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Random Matrices

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Abstract. Large complex systems tend to develop universal patterns that often represent their essential characteristics. For example, the cumulative effects of independent or weakly dependent random variables often yield the Gaussian universality class via the central limit theorem. For non-commutative random variables, e.g. matrices, the Gaussian behavior is often replaced by another universality class, commonly called random matrix statistics. Nearby eigenvalues are strongly correlated, and, remarkably, their correlation structure is universal, depending only on the symmetry type of the matrix. Even more surprisingly, this feature is not restricted to matrices; in fact Eugene Wigner, the pioneer of the field, discovered in the 1950s that distributions of the gaps between energy levels of complicated quantum systems universally follow the same random matrix statistics. This claim has never been rigorously proved for any realistic physical system but experimental data and extensive numerics leave no doubt as to its correctness. Since then random matrices have proved to be extremely useful phenomenological models in a wide range of applications beyond quantum physics that include number theory, statistics, neuroscience, population dynamics, wireless communication and mathematical finance. The ubiquity of random matrices in natural sciences is still a mystery, but recent years have witnessed a breakthrough in the mathematical description of the statistical structure of their spectrum. Random matrices and closely related areas such as log-gases have become an extremely active research area in probability theory.

This workshop brought together outstanding researchers from a variety of mathematical backgrounds whose areas of research are linked to random matrices. While there are strong links between their motivations, the techniques used by these researchers span a large swath of mathematics, ranging from purely algebraic techniques to stochastic analysis, classical probability theory, operator algebra, supersymmetry, orthogonal polynomials, etc.

Mathematics Subject Classification (2010): 60B20.

Introduction by the Organizers

The workshop focused on the latest progresses in random matrix theory.

A large part of the conference was devoted to the study of universality. In recent years, many models of random matrices were shown to belong to the same universality class than Gaussian ensembles. Yet, general sparse or structured random matrices still resist this analysis. The workshop started with an overview of the recent advances concerning the emblematic model of the adjacency matrix of d-regular graphs: Bauerschmidt recalled that even though universality is expected for all $d \geq 3$, recent results could "only" show flatness of the eigenvectors in the bulk for sufficiently large d and universal Tracy-Widom fluctuations of the extreme eigenvalues for d going to infinity fast enough with the dimension. This contrasts with the behavior of the spectrum of the adjacency matrix of Erdős-Rényi graphs whose largest eigenvalue fluctuates like a Gaussian variable. Knowles described recent progress on the later model in the critical regime where the average degree goes to infinity fast enough with the dimension and in particular gave a precise description of its eigenvectors both in the bulk and at the edge. Another long-standing open problem is to understand the universality class of band matrices while the width of the band varies. In a breakthrough series of papers, M. and T. Shcherbina proved that these matrices belong to the universality class of the Gaussian ensembles up to the critical width of the square root of the dimension. T. Shcherbina discussed the uses of supersymmetric methods to prove this result. Aggarwal discussed yet another model where the entries are alpha-stable laws: even though it is not in the universality class of the Gaussian ensembles in the sense that the eigenvectors have a different limiting law, he showed that for alpha big enough or the eigenvalues small enough, local fluctuations of the eigenvalues also follow the sine kernel. Moreover, in the middle of the bulk, the fluctuations of the eigenvectors converge in distribution.

Another important subject which was discussed in the conference concerns non-normal matrices. On the first day, Fyodorov introduced such models and discussed their condition number, which measures the Euclidean norms of their left and right eigenvectors. He showed that it is typically of order N and, once properly rescaled, converges in distribution for the so-called Elliptic model. Akemann put forward the universality of the eigenvalues and eigenvectors statistics in the quaternionic Ginibre ensemble. On the other hand, Zeitouni and Capitaine discussed the outliers of two different models of non-normal random matrices. Capitaine studied the outliers of general polynomials in deterministic and Ginibre matrices. On the other hand, Zeitouni considered Toeplitz matrices and described the outliers in terms of the zeroes of random analytic functions. More traditional universality questions for non-Hermitian matrices were presented by Cipolloni (edge universality at the boundary of the circular law) and Schröder (CLT for linear statistics of iid. random matrices). Optimal local law and accurate spectral radius estimates for non-Hermitian matrices with independent but non identical distributions were given by Alt, and similar questions for products of random matrices were presented by Jalowy. Renfrew studied general analytic functions of i.i.d. matrices with a

concrete biological application in mind: large systems of randomly coupled ODE's are standard tools both in theoretical neuroscience and in mathematical ecology. Tuning the coupling parameter to its critical value yields a polynomial decay of the solution with exponent that is sensitive to the symmetry class of the random matrix. While Renfrew's model was dynamical, Najim considered the equilibrium vector of a similar biological model and identified a sharp threshold for the coupling parameter to guarantee the positivity of the solution which is a basic criteria for feasibility of the model.

One of the central tools in random matrix theory is concentration inequalities. Youssef discussed concentration inequalities for the largest eigenvalue of matrix valued functions.

During the first day, Virág discussed the connection between random matrices and random geometries. This subject was also tackled by Najnudel who showed the close connection between Gaussian multiplicative chaos and random matrix theory.

The topics of Wednesday went beyond the standard universality questions and interesting new models were discussed. Benigni presented a random matrix model motivated by machine learning where the randomness enters in the matrix in a nonlinearly correlated way. The sophisticated and well developed resolvent methods immediately fail and only a technically quite challenging version of Wigner's original moment method can be used to find the analogue of the global Wigner semicircle law. This direction carries a lot of potential, partly its strong tie to machine learning, and is a completely uncharted territory.

Nemish considered random matrices with very different type of correlation: polynomials and even rational functions of i.i.d. matrices, and he proved local laws. An interesting physics application is to compute the density of transmission eigenvalues for a quantum dot coupled to a reservoir, extending Beenakker's prediction to much more general ensembles. Norms of polynomials in random matrices were also estimated in Parraud's talk who gave a new, strengthened concentration estimate for the deviation away from the result predicted by free probability theory. Schnelli talked about local versions and optimal speed of convergence in Voiculescu's fundamental theorem connecting addition of large matrices (with random relative bases) to the additive free convolution of two measures. Mai and Speicher presented yet another perspective of free probability theory: instead of trying to solve the Dyson equation that may be complicated for structured matrices, he was aiming at certain qualitative results like the absolute continuity of the density of states under very general structural conditions.

Finally, yet another physically motivated aspect of random matrices was discussed in several talks: connections to random Schrödinger operators. It is well known that random matrices can be viewed as quantum Hamiltonians with disorder. Most currently tractable models in random matrix theory concern mean field models, while random Schrödinger operators essentially depend on the nontrivial spatial structure of the underlying physical configuration space. Band matrices mentioned above are the most obvious bridges between these two communities,

but other aspects have also been presented. Rider explained the connection between universality of hard and soft edge laws for general β-ensembles and the stochastic Airy operator that is technically a one-dimensional Schrödinger operator with a white noise. Shamis gave a very general estimate on the sensitivity of the density of states of a random Schrödinger operator on the distribution of the random potential. She found that the Kantorovich-Rubinstein metric is the right concept that allows for a robust result with a simple proof that could be presented in its entirety in her talk. The very last talk of the meeting by Disertori was exploring very intriguing connections between lattice random Schrödinger operators, random walks with special history dependent jump rates and nonlinear sigma-models whose non-rigorous analysis relies on supersymmetric methods. These very striking relations between apparently more distant areas of probability theory and mathematical physics were the closing messages that everybody took home and hopefully some progress could be presented in the forthcoming Oberwolfach meeting on random matrices.

The role of the organizers was to keep the number of long talks to a fairly low number in order to leave as much time as possible for informal sessions and discussions between participants, but allow for additional short talks to learn about the work of young participants. In this way, the overwhelming majority of participants could present their work and everyone could be given an opportunity who wished to talk. We also looked at the weather forecast to advance the traditional mid-week hike (this time to Brandenkopf) to Tuesday afternoon instead of the indeed very rainy Wednesday.

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Workshop: Random Matrices

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Abstracts

Spectral Statistics for Lévy Matrices

Amol Aggarwal
(joint work with Patrick Lopatto, Jake Marcinek, Horng-Tzer Yau)

Fix $\alpha \in (0,2)$. A random variable Z is a *centered α-stable law* if

$$\mathbb{E}\big[\exp(\mathbf{i}tZ)\big] = \exp(-\sigma|t|^{\alpha}), \quad \text{for any } t \in \mathbb{R}, \text{ where } \quad \sigma = \sigma_{\alpha} = \frac{\pi}{2\sin\big(\frac{\alpha}{2}\big)\Gamma(\alpha)}.$$

Introduced by Cizeau–Bouchaud [7] in 1994, a *Lévy matrix* is an $N \times N$ real symmetric random matrix $\mathbf{H} = \{h_{ij}\}$ whose upper triangular entries $\{h_{ij}\}_{i \le j}$ are mutually independent and each have law $N^{-1/\alpha}Z$. The $N^{-1/\alpha}$-scaling here is different from the typical $N^{-1/2}$-scaling for Wigner matrices and ensures that row sums (and therefore eigenvalues) of $\mathbf{H}$ are typically of order 1. Denote the eigenvalues of $\mathbf{H}$ by $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_N$ and the corresponding unit eigenvectors by $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N$. Set $\mathbf{u}_i = \big(u_i(1), u_i(2), \ldots, u_i(N)\big) \in \mathbb{R}^N$ for each $i \in [1,N]$.

If $\mathbf{H}$ were from the Gaussian Orthogonal Ensemble (GOE), then the $\mathbf{u}_i$ would be uniformly distributed on the $(N-1)$-sphere, and so $N^{1/2}u_i(j)$ would converge to a Gaussian random variable. In [6], Bourgade–Yau showed that the same would be true if $\mathbf{H}$ were Wigner.

In [2] we show that the eigenvector entry fluctuations of the Lévy matrix $\mathbf{H}$ are not Gaussian. The following is an example of this result for the median eigenvector.

Theorem 1 ([2])**.** *Set $i = \lfloor \frac{N}{2} \rfloor$. For almost all $\alpha \in (0,2)$ and each index $j \in [1,N]$, $N^{1/2}u_i(j)$ converges in moments to the random variable $\mathcal{N}\Theta^{-1}$, where $\mathcal{N}$ is a centered Gaussian random variable of variance 1, and $\Theta > 0$ is an independent positive $\frac{\alpha}{2}$-stable law, characterized by its Laplace transform*

$$\mathbb{E}\big[\exp(-t\Theta)\big] = \exp\bigg(-\Gamma\Big(1+\frac{2}{\alpha}\Big)^{\alpha/2} t^{\alpha/2}\bigg), \quad \textit{for any } t \in \mathbb{R}_{>0}.$$

This theorem comprises one of many features of Lévy matrices that is not present in the Wigner setting. In [2] we also analyze non-median eigenvector entry statistics, as well as correlations between different entries and eigenvectors.

The above results on exact limiting statistics for eigenvector entries do not seem to have been predicted previously. Instead, [7] primarily posed predictions about the eigenvalue behavior of $\mathbf{H}$ and (de)localization of its eigenvectors. For instance, it predicted that the (global) empirical spectral distribution of $\mathbf{H}$ should converge to a deterministic measure μ_{α}, as N tends to ∞. This measure is not explicit but can be characterized through certain fixed point equations for its Stieltjes transform. In particular, it can be shown to exhibit a continuous density that exhibits an α-heavy tail (so, unlike the semicircle law, is not compactly supported). This prediction was later proven by Ben Arous and Guionnet in 2008 [3].

The main predictions of [7] were certain transitions in the eigenvector behavior and local spectral statistics of Lévy matrices. Their predictions are not fully

consistent with the more recent ones of Tarquini–Biroli–Tarzia [8], which can be summarized as follows. Below, a random unit vector $\mathbf{u} \in \mathbb{R}^N$ is *completely delocalized* if $\|\mathbf{u}\|_\infty < N^{\delta-1/2}$ with high probability, for any $\delta > 0$.

Prediction 2 ([8])**.** Fix real numbers $\alpha \in (0,2)$ and $E \in \mathbb{R}$, and an index $i \in [1,N]$ such that $\lim_{N\to\infty} \lambda_i = E$.

(1) If $\alpha \in [1,2)$, then $\mathbf{u}_i$ is completely delocalized and the local statistics of $\mathbf{H}$ around E converge to those of the GOE.
(2) If $\alpha \in (0,1)$, then there exists a *mobility edge* $E_\alpha > 0$ such that the following holds.
   (a) If $|E| < E_\alpha$, then $\mathbf{u}_i$ is completely delocalized and the local statistics of $\mathbf{H}$ around E converge to those of the GOE.
   (b) If $|E| > E_\alpha$, $\mathbf{u}_i$ is completely localized and the local statistics of $\mathbf{H}$ around E converge to those of a Poisson point process.

The mobility edge E_α is also called a *Mott* (or *Anderson*) *transition*, a fundamental concept in Anderson localization that has remained with little mathematical understanding for decades. Lévy matrices provide one of the few examples of a random matrix ensemble for which such a transition is believed to appear. Previous results towards the above predictions were established by Bordenave–Guionnet [5, 4], who showed partial forms of eigenvector (de)localization in the regimes listed above.

Our main results from [1] constitute complete eigenvector delocalization and convergence to GOE local statistics at nonzero E when $\alpha \in (1,2)$ and at sufficiently small E for almost all $\alpha \in (0,2)$.

Theorem 3 ([1, Theorem 2.4 and Theorem 2.5])**.** *Fix real numbers $\alpha \in (0,2)$ and $E \in \mathbb{R}$, and an index $i \in [1,N]$ such that $\lim_{N\to\infty} \lambda_i = E$.*

(1) *If $\alpha \in (1,2)$ and $E \neq 0$, then $\mathbf{u}_i$ is completely delocalized and the local statistics of $\mathbf{H}$ around E converge to those of the GOE.*
(2) *For almost all $\alpha \in (0,2)$, there exists a constant $c = c_\alpha > 0$ such that the same holds if $|E| < c_\alpha$.*

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Universality of bulk eigenvalues and eigenvectors of non-Hermitian random matrices: the quaternionic Ginibre ensemble

Gernot Akemann
(joint work with Y.-P. Förster, M. Kieburg, A. Mielke and T. Prosen)

The question of universality have been much less studied in the three ensembles of non-Hermitian random matrices introduced by Ginibre [6]. They are defined choosing i.i.d. Gaussian matrix elements J_{ij} from $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. We abbreviate these ensembles by GinOE, GinUE and GinSE, respectively. A Schur decomposition yields determinantal or Pfaffian point processes for the joint densities of complex eigenvalues of J of size $N \times N$. These read for the GinUE and GinSE [6]

$$P_N^{\rm GinUE}(z_1,\ldots,z_N) = C_N \prod_{i<j}^{N} |z_j - z_i|^2 \; e^{-\sum_{l=1}^N |z_l|^2},$$

$$P_N^{\rm GinSE}(z_1,\ldots,z_N) = C'_N \prod_{i<j}^{N} |z_j - z_i|^2 |z_j - z_i^*|^2 \prod_{n=1}^{N} |z_n - z_n^*|^2 e^{-2\sum_{l=1}^N |z_l|^2}.$$

We report new results on eigenvalue [2] and eigenvectors statistics [1] in the GinSE. The k-point eigenvalue correlation functions are defined as follows

$$R_{k,N}(z_1,\ldots,z_k) := \frac{N!}{(N-k)!} \int_{\mathbb{C}^{N-k}} d^2 z_{k+1} \cdots d^2 z_N P_N(z_1,\ldots,z_N) \; .$$

The global limiting spectral density approaches the circular law for all three Ginibre ensembles. For the local statistics one has to distinguish three regions of the spectrum: the real line, the bulk and the edge of the unit disc. Along the real line the limiting $R_{k,N}$ are known and differ in all three ensembles. In contrast, their local edge statistics was shown to agree [8, 5]. This raises the question about the bulk region, defined by the interior of the unit disc, at distance more than $O(1/\sqrt{N})$ away from the unit circle and from $[-1,1]$, where the following holds.

Theorem 1. *(Universal local bulk eigenvalue statistics) For any bulk point parametrised by $z_0 \in \mathbb{C} \setminus \mathbb{R}$, with $0 < |z_0| < 1$, it holds for $\xi_1,\ldots,\xi_k \in \mathbb{C}$ for fixed k*

$$\lim_{N\to\infty} R_{k,N}(\sqrt{N}z_0 + \xi_1, \ldots, \sqrt{N}z_0 + \xi_k) = \det_{1\le i,j\le k} \left[\exp\left[\xi_i \xi_j^* - \frac{1}{2}|\xi_i|^2 - \frac{1}{2}|\xi_j|^2 \right] \right].$$

The respective proofs for the GinUE (at the origin), GinOE and GinSE can be found in [6, 5, 2], and extend beyond the Gaussian ensembles [9]. This agreement among the three Ginibre ensembles is in stark contrast to the corresponding local statistics of the Hermitian Wigner-Dyson ensembles.

We turn to the eigenvector statistics, to partly answer the question if the agreement found above extends to these as well. In [7] the matrix of overlaps

$$\mathcal{O}_{ij} := L_i^\dagger \cdot L_j \; R_j^\dagger \cdot R_i \; , \quad i,j = 1,\ldots,N \; ,$$

was defined, combining scalar products among the left and right eigenvectors,

$$J R_i = z_i R_i \; , \quad L_i^\dagger J = z_i L_i^\dagger \; , \quad i,j = 1,\ldots,N.$$

In the GinSE the $2N$-dimensional complex representation of matrix J has further N complex conjugated eigenvalues $z_1^*, \ldots, z_N^*$ and corresponding eigenvectors, for which similar results can be derived [1]. It was shown in [7] for the GinUE that the conditional expectation of the off-diagonal overlap

$$\mathcal{O}_N(x_1, x_2) := \frac{1}{N^2}\mathbb{E} \sum_{k\neq l}^{N} \delta(x_1 - z_k)\delta(x_2 - z_l)\ \mathcal{O}_{kl}\ ,$$

can be expressed as an expectation over complex eigenvalues only,

$$\mathcal{O}_N(z_1, z_2) = \int_{\mathbb{C}^{N-2}} d^2z_3 \cdots d^2z_N\ \frac{-P_N(z_1, \ldots, z_N)}{|z_1 - z_2|^2} \prod_{l=3}^{N} \left[1 + \frac{1}{(z_1 - z_l)(z_2^* - z_l^*)}\right].$$

The same holds for the GinSE [1]. The heuristic arguments [7] for the global bulk asymptotic in the GinUE were made rigorous using probabilistic methods in [4]:

Theorem 2. *(Universal global bulk eigenvector statistics) For bulk points x_1, x_2 of distance much larger than $O(1/\sqrt{N})$ the following limit holds*

$$\lim_{N\to\infty} \mathcal{O}_N(x_1, x_2) = -\frac{1 - x_1x_2^*}{\pi^2|x_1 - x_2|^4}.$$

The same limit was shown heuristically for the GinSE [1] and is conjectured to hold for the GinOE. Further local limits were derived for the GinUE including the diagonal overlap $\mathcal{O}_{ii}$ [4, 3], and we expect these to equally agree among the three Ginibre ensembles in the bulk and at the edge.

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Spectral radius of random matrices with independent entries

Johannes Alt
(joint work with László Erdős and Torben Krüger)

We consider random matrices $X \in \mathbb{C}^{n\times n}$ with independent and centered entries. When n tends to infinity the eigenvalue density of X is well approximated by a deterministic probability measure [1, 3]. We denote this measure by μ. It has a rotationally symmetric, smooth density with respect to the Lebesgue measure on $\mathbb{C}$. The support of μ is a disk around the origin whose radius is the square root of the spectral radius $\varrho(S)$ of the variance matrix $S = (\mathbb{E}|x_{ij}|^2)_{i,j=1}^n$ of $X = (x_{ij})_{i,j=1}^n$.

In [2], we show that the spectral radius $\varrho(X)$ of X converges to the square root of the spectral radius of S. More precisely, we prove the following theorem.

Theorem 1. *For any $\varepsilon > 0$, we have that, with very high probability,*

$$\left|\varrho(X) - \sqrt{\varrho(S)}\right| \le n^{\varepsilon} n^{-1/2}. \tag{1}$$

Such estimate is new even for matrices with i.i.d. entries apart from the explicitly solvable Gaussian case, i.e. there was no result on the speed of convergence of the spectral radius prior to [2]. The convergence rate in (1) is optimal up to the factor n^{ε}.

We now briefly list our assumptions for Theorem 1. The variances $\mathbb{E}|x_{ij}|^2$ are required to be of order n^{-1} with uniform upper and lower bounds. We suppose that the random variables $\sqrt{n}x_{ij}$ have bounded moments of all orders and a density whose L^q-norm is at most of polynomial order in n for some $q \in [1, \infty]$. The latter condition is solely used to control the smallest singular value of X.

The main ingredient in [2] is the *local inhomogeneous circular law* in the edge regime, the vicinity of the circle of radius $\sqrt{\varrho(S)}$ centered at the origin. This is the boundary of the support of μ. The complementary bulk regime was treated in our previous work [1]. The local inhomogeneous circular law states that the eigenvalue density of X is well approximated by μ on all mesoscopic scales. These are all scales slightly larger than the typical distance of neighbouring eigenvalues.

As a consequence of the local inhomogeneous circular law we also obtain complete delocalization of all eigenvectors of X, a result first proved in [4].

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Large deviations for traces of Wigner matrices

Fanny Augeri

Understanding the large deviation behavior of the spectrum of Wigner matrices is particularly difficult. Except for integrable models, the large deviations of the empirical measure of the eigenvalues are unknown for Wigner matrices with sub-Gaussian coefficients. A more accessible question is the one of the large deviations of the traces of powers of Wigner matrices. It turns out that this question is closely connected to the emerging theory of nonlinear large deviations, introduced by Chatterjee and Dembo [2]. Using a different approach, we could improve the quantitative estimates of [2], and obtain as a consequence a large deviations principle for the traces of a certain class of Wigner matrices, which includes the case where the coefficients are Rademacher variables. More precisely, let X be a Wigner matrix, that is a random Hermitian matrix such that $(X_{i,j})_{1\leq i<j\leq n}$ and $(X_{i,i})_{i\leq n}$ are independent families of independent and identically distributed random variables which satisfy the following conditions:

$$\mathbb{E}X_{1,1} = \mathbb{E}X_{1,2} = 0,\ \mathbb{E}|X_{1,2}|^2 = 1.$$

Denote by $\mathcal{H}_n^{(\beta)}$ the set of symmetric matrices if $\beta = 1$ and Hermitian matrices if $\beta = 2$, of size n. We assume that X is with *sharp sub-Gaussian tails*, that is, for any $H \in \mathcal{H}_n^{(\beta)}$,

$$\mathbb{E}(e^{\mathrm{tr}(XH)}) \leq \mathbb{E}(e^{\mathrm{tr}(X\Gamma)}),$$

where Γ is a GOE matrix if $\beta = 1$ and GUE matrix if $\beta = 2$. In [1] we prove that for any $d \geq 3$, $(n^{-1}\mathrm{tr}(X/\sqrt{n})^d)_{n\in\mathbb{N}}$ satisfies a large deviations principle with speed $n^{1+\frac{2}{d}}$ and rate function J_d. If d is even, J_d is given by,

$$\forall x \in \mathrm{R},\ J_d(x) = \begin{cases} \frac{\beta}{4}\left(x - C_{d/2}\right)^{\frac{2}{d}} & \text{if } x \geq C_{d/2}, \\ +\infty & \text{otherwise,} \end{cases}$$

where $C_{d/2}$ denote the $(\frac{d}{2})^{\text{ th}}$ Catalan's number, and if d is odd,

$$\forall x \in \mathbb{R},\ J_d(x) = \frac{\beta}{4}\left|x\right|^{\frac{2}{d}}.$$

This result complements the large deviation universal behavior of the top eigenvalue of these Wigner matrices with sharp sub-Gaussian tails shown by Guionnet and Husson [3].

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Unitary, symplectic, and orthogonal moments of moments

Emma Bailey
(joint work with T. Assiotis, J. P. Keating)

For a matrix $A \in G(N)$, where $G(N) \in \{U(N), Sp(2N), SO(2N)\}$ is one of the classical compact groups, we write $P_N(A,\theta) = \det(I - A\exp(-i\theta))$ for its characteristic polynomial. The study of moments of $P_N(A,\theta)$, averaging over the matrix group, has connections to number theory, see for example work of Montgomery and Dyson, Katz and Sarnak [4], and Keating and Snaith [5]. In particular, for $A \in U(N)$, Keating and Snaith computed the 2βth moment of $|P_N(A,\theta)|$ for finite N, with $\Re(\beta) > -1/2$.

Of interest is the log-correlated behaviour of $|P_N(A,\theta)|$. In the theory of log-correlated Gaussian processes, the following 'moments of moments' play a vital role,

$$\mathrm{MoM}_{G(N)}(k,\beta) = \mathbb{E}_{A\in G(N)}\left[\left(\frac{1}{2\pi}\int_0^{2\pi} |P_N(A,\theta)|^{2\beta} d\theta\right)^k\right].$$

Such averages were used by Fyodorov and Keating [3] to make precise conjectures about the asymptotic growth of $\mathrm{MoM}_{U(N)}(k,\beta)$ as $N \to \infty$. Further, strong information about the moments of moments for general β would lead to results for the maximum of $\log|P_N(A,\theta)|$.

The results presented are as follows.

Theorem 1 (B.-Keating [2])**.** *Let $k, \beta \in \mathbb{N}$. Then $\mathrm{MoM}_{U(N)}(k,\beta)$ is a polynomial in N, and $\mathrm{MoM}_{U(N)}(k,\beta) = \mathfrak{c}_U(k,\beta)N^{k^2\beta^2-k+1}\left(1+O(N^{-1})\right)$, where a general formula for the coefficient $\mathfrak{c}_U(k,\beta)$ is given.*

Theorem 2 (Assiotis-B.-Keating [1])**.** *Let $k, \beta \in \mathbb{N}$. Both $\mathrm{MoM}_{Sp(2N)}(k,\beta)$ and $\mathrm{MoM}_{SO(2N)}(k,\beta)$ are polynomials in N. Further, the asymptotic growth of both moments of moments is calculated and differs from the unitary case. The leading order coefficient in both cases is shown to be strictly positive and is given as a volume of a convex region.*

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Overview of results on random regular graphs

Roland Bauerschmidt
(joint work with Jiaoyang Huang, Antti Knowles, Horng-Tzer Yau)

The random regular graph ensemble can be seen as the uniform measure on symmetric $N \times N$ adjacency matrices $A = (A_{ij})$ with entries in $\{0, 1\}$ satisfying the constraints $A_{ii} = 0$ and $\sum_k A_{ik} = d$ for all i, j. From the random matrix point of view, the random regular graph ensemble poses two main challenges: the hard constraint on the degree and sparsity when d becomes small compared to N. However, a particularly interesting aspect is that these two difficulties effectively cancel to a certain extent: the hard constraint provides a geometric stability when the degree becomes small which in turn are responsible for spectral stability. In my talk, I reported on results obtained over the last few years in which we observe this phenomenon in the sense that the analogous statements are false for Erdős–Rényi graphs of the same average degree.

First, for fixed degree $d \geq 3$, it is well known that a random regular graph has a local tree structure near most vertices and few cycles in large neighbourhoods of all vertices. This deterministic condition already implies (i) that the asymptotic spectral density is given by the Kesten–McKay Law with support in $[-2, 2]$ when appropriately normalised; (ii) that eigenfunctions cannot localise; and (iii) that, together with a spectral gap assumption, averages of eigenfunctions equidistribute (quantum ergodicity). On the other hand, it is also known that such deterministic assumptions do *not* imply that eigenfunctions are completely flat. However, for random regular graphs of degree $d \geq 10^{40}$ fixed, we proved in [3] that all bulk eigenfunctions are completely flat in the sense that the ℓ^∞-norm of any ℓ^2-normalised eigenfunction is at most $N^{-1/2}(\log N)^C$ which is the optimal scale. For Erdős–Rényi graphs of fixed average degree, on the other hand, it is well known and easy to see that localised eigenfunctions exist. Our results also include a local version of the Kesten–McKay Law down to spectral scale $N^{-1}(\log N)^C$ and other consequences. The conjecture that the local eigenvalue statistics are given by those of the GOE remains open for random regular graphs of fixed degree. (Earlier results prove such a statement when the degree grows with N, see [2].)

Second, I reported on results on the non-trivial extremal eigenvalues for the intermediate regime of degrees $N^\epsilon \leq d \leq N^{2/3-\epsilon}$ obtained recently in [1]. In this regime, we proved the bound $2 + o(1)$ on the non-trivial extremal eigenvalues, which is optimal on the macroscopic scale and improves on results results which obtained the bound $O(1)$ up to $d \leq N/2$ (see discussion and references in [1]). Our error $o(1)$ is in fact quantitative and gives the mesoscopically optimal bound $O(N^{-2/3+\epsilon})$ when $N^{2/9+\epsilon} \leq d \leq N^{2/3-\epsilon}$. In this regime, we also show that the microscopic fluctuations of the non-trivial eigenvalues are given by the Tracy–Widom$_1$ Law. This result is to be contrasted with [4], in which it was shown that the extremal eigenvalues have Gaussian fluctuations in the same regime of average degree.

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Eigenvalues of nonlinear models of random matrices

Lucas Benigni

(joint work with Sandrine Péché)

Machine learning has entered many, if not most, scientific fields in recent years [3]. However, the theoretical understanding of some of the tools used in such algorithms is not complete. In this talk, we aim to obtain theoretical result on a model of artificial feed-forward neural network. Consider $W \in \mathbb{R}^{n_1 \times n_0}$ and $X \in \mathbb{R}^{n_0 \times m}$ two matrices. The matrix W can be understood as a matrix of *weights* from a layer of the network while X would be the *input data*. We will also suppose that n_0, n_1 and m grow together to infinity. A nonlinearity will be introduced through an *activation function* f which will always be applied entrywise. The matrix model considered is the following

$$M = \frac{1}{m} f\left(\frac{WX}{\sqrt{n_0}}\right) f\left(\frac{WX}{\sqrt{n_0}}\right).$$

It consists of the data covariance matrix after the first layer of the network. Spectral properties of this matrix measure the performance of the learning procedure for this neural network. A possible idea to understand better such large complex systems, as all the dimensions tend to be very large, is to approximate the elements of the system by random variables as it is done in statistical physics and thermodynamics.

From a mathematical standpoint, a first result was given in [5] where the case of a deterministic data and independent Gaussian weights was studied. They showed that the empirical spectral distribution converges to a deterministic measure given by a quadratic equation of its Stieltjes transform. The same equation arises for sample covariance matrix with general population of the form TXX^*T^*.

In [2], we study the above model in the case where both the weights and data are random. This was first considered in the case of Gaussian entries in [7] where the authors showed nonrigorously that the asymptotic eigenvalue distribution was given in terms of a quartic self-consistent equation of the Stieltjes transform of the empirical spectral distribution. We obtain this result for independent sub-exponential entries and for an analytic activation function f proving that this

asymptotic deterministic measure is universal. The asymptotic measure only depends on f through two explicit parameters:

$$\theta_1 = \mathbb{E}\left[f(\mathcal{N}(0,1))^2\right] \quad \text{and} \quad \theta_2 = \mathbb{E}\left[f'(\mathcal{N}(0,1))\right]^2.$$

There are two extremal cases for the equation. Firstly, if $\theta_1 = \theta_2$, the equation becomes a cubic equation which corresponds to the linear case where $f = x$. This has been widely studied, see [1] for instance. Also, if $\theta_2 = 0$, the equation collapses to a quadratic equation and we recover the usual Marchenko–Pastur distribution [6]. Thus for a class of functions, the asymptotic eigenvalue distribution of M is the same as the one of $\frac{1}{m}ZZ^*$ for some i.i.d matrix $Z \in \mathbb{R}^{n_1 \times m}$. In general, the eigenvalue distribution is actually the same as the one of the matrix

$$\frac{1}{m}(\sqrt{\theta_2}WX + \sqrt{\theta_1 - \theta_2}Z)(\sqrt{\theta_2}WX + \sqrt{\theta_1 - \theta_2}Z)^*$$

and thus corresponds to the squared singular values of the additive convolution of a Ginibre and a product of two Ginibre. This linearization of the problem can be compared to the result of [4] on kernel matrices.

Our proof relies on the method of moments by expanding traces of powers of the matrix M and performing a careful counting of graphs. This method also allows us to obtain a result on the behavior of the largest eigenvalue. If we suppose that f is chosen so that $\mathbb{E}\left[f(\mathcal{N}(0,1))\right] = 0$, we are able to prove that the largest eigenvalue converges to the edge of the support of the asymptotic deterministic distribution.

In [7], the authors conjectured that the eigenvalue distribution would keep converging to the Marchenko–Pastur distribution after several layers if the activation function is such that $\theta_2 = 0$. This invariance of the spectral measure was also empirically seen as a benefit to the performance of the training of the network. We also showed this result for a finite number of layers L. First consider a family $(W^{(i)})_{1 \leq i \leq L}$ of independent random matrices such that $W^{(i)} \in \mathbb{R}^{n_{i+1} \times n_i}$. If one defines

$$Y^{(i+1)} = f\left(\frac{W^{(i)}Y^{(i)}}{\sqrt{n_i \theta_1(f)}}\right) \quad \text{with} \quad Y^{(1)} = f\left(\frac{WX}{\sqrt{n_0}}\right) \quad \text{and} \quad M^{(i)} = \frac{1}{m}Y^{(i)}Y^{(i)*},$$

and if f is such that $\theta_2 = 0$ then the asymptotic eigenvalue distribution of $M^{(L)}$ is again given by the Marchenko-Pastur distribution.

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Outlier eigenvalues for non-Hermitian polynomials in independent i.i.d. matrices and deterministic matrices

Mireille Capitaine
(joint work with S. Belinschi, C. Bordenave, G. Cébron)

Here are the matricial models we deal with. Let t and u be fixed nonzero integer numbers independent from N.

(A1) $(A_N^{(1)}, \dots, A_N^{(t)})$ is a $t-$tuple of $N \times N$ deterministic matrices such that
(1) for any $i = 1, \dots, t$,

$$\sup_N \|A_N^{(i)}\| < \infty,$$

where $\|\cdot\|$ denotes the spectral norm,
(2) $(A_N^{(1)}, \dots, A_N^{(t)})$ converges in $*$-distribution towards a t-tuple $a = (a^{(1)}, \dots, a^{(t)})$ in some $\mathcal{C}^*$-probability space $(\mathcal{A}, \phi)$ where ϕ is faithful and tracial.

(X1) We consider u independent $N \times N$ random matrices $X_N^{(v)} = [X_{ij}^{(v)}]_{i,j=1}^N$, $v = 1, \dots, u$, where, for each v, $[X_{ij}^{(v)}]_{i\geq 1, j\geq 1}$ is an infinite array of random variables such that $\{\sqrt{2}\Re(X_{ij}^{(v)}), \sqrt{2}\Im(X_{ij}^{(v)}), i \geq 1, j \geq 1\}$ are independent identically distributed centred random variables with variance 1 and finite fourth moment.

Let P be a polynomial in $t + u$ noncommutative indeterminates and define

$$M_N = P\left(\frac{X_N^{(1)}}{\sqrt{N}}, \dots, \frac{X_N^{(u)}}{\sqrt{N}}, A_N^{(1)}, \dots, A_N^{(t)}\right),$$

and

$$M_N^{(0)} = P(0_N, \dots, 0_N, A_N^{(1)}, \dots, A_N^{(t)}),$$

where 0_N denotes the $N \times N$ null matrix. Let $c = (c^{(1)}, \dots, c^{(u)})$ be a free noncommutative circular system in $(\mathcal{A}, \phi)$ which is free from $a = (a^{(1)}, \dots, a^{(t)})$. We are now interested by describing the individual eigenvalues of M_N outside a small neighborhood of the spectrum of $P(c, a)$ (denoted by $\text{spect}(P(c, a))$), that we call outliers. In the lineage of [2], our main result in [1] gives a sufficient condition to guarantee that outliers are stable in the sense that outliers of M_N coincide asymptotically with the eigenvalues of $M_N^{(0)}$ which are located outside the spectrum of $P(c, a)$.

Theorem 1. *Assume that hypotheses* (**A1**), (**X1**) *hold. Let* Γ *be a compact subset in the complement of the spectrum of* $P(c,a)$*. Assume moreover that*

(**A2**) *for* $k=1,\dots,t$, $A_N^{(k)} = (A_N^{(k)})' + (A_N^{(k)})''$,

where $(A_N^{(k)})''$ *has a bounded rank* $r_k(N) = O(1)$ *and* $\left((A_N^{(1)})', \dots, (A_N^{(t)})'\right)$ *satisfies*

- *for any* z *in* Γ*, there exists* $\eta_z > 0$ *such that for all* N *large enough, there is no singular value of*
$$P(0_N, \dots, 0_N, (A_N^{(1)})', \dots, (A_N^{(t)})') - zI_N$$
in $[0, \eta_z]$.
- *for any* $k = 1, \dots, t$,
$$\sup_N \|(A_N^{(k)})'\| < +\infty.$$

If for some $\epsilon > 0$*, for all large* N*,*

$$\min_{z\in\partial\Gamma} \left| \frac{\det(zI_N - P(0_N, \dots, 0_N, A_N^{(1)}, \dots, A_N^{(t)})}{\det(zI_N - P(0_N, \dots, 0_N, (A_N^{(1)})', \dots, (A_N^{(t)})')} \right| \geq \epsilon$$

then almost surely for all large N*, the numbers of eigenvalues of* $M_N^{(0)}$ *and* M_N *in* Γ *are equal.*

The next statement is an easy consequence of Theorem 1.

Corollary 1. *Assume that* (**X1**) *holds and that, for* $k = 1, \dots, t$, $A_N^{(k)}$ *are deterministic matrices with rank* $O(1)$ *and operator norm* $O(1)$*. Let* $\epsilon > 0$*, and suppose that for all sufficiently large* N*, there are no eigenvalues of* $M_N^{(0)}$ *in* $\{z \in \mathbb{C}, \epsilon < d(z, \mathrm{spect}(P(c,0))) < 3\epsilon\}$*, and there are* j *eigenvalues* $\lambda_1(M_N^{(0)}), \dots, \lambda_j(M_N^{(0)})$ *for some* $j = O(1)$ *in the region* $\{z \in \mathbb{C}, d(z, \mathrm{spect}(P(c,0))) \geq 3\epsilon\}$*. Then, a.s , for all large* N*, there are precisely* j *eigenvalues of* M_N *in* $\{z \in \mathbb{C}, d(z, \mathrm{spect}(P(c,0))) \geq 2\epsilon\}$ *and after labeling these eigenvalues properly,*

$$\max_{j\in J} |\lambda_j(M_N) - \lambda_j(M_N^{(0)})| \to_{N\to+\infty} 0.$$

To prove Theorem 1, we make use of a linearization procedure which brings the study of the polynomial back to that of the sum of matrices in a higher dimensional space. Then, this allows us to follow the approach of [2]. But for this purpose, we need to establish substantial operator-valued free probability results.

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Aspects of Coulomb gases

Djalil Chafaï
(joint work with many co-authors)

This talk gives an overview around various contributions to the asymptotic analysis of Coulomb gases (list of works below). Coulomb gases are exchangeable Gibbs measures on $(\mathbb{R}^d)^n$ modelling confinement as well as singular pair repulsion:

$$\mathrm{d}P(x_1,\dots,x_n) = \mathrm{e}^{-\beta_n(\sum_{i=1}^n V(x_i)+\frac{1}{2}\sum_{i\neq j} g(x_i-x_j))}\mathrm{d}x_1\cdots\mathrm{d}x_n$$

where the interaction is given by the Coulomb kernel

$$g = \begin{cases} \frac{1}{(d-2)|\cdot|^{d-2}} & \text{if } d\neq 2 \\ \log\frac{1}{|\cdot|} & \text{if } d = 2 \end{cases}.$$

They appear at various places in mathematical physics, in particular in random matrix theory for $d = 2$. They were extensively studied in the past decade, notably by Sylvia Serfaty and co-authors. After recalling the notion of electrostatic energy and equilibrium measure, we consider the notion of Coulomb gas, the associated large deviation principle, the law of large numbers, and the central limit theorem. We also discuss the quantitative version of this last statement, via concentration of measure inequalities. We then consider Langevin type stochastic dynamics associated to Coulomb gases, that can be used to simulate and to compute. The trend to the equilibrium of such dynamics is related to functional inequalities, which are in general of quite difficult access beyond convex cases. We also discuss the problem of conditioning Coulomb gases with respect to a linear statistics. Finally we discuss the connection between Coulomb gases and Wigner jelliums, and focus more specifically on a two-dimensional model for which a transition phenomenon occurs for the edge fluctuation, from Gumbel to heavy-tailed.

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Edge Universality for non-Hermitian Random Matrices

Giorgio Cipolloni
(joint work with László Erdős, Dominik Schröder)

We consider non-Hermitian matrices X with i.i.d. entries $x_{ab} = n^{-1/2}\chi$ such that $\mathbf{E}\chi = 0$, $\mathbf{E}|\chi|^2 = 1$. It is well known [6] that the empirical measure of the eigenvalues $\sigma_1, \dots, \sigma_n$ of X converges weakly to the *circular law*:

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f(\sigma_i) = \frac{1}{\pi}\int_{|z|\le 1} f(z)\, d^2 z. \tag{1}$$

The convergence in (1) has been extended to mesoscopic scales, i.e. for test functions $f(z) = n^{2a}\varphi(n^a(z - z_0))$ for any $0 \le a < 1/2$, $|z_0| \le 1$.

The *law of large number type* limit (1) does not hold for $a = 1/2$ since the eigenvalues of X fluctuate on a scale $n^{-1/2}$. On this scale it is conjectured that the local spectral statistics $\sum_i f(\sigma_i)$ converges to a universal distribution. In a recent work [3] we proved that the universality conjecture holds at the edge of the spectrum of X.

Theorem 1. *Let X be a matrix with centred i.i.d. entries of variance n^{-1}, and denote by $p_k^{(n)}$ the k-point correlation function of the eigenvalues of X. Then, there exits a universal correlation function $p_{\mathbf{z}}^{(\infty, Gin(\mathbf{F}))}$, with $\mathbf{F} = \mathbf{R}, \mathbf{C}$, such that for any fixed $k \ge 1$, and $z_1, \dots, z_k$ so that $|1 - |z_j|^2| \le n^{-1/2}$, $j = 1, \dots, k$, it holds*

$$\int_{\mathbf{C}^k} F(\mathbf{w}) \left[p_k^{(n)}\left(\mathbf{z} + \frac{\mathbf{w}}{\sqrt{n}}\right) - p_{\mathbf{z}}^{(\infty, Gin(\mathbf{F}))}(\mathbf{w}) \right] d\mathbf{w} = \mathcal{O}(n^{-c}), \tag{2}$$

for any compactly supported smooth function $F : \mathbf{C}^k \to \mathbf{C}$.

The correlation kernel $p_{\mathbf{z}}^{(\infty, \mathrm{Gin}(\mathbf{F}))}$ is the scaling limit of the Ginibre (Gaussian) correlation function and it is explicitly known for complex [5] and real [2] matrices X. This result is the non-Hermitian analogue of the Tracy-Widom Hermitian edge universality. The only known previous result is due to Tao and Vu [9] which proved that the non-Hermitian universality conjecture holds in the bulk and at the edge of the spectrum of X under the additional assumption that the first four moments of the entries of X (almost) match the respective Gaussian moments.

Investigating spectral statistics of non-Hermitian random matrices is considerably more challenging than Hermitian ones. There are two fundamental reasons for this: (i) the resolvent $(X - z)^{-1}$ of a non-normal matrix, unlike in the Hermitian case, is not effective to study eigenvalues near z; (ii) lack of a good analogue of the Dyson Brownian motion (DBM). The only useful expression to grasp non-Hermitian eigenvalues is Girko's celebrated formula

$$\sum_i \varphi(\sqrt{n}(\sigma_i - z_0)) = -\frac{1}{4\pi}\int_{\mathbf{C}} \Delta\varphi(\sqrt{n}(z - z_0)) \int_0^\infty \Im \mathrm{Tr} G^z(i\eta) d\eta dz, \tag{3}$$

with

$$H^z := \begin{pmatrix} 0 & X - z \\ X^* - \overline{z} & 0 \end{pmatrix}, \tag{4}$$

and $G^z(i\eta) := (H^z - i\eta)^{-1}$. The key difficulty with Girko's formula is that it requires a good lower bound on the smallest singular value of $X - z$.

Our proof of the edge universality circumvents DBM and it has two key ingredients. The first main input is an optimal local law for the resolvent of H^z [1] that allows for a comparison of the joint distribution of several resolvents of H^z with their Gaussian counterparts by following their evolution for long time under the natural Ornstein-Uhlenbeck (OU) flow for $\eta \geq n^{-3/4-\epsilon}$.

Our second key input is a lower tail estimate on the lowest singular value of $X-z$ when $|z| \approx 1$, or equivalently on the lowest eigenvalue λ_1 of $(X-z)(X-z)^*$. In the bulk ($|z| < 1$) classical smoothing inequalities [8], which implies no eigenvalues in $[0, n^{-2}]$, are optimal. On the other hand for $|z| > 1$ the smallest eigenvalue of $(X - z)(X - z)^*$ is bounded away from zero. In Theorem 2 we consider the transitional regime $|z| \approx 1$. With supersymmetric methods in [4], using *superbosonization* formula [7], we recently proved the following theorem.

Theorem 2. *Let X be a real or complex Ginibre matrix, and denote $\delta := 1 - |z|^2$ for any fixed $z \in \mathbf{C}$, then*

(5)

$$\mathbf{P}\Big(\lambda_1\big((X - z)(X - z)^*\big) \leq x \min\left\{\frac{1}{n^{3/2}}, \frac{1}{n^2\delta}\right\}\Big) \leq \begin{cases} x + \sqrt{x}e^{-n(\Im z)^2}, & X \sim \mathit{Gin}(\mathbf{R}) \\ x, & X \sim \mathit{Gin}(\mathbf{C}). \end{cases}$$

The bound (5) for $|z| = 1 + \mathcal{O}(n^{-1/2})$ controls λ_1 on the optimal $n^{-3/2}$ scale and thus excludes singular values in the regime $[n^{-2}, n^{-3/2-\epsilon}]$ that was inaccessible with other methods.

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Pseudospectra of structured random matrices

Nicholas Cook
(joint work with Alice Guionnet and Jonathan Husson)

In recent years, methods from high-dimensional geometry have been used to obtain quantitative bounds on the norm of the inverse of random matrices; see [8] for an overview. We describe two new results obtained by geometric methods: (1) spectral anti-concentration estimates for structured Hermitian random matrices, and (2) control on the pseudospectrum of quadratic polynomials in Ginibre matrices.

For our first result, consider H an $N \times N$ Hermitian random matrix with $\{H_{ij}\}_{i \leq j}$ independent, centered and sub-Gaussian with variances $\sigma_{ij}^2 \in [0,1]$. Denote by $\Sigma = (\sigma_{ij})_{i,j=1}^N$ the standard deviation profile. For an arbitrary fixed cutoff parameter $\sigma_0 \in (0,1]$, denote $\mathrm{spt}(\Sigma) = \{(i,j) \in [N]^2 : \sigma_{ij} \geq \sigma_0\}$. We say Σ is

- δ-**broadly connected** if $\forall I, J \subset [N]$ with $|I| + |J| \geq N$, $|\mathrm{spt}(\Sigma) \cap (I \times J)| \geq \delta |I||J|$ (this notion was introduced in [10]);
- δ-**robustly irreducible** if $\forall J \subset [N]$, $|\mathrm{spt}(\Sigma) \cap (J \times J^c)| \geq \delta |J||J^c|$.

The robust irreducibility condition can be interpreted as an expansion condition on the naturally associated graph that contains an edge $\{i,j\}$ whenever $\sigma_{ij} \geq \sigma_0$. Note that it permits the ESD of H to have an atom of size $\gtrsim 1$ at zero.

Theorem 1 ([3]). (1) *Fix $\delta > 0$ and suppose Σ is δ-broadly connected. Then with probability $1 - O(N^{-100})$, for any interval $\mathcal{I} \subset \mathbb{R}$ with length$(\mathcal{I}) \geq C\frac{\log N}{N}$ we have $\mu_{\frac{1}{\sqrt{N}}H}(\mathcal{I}) \lesssim_{\delta,\sigma_0}$ length$(\mathcal{I})$.*

(2) *Fix $\delta, \kappa > 0$ and suppose Σ is δ-robustly irreducible. Then with probability $1 - O(N^{-100})$, for any $\mathcal{I} \subset \mathbb{R} \setminus (-\kappa, \kappa)$ with length$(\mathcal{I}) \geq C\frac{\log N}{N}$ we have $\mu_{\frac{1}{\sqrt{N}}H}(\mathcal{I}) \lesssim_{\delta,\kappa,\sigma_0}$ length$(\mathcal{I})$.*

Under more restrictive hypotheses on Σ, such estimates controlling the ESDs by Lebesgue measure down to the optimal "mesoscopic" scale $N^{o(1)-1}$ follow immediately from a local law established by Ajanki, Erdős and Krüger in [1] by careful analysis of an associated *vector Dyson equation*; see also the notes [5].

For the proof, the key technical step is to show that the eigenvectors of H are delocalized in the "no gaps" sense (cf. [9]). The proof is flexible and can be adapted to other settings where the analysis of the Dyson equations is challenging. For instance, together with some ideas for the proof of Theorem 2 we can obtain analogous bounds for polynomials in independent Wigner matrices. This will be the subject of a forthcoming work. Local laws for certain classes of polynomials were obtained through analysis of the (matrix) Dyson equations in [2, 6].

For our next result, recall that the ε-pseudospectrum $\Lambda_\varepsilon(M)$ of a square matrix M is the set $\Lambda(M)$ of eigenvalues together with all $z \in \mathbb{C}$ for which $\|(M-z)^{-1}\|_{op} \geq 1/\varepsilon$. If M is normal then $\Lambda_\varepsilon(M)$ is simply the ε-neighborhood of Λ, whereas for non-normal matrices it can be considerably larger. The following gives control on the size of the pseudospectrum for certain polynomials in random matrices. Recall that a (complex) $N \times N$ *Ginibre matrix* X has iid entries $X_{ij} \sim \mathcal{N}_{\mathbb{C}}(0, 1/N)$.

Theorem 2 ([4]). *Let $m \geq 1$ and let p be a quadratic polynomial in non-commutative variables $x_1, \ldots, x_m$. Let $N \geq 2$ and $X_1, \ldots, X_m$ be iid $N \times N$ Ginibre matrices. Set $P = p(X_1, \ldots, X_m)$. For any $z \in \mathbb{C}$ and any $\varepsilon > 0$,*

$$\mathbb{P}\{z \in \Lambda_\varepsilon(P)\} \leq N^C \varepsilon^c + e^{-cN}$$

for constants $C, c > 0$ depending only on p.

Via the *linearization trick* used in [7] the proof reduces to studying the condition number of random matrices in $M_N(M_d(\mathbb{C}))$, where the $d \times d$ entries are independent random matrices of bounded dimension with correlated entries. Even when $d = 2$ the proof requires significant new ideas over the well-studied scalar-entry case.

Theorem 2 is key for the proof of the following result on the limiting ESDs:

Theorem 3 ([4]). *Fix $m \geq 1$ and a quadratic polynomial p in non-commutative variables $x_1, \ldots, x_m$. For each N let $X_1^{(N)}, \ldots, X_m^{(N)}$ be iid $N \times N$ Ginibre matrices. Set $P^{(N)} = p(X_1^{(N)}, \ldots, X_m^{(N)})$. Almost surely,*

$$\mu_{P^{(N)}} \to \nu_p \quad \text{weakly,}$$

where ν_p is the Brown measure for $p(c_1, \ldots, c_m)$ with $c_1, \ldots, c_m$ free circular elements of a non-commutative probability space.

The proof of Theorem 3 follows the standard Hermitization strategy, combining Theorem 2 with quantitative control on singular value distributions of $P^{(N)} - z$ that can be deduced from results in [7].

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Random matrices and history dependent stochastic processes

Margherita Disertori

We consider here the connection between a lattice random Schrödinger operator and certain stochastic processes with memory (cf. [4, 5, 6]). These processes have been under intensive study in the last ten years, due to their connection with a supersymmetric nonlinear sigma model (the so-called $H^{2|2}$) introduced by Zirnbauer in [1] as a toy model for random matrices in quantum diffusion.

A lattice random Schrödinger operator (RS). Let $\Lambda \subset \mathbb{Z}^d$ be a cube and consider the random matrix $H_\Lambda \in \mathbb{R}^{\Lambda\times\Lambda}$ of the form $H_\Lambda = -\Delta_\Lambda + \lambda\hat{V}$, where Δ_Λ is the lattice Laplacian, $\hat{V} = \mathrm{diag}(\{V_x\}_{x\in\Lambda})$ is a random diagonal matrix, and $\lambda > 0$ is a parameter encoding the strength of the disorder. Physical information can be inferred from the spectral properties of this operator as $\Lambda \to \mathbb{Z}^d$.

For a large class of random potentials V localization of the eigenfunctions has been proved in $d = 1$ for arbitrary disorder and in $d \geq 2$ for large disorder or at the band edge. A localization - delocalization transition has been proved on tree graphs, and is conjectured to hold on $\mathbb{Z}^d$, for $d \geq 3$. A detailed up-to-date review on the model, known results and tools can be found in the book by Aizenman and Warzel [7].

Here we consider the matrix $H_W := 2\hat{\beta} - WP$, where $P_{ij} = \mathbf{1}_{|i-j|=1}$ is the off-diagonal lattice Laplacian, $W > 0$ is a parameter and $\hat{\beta} = \mathrm{diag}(\{\beta_x\}_{x\in\Lambda})$ is a random diagonal matrix, with distribution

$$d\mathbb{P}(\beta) := \left(\tfrac{2}{\pi}\right)^{|\Lambda|/2} e^{\sum_{|i-j|=1} W} \mathbf{1}_{H_W>0} \frac{e^{-\sum_{j\in\Lambda}\beta_j}}{\det H_W} \prod_{j\in\Lambda} d\beta_j. \tag{1}$$

Some features:

- $\beta_x > 0$ almost surely for all $x \in \Lambda$ and the random variables β_i, β_j are independent for all $|i - j| \geq 2$.
- H_W can be written as a random Schrödinger operator as follows:
  $H_W = W(-\Delta + \frac{1}{W}\hat{V})$ with $V_x := 2\beta_x - 2dW$

(neglecting eventual boundary terms). Note that $\mathbb{E}[V_x] = 1$ is independent of W. This is not true for higher order correlations. For example $\mathrm{Var}(V_x) = \sqrt{2(1 + dW)}$.

A history dependent stochastic process (VRJP). Let $\Lambda \subset \mathbb{Z}^d$ be again a cube and consider the continuous time jump process $(Y_t)_{t\geq 0}$ on Λ with transition probability

$$\mathbb{P}(Y_{t+dt} = j \,|\, Y_t = i, (Y_s)_{s\leq t}) = \mathbf{1}_{|i-j|=1} w_{ij}(t)dt + o(dt)$$

where $w_{ij}(t) > 0$ is the jump rate. If $w_{ij} = c_{ij}$ is independent of t this is a Markov jump process (no memory). Here we consider the so-called vertex-reinforced jump process, introduced by Werner in 2000, with jump rate $w_{ij}(t) := W(1 + L_j(t))$, where $L_j(t) := \int_0^t \mathbf{1}_{Y_s=j} ds$ is the total time spent at j up to time t, and $W > 0$ is a parameter encoding the strength of the reinforcement. For this process a

recurrence/transience phase transition has been proved in $d \geq 3$ [2, 3]. The model is recurrent at $d = 2$ $\forall W$ [10].

Connection between RS and VRJP. Consider the time-changed process $(Z_s)_{s\geq 0}$ whose jump rate is given by $w_{ij}(s) := \frac{W}{2}\frac{\sqrt{1+T_j(s)}}{\sqrt{1+T_i(s)}}$, where $T_j(s)$ it the local time at j. In finite volume, this new process is a mixture of Markov jump processes [4]: $\mathbb{P}^Z(\cdot) = \mathbb{E}[\mathbb{P}_\Lambda^{w(u)}(\cdot)]$, where $w_{ij}(u) := e^{u_j-u_i}W/2$ and $u \in \mathbb{R}^\Lambda$ is a random field.

In particular, the generator of the Markov jump process is
$\mathcal{L}_{u,W} f(x) := \sum_{|y-x|=1}(f_x - f_y)We^{u_y-u_x}$.

This can be written as $\mathcal{L}_{u,W} = e^{\hat{u}} H_W e^{-\hat{u}}$, where $H_W = 2\hat{\beta} - WP$ and $2\beta_x(u) := \sum_{|y-x|=1} We^{u_y-u_x}$.

Performing a change of coordinate we obtain $d\mathbb{P}(u) = d\mathbb{P}(\beta)d\rho(u_{i_0})$, where i_0 is the starting point of the process and $d\mathbb{P}(\beta)$ is the probability measure given in (1) above.

This connection was explored by Zeng, Sabot and Tarres [8, 9]. In particular a connection was found between the ground state of H_W in infinite volume and recurrence/transience properties.

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Random Bipartite Biregular Graphs, Regular Hypergraphs, and Applications

Ioana Dumitriu
(joint work with Gerandy Brito, Kameron Decker Harris, Yizhe Zhu)

Spectra of random graphs and hypergraphs have recently been the focus of much attention in connection to network science and machine learning, where they have been used to study problems from community detection to matrix completion, and from pattern recognition to bioinformatics. One of their main properties to study, with wide applicability in the fields mentioned, is *expansion*; the key feature of expansion is fast random walk mixing on the graph in question.

The classical example of a very good expander is the random regular graph, for which all vertices have the same degree d. For connected, simple graphs, expansion has been known for a long time to be related to the *spectral gap* of the Laplacian matrix; if the graph is regular, the Laplacian matrix is only a shifted and scaled version of the adjacency matrix. The Alon conjecture, which held that the uniformly random regular graph was asymptotically almost surely a good expander, was proved in the early aughts by Joel Friedman, and recently (in 2015) reproved by Charles Bordenave. The latter's techniques include a sophisticated version of the method of moments applied not to the (symmetric) adjacency matrix, but to the Hashimoto (non-backtracking) operator, which can be related to the former via the Ihara-Bass formula.

The author, in collaboration with students and postdocs, has adapted Bordenave's proof to quasi-regular graphs (bipartite biregular and the more general k-frame models), and recently has discovered that these results translate well to regular k-uniform hypergraphs via a simple well-known bijection. The key here is proving spectral gap for biregular bipartite graphs, which can again be done by implicitly using the fact that the Laplacian matrix, just as was the case with regular graphs, is related to the adjacency matrix through a scale and shift. The spectrum of the adjacency matrix is once again related to the spectrum of the Hashimoto operator via the Ihara-Bass formula, and thus showing the spectral gap of the latter implies expansion of the former [1].

An elegant paper by Marina Meila and Yali Wan from 2015 introduced the idea of *preference frame* graph model, which covers a variety of random graphs, among them the regular ones, bipartite regular, and k-frames or *equitable* graphs as studied by Mark Newman et al. Meila and Wan showed that equitable graphs are expanders (specifically, that they have significant spectral gap) if and only if the component bipartite biregular graphs have spectral gap. By showing that almost all bipartite biregular random graphs have spectral gap, we have completed the proof [1]. Along the way, we also explored some other applications (e.g., matrix completion and Tanner codes).

Finally, the spectra of regular k-uniform hypergraphs has been defined in terms both of tensors (in the style of Joel Friedman and Avi Wigderson) and of associated families of adjacency matrices (Winnie Li et al.) We found that the later approach,

through its very simple bijection connection to bipartite biregular graphs, can be used to show directly that spectral expansion, rapid random walk mixing, and both vertex and edge expansion are properties which the regular uniform hypergraphs satisfied asymptotically almost surely [2].

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On eigenvector statistics in non-normal random matrices

Yan V. Fyodorov
(joint work with Woiciech Tarnowski)

Generically, non-selfadjoint square $N \times N$ random matrices X are non-normal, i.e. do not commute with their adjoint X^*, and diagonalizable. To each eigenvalue λ_i, real or complex, correspond two sets of eigenvectors: *left* $\mathbf{l}_i$ and *right* $\mathbf{r}_i$. The corresponding eigenproblems are $X\mathbf{r}_i = \lambda_i \mathbf{r}_i$ and $X^*\mathbf{l}_i = \lambda_i \mathbf{l}_i$. The two sets can always be chosen *bi*-orthogonal: $(\mathbf{l}_i^* \mathbf{r}_j) = \delta_{ij}$. The eigenvalue condition numbers $\kappa_i = \sqrt{(\mathbf{l}_i^* \mathbf{l}_i)(\mathbf{r}_i^* \mathbf{r}_i)} \geq 1$ can be arbitrary big, the large values signalling of the non-orthogonality of set of left $\mathbf{l}_i$ and right $\mathbf{r}_i$ eigenvectors and enhanced sensitivity of the associated eigenvalues against additive perturbations of the matrix entries: $X \to X + \epsilon P$. Note that $\kappa = 1$ only when X is normal. It is natural to ask how well-conditioned are eigenvalues of a ' typical' $N \times N$ non-normal matrix randomly chosen according to a probability measure or "ensemble". The simplest choice is to choose all entries to be i.i.d. normals $X_{j,k} \sim N^{-1/2}\mathcal{N}(0,1)$ (defining the so-called real Ginibre ensemble) or $\Re X_{j,k} \sim \Im X_{j,k} \sim N^{-1/2}\mathcal{N}(0,1/2)$ for the complex Ginibre ensemble. The study of the eigenvalue condition numbers has been initiated two decades ago by Chalker and Mehlig [1] who introduced the matrix of inner products $O_{ij} = (\mathbf{l}_i^* \mathbf{l}_j)(\mathbf{r}_j^* \mathbf{r}_i)$, which they called " eigenvector overlaps". The diagonal 'overlaps' are simply the squared eigenvalue condition numbers. They showed that for a typical eigenvalue of the complex Ginibre matrix the overlaps are of order of N bigger than for normal matrices, and also put forward a conjecture on the tail for the corresponding probability density: $\mathcal{P}(O_{ii}) \sim \mathcal{O}_{ii}^{-3}$. This conjecture has been settled recently by two different methods in [2] and [3] where $\mathcal{P}(O_{ii})$ has been explicitly found in a closed form for $N \to \infty$. In [3] also an explicit density of condition numbers for real eigenvalues of real Ginibre ensemble has been determined, as summarized in the following **Theorem 1**, see [2] and [3]:
Consider the (conditional) probability density $\mathcal{P}_N(z,t)$ of the (scaled) 'diagonal overlap' factor $t = (O_{ii} - 1)/N$ for the right and left eigenvectors corresponding to eigenvalues in the vicinity of a point $z = x + iy$ in the complex plane. Then

$$\lim_{N\to\infty} \mathcal{P}_N(z,t) = \frac{\langle\rho(z)\rangle}{t} e^{-\frac{O_1^{(\beta)}(z)}{t\langle\rho(z)\rangle}} \left(\frac{O_1^{(\beta)}(z)}{t\langle\rho(z)\rangle}\right)^{\beta}, \quad |z|<1. \tag{1}$$

where parameter $\beta = 1$ corresponds to the real eigenvalues of the real Ginibre matrices (in which case $z \in \mathbb{R}$) and $\beta = 2$ to the complex Ginibre case. Correspondingly, the limiting spectral density of real eigenvalues for $\beta = 1$ is $\langle\rho(z)\rangle = \frac{1}{\sqrt{2\pi}}$ for the interval $|z| < 1$, whereas the density of complex eigenvalues is $\langle\rho(z)\rangle = \pi^{-1}$ inside the unit circle $|z| < 1$, with $O_1^{(\beta=2)}(z) = \pi^{-1}(1-|z|^2)$ providing 'typical scale' value for the diagonal overlap, whereas $O_1^{(\beta=1)}(z) = \frac{1}{2\sqrt{2\pi}}(1-|z|^2)$.

More generally, one can consider the entries $X_{j,k}$ and $X_{k,j}$ of the matrix X to be correlated: e.g. in the real case $\mathbb{E}(X_{ij}X_{ji}) = \tau/N$ for $i \neq j$, where the parameter $\tau \in [0,1]$ tunes the degree of correlation interpolating between real Ginibre for $\tau = 0$ and real symmetric GOE matrices for $\tau = 1$. This defines the Elliptic (Gaussian) Ensembles, the name coming from the shape of the support for the eigenvalue density as $N \gg 1$, an ellipse with the vertical semiaxis $1-\tau$ and the horizontal semiaxis $1+\tau$. For such an ensemble one finds the following generalization:

Theorem 2 (Bulk scaling), see [4].
The (conditional) probability density function of the (scaled) 'diagonal overlap' factor $t = (O_{ii}-1)/N$ for the eigenvectors of the real Elliptic Ensemble corresponding to real eigenvalues in the interval $|z| < 1+\tau$ has asymptotically the form

$$\lim_{N\to\infty} \mathcal{P}_N(z,t) = \frac{\langle\rho_\tau(z)\rangle}{t} e^{-\frac{O_\tau^{(1)}(z)}{t\langle\rho_\tau(z)\rangle}} \left(\frac{O_\tau^{(1)}(z)}{t\langle\rho_\tau(z)\rangle}\right), \tag{2}$$

where $\langle\rho_\tau(z)\rangle = \frac{1}{\sqrt{2\pi(1-\tau^2)}}$ provides the asymptotic density of real eigenvalues and the typical scale value of the overlap is given by $O_\tau^{(1)}(z) = \frac{\sqrt{1-\tau^2}}{2\sqrt{2\pi}}\left(1-\frac{|z|^2}{(1+\tau)^2}\right)$.

When approaching the boundary $|z| = 1+\tau$ of the eigenvalue support the typical diagonal overlap $O_\tau^{(1)}(z)$ tends to zero. A more detailed investigation then shows that in the appropriate scaling vicinity of the boundary non-orthogonality becomes parametrically weaker, as the 'typical scale' value of the diagonal overlap O_{ii} becomes of the order $\sqrt{N}$. The expression for $\mathcal{P}_N(z,t)$ in this range of parameters can also be found after appropriate rescaling.

Yet another scaling regime occurs for $N \to \infty$ when τ approaches unity with the rate $O(N^{-1})$, so that the parameter $2N(1-\tau) = a^2$ is fixed. Such parameter therefore controls the deviation from the fully symmetric GOE matrices. In this regime of " almost symmetric" matrices already a finite fraction of order of N eigenvalues turn out to be typically real. Such regime turns out to be not only " weakly asymmetric", but also " weakly non-normal" as the typical value of the

diagonal overlap $t = O_{ii} - 1$ turns out to be of the order of unity, see [4] for more detail.

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Bulk eigenvalue fluctuations of sparse Erdős-Rényi graphs

Yukun He

Let us consider the Erdős-Rényi graph $\mathcal{G}(N,p)$. After rescaling and centering, its adjacency matrix H is an $N \times N$ real symmetric random matrix with independent upper triangular entries, and satisfies

$$\mathbb{E}H_{ij} = 0\,, \quad \mathbb{E}H_{ij}^2 = \frac{1}{N} \quad \text{and} \quad \mathbb{E}|H_{ij}|^k \asymp \frac{1}{N^{k/2}p^{(k-2)/2}}$$

for all $i,j \in \{1,2,...,N\}$. For a *dense graph* $p \asymp 1$, one has $\|H_{ij}\|_2 \asymp \|H_{ij}\|_k$, and H is a standard Wigner matrix. In practice, people are more interested in *spare graphs* $p \ll 1$, where in this case we have

$$\|H_{ij}\|_2 \ll \|H_{ij}\|_k\,,$$

i.e. the entries of H are no longer light-tailed.

There has been many works studying the spectral statistics of H. In the famous works [3, 9], it was shown that the k-point correlation function and distribution of eigenvalue gaps of a Wigner matrix H coincide with those of the Gaussian Orthogonal Ensemble (GOE). In terms of Erdős-Rényi graphs $\mathcal{G}(N,p)$, this corresponds to the regime $p \asymp 1$. Later in [2] the results were extended to the sparse regime $p \geq N^{-1/3+\varepsilon}$ and finally in [8] to the almost optimal regime $p \geq N^{-1+\varepsilon}$.

In addition to the k-point correlation function and distribution of eigenvalue gaps, another very natural quantity to study is the distribution of individual eigenvalues. This problem was first studied in [4] for Gaussian Unitary Ensembles (GUE), where it was proved that the bulk eigenvalue distributions are asymptotically Gaussian, and the fluctuations are on the scale $\log N/N$. Later in [7], a similar result was proved for GOE. Recently in [6, 1], it was shown that the fluctuation is universal among all Wigner matrices.

In this talk, we present a result in [5], where we study the bulk eigenvalue fluctuations for sparse Erdős-Rényi graph $\mathcal{G}(N,p)$, $p \in [N^{-1+\varepsilon}, N^{-\varepsilon}]$. Our result indicates that unlike Wigner matrices, the eigenvalues of sparse matrices fluctuate

on a much bigger scale $1/(N\sqrt{p})$. In addition, the eigenvalues fluctuate simultaneously: the correlation of two eigenvalues of the same/different sign is asymptotically 1/-1. In other words, we showed that the bulk eigenvalue fluctuations are Non-universal for sparse matrices.

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Large deviations for the largest eigenvalue of some sub-Gaussian matrices

Jonathan Husson

(joint work with Alice Guionnet, Fanny Augeri)

In random matrix theory, using the explicit formula for the joint distribution of the eigenvalues of the Gaussian ensembles, Ben Arous and Guionnet proved a large deviation principle for the empirical measure of GOE and GUE matrices [4] and Ben Arous, Dembo and Guionnet proved a similar result for the largest eigenvalue [3]. Outside of the case of Gaussian matrices, few proper large deviation principles are known for the empirical measure or for the largest eigenvalue (see for example the results of Augeri, Bordenave and Caputo for heavier than Gaussian tails [5, 1]).

This talk exposes a new large deviation principle for the largest eigenvalue of Wigner matrices whose entries satisfy a sharp sub-Gaussian bound [6] using a spherical integral studied by Guionnet and Maïda [7].

Remarkably, this large deviation principle exhibits the same rate function as in the Gaussian case. This result can also be extended for matrices with variance profiles [8] where this large deviation result is new even in the Gaussian case. Lastly, we look at the case where the sharpness of our sub-Gaussian hypothesis is relaxed in the real case [2]. In this last case instead of a complete LDP, we have

upper and lower local large deviation bounds that coincide for large values of the largest eigenvalue. In other words :

$$\lim_{\delta\to 0}\limsup_{N\to\infty} N^{-1}\ln \mathbb{P}[|\lambda_{\max}-x|\leq\delta] = \lim_{\delta\to 0}\liminf_{N\to\infty} N^{-1}\ln \mathbb{P}[|\lambda_{\max}-x|\leq\delta] = -I(x)$$

for some function I and x large enough. In this case however we can show that $I(x) < I_{GOE}(x)$.

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Rate of Convergence for non-Hermitian random matrices and their products

Jonas Jalowy

(joint work with Friedrich Götze)

Similar to the Circular Law, the empirical spectral distribution (ESD) of products of $m \in \mathbb{N}$ independent non-Hermitian random matrices $X^{(1)}, \dots, X^{(m)}$ with independent entries will converge to a deterministic limiting distribution μ_∞^m as the size n of the matrix tends to infinity, see [1]. We investigate the rate of convergence of the ESD $\mu_n^m = \frac{1}{n}\sum_{j=1}^n \delta_{\lambda_j(n^{-m/2}\mathbf{X})}$ of the product $\mathbf{X} = X^{(1)}\cdots X^{(m)}$ to the power of the Circular Law in terms of a uniform Kolmogorov-like distance.

The optimal rate of convergence is determined by products of Ginibre matrices. We derive explicit sharp bounds for the distance of μ_∞^m to the mean spectral distribution $\bar{\mu}_n^m = \mathbb{E}\mu_n^m$. By saddle-point approximation of a double contour integral representation, we obtain that the optimal rate is given by

$$\sup_{R>0}|\bar{\mu}_n^m(B_R(0)) - \mu_\infty^m(B_R(0))| \asymp 1/\sqrt{n}. \tag{1}$$

Inside the bulk, this rate of convergence is even faster, interestingly we even have an exponential rate in the case of the Circular Law, i.e. $m = 1$, see [2].

Secondly, we consider products of matrices with independent entries satisfying $\max_{i,j,q,n} \mathbb{E}\big|X_{ij}^{(q)}\big|^{4+\delta} < \infty$. We exploit techniques from local laws in order to show that for every $\tau, Q > 0$ we have nearly optimal rate of convergence

$$\mathbb{P}\Big(\sup_{B_R(z_0)} |(\mu_n^m - \mu_\infty^m)(B)| \lesssim \log^2 n/\sqrt{n} \Big) \geq 1 - n^{-Q}, \tag{2}$$

where the supremum runs over all balls that avoid the edge and the origin. The proof relies on a smoothing, a random grid approximation and a pointwise concentration for the log-potentials, which can be extracted from the local law [4].

Moreover, a similar approach applies to other models, where logarithmic potentials can be controlled. We prove a smoothing inequality for complex measures that quantitatively relates the uniform Kolmogorov-like distance to the concentration of logarithmic potentials. From this we deduce the same rate of convergence for the empirical measure of the roots of Weyl random polynomials and we show nearly optimal rate of convergence to the Circular Law in the classical 2-dimensional Kolmogorov distance.

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Eigenvalues and eigenvectors of critical Erdős-Reńyi graphs

Antti Knowles

(joint work with Johannes Alt, Raphaël Ducatez)

An (undirected simple) graph on N vertices can be characterized by its adjacency matrix $A = A^* = (A_{xy})_{x,y=1}^N \in \{0,1\}^{N \times N}$, where A_{xy} is 1 if x and y are adjacent and 0 otherwise. The eigenvalues and eigenvectors of A play a central role in spectral graph theory. By making A random, we obtain a natural construction of sparse random matrices. Such a random operator can also be interpreted as a random Hamiltonian, describing a particle whose hopping is restricted to edges of a sparse graph.

Arguably the simplest model of a random graph is the *Erdős-Rényi graph* $G(N,p)$, where each edge $\{x,y\}$ of the complete graph is kept with probability p independently of the others. This means that the upper-triangular entries $(A_{xy} : x \leq y)$ are independent Bernoulli(p) random variables. Here the parameter $p \equiv p_N$ can depend on N, and the graph is sparse if $p_N \to 0$ as $N \to \infty$. Despite its simple definition, the eigenvalues and eigenvectors of the adjacency matrix of

$G(N,p)$ exhibit very complex behaviour, depending on the scale of the parameter p. To discuss this behaviour in more detail, we parametrize $p = d/N$.

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$ denote the eigenvalues of the (conveniently normalized) adjacency matrix $d^{-1/2}A$. The most basic question about the distribution of the eigenvalues is the convergence of the empirical eigenvalue measure $\frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i}(\mathrm{x})$, which, with high probability, converges in distribution to Wigner's semicircle law $\varrho(x)\mathrm{x}$ with $\varrho(x) = \frac{1}{2\pi}\sqrt{(4-x^2)_+}$ if and only if $d \to \infty$ as $N \to \infty$. Note that this statement says nothing about the fluctuations or even locations of individual eigenvalues.

The expected behaviour of the eigenvalues and eigenvectors changes dramatically around the *critical scale* $d \asymp \log N$. Informally, the critical scale corresponds to the threshold below which the concentration of the degree sequence $D_1, \ldots, D_N$ fails, where $D_x = \sum_y A_{xy}$ is the degree of vertex x. Thus, for[1] $d \gg \log N$ all degrees are with high probability close to d, while for $d \ll \log N$ the degrees differ wildly. A famous manifestation of this phenomenon is the connectivity threshold for $G(N, d/N)$: for $d/\log N > 1$ the graph is with high probability connected, and for $d/\log N$ the graph has with high probability isolated vertices. It is well known that as long as $d > 1$, the graph has a unique giant component (of size of order N), while all other components are of order $O(\log N)$. Throughout the following, we only consider the giant component, discarding the smaller components.

In the recent works [1, 2], we analyse the eigenvalues and eigenvectors of $G(N, d/N)$ on the critical scale. We uncover a phase diagram consisting of localized and delocalized phases, which is analogous to the well-known conjectured phase diagram of the three-dimensional Anderson model at weak disorder [3]. For its statement, we introduce the function $\Lambda : [2,\infty) \to [2,\infty)$ defined by $\Lambda(\alpha) = \frac{\alpha}{\sqrt{\alpha-1}}$.

Theorem 1. *Suppose that $\sqrt{\log N} \ll d \leq N/2$. Denote by $\alpha_x = D_x/d$ the normalized degree of x. Then for any $\nu > 0$ the following statements hold with probability at least $1 - O_\nu(N^{-\nu})$.*

(1) ***Eigenvalue locations.*** *Let $\sigma \in S_N$ satisfy $\alpha_{\sigma(1)} \geq \alpha_{\sigma(2)} \geq \cdots \geq \alpha_{\sigma(N)}$. Let $L = \max\{l \geq 1 : \alpha_{\sigma(l)} \geq 2 + o(1)\}$ denote the expected number of eigenvalues larger than $2 + o(1)$. Then for $1 \leq l \leq L$ we have $|\lambda_{l+1} - \Lambda(\alpha_{\sigma(l)})| \leq o(1)$ and $|\lambda_{L+2} - 2| = o(1)$.*

(2) ***Eigenvector delocalization.*** *The eigenvector $\mathbf{u}_i$ is completely delocalized (i.e. $\|\mathbf{u}_i\|_\infty^2 \leq N^{-1+o(1)}$) if $d \geq C\log N$ for some constant C or if the associated eigenvalue λ_i satisfies $o(1) \leq |\lambda_i| \leq 2 - o(1)$.*

(3) ***Eigenvector localization.*** *Let $\lambda \geq 2 + o(1)$ be an eigenvalue with eigenvector $\mathbf{u} \in \mathbb{S}^{N-1}$. Define the set of* vertices in resonance with λ,

$$\mathcal{W}(\lambda) = \left\{x : \alpha_x \geq 2, |\Lambda(\alpha_x) - \lambda| = o(1)\right\}.$$

For $r \geq 1$ define the set of resonant balls $\mathcal{B}_r(\lambda) = \bigcup_{x\in\mathcal{W}(\lambda)} B_r(x)$. Then for $r \gg 1$ we have $\sum_{x\notin\mathcal{B}_r(\lambda)} u(x)^2 = o(1)$.

(All factors $o(1)$ are in fact quantitative functions of d.)

[1] Here $a \ll b$ or $b \gg a$ means that $a = o(b)$.

This result indicates that at criticality $G(N, d/N)$ provides a model of a random operator that exhibits analogous characteristics to the Anderson model in three dimensions. However, it seems to be more amenable to a rigorous analysis, informally because it tends to be more delocalized. This means not only that we can rigorously establish the existence of a delocalized phase, but also that in the localized phase the eigenvectors are localized around not one localization centre (as in Anderson localization), but around many (the resonant vertices $\mathcal{W}(\lambda)$). Heuristically, this difference may be understood by noting that $G(N, d/N)$ is much better connected than $\mathbb{Z}^3/N\mathbb{Z}^d$, with a diameter logarithmic in N instead of polynomial.

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Harmonic Means of Wishart Random Matrices

Asad Lodhia

(joint work with Keith Levin and Elizaveta Levina)

Let $\mathbf{W}_i = n^{-1}\mathbf{X}_i\mathbf{X}_i^*$ be a sequence of independent complex Wishart random matrices where $i = 1, \ldots, N$ and $\mathbf{X}_i$ are $p \times n$ dimensional. In this talk we present properties of the following matrix average

$$\mathbf{H} = N\big(\mathbf{W}_1^{-1} + \cdots + \mathbf{W}_N^{-1}\big)^{-1},$$

in the limit that p/n converges to $\gamma \in (0, 1)$. In particular, we focus on its effectiveness in estimating the population covariance $\mathbb{E}[\mathbf{W}_i] = \Sigma$ in Frobenius norm, operator norm and leading eigenvector recovery compared to

$$\mathbf{A} = \frac{1}{N}\sum_{i=1}^{N}\mathbf{W}_i.$$

We proved [1] there is an integer $N^*(\gamma)$ for which $N \leq N^*(\gamma)$ implies asymptotically almost surely $\|\mathbf{H} - \Sigma\|_{\mathrm{op}} < \|\mathbf{A} - \Sigma\|_{\mathrm{op}}$ for Σ with small condition number. We reinterpreted this phenomenon in [2] showing the above result implies if $\mathbf{A} = T^{-1}\mathbf{X}\mathbf{X}^*$ (where T is now Nn and $T \geq 2p$), subdividing $\mathbf{X}$ arbitrarily into two matrices $\mathbf{X}_1$ and $\mathbf{X}_2$ with asymptotically equal aspect ratios and computing $\mathbf{H}$ yields a matrix asymptotically almost surely closer to Σ in operator norm when Σ has a small condition number. Counter intuitively, the limit of the normalized Frobenius norm error $p^{-1}\|\mathbf{H} - \Sigma\|_{\mathrm{F}}^2$ is the same as the error for $\mathbf{A}$ when $\Sigma = \mathbf{I}$. Further, when $\Sigma = \mathbf{I} + \theta\mathbf{v} \otimes \mathbf{v}$, in spite of $\mathbf{H}$ having a better operator norm error,

the leading eigenvector of $\mathbf{H}$ is never a better estimator for $\mathbf{v}$ than the leading eigenvector of $\mathbf{A}$. Finally, for the case of real Gaussian entries for $\mathbf{X}_i$ with $N = 2$, we computed

$$\mathbb{E}[\mathbf{H}|\mathbf{A}] = \frac{n(2n-p)}{(2n-1)(n+1)}\mathbf{A},$$

which by Rao-Blackwell Theorem implies a scalar multiple of $\mathbf{A}$ improves on $\mathbf{H}$ in both operator and Frobenius norm error. This suggests that the above properties are related to those arising in well-known shrinkage estimators of Σ.

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Limit laws of random matrices beyond the Dyson equation

Tobias Mai
(joint work with Sheng Yin)

Operator-valued free probability has proved to provide a powerful machinery for the study of limit laws of various random matrix models. An instructive example is the following basic model: let $(X_1^{(N)}, \ldots, X_n^{(N)})$ be n-tuples of independent GUE matrices of size $N \times N$, fix n hermitian matrices $b_1, \ldots, b_n \in M_k(\mathbb{C})$, and consider $\mathbb{X}^{(N)} := b_1 \otimes X_1^{(N)} + \cdots + b_n \otimes X_n^{(N)}$; we are interested in the asymptotic eigenvalue distribution of $\mathbb{X}^{(N)}$ as $N \to \infty$. Since $(X_1^{(N)}, \ldots, X_n^{(N)})$ converges in distribution to a tuple $(X_1, \ldots, X_n)$ of freely independent semicircular operators, say in some tracial W^*-probability space $(\mathcal{M}, \tau)$, the asymptotic eigenvalue distribution of $\mathbb{X}^{(N)}$ is given as the spectral distribution $\mu_{\mathbb{X}}$ of the selfadjoint operator $\mathbb{X} := b_1 \otimes X_1 + \cdots + b_n \otimes X_n$ in $(M_k(\mathbb{C}) \otimes \mathcal{M}, \mathrm{tr}_k \otimes \tau)$. In fact, $\mathbb{X}$ is an operator-valued semicircular element with the covariance map $\mathcal{L} : M_k(\mathbb{C}) \to M_k(\mathbb{C})$ given by $\mathcal{L}(b) = b_1 b b_1 + \cdots + b_n b b_n$; in particular, its matrix-valued Cauchy transform $\mathbf{G}_{\mathbb{X}} : \mathbb{H}^+(M_k(\mathbb{C})) \to \mathbb{H}^-(M_k(\mathbb{C}))$ is uniquely determined by the *Dyson equation* $\mathbf{G}_{\mathbb{X}}(b)^{-1} = b - \mathcal{L}(\mathbf{G}_{\mathbb{X}}(b))$ for all $b \in \mathbb{H}^+(M_k(\mathbb{C}))$. While there are efficient algorithms to numerically compute $\mathbf{G}_{\mathbb{X}}$ and hence $\mu_{\mathbb{X}}$, qualitative statements are harder to achieve. The regularity properties of $\mu_{\mathbb{X}}$ obviously depend on the choice of matrices $b_1, \ldots, b_n$ and thus on the corresponding map $\mathcal{L}$. For $\mathcal{L}$ which are *flat*, i.e., if there is $c > 0$ so that $c^{-1}\,\mathrm{tr}_k(b)\mathbf{1}_k \geq \mathcal{L}(b) \geq c\,\mathrm{tr}_k(b)\mathbf{1}_k$ holds for every positive semi-definite matrix $b \in M_k(\mathbb{C})$, a detailed analysis of the Dyson equation has led to a deep understanding of $\mu_{\mathbb{X}}$; see [1, 2]. In fact, [1] goes beyond the matricial case and addresses solutions $\mathbf{G} : \mathbb{H}^+(\mathcal{B}) \to \mathbb{H}^-(\mathcal{B})$ of the Dyson equation for positive maps $\mathcal{L} : \mathcal{B} \to \mathcal{B}$, where the von Neumann algebra $\mathcal{B}$ takes over the role of $M_k(\mathbb{C})$. Recently, by utilizing Voiculescu's L^2-theory of free differential

operators [4], we established absolute continuity of $\mu_{\mathbb{X}}$ and Hölder continuity of its cumulative distribution function – the former even under the less restrictive assumption that $\mathcal{L}$ is *nowhere rank decreasing*, meaning that there is no positive semi-definite $b \in M_k(\mathbb{C})$ such that $\operatorname{rank}(\mathcal{L}(b)) < \operatorname{rank}(b)$. Since this approach does not rely on the Dyson equation, it works for $(X_1^{(N)}, \ldots, X_n^{(N)})$ with more general limit laws, but is limited so far to the matricial setup. It is work in progress to close this gap with the aid of Shlyakhtenko's operator-valued extension [3] of [4].

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Positive solutions for large random linear systems

Jamal Najim

(joint work with Pierre Bizeul)

Consider a large linear system where A_n is a $n \times n$ matrix with independent real standard Gaussian entries, $\mathbf{1}_n$ is a $n \times 1$ vector of ones and with unknown the vector $\boldsymbol{x}_n = (x_k)_{k\in[n]}$ satisfying

$$\boldsymbol{x}_n = \mathbf{1}_n + \frac{1}{\alpha_n\sqrt{n}} A_n \boldsymbol{x}_n \ . \tag{1}$$

We investigate the (componentwise) positivity of the solution $\boldsymbol{x}_n$ depending on the scaling factor α_n as the dimension n goes to ∞. We prove the following:

Theorem. Let $\alpha_n \xrightarrow[n\to\infty]{} \infty$ and denote by $\alpha_n^* = \sqrt{2\log n}$. Let $\boldsymbol{x}_n = (x_k)_{k\in[n]}$ be the solution of (1).

(1) If there exists $\varepsilon > 0$ such that eventually $\alpha_n \le (1-\varepsilon)\alpha_n^*$ then

$$\mathbb{P}\left\{\min_{k\in[n]} x_k > 0\right\} \xrightarrow[n\to\infty]{} 0\,.$$

(2) If there exists $\varepsilon > 0$ such that eventually $\alpha_n \ge (1+\varepsilon)\alpha_n^*$ then

$$\mathbb{P}\left\{\min_{k\in[n]} x_k > 0\right\} \xrightarrow[n\to\infty]{} 1\,.$$

Such linear systems arise as solutions at equilibrium of large Lotka-Volterra systems of differential equations, widely used to describe large biological communities with interactions such as foodwebs for instance.

Consider for instance a given foodweb and denote by $\boldsymbol{x}_n(t) = (x_k(t))_{k\in[n]}$ the vector of abundances of the various species within the foodweb at time t. A standard way to connect the various abundances is via a Lotka-Volterra system of equations

$$\frac{dx_k(t)}{dt} = x_k(t)\left(1 - x_k(t) + \frac{1}{\alpha_n\sqrt{n}}\sum_{\ell\in[n]} A_{k\ell}x_\ell(t)\right) \qquad \text{for} \quad k \in [n]\,, \tag{2}$$

where the interactions $(A_{k\ell})$ can be modeled as random in the absence of any prior information. Here, the $A_{k\ell}$'s are assumed to be i.i.d. $\mathcal{N}(0,1)$. At the equilibrium $\frac{d\boldsymbol{x}_n}{dt} = 0$, the abundance vector $\boldsymbol{x}_n$ is a solution of (1) and a key issue is the existence of a *feasible* solution, that is a solution $\boldsymbol{x}_n$ where all the x_k's are positive.

In the domain of positivity of the solution $\boldsymbol{x}_n$, we establish that the Lotka-Volterra system of differential equations whose solution at equilibrium is precisely $\boldsymbol{x}_n$ is stable in the sense that its Jacobian

$$\mathcal{J}(\boldsymbol{x}_n) = \mathrm{diag}(\boldsymbol{x}_n)\left(-I_n + \frac{1}{\alpha_n\sqrt{n}}A_n\boldsymbol{x}_n\right)$$

has all its eigenvalues with negative real part with probability tending to one. We prove the following:

Corollary. Let $\boldsymbol{x}_n = (x_k)_{k\in[n]}$ be the solution of (1). Denote by $\boldsymbol{\ell}^+ = \limsup_{n\to\infty} \frac{\sqrt{2\log n}}{\alpha_n}$ and assume that $\boldsymbol{\ell}^+ < 1$. Denote by $\mathcal{S}_n$ the spectrum of $\mathcal{J}(\boldsymbol{x}_n)$ and let $\lambda \in \mathcal{S}_n$. Then

$$\max_{\lambda\in\mathcal{S}_n} \mathrm{Re}(\lambda) \;\le\; -(1-\boldsymbol{\ell}^+) + o_P(1)\,.$$

Our results shed a new light and complement the understanding of feasibility and stability issues for large biological communities with interaction.

A preprint arXiv:1904.04559 corresponding to this presentation is available.

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Gaussian Multiplicative Chaos and random matrix theory

Joseph Najnudel
(joint work with Reda Chhaibi)

In our joint paper with Chhaibi [2], we identify an *equality* between two objects arising from different contexts of mathematical physics: Kahane's Gaussian Multiplicative Chaos (GMC^γ) on the circle, and the Circular Beta Ensemble ($C\beta E$) from Random Matrix Theory.

The Gaussian Multiplicative Chaos has been introduced by Kahane [4] and studied in more detail by a number of authors, in particular Rhodes and Vargas [7]. This object has connection with theoretical physics, in particular turbulence and Liouville quantum gravity. It can be defined as follows: we start with a centered Gaussian field G on the unit disc, whose covariance structure is given by

$$\mathbb{E}[G(w)G(z)] = -2\log|1-w\bar{z}|$$

Then, for $\gamma > 0$, $r \in [0,1)$, we consider the random measure GMC_r^γ on the unit circle $\mathbb{U}$, such that

$$\frac{dGMC_r^\gamma}{d\mu^0}(z) := \frac{e^{\gamma G(rz)}}{\mathbb{E}[e^{\gamma G(rz)}]},$$

where μ^0 is the uniform probability measure on $\mathbb{U}$. Then, in the so-called *subcritical phase* $\gamma < 1$, there exists a random measure GMC^γ, which defines the Gaussian Multiplicative Chaos, such that for all smooth test functions f on the unit circle,

$$\int_{\mathbb{U}} f\ dGMC_r^\gamma \underset{r\to 1}{\longrightarrow} \int_{\mathbb{U}} f\ dGMC^\gamma$$

in L^1. In the *critical phase* $\gamma = 1$, letting $r \to 1$ gives an almost sure convergence to zero, and then another normalization is needed to define GMC^γ as a non-trivial

object. Aru, Powell and Sepúlveda [1] have proven the existence of a non-zero measure GMC^1 on $\mathbb{U}$, such that for a smooth test function f,

$$\frac{1}{1-\gamma}\int_{\mathbb{U}} f\ dGMC^{\gamma} \underset{r\to 1}{\longrightarrow} \int_{\mathbb{U}} f\ dGMC^{1}.$$

in probability. This defines the critical Gaussian Multiplicative Chaos, for which other constructions have been described, all giving random measures with the same law. In the *supercritical phase* $\gamma > 1$, some constructions have been made, which do not all agree with each other.

The random measure GMC^{γ} can be written as its total mass $GMC^{\gamma}(\mathbb{U})$, multiplied by a random probability measure μ^{γ}. We can then consider the random sequence $(\Phi_k)_{k\geq 0}$ of Orthogonal Polynomials on the Unit Circle (OPUC) with respect to this random probability measure: see [8] for a general introduction to the theory of OPUC. The Verblunsky coefficients associated to $(\Phi_k)_{k\geq 0}$ forms a random sequence $(\alpha_k)_{k\geq 0}$ of complex numbers of modulus strictly smaller than 1. A question one can ask concerns the joint distribution of these Verblunsky coefficients. In the case where $\gamma \leq 1$, i.e. the subcritical and the critical phase, we answer to this question in our paper with Chhaibi [2]: $(\alpha_k)_{k\geq 0}$ are independent, their arguments are uniform on $[0, 2\pi)$ and $|\alpha_k|^2$ is Beta-distributed with parameters 1 and $(k+1)/\gamma^2$. For $n \geq 1$, the measure μ^{γ} can be naturally approximated by a random measure μ_n^{γ} supported by n points: we take the measure whose finite sequence of Verblunsky coefficients is $(\alpha_0, \alpha_1, \ldots, \alpha_{n-2}, \alpha_{n-1}/|\alpha_{n-1}|)$. Now, Killip and Nenciu [5] have proven that the law of the support of the measure μ_n^{γ} corresponds to the *Circular β Ensemble* in random matrix theory: i.e. n random points on the unit circle whose joint probability distribution has density proportional to

$$\prod_{1\leq i<j\leq n} |z_i - z_j|^{\beta}$$

with respect to the uniform measure on $\mathbb{U}^n$. Here, the parameter β is given by $\beta = 2/\gamma^2$, or equivalently $\gamma = \sqrt{2/\beta}$: note that $\beta > 2$ in the subcritical phase and $\beta = 2$ in the critical phase. In the critical phase $\gamma = 1$, $\beta = 2$, the support of μ_n^{γ} is then distributed like the *Circular Unitary Ensemble*, i.e. the eigenvalues of a Haar-distributed unitary matrix of order n.

The construction described just above gives a way to recover the Circular Beta Ensemble of all orders n from the Gaussian Multiplicative Chaos. Moreover, the total measure $GMC^{\gamma}(\mathbb{U})$ has an explicit expression in terms of the Verblunsky coefficients $(\alpha_k)_{k\geq 0}$, which implies that it is distributed like the negative power $-\gamma^2$ of an exponential random variable. This solves a conjecture by Fyodorov and Bouchaud [3], proven in another way by Remy [6]. Our main result also provides an explicit description of the distribution of the Fourier coefficients of the Gaussian Multiplicative Chaos, via their general expression in terms of the Verblunsky coefficients.

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Rational functions of Wigner matrices and scattering in quantum dots

Yuriy Nemish

(joint work with László Erdős and Torben Krüger)

We consider a random matrix model for scattering in quantum dots in the regime when the number of coupling channels is much smaller or comparable to the size of the quantum dot. The model was introduced by Beenakker [1] and defines the *scattering matrix* of a quantum dot of size M coupled to electron reservoirs via $2N$ channels as

$$S(E) := I - 2\pi \mathrm{i}\, W^*(E \cdot I - H + \mathrm{i}\pi W W^*)^{-1} W \in \mathbb{C}^{2N \times 2N},$$

where H is an $M \times M$ Wigner matrix, W is an $M \times 2N$ matrix with i.i.d. entries and $E \in \mathbb{R}$ is the energy parameter. We split the scattering matrix into reflection and transmission components, $S(E) = \begin{pmatrix} R & T' \\ T & R' \end{pmatrix}$ with $T \in \mathbb{C}^{N \times N}$, and we study the distribution of the *transmission eigenvalues* of the model, i.e., the eigenvalues of TT^*, in the limit $M, N \to \infty$.

The question was previously investigated in [1, 2] in the regime $\phi := N/M \to 0$ for the model with Gaussian entries relying heavily on phenomenological and non-rigorous arguments. We revisited this problem [3] in the setting when $\phi \to 0$ or $N \sim M$, allowing general distributions for the matrix elements of the Hamiltonian of the quantum dot and the coupling matrix.

To access this level of generality we developed the theory of *global and local laws* on the spectral density of a large class of *noncommutative rational expressions in large random matrices* with i.i.d. entries. We provided the characterization of the limiting density of states for the transmission eigenvalues in terms of the solution to the Dyson equation in the most general setting. For arbitrary parameters of the model, we obtained the explicit formulas for the density of states in the regime $\phi \to 0$, as well as the asymptotic behavior at the singularities for general $\phi \in (0, 1]$.

In particular, the rigorous proof of the limiting transmission eigenvalue density from [1, 2] can be obtained from our result by making the special choice of model parameters. The comprehensive study of the (approximated) Dyson equations related to the above model made use of the analysis conducted in the previous works [4, 5].

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On the operator norm of non-commutative polynomials in deterministic matrices and iid GUE matrices

Félix Parraud

(joint work with Benoît Collins, Alice Guionnet)

Let $X^N = (X_1^N, \dots, X_d^N)$ be a d-tuple of $N \times N$ independent GUE random matrices and Z^{NM} be any family of deterministic matrices in $\mathbb{M}_N(\mathbb{C}) \otimes \mathbb{M}_M(\mathbb{C})$. Let P be a self-adjoint non-commutative polynomial, seminal work of Voiculescu [1] shows that the empirical measure converges towards a deterministic measure defined thanks to the free probability theory. Let now f be a smooth function, the main technical result that we present in this talk is a precise bound of the difference between the expectation of

$$\frac{1}{MN} \mathrm{Tr}_{\mathbb{M}_N(\mathbb{C})} \otimes \mathrm{Tr}_{\mathbb{M}_M(\mathbb{C})} \left(f(P(X^N \otimes I_M, Z^{NM})) \right) \ ,$$

and its limit when N goes to infinity. If f is six times differentiable, we show that it is bounded by $M^2 \|f\|_{\mathcal{C}^6} N^{-2}$ uniformly in M, N and f. As a corollary we obtain a new proof of a result of Haagerup and Thorbjørnsen in [2] later developed by Male in [3] which stated sufficient conditions for the operator norm of a polynomial evaluated in (X^N, Z^{NM}, Z^{NM*}) to converge almost surely towards its free limit. Restricting ourselves to polynomials in independent GUE matrices, we give concentration estimates on the largest eigenvalue of those polynomials around their free limit. A direct consequence of those inequalities is that there exists some $\beta > 0$, that we will define in the paper, such that for any $\varepsilon_1 < (3+\beta)^{-1}$ and $\varepsilon_2 < 1/4$, almost surely for N large enough,

$$-\frac{1}{N^{\varepsilon_1}} \leq \|P(X^N)\| - \|P(x)\| \leq \frac{1}{N^{\varepsilon_2}} .$$

Finally if X^N and Y^{M_N} are independent and $M_N = o(N^{1/3})$, then almost surely, the norm of any polynomial in $(X^N \otimes I_{M_N}, I_N \otimes Y^{M_N})$ converges almost surely towards its free limit. This result is an improvement of a Theorem of Pisier in [4], who was himself using estimates from Haagerup and Thorbjørnsen, where M_N had size $o(N^{1/4})$.

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Eigenvalues of random non-Hermitian matrices and randomly coupled differential equations

David Renfrew

(joint work with Torben Krüger, Laszlo Erdős)

We consider the randomly coupled linear differential equation

$$\partial_t u_t = -u_t + gXu_t \tag{1}$$

with u_t an N dimensional vector, X a random matrix, and g a positive coupling constant. We compute properties of the solution, u_t, by studying the trace of functions of X multiplied by functions of its adjoint. Our main result shows that when the differential equation is critically coupled, the solution decays polynomially, at a slower rate than in the analogous Hermitian case. Equation (1) has been used for modeling in theoretical neuroscience and mathematical ecology, see for instance [1, 2, 6, 7, 8]. However, in these works the often unrealistic assumption that the entries have identical distributions is made. Such an assumption can not incorporate spatial information or different species/cells types.

In [3, 4], we incorporate relevant features into mathematical models by allowing the entries of X to have different variances, as well as non-trivial correlations. In particular, large blocks of the matrix can be zero. We give particular attention to the case where the ij and ji entry are correlated, and the entries are otherwise independent. In such a model, the correlation coefficient corresponds to suppression or promotion of reciprocal connections.

The first step in understanding the behavior of u_t is determine the location of the spectrum of X, then the parameter g can be tuned to balance the two

terms on the right side of (1). We accomplish this by consider the Hermitized resolvent $R(z,\zeta) = \begin{pmatrix} -z & X-\zeta \\ X^*-\zeta & -z \end{pmatrix}$, and show that for large N the spectrum of X concentrates on a deterministic set, which we call the *self-consistent spectrum*. The complement of the self-consistent spectrum is determined by the ζ for which the diagonal entries of $R(z,\zeta)$ vanish as z is taken to zero.

Our main tool in understanding the resolvent, $R(z,\zeta)$, is its associated matrix Dyson equation (MDE). The optimal local law in [5] states that: given a centered self-adjoint random matrix Z and a deterministic matrix A, satisfying certain conditions, the matrix $(Z-A)^{-1}$ approximately solves the equation $-M^{-1} = \Sigma(M) + A$, where $\Sigma(M) = \mathbb{E}[ZMZ]$.

Having a good understanding of the location of the spectrum we then consider quantitative features of the solution to (1). When starting with a random initial condition uniformly distributed on the unit sphere the solution, after taking expectation with respect to the initial conditions, is given by

$$\mathbb{E}_{u_0}\|u_t\|^2 = \mathrm{tr}_N(e^{(gX-I)t}e^{(gX^*-I)t}). \tag{2}$$

For a large class of covariances between the entries of X, we show the following universal decay rate:

$$\mathbb{E}_{u_0}\|u_t\|^2 \sim \frac{e^{2t[g\zeta^*-1]}}{\sqrt{t}},$$

where ζ^* is the rightmost point of the self-consistent spectrum of X. In particular, when g is chosen to be $1/\zeta^*$, the solution decays like $t^{-1/2}$. The analogous solution when X is taken to be a random Hermitian matrix, decays like $t^{-3/2}$.

In order compute the quantity in (2), we consider the more general problem computing $\mathrm{tr}_N(f(X)g(X^*))$, for analytic test functions f and g and show that with high probability

$$\mathrm{tr}_N f(X)g(X^*) \sim \left(\frac{1}{2\pi \mathrm{i}}\right)^2 \oint_\gamma dz_1 \oint_{\overline{\gamma}} d\overline{z}_2\, f(z_1)g(\overline{z}_2) \times K(z_1,\overline{z}_2) \tag{3}$$

Where the kernel, $K(z_1,\overline{z}_2)$, is computed from just the covariance structure of the entries of X.

This theorem follows by Cauchy's integral formula and proving the convergence of the trace of the product of the resolvents, $\mathrm{tr}_N((X-\zeta_1)^{-1}(X^*-\zeta_2)^{-1})$ to the corresponding kernel. In order to compute this product, we introduce the following novel linearization scheme which allows the products of resolvents to be computed from the knowledge of the individual resolvents $(X-\zeta_1)^{-1}$ and $(X^*-\zeta_2)^{-1}$.

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A general beta crossover ensemble

Brian Rider
(joint work with Jose A. Ramírez)

The general β soft and hard edge laws are known to be described by the Stochastic Airy and Bessel Operators. These read:

$$\mathrm{SAO}_\beta = -\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}} b'(x), \tag{1}$$

and

$$\mathrm{SBO}_{\beta,a} = -\, e^{(a+1)x + \frac{2}{\sqrt{\beta}} b(x)} \frac{d}{dx} e^{-ax - \frac{2}{\sqrt{\beta}} b(x)} \frac{d}{dx}. \tag{2}$$

In both cases $x \mapsto b(x)$ is a standard Brownian motion, and both operators act on $\mathbb{R}_+$ with a Dirichlet boundary condition at the origin. See [3] and [4]. These operator limits were first established via the well known random Jacobi β-Hermite and β-Laguerre ensembles. Extending the picture to more general log-gases, [2] and [5] established universality for SAO_β and $\mathrm{SBO}_{\beta,a}$ when the underlying potential is a convex polynomial.

Moving away for convex potentials one can find "irregular" behavior. No such model has been analyzed outside of $\beta = 2$. Here we consider a general β version of a Laguerre type ensemble introduced by Claeys-Kuijlaars which exhibits a crossover behavior at its lower edge [1]. There they find a limiting correlation kernel built out of the ψ-functions for Painlevé II, which interpolates between the usual Airy and Bessel kernels that describe the $\beta = 2$ soft and hard edges.

The model can be defined by prescribing a joint density for $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}_+^n$ proportional to

$$\exp\left(-\frac{\beta N}{4} \sum_{j=1}^n (\lambda_j - 2)^2\right) \prod_{j=1}^n \lambda_j^{\frac{\beta}{2}(a+1)-1} \prod_{j<k} |\lambda_j - \lambda_k|^\beta. \tag{3}$$

Here $\beta > 0$, $a > -1$, and N is defined by the double scaling relation $\frac{N}{n} = 1 + sn^{-2/3}$. There is again a random Jacobi matrix T_n with (3) as its eigenvalue density. Importantly though its Gibbsian prescription possesses a non-convex Hamiltonian. Still, we establish the convergence

$$\lim_{n\to\infty} n^{2/3} T_n \Rightarrow \tau_{\beta,a,s},$$

to a differential operator that interpolates between SAO_β and $\mathrm{SBO}_{\beta,a}$.

To explain, first introduce a version of the "soft-edge" diffusion $t \mapsto z_t = z_{t;\beta,s}$: with $\beta > 0$ and $s \in \mathbb{R}$,

$$dz_t = \frac{2}{\sqrt{\beta}} db_t + (s + t - z_t^2)dt, \quad z_0 = 0, \tag{4}$$

and killed at the explosion time $\tau_{-\infty} = \inf\{t > 0 : z_t = -\infty\}$. Note that (4) is the Riccati transform of SAO_β (with s playing the role of the spectral parameter). Introduce as well $t \mapsto \bar{z}_t$, the process absolutely continuous to z_t over all short fields via,

$$\frac{d\bar{P}}{dP}\Big|_{\mathcal{F}_{[0,T]}} = e^{\alpha \int_0^T z_t dt} \mathbf{1}_{\{T<\tau_{-\infty}\}}, \quad \alpha = (\frac{\beta}{2}(a+1) - 1).$$

With these ingredients we can state:

Theorem. *The "hard-meets-soft-edge" operator has the form*

$$\tau_{\beta,a,s} = -\frac{d^2}{dt^2} + Z_t^2 - Z_t', \quad \textit{acting on } L^2(\mathbb{R}_+).$$

Here Z_t is characterized to the following extent:

$$\begin{aligned} &Z_t \textit{ is Markovian}, Z_0 = 0 \textit{ and } Z_t \to +\infty \textit{ a.s. as } t \to +\infty, \\ &Z_t \textit{ has the same bridge measures as } \bar{z}_t. \end{aligned} \tag{5}$$

There remains an important uniqueness question - there is no shortage of diffusions with the same bridges as $\bar{z}$. Certainly any h-transform,

$$dZ_t = \frac{2}{\sqrt{\beta}} db_t + \left(s + t - Z_t^2 + \frac{4}{\beta} \frac{h'(t+s, Z_t)}{h(t+s, Z_t)} \right) dt, \tag{6}$$

with h a (non-negative) space-time harmonic function for the $\bar{z}$ process,

$$0 = \frac{\partial h}{\partial t} + \frac{2}{\beta} \frac{\partial^2 h}{\partial z^2} + \left(t - z^2\right) \frac{\partial h}{\partial z} + \alpha z h, \tag{7}$$

is a candidate. We claim that the boundary condition "at infinity" in line one of (5) pins everything down.

Conjecture. *The space-time Martin boundary of $\bar{z}_t$ has a unique point at infinity, and Z_t is defined by* (6) *where $h = h_{\beta,a}$ is the corresponding solution to* (7).

When there is no creation of mass ($\alpha = 0$) things take a particularly nice form:

Corollary. *Granted the conjecture, when $\alpha = 0$ it holds that $h_\beta(z,t) = P_{z,t}(\tau_{-\infty}(z) = +\infty)$. That is, in this case Z_t is simply* (4) *conditioned never to explode.*

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Local law and CLT for linear eigenvalue statistics of free addition of matrices

Kevin Schnelli

(joint work with Zhigang Bao, László Erdős, Yuanyuan Xu)

Consider the sum of two large Hermitian matrices with a Haar unitary conjugation bringing them into a general relative position: Let A_N and B_N be N by N deterministic symmetric matrices. Let U_N be a unitary random matrix distributed according to Haar measure on $U(N)$. We are then interested in the random matrix

$$H_N = A_N + U_N B_N U_N^* \,. \tag{1}$$

The matrix H_N is the simplest version of a *unitary multi-matrix model* and is often referred to as the *free sum* of A_N and B_N. If the empirical eigenvalue distributions of A_N and B_N are asymptotically, as N tends to infinity, given by probability measures μ_α and μ_β, then the empirical eigenvalue distribution of H_N is asymptotically given by the *free additive convolution* of μ_α and μ_β. This result was first established in a seminal work of Voiculescu [4].

Given this result it is natural to ask for the smallest possible scale so that the local eigenvalue distribution on that scale converges as N tends to infinity. Put differently, what is the smallest scale on which the Green function or resolvent of H_N is determined by the free additive convolution and a *local law* holds.

In the first part of the talk, we reported on joint work with Bao and Erdős [1, 2] where we established an optimal local law for the Green function in the regular bulk, the part of the spectrum where the free convolution measure admits a continuous and bounded density. We established the convergence of the Green function to the Stieltjes transform of the free additive convolution measure down to local scales just above the typical eigenvalue spacing at a speed order N^{-1}.

In the second part of the talk, we reported on on-going work with Bao and Xu [3] on fluctuations of the linear eigenvalue statistics for the ensemble (1). We establish a CLT for the linear eigenvalue statistics on mesoscopic scales within the regular bulk and show that the limiting variance agrees with the (universal) variance from the Gaussian unitary ensemble.

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Fluctuations in the Circular Law: Central Limit Theorem for the Linear Statistics of non-Hermitian Random Matrices

Dominik Schröder

(joint work with Giorgio Cipolloni, László Erdős)

We consider $n \times n$ non-Hermitian random matrices X with complex, independent, identically distributed centred entries, and eigenvalues $\sigma_1, \dots, \sigma_n$, and denote their linear statistics by $L_n(f) := \sum_{k=1}^n f(\sigma_k)$. Girko's circular law [5] states that L_n converges, in probability, to the uniform distribution on the unit disk, i.e.

$$L_n(f) \to \frac{1}{\pi} \int_{\mathbf{D}} f(z) \, \mathrm{d}^2 z, \quad \mathbf{D} := \{z : |z| < 1\}.$$

Our main result is the following central limit theorem for $L_n(f)$.

Theorem 1. *Assume that the entries x_{ab} are independent and distributed as $x_{ab} \overset{\mathrm{d}}{=} n^{-1/2}\chi$ with $\mathbf{E}\,\chi = 0 = \mathbf{E}\,\chi^2$, $\mathbf{E}|\chi|^2 = 1$ and fourth cumulant $\kappa_4 := \mathbf{E}|\chi|^4 - 2$, such that high moments of χ exist. Then for test functions $f \in H^{2+\epsilon}(\mathbf{C})$ the centred linear statistics converge $L_n(f) - \mathbf{E}\,L_n(f) \Rightarrow L(f)$ to a centred Gaussian with*

$$\mathbf{E}|L(f)|^2 = \frac{1}{4\pi}\|\nabla f\|^2_{L^2(\mathbf{D})} + \frac{1}{2}\|f\|^2_{\dot{H}^{1/2}(\partial\mathbf{D})} + \kappa_4 \left| \frac{1}{\pi} \int_{\mathbf{D}} f(z) \, \mathrm{d}^2 z - \frac{1}{2\pi} \int_0^{2\pi} f(e^{\mathrm{i}\theta}) \, \mathrm{d}\theta \right|.$$

Previously the analogue of Theorem 1 was known for analytic test functions [7], or entry distributions χ which are either Gaussian [8], or at least match the Gaussian up to the first four moments [9]. In particular, the dependence on the fourth cumulant κ_4 escaped all previous works since its coefficient vanishes in the case of analytic functions.

Our proof relies on Girko's Hermitisation formula

$$L_n(f) = -\frac{1}{4\pi} \int_{\mathbf{C}} \Delta f(z) \int_0^\infty \Im \operatorname{Tr} G^z(i\eta) \, \mathrm{d}\eta \, \mathrm{d}^2 z, \tag{1}$$

which relates the spectrum of the non-Hermitian matrix X to the resolvent

$$G^z(\mathrm{i}\eta) := (H^z - \mathrm{i}\eta)^{-1}, \quad H^z := \begin{pmatrix} 0 & X - z \\ (X-z)^* & 0 \end{pmatrix}$$

of the Hermitian matrix H^z. The regime $\eta \le n^{-1-\epsilon}$ in (1) can be controlled via smoothing estimates [10] which, in high probability, allow to exclude singular

values $|\lambda_i^z| \le n^{-1-\epsilon}$ of $X-z$. For the regime $\eta \ge n^{-1+\epsilon}$ in (1) so called *local laws* [3, 1] for the resolvent G^z are applicable which provide a deterministic approximation $m^z(\mathrm{i}\eta)$ of the resolvent,

$$\left|\frac{1}{n}\operatorname{Tr}\big[G^z(\mathrm{i}\eta) - m^z(\mathrm{i}\eta)\big]\right| \le \frac{n^\epsilon}{n\eta}, \tag{2}$$

that can be obtained by solving the cubic *Dyson equation*

$$-\frac{1}{m^z(\mathrm{i}\eta)} = m^z(\mathrm{i}\eta) + \mathrm{i}\eta + \frac{|z|^2}{m^z(\mathrm{i}\eta) + \mathrm{i}\eta}, \quad \Im m^z(\mathrm{i}\eta) > 0, \quad \eta > 0.$$

The local law (2) allows to explicitly obtain a Wick-type theorem for resolvents.

Proposition 2. *Assume that $\eta_1, \dots, \eta_k \ge n^{-1+\epsilon}$, and that $z_1, \dots, z_k \in \mathbf{C}$ are such that $|z_k - z_l| \ge n^{-\delta}$. Then the resolvents $G^{z_1}(\mathrm{i}\eta_1), \dots, G^{z_k}(\mathrm{i}\eta_k)$ satisfy*

$$\mathbf{E}\prod_i \operatorname{Tr}\big[G^{z_i}(\mathrm{i}\eta_i) - \mathbf{E}\,G^{z_i}(\mathrm{i}\eta_i)\big] \approx \sum_{P\in\mathrm{Pairings}([k])} \prod_{\{i,j\}\in P} \prod_{\{i,j\}\in P} \frac{V_{i,j} + \kappa_4 U_i U_j}{2}$$

for some explicit $V_{i,j} = V(m^{z_i}, m^{z_j})$ and $U_i = U(m^{z_i})$.

For the intermediate regime $\eta \in [n^{-1-\epsilon}, n^{-1+\epsilon}]$ in (1) we establish the following asymptotic independence of resolvents.

Proposition 3. *Assume that $\eta_1, \dots, \eta_k \in [n^{-1-\epsilon}, n^{-1+\epsilon}]$, and that $z_1, \dots, z_k \in \mathbf{C}$ are such that $|z_k - z_l| \ge n^{-\delta}$. Then the resolvents are asymptotically independent,*

$$\mathbf{E}\prod_{i=1}^k \operatorname{Tr} G^{z_l}(\mathrm{i}\eta_l) \approx \prod_{i=1}^k \mathbf{E}\operatorname{Tr} G^{z_l}(\mathrm{i}\eta_l).$$

The main technical input for both propositions is a local law $G^{z_1}G^{z_2} \approx M^{z_1,z_2}$ for products of resolvents with mesoscopically separated $|z_1 - z_2| \ge n^{-\delta}$. This local law allows to bound the *singular vector overlaps* which in turn allows to show that along the stochastic flow generated by adding a small time-dependent Gaussian component to the original matrix, the small singular values $\lambda_i^{z_1}, \lambda_j^{z_2}$ become asymptotically independent. This step relies on the analysis of the *Dyson Brownian motion* developed in the recent years [2, 4, 6].

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Some applications of the Ky Fan inequality to random (and almost periodic) operators

Mira Shamis

The talk is based on the recent preprint [4].

In the recent works [2, 3] Hislop and Marx studied the dependence of the integrated density of states of random Schrödinger operators on the distribution of the potential. The first work, [2], is devoted to the discrete case: $H = -\Delta + V$ acts on $\ell^2(\mathbb{Z}^d)$ via

$$(H\psi)(n) = \sum_{m \text{ is adjacent to } n} (\psi(n) - \psi(m)) + V(n)\psi(n),$$

where $\{V(n)\}_{n\in\mathbb{Z}^d}$ are independent identically distributed random variables sampled from a probability distribution μ. Let ρ_μ be the density of states measure corresponding to the operator H. Hislop and Marx showed that if $\operatorname{supp}\mu, \operatorname{supp}\tilde{\mu} \subset [-A, A]$, then

$$\mathrm{d_{KR}}(\rho_\mu, \rho_{\tilde{\mu}}) \le C_A \mathrm{d_{KR}}(\mu, \tilde{\mu})^{\frac{1}{1+2d}},$$

where $\mathrm{d_{KR}}$ is the Kantorovich-Rubinstein metric, and $C_A > 0$ is a constant that depends only on A. In the same setting we prove the following.

Theorem 1. *For arbitrary probability measures $\mu, \tilde{\mu}$*

$$\mathrm{d_{KR}}(\rho_\mu, \rho_{\tilde{\mu}}) \le \mathrm{d_{KR}}(\mu, \tilde{\mu}).$$

Note that the power 1 as well as the prefactor 1 are optimal in general.

The work [3] is devoted to the continual setting: $H = -\Delta + V$ is a random Schrödinger operator on $L^2(\mathbb{R}^d)$, where

$$V(x) = \sum_{n\in\mathbb{Z}^d} v_n u(x-n),$$

where $\{v_n\}_{n\in\mathbb{Z}^d}$ are independent identically distributed random variables sampled from a probability distribution μ; $u \in C_c(\mathbb{R}^d)$ is real-valued continuous compactly supported function. Let ρ_μ be the density of states measure corresponding to the operator H. Hislop and Marx prove the following. Assume that $0 \le u(x) \le 1$, $u \in C_c^k(\mathbb{R}^d)$ with compact support in a neighborhood of the origin, where the degree of the regularity $k > \max\left\{d-2; \frac{4+2d}{3}\right\}$ depends on the dimension. Assume that

$\operatorname{supp}\mu, \operatorname{supp}\tilde{\mu} \subset [-A, A]$. Then, there exists $\delta = \delta(d) > 0$ and $M(d) = M \in \mathbb{N}$ such that for any $\mathrm{d_{KR}}(\mu, \tilde{\mu}) < \delta$ and all $f \in C_c^M(\mathbb{R}), \operatorname{supp} f \in [-r, r], r \geq 1$

$$\left| \int f d\rho_\mu - \int f d\rho_{\tilde{\mu}} \right| \leq C_{k,r,M} \mathrm{d_{KR}}(\mu, \tilde{\mu})^{\frac{1}{1+d}},$$

where $C_{k,r,M} > 0$ is a constant that depends on k, r, and M.

In this setting we prove the following.

Theorem 2. *Suppose that* $\operatorname{supp}\mu, \operatorname{supp}\tilde{\mu} \subset \mathbb{R}_+$ *and* $u \geq 0$. *If* $\alpha > \frac{d}{2} - 1$, *then*

$$\left| \int f \left(\frac{1}{(1+E)^\alpha} \right) (d\rho_\mu(E) - d\rho_{\tilde{\mu}}(E)) \right| \leq C(d, u, \alpha) \mathrm{d_{KR}}(\mu, \tilde{\mu}), \tag{1}$$

for any 1*-Lipschitz function* f *for which* $\int f \left(\frac{1}{(1+E)^\alpha} \right) d\rho_\mu(E)$ *converges.*

In general, the condition $\alpha > \frac{d}{2} - 1$ can not be relaxed.

The proofs of Theorem 1 and Theorem 2 are based on the Ky Fan inequality.

We also present an application of the Ky Fan inequality to quasiperiodic operators, which is part of an ongoing work in progress, joint with A. Avila, Y. Last, and Q. Zhou.

Let $H_{\alpha,\theta} = -\Delta + V_{\alpha,\theta}$ on $\ell^2(\mathbb{Z})$, where

$$V_{\alpha,\theta}(n) = \phi(2\pi\alpha n + \theta), \quad \theta \in [0, 2\pi),$$

$\phi : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function. Denote

$$\|\phi\|_{\mathrm{Lip}} = \max \frac{|\phi(x) - \phi(y)|}{|x - y|}.$$

Denote by ρ_α the θ-averaged density of states measure corresponding to $H_{\alpha,\theta}$.

Theorem 3. *For any* $\alpha, \tilde{\alpha} \in \mathbb{R}$

$$\mathrm{d_{KR}}(\rho_\alpha, \rho_{\tilde{\alpha}}) \leq \sqrt{4\pi \|\phi\|_{\mathrm{Lip}} |\alpha - \tilde{\alpha}|}.$$

For comparison, Avron, van Mouche and Simon [1] showed that

$$\mathrm{d_H}(\operatorname{supp}\rho_\alpha, \operatorname{supp}\rho_{\tilde{\alpha}}) \leq 6\sqrt{\|\phi\|_{\mathrm{Lip}} |\alpha - \tilde{\alpha}|}\ ,$$

where $\mathrm{d_H}$ is the Hausdorff metric.

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Universality for random band matrices

Tatyana Shcherbina
(joint work with Mariya Shcherbina)

Random band matrices (RBM) are natural intermediate models to study eigenvalue statistics and quantum propagation in disordered systems, since they interpolate between mean-field Wigner matrices (Hermitian or real symmetric matrices with i.i.d. random entries) and random Schrödinger operators, which have only a random diagonal potential in addition to the deterministic Laplacian on a box in $\mathbb{Z}^d$. RBM can be defined as Hermitian (or real symmetric) $N \times N$ matrices H whose entries H_{ij} are independent random variables with mean zero such that

$$\mathbf{E}\{H_{ij}H_{lk}\} = \delta_{ik}\delta_{jl}J_{ij}, \tag{1}$$

where J_{ij} is a function which is zero or decrease sufficiently fast for $|i-j| \geq W$.

The key physical parameter of RBM is the localization length, which describes the length scale of the eigenvectors. The system is called delocalized if in the bulk of spectrum the localization length is comparable with the system size, and it is called localized otherwise.

The questions of the localization length are closely related to the universality conjecture of the random matrix theory. According to the Wigner – Dyson universality conjecture, the local behaviour of the eigenvalues does not depend on the matrix probability law (ensemble) and is determined only by the symmetry type of matrices. For example, the conjecture states that for Hermitian random matrices in the bulk of the spectrum

$$\frac{1}{(N\rho(E))^k} R_k\left(E + \frac{\xi_1}{\rho(E)\,N}, \dots, E + \frac{\xi_k}{\rho(E)\,N}\right) \xrightarrow{w} \det\left\{\frac{\sin\pi(\xi_i - \xi_j)}{\pi(\xi_i - \xi_j)}\right\}_{i,j=1}^{k}, \tag{2}$$

as $N \to \infty$, where R_k is a k-point correlation function (k is fixed) and $\rho(E)$ is a density of states. This means that the limit coincides with that for GUE.

One of the main long standing problem in the field is to prove a fundamental physical conjecture [4] formulated in late 80th. It states that the eigenvectors of $N \times N$ RBM are completely delocalized and the local spectral statistics governed by the Wigner-Dyson statistics (2) for large bandwidth $W \gg \sqrt{N}$, and by Poisson statistics for a small $W \ll \sqrt{N}$ (with exponentially localized eigenvectors).

Despite numerous attempts, so far there were only a few partial results about the local spectral statistics of RBM on the mathematical level of rigour. Localization of eigenvectors in the bulk of the spectrum was first shown for $W \ll N^{1/8}$ [5]. On the other side, by a development of the classical Erdős-Schlein-Yau approach to Wigner matrices, there were obtained some results where the weaker form of delocalization was proved: $W \gg N^{6/7}$ [2] , $W \gg N^{4/5}$ [3]. The combination of this approach with the new ideas based on quantum unique ergodicity were developed in [1] to obtain bulk universality and delocalization in the range $W \gg N^{3/4}$.

A completely different approach to the problem is based on the supersymmetric (SUSY) techniques. SUSY allows to obtain a representation for the main spectral

characteristics (such as density of states, correlation functions, etc) as an integrals containing both complex and Grassmann variables. The rigorous analysis of such integral representation usually is very complicated, but the specific form of the covariance J in (1) allows to combine SUSY techniques with a transfer matrix approach. This allows to prove universality (2) for the second correlation function of some specific types of Gaussian 1d RBM up to the optimal scale $W \gg \sqrt{N}$, which is the main result of the presentation (see [9]).

The supersymmetric transfer matrix formalism in this context was first suggested by Efetov, but its rigorous application to RBM is quite difficult due to the complicated structure of the corresponding transfer operator. So during the last years the techniques which led to the main result were developed step by step. First we applied it in [6] to obtain the precise estimate for the density of state. Then in [7] to prove the transition around $W \sim \sqrt{N}$ on the level of characteristic polynomials (see also [10]). The next crucial step was done in [8], where we applied the techniques to the so-called sigma-model approximation, which is often used by physicists to study complicated statistical mechanics systems. The mechanism of the crossover for the sigma-model is essentially the same as for the correlation functions of characteristic polynomials, but the structure of the transfer operator is much more complicated: it is a 6×6 matrix whose entries are kernels depending on two unitary 2×2 and two hyperbolic 2×2 matrices. In the case of the second correlation function the size of the transfer operator $\mathcal{K}$ becomes 70×70, and so the spectral analysis of $\mathcal{K}$ provides serious structural problems. The key idea is to prove that the main part of $\mathcal{K}$ is still 6×6 matrix kernel appeared in the transfer operator corresponding to the sigma-model approximation.

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Regularity of non-commutative distributions and random matrices

Roland Speicher
(joint work with Tobias Mai, Sheng Yin)

In free probability theory we like to go to the limit. For n selfadjoint (random) $N \times N$-matrices $X_1^{(N)}, \ldots, X_n^{(N)}$ and n operators $X_1, \ldots, X_n$, living in some C^*- or von Neumann algebra, equipped with a trace τ, we say that $(X_1^{(N)}, \ldots, X_n^{(N)})$ converges in distribution to $(X_1, \ldots, X_n)$ if we have the convergence of all moments (where tr denotes the normalized trace of matrices)

$$\lim_{N \to \infty} \mathrm{tr}[X_{i_1}^{(N)} \cdots X_{i_k}^{(N)}] = \tau(X_{i_1} \cdots X_{i_k}).$$

A basic example for this is the convergence in distribution of independent GUE to free semicircular variables.

The collection $\{\tau(X_{i_1} \cdots X_{i_k}) \mid k \in \mathbb{N}, 1 \leq i_1, \ldots, i_k \leq n\}$ of all moments constitutes the non-commutative distribution of $(X_1, \ldots, X_n)$ and we are interested in getting a better analytic understanding of this (in particular, in regard of regularity questions), for large classes of operators. As there is no direct analytic object encoding this non-commutative distribution (in particular, the non-commutative moments cannot be identified with the moments of a probability measure in $\mathbb{R}^n$), we try to understand the non-commutative distribution of $(X_1, \ldots, X_n)$ via the understanding of the ordinary distributions of sufficiently many "test functions" $f(X_1, \ldots, X_n)$. This touches of course upon the question: what are functions in non-commuting variables? At the moment there are three classes of functions of which one can make sense and where we have some results on the regularity of their distributions, namely: polynomials, matrices of polynomials, and rational functions. If such a function f is selfadjoint then we have on the side of matrices the eigenvalue distribution of $f(X_1^{(N)}, \ldots, X_n^{(N)})$ and on the side of the limit operators the distribution of $f(X_1, \ldots, X_n)$ and one expects that the former should converge to the latter, in the weak topology for probability measures.

In the cases of polynomials and of matrices of polynomials it is quite clear that the convergence of the non-commutative distribution implies indeed this weak convergence. This means in particular that results which we derive on the distribution of the limit operators are indeed results about the asymptotic eigenvalue distribution of the corresponding random matrices.

In the case of rational functions one has to address the problem that our matrices or operators might not belong to the domain of the considered rational function. For nice ensembles of random matrices (like independent GUE) one expects that any rational function of them should almost surely make sense, and on the side of the limit operators one of our main results is that indeed any rational function in free semicirculars makes always sense as an unbounded operator. (Actually, we have this for much larger classes of limit operators.) Whether, for a rational function r, one also has that the eigenvalue distribution of $r(X_1^{(N)}, \ldots, X_n^{(N)})$ converges to the distribution of $r(X_1, \ldots, X_n)$ is an open question.

Here is our main result on the regularity properties of the limit operators. The free skew field denotes there the universal field of fractions of the non-commutative polynomials, which had been extensively developed in the work of Cohn, see, e.g., [1].

Theorem [4]**:** Let (M, τ) be a tracial W^*-probability space (i.e., M is a von Neumann algebra, and τ a faithful normal trace). Denote by $\mathcal{A}$ the $*$-algebra of unbounded closed and densely defined operators affiliated to M. Consider a tuple $X = (X_1, \ldots, X_n) \in M^n$ of operators in M. Then the following are equivalent.

- The division closure of $X_1, \ldots, X_n$ in $\mathcal{A}$ is the free skew field, i.e.,

$$\mathbb{C}\langle x_1, \ldots, x_n \rangle \to \mathcal{A}, \qquad x_i \mapsto X_i, \quad 1 \mapsto 1,$$

  extends to a an injective homomorphism on the free skew field.
- Matrices over polynomials in $X_1, \ldots, X_n$ are invertible as matrices in unbounded operators if and only if those matrices are invertible over the free skew field (which is the case if the matrix cannot be decomposed as a product of strictly rectangular matrices over the polynomials).
- $\Delta(X_1, \ldots, X_n) = n$; Δ is here a quantity introduced by Connes and Shlyakhtenko in [2] and the above requirement on Δ being maximal means concretely the following: the only finite rank operators $T_1, \ldots, T_n$ in $L^2(M, \tau)$ which satisfy the commutator relation $\sum_{k=1}^n [T_k, X_k] = 0$ are the trivial ones $T_1 = \cdots = T_n = 0$.

This yields as a corollary that for operators $X_1, \ldots, X_n$ with maximal Δ any rational function $r(X_1, \ldots, X_n)$ makes sense as an unbounded operator and, for $r \neq 0$, has no zero-divisors; for non-constant and selfadjoint r this means that $r(X_1, \ldots, X_n)$ has no atoms in its distribution. Since the maximality of Δ is implied by the same kind of maximality for free entropy dimension or by the existence of dual systems, free probability has a lot of tools to decide on the maximality of Δ. In particular, our results also include the result of Linnell [3] that the division algebra of the generators of the free group (i.e., free Haar unitaries) is the free skew field.

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Local Laws for Sample Covariance Sparse Matrices

Alexander Tikhomirov
(joint work with Friedrich Götze and Dmitry Timushev)

We explain some recent results on local laws for sample covariance sparse matrices. We consider a random $m \times n$ matrix $\mathbf{X} = [X_{jk}\xi_{jk}]$ where X_{jk} are i.i.d. random variables with mean zero and unit variance and ξ_{jk} are i.i.d. Bernoulli random variables with $\mathbf{E}\xi_{jk} = p_n$, $1 \le j \le m$, $1 \le k \le n$. We additionally suppose that $\mathbf{E}|X_{11}|^{4+\delta} =: \mu_{4+\delta} < \infty$ for some $\delta > 0$. Assuming that $np_n >> \log^{\frac{2}{\kappa}} n$ with $\kappa = \frac{\delta}{2(4+\delta)}$, we show that with high probability the typical distance between the Stieltjes transform of the empirical spectral distribution (ESD) of the matrix $\mathbf{W} = (np_n)^{-1}\mathbf{X}\mathbf{X}^*$ ($\mathbf{A}^*$ denotes the complex conjugate of matrix $\mathbf{A}$) and Marchenko – Pastur law with parameter $y = n/m$ is of order $((nv)^{-1} + (np_n)^{-1})\log n$, where v denotes the distance to the real line in the complex plane. We apply this result to the estimation of the rate of the ESD convergence to the Marchenko – Pastur distribution as well as to the rigidity of the eigenvalues and the eigenvector delocalization. Let $s_1^2 \ge \ldots \ge s_n^2$ be eigenvalues of matrix $\mathbf{W}$. We shall consider the symmetrized empirical spectral distribution function

$$F_n(x) = \frac{1}{2n}\sum_{j=1}^{n}\Big(\mathbb{I}\{s_j \le x\} + \mathbb{I}\{-s_j \le x\}\Big).$$

Define as well the symmetrized Marchenko-Pastur distribution G_y with density

$$g_y(x) = \frac{1}{\pi y x}\sqrt{[(x^2-a^2)(b^2-x^2)]_+},$$

where $a = 1-\sqrt{y}$, $b = 1+\sqrt{y}$ and $[d]_+ = \max\{d, 0\}$. The main results are the following.

Theorem 1. *Let $\mathbf{E}\,X_{jk} = 0$ and $\mathbf{E}\,|X_{jk}|^2 = 1$. Assume that for any $j, k \ge 1$ and for some $\delta > 0$*

$$\mathbf{E}\,|X_{jk}|^{4+\delta} \le C < \infty,$$

and there exists a positive constant B, s.t.

$$np_n \ge B\log^{\frac{2}{\kappa}} n,$$

where $\kappa = \frac{\delta}{2(4+\delta)}$. Then for any $z = u + iv$ and $\gamma > 0$ s.t. $v \ge v_0 := c\log^4 n$ and $\min\{u - (1-\sqrt{y}), 1+\sqrt{y}-u\} \ge \gamma$ there exists a constant C depending on c and γ s.t. with high probability

$$|m_n(z) - S_y(z)| \le C\log n\Big(\frac{1}{nv} + \frac{1}{np_n}\Big).$$

Here $m_n(z)$ denotes symmetrized Stieltjes transform of ESD of matrix $\mathbf{W}$ and $S_y(z)$ denotes the Stieltjes transform of symmetrized Marchenko-Pastur distribution with parameter y.

Theorem 2. *Assuming additionally to the condition of Theorem 1 that*

$$|X_{jk}| \le C(np_n)^{\frac{1}{2}-\kappa},$$

we get for $z = u + iv$ *with* $(1 - \sqrt{y} - v)_+ \le |u| \le 1 + \sqrt{y} + v$ *and* $v \ge v_0$ *with high probability*

$$|m_n(z) - S_y(z)| \le C \log n \left(\frac{1}{nv} + \min \left\{ \frac{1}{np_n \sqrt{\gamma + v}}, \frac{1}{\sqrt{np_n}} \right\} \right),$$

where $\gamma = \min\{|1 - \sqrt{y} - |u||, |1 + \sqrt{y} - |u||\}$. *Moreover, for any* u_0 *there exists a constant* C *depending on* u_0 *s.t. for* $z = u + iv$ *with* $|u| \le u_0$

$$|\mathrm{Im} m_n(z) - \mathrm{Im} S_y(z)| \le C \log n \left(\frac{1}{nv} + \min \left\{ \frac{1}{np_n \sqrt{\gamma + v}}, \frac{1}{\sqrt{np_n}} \right\} \right)$$

From Theorem 1 it follows that

Corollary 1. *Under conditions of Theorem 1 there exists a constant* C *depending on* δ *and* $\mu_{4+\delta}$ *such that with high probability*

$$\sup_x |F_n(x) - G_y(x)| \le C \log n/(np_n).$$

Let γ_j be the quantile of distribution function $G_y(x)$ of order j/n and let $\mathbf{u}_j = (u_{j1}, \dots, u_{jn})$ be the eigenvector of matrix $\mathbf{W}$.

Corollary 2. *Under conditions of Theorem 1 for* $j \in [Cp_n^{-1} \log n, n - Cp_n^{-1}]$ *the following inequality holds with high probability*

$$|s_j - \gamma_j| \le C \log n/(np_n).$$

From Theorem 2 it follows

Corollary 3. *Under conditions of Theorem 2 there exists a constant* C *such that with high probability*

$$\max_{1 \le j,k \le n} \{|u_{jk}|\} \le C \log^2 n/\sqrt{n}.$$

The similar and stronger results for Wigner sparse matrices where obtained in several papers of Erdös, Knowles, Yau and co-authors, see [1], [2].

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Critical Behavior of Non-Intersecting Brownian Motions

Martin Venker
(joint work with Tom Claeys, Thorsten Neuschel)

Non-intersecting Brownian motions (NIBM) constitute one of the most important dynamical models of random eigenvalues. They may be obtained either by conditioning n independent Brownian motions on no intersection for all times or as process $(X(t))_{t\geq 0}$, $X_1(t) \leq \cdots \leq X_n(t)$, of eigenvalues of $n^{-1/2}M(t)$, where $(M(t))_{t\geq 0}$ is an $n \times n$ Hermitian matrix-valued Brownian motion. We consider special deterministic initial configurations $X(0)$ mimicking a measure whose density vanishes at an isolated point. More precisely, we assume that $\mu_n := n^{-1}\sum_{j=1}^n \delta_{X_j(0)}$ converges for $n \to \infty$ weakly to a probability measure μ which admits a continuous density ψ that vanishes at some point x^* as $\psi(x) \sim |x - x^*|^\kappa$, $x \to x^*$. For a vanishing order κ larger than 1, this setting naturally leads to the initial points $X(0)$ being split into two bulks of particles which merge at a certain critical time. In fact, defining the evolution of the initial critical point x^* as

$$x^*(t) := x^* + t\int \frac{\psi(s)ds}{x^* - s},$$

and denoting by ψ_t the evolved density (the density of the additive free convolution of μ with a certain semicircle distribution), we have $\psi_t(x^*(t)) = 0$ for $0 \leq t \leq t_{\mathrm{cr}}$ and $\psi_t(x^*(t)) > 0$ for $t > t_{\mathrm{cr}}$ not too large, where the critical time t_{cr} is defined as

$$t_{\mathrm{cr}} := \left(\int \frac{\psi(s)ds}{(x^* - s)^2}\right)^{-1}$$

with the convention of it being 0 in case of divergence of the integral. Of special interest are the correlations of particles around the merging point $x^*(t_{\mathrm{cr}})$ at the critical time. In [4], the case of a weak vanishing $0 \leq \kappa < 1$ was studied, in which case the critical time is 0. We found under some rigidity assumptions typical limiting bulk correlations given by the sine kernel, as long as $t \gg (\frac{(\log n)^{1+\rho}}{n})^{\frac{1-\kappa}{1+\kappa}}$ for some $\rho > 0$. We also saw in [4] that correlations are trivial for times well before that threshold. The talk mainly reported about the findings of [5] which studies the strong vanishing case $\kappa > 1$ where the critical time is positive. The behavior of the evolved density at the critical time and around the critical point, has already been studied for integers $\kappa \geq 4$ in [3]. There it was proven that if the integral

$$\int \frac{\psi(s)ds}{(x^* - s)^3} \tag{1}$$

is non-zero, then $\psi_{t_{\mathrm{cr}}}$ vanishes as a square-root if $x^*(t_{\mathrm{cr}})$ is approached from one side and has a higher order vanishing from the other side. If the integral (1) is 0, then $\psi_{t_{\mathrm{cr}}}$ has a cusp singularity at $x^*(t_{\mathrm{cr}})$ with cubic-root vanishing from both sides.

In [5], we studied the local statistics at criticality. We found that, under appropriate rescaling, two different limiting point processes arise at the critical point and the critical time. If $\kappa > 2$ and the integral (1) is non-zero, say positive, then

the limiting space-time correlations around $(x^*(t_{\mathrm{cr}}), t_{\mathrm{cr}})$ are (under a few technical conditions) given by the Airy line ensemble, provided the KPZ like 1-2-3 rescaling $t_{\mathrm{cr}} \to t_n(\tau) := t_{\mathrm{cr}} + \frac{\tau}{n^{-1/3}}$, $x^*(t_{\mathrm{cr}}) \to x^*(t_n(\tau)) + \frac{u}{n^{-2/3}}$ is performed. To be more precise, recall that the NIBM form a determinantal ensemble in the sense that its space-time correlation functions can be written as determinants of a matrix (whose dimension depends on the dimension of the correlation function) which has entries given by a kernel $K_{n,s,t}(u,v)$. Here s,t and u,v are time and space variables, respectively. Then we show for some explicit constants $\epsilon, c_2 > 0$, uniformly for u, v, τ_1, τ_2 in compacts

$$\begin{aligned}&\frac{1}{c_2 n^{2/3}} K_{n,t_n(\tau_1),t_n(\tau_2)}\left(x^*(t_n(\tau_1)) + \frac{u}{c_2 n^{2/3}}, x^*(t_n(\tau_2)) + \frac{v}{c_2 n^{2/3}}\right)\\ &= \mathbb{K}^{\mathrm{Ai}}_{\tau_1,\tau_2}(u,v) + \mathcal{O}\left(n^{-\epsilon}\right),\end{aligned}$$

where $\mathbb{K}^{\mathrm{Ai}}$ denotes the extended Airy kernel.

If $\kappa > 3$ and the integral (1) is zero, then the limiting correlations are given by the Pearcey process if a rescaling $t_{\mathrm{cr}} \to t_n(\tau) := t_{\mathrm{cr}} + \frac{\tau}{n^{-1/2}}$, $x^*(t_{\mathrm{cr}}) \to x^*(t_n(\tau)) + \frac{u}{n^{-3/4}}$ is applied. This complements results in [1] and [6, 2] on Pearcey universality.

While the Pearcey process is expected to appear at the merging, the appearance of the Airy line ensemble is surprising. We explain this occurrence by showing that in the case of the integral (1) being non-zero, at one side of the evolved critical point $x^*(t_{\mathrm{cr}})$ and at the critical time, a mesoscopic gap is present that is larger than the typical edge spacings $n^{-2/3}$, implying that particles separated by this gap are effectively uncorrelated and therefore the typical edge limits appear. We also identified a particle just below or above the gap having limiting space-time fluctuations given by the Airy_2 process.

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Random planar geometry

Balint Virág

Non-critical random versions of Euclidean geometry seem to share many common properties. A similar metric structure is present in directed last an first passage percolation in two dimensions. Many growth models and particle systems share a hidden planar metric structure. I will talk about the recently constructed (in many cases conjectured) common scaling limit, the directed landscape.

On an almost sure Weyl law for quantized tori

Martin Vogel

The Toeplitz quantization (also known as the Berezin-Toeplitz quantization) of a complex-valued function on the $2d$-dimensional torus $\mathbb{T}^{2d} = \mathbb{R}^{2d}/\mathbb{Z}^{2d}$ maps a smooth function to an $N^d \times N^d$ matrix

$$C^\infty(\mathbb{T}^{2d}) \ni p \mapsto p_N \in \mathcal{L}(\mathbb{C}^{N^d}, \mathbb{C}^{N^d}). \tag{1}$$

In the case of $\mathbb{T}^2 = S^1_x \times S^1_\xi$, the operators p_N are also referred to as *twisted Toeplitz matrices* [3], and

$$f = f(x) \mapsto f_N = \mathrm{diag}(f(l/N); l = 0, \dots, N-1),$$
$$g = g(\xi) \mapsto g_N = \mathcal{F}_N^* \,\mathrm{diag}(g(l/N); l = 0, \dots, N-1)\mathcal{F}_N,$$

where $\mathcal{F}_N^* = N^{-1/2}(\exp(2\pi i k\ell/N))_{0\le k,\ell\le N-1}$ is the discrete Fourier transform. In [2] Christiansen and Zworski proved that the expected number of eigenvalues in some compact set $\Omega \subset \mathbb{C}$ with smooth boundary, of small complex Gaussian random perturbations $N^{-\gamma}Q_N$ of p_N satisfies a Weyl law,

$$\mathbb{E}\left(|\mathrm{Spec}(p_N + N^{-\gamma}Q_N) \cap \Omega|\right) \sim N^d \mathrm{vol}_{\mathbb{T}^{2d}}(p^{-1}(\Omega)), \quad \gamma > d + 1/2.$$

provided that there exists a $\kappa \in]1/2, 1]$ such that

$$V_z(t) = \mathrm{Vol}\{\rho \in \mathbb{T}^{2d}; |p(\rho) - z|^2 \le t\} = \mathcal{O}(t^\kappa),\ 0 \le t \ll 1, \tag{2}$$

uniformly for z in a neighbourhood of $\partial\Omega$. In [2] it was conjectured that the empirical measure of eigenvalues μ_N of such random perturbations converges almost surely to the push-forward of the measure induced by symplectic volume form $\sigma^n/n!$ on $\mathbb{T}^{2d}$ under the symbol p, i.e.

$$\mu_N = N^{-d} \sum_{\lambda \in \mathrm{Spec}(p_N + N^{-\gamma}Q_N)} \delta_\lambda \rightharpoonup p_*(\sigma^n/n!), \quad N \to \infty, \tag{3}$$

In this talk we present a recent result [6] showing that for an open relatively compact simply connected set $\Omega \subset \mathbb{C}$ satisfying (2) for some $\kappa \in]0, 1]$, the following holds : Let Q_ω be a random $N^d \times N^d$-matrix whose entries are independent copies of a random variable q with mean 0, variance 1 and bounded fourth moment. Let $\delta_0 > 0$, and let $C > 0$ be sufficiently large, then for every $\tau > 0$

$$\left| \#(\mathrm{Spec}(p_N + C^{-1}N^{-d/2-\delta_0}Q_\omega) \cap \Omega) - N^d \int_{p^{-1}(\Omega)} d\rho \right| \le o(N^d)$$

with probability $\geq 1-\mathcal{O}(N^{-(1-\tau)\delta_0})$. This result can be used in particular to show the conjectured convergence (3), see [6]. Related results can be found in [1, 4, 5].

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Matrix Poincaré inequalities and concentration

Pierre Youssef

(joint work with R. Aoun, M. Banna)

Matrix concentration inequalities are noncommutative extensions of their scalar counterpart and have been extensively developed in the last decade. In this direction, many papers were devoted to extending scalar methods for deriving concentration inequalities in the matrix setting [2]. In this talk, we focus on Poincaré inequalities and aim to implement a general procedure turning a *matrix* Poincaré inequality into a concentration inequality. In the scalar case, such a procedure is by now standard.

Given a probability measure μ on some polish space Ω and $f : \Omega \to \mathcal{H}$, the variance of f is given by

$$\mathrm{Var}_\mu(f) = \mathbb{E}_\mu[f^2] - (\mathbb{E}_\mu f)^2.$$

We will say that μ satisfies a *matrix Poincaré inequality* with constant α and *matrix Markov generator* $\mathcal{L}$ if for any $f : \Omega \to \mathcal{H}$ we have

$$\mathrm{Var}_\mu(f) \preceq \alpha\mathcal{E}(f),$$

where $\mathcal{E}(f) = -\mathcal{E}[f\mathcal{L}f]$ is the *matrix Dirichlet form* associated with $\mathcal{L}$ and $\preceq$ refers to the positive semi-definite ordering.

We show that any probability measure satisfying a Matrix Poincaré inequality with respect to some reversible Markov generator satisfies an exponential matrix concentration inequality depending on the associated matrix carré du champ operator.

Theorem([1]) *Let μ be a probability measure on some polish space Ω. Suppose that μ satisfies a matrix Poincaré inequality with constant α and matrix Markov*

generator $\mathcal{L}$ reversible with respect to μ. Then, for any $f : \Omega \to \mathcal{H}$ and any $t \geq 0$, we have

$$\mu\Big(\lambda_{max}\big(f - \mathcal{E}f\big) \geq t\Big) \leq d \exp\left(- \frac{t^2}{2\big(2\alpha v_f + t\sqrt{\alpha v_f}\big)}\right),$$

where $v_f = \big\|\|\Gamma(f)\|\big\|_{L_\infty}$ with Γ being the matrix carré du champ operator associated with $\mathcal{L}$ and $\|\cdot\|$ the operator norm.

The proof of this theorem hides many challenging obstacles caused by non-commutativity, and gives rise to new matrix trace inequalities relating the matrix Dirichlet form and the Laplace transform.

We illustrate the strength of this general machinery on several examples. To this aim, we derive matrix Poincaré inequalities for the Gaussian measure in $\mathbb{R}^n$, for general product measures, and for probability measures on $\{0,1\}^n$ satisfying a form of weak dependence known as the *Stochastic Covering Property*, and deduce corresponding matrix concentration inequalities.

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Small perturbations of Toeplitz matrices: convergence and outliers

Ofer Zeitouni

(joint work with Anirban Basak, Elliot Paquette)

Consider an $N \times N$ Toeplitz matrix T_N with symbol $\mathbf{a}(\lambda) := \sum_{\ell=-d_2}^{d_1} a_\ell \lambda^\ell$, perturbed by an additive noise matrix $N^{-\gamma} E_N$, where the entries of E_N are centered i.i.d. complex random variables of unit variance and $\gamma > 1/2$. Several groups [1, 2, 5, 6, 7] have shown that, under appropriate assumptions, the empirical measure of eigenvalues of the perturbed matrix converges weakly, as $N \to \infty$, to the law of $\mathbf{a}(U)$, where U is distributed uniformly on $\mathbb{S}^1$. The talk reviews this, and then turns to the study of outliers, i.e. eigenvalues that are at a positive (N-independent) distance from $\mathbf{a}(\mathbb{S}^1)$. We describe the results of [3], which assume i.i.d. entries of E_N possessing a density in the complex plane. First, we show that there are no outliers outside $\operatorname{spec} T(\mathbf{a})$, the spectrum of the limiting Toeplitz operator, with probability approaching one, as $N \to \infty$. (This could also be deduced from the results of [6].) We then describe the outliers field in different subsets of $\operatorname{spec} T(\mathbf{a}) \setminus \mathbf{a}(\mathbb{S}^1)$. It turns out that the process of outliers converges to a point process described by the zero set of certain random analytic functions. The limiting random analytic functions can be expressed as linear combinations of the determinants of finite sub-matrices of an infinite dimensional matrix, whose entries are i.i.d. having the same law as that of E_N. The coefficients in the linear combination depend on the roots of the polynomial $P_{z,\mathbf{a}}(\lambda) := (\mathbf{a}(\lambda) - z)\lambda^{d_2} = 0$ and

semi-standard Young Tableaux with shapes determined by the number of roots of $P_{z,\mathbf{a}}(\lambda) = 0$ that are greater than one in moduli.

While the description of the limiting random analytic function (appearing in [3]) is complicated (and not universal!), the different regions where transition from one analytic function to the other occur admit a simple description, as follows. Set $P_{z,\mathbf{a}}(\lambda) := \lambda^{d_2}(\mathbf{a}(\lambda) - z)$. Writing $d := d_1 + d_2$, let $\{-\lambda_\ell(z)\}_{\ell=1}^d$ be the roots of the equation $P_{z,\mathbf{a}}(\lambda) = 0$ arranged in an non-increasing order of their moduli. For $\mathfrak{d}$ an integer such that $-d_2 \leq \mathfrak{d} \leq d_1$, set

$$\begin{aligned}\mathcal{S}_{\mathfrak{d}} :=& \{z \in \mathbb{C} \setminus \mathbf{a}(\mathbb{S}^1) : d_0(z) = d_1 - \mathfrak{d}, \\ & \text{where } d_0(z) \text{ such that } |\lambda_{d_0(z)}(z)| > 1 > |\lambda_{d_0(z)+1}(z)|\},\end{aligned}$$

where for convenience we set $\lambda_{d+1}(z) = 0$ and $\lambda_0(z) = \infty$ for all $z \in \mathbb{C}$. The region $\mathcal{S}_0$ coincides with $\big(\operatorname{spec} T(\mathbf{a})\big)^c$. Each of the other regions carries a limiting random analytic function whose zeroes describe the limiting field of outliers.

In the special case of the "maximal nilpotent matrix" $T_N(i,j) = \mathbf{1}_{j=i+1}$, we have that $\mathfrak{d} = 1$ and $\mathcal{S}_1 = \{z \in \mathbb{C} : |z| = 1\}$. In that case, the outlier field in $\mathcal{S}_1$ converges to the zero field of the random analytic function $F(z) = \sum_{k=0}^\infty z^k g_k \sqrt{k+1}$. where g_k are iid variables of the same law as the entries of E_N. In the complex Gaussian case, this is the *hyperbolic Gaussian analytic function*,and the first intensity of its zeroes is

$$\rho_1(dz) := \frac{2}{\pi(1-|z|^2)^2}\mathbf{1}_{\{|z|<1\}}dz.$$

In this particular case, the intensity (but not the limiting law) was computed earlier in [8], and also appeared as limit of an outlier field in a different context [4].

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