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Dynamische Systeme

Organised by
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Abstract. This workshop continued the biannual series at Oberwolfach on Dynamical Systems that started as the “Moser & Zehnder meeting” in 1981. The main theme of the workshop were the new results and developments in the area of classical dynamical systems, in particular in celestial mechanics and Hamiltonian systems. Among the main topics were new results on Arnold diffusion, new global results on symplectic fixed point theory and the dynamics on Hamiltonian energy surfaces. A high point was Ginzburg's solution of the Conley conjecture for aspherical symplectic manifolds generalizing recent results by N. Hinston. Another highlight was Mather's report on Aubry Sets in Small Perturbations of Integrable Systems.

Mathematics Subject Classification (2000): 37xx, 53D.

Introduction by the Organisers

This workshop, organised by Hakan Eliasson (Paris), Helmut Hofer (New York), and Jean-Christophe Yoccoz (Paris) continued the biannual series at Oberwolfach on Dynamical Systems that started as the “Moser & Zehnder meeting” in 1981. The workshop was attended by more than 50 participants from 12 countries. The main themes of the workshop were the new results and developments in the area of classical dynamical systems, in particular in celestial mechanics and Hamiltonian systems. The workshop covers a large area of dynamical systems and the following samples give an idea about the scope. The topic of Arnold Diffusion was treated in great detail by talks of M. Levi and J. Mather. In the classical field of celestial mechanics new insight has been gained about two interesting families of relative periodic solutions of the spatial n-body problem (the P_{12}-family and the hip-hop-family) as they share the property of being global continuations of Lyapunov families which bifurcate from a relative equilibrium solution in the direction orthogonal to the plane of motion (A. Chenciner). K. Kuperberg reported on the construction of flows on three-manifolds where every nonconstant trajectory is wild in a sense related to the Artin-Fox example of an exotic arc in Euclidean three-space. Other results were concerned with one of the main problems in Hamiltonian dynamic, namely the stability of motions in nearly-integrable systems (L. Niedermann). Y. Pesin outlined the construction of hyperbolic volume-preserving flows on manifolds of dimension at least three and V. Ginzburg described the recent developments concerning the Conley Conjecture for periodic points of Hamiltonian symplectic maps. John Franks described his recent results about group actions on surfaces. In addition several talks covered new Floer-theoretic methods in the study of Hamiltonian systems and it will be interesting to see in the future how these symplectic methods can be merged with the more classical dynamical systems methods.

Workshop: Dynamische Systeme

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Abstracts

The Conley Conjecture

Viktor Ginzburg

In 1984 Conley conjectured that a Hamiltonian diffeomorphism of a torus has infinitely many periodic points or, more precisely, such a diffeomorphism with finitely many fixed points has simple periodic points of arbitrarily large period. This fact has been recently proved by Hingston, [Hi]. Similar results for Hamiltonian diffeomorphisms of surfaces of positive genus were established by Franks and Handel, [FH]. Tori and such surfaces are particular examples of symplectically aspherical manifolds, as are all symplectic manifolds with zero second homotopy group. Of course, one can expect the Conley conjecture to be true for a general closed, symplectically aspherical manifold and numerous partial results to this effect have been proved in the context of symplectic topology. For instance, Salamon and Zehnder established in [SZ] this generalized conjecture under a suitable non-degeneracy assumption on the fixed points of the diffeomorphism by controlling the growth of Conley–Zehnder indices of periodic points.

In [Gi] the Conley conjecture is proved for an arbitrary closed, symplectically aspherical manifold. The proof is a combination of the index control method from [SZ] and an action control method based on a Floer homological calculation similar to that carried out in [GG].

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Arnold Diffusion

Mark Levi
(joint work with Vadim Kaloshin)

We present what is perhaps the simplest possible geometrical picture explaining the mechanism of Arnold diffusion. The example is given in terms of a geodesic flow on the 3-torus in the metric arbitrarily close to Euclidean, or equivalently, in terms of the particle in an arbitrarily weak periodic potential in $\mathbb{R}^3$. Arnold diffusion amounts to the existence of orbits which turn by $O(1)$ no matter how weak the potential is.

Topology of closed geodesics on symmetric Finsler surfaces

Sigurd Angenent

Let M be a compact orientable surface without boundary, and let $L : TM \to \mathbb{R}$ be a symmetric Finsler metric on M. Thus, $L|T_xM$ is a norm on each tangent space whose unit ball is a strictly convex symmetric set. We assume that L is smooth (except on the zero section). A *geodesic* for (M, L) is a critical point of the Finsler length functional

$$S(\gamma) = \int L(\gamma(z), \gamma'(z))\, dz.$$

This functional is well defined on the space

$$\Omega = \frac{\{C^2 \text{ immersions } S^1 \to M\}}{\{C^2 \text{ reparametrizations } h : S^1 \to S^1\}}.$$

One can construct a gradient flow for the Finsler length on Ω by choosing a smooth Riemannian background metric g on M. If one lets immersed curves evolve by

$$v = \kappa_g + f(\gamma, T), \tag{1}$$

then one has

$$\frac{dS(\gamma(t)}{dt} = -\int_{\gamma(t)} B(\gamma, T) v^2 ds. \tag{2}$$

Here v and κ_g are the normal velocity and geodesic curvature of the curves $\gamma(t)$ (measured using g), N and T are the unit normal and tangent vectors to γ, and finally,

$$f(\gamma, T) = \frac{A(\gamma, T)}{B(\gamma, T)},$$
$$A(\gamma, T) = g\big(N, L_{vx}(\gamma, T) \cdot T - L_x(\gamma, T)\big),$$
$$B(\gamma, T) = g\big(N, L_{vv} \cdot N\big)$$

Convexity of the Finsler metric guarantees that $B(\gamma, T) > 0$, and symmetry of the Finsler metric implies that f is odd:

$$f(\gamma, -T) = -f(\gamma, T). \tag{3}$$

Jeffrey Oaks [2] has proved that a solution of (1) exists for all C^2 initial curves. Moreover, he showed that if such a solution becomes singular at a finite time T, then the curves $\gamma(t)$ contain a loop whose area becomes arbitrarily small as $t \nearrow T$; if the solution exists for all $t > 0$, then its curvature remains bounded, and its ω-limit set consists of closed geodesics for the Finsler metric.

Another consequence of the symmetry of L (i.e. of (3)) is that the number of self intersections of a solution of (1) cannot increase with time, and must in fact drop whenever the curve $\gamma(t)$ develops a self-tangency.

As a consequence, the Lyusternik-Schirelman theorem can be extended from the Riemannian to the symmetric Finsler case: *any smooth symmetric Finsler metric on S^2 admits at least three closed simple geodesics.*

One can also extend the arguments from [1] to prove

Theorem. *If a surface M with a symmetric Finsler metric L admits a simple closed geodesic γ with rotation number $\omega \neq 1$, then for every $\frac{p}{q}$ between 1 and ω there is a closed geodesic on (M, L) which is a (p, q) satellite of γ (in the sense of Poincaré* [3]*).*

The main obstacle in extending the arguments of [1] is the use of the Gauss-Bonnet formula when estimating the rate with which small loops vanish. In doing this for the new flow one must estimate integrals of the form

$$I = \int_{\text{loop}} f(\gamma, T) ds$$

over (small) closed loops. This is done by observing that if the enclosed area of the loop is small, then either the loop is short (so the integral I is small), or else the loop consists mostly of parallel strands with opposite orientation so that the oddness (3) causes cancellation in the integral I which therefore again ends up being small.

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Global continuation of relative equilibria and action minimization

Alain Chenciner

(joint work with Jacques Féjoz)

The P_{12}-family and the Hip-Hop family are two families of relative periodic solutions of the spatial n-body problem (respectively $n = 3$ and $n = 4$). They share the property of being global continuations of Lyapunov families which bifurcate from a relative equilibrium solution in the direction orthogonal to the plane of motion. The first one, discovered by C. Marchal, connects the Lagrange equilateral relative equilibrium of three equal masses to the figure eight solution while the second one connects the square relative equilibrium of four equal masses to the Hip-Hop solution. In both cases the global continuation minimizes the action among paths of configurations which, in an appropriate rotating frame, become loops sharing a discrete symmetry. These symmetries originate from the symmetries of the solutions of the variational equation along the relative equilibrium. For the regular n-gon relative equilibrium, the solutions of the "vertical variational equation" are known explicitly and their symmetries are easily analyzed. This leads to representations of $D_n \times \mathbb{Z}/2\mathbb{Z}$ (where D_n is the dihedral group with $2n$ elements) in the space of loops of relative n-body configurations which, as the examples above suggest, are of a different nature according to the parity of n. The corresponding local Lyapunov families, when unique, inherit these discrete symmetries when looked at in appropriately chosen families of rotating frames and one can ask for their minimizing properties. In the good cases, (locally) minimizing the Lagrangian action in a space of symmetric loops of configurations may then lead to global continuation of the Lyapunov families.

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Hamiltonian Stability and Morse-Sard Theory

Laurent Niederman

One of the main problem in Hamiltonian dynamic is the stability of motions in nearly-integrable systems (for example : the n-body planetary problem).

According to the theorem of Liouville-Arnold, under general topological conditions, a Hamiltonian system integrable by quadrature can be reduced to a system defined over the cotangent bundle $T^*\mathbb{T}^n$ of the $n-$dimensional torus.

Hence, an analytic nearly-integrable system can be reduced to :

$$\mathcal{H}(\varepsilon, I, \theta) = h_0(I) + \varepsilon f(I, \theta) \text{ for } h_0 \in \mathcal{C}^\omega(U, \mathbb{R}) \text{ and } f \in \mathcal{C}^\omega(U \times \mathbb{T}^n, \mathbb{R}) \qquad (1)$$

where $(I, \theta) \in \mathbb{R}^n \times \mathbb{T}^n$ are the action angle variables of the unperturbed Hamiltonian.

An important tool of investigation in this context is the construction of normal forms which yields two kinds of theorems :

i) Results of stability over infinite times provided by Kolmogorov-Arnold-Moser theory which are valid for solutions with initial conditions in a Cantor set of large measure. But no information are given for trajectories lying outside of this set and in systems with more than three degrees of freedom, a strong instability can occur under an arbitrary small perturbation (the Arnold instability). Russmann has given a minimal non degeneracy condition on the unperturbed Hamiltonian to ensure the persistence of invariant tori under perturbation. Namely, the image of the gradient map associated to the integrable Hamiltonian should not be included in an hyperplane and this condition is generic among real analytic numerical functions.

ii) Nekhorochev ([1]) has proved global results of stability over open sets which complete KAM theory and ensure that under generic assumptions Arnold diffusion can only occur over very long times, namely we look at :

Definition 1. (exponential stability) *Consider an open set $\Omega \subset \mathbb{R}^n$, an analytic integrable Hamiltonian $h : \Omega \longrightarrow \mathbb{R}$ and action-angle variables $(I, \varphi) \in \Omega \times \mathbb{T}^n$ where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. For an arbitrary $\rho > 0$, let $\mathcal{O}_\rho$ be the space of analytic functions over a complex neighborhood $\Omega_\rho \subset \mathbb{C}^{2n}$ of size ρ around $\Omega \times \mathbb{T}^n$ equipped with the supremum norm $||.||_\rho$ over Ω_ρ. We say that the Hamiltonian h is exponentially stable over an open set $\widetilde{\Omega} \subset \Omega$ if there exists positive constants ρ, C_1, C_2, a, b and ε_0 which depend only on h and $\widetilde{\Omega}$ such that :*

i) $h \in \mathcal{O}_\rho$.

ii) For any function $\mathcal{H}(I, \varphi) \in \mathcal{O}_\rho$ such that $||\mathcal{H} - h||_\rho = \varepsilon < \varepsilon_0$, an arbitrary solution $(I(t), \varphi(t))$ of the Hamiltonian system associated to $\mathcal{H}$ with an initial action $I(t_0)$ in $\widetilde{\Omega}$ is defined over a time $\exp(C_2/\varepsilon^a)$ and satisfies :

$$||I(t) - I(t_0)|| \leq C_1 \varepsilon^b \text{ for } |t - t_0| \leq \exp(C_2/\varepsilon^a) \qquad (\mathcal{E})$$

a and b are called stability exponents.

Remark 2. Along the same lines, the previous definition can be extended to an integrable Hamiltonian in the Gevrey class.

Here, we prove that such a property of stability is generic according to :

Theorem 3. (Genericity of exponential stability, [2])
Consider an arbitrary real analytic integrable Hamiltonian h defined on a neighbourhood of the closed ball $\overline{B}_R^{(n)}$ of radius R centered at the origin in $\mathbb{R}^n$. For almost any $\Omega \in \mathbb{R}^n$, the integrable Hamiltonian $h_\Omega(x) = h(I) - \Omega.I$ is exponentially stable with the exponents :

$$a = \frac{b}{2+n^2} \text{ and } b = \frac{1}{2(2+(2n)^n)}.$$

In order to introduce the problem, we begin by a typical example of *non-exponentially* stable integrable Hamiltonian : $h(I_1, I_2) = I_1^2 - I_2^2$. Indeed, a solution of the perturbed system governed by $h(I_1, I_2) + \varepsilon \sin(I_1 + I_2)$ with an initial actions located on the first diagonal ($I_1(0) = I_2(0)$) admits a drift of the actions $(I_1(t), I_2(t))$ on a segment of length 1 over a timespan of order $1/\varepsilon$. Actually, with this example, we have the fastest possible drift of the action variables according to the magnitude ε of the perturbation.

The important feature in this example which has to be avoided in order to ensure exponential stability is the fact the gradient $\nabla h(I_1, I_1)$ remains orthogonal to the first diagonal. Equivalently, the gradient of the restriction of h on this first diagonal is identically zero.

Nekhorochev ([1]) have introduced the class of *steep* functions where this problem is avoided. The property of steepness is a quantitative condition of transversality and the steep functions can be characterized by the following simple geometric criterion which is proved with theorems of real subanalytic geometry :

Theorem 4. ([3])
A real analytic real valued function is steep (according to Nekhorochev) if and only its restriction to any affine subspace admits only isolated critical points.

In this setting, Nekhorochev proved the following :

Theorem 5. ([1])
If h is real analytic, does not admit critical points, non-degenerate ($|\nabla^2 h(I)| \neq 0$ for any $I \in \Omega$) and steep then h is exponentially stable.

The fundamental difference between our result of stability and the generic theorems of stability which can be ensured with Nekhorochev's original work is the *fixed* value of the exponents a and b in our theorem 3.

Indeed, the set of steep functions is generic among sufficiently smooth functions. For instance, we have seen that the function $x^2 - y^2$ is not steep but it can be easily showed that $x^2 - y^2 + x^3$ is steep. Actually, a given function can be transformed in a steep function by adding higher order terms (Nekhorochev [1]) but the order of contact of the considered manifolds can be high and theorem 5. is valid with small exponents of stability. Hence, the initial theorem of Nekhorochev allows to find a generic set of exponentially stable integrable Hamiltonians but with exponents of stability which can be arbitrary *small.*

Here, according to our theorem 3, *fixed* stability exponents are obtained on a *measure-theoretic* generic set. Actually, we exhibit a set of exponentially stable integrable Hamiltonian which is *prevalent* according to the terminology of Hunt, Sauer and Yorke or Kaloshin.

Our main theorem 3. is proved thanks to a result of exponential stability under a strictly weaker assumption than steepness which involves *only* affine subspaces spanned by integer vectors (the rational subspaces). For instance, $h(I_1, I_2) = I_1^2 - \delta I_2^2$ where δ is the square of a Diophantine number is not steep but nevertheless exponentially stable.

Then we consider a class of functions which satisfy our weak condition of steepness but with *fixed* orders of contact of the considered manifolds. We show that this set is prevalent among sufficiently smooth functions defined over a relatively-compact subset in $\mathbb{R}^n$.

Actually, by an application of the usual Sard's theorem, one can see easily that the Morse functions are prevalent in the Banach space $\left(\mathcal{C}^2\left(\overline{B}_R^{(n)}, \mathbb{R}\right), ||.||_{\mathcal{C}^2}\right)$ where $\overline{B}_R^{(n)}$ is the closed ball of radius R centered at the origin in $\mathbb{R}^n$.

Here, we follow the same kind of reasonings but we have to substitute the Sard's theorem by a quantitative Morse-Sard theory developed by Yomdin ([4]).

One can notice that this later theory is valid for sufficiently smooth functions and does not require analyticity. This last point is only needed to ensure exponentially long times of stability. Hence, it would be natural to look for generic results of stability for non-analytic but smooth enough quasi-integrable Hamiltonian systems. In this case, one should obtain polynomially long times of stability.

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Aubry Sets in Small Perturbations of Integrable Systems

John N. Mather

I announced results related to the problem of Arnold diffusion at a conference in Russia in 2002 ([1]). These concerned a Lagrangian $L(\theta, \dot\theta, t) = l_0(\dot\theta) + \epsilon P(\theta, \dot\theta, t)$ where $\theta = (\theta_1, \theta_2) \in \mathbb{T}^2$ and $d^2 l_0 > 0$. Thus, L is a Lagrangian in $2\frac{1}{2}$ degrees of freedom and it is a small perturbation of the integrable system l_0. The proof of these results involves finding orbits for which $\dot\theta$ moves approximately along a resonant line segment Γ. The resonance condition of Γ is that there exists

$(k_0, k_1, k_2) \in \mathbb{Z}^3$ with $k_0 \neq 0$ such that $(\omega_1, \omega_2) \in \Gamma$ implies that $k_0 + k_1\omega_1 + d_2\omega_2 = 0$.

A substancial part (now mostly written) of the proof consists of a discussion of the Aubry sets Au_c for $c \in Ch$, where Ch is a connected open set in a const $\times \sqrt{\epsilon}$-neighborhood of $dl_0(\Gamma)$ and Ch contains $dl_0(P)$ and $dl_0(Q)$, where P and Q are the endpoints of the interval. This discussion is valid for $0 < \epsilon < \epsilon_0$, where ϵ_0 is a sufficiently small positive number. For $c \in Ch$, the structure of the Aubry sets closely resembles the structure of the Aubry sets associated to a twist mapping.

This permits the proof of the existence of diffusing orbits as minimizers (subject to suitable auxiliary conditions) of $L - c$ on large segments. For different segments one minimizes $L - c$ for different c. One always chooses $c \in Ch$. On neighboring segments, one chooses the corresponding c's to be close. For large negative values of the time-parameter, one chooses c to be $dl_0(P)$; for large positive values, $dl_0(Q)$. The resulting orbit has $\dot{\theta}$ near P for large negative values of the time-parameter and near Q for large positive values.

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Codimension one laminations and the structure of minimizing closed normal currents

Victor Bangert

On a compact Riemannian manifold (M^n, g) the "minimal volume of a real homology class" defines a natural norm on the real homology $H_*(M, \mathbb{R})$. This minimal volume can be defined for $h \in H_q(M, \mathbb{R})$ by

$$S(h) := \inf \left\{ \sum |\lambda_i| \mathrm{vol}_q^g(\sigma_i) \mid \sum \lambda_i \sigma_i \text{ a real Lipschitz } q\text{-cycle representing } h \right\}$$

The norm S is dual to the "comass norm" S^* defined on the de Rham cohomology $H^q_{dR}(M) \simeq H_q(M, \mathbb{R})^*$ by

$$S^*(\alpha) = \inf\{\| \omega \|_\infty | \ \omega \in \Omega^q M, d\omega = 0, [\omega] = \alpha\}$$

where

$$\| \omega \|_\infty = \max\{\omega_x(e_1, \ldots, e_q) | x \in M, e_i \in TM_x, |e_i| \leq 1\}$$

In a more general setting these notions go back to H. Federer [Fe], and S was baptized "stable norm" by M. Gromov.

In general, the infimum defining the stable norm S will not be attained by a Lipschitz cycle. The following natural problems arise:

1) Existence and properties of minimizing representatives of h, i.e. of representatives whose volume realizes the stable norm of h.
2) What can be said about the shape of the norm ball

$$B_q^g = \{h \in H_q(M, \mathbb{R}) \mid S(h) \leq 1\}?$$

3) Rigidity (e.g., does differentiability of B_1^g imply that (M, g) is a flat torus?).

In the case $q = 1$ these problems are closely related to J. Mather's theory of minimal measures, see [Ma] and [Ba]. For $2 \leq q \leq n-2$ most of the fundamental problems are open. It was J. Moser [Mo] who - in a different context - noticed that the codimension one case, i.e. $q = n-1$, is a natural generalization of the Aubry-Mather theory of minimizing orbits of monotone twist maps. In ongoing work with F. Auer [AB1,2] we study properties of minimizers in this codimension one case.

The basic result is that a minimizer in a class $h \in H_{n-1}(M, \mathbb{R})$ is a measured lamination of M by oriented minimal hypersurfaces that are injectively immersed into M and possibly have a small singular set. Using this the following result from [AB3] on the differentiability of the stable norm S was presented

Theorem 1 (F. Auer, V. Bangert). *For $h \in H_{n-1}(M, \mathbb{R})$ let $V(h)$ denote the smallest subspace of $H_{n-1}(M, \mathbb{R})$ that contains h and is generated by integer classes. Then $S|_{V(h)}$ is differentiable at h.*

In the recent thesis by H. Junginger-Gestrich [Ju] it is shown that for $M = \mathbb{T}^n$ this result is optimal, in the sense that for a large open set of metrics on $\mathbb{T}^n$ the stable norm is only differentiable in the directions given in the theorem.

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Projective dynamics of a classical particle or a multiparticle system

Alain Albouy

(1) *If two Riemannian metrics on a manifold have the same pre-geodesics (i.e. unparametrized geodesics), then their geodesic flows are integrable.* This statement needs some more hypotheses but it is however quite striking. It was discussed in 1998 by Matveev and Topalov [9], and independently by Tabachnikov (see [11]), and applied to the geodesic flow on the n-dimensional ellipsoid. The discussion is based on the paper [6] by Levi-Civita.

(2) *If a Newton system, i.e. a system of the form $\ddot{q} = f(q)$, with $q \in \Omega \subset \mathbb{R}^n$, $f : \Omega \to \mathbb{R}^n$ being a smooth function, possesses two quadratic first integrals then it is integrable.* Again the statement is astonishing, and it requires some technical hypotheses (some seem generic but are not often satisfied in the examples while others seem quite unlikely to happen but do happen). It is due to Lundmark's thesis in 1999 (see e.g. [7]).

(3) *The geodesic flow on the ellipsoid is after a change of time the Neumann problem on this ellipsoid seen as a sphere* (i.e. choosing the Euclidean structure that makes the ellipsoid a sphere). More precisely it is a energy level of Neumann's problem. This is due to Knoerrer [5].

(4) *Appell's central projection sends Neumann's problem onto a Newton system*, if we define this projection as follows. The particle moves on a hypersphere under a quadratic potential (Neumann). We choose a hyperplane not passing through the center of the sphere, and project on it the particle motion using the central projection from the center of the sphere. Finally we apply Appell's change of time [2].

Whatever be the hyperplane, the projected system possesses two quadratic first integrals satisfying Lundmark's hypotheses. Thus Statements (3) (4) and (2) give an elegant way to reach the main example of Statement (1): the geodesic flow on the ellipsoid.

(4') If this hyperplane is parallel to a coordinate hyperplane for the coordinates diagonalizing the quadratic potential, the projected system is naturally Hamiltonian. It is number -1 in the bi-infinite Jacobi family of separable potentials, defined by Rauch-Wojciechowski [12]. This integrable Newton system was also noticed by Appell [3].

Levi-Civita was trying to extend Appell's surprising transformation from the projectively flat to the curved framework. Levi-Civita paper was used by dozens of authors while Appell was being forgotten (we don't know any mention of his transformation in the period 1952–2002). If Appell's transformation was quite unpopular, it is maybe because it is not symplectic, it does not respect the time parameter, and the function called Energy before transformation has nothing to do with a possible energy after transformation. However, we developed in [1] very elementary and concrete consequences of Appell's remark. As an example, we give the simplest way to find the Hamiltonian of the projected system (4').

Neumann's Hamiltonian on the sphere $\|q\| = 1$, where $q \in \mathbb{R}^{n+1}$, with potential $U(q) = \langle aq, q\rangle/2$ is

$$H = \frac{1}{2}\big(\|q\|^2\|p\|^2 - \langle q, p\rangle^2\big) + \frac{1}{2}\langle aq, q\rangle.$$

In a base where the symmetric matrix a is diagonal with diagonal $(a_0, \dots, a_n)$ one of the Uhlenbeck-Devaney first integrals is

$$F_0 = \sum_{i=1}^{n} \frac{(q_0p_i - q_ip_0)^2}{a_0 - a_i} + q_0^2.$$

Projective dynamics (a possible name for considerations around Appell's central projection) teaches us that there is a unique homogeneous form for each of these first integrals. We find it using [1]:

$$\tilde{H} = \frac{1}{2}\big(\|q\|^2\|p\|^2 - \langle q, p\rangle^2\big) + \frac{1}{2}\frac{\langle aq, q\rangle}{\|q\|^2}. \qquad \tilde{F}_0 = \sum_{i=1}^{n} \frac{(q_0p_i - q_ip_0)^2}{a_0 - a_i} + \frac{q_0^2}{\|q\|^2}.$$

Note that we did not need to change the velocity dependent term of these first integrals. We were lucky: in general the homogeneization of this term requires a computation. For example, if we had written above H in the simpler way $H = (\|p\|^2 + \langle aq, q\rangle)/2$, the deduction of $\tilde{H}$ would require a computation. We took the expressions of H and F_0 in Moser's papers (e.g. [8]) but only readers who are familiar with Moser's *constrained Hamiltonian systems* can understand why he expressed H in this complicated way. Moser happened to write the homogeneous form of the velocity dependent term, and his motivations seem unrelated to projective dynamics.

The operation opposite to homogeneization is restriction. If we restrict $\tilde{H}$ and $\tilde{F}_0$ to the sphere $\|q\| = 1$ we find H and F_0. If we restrict them to $q_0 = 1$, together with the associated tangent condition $p_0 = 0$, we find:

$$\bar{H} = \frac{1}{2}\big((1 + q_1^2 + \dots + q_n^2)(p_1^2 + \dots + p_n^2) - (q_1p_1 + \cdots)^2\big) + \frac{a_0 + a_1q_1^2 + \dots + a_nq_n^2}{2(1 + q_1^2 + \dots + q_n^2)}$$

$$\bar{F}_0 = \sum_{i=1}^{n} \frac{p_i^2}{a_0 - a_i} + \frac{1}{1 + q_1^2 + \dots + q_n^2}.$$

In these expressions we make $p_i = \dot{q}_i$ and they become the first integrals of some Newton system. There is a unique Newton system having $\bar{F}_0$ as a first integral and this system is the Hamiltonian system associated with the Hamiltonian $\bar{F}_0/2$ expressed in the momenta $P_i = p_i/(a_0 - a_i)$. This is Appell's or Rauch-Wojciechowski's system. We see that H was the Hamiltonian, and F_0 just a quadratic first integral, and now $\bar{H}$ is just a quadratic first integral while $\bar{F}_0/2$ is the Hamiltonian. In the terminology of Magri's school the system is quasi-bi-Hamiltonian ("quasi" because time is changed, see [10]).

I wish to thank A. Borisov, I. Mamaev (see [4]), G. Falqui, H. Lundmark and S. Rauch-Wojciechowski for recent discussions contributing to this work.

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Global Fixed Points for Group Actions and Morita's Theorem

John Franks

This talk concerned the existence of global fixed points for certain smooth group actions on surfaces.

Theorem 1 (Franks, Handel, Parwani [1]). *Let $\mathcal{G}$ be an abelian subgroup of $\mathit{Diff}^1_0(\mathbb{R}^2)$ with the property that there is a compact $\mathcal{G}$ invariant subset of $\mathbb{R}^2$. Then there is a point $x \in \mathbb{R}^2$ such that $g(x) = x$ for all g in $\mathcal{G}$.*

Theorem 2 (F, Handel, Parwani [1]). *Let $\mathcal{G}$ be an abelian subgroup of $Diff^1_0(S^2)$ Then there is a subgroup $\mathcal{G}_0$ of $\mathcal{G}$ of index at most two and a point $x \in S^2$ such that $g(x) = x$ for all g in $\mathcal{G}_0$.*

Theorem 2 was previously proved by M. Handel [3] for groups generated by two elements.

Theorem 3 (Franks, Handel, Parwani [2]). *Suppose S is a closed oriented surface of genus at least two and that $\mathcal{F}$ is an abelian subgroup of $\mathit{Diff}_0(S)$ Then the set of contractible fixed points, $\mathit{Fix}_c(\mathcal{F})$, is non-empty. In particular $\mathit{Fix}(\mathcal{F})$ is non-empty.*

Theorem 4 (Franks, Handel, Parwani[2]). *Suppose S is a closed oriented surface of genus at least two and that $\mathcal{F}$ is an abelian subgroup of $\mathit{Diff}(S)$. Then $\mathcal{F}$ has a finite index subgroup $\mathcal{F}_0$ such that $\mathit{Fix}(\mathcal{F}_0)$ is non-empty.*

The Mapping Class Group Lifting problem

The *mapping class group* MCG(S) of a surface S with genus g is the group of isotopy classes of orientation preserving homeomorphisms of S. We note

- MCG(S^2) $\cong \{1\}$
- MCG(T^2) $\cong SL(2, \mathbb{Z})$

There is a natural homomorphism Homeo(S) $\to$ MCG(S). A *lift* of a subgroup Γ of MCG(S) is a homomorphism $\mathcal{L} : \Gamma \to$ Homeo(S) so that the composition $\Gamma \to$ Homeo(S) $\to$ MCG(S) is the inclusion.

Question: Which subgroups of MCG(S) lift to Homeo(S)[Diff(S)]?
For genus $g = 1$, MCG(S) lifts to Diff(S) so assume that $g \geq 2$.

- Any free group
- Any free abelian group
- Any finite group [Kerckhoff]
- MCG(S) does not lift to Diff(S) for $g \geq 5$ [Morita]
- MCG(S) does not lift to Homeo(S) for $g \geq 6$ [Markovic]

This talk was largely devoted to a preliminary report on joint work with Michael Handel proving the following.

Theorem 5. *MCG(S) does not lift to Diff(S) for $g \geq 3$.*

The proof is by contradiction and involves finding a global fixed point for an action of a mapping class group. The following result of Thurston and its corollary below are applied to obtain a contradiction.

Theorem 6 (Thurston Stability[4])**.** *If G is a finitely generated non-trivial subgroup of Diff(M^n) and if there exists $x \in$ Fix(G) such that $Dg_x = Id$ for all $g \in G$ then there is a non-trivial homomorphism from G to $\mathbb{R}$.*

Corollary 1. *If G is a finitely generated non-trivial subgroup of Diff(M^2) and if there exists an accumulation point $x \in$ Fix(G) then there is a non-trivial homomorphism from G to $\mathbb{R}$.*

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Topological Methods in Classical Scattering

Andreas Knauf

In potential scattering on $\mathbb{R}^d$ one considers the solutions of the Hamiltonian equations for the Hamiltonian function $H(p,q) = \frac{1}{2}\|p\|^2 + V(q)$, on the energy surfaces $\Sigma_E := H^{-1}(E)$, $E > 0$. The (long range) potential V and its derivatives are assumed to decay at spatial infinity, see e.g. [DG].

An orbit is called *scattering* if $\lim_{|t|\to\infty} \|q(t)\| = \infty$ and *bounded* if

$$\limsup_{|t|\to\infty} \|q(t)\| < \infty.$$

Finally, *trapped* orbits go to infinity in one time direction and stay bounded in the other time direction. These three cases lead to a partition

$$\Sigma_E = b_E \,\dot\cup\, t_E \,\dot\cup\, s_E.$$

The set $\mathcal{NT} := \{E > 0 \mid t_E = \emptyset\}$ of *nontrapping energies* is open.

Although the union t_E of trapped orbits is of Liouville measure zero, in quantum scattering they give rise to so-called quantum resonances. In the paper [KK2] with M. Krapf we give a criterion for their existence, namely

Theorem 1. [KK2] *For smooth potentials V, if $V^{-1}(E) \subset \mathbb{R}^d$ is neither empty nor homeomorphic to the sphere S^{d-1}, then $E \notin \mathcal{NT}$.*

The proof is based on relative homotopy groups and the h-cobordism theorem.

Next we consider nontrapping energies $E \in \mathcal{NT}$. Then asymptotically the solutions have the form of straight lines and can thus be parametrized by a point in the cotangent bundle $N := T^*S^{d-1}$. Dynamics induces symplectic diffeomorphisms

$$S_E : N \to N \qquad (E \in \mathcal{NT}).$$

In [Kn1] this *scattering map* was used to define a topological index, $\deg(E) \in \mathbb{Z}$. In examples of centrally symmetric V all values ≤ 1 were shown to occur.

This is illustrated in Figure 1.

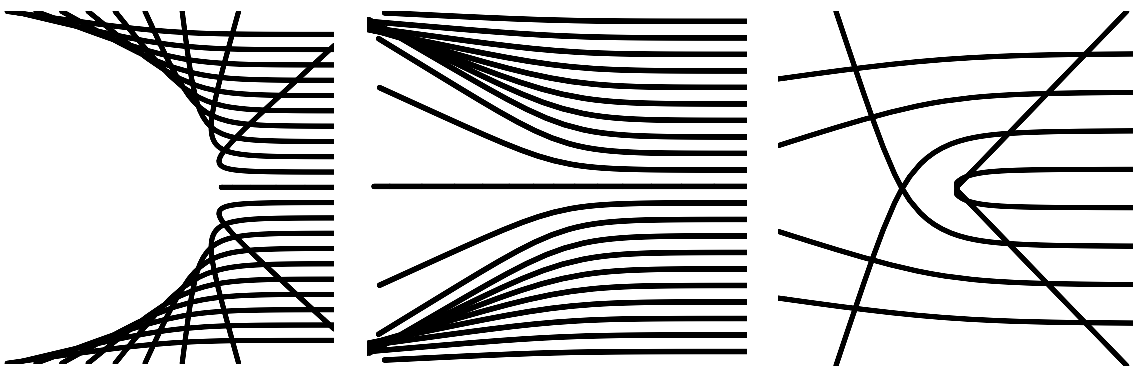

Figure 1. 2D scattering with degrees 1 (left), 0 (center) and -1 (right)

These results were now generalized to arbitrary long range potentials, including the singular ones of the form

$$V(q) = \frac{-Z}{\|q-s\|^\alpha} + W(q). \tag{1}$$

with $Z > 0$, $\alpha > 0$ and $W \in C^2(\mathbb{R}^d, \mathbb{R})$. Precisely for $\alpha = 2n/(n+1)$, $n \in \mathbb{N}$ the motion can be regularized to yield a smooth complete flow on a symplectic manifold (P, ω) of dimension $2d$.

Theorem 2. [KK2] *For all dimensions $d \geq 2$ and energies $E \in \mathcal{NT}$*

(1) *we have in the case of smooth potentials V:*
   (a) *if $V^{-1}(E) = \emptyset$, then* $\deg(E) = 0$.
   (b) *if $V^{-1}(E) \cong S^{d-1}$, then* $\deg(E) = 1$.
(2) *we have in the case of singular potentials V, with $\alpha = 2n/(n+1)$ for $n \in \mathbb{N}$*

$$\deg(E) = \begin{cases} -n & , \ d \text{ even} \\ \frac{1-(-1)^n}{2} & , \ d \text{ odd.} \end{cases} \tag{2}$$

The proof is based on homotopy arguments, respectively in the singular case the calculation of an Euler number (the energy surface being a sphere bundle over configuration space $\mathbb{R}^d$).

The index $\deg(E)$ is also related to the folding of the Lagrange manifold of given asymptotic momentum over configuration space.

This index can be used to imbed symbolic dynamics for scattering in a potential $V = V_1 + \ldots + V_k$, where the V_i were assumed to carry non-zero degree, and to have *non-shadowing* supports (no line meeting more then two supports). More precisely, for any bi-infinite sequence a in

$$\left\{a \in \{1, \ldots, k\}^{\mathbb{Z}} \mid a_l \neq a_{l+1}\right\},$$

there exists an orbit of energy E, visiting the supports of the V_i in the succession prescribed by a.

This work is part of an ongoing project striving to introduce new topological and geometric concepts in classical [BN, DG, KK1, Kn1, Kn2, KK3, KT] and quantum [CJK, DG, GK] scattering, in particular in celestial mechanics [Kn3].

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Wild Dynamics

Krystyna Kuperberg

By a *flow* we understand an $\mathbb{R}$-action. A semi-trajectory in a flow defined on a 3-manifold is *wild* if its closure is a wild arc. A trajectory is *2-wild* if both of its two semi-trajectories are wild.

An example of a wild arc in $\mathbb{R}^3$, an arc embedded in $\mathbb{R}^3$ in such a way that there is no ambient homeomorphism of $\mathbb{R}^3$ taking the arc onto a straight segment, was first given by R. Fox and E. Artin.

We prove the following:

Theorem 1. Every boundaryless 3-manifold admits a flow with a discrete set of fixed points and such that every non-trivial trajectory is 2-wild.

Theorem 2. Every closed 3-manifold admits a flow with exactly one fixed point and such that every non-trivial trajectory is homoclinic and 2-wild.

Both proofs are based on the construction of a *wild-arc ridge*, a vector field $\mathcal{W}$ defined on a compact 3-manifold W with boundary. Denote by F the set $\mathbb{R} \times [-1,1] \times [-1,1]$ in $\mathbb{R}^3$. Denote by W the two-point compactification of F, $F \cup \{p, q\}$, and keep the notation for the coordinates of points in F. The important properties of $\mathcal{W}$ are:

(1) $\mathcal{W}(p) = \mathcal{W}(q) = \vec{0}$ and there are no other singular points.
(2) $\mathcal{W}$ is vertical with positive z-component on ∂F.
(3) $\mathcal{W}$ satisfies the *matched ends* condition, *i.e.*, if a trajectory passes through points $(x_1, y_1, -1)$ and $(x_2, y_2, 1)$, then $(x_1, y_1) = (x_2, y_2)$.
(4) There is a set S with non-empty interior in the bottom boundary of W such that the positive semi-trajectory of a point in S is contained in F.
(5) The semi-trajectories in F that do not intersect ∂F are wild.

Singularities and mixing in multidimensional dispersing billiards

Péter Bálint
(joint work with Imre Péter Tóth)

Let us consider a domain in the d-dimensional flat torus $\mathbb{Q} \subset \mathbb{T}^d$, and a point particle that travels uniformly (follows straight lines with constant speed) within $\mathbb{Q}$, and bounces off the boundary (the scatterers) via elastic collisions (angle of incidence is equal to the angle of reflection). We investigate the resulting dynamics in discrete time, that is, collision to collision. As the length of the velocity is an integral of motion, the phase space, to be denoted by M, is a $2d-2$ dimensional manifold, a hemisphere bundle with base $\partial\mathbb{Q}$ – configurations – and hemisphere fibres – the possible outgoing velocities of unit length. The billiard map $T : M \to M$ has a natural invariant measure μ, absolutely continuous with respect to Lebesgue measure on M. The case $d = 2$ is often referred to as planar, while the technically much more involved $d \geq 3$ as multidimensional. For further material on billiards in general, and on multidimensional dipersing billiards (see below) in particular see [CM] and [BCST], respectively, and references in these works.

Dynamical properties of the billiard map are mainly determined by the shape of the boundary $\partial\mathbb{Q}$. We make the following assumptions:

(i) $\partial\mathbb{Q}$ is assumed to be a finite collection of pairwise disjoint, compact $d-1$ dimensional C^3-smooth submanifolds in $\mathbb{T}^d$. This implies, in particular, that it is possible to define the curvature operator, or second fundamental form K in any point of $\partial\mathbb{Q}$. K should be understood as the second fundamental form for the relevant one codimensional submanifold(s) with (unit) normal vectors pointing inward $\mathbb{Q}$. The billiard is **strictly dispersing**. That is, the boundary components, as viewed from the exteriour, are stricly convex. In other words, the operator K is positive definite on $\partial\mathbb{Q}$.

(ii) Given a phase point $x = (q, v) \in M$ we may consider the *free flight function*: $\tau(x)$ measures the distance along the straight line that starts out of $q \in \partial\mathbb{Q}$ in the direction of v until it reaches $\partial\mathbb{Q}$ again. We assume that **the horizon is finite**; there is a positive constant $\tau_{max} < \infty$, depending only on the billiard domain, such that for any phase point $x \in M$: $\tau(x) \leq \tau_{max}$. Note that, according to our assumption of disjoint and compact boundary pieces (*lack of corner points*), there is also a lower bound on the free flight function, there exists a constant $\tau_{min} > 0$, depending only on the billiard domain, such that for any phase point $x \in M$: $\tau(x) \geq \tau_{\min}$.

To summarize briefly properties (i) and (ii), one can say that we consider dispersing billiards with finite horizon. Our third assumption is more technical, thus we need a little more formulation.

The main consequence of dispersivity is the hyperbolicity of billiard dynamics: diverging wavefronts, when scattered on the boundary $\partial\mathbb{Q}$, remain diverging, which implies the presence of an invariant unstable cone field in the tangent bundle, $C^u \subset \mathcal{T}M$. Moreover, because of strict dispersivity, *hyperbolicity is uniform*, that is, there exists a constant $\Lambda > 1$ such that

$$|T^n dx| \geq c\Lambda^n |dx| \tag{1}$$

for every $dx \in C^u_x$, for some $c > 0$, uniformly for $x \in M$.

In this sense dispersing billiard maps resemble to Anosov diffeomorphisms, however, there is an important difference: T has *singularities*. For domains satisfying properties (i)-(ii) above, T is discontinuous precisely at the preimages of tangential reflections, that is, at the preimage of the boundary ∂M. We shall denote this singularity set by $S = \partial M \cup T^{-1}\partial M$, which is a finite collection of 1 codimensional submanifolds. The discontinuity set for higher iterates T^n, $n \geq 1$, is denoted as $S^n = \cup_{i=0}^{n-1} T^{-i} S$.

The ergodic properties of dispersing billiards are determined by the interplay of hyperbolicity and singularities. Here we concentrate on ergodicity with respect to the natural measure μ, and exponential decay of correlations in the following sense. We say that the dynamical system (M, T, μ) has exponential decay of correlations (EDC), if for every $f, g : M \to \mathbb{R}$ Hölder-continuous pair of functions there exist constants $C < \infty$ and $a > 0$ such that for every $n \in \mathbb{N}$

$$\left| \int_M f(x) g(T^n x) d\mu(x) - \int_M f(x) d\mu(x) \int_M g(x) d\mu(x) \right| \leq C e^{-an}.$$

Our third assumption is related to a quantity characterizing the local combinatorics of the singularity set S^n, the so-called complexity. Consider first for fixed $n \geq 0$ and $x \in M$

$$K_n(x, \varepsilon) = \#\{\text{ Connected components of } B_\varepsilon(x) \cap (M \setminus S^n)\},$$

where $B_\varepsilon(x)$ denotes the ε neighborhood of x in M. $K_n(x, \varepsilon)$ counts the number of pieces into which the n-step singularity set partitions the phase space near x. Then define

$$K_n = \sup_{x \in M} \left(\lim_{\varepsilon \to 0} K_n(x, \varepsilon) \right).$$

We say that the billiard domain $\mathbb{Q}$ satisfies the *finite complexity condition* if there exists a constant $K < \infty$ such that $K_n \leq K$ for any $n \geq 0$. $\mathbb{Q}$ satistfies the *sub-exponential complexity condition* if $K_n = o(\Lambda^n)$, where Λ is the constant of minimum expansion, cf. Formula (1) above.

(iii) Let us assume that $\mathbb{Q}$ satisfies the **sub-exponential complexity** condition.

Now we can state the main result.

Theorem. Let us consider a multidimensional ($d \geq 3$) billiard domain that satisfies (i), (ii) and (iii) above. Then the billiard map (M, T, μ) is ergodic ([BBT]) and has exponential decay of correlations ([BT]).

Our result on EDC generalizes a Theorem from [Y], where Young obtained the same property in the $d = 2$ case. Actually, our proof also follows her approach, that is, a tower construction with exponential return time statistics. For this, we need to perform a careful analysis of the growth and regularity properties of local unstable manifolds, which has a turned out to be technically much more complicated in the multidimensional case.

Our assumption on complexity is a different issue: requiring such a property on the local combinatorics of the singularity set seems to be unavoidable to obtain exponential mixing. It is worth noting that different versions of this condition are very common in the literature of expanding and hyperbolic systems with singularities, see, for example, the works [S] or [DL].

As for the complexity of the singularity set in dispersing billiards with finite horizon, we have the following picture.

Bunimovich proved (see [CY] for a modern presentation) that for $d = 2$ finite horizon dispersing billiards complexity can grow at most linearly. In particular, if a planar billiard domain satisfies properties (i) and (ii), property (iii) is automatic.

In contrast, there exists an example of a multidimensional domain $\mathbb{Q}$ for which complexity grows exponentially. Thus, our Theorem above does not apply to all finite horizon dispersing billiard systems. Nonetheless, we have the following conjecture.

Conjecture. Let us consider the collection of all possible finite horizon dispersing billiard configurations $\partial\mathbb{Q}$, that is, those satisfying conditions (i) and (ii) above, and endow this set with the C^3 topology. The subset of configurations for which the finite complexity conditon holds is generic (dense and G_δ).

To be more precise, we conjecture that in a typical configuration no phase point $x \in M$ can have a trajectory that collides tangentially more than $2d - 2$ times. The rough explanation for this conjecture is as follows: tangency is a codimension 1 phenomenon, the set of double tangencies has codimension two, and so on. Observe that $\dim M = 2d - 2$, thus $2d - 2$ tangencies along the same trajectory seems to be a zero dimensional – atypical – property.

So far we have managed to make this argument precise, and thus completed the proof of the above conjecture only in the $d = 2$ case. Certain details are still to be worked out in the technically more complicated multidimensional situation.

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Closed characteristics on compact convex hypersurfaces in $\mathbb{R}^6$

Yiming Long
(joint work with Wei Wang, Xijun Hu)

Let Σ be a fixed C^3 compact convex hypersurface in $\mathbf{R}^{2n}$, i.e., Σ is the boundary of a compact and strictly convex region U in $\mathbf{R}^{2n}$. We denote the set of all such hypersurfaces by $\mathcal{H}(2n)$. Without loss of generality, we suppose U contains the origin. We consider closed characteristics (τ, y) on Σ, which are solutions of the following problem

$$\begin{cases} \dot{y} = JN_\Sigma(y), \\ y(\tau) = y(0), \end{cases} \tag{1}$$

where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, I_n is the identity matrix in $\mathbf{R}^n$, $\tau > 0$, $N_\Sigma(y)$ is the outward normal vector of Σ at y normalized by the condition $N_\Sigma(y) \cdot y = 1$. Here $a \cdot b$ denotes the standard inner product of $a, b \in \mathbf{R}^{2n}$. A closed characteristic (τ, y) is *prime*, if τ is the minimal period of y. Two closed characteristics (τ, y) and (σ, z) are *geometrically distinct*, if $y(\mathbf{R}) \neq z(\mathbf{R})$. We denote by $\mathbf{T}(\Sigma)$ the set of all geometrically distinct closed characteristics on Σ.

There is a long standing conjecture on the number of closed characteristics on compact convex hypersurfaces in $\mathbf{R}^{2n}$:

$$^\#\mathbf{T}(\Sigma) \geq n, \qquad \forall\, \Sigma \in \mathcal{H}(2n). \tag{2}$$

Since the pioneering works [Rab] of P. Rabinowitz and [Wei] of A. Weinstein in 1978 on the existence of at least one closed characteristic on every hypersurface in $\mathcal{H}(2n)$, the existence of multiple closed characteristics on $\Sigma \in \mathcal{H}(2n)$ has been deeply studied by many mathematicians. When $n \geq 2$, besides many results under pinching conditions, in 1987-1988 I. Ekeland-L. Lassoued, I. Ekeland-H. Hofer, and A. Szulkin (cf. [EkL], [EkH], [Szu]) proved $^\#\mathbf{T}(\Sigma) \geq 2$ for every $\Sigma \in \mathcal{H}(2n)$. In [LoZ] of 2002, Y. Long and C. Zhu further proved $^\#\mathbf{T}(\Sigma) \geq [\frac{n}{2}] + 1$ for every $\Sigma \in \mathcal{H}(2n)$, where we denote by $[a] \equiv \max\{k \in \mathbf{Z} \,|\, k \leq a\}$. Note that this estimate yields still only at least 2 closed characteristics when $n = 3$. For more references on this topic we refer to [Lo2]. The following recent result proved by Wei Wang, Xijun Hu and the author in [WHL] gives a confirmed answer to the conjecture (2) for $n = 3$.

Theorem 1. ([WHL]) *There holds* $^\#\mathbf{T}(\Sigma) \geq 3$ *for every* $\Sigma \in \mathcal{H}(6)$.

One of the main ingredients of our proof of this theorem is the following new resonance identity on closed characteristics.

Theorem 2. ([WHL]) *Suppose* $\Sigma \in \mathcal{H}(2n)$ *satisfies* $^\#\mathbf{T}(\Sigma) < +\infty$. *Denote all the geometrically distinct closed characteristics by* $\{(\tau_j,\ y_j)\}_{1\leq j\leq k}$. *Then the following identity holds*

$$\sum_{1\leq j\leq k} \frac{\hat{\chi}(y_j)}{\hat{i}(y_j)} = \frac{1}{2}, \tag{3}$$

where $\hat{i}(y_j) \equiv \lim_{m\to\infty} i(y_j^m)/m \in \mathbf{R}$ *is the mean index of* y_j, $\hat{\chi}(y_j) \in \mathbf{Q}$ *is the average Euler characteristic given by*

$$\hat{\chi}(y) = \frac{1}{K(y)} \sum_{\substack{1\le m\le K(y)\\ 0\le l\le 2n-2}} (-1)^{i(y^m)+l} k_l(y^m), \tag{4}$$

$K(y) \in \mathbf{N}$ *is the minimal period of critical modules with* $\mathbf{Q}$*-coefficients of iterations of* y, $i(y^m)$ *is the Morse index of the Clarke-Ekeland dual-action functional at the* m*-th iteration* y^m *of* y, $k_l(y^m)$ *is the dimension of the* l*-th critical module with* $\mathbf{Q}$*-coefficients of* y^m.

Note that such a resonance condition was conjectured by I. Ekeland in [Eke] of 1984. When all the closed characteristics on $\Sigma \in \mathcal{H}(2n)$ together with their iterations are nondegenerate, i.e., 1 is a Floquet multiplier of them of precisely algebraic multiplicity 2, this identity was obtained by C. Viterbo in [Vit] of 1989 for star-shaped hypersurfaces.

Note that in [HWZ] of 1998, H. Hofer-K. Wysocki-E. Zehnder proved that ${}^{\#}\mathbf{T}(\Sigma) = 2$ or ∞ holds for every $\Sigma \in \mathcal{H}(4)$. In [Lo1] of 2000, Y. Long proved further that $\Sigma \in \mathcal{H}(4)$ and ${}^{\#}\mathbf{T}(\Sigma) = 2$ imply that both of the closed characteristics must be elliptic, i.e., each of them possesses four Floquet multipliers with two 1s and the other two locate on the unit circle too. Now as a by-product of our Theorem 2 we obtain a stronger result:

Theorem 3. ([WHL]) *Let* $\Sigma \in \mathcal{H}(4)$ *satisfy* ${}^{\#}\mathbf{T}(\Sigma) = 2$. *Then both of the closed characteristics must be irrationally elliptic, i.e., each of them possesses four Floquet multipliers with two* 1*s and the other two located on the unit circle with rotation angles being irrational multiples of* π.

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Floer caps and length minimizing Hamiltonian paths

Ely Kerman

1. Introduction

Let (M, ω) be a closed symplectic manifold. A smooth time-periodic Hamiltonian function $H\colon \mathbb{R}/\mathbb{Z} \times M \to \mathbb{R}$ determines a Hamiltonian flow ϕ_H^t on M. This flow can also be viewed as a path in the group of Hamiltonian diffeomorphisms of (M, ω). The metric properties of this path, with respect to the Hofer metric, are intimately related to the dynamics of the flow. In particular, playing a role similar to the conjugate points of a Riemannian geodesic, certain periodic orbits of the flow are required to exist in order for the path ϕ_H^t to not minimize the Hofer length. This phenomenon was first discovered by Hofer in [Ho2] for Hamiltonian flows on symplectic vector spaces generated by time-independent Hamiltonian functions. The relevant result from [Ho2] was later extended to all compact symplectic manifolds by McDuff and Slimowitz in [McDSl].

In the present work, Floer theory is used to study this relation between dynamics and Hofer's geometry for Hamiltonian flows on compact symplectic manifolds which are generated by general, time-dependent Hamiltonian functions. In particular, the length minimizing properties of a sufficiently short Hamiltonian path are related to the properties and number of its 1-periodic orbits.

2. Hofer's length functional

For every path of Hamiltonian diffeomorphisms, ψ_t, there is a function H in $C^\infty(\mathbb{R}/\mathbb{Z} \times M)$ such that $\phi_H^t \circ \psi_0 = \psi_t$. This generating Hamiltonian can be chosen uniquely if one imposes the normalization condition $\int_M H_t \, \omega^n = 0$, where n is half of the dimension of M and $H_t(p) = H(t,p)$. Following [Ho1], one can use these unique generating Hamiltonians to define a length functional on the group of Hamiltonian diffeomorphisms as follows

$$\begin{aligned} \text{length}(\psi_t) &= \|H\| \\ &= \int_0^1 \max_M H_t \; dt - \int_0^1 \min_M H_t \; dt \\ &= \|H\|^+ + \|H\|^-. \end{aligned}$$

Both $\|H\|^+$ and $\|H\|^-$ provide different measures of the length of ψ_t, called the positive and negative Hofer lengths, respectively.

Let $[\psi_t]$ be the class of Hamiltonian paths which are homotopic to ψ_t relative to its endpoints. Denote the set of normalized Hamiltonians which generate the paths in $[\psi_t]$ by $C_0^\infty([\psi_t])$. The Hofer seminorm of $[\psi_t]$ is then defined by

$$\rho_{\mathrm{H}}([\psi_t]) = \inf\{\|H\| \mid H \in C_0^\infty([\psi_t])\}.$$

The positive and negative Hofer seminorms of $[\psi_t]$ are defined similarly as

$$\rho^\pm([\psi_t]) = \inf\{\|H\|^\pm \mid H \in C_0^\infty([\psi_t])\}.$$

Since these seminorms are bi-invariant, we need only consider paths which start at the identity and hence have the form ϕ_H^t. We say that ϕ_H^t minimizes the Hofer length in its homotopy class if there is no path in $[\phi_H^t]$ with a smaller Hofer length, i.e., $\|H\| = \rho_{\mathrm{H}}([\phi_H^t])$. The notion of a path which minimizes the negative or positive Hofer length in its homotopy class is defined analogously.

3. Main results

If ϕ_H^t does not minimize ρ_H in its homotopy class, then it also fails to minimize ρ^+ or ρ^-. For this reason, we formulate our results for the one-sided seminorms. They are stated for the positive Hofer seminorm. The corresponding results for ρ^- are entirely similar.

For an ω-compatible almost complex structure J on M, let $\hbar(J)$ be the infimum over the symplectic areas of nonconstant J-holomorphic spheres in M. The Floer theoretic methods used in this work require us to consider paths whose Hofer length is less than the strictly positive quantity

$$\hbar = \sup_J \hbar(J).$$

Theorem 1. Let H be a nondegenerate Hamiltonian such that $\|H\| < \hbar$. If ϕ_H^t does not minimize the positive Hofer length in its homotopy class, then there are at least $\mathrm{rank}(\mathrm{H}(M;\mathbb{Z}))$ contractible 1-periodic orbits x_j of H which admit spanning disks u_j such that, for each j, the Conley-Zehner index $\mu_{CZ}(x_j, u_j)$ lies in the interval $[-n, n]$, and the symplectic action $\mathcal{A}_H(x_j, u_j)$ lies in the interval $[-\|H\|^-, \|H\|^+)$.

Here, H is said to be nondegenerate if the contractible 1-periodic orbits of ϕ_H^t are nondegenerate. A spanning disc for a 1-periodic orbit x, is a map u from the unit disc $D^2 \subset \mathbb{C}$ such that $u(e^{2\pi i t}) = x(t)$. As well, the symplectic action of a 1-periodic orbit x with respect to a spanning disc u is defined by

$$\mathcal{A}_H(x, u) = \int_0^1 H(t, x(t))\, dt - \int_{D^2} u^*\omega.$$

For a general, possibly degenerate, Hamiltonian we also prove:

Theorem 2. Let H be a Hamiltonian function with $\|H\| < \hbar$. If ϕ_H^t does not minimize the positive Hofer length in its homotopy class, then there is a contractible 1-periodic orbits y of H which admits a spanning disk w such that

$$-\|H\|^- \leq \mathcal{A}_H(y,w) < \|H\|^+.$$

Theorem 1 can be refined for Hamiltonians which have the properties known to be necessary to generate a length minimizing path. A Hamiltonian H is called quasi-autonomous if it has at least one fixed global maximum $P \in M$ and one fixed global minimum $Q \in M$. That is,

$$H(t,P) \geq H(t,p) \geq H(t,Q), \text{ for all } p \text{ in } M.$$

By the work of Bialy-Polterovich in [BP] and Lalonde-McDuff in [LMcD], it is known that if the path generated by H is length minimizing in its homotopy class, then H must be quasi-autonomous.

A symplectic manifold (M,ω) is said to be spherically rational if the quantity

$$r(M,\omega) = \inf_{A \in \pi_2(M)} \{|\omega(A)| \mid |\omega(A)| > 0\}.$$

is strictly positive. In this case, $r(M,\omega) \leq \hbar$ and we prove the following result.

Theorem 3. Let H be a nondegenerate Hamiltonian on a spherically rational symplectic manifold (M,ω) such that H is quasi-autonomous and $\|H\| < r(M,\omega)$. If ϕ_H^t does not minimize the Hofer length in its homotopy class, then H has at least $\mathrm{rank}(\mathrm{H}(M;\mathbb{Q})) + 2$ contractible 1-periodic orbits.

4. A final question

The Arnold conjecture implies the existence of at least $\mathrm{rank}(\mathrm{H}(M;\mathbb{Q}))$ contractible 1-periodic orbits of the flow generated by a nondegenerate Hamiltonian. The two *extra* orbits detected in Theorem 3 were also found in the case of symplectically aspherical manifolds in [KL]. One is lead by these results, as well as the previously mentioned work of Hofer and McDuff-Slimowitz, to the the following question:

If the path generated by a nondegenerate Hamiltonian does not minimize the (positive/negative) Hofer length (in its homotopy class), must it have at least $\mathrm{rank}(\mathrm{H}(M;\mathbb{Q})) + 2$ contractible 1-periodic orbits?

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Smooth linearization of commuting circle diffeomorphisms

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(joint work with Kostantin Khanin)

We show that a finite number of commuting diffeomorphisms with simultaneously Diophantine rotation numbers are smoothly conjugated to roations. This is a joint work with Kostantin Khanin.

1. Introduction

The problem of smooth linearization of commuting circle diffeomorphisms was raised by Moser in [10] in connection with the holonomy group of certain foliations with codimension 1. Using the rapidly convergent Nash-Moser iteration scheme he proved that if the rotation numbers of the diffeomorphisms satisfy a simultaneous Diophantine condition and if the diffeomorphisms are in some C^∞ neighborhood of the corresponding rotations (the neighborhood beign imposed by the constants appearing in the arithmetic condition, as usual in perturbative KAM theorems) then they are C^∞-linearizable, that is, C^∞-conjugated to rotations.

In terms of small divisors, the latter result presented a new and striking phenomenon: if d is the number of commuting diffeomorphisms, the rotation numbers of some or of all the diffeomorphisms may well be non-Diophantine, but still, the full $\mathbb{Z}^d$-action is smoothly linearizable due to the absence of simultaneous resonances. Further, Moser showed in his paper that this new phenomenon is a *genuine* one in the sense that the problem cannot be reduced to that of a single diffeomorphism with a Diophantine frequency. Indeed, it is shown that there exist numbers $\theta_1, \ldots, \theta_d$ that are simultaneously Diophantine but such that for all linearly independent vectors $a, b \in \mathbb{Z}^{d+1}$, the ratios $(a_0+a_1\theta_1+\ldots+a_d\theta_d)/(b_0+b_1\theta_1+\ldots+b_d\theta_d)$ are Liouville numbers. Moreover, this shows that the theory for individual circle maps does not suffice to yield smooth linearization.

According to Moser, the problem of linearizing commuting circle diffeomorphisms could be regarded as a model problem where KAM techniques can be applied to an overdetermined system (due to the commutation relations). This assertion could again be confirmed by the recent work [2] where local rigidity of some higher rank abelian groups was established using a KAM scheme for an overdetermined system.

At the time Moser was writing his paper, the *global* theory of linearization for circle diffeomorphisms (Herman's theory) was already known for a while. A highlight result is that a diffeomoprhism with a Diophantine rotation number is smoothly linearizable (without a *local* condition of closeness to a rotation). The proof of the first global smooth linearization theorem given by Herman ([5]), as well as all the subsequent different proofs and generalizations ([11], [8],[9],[6],[7],[12]), extensively used the Gauss algorithm of continued fractions that yields the best rational approximations for a real number.

As pointed out in Moser's paper, one of the reasons why the related global problem for a commuting family of diffeomorphisms with rotation numbers satisfying a simultaneous Diophantine condition is difficult to tackle, is due precisely to the absence of an analogue of the one dimensional continuous fractions algorithm in the case of simultaneous approximations of several numbers (by rationals with the same denominator). Although, in certain sense such algorithms were later developed and even used in the KAM setting, our approach is based on different ideas.

Moser asked *under which conditions on the rotation numbers of* n *smooth commuting circle diffeomorphisms can one assert the existence of a smooth invariant measure* μ*? In particular is the simultaneous Diophantine condition sufficient?* Here, we answer this question positively (Theorem 1, the existence of a smooth invariant measure being an equivalent statement to smooth conjugacy). On the other hand, it is not hard to see that the same arithmetic condition is optimal (even for the local problem) in the sense given by Remark 1.

Before we state our results and discuss the plan of the proof, we give a brief summary of the linearization theory of single circle diffeomorphisms on which our proof relies.

We denote the circle by $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and by $\mathrm{Diff}^r_+(\mathbb{T})$, $r \in [0, +\infty] \cup \{\omega\}$, the group of orientation preserving diffeomorphisms of the circle of class C^r or real analytic. We represent the lifts of these diffeomorphisms as elements of $D^r(\mathbb{T})$, the group of C^r-diffeomorphisms $\tilde{f}$ of the real line such that $f - \mathrm{Id}_{\mathbb{R}}$ is $\mathbb{Z}$-periodic.

Following Poincaré, one can define the rotation number of a circle homeomorphism f as the uniform limit

$$\rho_f = \lim_{j\to\infty} \frac{\tilde{f}^j(x) - x}{j} \mathrm{mod}[1],$$

where $\tilde{f}^j$ ($j \in \mathbb{Z}$) denote the iterates of a lift of f. A rotation map of the circle with angle θ, that we denote by $R_\theta : x \mapsto x + \theta$, has clearly a rotation number equal to θ. Poincaré raised the problem of comparing the dynamics of a homeomorphism of the circle with rotation number θ to the simple rotation R_θ.

A classical result of Denjoy (1932) asserts that if $\rho_f = \theta$ is irrational (not in $\mathbb{Q}$) and if f is of class C^1 with the derivative Df of bounded variations then f is topologically conjugated to R_θ, i.e. there exists a circle homeomorphism h such that $h \circ f \circ h^{-1} = R_\theta$.

The first result asserting regularity of the conjugation of a circle diffeomorphism to a rotation was obtained by Arnol'd in the real analytic case: if the rotation number of a real analytic diffeomorphism satisfies certain Diophantine conditions and if the diffeomorphism is sufficiently close to a rotation, then the conjugation is analytic. This result has been proven using KAM approach. The general idea, that is due to Kolmogorov, is to use a quadratic Newton approximation method to show that if we start with a map sufficiently close to the rotation it is possible to compose successive conjugations and get closer and closer to the rotation while the successive conjugating maps tend rapidly to the Identity. The Diophantine condition is used to control the loss of differentiability in the linearized equation which allows to compensate this loss at each step of the algorithm due to its quadratic convergence. Applying the same Newton scheme in the C^∞ setting is essentially due to Moser.

At the same time, Arnol'd also gave examples of real analytic diffeomorphisms with irrational rotation numbers for which the conjugating maps are not even absolutely continuous, thus showing that the small divisors effect was inherent to the regularity problem of the conjugation. Herman also showed that there exist "pathological" examples for any non-Diophantine irrational (i.e. Liouville) rotation number (see [5, chap. XI], see also [4]).

A crucial conjecture was that, to the contrary, the hypothesis of closeness to rotations should not be necessary for smooth linearization, that is, any smooth diffeomorphism of the circle with a Diophantine rotation number must be smoothly conjugated to a rotation. This *global* statement was finally proved by Herman in [5] for almost every rotation number, and later on by Yoccoz in [11] for all Diophantine numbers. In the end of the 80's two different approaches to Herman's *global theory* were developed by Khanin, Sinai ([8], [9]) and Katznelson, Ornstein ([6], [7]). These approaches give sharp results on the smootheness of the conjugacy in the case of diffeomorphisms of finite and low smoothness. In principle, it should be possible to study the case of commuting diffeomorphisms by all the three method. In the present paper, we focus on the C^∞ and C^ω case and use the classical Herman-Yoccoz approach.

Herman, and Yoccoz, developed a powerful machinery giving sharp estimates on the derivatives growth for the iterates of circle diffeomorphisms, the essential criterion for the C^r regularity of the conjugation of a C^k diffeomorphism f, $k \geq r \geq 1$, being the fact that the family of iterates (f^n) should be bounded in the C^r topology. The Herman-Yoccoz estimates on the growth of derivatives of the iterates of f will be crucial for us in all the paper.

2. **Results**

For $\theta \in \mathbb{T}$ and $r \in [1, +\infty] \cup \{\omega\}$, we denote by $\mathcal{D}^r_\theta$ the subset of $\mathrm{Diff}^\infty_+(\mathbb{T})$ of diffeomorphisms having rotation number θ.

Let $d \in \mathbb{N}$, $d \geq 2$, and assume that $(\theta_1, \ldots, \theta_d) \in \mathbb{T}^d$ are such that there exist $\nu > 0$ and $C > 0$ such that for each $k \in \mathbb{Z}^*$,

$$\max(\|k\theta_1\|, \ldots, \|k\theta_d\|) \geq C|k|^{-\nu}. \tag{1}$$

Finally, we say that a family of circle diffeomorphisms $(f_1, \ldots, f_d)$ is *commuting* if $f_i \circ f_j = f_j \circ f_i$ for all $1 \leq i \leq j \leq p$. Note that if h is a homeomorphism of the circle such that $h \circ f_1 \circ h^{-1} = R_{\theta_1}$, then for every $j \leq p$ we have that $h \circ f_j \circ h^{-1}$ commutes with R_{θ_1}, from which it is easy to see that $h \circ f_j \circ h^{-1} = R_{\theta_j}$. Hence, for $r \geq 2$, Denjoy theory gives a homeomorphism that conjugates every f_j to the corresponding rotation. Here, we prove the following.

Theorem 1. *Assume that $\theta_1, \ldots, \theta_d$ satisfy (1) and let $f_i \in \mathcal{D}^{\infty}_{\theta_i}$, $i = 1, \ldots, p$. If a family $(f_1, \ldots, f_d)$ is commuting then, there exists $h \in \mathrm{Diff}^{\infty}_{+}(\mathbb{T})$, such that for each $1 \leq i \leq p$, $h \circ f_i \circ h^{-1} = R_{\theta_i}$.*

Remark 1. Using Liouvillean constructions (constructions by successive conjugations) we see that the above sufficient arithmetic condition is also necessary to guarantee some regularity on the conujugating homeomorphism h (essentially unique, up to translation). There is indeed a sharp dichotomy with the above statement in the case when the arithmetic condition (1) is not satisfied (see for example [5, chap. XI] and [4] where the same techniques producing a single diffeomorphism readily apply to our context): *Assume that $\theta_1, \ldots, \theta_d$ do not satisfy (1), then there exist $f_i \in \mathcal{D}^{\infty}_{\theta_i}$, $i = 1, \ldots, p$ such that a family $(f_1, \ldots, f_d)$ is commuting and such that the conjugating homeomorphism of the maps f_i to the rotations R_{θ_i} is not absolutely continuous.*

As a corollary of Theorem 1 and of the local theorem (on commuting diffeomorphisms) of Moser in the real anlytic category [10] we have by the same techniques as in [5, chap. XI. 6]:

Corollary 1. *Assume that $\theta_1, \ldots, \theta_d$ satisfy (1) and let $f_i \in \mathcal{D}^{\omega}_{\theta_i}$, $i = 1, \ldots, p$. If $(f_1, \ldots, f_d)$ is commuting then, there exists $h \in \mathrm{Diff}^{\omega}_{+}(\mathbb{T})$, such that for each $1 \leq i \leq p$, $h \circ f_i \circ h^{-1} = R_{\theta_i}$.*

In the analytic setting the condition (1) is not optimal although it is necessary to impose some arithmetic condition. It is possible to show that in the case when the rotation numbers $(\theta_1, \ldots, \theta_d) \in \mathbb{T}^p$ are such that there exist $a \in (0, 1)$ and infinitely many $k \in \mathbb{N}$ satisfying

$$\max(\|k\theta_1\|, \ldots, \|k\theta_d\|) \leq a^k$$

then it is possible to construct a commuting family $(f_1, \ldots, f_d) \in \mathcal{D}^{\omega}_{\theta_1} \times \ldots \times \mathcal{D}^{\omega}_{\theta_d}$ such that the conjugating homeomorphism of the maps f_i to the rotations R_{θ_i} is not absolutely continuous.

It is a delicate problem however to find the optimal arithmetic condition under which any commuting family of real analytic diffeomorphisms will be linearizable in the real analytic category. For a single real analytic diffeomorphism, the optimal condition was obtained by Yoccoz in [12].

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PS–overtwisted contact manifolds are algebraically overtwisted

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(joint work with Frédéric Bourgeois)

The plastikstufe [Nie06] is an attempt to generalize the overtwisted disk to higher dimensional contact topology. Since it is unclear whether the notion is general enough, we call contact manifolds containing a plastikstufe *PS–overtwisted* (instead of just calling them *overtwisted*).

Over the last two years several indications have been collected that give some justifications for the definition of the plastikstufe: *PS*–overtwisted manifolds are non fillable [Nie06], and after the first closed higher dimensional examples of such manifolds were found [Pre06], it was not difficult to convert any contact structure into one that is *PS*–overtwisted [KN07]. Recently the Weinstein conjecture has been shown to hold for these structures [AH07].

In the work sketched here, we show that a PS–overtwisted manifold has vanishing contact homology (a manifold with trivial contact homology is called *algebraically overtwisted*). In fact, we seem to be able to prove that symplectic field theory vanishes for such manifolds, extending the well known result for dimension 3 [Yau06], and giving virtually the first explicit computations of symplectic field theory.

Sketch of the proof. Contact homology is the homology of a differential graded algebra $(\mathcal{A}, \partial)$ with $\mathbb{1}$–element. The vanishing of $H_*(\mathcal{A}, \partial)$ is equivalent to the exactness of the $\mathbb{1}$–element. The aim of our proof is thus to show that the $\mathbb{1}$–element of the differential graded algebra $\mathcal{A}$ is exact. Recall that the algebra $\mathcal{A}$ is generated by linear combinations of abstract products of closed Reeb orbits $\{\gamma_j\}$, and the boundary operator is given by

$$\partial\gamma = \sum \big(\#\mathcal{M}_0^A(\gamma; \gamma_{a_1}, \dots, \gamma_{a_m})\big)\, e^A \gamma_{a_1} * \cdots * \gamma_{a_m} \ ,$$

where $\mathcal{M}^A(\gamma; \gamma_{a_1}, \dots, \gamma_{a_m})$ is the moduli space of the $(n+1)$–times punctured holomorphic spheres in the symplectization W of the contact manifold such that the first puncture converges in a certain sense to the closed Reeb orbit $\{+\infty\} \times \gamma$, and for each orbit γ_{a_j} there is a puncture converging to $\{-\infty\} \times \gamma_{a_j}$. The symbol $\#\mathcal{M}_0^A(\gamma; \gamma_{a_1}, \dots, \gamma_{a_m})$ denotes a rational number that counts the 0–dimensional components of $\mathcal{M}^A(\gamma; \gamma_{a_1}, \dots, \gamma_{a_m})$ taking into account orientations and the order of the automorphism group. Note that the "empty product" of closed Reeb orbits corresponds to the $\mathbb{1}$–element in $\mathcal{A}$ and we also have to include in the definition of $\partial\gamma$ the term

$$\big(\#\mathcal{M}_0^A(\gamma; \emptyset)\big) \cdot \emptyset = \big(\#\mathcal{M}_0^A(\gamma; \emptyset)\big) \cdot \mathbb{1}$$

in the summation. The elements in $\mathcal{M}^A(\gamma; \emptyset)$ are called *finite energy planes*, and if such an element lies in a 0–dimensional moduli space, it is called a *rigid* finite energy plane.

We have to find a finite combination of closed Reeb orbits

$$\sigma = \sum_{j \in I} a_j \Gamma_j \ ,$$

where Γ_j is a formal product $\gamma_1 * \cdots * \gamma_m$ of closed Reeb orbits, such that

$$\partial\sigma = \mathbb{1} \ .$$

Our proof can now be sketched like this: In a first step we find a closed Reeb orbit γ_0 that bounds a rigid finite energy plane. Existence of such an orbit follows Hofer's argument in the proof of the Weinstein conjecture for overtwisted 3–manifolds [Hof93] (for higher dimensions [AH07] respectively). Regard the manifold M as the 0–level set in W. Then there is a 1–dimensional family of holomorphic disks, the so-called Bishop family, living in the "lower half" of the symplectization W, and having its boundary on the plastikstufe. The moduli space is a compact closed interval. On one of its ends, the disks collapse to a point on the singular set of $\mathcal{PS}(S)$, and on the other one some kind of bubbling has to occur. The only type of bubbling that is possible in this situation is that the disks grow deeper and deeper into the negative direction finally breaking as a punctured disk u_C in $\big(W, \mathcal{PS}(S)\big)$ that goes from the plastikstufe to a closed Reeb orbit $\{-\infty\} \times \gamma_0$, and a rigid finite energy plane u_0 in W that is bounded on the top by $\{+\infty\} \times \gamma_0$ (see Figure 1).

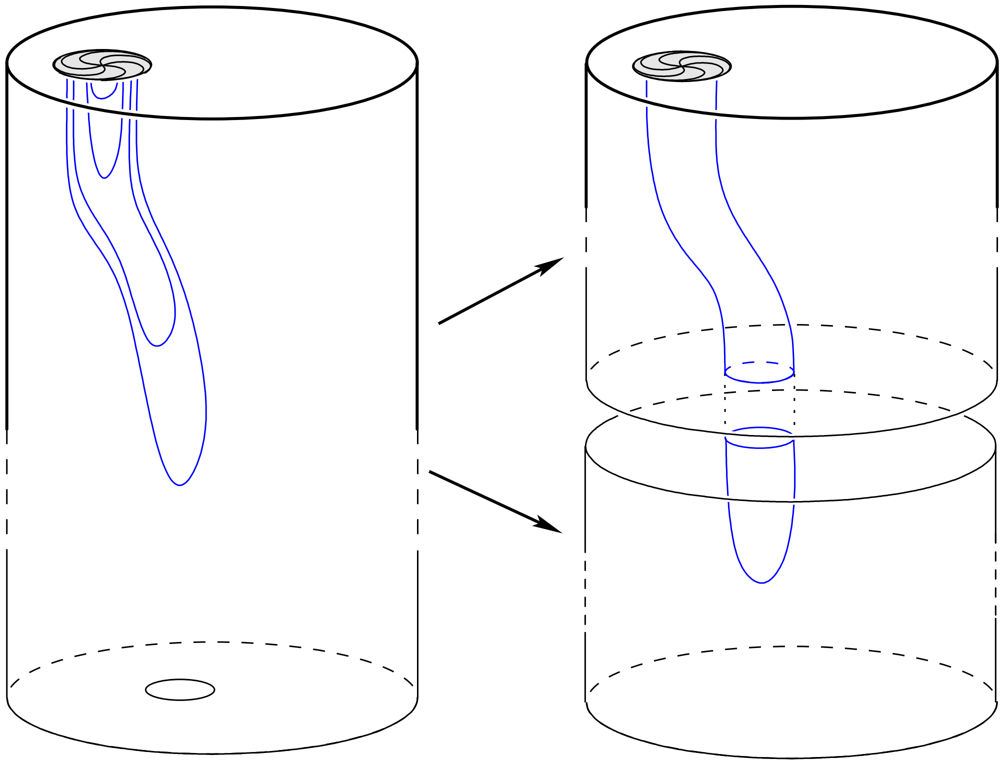

FIGURE 1. The disks in the Bishop family start as a point on the singular set S of the plastikstufe $\mathcal{PS}(S)$. They grow down into the symplectization until they finally break into a punctured disk u_C ending asymptotically at the Reeb orbit γ_0 and a finite energy plane u_0 having γ_0 as its top boundary.

If there was no other rigid punctured sphere having γ_0 as the only top puncture, then the proof would finish here, because then $\partial\gamma_0 = \pm\mathbb{1}$. Unfortunately, this is in general not the case. So assume there are other rigid punctured holomorphic spheres $u_1, \dots, u_N$ in W having γ_0 as the only top boundary (N is finite because the moduli space is a discrete compact set). Let u_1 have the closed Reeb orbits $\gamma_1, \dots, \gamma_m$ as bottom punctures. We can glue u_1 and u_C to obtain a 1–dimensional moduli space of punctured holomorphic disks in $\big(W, \mathcal{PS}(S)\big)$, whose boundary sits on the plastikstufe and whose punctures converge asymptotically to $\{-\infty\}\times\gamma_1, \dots, \{-\infty\}\times\gamma_m$ (see Figure 2). This moduli space can also be naturally compactified, and becomes this way a closed interval. Both of its ends correspond to breaking. The left boundary point of the interval represents the breaking into the curves we glued together, i.e., into u_C and the punctured sphere u_1. The other end breaks into a single punctured sphere u_1' and a collection of vertical cylinders in one level of W and a punctured disk u_C' lying one level higher. The boundary of the disk u'_C sits on the plastikstufe and its punctures converge to closed Reeb orbits $\{-\infty\} \times \gamma'_1, \dots, \{-\infty\} \times \gamma'_k$ at the bottom. The vertical cylinders and the sphere u'_j connect at the orbits $\{+\infty\} \times \gamma'_1, \dots, \{+\infty\} \times \gamma'_k$ to u'_C and converge at the bottom punctures to the orbits $\{-\infty\} \times \gamma_1, \dots, \{-\infty\} \times \gamma_m$. The reason why the holomorphic curve in the lower part of the breaking consists of a single non trivial element is that otherwise the dimensions of the bubbled moduli space would be larger than 0, because disconnected components could be moved independently against each other increasing the dimension.

When applying the boundary operator ∂ to the sum of the element $\gamma_0 \in \mathcal{A}$ and the product $\gamma'_1 * \cdots * \gamma'_k$, we do not find terms of the form $\gamma'_1 * \cdots * \gamma'_k$ in the result, because the punctured spheres u_j and u'_j represent points with different orientation in the moduli space. By repeating the gluing argument first for all curves $u_1, \dots, u_N$, and collecting the elements corresponding to the second boundary of the 1–dimensional moduli spaces, we obtain a term $\sigma_0 = \gamma_0 + \sum \gamma'_{j_1} * \cdots * \gamma'_{j_{k_j}}$. In the boundary $\partial\sigma_0$, we have succeeded in canceling out all the contributions from γ_0 with the exception of the $\mathbb{1}$–element. Unfortunately, the "correction terms" $\gamma'_{j_1} * \cdots * \gamma'_{j_{k_j}}$ may give new undesired terms in the boundary $\partial\sigma_0$. But each of these elements can be dealt with by repeating analogous steps as above, and after a finite number of applications of this method, we arrive at a collection of elements, whose boundary is finally just the $\mathbb{1}$–element.

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Think global, act local - a new aproach to Gromov compactness for pseudo-holomorphic curves.

Joel Fish

Since their introduction by Gromov, pseudo-holomorphic curves have been studied as maps from closed Riemann surfaces into almost complex manifolds with a taming symplectic form. This parameterized view has lead to a number of versions of Gromov compactness which are quite global in nature. For instance, in order to obtain convergence of a sequence of pseudo-holomorphic curves mapping into a family of symplectic manifolds, typically one must first assume the family

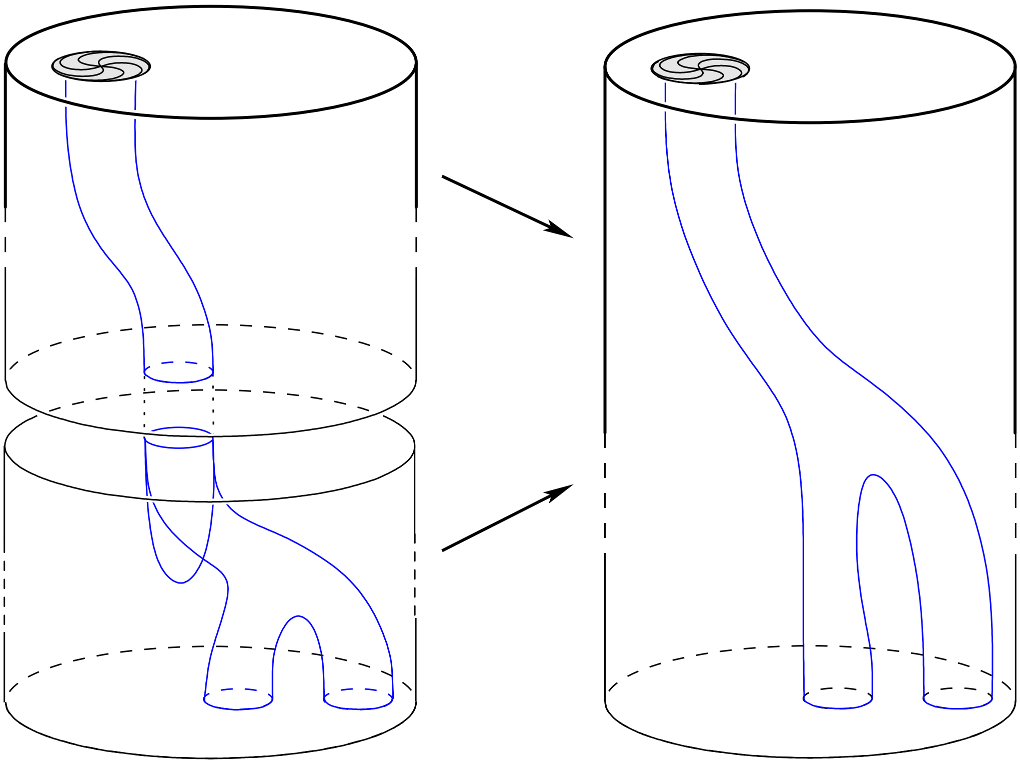

FIGURE 2. If γ_0 bounds several rigid punctured curves, we can perform gluing of any of these new planes to the punctured disk u_C to obtain a 1–dimensional moduli space of (punctured) holomorphic disks.

has uniform global bounds on geometric quantities like curvature, injectivity radius, energy threshold, etc. This talk will focus on a new approach to Gromov's compactness theorem, in which the curves are treated as generalized (unparameterized) surfaces. In particular, we prove a local compactness theorem which is useful when considering a family of target manifolds which develop unbounded geometry. This result recovers for instance compactness in the standard "stretching the neck" construction. Furthermore we will also provide applications of the local result to families of connected sums of contact manifolds in which the connecting handle degenerates to a point.

KAM-Liouville Theory for quasi-periodic cocycles

RAPHAEL KRIKORIAN

In this joint work with Bassam Fayad (CNRS, Paris 13) we extend the reducibility theory of cocycles of the form $(\alpha, A) : \mathbb{R}/\mathbb{Z} \times SL(2,\mathbb{R}) \to \mathbb{R}/\mathbb{Z} \times SL(2,\mathbb{R})$, $(\theta, y) \mapsto (\theta+\alpha, A(\theta)y)$, $A \in C^\infty(\mathbb{R}/\mathbb{Z}, SL(2,\mathbb{R}))$ to the case where α is of Liouville type ($q_{n+1} \geq q_n^n$ infinitely many times). We prove that such a C^∞ cocycle which is

C^0 (resp. L^2) conjugated to a cocycle of rotations is C^∞-approximated by rotation-reducible cocycles (i.e. $B_n(\theta+\alpha)R_n(\theta)B_n(\theta)^{-1}$, $R_n \in C^\infty(\mathbb{R}/\mathbb{Z}, SO(2,\mathbb{R}))$, $B_n \in C^\infty(\mathbb{R}/\mathbb{Z}, SL(2,\mathbb{R}))$) provided α is Liouville (resp. super-Liouville) and the fibered rotation of the cocycle satisfies some diophantine property (of full Lebesgue measure).

We also prove (when α is super-Liouville) that a C^∞ cocycle close to constants and whose rotation number satisfies a diophantine property is rotation reducible. The proof of this theorem (which is an extension of theorems of Dinaburg-Sinai and Eliasson to the Liouville case) uses a renormalization scheme and KAM-type inductive procedure, which is quite surprising in that context.

Towards an Aubry-Mather theory for PDEs

Paul Rabinowitz

J. Moser and subsequently V. Bangert initiated an Aubry-Mather theory for a family of quasilinear elliptic PDEs. We survey a further collection of such results for the special case

$$-\Delta u + F_u(x,u) = 0, \quad x \in \mathbb{R}^n \tag{1}$$

where $F \in C^2(\mathbb{T}^{n+1}, \mathbb{R})$. Of particular interest are a large variety of spatially heteroclinic and homoclinic solutions obtained as local minima of a suitably renormalized functional associated with (1).

Existence of Hyperbolic Flows on Smooth Manifolds

Ya. Pesin

It has been a long-standing problem whether any smooth compact Riemannian manifold admits a hyperbolic volume preserving dynamical system with discrete time (in the case dimM ≥ 2) or continuous (in the case dimM ≥ 3) time. The affirmative solution of this problem is given by the following two theorems.

Theorem 1 (Dolgopyat, Pesin)**.** *Given a smooth compact Riemannian manifold of* dimM ≥ 2*, there exists a* C^∞ *volume preserving ergodic (indeed, Bernoulli) diffeomorphism* f*, which has nonzero Lyapunov exponents almost everywhere.*

Theorem 2 (Hu, Pesin, Talitskaya)**.** *Given a smooth compact Riemannian manifold* M *of* dimM ≥ 3*, there exists a* C^∞ *volume preserving ergodic (indeed, Bernoulli) flow* ϕ_t*, which has nonzero Lyapunov exponents almost everywhere (except for the direction of the flow).*

In the talk I will sketch the proof of the second theorem (the continuous-time case). The starting point of the construction (in fact, in both discrete- and continuous-time cases) is a special C^∞ volume preserving ergodic diffeomorphism g of the two dimensional disk with nonzero Lyapunov exponents, constructed by Katok. It is the identity on the boundary and all its derivatives of any order vanish on the boundary. We then show that this diffeomorphism is diffeotopic to the identity map of the disk. The proof then goes by constructing a smooth divergence free vector field X on certain smooth (of class C^∞) manifold K (of the same dimension as M) with the boundary. The vector field X vanishes on the boundary along with its derivatives. Then one can use results by Brin and Katok that allow one to embed this manifold into M and carry over the vector field X in the desired one.

I will also discuss a still open and challenging problem in the hyperbolic theory, closely related to the above theorems, of whether any sufficiently small perturbation (in the $C^{1+\alpha}$ topology) of a volume preserving diffeomorphism with nonzero Lyapunov exponents possesses nonzero exponents on a set of positive (but not necessarily full) volume.

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KAM method in problems of rigidity for algebraic actions of higher rank abelian groups

Anatole Katok

The first successful application of a KAM-type iteration scheme to this kind of problem appeared three years ago in the joint work with Danijela Damjanovic. In that work the algebraic action in question is an action by ergodic partially hyperbolic but not hyperbolic automorphisms of a torus. Neither individual elements nor the action locally are stable and the a priori regularity/non-stationary normal form method cannot be applied. Applicability of the KAM scheme is based on vanishing of the obstructions to solutions to the linearized conjugacy equations ("the higher rank trick"), tame estimates for solutions of those equations and the estimates of the second cohomology based on a joint work with S. Katok.

Most of this talk is dedicated to the new and rather surprising application to rigidity of totally non-hyperbolic, in fact parabolic, actions on homogeneous spaces of some semisimple Lie groups. Specifically one considers the action of the unipotent upper-triangular subgroups by left translations on right factors by irreducible cocompact lattices of the following

$$SL(2,\mathbb{R}) \times SL(2,\mathbb{R}), SL(2,\mathbb{R} \times SL(2,\mathbb{C}), SL(2,\mathbb{C})$$

The vanishing of the obstructions and construction of solutions for linearized equations in these cases was estabilished by David Mieszkowski in his 2006 PhD thesis in [1]. An interesting peculiarity appears in the $SL(2,\mathbb{C})$ case due to exceptional behaviour of a particular irreducible representation which is responsible for the first Betti number of $SL(2,\mathbb{C})/\Gamma$.

Starting from this, jointly with Damjanovic, we estabilish that in a general position parametric family of perturbations there is an action conjugate to the original unipotent action. Here the number of parameters needed corresponds to the codimension of the unipotent action among the homogeneous ones, e.g., it is equal to 2 for $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})/\Gamma$.

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