C} , L is not semi-ample. Indeed $L|_{2\Delta}$ is not torsion [3]. On the other hand, if we define $L' = \omega_X(2\Delta)$, then L' is big, nef and semi-ample, and it defines a morphism $f: X \rightarrow S$ which contracts Δ . So, in particular, over \mathbb{C} we have that $L|_F$ is semi-ample for any fibre F of f , but $L \otimes f^*(A)$ is not semi-ample for any line bundle A on S .

The main result in [2] is to show that this cannot happen over a field of positive characteristic:

Theorem 1 ([2]). *Let $f: X \rightarrow S$ be a projective morphism of noetherian \mathbb{F}_p -schemes. Let L be an invertible sheaf on X . Assume that $L|_{X_s}$ is semi-ample for any point $s \in S$, where X_s denotes the fibre of f over s .*

Then L is f -semi-ample.

Recall that L is f -semi-ample if there exists a positive integer m such that the natural map

$$f^* f_* L^{\otimes m} \rightarrow L^{\otimes m}$$

is surjective.

Note that we are not assuming that f is birational or that X is normal. Indeed, X is just a noetherian scheme (possibly of infinite dimension).

On the other hand, we do need to consider all the scheme-theoretical points of S (it is not enough to consider closed points) as otherwise the result does not hold: E.g. let E be an elliptic curve over $\overline{\mathbb{F}}_p$, let $X := E \times E$, $S := E$ and let $f = p_1: X \rightarrow S$ be the first projection. Let $L := \mathcal{O}_X(\Delta - Z)$, where Δ is the diagonal divisor of $X = E \times E$ and $Z := E \times \{Q\}$ for a closed point $Q \in E$. Then L is f -nef but not f -semi-ample (indeed $(L + f^*A)^2 = -2$ for any A). Thus,

$L|_{X_s}$ is semi-ample for any closed point $s \in S$ since our base field is $\overline{\mathbb{F}}_p$. Thus, Theorem 1 implies that $L|_{X_\xi}$ is not semi-ample for the generic point ξ of S .

If the field k is uncountable, we have:

Theorem 2 ([2]). *Let $f: X \rightarrow S$ be a projective morphism of schemes of finite type over an uncountable field of positive characteristic. Let L be an invertible sheaf on X . Assume that $L|_{X_s}$ is semi-ample for any closed point $s \in S$, where X_s denotes the fibre of f over s .*

Then L is f -semi-ample.

Note that if L is f -numerically trivial then the result follows also from [1], using different methods.

2. IDEA OF THE PROOF

Let X, S, f and L be as in Theorem 1. By base change, we may assume that X has finite dimension n . Thus, we may proceed by induction on n . Assuming that the Theorem holds in dimension $n - 1$ and assuming that X is normal, then the result is a consequence of Keel’s Theorem mentioned above. On the other hand, even assuming that X is normal, we do need to assume the Theorem in full generality in lower dimension. Thus, the main difficulty of the proof is to study what happens after considering the normalization.

A line bundle L is said to be endowed with a map (EWM) over S if there exists a morphism $g: X \rightarrow Z$ onto an algebraic space Z over S such that $L^{\dim V} \cdot V = 0$ if and only if $\dim g(V) < \dim V$.

Kollár and Keel showed that there exists a line bundle L on a variety X and a finite morphism $f: Y \rightarrow X$ such that f^*L is EWM but L is not EWM.

On the other hand we prove:

Theorem 3 ([2]). *Let S be a noetherian \mathbb{F}_p -scheme. Let $f: Y \rightarrow X$ be a finite surjective S -morphism of reduced algebraic spaces proper over S . Let L be an invertible sheaf on X which is nef over S .*

Then L is EWM over S if and only if

- a) $L|_Y$ is EWM over S , and
- b) there exists a positive integer m_0 such that for any geometric point $s \in S$, the $L|_{X_s}$ -equivalence relation on X_s is bounded by m_0

Recall that $x, y \in X_s$ are $L|_{X_s}$ -equivalent if there exist curves C_1, \dots, C_q such that $x, y \in \cup C_i$ and $L \cdot C_i = 0$. The equivalence is bounded by m_0 if we can always find C_1, \dots, C_q so that $q < m_0$.

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Stable Rationality in Families of Threefolds

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(joint work with Yuri Tschinkel)

This is work in progress to exhibit smooth projective stably rational threefolds that deform to varieties that are not stably rational. Thus stable rationality is not a deformation invariant in dimension three.

A complex variety V is *stably rational* if the product $V \times \mathbb{P}^r$ is rational for some r . The first stably rational non-rational varieties were found in by Beauville, Colliot-Thélène, Sansuc, and Swinnerton Dyer [1]. They offered two related classes of examples. The first is Châtelet surfaces, defined over a field k by

$$\{y^2 - az^2 = P(x)\} \subset \mathbf{A}^3,$$

where $P(x) \in k[x]$ is a cubic polynomial with Galois group \mathfrak{S}_3 and discriminant a . These are stably rational but non-rational over k . Geometrically, they admit conic bundle fibrations

$$\varphi : \tilde{V} \rightarrow \mathbb{P}_x^1,$$

with four degenerate fibers corresponding to the roots of $P(x)$ and $x = \infty$. Passing to $k = \mathbb{C}(t)$, we obtain smooth projective threefolds with fibrations

$$(1) \quad \mathcal{X}_0 \xrightarrow{\phi_0} \mathcal{S}_0 \xrightarrow{\rho} \mathbb{P}_t^1.$$

Here \mathcal{S}_0 is a smooth projective surface birationally ruled over \mathbb{P}_t^1 , and $\mathcal{X}_0 \rightarrow \mathcal{S}_0$ is a conic fibration degenerate over a curve

$$(2) \quad D_0 = C \cup R \subset \mathcal{S}_0,$$

where C is a trisection and R a section of ρ . (These correspond to $P(x) = 0$ and $x = \infty$ respectively.) The variety \mathcal{X}_0 is stably rational over $\mathbb{C}(t)$ and thus over \mathbb{C} . The Clemens-Griffiths theory of intermediate Jacobians shows it is often non-rational.

Voisin's technique of decomposition of the diagonal [7], refined by Colliot-Thélène, Pirutka [2] and others, is a powerful tool for proving that varieties are not stably rational. It is the key to showing that stable rationality is not a deformation invariant of smooth projective complex varieties of dimension at least four [4]. The case of dimension three was left open, although Nicaise and Shinder have shown that stable rationality is closed under specialization in families of smooth projective varieties of arbitrary dimension [6].

Here are the elements of the construction of a degeneration of smooth projective threefolds $\mathcal{X} \rightsquigarrow \mathcal{X}_0$ with \mathcal{X}_0 stably rational but \mathcal{X} not stably rational.

First, we use an extension of the class of Châtelet surfaces analyzed in [5]. We consider all degree four del Pezzo surfaces with conic fibrations

$$\widetilde{W} \rightarrow \mathbb{P}^1$$

over k , admitting the same Galois structure as above. Precisely, the Galois actions on the Picard groups of \widetilde{V} and \widetilde{W} are equivalent. These depend on *two* parameters rather than the one parameter governing Châtelet surfaces. Nevertheless, the new surfaces remain birational over k to Châtelet surfaces and thus are stably rational.

From now on, take $k = \mathbb{C}(t)$ and seek towers (1) associated with the generalizations of Châtelet surfaces discussed above.

We analyze in general terms the possible branching data (2) for our surfaces. We take $f : C \rightarrow \mathbb{P}^1$ to be an arbitrary simply branched triple cover of genus g , and $p_1, \dots, p_{2g+4} \in C$ the points residual to ramification points. Consider

$$D_0 = C \cup_{p_i=f(p_i)} R, \quad R \simeq \mathbb{P}^1,$$

where we glue the residual points and their images in \mathbb{P}^1 . Note the induced degree four morphism $g_0 : D_0 \rightarrow \mathbb{P}^1$. The discriminant double cover induces an admissible cover

$$\widetilde{D}_0 \rightarrow D_0$$

which we will use to encode the degeneracy data of conic bundles.

The third step is to construct embeddings

$$D_0 \hookrightarrow \mathcal{S}_0 \rightarrow \mathbb{P}^1$$

of D_0 into a birationally ruled surface that induces g_0 . We take $g = 1$ and \mathcal{S}_0 to be the blow up of \mathbb{P}^2 at four points, three of whom are collinear. Here $D_0 \in |-2K_{\mathcal{S}_0}|$, i.e., is bi-anticanonical.

The next step is to construct conic bundles $\mathcal{X}_0 \rightarrow \mathcal{S}_0$ with the degeneracy (ramification) data $(g_0 : D_0 \rightarrow \mathbb{P}^1, \widetilde{D}_0 \rightarrow D_0)$. The main technical challenge is to do this in such a way that everything deforms in families.

Indeed, consider pairs (\mathcal{S}, D) where \mathcal{S} is a quintic del Pezzo surface and $D \in |-2K_{\mathcal{S}}|$ is a general bi-anticanonical divisor. We can clearly specialize

$$(\mathcal{S}, D) \rightsquigarrow (\mathcal{S}_0, D_0)$$

but we'd also like a conic bundle $\mathcal{X} \rightarrow \mathcal{S}$ degenerate over D such that

$$\mathcal{X} \rightsquigarrow \mathcal{X}_0.$$

The final step is to prove that \mathcal{X} is not stably rational. The decomposition of the diagonal technique has been implemented for conic bundles over rational surface [3]. For our application, we specialize

$$D \rightsquigarrow D_1 \cup D_2, \quad D_1, D_2 \in |-K_{\mathcal{S}}|$$

to a union of two smooth elliptic curves. It follows that \mathcal{X} fails to admit a decomposition of the diagonal and thus is not stably rational.

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NC deformations of simple collections

YUJIRO KAWAMATA

The usual deformation theory of modules over commutative Artin rings ([5]) is generalized to that over NC associative Artin rings in a parallel way ([4]), where NC means “not necessarily commutative”:

Definition 1 ([4]). (1) Fix a field k and a positive integer r . (Art_r) is the category of associative k -algebras R with k -algebra homomorphisms

$$k^r \xrightarrow{f} R \xrightarrow{g} k^r$$

such that $gf = \text{id}$, where k^r is the product ring, and such that the two-sided ideal $M = \text{Ker}(g)$ is nilpotent. Denote $M_i = \text{Ker}(p_i g)$ for the i -th projection p_i , so that $M = \bigcap_{i=1}^r M_i$.

(2) Let $F = \bigoplus_{i=1}^r F_i$ be a sum of coherent sheaves on a k -scheme X . An r -pointed NC deformation of F over $R \in (\text{Art}_r)$ is a pair (F_R, ϕ) , where F_R is a left $R \otimes_k \mathcal{O}_X$ -module which is R -flat, and $\phi : R/M \otimes_R F_R \rightarrow F$ is an isomorphism. An r -pointed NC deformation functor $\text{Def}_F : (\text{Art}_r) \rightarrow (\text{Set})$ assigns an algebra R to the set of isomorphism classes of NC deformations over R .

The paper [3] was inspired by a recent work by Donovan and Wemyss on the NC deformations of the exceptional curves of the 3-fold small birational contractions [2].

We consider deformation of simple collections:

Definition 2 ([3]). A sum of sheaves $F = \bigoplus_{i=1}^r F_i$ is said to be a *simple collection* if

$$\text{Hom}(F_i, F_j) \cong \begin{cases} k & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3 ([3]). *A versal NC deformation of a simple collection $F = \bigoplus_{i=1}^r F_i$ coincides with the projective system of sheaves $F^{(n)}$ obtained by a sequence of universal extensions*

$$0 \rightarrow \bigoplus_{i=1}^r \text{Ext}^1(F^{(n)}, F_i) \rightarrow F^{(n+1)} \rightarrow F^{(n)} \rightarrow 0$$

where the base ring of the NC deformations coincides with NC algebras $R^{(n)} = \text{End}(F^{(n)})$.

If the sequence of universal extensions terminates after finitely many steps, then we obtain a deformation F_R such that $\text{Ext}^1(F^{(n)}, F) = 0$:

Definition 4 ([3]). Let F_R be a versal r -pointed NC deformation of F over $R \in (\text{Art}_r)$.

(1) F_R is said to be a *relative exceptional object* over R if $\text{Ext}^p(F^{(n)}, F) = 0$ for all $p > 0$.

(2) Let d be a positive integer. Assume in addition that $D^b(\text{coh}(X))$ has a Serre functor S . F_R is said to be a *relative d -spherical object* over R if

$$\text{Ext}^p(F^{(n)}, F_i) \cong \begin{cases} R/M_i & \text{if } p = 0 \\ R/M_{\sigma(i)} & \text{if } p = d \\ 0 & \text{otherwise} \end{cases}$$

for a permutation $\sigma \in \mathfrak{S}_r$ and if $S(F_i) \cong F_i[d]$.

Theorem 5 ([3]). *Let F_R be a relative exceptional object over R . Assume in addition that X is quasi-projective, the support of F_R is projective and that F_R is a perfect complex. Then the triangulated subcategory $\langle F_R \rangle \subset D^b(\text{coh}(X))$ is equivalent to $D^b(\text{mod-}R)$ and there is a semi-orthogonal decomposition*

$$D^b(\text{coh}(X)) = \langle \langle F_R \rangle^\perp, \langle F_R \rangle \rangle.$$

Theorem 6 ([3]). *Let F_R be a relative d -spherical object over R . Assume in addition that X is smooth and quasi-projective and the support of F_R is projective. Then the following hold:*

- (1) R is reflexive, i.e., $\text{Hom}_k(R, k)$ is a free right R -module of rank 1.
- (2) The functor $\Phi : D^b(\text{mod-}R) \rightarrow D^b(\text{coh}(X))$ given by $\Phi(\bullet) = \bullet \otimes_R F_R$ is a spherical functor in the sense of [1].

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Rational singularities

SÁNDOR J. KOVÁCS

The importance of rational singularities has been demonstrated for decades through various applications, although most of these applications are in characteristic zero. A major problem standing in the way of effective use of rational singularities in positive characteristic is that their definition depends on the existence of a resolution of singularities.

The main goal of this talk is to propose a new definition of rational singularities that does not even mention resolutions, but which is equivalent to the usual definition when resolutions exist.

Definition [Kov17b] A scheme Y is said to have *rational singularities*, if

- (1) Y is an excellent normal Cohen-Macaulay scheme that admits a dualizing complex, and
- (2) for every excellent Cohen-Macaulay scheme X , and every $f : X \rightarrow Y$ locally projective birational morphism, the induced natural morphism $\mathcal{O}_Y \xrightarrow{\sim} Rf_*\mathcal{O}_X$ is an isomorphism.

Note that a priori it is not evident that this is satisfied by even smooth complex varieties. This definition is reminiscent of Lipman-Tessier's definition of *pseudo-rational singularities* [LT81], but the relationship between the two definitions are not obvious.

The following is the main result concerning this proposed new definition.

Theorem [Kov17b] *Let X and Y be excellent Cohen-Macaulay schemes and $f : X \rightarrow Y$ a locally projective birational morphism. If Y has pseudo-rational singularities, then*

$$\mathcal{O}_Y \simeq f_*\mathcal{O}_X, f_*\omega_X \simeq \omega_Y, \text{ and } R^i f_*\mathcal{O}_X = 0 \text{ and } R^i f_*\omega_X = 0 \text{ for } i > 0.$$

This result has several applications. In particular, it implies the equivalence of rational and pseudo-rational singularities as well as the equivalence of this new definition with the “traditional” definition, which we will call *resolution-rational* here. More precisely, we have the following implications (cf. [Kov17b, (1.4.1)]):

$$\begin{aligned} \text{regular} &\implies \text{rational} \\ \text{regular} &\implies \text{resolution-rational} \implies \text{pseudo-rational} \\ \text{rational} &\iff \text{pseudo-rational} \end{aligned}$$

and if there exists a resolution of singularities, then also

$$\text{rational} \iff \text{resolution-rational}$$

An important application of rational singularities in the minimal model program stemming from the fact that klt singularities (cf. [Kol13]) are rational in characteristic zero [Elk81]. Unfortunately, this fails in positive characteristic [Yas14, CT16a, CT16b, Kov17a, Ber17, Tot17, Yas17], but using the above theorem one can prove that Cohen-Macaulay klt singularities are rational in arbitrary characteristic:

Theorem [Kov17b] *Let W be a Cohen-Macaulay klt scheme. Then W has rational singularities and hence if W admits a resolution of singularities, then it also has resolution-rational singularities.*

Note that Hacon and Witaszek [HW17] recently proved that for large characteristics and assuming that $\dim W = 3$ this already holds without the Cohen-Macaulay assumption. However, assuming some lower bound on the characteristic is necessary by [Kov17a, Ber17, Tot17, Yas17].

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D-equivalence and L-equivalence

ALEXANDER KUZNETSOV

Let X and Y be smooth projective varieties over a field \mathbb{k} of characteristic 0.

Definition 1. We say that X and Y are *D-equivalent* if the bounded derived categories of coherent sheaves on X and Y are equivalent as triangulated categories, i.e., $\mathbf{D}(X) \cong \mathbf{D}(Y)$.

Let $K_0(\text{Var}/\mathbb{k})$ be the Grothendieck ring of \mathbb{k} -varieties; it is generated by the classes $[X]$ of all (not necessarily smooth or projective) \mathbb{k} -varieties subject to relation $[X] = [Z] + [X \setminus Z]$ for $Z \subset X$ a closed subset. Multiplication in $K_0(\text{Var}/\mathbb{k})$ is defined by $[X] \cdot [Y] = [X \times_{\mathbb{k}} Y]$. We denote by

$$\mathbb{L} = [\mathbb{A}_{\mathbb{k}}^1],$$

the class of an affine line.

Definition 2. We say that X and Y are *L-equivalent* if

$$([X] - [Y])\mathbb{L}^r = 0$$

in $K_0(\text{Var}/\mathbb{k})$ for some $r \geq 0$. We say that X and Y are *trivially L-equivalent* if $[X] = [Y]$ in $K_0(\text{Var}/\mathbb{k})$.

It was known for a long time that the ring $K_0(\text{Var}/\mathbb{k})$ is not a domain, but it was not known whether \mathbb{L} is a zero-divisor. In particular, if there are L-equivalent varieties that are not trivially L-equivalent. A recent paper [GS14] of Galkin and Shinder gave an additional motivation for this question (it was shown that if \mathbb{L} is not a zero divisor then a very general cubic fourfold in \mathbb{P}^5 is irrational), and soon afterwards Lev Borisov found the first non-trivial example of L-equivalence.

Example 3. Let $X = \text{Gr}(2, 7) \cap \mathbb{P}^{13} \subset \mathbb{P}^{20}$ and $Y = \text{Pf}(7) \cap \mathbb{P}^6 \subset \check{\mathbb{P}}^{20}$ be a pair of mutually orthogonal smooth linear sections of the Grassmannian $\text{Gr}(2, 7) \subset \mathbb{P}^{20}$ in its Plücker embedding, and of its projectively dual variety $\text{Pf}(7) = \text{Gr}(2, 7)^\vee \subset \check{\mathbb{P}}^{20}$ in the dual projective space.

Theorem 4 ([Bor14, Mar16]). *The varieties X and Y are L-equivalent Calabi–Yau 3-folds:*

$$([X] - [Y])\mathbb{L}^6 = 0.$$

Moreover, they are not trivially L-equivalent.

The reason why Borisov looked at this particular pair of varieties is the fact that they are known to be D-equivalent [BC09, Kuz06] and not birational. The L-equivalence relation came out as a byproduct of some computations with this example. The Calabi–Yau property and non-birationality in this example are important for proving that the L-equivalence is not trivial (the proof is based on a wonderful result of Larsen and Lunts [LL03] about the structure of the quotient ring $K_0(\text{Var}/\mathbb{k})/(\mathbb{L})$).

This example suggests the following:

Conjecture 5 ([KS16]). *Let X and Y be a pair of smooth projective and simply connected D-equivalent varieties. Then X is L-equivalent to Y .*

By now there is a number of examples supporting this conjecture (for more examples see [KS16]):

- (1) Let X and Y be a pair of Calabi–Yau 3-folds obtained from G_2 -geometry [IMOU16a]. Then $\mathbf{D}(X) \cong \mathbf{D}(Y)$ by [Kuz16b] and $([X] - [Y])\mathbb{L} = 0$ by [IMOU16a]. In fact, pairs (X, Y) of this example are specializations of pairs from the Grassmannian–Pfaffian example of Borisov, but the relation they satisfy is more sharp.
- (2) Let $X = \text{Gr}(2, 5) \cap g(\text{Gr}(2, 5)) \subset \mathbb{P}^9$ and $Y = \text{Gr}(2, 5)^\vee \cap g^T(\text{Gr}(2, 5)^\vee) \subset \check{\mathbb{P}}^9$ for a general automorphism $g \in \text{PGL}_{10}(\mathbb{k})$. Then $\mathbf{D}(X) \cong \mathbf{D}(Y)$ by [KP17] and $([X] - [Y])\mathbb{L}^4 = 0$ by [BCP17].
- (3) Let $X = \text{OGr}_+(5, 10) \cap g(\text{OGr}_+(5, 10)) \subset \mathbb{P}^{15}$ and $Y = \text{OGr}_-(5, 10) \cap g^T(\text{OGr}_-(5, 10)) \subset \check{\mathbb{P}}^{15}$ for general $g \in \text{PGL}_{16}(\mathbb{k})$. Then $\mathbf{D}(X) \cong \mathbf{D}(Y)$ by [KP17] and $([X] - [Y])\mathbb{L}^7 = 0$ by [M17].

All the above examples are Calabi–Yau varieties. There is also a number of examples with K3 surfaces.

- (4) Let $X = Q_1 \cap Q_2 \cap Q_3 \subset \mathbb{P}^5$ be a smooth complete intersection of three quadrics and let $Y \rightarrow \mathbb{P}^2$ be the associated smooth double covering branched over the discriminant sextic curve. Then X and Y are K3 surfaces of degrees 8 and 2 respectively. There is a natural Brauer class $\alpha \in \text{Br}(Y)$ such that $\mathbf{D}(X) \cong \mathbf{D}(Y, \alpha)$. On the other hand, if $X(\mathbb{k}) \neq \emptyset$ and α is trivial, one has $([X] - [Y])\mathbb{L} = 0$ by [KS16]. In general this is not a trivial L-equivalence.
- (5) Let $X = \text{OGr}_+(5, 10) \cap \mathbb{P}^7$ and $Y = \text{OGr}_-(5, 10) \cap \mathbb{P}^7$ be mutually orthogonal linear sections. Then both X and Y are K3 surfaces of degree 14 such that $\mathbf{D}(X) \cong \mathbf{D}(Y)$ and by [IMOU16b] one has $([X] - [Y])\mathbb{L}^3 = 0$, while by [HL16] one has $([X] - [Y])\mathbb{L} = 0$.

The simple connectedness assumption in the conjecture is necessary because otherwise there are counterexamples among abelian varieties, see [IMOU16a, E17]¹.

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Geometric Manin's Conjecture and rational curves

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(joint work with Sho Tanimoto)

Let X be a smooth projective Fano variety over the complex numbers. The celebrated bend-and-break theorem by Mori tells us that the negative curvature of the canonical divisor K_X leads to the existence of rational curves on X . This relationship is *quantified* by Manin's Conjecture: the amount of curvature of K_X should govern the number of rational curves on X . The goal of the talk is to explain recent partial progress toward this conjecture. In the three sections below, we will discuss (1) how to quantify curvature, (2) explicit results on families of rational curves, and (3) a precise formulation of Manin's Conjecture.

1. QUANTIFYING CURVATURE

Let X be a smooth projective variety and let L be a big and nef \mathbb{Q} -divisor on X . In a loose sense, the following quantities compare the curvature of K_X against the curvature of L . Set

$$a(X, L) = \min\{t \in \mathbb{R} \mid t[L] + [K_X] \in \overline{\text{Eff}}^1(X)\}$$

and

$$b(X, L) = \text{the codimension of the minimal supported face of } \overline{\text{Eff}}^1(X) \text{ containing the numerical class } a(X, L)[L] + [K_X].$$

When X is singular, we define these invariants by pulling back L to a resolution of X .

The a and b invariants are the primary inputs into Manin's Conjecture (over any field). Thus, it is crucial to understand their geometric and structural properties. This is a topic of active ongoing research; we will mainly need one such result.

Theorem 1.1 ([HJ17] Theorem 1.1). *Let X be a smooth uniruled projective variety and let L be a big and nef \mathbb{Q} -divisor on X . Then there is a proper closed subset $V \subsetneq X$ such that any subvariety Y with $a(Y, L|_Y) > a(X, L)$ is contained in V .*

2. FAMILIES OF RATIONAL CURVES

Let X be a smooth projective variety and let $\text{Mor}(\mathbb{P}^1, X)$ denote the parameter space of morphisms of rational curves to X . We will be interested in understanding the most basic features of $\text{Mor}(\mathbb{P}^1, X)$: what is its dimension? How many components does it have? Such questions have been intensively studied for special kinds of varieties (hypersurfaces, toric varieties, and homogeneous spaces), but we are interested in understanding a picture for arbitrary Fano varieties.

Our first result describes the expected dimension of parameter spaces of rational curves on Fano varieties.

Theorem 2.1 ([LT17]). *Let X be a smooth projective weak Fano variety and set $L = -K_X$. Let $V \subsetneq X$ be the proper closed subset which is the Zariski closure of all subvarieties Y such that $a(Y, L|_Y) > a(X, L)$. Then any component of $\text{Mor}(\mathbb{P}^1, X)$ parametrizing a curve not contained in V will have the expected dimension and will parametrize a dominant family of curves.*

Assuming standard conjectures about rational curves, the converse implication is also true: a subvariety with higher a -value will contain families of rational curves with dimension higher than the expected dimension in X . In this way the a -invariant should completely control the expected dimension of components of $\text{Mor}(\mathbb{P}^1, X)$.

Theorem 2.1 is significant for two reasons. The first is that V is a proper closed subset of X (as explained above). The second is that Theorem 2.1 gives an explicit description of the closed set in terms of the a -invariant. This description can be effectively used to prove the non-existence of families of rational curves of larger-than-expected dimension.

Example 2.2. Suppose that X is a smooth cubic hypersurface of dimension ≥ 3 , a smooth quartic hypersurface of dimension ≥ 5 , or a smooth del Pezzo variety of Picard rank 1 and dimension ≥ 3 . Then every family of rational curves on X is dominant and has the expected dimension.

Just as the a -values describe dimension, the b -values should capture the number of components of $\text{Mor}(\mathbb{P}^1, X)$. [LT17] makes partial progress toward the following conjecture:

Conjecture 2.3. Fix an ample divisor L . The number of dominant components of $\text{Mor}(\mathbb{P}^1, X)$ of L -degree at most d is bounded above by a polynomial in d .

We expect the degree of this polynomial to reflect the behavior of the b -invariant in a precise way.

3. MANIN’S CONJECTURE

Finally, we explain how to formulate Manin’s Conjecture. Over an algebraically closed field one can not actually “count curves”; instead, we define a counting function which encapsulates the discrete invariants of $\text{Mor}(\mathbb{P}^1, X)$. For the sake of simplicity I will completely ignore a lattice issue in this talk. That is, I will give the correct definitions for varieties X admitting a curve class C such that $K_X \cdot C = -1$.

Let X be a smooth projective Fano variety and let L be a big and nef \mathbb{Q} -divisor on X . Define

$$\overline{N}(X, q, d) = \sum_{i=1}^d \sum_{W \in M_d} q^{\dim W}$$

where M_d is the set of dominant components of $\text{Mor}(\mathbb{P}^1, X)$ parametrizing curves of anticanonical degree d . Loosely speaking, Manin’s Conjecture predicts an asymptotic formula

$$\overline{N}(X, q, d) \sim Cq^d d^{\rho(X)-1}$$

Here there is an implicit $a(X, -K_X) = 1$ in the exponent of q and an implicit $b(X, -K_X) = \rho(X)$ in the exponent of d . However, this statement can't possibly be correct due to the presence of "accumulating subvarieties" where there are families of rational curves of higher than the expected dimension. So we need a systematic way of discounting the contributions of such varieties.

Definition 3.1. Let $f : Y \rightarrow X$ be a morphism of projective varieties that is generically finite onto its image. We say that f is a breaking morphism if

$$(a(Y, f^*L), b(Y, f^*L)) > (a(X, L), b(X, L))$$

in the lexicographic order.

Note that for a breaking map $f : Y \rightarrow X$ the counting function for rational curves on Y has larger expected growth rate than the counting function for rational curves on X . In other words, we must remove all contributions of rational curves from Y to obtain an internally-consistent version of Manin's Conjecture. First, we want to know that we have not removed "too many" curves by discounting these contributions.

Conjecture 3.2. Let X be a smooth uniruled projective variety. There is a finite set of breaking maps $f_i : Y_i \rightarrow X$ such that any breaking map factors rationally through one of the f_i .

This is a geometric version of the well-known "thinness" criterion commonly used in the number-theoretic setting.

Next we define a new counting function

$$N(X, q, d) = \sum_{i=1}^d \sum_{W \in M_d} q^{\dim W}$$

where M_d is the set of dominant components of $\text{Mor}(\mathbb{P}^1, X)$ parametrizing curves of anticanonical degree d such that the family map $s : \mathcal{C} \rightarrow X$ does not rationally factor through a breaking map $f : Y \rightarrow X$.

Conjecture 3.3. Let X be a smooth projective Fano variety. Then as $d \rightarrow \infty$ we have

$$N(X, q, d) \sim Cq^d d^{\rho(X)-1}$$

for some positive constant C .

We prove a somewhat weaker upper bound on the behavior of the counting function:

Theorem 3.4 ([LT17]). *Let X be a smooth projective Fano variety. Fix $\epsilon > 0$; then for sufficiently large q*

$$N(X, q, d) = O\left(q^{B(1+\epsilon)}\right).$$

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**A projective variety with discrete, non-finitely generated
automorphism group**

JOHN LESIEUTRE

Suppose that X is a projective variety over a field K . The automorphism group of X has the structure of a group scheme of locally finite type over K . We write $\text{Aut}^0(X)$ for the connected component of the identity in $\text{Aut}(X)$, and let $\pi_0(\text{Aut}(X)) = (\text{Aut}(X)/\text{Aut}^0(X))_{\bar{K}}$ be the group of geometric components.

It is possible that $\text{Aut}(X)$ has infinitely many components, so that the group $\pi_0(\text{Aut}(X))$ is infinite. For a simple example, let E be an elliptic curve over K , and consider the abelian surface $X = E \times E$. There is a natural action of $\text{GL}(2, \mathbf{Z})$ on X , giving rise to an inclusion $\text{GL}(2, \mathbf{Z}) \subset \pi_0(\text{Aut}(X))$.

Brion [3] has showed that any connected algebraic group can be realized as $\text{Aut}^0(X)$ for some X , and it is natural to wonder what can be said about the group $\pi_0(\text{Aut}(X))$. This is a countable group, but there are a variety of constraints on its structure. For example, pulling back divisor classes gives rise to an action of $\text{Aut}(X)$ on $N^1(X)$, the (finite rank) group of classes of divisors on X modulo numerical equivalence. Any element of $\text{Aut}^0(X)$ acts trivially on $N^1(X)$, but according to a result of Lieberman and Fujiki, the kernel of the induced map $\pi_0(\text{Aut}(X)) \rightarrow \text{GL}(N^1(X))$ is finite. In particular, a quotient of $\pi_0(\text{Aut}(X))$ by a finite subgroup embeds into some $\text{GL}(n, \mathbf{Z})$. This is a nontrivial constraint, for it implies that there is a uniform bound on the orders of the torsion elements of $\pi_0(\text{Aut}(X))$. In particular, the group $\bigoplus_{m \geq 2} (\mathbf{Z}/m\mathbf{Z})$ can not be realized as $\pi_0(\text{Aut}(X))$ for any projective variety X over a field.

Given that the group $\pi_0(\text{Aut}(X))$ is a countable group, perhaps the most natural question to ask is whether this group is always finitely generated.

Question 1 (Mazur, [1]). Suppose that X is a projective variety over K . Must the group $\pi_0(\text{Aut}(X))$ be finitely generated?

The main result of this talk was to show that the answer to this question is negative.

Theorem 1 ([2]). *Let K be a field that is either of characteristic 0, or of characteristic $p > 0$ but not algebraic over \mathbf{F}_p . Then there exists a smooth, projective variety over K for which the group $\pi_0(\text{Aut}(X))$ is not finitely generated.*

The theorem follows not from a highbrow existence proof, but by exhibiting an explicit six-dimensional variety for which finite generation fails to hold. The

strategy is to start with a variety X with large but (probably) finitely generated automorphism group, and then to pass to a suitable blow-up of X .

Suppose that X is a smooth, projective variety, and that $V \subset X$ is a smooth subvariety of X . If $\phi : X \rightarrow X$ is an automorphism of X satisfying $\phi(V) = V$, then ϕ lifts to an automorphism $\tilde{\phi} : \text{Bl}_V(X) \rightarrow \text{Bl}_V(X)$, where $\text{Bl}_V(X)$ is the blow-up of X along V . Writing $\text{Aut}(X; V) = \{\phi \in \text{Aut}(X) : \phi(V) = V\}$, we obtain a homomorphism $\text{Aut}(X; V) \rightarrow \text{Aut}(\text{Bl}_V(X))$.

In general, this map is not surjective: a simple example is provided by taking $X = \mathbf{P}^2$ and V the union of three non-collinear points: then $\text{Aut}(\text{Bl}_V(X))$ admits an automorphism not lifted from X , namely the map induced by the standard Cremona involution centered at the three points.

There are now two steps to construct a variety with non-finitely generated automorphism group:

- (1) Find a variety X and subvariety V for which the group $\text{Aut}(X; V) \subset \text{Aut}(X)$ is not finitely generated (recall that a subgroup of a finitely generated group need not be finitely generated!)
- (2) Show that for this X and V , the map $\text{Aut}(X; V) \rightarrow \text{Aut}(\text{Bl}_V(X))$ is surjective, so that $\text{Aut}(\text{Bl}_V(X))$ is not finitely generated.

The second of these is straightforward: roughly speaking, the map is guaranteed to be surjective as long as the codimension of V is sufficiently large. The first is more interesting, and the construction of X is based on the study of the automorphism groups of certain Coble rational surfaces, a classically well-studied class of blow-ups of the plane. Ultimately, the variety X we obtain is six-dimensional, formed as product of two rational surfaces and a surface of general type.

The arithmetic impetus towards questions about the automorphism group arises from the fact that this group determines the forms of a variety over extension fields. If L/K is a Galois extension, then a K -variety X' is called an L/K -form of X if $X_L \cong X'_L$. The set of L/K -forms of X , up to isomorphism over K is classified by the Galois cohomology set $H^1(\text{Gal}(L/K), \text{Aut}(X_L))$. The second main result of my talk was that even when L/K is a quadratic extension, the set of L/K forms can be infinite.

Theorem 2 ([2]). *Let K be a field that is either of characteristic 0, or of characteristic $p > 0$ but not algebraic over \mathbf{F}_p . Suppose that L/K is a separable quadratic extension. Then there exists a projective K -variety X' with infinitely many L/K -forms.*

When $L = \mathbf{C}$ and $K = \mathbf{R}$, this provides an example of a variety with infinitely many real forms.

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Nef cones of hyperkähler fourfolds and applications

EMANUELE MACRÌ

(joint work with Olivier Debarre)

We study smooth projective hyperkähler fourfolds which are deformations of Hilbert squares of K3 surfaces and are equipped with a polarization of fixed degree and divisibility. These are parametrized by a quasi-projective irreducible 20-dimensional moduli space and Verbitsky’s Torelli Theorem implies that their period map is an open embedding. Our main result is that the complement of the image of the period map is a finite union of explicit Heegner divisors that we describe. The key technical ingredient is the description of the nef and movable cone for projective hyperkähler manifolds by Bayer, Hassett, Tschinkel, and Mongardi.

We will sketch two applications. First of all, we present a new short proof (by Bayer and Mongardi) for the celebrated result by Laza and Looijenga on the image of the period map for cubic fourfolds. As second application, we show that infinitely many Heegner divisors in a given period space have the property that their general points correspond to fourfolds which are isomorphic to Hilbert squares of a K3 surfaces.

All results are based on joint work with Olivier Debarre.

1. INTRODUCTION

Let X be a smooth projective hyperkähler fourfold which is deformation-equivalent to the Hilbert square of a K3 surface (one says that X is of $\text{K3}^{[2]}$ -type). The abelian group $H^2(X, \mathbf{Z})$ is free of rank 23 and it is equipped the *Beauville–Bogomolov–Fujiki form* q_X , a non-degenerate \mathbf{Z} -valued quadratic form of signature $(3, 20)$ ([Be83, Théorème 5]). A *polarization* H on X is the class of an ample line bundle on X that is primitive (i.e., non-divisible) in the group $H^2(X, \mathbf{Z})$. The *square* of H is the positive even integer $2n := q_X(H)$ and its *divisibility* is the integer $\gamma \in \{1, 2\}$ such that $H \cdot H^2(X, \mathbf{Z}) = \gamma\mathbf{Z}$ (the case $\gamma = 2$ only occurs when $n \equiv -1 \pmod{4}$).

Smooth polarized hyperkähler fourfolds (X, H) of $\text{K3}^{[2]}$ -type of degree $2n$ and divisibility γ admit an irreducible quasi-projective coarse moduli space $\mathcal{M}_{2n}^{(\gamma)}$ of dimension 20. The period map

$$\wp_{2n}^{(\gamma)} : \mathcal{M}_{2n}^{(\gamma)} \longrightarrow \mathcal{P}_{2n}^{(\gamma)}$$

is algebraic and it is an open embedding by Verbitsky’s Torelli Theorem ([Ve13], [GHS13, Theorem 3.14], [Ma11]). Our main result is the following (see Theorem 2.2):

Theorem 1.1 (Bayer; Debarre-Macri; Amerik-Verbitsky). *The image of $\wp_{2n}^{(\gamma)}$ is the complement of a finite union of explicit Heegner divisors.*

The main ingredient in the proof is the explicit determination of the nef and movable cones of smooth projective hyperkähler fourfolds of $K3^{[2]}$ -type (see Theorem 2.1). This is a simple consequence of previous results by Markman ([Ma11]), Bayer–Macri ([BM14]), Bayer–Hassett–Tschinkel ([BHT15]), and Mongardi ([Mo15]).

Since the nef and movable cones can be described in any dimension, Theorem 1.1 extends with some modifications to smooth projective hyperkähler manifolds of $K3^{[n]}$ -type, though the description of the image of the period map is less explicit. We will use this generalization, together with a *strange duality* statement and a construction by Lehn–Lehn–Sorger–van Straten ([LLSvS17]) to give a new proof (by Bayer and Mongardi) of the description of the image of the period map for cubic fourfolds ([La10, Lo09]).

The *Noether–Lefschetz locus* is the inverse image by the period map in $\mathcal{M}_{2n}^{(\gamma)}$ of the union of all Heegner divisors. As a second application, we can study birational isomorphisms between some of its irreducible components. In particular, we show that points corresponding to Hilbert squares of $K3$ surfaces are dense in the moduli spaces $\mathcal{M}_{2n}^{(\gamma)}$.

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2. NEF CONES OF HYPERKÄHLER FOURFOLDS

Cones of divisors on projective hyperkähler manifolds of $K3^{[n]}$ -type were described in [BHT15, BM14, Ma11, Mo15]. When $n = 2$, these results take a very special form.

Let X be a projective hyperkähler fourfold of $K3^{[2]}$ -type. The *positive cone*

$$\text{Pos}(X) \subset \text{Pic}(X) \otimes \mathbb{R}$$

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is the connected component of the open subset $\{x \in \text{Pic}(X) \otimes \mathbb{R} \mid x^2 > 0\}$ containing the class of an ample divisor. The *movable cone*

$$\text{Mov}(X) \subset \text{Pic}(X) \otimes \mathbb{R}$$

is the (not necessarily open nor closed) convex cone generated by classes of movable divisors (i.e., those divisors whose base locus has codimension at least 2). We have inclusions $\text{Int}(\text{Mov}(X)) \subset \text{Pos}(X)$ of the interior of the movable cone into the positive cone, and $\text{Amp}(X) \subset \text{Mov}(X)$ of the ample cone into the movable cone.

We set

$$\begin{aligned} \text{Div}_X &:= \{a \in \text{Pic}(X) \mid a^2 = -2\}, \\ \text{Flop}_X &:= \{a \in \text{Pic}(X) \mid a^2 = -10, \div_{H^2(X, \mathbf{Z})}(a) = 2\}. \end{aligned}$$

Given a divisor class $a \in \text{Pic}(X) \otimes \mathbb{R}$, we denote by H_a the hyperplane

$$H_a := \{x \in \text{Pic}(X) \otimes \mathbb{R} \mid x \cdot a = 0\}.$$

Theorem 2.1 ([DM17, Theorem 5.1]). *Let X be a hyperkähler fourfold of $\text{K3}^{[2]}$ -type.*

(a) *The interior $\text{Int}(\text{Mov}(X))$ of the movable cone is the connected component of*

$$\text{Pos}(X) \setminus \bigcup_{a \in \text{Div}_X} H_a$$

that contains the class of an ample divisor.

(b) *The ample cone $\text{Amp}(X)$ is the connected component of*

$$\text{Int}(\text{Mov}(X)) \setminus \bigcup_{a \in \text{Flop}_X} H_a$$

that contains the class of an ample divisor.

Theorem 2.1 easily implies our main result on the images of the period maps. We first recall the definition of the Heegner divisors. Let us denote

$$\Lambda_{\text{K3}^{[2]}} := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus I_1(-2)$$

and let h_0 be a class with square $2n$ and divisibility γ (all classes with the same square and divisibility are in the same $O(\Lambda_{\text{K3}^{[2]}})$ -orbit, by Eichler’s criterion [GHS10, Lemma 3.5]). Let K be a primitive, rank-2, signature-(1, 1) sublattice of $\Lambda_{\text{K3}^{[2]}}$ containing the class h_0 . The codimension-2 subspace $\mathbf{P}(K^\perp \otimes \mathbf{C})$ in $\mathbf{P}(\Lambda_{\text{K3}^{[2]}} \otimes \mathbf{C})$ cuts out an irreducible hypersurface in the period domain $\mathcal{P}_{2n}^{(\gamma)}$, which will be denoted by $\mathcal{D}_{2n, K}^{(\gamma)}$ and called a *Heegner divisor*. For each positive integer d , the union

$$\mathcal{D}_{2n, d}^{(\gamma)} := \bigcup_{\text{disc}(K^\perp) = -d} \mathcal{D}_{2n, K}^{(\gamma)} \subset \mathcal{P}_{2n}^{(\gamma)}$$

of Heegner divisors is finite, hence it is either empty or of pure codimension 1. Following Hassett ([Ha00]), we say that the polarized hyperkähler fourfolds whose period point is in $\mathcal{D}_{2n, d}^{(\gamma)}$ are *special of discriminant d* (the lattice K^\perp has signature $(2, 19)$, hence d is positive). We use the notation $\mathcal{C}_{2n, d}^{(\gamma)} := (\wp_{2n}^{(\gamma)})^{-1}(\mathcal{D}_{2n, d}^{(\gamma)}) \subset \mathcal{M}_{2n}^{(\gamma)}$.

Theorem 2.2 ([DM17, Theorem 6.1]). *Let n be a positive integer and let $\gamma \in \{1, 2\}$. The image of the period map*

$$\wp_{2n}^{(\gamma)} : \mathcal{M}_{2n}^{(\gamma)} \longrightarrow \mathcal{P}_{2n}^{(\gamma)}$$

is exactly the complement of finitely many Heegner divisors. More precisely, these Heegner divisors are

- if $\gamma = 1$,
 - some irreducible components of the hypersurface $\mathcal{D}_{2n,2n}^{(1)}$ (two components if $n \equiv 0$ or $1 \pmod{4}$, one component if $n \equiv 2$ or $3 \pmod{4}$);
 - one irreducible component of the hypersurface $\mathcal{D}_{2n,8n}^{(1)}$;
 - one irreducible component of the hypersurface $\mathcal{D}_{2n,10n}^{(1)}$;
 - and, if $n = 5^{2\alpha+1}n''$, with $\alpha \geq 0$ and $n'' \equiv \pm 1 \pmod{5}$, some irreducible components of the hypersurface $\mathcal{D}_{2n,2n/5}^{(1)}$;
- if $\gamma = 2$ (and $n \equiv -1 \pmod{4}$), one irreducible component of the hypersurface $\mathcal{D}_{2n,2n}^{(2)}$.

Remark 2.3. When n is square-free (so in particular $n \not\equiv 0 \pmod{4}$),

- the hypersurface $\mathcal{D}_{2n,2n}^{(1)}$ has two components if $n \equiv 1 \pmod{4}$, one component otherwise;
- the hypersurface $\mathcal{D}_{2n,8n}^{(1)}$ has two components if $n \equiv -1 \pmod{4}$, one component otherwise;
- the hypersurface $\mathcal{D}_{2n,10n}^{(1)}$ has two components if $n \equiv 1 \pmod{4}$, one component otherwise;
- the hypersurface $\mathcal{D}_{2n,2n}^{(2)}$ is irreducible (when $n \equiv -1 \pmod{4}$).

Idea of the proof of Theorem 2.2. First of all, one uses Huybrechts' surjectivity of the period map for compact complex analytic hyperkähler manifolds ([Hu99]), to find a manifold M' with the given period and for which the class h_0 is algebraic. Since $h_0^2 = 2n > 0$, by Huybrechts' Projectivity Criterion (still [Hu99]), M' is projective. By using the Zariski decomposition ([Bo04] and [Ma11, Lemma 6.22]), by acting with a subgroup of the group of Hodge isometries of $\Lambda_{K3[2]}$, we can assume that the class h_0 is movable. Then, there exists a projective birational model M for which h_0 is nef. The conclusion follows then by using Theorem 2.1 to explicitly describe when h_0 is ample. \square

Remark 2.4. Let $n \geq 1$ and $m \geq 2$ be positive integers. In general we can define moduli spaces of polarized hyperkähler manifolds of dimension $2m$, together with a polarization of degree $2n$ and divisibility γ (and with fixed orbit-type, if $m \geq 4$; see [De17] for more details about this). By using a general version of Theorem 2.1 in any dimension, Theorem 2.2 also holds in any dimension, namely, the image of the period map for polarized hyperkähler manifolds is always the complement of a finite number of Heegner divisors (notice that the moduli space may not be connected anymore, though the Torelli Theorem still implies that the restriction of the period map to each connected component is an open embedding). Actually,

as a consequence of general results by Amerik and Verbitsy ([AK15]), this holds for any projective hyperkähler manifold, not necessarily of $K3^{[m]}$ -type.

We will consider only a special case. Assume that either $\gamma = 1$ or $\gamma = 2$ and $n + m \equiv 1 \pmod{4}$. Then the moduli space ${}^m\mathcal{M}_{2n}^{(\gamma)}$ and ${}^{n+1}\mathcal{M}_{2m-2}^{(\gamma)}$ are both irreducible ([GHS10, Ap15]). In such a case, there is a *strange duality* birational isomorphism between the two moduli spaces, which is compatible with a natural isomorphism between the period domains. This duality will be very important in one of the applications of Theorem 2.2 we will describe in the next section. When the hyperkähler manifolds are actually moduli spaces of stable objects on a certain K3 category (e.g., in the case of the manifolds coming from cubic fourfolds, as in the next section; see [BLMS17]), then this strange duality is an instance of Le Potier's Strange Duality, and the birational map described below should also have (conjecturally) the extra property that the spaces of sections of the two polarizations should be naturally isomorphic (up to taking the dual vector space).

3. APPLICATIONS

We will discuss now two applications of Theorem 2.1 and Theorem 2.2.

3.1. The period map for cubic fourfolds. Let \mathcal{M}_{cub} be the moduli space of smooth projective cubic fourfolds. We can consider the period map, and on the period domain there are Heegner divisors (generally denoted by \mathcal{C}_d). Actually, in the notation of the previous section, the period of a cubic fourfold W can be naturally identified with the period of its Fano variety of lines $F(W)$; the variety $F(W)$ is a polarized hyperkähler fourfold of $K3^{[2]}$ -type endowed with the Plücker polarization of degree 6 and divisibility 2. In our previous notation the Heegner divisors \mathcal{C}_d would then correspond to $\mathcal{D}_{6,d}^2$.

Theorem 3.1 (Laza; Looijenga). *The image of the period map for cubic fourfolds is the complement of the divisors \mathcal{C}_2 and \mathcal{C}_6 .*

Proof. Idea of the proof by following Bayer and Mongardi, when W does not contain a plane.

Let W be a cubic fourfold which does not contain a plane. We look at the hyperkähler eightfold $X(W)$ constructed by Lehn-Lehn-Sorger-van Straten ([LLSvS17]). The key property of $X(W)$ is that it carries a polarization of degree 2 and divisibility 2. There is an associated anti-symplectic involution and the fixed locus has exactly two (smooth projective fourfolds) connected components, one of them is the cubic itself. From our viewpoint, the varieties $F(W)$ and $X(W)$ are dual, in the sense of Remark 2.4. We can also compute the periods, and these are all compatible.

To prove the theorem, by the Torelli Theorem, any element in the moduli space ${}^4\mathcal{M}_2^{(2)}$ has a regular involution. By varying the hyperkähler eightfold in the moduli space, the fixed locus gives a smooth family. In particular, since the deformation of a cubic fourfold is a cubic fourfold as well, it means that for any $X \in {}^4\mathcal{M}_2^{(2)}$, there exists a cubic fourfold $W_X \subset X$ in the fixed locus of the involution, and

the periods of W_X and X are compatible via duality. An easy computation along the lines of Theorem 2.2 shows that the period map for ${}^4\mathcal{M}_2^{(2)}$ via duality exactly avoids the divisors $\mathcal{C}_2, \mathcal{C}_6,$ and \mathcal{C}_8 . Since cubic fourfolds in the divisor \mathcal{C}_8 are those containing a plane, this prove the theorem except in this case. \square

3.2. Unexpected isomorphisms. As another application, we can understand when the general member of one of these Heegner divisors is isomorphic to a Hilbert square of a K3 surface. First of all, we need the following particular case of Theorem 2.1. Let us briefly review Pell-type equations (see [Na64, Chapter VI]). Given non-zero integers e and t with $e > 0$, we denote by $\mathcal{P}_e(t)$ the equation

$$a^2 - eb^2 = t,$$

where a and b are integers. A solution (a, b) of this equation is called positive if $a > 0$ and $b > 0$. A positive solution with minimal a is called the minimal solution; it is also the positive solution (a, b) for which the ratio a/b is minimal when $t < 0$, maximal when $t > 0$.

Example 3.2 ([BM14, Proposition 13.1 and Lemma 13.3]²). Let (S, L) be a polarized K3 surface such that $\text{Pic}(S) = \mathbf{Z}L$ and $L^2 = 2e$. Then $\text{Pic}(S^{[2]}) = \mathbf{Z}L_2 \oplus \mathbf{Z}\delta$, where L_2 is the class on $S^{[2]}$ induced by L and 2δ is the class of the divisor in $S^{[2]}$ that parametrizes non-reduced length-2 subschemes of S ([Be83, Remarque, p. 768]).

Cones of divisors on $S^{[2]}$ can be described as follows.

- (a) The extremal rays of the (closed) movable cone $\text{Mov}(S^{[2]})$ are spanned by L_2 and $L_2 - \mu_e\delta$, where
 - if e is a perfect square, $\mu_e = \sqrt{e}$;
 - if e is not a perfect square and (a_1, b_1) is the minimal solution of the equation $\mathcal{P}_e(1)$, $\mu_e = e\frac{b_1}{a_1}$.
- (b) The extremal rays of the nef cone $\text{Nef}(S^{[2]})$ are spanned by L_2 and $L_2 - \nu_e\delta$, where
 - if the equation $\mathcal{P}_{4e}(5)$ is not solvable, $\nu_e = \mu_e$;
 - if the equation $\mathcal{P}_{4e}(5)$ is solvable and (a_5, b_5) is its minimal solution, $\nu_e = 2e\frac{b_5}{a_5}$.³

Proposition 3.3 ([DM17, Proposition 7.1]). *Let n and e be positive integers. Assume that the equation $\mathcal{P}_e(-n)$ has a positive solution (a, b) that satisfies the conditions*

$$(1) \quad \frac{a}{b} < \nu_e \quad \text{and} \quad \text{gcd}(a, b) = 1.$$

If \mathcal{K}_{2e} is the moduli space of polarized K3 surfaces of degree $2e$, the rational map

$$\begin{aligned} \varpi: \mathcal{K}_{2e} &\dashrightarrow \mathcal{M}_{2n}^{(\gamma)} \\ (S, L) &\longmapsto (S^{[2]}, bL_2 - a\delta), \end{aligned}$$

²Parts of the results of this example were first proved in [HT09, Theorem 22] and the rationality of the nef cone was also proved, by very different methods, in [Og14, Corollary 5.2].

³There is a typo in [BM14, Lemma 13.3(b)]: one should replace d with $2d$.

where $\gamma = 2$ if b is even, and $\gamma = 1$ if b is odd, induces a birational isomorphism onto an irreducible component of $\mathcal{C}_{2n,2e}^{(\gamma)}$.

By using a deep result by Clozel and Ullmo ([CU05]), we deduce then the following.

Proposition 3.4 ([DM17, Proposition 7.9]). *Let n be a positive integer. There are infinitely many distinct hypersurfaces in the moduli spaces $\mathcal{M}_{2n}^{(1)}$, and $\mathcal{M}_{2n}^{(2)}$ if $n \equiv -1 \pmod{4}$, whose general points correspond to Hilbert squares of K3 surfaces. In both cases, the union of these hypersurfaces is dense in the moduli space for the euclidean topology.*

Similar statements hold for other classes of hyperkähler fourfolds, for example double EPW sextics ([O’G08]) or Fano varieties of lines on a cubic fourfold (see [DM17, Section 7]).

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Half-twists and the Tate conjecture

BEN MOONEN

Consider a field K that is finitely generated over \mathbf{Q} , and a smooth projective scheme X over K . If ℓ is a prime number, the Galois group $\Gamma_K = \text{Gal}(\bar{K}/K)$ acts continuously on the ℓ -adic cohomology groups $H^i(X)(n) = H^i(X_{\bar{K}}, \mathbf{Q}_{\ell}(n))$. A class $\xi \in H^{2i}(X)(i)$ is called a *Tate class* if its stabilizer is an open subgroup of Γ_K . There are the following two conjectures:

- : (S) For every X/K , i and n as above, $H^i(X)(n)$ is semisimple as a representation of Γ_K .
- : (T) For every X/K as above and $i \geq 0$, the cycle class map

$$\text{cl}_{\ell}: \text{CH}^i(X_{\bar{K}}) \otimes \mathbf{Q}_{\ell} \rightarrow \{\text{Tate classes in } H^{2i}(X)(i)\}$$

is surjective.

By the (strong version of the) Tate conjecture one usually means the combination of (S) and (T). For a discussion of recent progress on the Tate conjecture we recommend Totaro's article [7]. We have recently proved the following result; see [6].

Theorem 1. — *Conjecture (T) implies Conjecture (S).*

The proof makes essential use of the fact that we have a semisimple Tannakian category of motives $\text{Mot}(K)$ over K . (We work with motives in the sense of André [1], though one could also use motives for absolute Hodge classes as defined by Deligne [2].) In the proof there is first a reduction step to the case $K = \mathbf{Q}$ and $n = 0$. If $V \subset H^i(X)$ is a $\Gamma_{\mathbf{Q}}$ -submodule of dimension m then $\wedge^m V$ is a 1-dimensional Galois representation that occurs in $H^{mi}(X^m)$. The Galois action on this line is given by a character $\psi: \Gamma_{\mathbf{Q}} \rightarrow \mathbf{Z}_{\ell}^{\times}$. By making use of p -adic Hodge theory, one shows that there is an open subgroup $\Gamma' \subset \Gamma_{\mathbf{Q}}$ on which ψ is an integral power of the ℓ -adic cyclotomic character. Conjecture (T) then gives that, working over some number field, $\wedge^m V \subset \wedge^m H^i(X)$ is a motivated subspace, i.e., it is stable under the action of the motivic Galois group. This implies that $V \subset H^i(X)$ is a

motivated subspace, and as $\text{Mot}(K)$ is a semisimple category, $H^i(X) = V \oplus V'$ as motives, and hence also as Galois representations.

A second result that illustrates how the theory of motives can be used to prove results about the Tate conjecture, uses a technique called “half-twisting” that was originally introduced by van Geemen [8] in the context of Hodge theory. I have given a different formulation of this that is better suited for motivic applications. As a concrete example, we have the following result.

Theorem 2. — *Let $X \subset \mathbf{P}^3$ be a non-singular surface given by an equation of the form $F_5(x_0, x_1, x_2) + x_3^5 = 0$ (in characteristic 0). Then the Tate conjecture for X is true.*

The proof is based on the idea that there exist abelian varieties A and B with an action of $\mathbf{Q}(\zeta_5)$ such that

$$(1) \quad H_{\text{prim}}^2(X) \cong H^1(A) \otimes_{\mathbf{Q}(\zeta_5)} H^1(B),$$

as Hodge structures. More precisely, B is an abelian surface of CM type, with CM field $\mathbf{Q}(\zeta_5)$; the functor $-\otimes_{\mathbf{Q}(\zeta_5)} H^1(B)$ is what we refer to as “half twisting”. (Note that it preserves the dimension, and increases the weight by 1.) By working in a family one proves, using results of Deligne [2] and André [1], that the relation (1) is even true on the level of motives. Hence one gets the corresponding relation for ℓ -adic cohomology (viewed as Galois representations), and using the results of Faltings about the Tate conjecture for homomorphisms between abelian varieties we obtain the theorem. Further details about this example appear in the last section of [5]. For other applications of half-twisting we refer to [3] and [4].

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Intersections of two Grassmannians in \mathbf{P}^9

ALEXANDER PERRY

(joint work with Lev Borisov and Andrei Căldăraru)

We work over an algebraically closed field k of characteristic 0. Let V be a 5-dimensional vector space over k , and let $W = \wedge^2 V$. We consider intersections of the form

$$X = g_1(\mathrm{Gr}(2, V)) \cap g_2(\mathrm{Gr}(2, V)) \subset \mathbf{P}(W),$$

where $g_1, g_2 \in \mathrm{PGL}(W)$ and $\mathrm{Gr}(2, V) \subset \mathbf{P}(W)$ via the Plücker embedding. When smooth of expected dimension, X is a Calabi–Yau threefold with Hodge numbers

$$h^{1,1}(X) = 1, \quad h^{1,2}(X) = 51.$$

These varieties were previously studied in works of Gross–Popescu [4], G. Kapustka [6], M. Kapustka [7], and Kanazawa [5], after whom we call X a *GP K^3 threefold*.

The elements $g_1, g_2 \in \mathrm{PGL}(W)$ determine another intersection of the same type, in the dual projective space:

$$Y = g_1^{-T}(\mathrm{Gr}(2, V^\vee)) \cap g_2^{-T}(\mathrm{Gr}(2, V^\vee)) \subset \mathbf{P}(W^\vee),$$

where $g_i^{-T} = (g_i^{-1})^\vee: \mathbf{P}(W^\vee) \rightarrow \mathbf{P}(W^\vee)$ is the inverse transpose of g_i . The variety X is a smooth threefold if and only if Y is. In this case, X and Y are smooth deformation equivalent Calabi–Yau threefolds, which we call *GP K^3 double mirrors*. This terminology is motivated by the following result, which appears as an example in forthcoming joint work with Alexander Kuznetsov.

Theorem 1 ([8]). *If X and Y are GP K^3 double mirrors, then there is an equivalence $D^b(X) \simeq D^b(Y)$ of bounded derived categories of coherent sheaves.*

Our main result says that, nonetheless, X and Y are typically not birational.

Theorem 2 ([3]). *For generic $g_1, g_2 \in \mathrm{PGL}(W)$, the varieties X and Y are not birational.*

Theorem 2 was also independently proved by John Ottem and Jørgen Rennemo [9]. Before explaining the main idea of our proof, we discuss some applications and auxiliary results.

Applications. Generic GP K^3 double mirrors give the first example of deformation equivalent, derived equivalent, but non-birational Calabi–Yau threefolds. By an observation from [1], a derived equivalence of complex Calabi–Yau threefolds induces an isomorphism of integral polarized Hodge structures on third cohomology. Thus we obtain:

Corollary 3 ([3]). *Generic complex GP K^3 double mirrors give a counterexample to the birational Torelli problem for Calabi–Yau threefolds.*

Previously, Szendrői [10] showed the usual Torelli problem fails for Calabi–Yau threefolds, but the birational version was open until our result.

As a second application of Theorem 2, we prove the following.

Theorem 4 ([3]). *If X and Y are GPK^3 double mirrors, then:*

(1) *In the Grothendieck ring $K_0(\text{Var}/k)$ of k -varieties, we have*

$$([X] - [Y])\mathbf{L}^4 = 0,$$

where $\mathbf{L} = [\mathbf{A}^1]$ is the class of the affine line.

(2) *If the elements $g_1, g_2 \in \text{PGL}(W)$ defining X and Y are generic, then*

$$[X] \neq [Y].$$

This adds to the growing list of examples, begun by [2], of derived equivalent varieties whose difference in the Grothendieck ring is annihilated by a power of \mathbf{L} . Part (1) is proved by studying a certain incidence correspondence, and part (2) is an easy consequence of Theorem 2.

Geometry and moduli of GPK^3 threefolds. Our proof of Theorem 2 involves several independently interesting results on the geometry and moduli of GPK^3 threefolds. The main result about the geometry of these threefolds is the following.

Proposition 5 ([3]). *The two $\text{Gr}(2, V)$ translates containing a GPK^3 threefold X are unique.*

This is proved by studying the restriction to X of the normal bundles of the translates $g_i(\text{Gr}(2, V)) \subset \mathbf{P}(W)$; the key insight is that these are slope stable vector bundles on X , whose isomorphism class determines $g_i(\text{Gr}(2, V)) \subset \mathbf{P}(W)$. Using Proposition 5, we obtain an explicit description of the automorphism group of X .

In terms of moduli, we consider two spaces: the *moduli stack \mathcal{N} of GPK^3 data*, defined as a $\mathbf{Z}/2 \times \text{PGL}(W)$ -quotient of the space of pairs of $\text{Gr}(2, V)$ translates in $\mathbf{P}(W)$ whose intersection is a smooth threefold (where $\mathbf{Z}/2$ swaps the two translates); and the *moduli stack \mathcal{M} of GPK^3 threefolds*, defined as a $\text{PGL}(W)$ -quotient of an open subscheme of the appropriate Hilbert scheme. There is a natural morphism $f: \mathcal{N} \rightarrow \mathcal{M}$ given pointwise by intersecting the two $\text{Gr}(2, V)$ translates.

Theorem 6 ([3]). *The morphism $f: \mathcal{N} \rightarrow \mathcal{M}$ is an open immersion of smooth separated Deligne–Mumford stacks of finite type over k .*

For this, the main step is showing that the derivative of f at any point is an isomorphism.

Theorem 7 ([3]). *The automorphism group of any geometric point $s \in \mathcal{N}$ acts faithfully on the tangent space $T_s\mathcal{N}$. Moreover, if $1 \neq \gamma \in \text{Aut}_{\mathcal{N}}(s)$ is an involution, then the trace of the induced element $\gamma_* \in \text{GL}(T_s\mathcal{N})$ satisfies*

$$\text{tr}(\gamma_*) \in \{3, 1, -3, -5, -13, -15, -35\}.$$

This is proved by a careful analysis of the eigenvalues of the action on $T_s\mathcal{N}$, which uses our description of the automorphism groups of GPK^3 threefolds.

The involution of $\text{PGL}(W) \times \text{PGL}(W)$ given by $(g_1, g_2) \mapsto (g_1^{-T}, g_2^{-T})$ descends to the *double mirror involution* $\tau: \mathcal{N} \rightarrow \mathcal{N}$. In these terms, our proof of Theorem 2 boils down to the following infinitesimal claim: there exists a fixed point $s \in \mathcal{N}$

of τ such that the derivative $d_s\tau \in \mathrm{GL}(T_s\mathcal{N})$ is not contained in the image of the homomorphism $\mathrm{Aut}_{\mathcal{N}}(s) \rightarrow \mathrm{GL}(T_s\mathcal{N})$. For this, we exhibit an explicit fixed point s such that $\mathrm{tr}(d_s\tau)$ does not occur in the list of traces from Theorem 7.

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Abelian varieties associated to hyperkähler varieties of Kummer type

KIERAN G. O'GRADY

Let X be a hyperkähler manifold, deformation equivalent to a generalized Kummer variety (following established terminology, we say that X is of *Kummer type*) of dimension at least 4. Then $b^3(X) = 8$, by well-known formulae of Göttsche, and of course $H^{3,0}(X) = 0$. Thus

$$J^3(X) = H^3(X)/(H^{2,1}(X) + H^3(X; \mathbb{Z}))$$

is a 4 dimensional compact complex torus. If X is projective, and L is an ample line bundle on X , then $J^3(X)$ is an abelian 4-fold (all of $H^3(X)$ is primitive because $H^1(X) = 0$), and we let Θ_L be the polarization defined by L . Below is our main result regarding $J^3(X)$.

Theorem 1 (With a Caveat, see below.). *Let X be a hyperkähler variety of Kummer type, of dimension $2n$, and let L be an ample line bundle on X . Then $(J^3(X), \Theta_L)$ is of Weil type, with an inclusion*

$$\mathbb{Q}\sqrt{-2(n+1)q_X(L)} \subset \mathrm{End}(J^3(X), \Theta_L)_{\mathbb{Q}},$$

where $q_X(L)$ is the value of the Beauville-Bogomolov-Fujiki (BBF) quadratic form on $c_1(L)$. The abelian varieties $(J^3(X), \Theta_L)$, for variable (X, L) , give all abelian fourfolds of Weil type with fixed numerical characters. Moreover, the Kuga-Satake variety $\mathrm{KS}(X, L)$ is isogenous to $J^3(X)^4$.

The main idea involved in the proof is simple, and goes as follows. Let X be as above. The BBF bilinear form $(,)$ defines an isomorphism $H^2(X) \rightarrow H^2(X)^\vee$, which is invertible because $(,)$ is non degenerate. The inverse $H^2(X)^\vee \rightarrow H^2(X)$ defines an element in $\text{Sym}^2 H^2(X)$, whose image by the cup-product map is a class in $H_{\mathbb{Q}}^{2,2}(X)$ that we denote by q^\vee . The class $\bar{q} := 2(n+1)q^\vee$ belongs to $H_{\mathbb{Z}}^{2,2}(X)$. Let $\phi: \bigwedge^2 H^3(X) \rightarrow H^2(X)^\vee$ be the composition of the map

$$\begin{aligned} \bigwedge^2 H^3(X) &\longrightarrow H^{4n-2}(X) \\ \alpha \wedge \beta &\mapsto \alpha \cup \beta \cup \bar{q}^{n-2} \end{aligned}$$

and the isomorphism $H^{4n-2}(X) \rightarrow H^2(X)^\vee$ defined by cup product. Then $\bigwedge^2 H^{2,1}(X)$ maps to the one dimensional space which annihilates $F^1 H^2(X)$. This is strong condition, since $H^{2,1}(X)$ is 4 dimensional, and in fact one can reconstruct the Hodge structure on $H^3(X)$ from the Hodge structure on $H^2(X)$, *provided ϕ is non zero*. For the moment being, we have checked that ϕ is non zero only if $n = 2$, i.e. $\dim X = 4$. Thus, for now Theorem 1 is proved only for $\dim X = 4$.

On the projectivity of the moduli space of stable varieties in characteristic $p > 5$

ZSOLT PATAKFALVI

Stable varieties are the natural higher dimensional generalizations of stable curves in two aspects:

- (1) The (partially conjectural) *moduli space $\overline{M}_{n,v}$ of stable varieties of dimension $n > 0$ and volume $v > 0$* contains an open locus $M_{n,v}^{\text{can}}$ classifying (moduli the Minimal Model Program) birational equivalence classes of smooth, projective varieties of dimension n and volume v . This latter open set specializes to the moduli space of smooth curves of genus g in dimension 1 (by setting $v = 2g - 2$).
- (2) $\overline{M}_{n,v}$ provides the “most natural” compactification of the above open locus, so that the methods of projective algebraic geometry can be applied to the moduli space itself.

Unlike for stable curves, in dimension at least 2 the state of the construction of the coarse moduli space $\overline{M}_{n,v}$ of stable varieties of dimension n and fixed volume $v > 0$ largely depends on what generality one considers:

- (1) In characteristic 0, $\overline{M}_{n,v}$ (and even its log-versions up to a little issue concerning the nilpotent structure) is known to exist [KSB88, Kol90, Ale94, Vie95, HK04, Kar00, AH11, Kol08, Kol13a, Kol13b, Fuj12, HMX14, Kol17, KP17].
- (2) In other situations (mixed and positive equicharacteristic), a few steps of the construction are known in any dimensions ([Kol08]), however the existence of the coarse moduli space is not known in any sense in arbitrary dimensions.

- (3) Nevertheless, in equicharacteristic $p > 5$, the moduli $\overline{\mathcal{M}}_{2,v}$ of stable surfaces is known to exist as a separated algebraic space.

In this talk we present different results concerning the construction of $\overline{\mathcal{M}}_{2,v}$. The first one concerns the positive equicharacteristic case:

Theorem 1. *Let $v > 0$ be a rational number, let k be an algebraically closed field with $\text{char}(k) = p > 5$, and let $\overline{\mathcal{M}}_{2,v}$ be the coarse moduli space of the moduli stack of stable surfaces of volume v (which is known to exist as a separated algebraic space of finite type over k). Then, every proper closed sub-algebraic space \overline{M} of $\overline{\mathcal{M}}_{2,v}$ is a projective scheme over k .*

We need a conditional statement for the mixed characteristic implication, as in that case even the algebraic space structure of the coarse moduli space is unknown. We state this below:

Theorem 2. *Fix a rational number $v > 0$. Let $\overline{\mathcal{M}}_{2,v}$ denote the moduli stack of stable surfaces of volume v and let (I) and (L) be the properties of $\overline{\mathcal{M}}_{2,v}$ defined in Definition 5 (intuitively meaning: existence of (L) imits, and (I) nversion of adjunction, where the latter is a deformation property of the singularities of stable varieties).*

- (1) *If $\overline{\mathcal{M}}_{2,v} \otimes_{\mathbb{Z}} \mathbb{Z}[1/30]$ satisfies (I) and (L) , then $\overline{\mathcal{M}}_{2,v} \otimes_{\mathbb{Z}} \mathbb{Z}[1/30]$ admits a projective coarse moduli space over $\mathbb{Z}[1/30]$.*
- (2) *If k is an algebraically closed field of characteristic $p > 5$, and $\overline{\mathcal{M}}_{2,v} \otimes_{\mathbb{Z}} k$ satisfies (L) , then $\overline{\mathcal{M}}_{2,v} \otimes_{\mathbb{Z}} k$ admits a projective coarse moduli space over k .*

The two main ingredients are Theorem 3, and some folklore results about when $\overline{\mathcal{M}}_{2,v}$ admits a structure of a separated Artin stack of finite type with finite diagonal.

Theorem 3. *If $f : X \rightarrow T$ is a family of stable surfaces of maximal variation with a normal, projective base over an algebraically closed field k of characteristic $p > 5$, then for all divisible enough integer $r > 0$, $\det f_* \mathcal{O}_X(rK_{X/T})$ is a big line bundle. Here maximal variation means that general isomorphism classes of the fibers are finite. If all isomorphism classes of the fibers are finite, then $\det f_* \mathcal{O}_X(rK_{X/T})$ is ample.*

Theorem 3 is deduced using the “ampleness lemma method” from Theorem 4.

Theorem 4. *Let $f : (X, D) \rightarrow T$ be a family of stable log-surfaces over a proper, normal base scheme of finite type over an algebraically closed field k of characteristic $p > 5$ such that the coefficients of D are greater than $5/6$. Then for every divisible enough integer $r > 0$, $f_* \mathcal{O}_X(r(K_{X/T} + D))$ is a nef vector bundle.*

We also remark on the appearance of the boundary divisor D in Theorem 4. In fact, for the application to the previous theorems, one does not need a boundary divisor. We still include it in the statement of Theorem 4, as we obtain this generality almost freely during the proof of the boundary free version. However,

we are not able to use it to prove log-versions of the above theorems, as in the proof of the logarithmic projectivity in [KP17], arbitrary dimensional semi-positivity theorems were used in characteristic 0 (notable the last 3 lines of [KP17, page 995]). From this we are unfortunately very far in positive characteristic. We also remark, that the bound $\frac{5}{6}$ on the coefficients of D appear in Theorem 4, as this is the largest log canonical threshold on surfaces, which is smaller than 1.

Lastly, we state the conditions used in the statement of Theorem 2. The first one requires that for certain small deformations (that is, for the ones having \mathbb{Q} -Cartier relative canonical), the singularities of stable varieties deform. The second one requires that at least one stable limit exists.

Definition 5. Let $v > 0$ be a rational number. Let S be a base-scheme, which is either an open set of $\text{Spec } \mathbb{Z}$ or an algebraically closed field k of characteristic $p > 0$.

- (1) We say that (I) is known for $\overline{\mathcal{M}}_{2,v} \otimes_{\mathbb{Z}} S$, if whenever we are given:
 - (a) an affine, normal, 1-dimensional scheme T of finite type over S ,
 - (b) a flat, projective morphism of finite type $f : X \rightarrow T$ with geometrically demi-normal surface fibers, such that $K_{X/T}$ is \mathbb{Q} -Cartier, and
 - (c) a closed point $t \in T$ such that $X_{\bar{t}}$ is a stable surface of volume v ,
 then X is semi-log canonical in a neighborhood of X_t .
- (2) We say that (L) is known for $\overline{\mathcal{M}}_{2,v} \otimes_{\mathbb{Z}} S$, if whenever we are given:
 - (a) an affine, normal, 1-dimensional scheme T of finite type over S ,
 - (b) a fixed closed point $t \in T$, for which we set $T^0 := T \setminus \{t\}$, and
 - (c) $f^0 : X^0 \rightarrow T^0$ a family of stable surfaces of volume v ,
 then there is a family $f : X \rightarrow T$ of stable surfaces of volume v extending f .

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Hodge ideals for \mathbb{Q} -divisors

MIHNEA POPA

(joint work with Mircea Mustața)

This describes a joint project with Mircea Mustața, from University of Michigan. Our work [MP1], [MP2], [MP3] has been devoted to the study of what we call *Hodge ideals* associated to effective divisors, an extension of the theory of multiplier ideals in birational geometry. They arise as an application of Morihiko Saito’s theory of mixed Hodge modules.

The program is completed in the case of reduced divisors. Let X be a smooth complex variety of dimension n , and D a reduced effective divisor on X . The left \mathcal{D}_X -module

$$\mathcal{O}_X(*D) = \bigcup_{k \geq 0} \mathcal{O}_X(kD)$$

of functions with arbitrary poles along D underlies the mixed Hodge module $j_* \mathbb{Q}_U^H[n]$, where $U = X \setminus D$ and $j : U \hookrightarrow X$ is the inclusion map. It therefore comes equipped with a Hodge filtration $F_\bullet \mathcal{O}_X(*D)$. Saito showed that this filtration is contained in the pole order filtration, namely

$$F_k \mathcal{O}_X(*D) \subseteq \mathcal{O}_X((k+1)D) \quad \text{for all } k \geq 0,$$

and the problem of how far these are from being different is of interest both in the study of the singularities of D and in understanding the Hodge structure on the cohomology of the complement $H^\bullet(U, \mathbb{C})$. The inclusion above leads to defining for each $k \geq 0$ a coherent sheaf of ideals $I_k(D) \subseteq \mathcal{O}_X$ by the formula

$$F_k \mathcal{O}_X(*D) = \mathcal{O}_X((k+1)D) \otimes I_k(D).$$

In our work, we study and apply the theory of these ideals using birational geometry methods; this involves redefining them by means of log-resolutions.

One can loosely summarize the main results of [MP1] and [MP2] as follows:

Theorem 1. [MP1] *Given a reduced effective divisor D on a smooth complex variety X , there exists a sequence of ideal sheaves $I_k(D)$ with $k \geq 0$, of Hodge theoretic origin, such that:*

(i) $I_0(D)$ is the multiplier ideal $\mathcal{I}((1-\epsilon)D)$, and there are inclusions

$$\cdots I_k(D) \subseteq \cdots \subseteq I_1(D) \subseteq I_0(D).$$

- (ii) $I_k(D)$ are all trivial if and only if D is smooth.
- (iii) If any $I_k(D)$ is trivial for $k \geq 1$, then D has rational singularities. (Note that by definition $I_0(D)$ is trivial if and only if the pair (X, D) has log canonical singularities.) More precisely, $I_1(D) \subseteq \text{Adj}(D)$, the adjoint ideal of D .
- (iv) There are non-triviality criteria for $I_k(D)$ at a point $x \in D$ in terms of the multiplicity of D at x .
- (v) $I_k(D)$ satisfy a vanishing theorem analogous to Nadel Vanishing for multiplier ideals.
- (vi) $I_k(D)$ determine Deligne's Hodge filtration on the singular cohomology $H^\bullet(U, \mathbb{C})$, where $U = X \setminus D$, via a Hodge-to-de Rham type spectral sequence.
- (vii) The $I_k(D)$ satisfy analogues of the restriction, subadditivity and semicontinuity theorems for multiplier ideals.

Here are some examples of concrete applications that come out of this:

- Solution to a conjecture on the multiplicities of points on theta divisors on principally polarized abelian varieties in the case of isolated singularities, improving in this case a result of Kollár.
- An effective bound for how far the Hodge filtration coincides with the pole order filtration on the cohomology $H^\bullet(U, \mathbb{C})$ of the complement, refining results of Deligne-Dimca.
- Effective bounds for the degrees of hypersurfaces on which isolated singular points on a hypersurface in \mathbb{P}^n of given degree impose independent conditions, extending a result of Severi for nodal surfaces in \mathbb{P}^3 .

The case of arbitrary \mathbb{Q} -divisors requires a somewhat more technical setting, where the \mathcal{D} -modules we consider are only direct summands of \mathcal{D} -modules underlying mixed Hodge modules. Let D be an effective \mathbb{Q} -divisor on X , with support Z . We denote $U = X \setminus Z$ and let $j: U \hookrightarrow X$ be the inclusion map. Locally we can assume that $D = \alpha \cdot \text{div}(h)$ for some nonzero $h \in \mathcal{O}_X(X)$ and $\alpha \in \mathbb{Q}_{>0}$. We denote $\beta = 1 - \alpha$. To this data one associates the left \mathcal{D}_X -module $\mathcal{M}(h^\beta)$, a rank 1 free $\mathcal{O}_X(*Z)$ -module with generator the symbol h^β , on which a derivation D of \mathcal{O}_X acts via the rule

$$D(wh^\beta) := \left(D(w) + w \frac{\beta \cdot D(h)}{h} \right) h^\beta.$$

The case $\beta = 0$ is the localization $\mathcal{O}_X(*Z)$ considered above.

This \mathcal{D} -module does not necessarily underlie a Hodge module itself. It is however a filtered direct summand of one such, via an analogue of cyclic covering constructions, and therefore again comes endowed with a Hodge filtration $F_\bullet \mathcal{M}(h^\beta)$. As above, one has an inclusion

$$F_k \mathcal{M}(h^\beta) \subseteq \mathcal{O}_X((k+1)Z)h^\beta,$$

and so for each $k \geq 0$, the k -th Hodge ideal associated to the \mathbb{Q} -divisor D is defined by

$$F_k \mathcal{M}(h^\beta) = I_k(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X((k+1)Z)h^\beta.$$

It is standard to check that the definition of these ideals is independent of the choice of α and h , and therefore makes sense globally on X . The reduced case corresponds to the value $\beta = 0$.

The bulk of the lecture was devoted to explaining that even in the \mathbb{Q} -divisor case, the ideals $I_k(D)$ satisfy many of the properties listed in Theorem 1, or suitable replacements. However, in this case, the theory of mixed Hodge modules comes into play much more prominently; more precise, some of these results are deduced after establishing a connection between Hodge ideals and the microlocal V -filtration, which was first noted by M. Saito [Sa] in the case of reduced divisors.

Theorem 2. [Sa], [MP3] *For every $k \geq 0$ we have*

$$I_k(D) = \tilde{V}^{k+\alpha} \mathcal{O}_X \pmod{I_D},$$

where I_D is the ideal of D , and $\tilde{V}^{k+\alpha} \mathcal{O}_X$ is the microlocal V -filtration on the structure sheaf of X .

The theorem is proved using the regular and quasi-unipotent property of filtered \mathcal{D} -modules underlying mixed Hodge modules, which describes a close interaction between the Hodge filtration and the V -filtration along D . Its usefulness also stems from the connection between the (microlocal) V -filtration and the Bernstein-Sato polynomial of D and its roots. Indeed, it allows us to give a lower bound for the microlocal log canonical threshold of D (the negative of the largest root of the Bernstein-Sato polynomial that is different from -1) in terms of discrepancies on any fixed log resolution of the pair (X, D) , partially answering a question of Lichtin.

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Complex ball quotients and moduli spaces of some irreducible holomorphic symplectic fourfolds

ALESSANDRA SARTI

The aim of the talk was to show a relation between the moduli space of some IHS fourfolds carrying a non-symplectic automorphism of order three and the moduli space of smooth cubic 3-folds, that was described by Allcock, Carlson and Toledo in a famous paper of 2011, [1].

1. NON-SYMPLECTIC AUTOMORPHISMS ON IHS MANIFOLDS

We start by recalling the following:

Definition 1.1. An irreducible holomorphic symplectic (IHS) manifold X is a compact, complex, Kähler manifold which is simply connected and admits a unique (up to scalar multiplication) everywhere non-degenerate holomorphic 2-form.

Assume that X is equivalent by deformation to the Hilbert scheme of n points $\text{Hilb}^{[n]}(S)$, where S is a K3 surface (we will say for simplicity that X is of type $K3^{[n]}$), let $\sigma \in \text{Aut}(X)$ be an automorphism, and assume that σ has prime order p . This induces an action on $H^{2,0}(X) = \mathbb{C}\omega_X$. If $\sigma^*\omega_X = \zeta\omega_X$, ζ a primitive p -root of unity, we say that σ acts *non-symplectically* on X .

Recall that by using the Beauville-Bogomolov-Fujiki (BBF) quadratic form on the second cohomology with integer coefficients we have an isometry $H^2(X, \mathbb{Z}) = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2(n-1) \rangle$, where U is the hyperbolic plane and $E_8(-1)$ is the even negative definite lattice associated to the root system E_8 . So $H^2(X, \mathbb{Z})$ is a lattice of signature $(3, 20)$ and an automorphism $\sigma \in \text{Aut}(X)$ induces an isometry of $H^2(X, \mathbb{Z})$. We can consider two important sublattices

$$T = H^2(X, \mathbb{Z}) = \{x \in H^2(X, \mathbb{Z}) \mid \sigma^*(x) = x\}, \quad S = T^\perp \cap H^2(X, \mathbb{Z}),$$

we call T the *invariant sublattice* and one can easily show that

$$T \subset \text{NS}(X), \quad \text{Trans}_X \subset S$$

where $\text{NS}(X)$ is the *Néron-Severi group* of X and Trans_X the *transcendental lattice* of X . We recall that since X is projective (see [2]) then $\text{sgn}(T) = (1, \rho - 1)$ and $\text{sgn}(S) = (2, 21 - \rho)$, where $\rho = \text{rank } T$. These two lattices play an important role if one wants to classify automorphisms, in fact one starts by classifying S and T . This was done in the case of $n = 2$ by Boissière, Camere, Sarti and Tari (see [4], [10]) and for $n > 2$ it is a work in progress by Camere and Cattaneo (see [6]).

Properties of S . Let X be of type $K3^{[2]}$ (i.e. X is an IHS fourfold) carrying a non-symplectic automorphism σ of prime order p , then

$$(1) \quad \text{rank } S = m(p - 1), \quad m \in \mathbb{Z}_{>0}$$

in fact the action of σ^* on $S \otimes \mathbb{C}$ is by primitive roots of unity, and the discriminant group satisfies

$$(2) \quad S^\vee / S \cong (\mathbb{Z}/p\mathbb{Z})^{\oplus a}, \quad a \in \mathbb{Z}_{\geq 0}$$

where $S^\vee = \{v \in S \otimes \mathbb{Q} \mid (v, z) \in \mathbb{Z}, \forall z \in S\}$.

2. A KEY EXAMPLE

Let $V \subset \mathbb{P}^5$ be a smooth cubic 4-fold, and assume that V has equation

$$x_5^3 + f_3(x_0, \dots, x_4) = 0$$

so that V is the triple cover of \mathbb{P}^4 ramified on a smooth cubic threefold $C : \{f_3(x_0, \dots, x_4) = 0\}$. The covering automorphism is

$$\sigma : \mathbb{P}^5 \longrightarrow \mathbb{P}^5, (x_0 : \dots : x_4 : x_5) \mapsto (x_0 : \dots : x_4 : \zeta x_5)$$

with $\zeta = e^{\frac{2\pi i}{3}}$ so that σ has order 3. We consider now the Fano variety of lines

$$F(V) = \{l \in Gr(1, 5) \mid l \subset V\}$$

where $Gr(1, 5)$ is the Grassmannian variety of lines of \mathbb{P}^5 . It was shown by Beauville and Donagi (see [3]) that $F(V)$ is of $K3^{[2]}$ type and it is $\langle 6 \rangle$ -polarized ample (i.e. there is a primitive embedding of the rank one lattice $\langle 6 \rangle$ in $NS(F(V))$ so that the image contains an ample class). The automorphism σ on V induces an automorphism $\bar{\sigma}$ on $F(V)$ of the same order and one can show that $\bar{\sigma}$ acts non-symplectically. Moreover the fixed locus $F(V)^{\bar{\sigma}}$ is the Fano surface of lines $F(C)$ of the smooth cubic threefold C , this is a surface of general type with Hodge numbers: $h^{1,0} = h^{0,1} = 5$, $h^{2,0} = h^{0,2} = 10$, $h^{1,1} = 25$. By using the topology of the fixed locus one computes in the formulas (1), (2) that $m = 11$ and $a = 1$, so that combining with results on lattices by Nikulin (see [8]) and by Rudakov-Shafarevich (see [9]) one computes that

$$(3) \quad S = U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus A_2(-1), \quad T = \langle 6 \rangle$$

where $A_2(-1)$ is the negative definite lattice associated to the root system A_2 .

3. A RELATION BETWEEN V , S AND $C = F(V)^{\bar{\sigma}}$

For $V \subset \mathbb{P}^5$ a smooth cubic 4-fold, recall that $H^4(V, \mathbb{Z})$ is a lattice of signature $(21, 2)$ which is odd, unimodular (see [7]), i.e. with the usual intersection pairing it is isometric to $\langle 1 \rangle^{\oplus 21} \oplus \langle -1 \rangle^{\oplus 2}$. Let $h \in H^2(\mathbb{P}^5, \mathbb{Z})$ be the class of a hyperplane and define $\theta(V) := h|_V \in H^4(V, \mathbb{Z})$ then one computes $\theta(V)^2 = 3$. Take now the *primitive cohomology*

$$H_0^4(V, \mathbb{Z}) := \theta(V)^\perp \cap H^4(V, \mathbb{Z}) \cong S(-1)$$

where the last isometry is shown by Hassett in [7] and S is the lattice we defined in (3). Recall that $H^{3,1}(V)$ is one-dimensional generated by a $(3, 1)$ -form, that we denote by v . In the case of a cubic 4-fold V as in the example above, one checks that $\sigma(v) = \zeta v$, with $\zeta = e^{\frac{2\pi i}{3}}$, so that

$$H^{3,1}(V) \subset H_0^4(V, \mathbb{Z})_\zeta \cong S(-1)_\zeta$$

where the last two spaces denote the eigenspaces for the eigenvalue ζ of the action of σ on $H_0^4(V, \mathbb{Z}) \otimes \mathbb{C}$ respectively $S(-1) \otimes \mathbb{C}$. Recall that by [1] the one-dimensional space $H^{3,1}(V)$ is called the *period point* of the smooth cubic threefold C .

Let us now go back to IHS fourfolds with non-symplectic automorphism. Consider $H^{2,0}(F(V)) = \mathbb{C}\omega_{F(V)}$, since $\bar{\sigma}^*\omega_{F(V)} = \zeta\omega_{F(V)}$ then $\omega_{F(V)} \in H^2(F(V), \mathbb{Z})_\zeta = S_\zeta \subset S \otimes \mathbb{C}$ where again $H^2(F(V), \mathbb{Z})_\zeta$ and S_ζ denotes the eigenspaces with respect to ζ for the action of $\bar{\sigma}$. More precisely the period point belongs to

$$\omega_{F(V)} \in \{[\omega] \in \mathbb{P}(S_\zeta) \mid (\omega, \bar{\omega}) > 0\}$$

which is an open analytic subset in a 10-dimensional complex projective space. The BBF-quadratic form restricts to an hermitian form on S_ζ of signature $(1, 10)$ so that by an easy computation one can see that in fact $\omega_{F(V)}$ belongs to a 10-dimensional complex ball, that we denote by \mathbb{B}_{10} . On the other hand not any

point in \mathbb{B}_{10} gives an IHS manifolds X of type $K3^{[2]}$ with a non-symplectic automorphism of order three. So

$$\omega_{F(V)} \in \frac{\mathbb{B}_{10} \setminus \mathcal{H}}{\Gamma} =: \Omega_{10}$$

where \mathcal{H} is a hyperplanes arrangement and Γ is an arithmetic subgroup of the group of the isometries of the lattice S . In fact Ω_{10} can be identified with the moduli space $\mathcal{M}_{(6)}^{\rho, \zeta}$ of IHS of $K3^{[2]}$ type where:

- we fix the representation $\rho : \mathbb{Z}/3\mathbb{Z} \rightarrow O(\Lambda)$, that fixes the action of the automorphism on the $K3^{[2]}$ -lattice $\Lambda := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2 \rangle$,
- we fix the embedding of the lattice $\langle 6 \rangle$ in Λ so that the orthogonal complement is S ,
- we fix the action on the holomorphic two form of the IHS manifold of $K3^{[2]}$ type as the multiplication by ζ .

Observe that a priori not all X in this moduli space are of the type $F(V)$ with the non-symplectic automorphism $\bar{\sigma}$.

We denote now by \mathcal{C}_3^{sm} the moduli space of smooth cubic threefolds as described in [1] and we have the following result obtained in a work in progress in collaboration with S. Boissière and C. Camere, [5]:

Theorem 1. (1) *We have an isomorphism of moduli spaces: $\mathcal{C}_3^{sm} \cong \mathcal{M}_{(6)}^{\rho, \zeta}$.*

- (2) *Let X be of type $K3^{[2]}$ and let $\langle 6 \rangle \hookrightarrow \text{NS}(X)$ be an ample polarization. Then X admits a non-symplectic automorphism of order three with invariant lattice $\langle 6 \rangle$ if and only if $X \cong F(V)$, and V is the triple cover of \mathbb{P}^4 ramified on a smooth cubic threefold and the automorphism is induced by the covering automorphism.*

The proof uses the results of [1] and in particular the description of the period point of a smooth cubic threefold that we recalled above, the study of the hyperplane arrangement \mathcal{H} and the study of the group Γ .

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***F*-rational mod *p* implies rational singularities in mixed characteristic**

KARL SCHWEDE

(joint work with Linquan Ma)

Suppose that (R, \mathfrak{m}) is a local domain essentially of finite type over \mathbb{C} . R. Elkik proved the following.

Theorem 1 ([Elk78]). *If $0 \neq f \in R$ is such that R/fR has rational singularities, then R has rational singularities.*

We provide a proof here for inspiration.

Proof. Note that since R/fR is normal and Cohen-Macaulay, we can assume that R is normal and Cohen-Macaulay. Let $X = \text{Spec } R$ and let $\pi : Y \rightarrow X$ be a resolution of singularities. Recall that R has rational singularities if and only if R is Cohen-Macaulay and $\pi_*\omega_Y = \omega_X$. Let $H = \text{Spec}(R/fR) \subseteq X$ and let \tilde{H} denote the strict transform of H under π . Applying the functor $\mathbf{R}\Gamma_{\mathfrak{m}}$ and cohomology to the short exact sequence $0 \rightarrow \mathcal{O}_X \xrightarrow{\cdot f} \mathcal{O}_X \rightarrow \mathcal{O}_H \rightarrow 0$, as well as to the corresponding sequence describing $\overline{H} = \pi^*H$ we have the following diagram:

$$\begin{array}{ccccccc}
 H_{\mathfrak{m}}^{d-1}(\mathcal{O}_X) & \longrightarrow & H_{\mathfrak{m}}^{d-1}(\mathcal{O}_H) & \xrightarrow{\phi} & H_{\mathfrak{m}}^d(\mathcal{O}_X) & \xrightarrow{\cdot f} & H_{\mathfrak{m}}^d(\mathcal{O}_X) & \longrightarrow \\
 \downarrow & & \downarrow \alpha & & \downarrow \beta & & \downarrow & \\
 \mathbb{H}_{\mathfrak{m}}^{d-1}(\mathbf{R}\pi_*\mathcal{O}_Y) & \longrightarrow & \mathbb{H}_{\mathfrak{m}}^{d-1}(\mathbf{R}\pi_*\mathcal{O}_{\overline{H}}) & \xrightarrow{\psi} & \mathbb{H}_{\mathfrak{m}}^d(\mathbf{R}\pi_*\mathcal{O}_Y) & \xrightarrow{\cdot f} & \mathbb{H}_{\mathfrak{m}}^d(\mathbf{R}\pi_*\mathcal{O}_Y) & \longrightarrow \\
 & & \downarrow \gamma & & & & & \\
 & & \mathbb{H}_{\mathfrak{m}}^{d-1}(\mathbf{R}\pi_*\mathcal{O}_{\tilde{H}}) & & & & &
 \end{array}$$

where $d = \dim R = \dim X$. It suffices to show that the map labeled β is injective since $H_{\mathfrak{m}}^d(\mathcal{O}_X) \rightarrow \mathbb{H}_{\mathfrak{m}}^d(\mathbf{R}\pi_*\mathcal{O}_Y)$ is Matlis/local-dual to $\pi_*\omega_Y \rightarrow \omega_X$. Note that $\mathbb{H}_{\mathfrak{m}}^{d-1}(\mathbf{R}\pi_*\mathcal{O}_Y) = 0$ by the Matlis/local-dual version of Grauert-Riemenschneider vanishing and hence that ψ injects. Likewise, the Matlis/local dual of the assertion that $\pi_*\omega_{\tilde{H}} = \omega_H$ is that $\gamma \circ \alpha$ is an isomorphism, and hence we know that α injects.

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Now, choose $0 \neq x \in H_m^d(\mathcal{O}_X)$ such that $fx = 0$. It suffices to show that $\beta(x) \neq 0$. However $x = \phi(y)$ for some $0 \neq y \in H_m^{d-1}(\mathcal{O}_H)$. Thus $\alpha(y) \neq 0$ since α is injective. Likewise $0 \neq \psi(\alpha(y)) = \beta(x)$ as desired. \square

In characteristic $p > 0$, there is an analog of rational singularities. Suppose that (R, \mathfrak{m}) is a d -dimensional local ring of characteristic $p > 0$. Then we say that R has *F-rational singularities* if R is Cohen-Macaulay and if $N \subseteq H_m^d(R)$ is a submodule such that

$$F(N) \subseteq N,$$

where $F : H_m^d(R) \rightarrow H_m^d(R)$ is the Frobenius action, then $N = 0$ or $N = H_m^d(R)$. Note that this implies that if $Y \rightarrow X = \text{Spec } R$ is any map of schemes, then induced the map $H_m^d(R) \xrightarrow{\kappa} \mathbb{H}_m^d(\mathbf{R}\pi_*\mathcal{O}_Y)$ is either injective or zero, since $K = \ker \kappa$ satisfies $F(K) \subseteq K$.

K. Smith was the first to make all the observations above and used them to prove the following Theorem relating *F-rational* and rational singularities (also see [MS97, Har98] for the converse):

Theorem 2 ([Smi97]). *Suppose (R, \mathfrak{m}) is a local domain of finite type over \mathbb{Q} and R_p is a family of mod p reductions over \mathbb{Z} . Then if the mod- p reductions R_p are *F-rational* for a Zariski dense set of primes $(p) \in \text{Spec } \mathbb{Z}$, then R has rational singularities in characteristic zero.*

Proof. The idea is to reduce a resolution of singularities $\pi : Y \rightarrow X$ to characteristic $p \gg 0$ to obtain $\pi_p : Y_p \rightarrow X_p$. We have that $H_m^d(\mathcal{O}_{X_p}) \rightarrow \mathbb{H}_m^d(\mathbf{R}(\pi_p)_*\mathcal{O}_{Y_p})$ injects since it is dual to $\pi_*\omega_{Y_p} \rightarrow \omega_{X_p}$ and hence nonzero. Thus $\pi_*\omega_Y \rightarrow \omega_X$ surjects in characteristic zero. \square

Note, that one really does not need R_p to be *F-rational* for a Zariski dense set of primes, one really needs only check a single prime large enough. Unfortunately, to find that prime one has to compute $(\pi_{\mathbb{Z}})_*\omega_{Y_{\mathbb{Z}}}$ in mixed characteristic, and if you can do that, you can presumably compute $\pi_*\omega_Y$ in characteristic zero (and so you can already check whether or not X has rational singularities).

In our work, we show the following linking the results of Elkik and Smith.

Theorem 3 ([MS17]). *Suppose (R, \mathfrak{m}) is a d -dimensional local ring of mixed characteristic $(0, p)$ such that R/pR has *F-rational singularities*. Then R has *pseudo-rational singularities*. This means that R is Cohen-Macaulay and that for any $\pi : Y \rightarrow X = \text{Spec } R$ such that π is proper and birational, that $H_m^d(\mathcal{O}_X) \rightarrow \mathbb{H}_m^d(\mathbf{R}\pi_*\mathcal{O}_Y)$ is injective (or dually that $\pi_*\omega_Y \rightarrow \omega_X$ is surjective).*

In fact, we prove a stronger theorem, we show that R has *big-Cohen-Macaulay rational singularities* which means that for any big-Cohen-Macaulay R -algebra B , we have that $H_m^d(R) \rightarrow H_m^d(B)$ is injective, which implies the result for $\pi : Y \rightarrow X$ proper and birational via arguments similar to [Ma15]. Note big Cohen-Macaulay algebras exist by [And16], also see [HM17]. One then runs an argument similar to the one in Elkik’s result above, replacing $\mathbf{R}\pi_*\mathcal{O}_Y$ with B and $\mathbf{R}\pi_*\mathcal{O}_{\overline{H}}$ with B/pB . Note that $H_m^i(B) = 0$ for $i < \dim R$ since B is Big-Cohen-Macaulay replacing the

use of the dual Grauert-Riemenschneider vanishing. On the other hand, the map $H_{\mathfrak{m}}^{d-1}(R/pR) \rightarrow H_{\mathfrak{m}}^{d-1}(B/pB)$ is injective since R/pR has F -rational singularities, this replaces the injectivity of the map α .

As an application, by simply localizing, we obtain the following.

Corollary 4 ([MS17]). *Suppose that (R, \mathfrak{m}) is a local ring essentially of finite type over \mathbb{Q} such that it has a family of characteristic $p > 0$ reductions (R_p, \mathfrak{m}_p) over \mathbb{Z} . If one of the characteristic $p > 0$ reductions R_p is an F -rational local ring, then R has rational singularities in characteristic zero.*

Note one can check whether a ring is F -rational using the package `TestIdeals.m2` in Macaulay2 [GS]. Related results for log canonical thresholds are obtained by very different means in [Zhu17, Corollary 4.1].

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Degeneracy loci, virtual cycles and nested Hilbert schemes

RICHARD THOMAS

(joint work with Amin Gholampour)

Moduli spaces in algebraic geometry are often singular of too high a dimension. In good situations — when they admit a *perfect obstruction theory* [BF] of the correct “virtual” dimension — they still admit a fundamental cycle (called the virtual cycle) of the *correct* dimension. This has good properties which ensure that one can integrate against it to define enumerative invariants which are deformation invariant.

The prototype of a scheme Z with perfect obstruction theory is the zero locus of a section of a vector bundle E on a smooth ambient variety A . In this case the virtual dimension is $\dim A - \operatorname{rk} E$ — the number of unknowns minus the number of equations. *All* perfect obstruction theories are *locally* of this form. When

a perfect obstruction theory takes this form *globally*, the natural virtual cycle pushes forward to what we might expect, namely the Euler class of the bundle. This facilitates computations, but is unfortunately extremely rare. (Instead one usually has to use a combination of degeneration and localisation, which makes computation very difficult.)

In [GT] we give another prototype of a perfect obstruction theory generalising zero loci, namely *degeneracy loci*. We show the deepest degeneracy locus (the locus where the rank drops lowest) of a map of vector bundles over a smooth ambient scheme carries a natural perfect obstruction theory.

Again, when this can be done *globally*, it allows us to express integrals over the virtual cycle in terms of integrals over the ambient space. Here the Euler class is replaced by a combination of Chern classes known as the Thom-Porteous formula.

Fix a smooth projective surface S . We show that the perfect obstruction theory relevant to the “reduced DT theory” [GSY1, GSY2] and the Vafa-Witten theory [TT1] of S takes this form. This allows us to compute Vafa-Witten invariants.

The simplest example is provided by the 2-step nested Hilbert scheme of points on S ,

$$S^{[n_1, n_2]} := \{I_1 \subseteq I_2 \subseteq \mathcal{O}_S : \text{length}(\mathcal{O}_S/I_i) = n_i\}.$$

This lies in the ambient space $S^{[n_1]} \times S^{[n_2]}$ as the locus of points (I_1, I_2) for which there is a nonzero map $\text{Hom}_S(I_1, I_2) \neq 0$. Thus it can be seen as the degeneracy locus of the complex of vector bundles

$$(1) \quad R\mathcal{H}om_\pi(\mathcal{I}_1, \mathcal{I}_2) \quad \text{over} \quad S^{[n_1]} \times S^{[n_2]}$$

which, when restricted to the point (I_1, I_2) , computes $\text{Ext}_S^*(I_1, I_2)$. (For precise statements and a description of all notation, see [GT].)

When $H^{0,1}(S) = 0 = H^{0,2}(S)$ this complex is 2-term, so our theory endows it with a virtual cycle whose pushforward is described by the Thom-Porteous formula via

$$(2) \quad \iota_* [S^{[n_1, n_2]}]^{\text{vir}} = c_{n_1+n_2}(R\mathcal{H}om_\pi(\mathcal{I}_1, \mathcal{I}_2)[1])$$

in the dimension n_1+n_2 Chow ring (or degree $2n_1+2n_2$ homology) of $S^{[n_1]} \times S^{[n_2]}$. There are similar results for k -step nested Hilbert schemes, and when we allow curves as our subschemes.

When either of $H^{0,1}(S)$ or $H^{0,2}(S)$ is nonzero we have to modify the complex (1) by splitting off $H^{\geq 1}(\mathcal{O}_S)$ terms. Since they do *not* split off on $S^{[n_1]} \times S^{[n_2]}$ we use a Jouanolou-type trick, pulling back to an affine bundle over $S^{[n_1]} \times S^{[n_2]}$ on which they do split. The affine bundle has the same Chow groups as the base, so this suffices for our purposes. Hence we prove a result like (2) for *all* surfaces S .

Considering $c_{n_1+n_2}(R\mathcal{H}om_\pi(\mathcal{I}_1, \mathcal{I}_2)[1])$ as a map

$$H^*(S^{[n_1]}) \rightarrow H^{*+2n_2-2n_1}(S^{[n_1]}),$$

it is a “Carlsson-Okounkov operator”. Carlsson-Okounkov [CO] calculate it in terms of Grojnowski-Nakajima operators. So the upshot is that we can calculate

Vafa-Witten invariants in terms of these operators on punctual Hilbert schemes. Our methods also allow us to reprove and generalise vanishing results of Carlsson-Okounkov for the higher Chern classes of (1).

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The failure of Kodaira vanishing for Fano varieties, and terminal singularities that are not Cohen-Macaulay

BURT TOTARO

The Kodaira vanishing theorem says that for a smooth projective variety X over a field of characteristic zero and an ample line bundle L on X , we have

$$H^i(X, K_X + L) = 0$$

for all $i > 0$. (Here K_X denotes the canonical line bundle, and we use additive notation for line bundles.) This result and its generalizations are fundamental for the classification of algebraic varieties. Unfortunately, Raynaud showed that Kodaira vanishing fails already for surfaces in every characteristic $p > 0$ [6].

For the minimal model program (MMP), it has been especially important to find out whether Kodaira vanishing holds for Fano varieties (varieties with $-K_X$ ample). By taking cones, this is related to the question of whether the singularities occurring in the MMP (klt, canonical, and so on) have the good properties familiar from characteristic zero (such as Cohen-Macaulayness and rational singularities). For example, klt surface singularities in characteristic $p > 5$ are strongly F -regular and hence Cohen-Macaulay; this is a key reason why the MMP for 3-folds is known only in characteristic $p > 5$ (or zero) [3, Theorem 3.1].

There seems to be only one example in the literature of a smooth Fano variety for which Kodaira vanishing fails: a 6-dimensional Fano in characteristic 2 found by Haboush and Lauritzen. (This is [1, section 6, Example 4] or [5, section 2]. Both papers give examples of the failure of Kodaira vanishing in any characteristic, but the variety they consider is Fano only in characteristic 2.) By taking a cone over Haboush-Lauritzen’s variety, Kovács gave the first example of a canonical singularity which is not Cohen-Macaulay; it has dimension 7 and characteristic 2 [4].

Haboush and Lauritzen's example is a projective homogeneous variety $X = G/P$ with non-reduced stabilizer group scheme P . This is a typical class of varieties that exist only in positive characteristic. In more detail, X is a smooth projective variety which is the image of a flag variety G/P_{red} for a reductive group G under a finite purely inseparable morphism. Most projective homogeneous varieties with non-reduced stabilizer group (apart from the familiar flag varieties) are not Fano. However, a point that seems to have been overlooked is that there is an infinite class of "nontrivial" homogeneous varieties which are Fano. It turns out that Kodaira vanishing often fails on these varieties. As a result, we show for the first time that Kodaira vanishing can fail for smooth Fano varieties in any characteristic $p > 0$:

Theorem 1. *Let p be a prime number. Then there is a smooth Fano variety X over \mathbf{F}_p , of dimension 5 for $p = 2$, or dimension $2p - 1$ for $p > 2$, such that Kodaira vanishing fails for some ample line bundle L on X . More precisely, $H^1(X, K_X + L) \neq 0$.*

Theorem 2. *Let p be a prime number greater than 2. Then there is a smooth Fano variety X over \mathbf{F}_p of dimension $2p + 1$ such that $-K_X$ is divisible by 2 in the Picard group, $-K_X = 2L$, and Kodaira vanishing fails for the ample line bundle $3L$. More precisely, $H^1(X, L) \neq 0$.*

Taking a cone yields examples of terminal singularities which are not Cohen-Macaulay:

Corollary 3. *Let p be a prime number greater than 2. Then there is a terminal singularity of dimension $2p + 2$ over \mathbf{F}_p which is not Cohen-Macaulay.*

After these results were announced, Takehiko Yasuda used quotient singularities to exhibit lower-dimensional examples of terminal singularities which are not Cohen-Macaulay, improving the klt examples in his paper [8]. He presented his results at the same Oberwolfach meeting. In particular, Yasuda found terminal singularities of dimension 6 and characteristic 2, dimension 5 and characteristic 3, and dimension 4 and characteristic 5.

A natural question in this area is: for each positive integer n , is there a number $p_0(n)$ such that every klt singularity over a field of characteristic $p \geq p_0(n)$ is Cohen-Macaulay? Hacon and Witaszek proved this in dimension 3, but I am not sure what to expect in higher dimensions. A striking feature of their proof is that no explicit choice of the number $p_0(3)$ is known [2].

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Non-CM (log) terminal singularities and moduli of formal torsors

TAKEHIKO YASUDA

The first subject of the talk was about the quotient variety X associated to a linear representation V of the cyclic group of order p over a perfect field of characteristic $p > 0$. The variety is normal, \mathbb{Q} -factorial and 1-Gorenstein, but not necessarily Cohen-Macaulay. The representation uniquely decomposes into indecomposable summands of dimensions from 1 to p . We define an invariant D_V by

$$D_V := \sum_i \frac{(d_i - 1)d_i}{2},$$

where d_i are the dimensions of summands. At the beginning of the talk, I introduced the following result from [12]:

Theorem. Suppose that $D_V \geq 2$, or equivalently that the fixed point locus V^G has codimension ≥ 2 . If $D_V \geq p$, then X is canonical. (Note that this is equivalent to that X is log terminal since X is 1-Gorenstein.) Moreover, if X has a log resolution, then the converse is also true.

On the other hand, it had been known since 1980 when X is Cohen-Macaulay: Ellingsrud and Skjelbred [3] proved that X is Cohen-Macaulay if and only if $\text{codim } V^G \leq 2$. Combining the two results, we get the first examples of log terminal (even canonical) but not Cohen-Macaulay singularities in positive characteristics. As related results, I mentioned other constructions of such singularities ([4, 2, 9, 1]). I mentioned also that Hacon and Witaszek [5] proved that in sufficiently large characteristics, three-dimensional log terminal singularities are Cohen-Macaulay.

Prior to my talk, I heard in Sándor Kovács’s talk at this workshop that Burt Totaro had constructed terminal but not Cohen-Macaulay singularities (afterward Totaro gave a talk on this construction, right after my talk). Then I realized that results in [12] easily imply that the quotient variety X as above is often terminal (but not Cohen-Macaulay again from [3]). I presented the following more precise result, which I obtained during the workshop:

Theorem ([15]). Suppose that $D_V \geq p$. Following the notation of [8, 7], we have

$$\text{discrep}(\text{center} \subset X_{\text{sing}}; X) \geq \frac{2D_V}{p} - 2.$$

In particular, X is terminal if $D_V > p$.

Honestly, in my talk I wrote $\text{discrep}(X)$ instead of $\text{discrep}(\text{center} \subset X_{\text{sing}}; X)$, which was not precise. From this theorem, for any $p > 0$, there exists V such that the associated quotient variety X is terminal but not Cohen-Macaulay.

Then I explained the outline of the proofs of the two theorems. The point is that we can explicitly compute the stringy motivic invariant of X by using the moduli space of Artin-Schreier extensions of $k((t))$, the field of Laurent power series. More precisely, for $G = \mathbb{Z}/p$, the cyclic group of order p , we let Δ_G denote the moduli space of G -torsors over $\text{Spec } k((t))$. The Artin-Schreier theory says that this space is an infinite dimensional affine space. We can define a function $s: \Delta_G \rightarrow \mathbb{Z}$ depending on the given representation, whose fibers are finite dimensional varieties. From [12], we have the following formula for the stringy motivic invariant of X :

$$(1) \quad M_{\text{st}}(X) = \int_{\Delta_G} \mathbb{L}^s := \sum_{a \in \mathbb{Z}} [s^{-1}(a)] \mathbb{L}^a.$$

The right hand side can be explicitly computed, thanks to the Artin-Schreier theory. Since the stringy invariant has information on discrepancies, we can deduce the above theorems.

The second part of the talk was about the moduli space Δ_G for more general finite groups G , based on the speaker's joint work with Fabio Tonini. In [13, 14], the author conjectured that equality (1) holds with s suitably defined. However, to have this conjecture making sense, we first need to construct the moduli space Δ_G for general G , which has not been done in full generality yet. Roughly speaking, our moduli space should represent the functor

$$(\text{Affine schemes}/k) \rightarrow (\text{Sets}), B \mapsto \{G\text{-torsors over Spec } B((t))\}.$$

Since torsors have non-trivial automorphisms, the moduli space should actually be a stack. We may view Δ_G as the “Weil restriction” of the stack $BG = [\text{Spec } k/G]$ with respect to the field extension $k((t))/k$.

An important result in this direction is Harbater's one [6]. He constructed the *coarse* moduli space of *pointed* G -torsors over $\text{Spec } k((t))$ when G is a p -group and the base field k is algebraically closed and proved that it is isomorphic to the infinite-dimensional affine space, more precisely, the limit

$$\varinjlim_n \mathbb{A}_k^n,$$

where the transition map $\mathbb{A}_k^n \rightarrow \mathbb{A}_k^{n+1}$ is the composition of the standard embedding $\mathbb{A}_k^n \hookrightarrow \mathbb{A}_k^{n+1}$ and the Frobenius map of \mathbb{A}_k^{n+1} .

To have the fine moduli space in a more general situation, we define Δ_G as the category fibered in groupoids over the category of affine k -schemes such that the fiber category over $\text{Spec } B$ is the category of G -torsors over $\text{Spec } B((t))$. It turns out that this category is too big. We shrink it by imposing a constraint on torsors about “local triviality over B ”, getting a subcategory denoted by Δ_G^* . When B is a field, then the constraint becomes trivial and we have $\Delta_G(\text{Spec } B) = \Delta_G^*(\text{Spec } B)$. I presented the following result:

Theorem ([10]). Let k be an arbitrary field of characteristic $p > 0$ and G a finite group of the form $H \rtimes C$ with H a p -group and C a tame cyclic group. Then Δ_G^* is isomorphic to the direct limit $\varinjlim \mathcal{X}_n$ such that

- (1) each \mathcal{X}_n is a separated Deligne-Mumford stack of finite type over k ,
- (2) transition maps $\mathcal{X}_n \rightarrow \mathcal{X}_{n+1}$ are finite and universally injective,
- (3) there is a compatible system of étale atlases $X_n \rightarrow \mathcal{X}_n$ with X_n affine schemes.

Groups of the above form are important, because the Galois group of any Galois extension of $k((t))$ with k algebraically closed is always of this form.

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