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Interplay of Analysis and Probability in Physics

Organised by
Wolfgang König, Leipzig
Peter Mörters, Bath
Johannes Zimmer, Bath

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Abstract. It is widely recognised that stochastic effects need to be included in the modelling of many physical systems, while reciprocally the sciences provide a challenging potential area of application of stochastic processes. This creates an increasing need to combine analytic and stochastic techniques. The aim of this workshop was to address this need and contribute to the efforts to surmount the language barrier between analysts and probabilists by stimulating and encouraging exchange and joint research between the two communities. The focus of the workshop was on recent and currently emerging progress in the investigation of complex physical systems, using a combination of analytical and stochastic methods.

Mathematics Subject Classification (2000): 60xx, 70xx, 74xx, 82xx.

Introduction by the Organisers

This workshop brought together researchers working in analysis and probability, stimulated an intensive exchange of ideas between the respective communities, and helped identify open problems at the boundaries and intersections of these areas.

Various approaches to scale-bridging were the main theme of the first day. Particular problems included the derivation of the thermal conductivity of a diffusion process from an atomistic model (Stefano Olla), the development of a coarse-graining technique to calculate canonical ensemble averages efficiently (Frédéric Legoll), and a potential-theoretic approach to the analysis of the transition from the metastable state to the stable state for a lattice gas under the Kawasaki dynamics (Anton Bovier). Jürgen Gärtner and Jean-Dominique Deuschel discussed a range of models interacting with a random disorder.

Spectral properties of large random matrices were in the focus of the morning session on Tuesday, in particular in the talks by Laszlo Erdős and Amir

Dembo. The afternoon was devoted to pinning problems, both from the probabilistic view points for polymer models (Giambattista Giacomin and Hubert Lacoin), and martensitic materials with imperfections (Patrick Dondl).

Wednesday started with a review on recent progress of optimal transportation and gradient flows (Karl-Theordor Sturm). This was continued on Friday with a derivation of a particle model of the Ricci flow on a manifold (Robert Philipowski). Existence of Gibbs measures was another central theme of Wednesday. The existence and uniqueness of the Gibbs measure for point processes with a Hamiltonian depending on nearest-neighbour triples was investigated (Hans-Otto Georgii). A highlight of the conference was the analysis of lattice gradient models with non-convex interaction energy. Stefan Adams explained, using rigorous renormalisation group theory, that the free energy is convex for sufficiently low temperatures.

The talks of Thursday morning had scaling limits as one focus. This included a mean-field approach to clustering (Barbara Niethammer), the sharp interface limit of perturbed Allen-Cahn equations (Matthias Röger), and a detailed investigation of scaling limits of pinned random walks when the rate-function has a non-unique minimiser (Tadahisa Funaki). In the afternoon, Dirk Hundertmark presented the analysis of properties of soliton solutions for a nonlinear Schrödinger equation with relevance to dispersion management.

On Friday, the topic of systems interacting with a random disorder was taken up again, covering ageing in the parabolic Anderson model (Marcel Ortgiese) and an interface model with general potentials (Takao Nishikawa).

Besides the programme of altogether 25 talks of length between 30 and 45 minutes there was plenty of time for informal discussion and interaction between participants. This time was well-used with most discussions involving members of both the probability and analysis community. Beyond the mathematical discussion, several participants used the opportunity to discuss future collaborations in various international networks currently emerging in this area.

There were 40 participants in total, 16 from Germany, 16 from other European countries (Switzerland, Italy, Spain, France and Britain), 5 from the US and 3 from Japan. All career stages were represented.

The organisers thank the NSF for funding the participation of speakers from the US and Nadia Sidorova for collecting the extended abstracts.

Workshop: Interplay of Analysis and Probability in Physics

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Abstracts

Heat transport: a weak coupling approach

Stefano Olla
(joint work with Carlangelo Liverani)

Let us consider a region $\Lambda \subset \mathbb{Z}^d$, set $N = |\Lambda|$, the number of sites in Λ. At each site we have a ν-dimensional, $\nu \geq 2$, nonlinear oscillator and we assume that such oscillators interact weakly via a non-liner potential. Such a situation is described by the following Hamiltonian in the variables $(q_i, p_i)_{i\in\Lambda} \in \mathbb{R}^{2\nu N}$

$$H_\varepsilon^\Lambda := \sum_{i\in\Lambda} \frac{1}{2}\|p_i\|^2 + \sum_{i\in\Lambda} U(q_i) + \varepsilon \sum_{|i-j|=1} V(q_i - q_j),$$

where $U, V \in \mathcal{C}^\infty(\mathbb{R}^\nu, \mathbb{R})$. We assume the potential U is a strictly convex with $U(0) = 0$ and $\nabla U(0) = 0$, moreover it is radially symmetric ($U(q) = \bar{U}(\|q\|)$ with $c^{-1} \leq \bar{U}'' \leq c$ for some finite positive constant c. We assume $V'(q)^2 \leq CU(q)$. For simplicity of notations, we choose $\nu = 2$. In addition to the hamiltonian dynamics, we consider random forces that conserve the kinetic energy of each atom, given by independent diffusions on the spheres $\|p_i\|^2 = cost$). In order to define such diffusions, consider the vector fields

$$X_i := p_i^1 \partial_{p_i^2} - p_i^2 \partial_{p_i^1} =: Jp_i \cdot \partial_{p_i},$$

and the second order operator

$$S = \sum_{i\in\Lambda} X_i^2$$

The generator of the process is given by

$$L_{\varepsilon,\Lambda} =: A_\varepsilon + \sigma^2 S$$

where $A_\varepsilon = \{H_\varepsilon^\Lambda, \cdot\}$, the usual Hamiltonian operator. The stationary (equilibrium) probability measures for the dynamics are given by the Gibbs measures at (any) temperature β^{-1}, defined by

$$m_\varepsilon^\beta(dq, dq) = Z_\varepsilon(\beta) e^{-\beta H_\varepsilon^\Lambda(q,p)} dpdq.$$

As reference measure we pick the one corresponding to $\beta = 1$ and we denote it by m_ε. Notice that for ε small enough, m_ε and the product measure m_0 are equivalent. We assume that the system is started in an initial distribution $d\nu_0 := F_\varepsilon dm_\varepsilon = F_0 dm_0$, with $F \in L^2(\mathbb{R}^{4N}, m_0)$.

$L_{\varepsilon,\Lambda}$ is the generator of a contraction semigroup P_ε^t in $L^2(\mathbb{R}^{2N\nu}, m_\varepsilon)$ with stationary measure m_ε and $P_\varepsilon^t H_\varepsilon^\Lambda = H_\varepsilon^\Lambda$, for all $t \in \mathbb{R}_+$.

The single particles energies are

$$\mathcal{E}_i^\varepsilon(q,p) = \frac{1}{2}\|p_i\|^2 + U(q_i) + \frac{1}{2d}\varepsilon \sum_{|i-j|=1} V(q_i - q_j).$$

The time evolution of these energies is given by:

$$\frac{d\mathcal{E}_i^\varepsilon}{dt} = \varepsilon \sum_{|i-k|=1} j_{i,k} \tag{1}$$

where the energy currents are defined by

$$j_{i,k} = \frac{1}{2d}\nabla V(q_i - q_k)\cdot(p_i + p_k) \tag{2}$$

Note that $j_{i,k} = -j_{k,i}$ and that this is a function only of q_i, p_i, q_k, p_k.

If $\varepsilon = 0$ the dynamics is given by non-interacting oscillators, and consequently the energy of each oscillator is a conserved quantity. So for $\varepsilon = 0$ there is a family of equilibrium measure parametrized by the vector $\underline{a} = (a_i)_{i\in\Lambda}$ of the energy of each oscillator. This is given by $\mu_{\underline{a}}^\Lambda$, the Liouville measure associated to the Hamiltonian flow H_0^Λ on the surface

$$\Sigma_{\underline{a}} := \{q, p \ : \ a_i = \mathcal{E}_i^0(q,p) = \frac{1}{2}\|p_i\|^2 + U(q_i)\} = \bigotimes_{i\in\Lambda} \Sigma_{a_i}.$$

This is also called microcanonical measure. Clearly, letting μ_a be the Liouville measure on the 3 dimensional surface Σ_a, we have $\mu_{\underline{a}}^\Lambda = \otimes_{i\in\Lambda}\mu_{a_i}$. By the symmetry between p and $-p$ it follows that $\mu_{\underline{a}}(j_{i,k}) = 0$ for each $\underline{a}$.

We are interested in the stochastic process of the time rescaled energies

$$\boldsymbol{\mathcal{E}}_i^\varepsilon(t) = \mathcal{E}_i^\varepsilon(q(\varepsilon^{-2}t), p(\varepsilon^{-2}t)).$$

In order to define the parameters of the macroscopic evolution, consider the dynamics of 2 non-interacting oscillators ($\varepsilon = 0$), each starting with the microcanonical distribution with corresponding energy a_1 and a_2. Let us denote by $\mathbb{E}_{a_1,a_2}(\cdot)$ the corresponding expectation in this equilibrium measure. This permits us to define the following positive function on $\mathbb{R}_+^2$

$$\gamma^2(a_1,a_2) = \int_0^\infty \mathbb{E}_{a_1,a_2}\left(j_{1,2}(\omega_t) j_{1,2}(\omega_0)\right)\, dt \tag{3}$$

We prove that $\gamma^2 \in \mathcal{C}^\infty(\mathbb{R}_+^2)$. Notice that γ^2 is a symmetric function of a_1, a_2. Correspondingly we define the *macroscopic* current by the antisymmetric function

$$\alpha(a_1,a_2) = \sigma^2(\partial_{a_1} - \partial_{a_2})\gamma^2(a_1,a_2). \tag{4}$$

Here is our main result:

Theorem 1. *In the limit $\varepsilon \to 0$, the process $\{\boldsymbol{\mathcal{E}}_i^\varepsilon\}_{i\in\Lambda}$ converges, in law, to the stochastic process $\{\boldsymbol{\mathcal{E}}_i\}_{i\in\Lambda}$ determined by the stochastic differential equations*

$$d\boldsymbol{\mathcal{E}}_i = \sum_{|i-k|=1} \alpha(\boldsymbol{\mathcal{E}}_i, \boldsymbol{\mathcal{E}}_k)dt + \sum_{|i-k|=1} \sigma\gamma(\boldsymbol{\mathcal{E}}_i, \boldsymbol{\mathcal{E}}_k) dB_{\{i,k\}} \tag{4}$$

where $B_{\{i,k\}} = -B_{\{k,i\}}$ are independent standard Brownian motions.

Notice that the generator of this diffusion on $\mathbb{R}_+^\Lambda$ is given by

$$\mathcal{L} = \sum_{i\in\Lambda} \sum_{|k-i|=1} \left(\sigma^2\gamma(\boldsymbol{\mathcal{E}}_i,\boldsymbol{\mathcal{E}}_k)^2(\partial_{\boldsymbol{\mathcal{E}}_i} - \partial_{\boldsymbol{\mathcal{E}}_k})^2 + \alpha(\boldsymbol{\mathcal{E}}_i,\boldsymbol{\mathcal{E}}_k)(\partial_{\boldsymbol{\mathcal{E}}_i} - \partial_{\boldsymbol{\mathcal{E}}_k})\right) \tag{5}$$

and, for any inverse temperature $\beta > 0$, the product probability measure

$$\prod_{i\in\Lambda} \beta e^{-\beta\boldsymbol{\mathcal{E}}_k} \tag{6}$$

is stationary and reversible for the diffusion generated by (5).

By (4) we can rewrite the generator as

$$\mathcal{L} = \sigma^2 \sum_{\substack{i,k\in\Lambda,\\ |k-i|=1}} (\partial_{\boldsymbol{\mathcal{E}}_i} - \partial_{\boldsymbol{\mathcal{E}}_k})\gamma^2(\boldsymbol{\mathcal{E}}_i,\boldsymbol{\mathcal{E}}_k)(\partial_{\boldsymbol{\mathcal{E}}_i} - \partial_{\boldsymbol{\mathcal{E}}_k}) \tag{7}$$

The process (1) is close the the one studied by Varadhan in [1]. In this paper Varadhan proves an *hydrodynamic limit*, i. e. that under certain condition on the initial distribution, for any test function G on $\mathbb{R}^d$, the convergence

$$\lim_{N\to\infty} \frac{1}{N^d} \sum_i G(i/N)\boldsymbol{\mathcal{E}}_i(N^2t) = \int G(y)u(y,t)dy \tag{8}$$

where $u(y,t)$ is the solution of a nonlinear heat equation

$$\partial_t u = \nabla D(u)\nabla u \tag{9}$$

for a properly defined diffusion matrix $D(u)$ depending on the corresponding equilibrium dynamics defined by (7).

Yet our case it is not covered by such result (due to the degenacy at zero of the diffusion coefficients and the non strict convexity of the potential of the invariant measure). In any case the extension of Varadhan's work to the present case would allow to obtain the *heat equation* in the present setting via a diffusive limit. We plan to work on such an extension in the future.

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Finite temperature coarse-graining of one-dimensional atomistic systems: some simple cases

Frédéric Legoll
(joint work with Xavier Blanc, Claude Le Bris and Carsten Patz)

In this work, we consider the derivation of reduced models for discrete systems, along with the design of efficient computational approaches, in a constant temperature setting. In short, our aim is to use standard asymptotic tools of probability (such as Large Deviations Principles) to design a computational strategy.

Consider an atomistic system consisting of N particles, at positions $X = \left(X^1, \ldots, X^N\right) \in \mathbb{R}^{dN}$, where d is the space dimension (d=1, 2 or 3). Provide this system with an energy $V(X) = V\left(X^1, \ldots, X^N\right)$ and allow the particles to sample $\mathbb{R}^d$. The finite temperature thermodynamical properties of the material are obtained from canonical ensemble averages,

$$\langle A \rangle = \frac{\displaystyle\int_{\mathbb{R}^{dN}} A(X)\, \exp(-\beta V(X))\, dX}{\displaystyle\int_{\mathbb{R}^{dN}} \exp(-\beta V(X))\, dX}, \tag{1}$$

where A is the observable of interest and $\beta = 1/(k_B T)$ is the inverse temperature. Computing such canonical averages is a standard task of computational materials science. Of course, the major difficulty comes from the N-fold integral, where N, the number of particles, is extremely large. One possible method is to compute (1) as a long-time average

$$\langle A \rangle = \lim_{T \to +\infty} \frac{1}{T} \int_0^T A(X_t)\, dt \tag{2}$$

along the trajectory generated by the stochastic differential equation

$$dX_t = -\nabla V(X_t)\, dt + \sqrt{2\beta^{-1}}\, dW_t, \tag{3}$$

where W_t is a standard dN-dimensional Brownian motion.

It is often the case that observables of interest do not depend on the positions of *all* the atoms, but only on *some* of them (for instance, because these atoms are located in a region of interest, where some particular phenomenon occurs). We assume that this set of interesting atoms (also called *repatoms*) is given *a priori*, and we denote by X_r their positions. We hence write

$$X = \left(X^1, \ldots, X^N\right) = (X_r, X_c), \quad X_r \in \mathbb{R}^{dN_r}, \quad X_c \in \mathbb{R}^{dN_c}, \quad N = N_r + N_c,$$

and our aim is to compute (1) for such observables, that is

$$\langle A \rangle = Z^{-1} \int_{\mathbb{R}^{dN}} A(X_r)\, \exp(-\beta V(X))\, dX \tag{4}$$

where $Z = \displaystyle\int_{\mathbb{R}^{dN}} \exp(-\beta V(X))\, dX$, by a cheaper method than (2)-(3).

Another question of interest concerns the free energy of the reduced system,

$$A(X_r) = -\frac{1}{\beta} \ln \int_{\mathbb{R}^{dN_c}} \exp(-\beta V(X_r, X_c))\, dX_c.$$

When $N_c \to +\infty$, this energy diverges. The meaningful quantity is the free energy per (removed) particle, $A(X_r)/N_c$. Can this quantity be efficiently computed, in the limit $N_c \to +\infty$?

In [1], we have addressed these question in a one-dimensional setting, using a thermodynamic limit approach (that is, we consider the limit $N_c \to +\infty$). It turns out that actually only the structure of the physical system needs to be one-dimensional: the space in which the atoms vary may be $\mathbb{R}^d$, $d \geq 1$. Our strategy hence applies to chain-like systems, such as polymers. The output of our study is an efficient, and apparently new, computational strategy, whose accuracy is grounded on standard probability theory arguments.

We have considered the case of next-to-nearest-neighbour interactions:

$$V(X) = \sum_{i=1}^{N-1} U_1\left(X^{i+1} - X^i\right) + \sum_{i=1}^{N-2} U_2\left(X^{i+2} - X^i\right).$$

The simpler case of nearest-neighbour interactions corresponds to $U_2 \equiv 0$. In the case of a unique repatom (generalization to the case of several repatoms is easy), it turns out that the average (4) can be recast as an expectation value:

$$\langle A \rangle = \mathbb{E}\left[A\left(\frac{1}{N}\sum_{i=1}^{N} Y_i\right)\right]$$

for random variables Y_i that are a realization of a Markov chain. Hence, using a Law of Large Numbers argument, we can compute a good approximation of the average (4). In turn, limits of free energies can be handled with classical Large Deviations arguments.

Such an approach may also be considered as a first step toward the numerical analysis of methods commonly used in practice [2], and the assessment of the simplifying assumptions upon which they rely.

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Kawasaki dynamics in large volumes

Anton Bovier
(joint work with F. den Hollander and C. Spitoni)

In this talk I report on recent work on metastability in large volumes at low temperatures for conservative (Kawasaki) dynamics of a lattice gas. Let β denote the inverse temperature and let $\Lambda_\beta \subset \mathbb{Z}^2$ be a square box with periodic boundary conditions such that $\lim_{\beta\to\infty} |\Lambda_\beta| = \infty$. We run the dynamics on Λ_β starting from a random initial configuration where all the droplets (= clusters of plus-spins, respectively, clusters of particles) are small. For large β, and for interaction parameters that correspond to the metastable regime, we investigate how the transition from the metastable state (with only small droplets) to the stable state (with one or more large droplets) takes place under the dynamics. This transition is triggered by the appearance of a single *critical droplet* somewhere in Λ_β. Using potential-theoretic methods, we compute the *average nucleation time* (= the first time a critical droplet appears and starts growing) up to a multiplicative factor that tends to one as $\beta \to \infty$. It turns out that this time grows as $Ke^{\Gamma\beta}/|\Lambda_\beta|$ for Glauber dynamics and $K\beta e^{\Gamma\beta}/|\Lambda_\beta|$ for Kawasaki dynamics, where Γ is the local canonical, respectively, grand-canonical energy to create a critical droplet and K is a constant reflecting the geometry of the critical droplet provided these times tend to infinity (which puts a growth restriction on $|\Lambda_\beta|$). The fact that the average nucleation time is inversely proportional to $|\Lambda_\beta|$ is referred to as *homogeneous nucleation*, because it says that the critical droplet for the transition appears essentially independently in small boxes that partition Λ_β.

A few words concerning the history of the problem. To our knowledge, the first rigorous results on Kawasaki dynamics were obtained in a series of papers by den Hollander, Olivieri, and Scoppola, partly with Nardi [7, 6]. In these papers a simplified version of the dynamics was considered: there is a finite "reaction" zone, Λ, which particles enter at a rate $\exp(-\beta\Delta)$, and leave with rate one. Within the reaction zone particles perform conservative dynamics with an dissociation energy U per bond. Again the limit $\beta \uparrow \infty$ was considered. In this situation one is close to the setting of a reversible Markov chain with finite state space and "exponentially small transition probabilities", even though there are simple random walk moves that happen with probabilities neither close to zero nor close to one. Nonetheless, the authors succeeded to use large deviation techniques, in combination with a renormalization procedure, to show results of the exponential asymptotics of the times, $\tau_{\blacksquare}$, the first time the box is completely filled with particles, of the form

$$\lim_{\beta\uparrow\infty} \mathbb{P}_{\square}\left[e^{\beta(\Gamma_d-\delta)} \le \tau_{\blacksquare} \le e^{\beta(\Gamma_d+\delta)}\right] = 1, \text{a.s}, \tag{1}$$

for all $\delta > 0$. Here $\square$ represents the configuration where the box Λ is empty.

For the same model, Bovier, den Hollander, and Nardi [4] obtained more precise results using the potential theoretic approach to metastability (for a review see

[3]. Their results (in dimension two) were of the form

$$\mathbb{E}_{\square}\tau_{\blacksquare} = K_2 \exp\left(\beta\Gamma_2^*\right)\left(1 + e^{-O(\beta)})\right) \tag{2}$$

where

$$K_2 = \frac{1}{4\pi N(\ell_c)}\frac{\ln|\Lambda|}{\Lambda}$$

and

$$N(\ell_c) = \frac{1}{3}(\ell_c - 1)\ell_c^2(\ell_c + 1)$$

plus exponentiality of $\tau_{\blacksquare}/\mathbb{E}_{\square}\tau_{\blacksquare}$.

These results are relatively simple consequences of the general theory once a fine understanding of the structure of the energy landscape near the critical droplet is obtained. It is crucial to use the fact that entropy of paths can almost completely be neglected. This makes, in particular, the computation of lower bounds on capacities relatively easy.

In the situation when Λ grows exponentially with β, and now external reservoir is present, a number of new difficulties have to be overcome. In particular, one has to understand how entropy is produced dynamically.

The key new ingredient that we used to overcome these difficulties is a variational principle, due to Berman and Konsowa [1], that allows to express capacities in reversible Markov chains in terms of a very convenient supremum over loop-free unit flows:

$$\mathrm{cap}(A,B) = \sup_{f\in\mathbb{U}_{A,B}} \mathbb{E}_{\mathcal{X}}^{f}\left[\sum_{e\in\mathcal{X}}\frac{f(e)}{\mathbb{Q}(e_a)p(e)}\right]^{-1}, \tag{3}$$

where $\mathbb{E}^f$ is the law of a directed Markov chain with transition probabilities proportional to the flow (see [2, 5, 3]).

Together with standard formulas, this allows us to prove

Theorem.

- $U < \Delta < 2\Delta$ *and*
- $\lim_{\beta\to\infty}|\Lambda_\beta|\,\rho_\beta = \infty, \qquad \lim_{\beta\to\infty}|\Lambda_\beta|\,e^{-\beta\Gamma_2^*} = 0.$

Then, for $L < \ell_c \le M \le 2\ell_c - 1$,

$$\lim_{\beta\to\infty}|\Lambda_\beta|\,\frac{4\pi}{\beta\Delta}\,e^{-\beta\Gamma_2^*}\mathbb{E}_{\nu_{\mathcal{S}_L}}\left(\tau_{\mathcal{D}_M}\right) = \frac{1}{N}$$

where

$$\mathcal{S}_L = \{\sigma\in\mathcal{S}\colon \textit{ no box of size } L_\beta \textit{ contains more than } L \textit{ particles in } \Lambda_\beta\}$$

and

$$\mathcal{D}_M = \{\sigma\in\mathcal{S}\colon \exists \textit{ a full } M\times M\textit{-square in } \Lambda_\beta\}$$

and $\nu_{\mathcal{S}_L}$ *is a particular probability measure on* $\mathcal{S}_L$ *(the "last exit biased measure").*

In this theorem the starting measure $\nu_{\mathcal{S}_L}$ is a particular measure, related to the harmonic measure, on the set $\mathcal{L}_L$ of configurations that do not contain any where agglomerations of more then L particles; in particular, no critical droplets are present in this set. The target configuration, $\mathcal{D}_M$, is the set of configurations that contain at least one $M \times M$ square that is occupied by particles, with $2\ell_c > M > \ell_c$. It would be desirable to prove results for more general initial distributions and $\mathcal{D}_M$ replaced by configurations carrying the equilibrium Gibbs measure, but technical problems at various levels still have to be overcome to reach this aim.

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Invariance principle for the random conductance model with unbounded conductances

Jean-Dominique Deuschel

(joint work with Martin Barlow)

We study two types of continuous time random walk $\{X_t, t \geq 0\}$ in an environment of i.i.d. random conductances

$$\mu_e \in [1, \infty), \quad e = \{x, y\}, x, y \in \mathbb{Z}^d, \quad |x - y| = 1 :$$

the variable speed random walk, generated by

$$L_V f(x) = \sum_y \mu_{\{x,y\}}(f(y) - f(x))$$

and the constant speed random walk with generator

$$L_C f(x) = \frac{1}{\mu_x} \sum_y \mu_{\{x,y\}}(f(y) - f(x)),$$

where $\mu_x = \sum_y \mu_{\{x,y\}}$.

We consider the case where the conductances are bounded from below

$$1 \leq \mu_e < \infty$$

but not from above, so that we even allow $\mathbb{E}[\mu_e] = \infty$. The bounded case with

$$0 < c_1 < \mu_e < c_2 < \infty$$

has been solved in [6], whereas the degenerate case but bounded above is treated in both [3] and [5], see also [2] and [4] for random walks on the supercritical percolation cluster.

We first obtain heat kernel Gaussian bounds for the variable speed random walk. These bounds are very similar to the ones obtained in [1] for the random walk on the super critical percolation cluster, that is they hold up to a random time τ with stretched exponential tails.

Next we prove a quenched invariance principle for the rescaled process

$$X_t^\epsilon = \epsilon X_{t/\epsilon^2}, \quad t \geq 0.$$

That is, for $\mathbb{P}$ almost all environments, X^ϵ converges weakly as $\epsilon \to 0$ to a Brownian motion with variances σ_V^2 and σ_C^2 for the variable speed, respectively constant speed random walk. The variance $\sigma_V^2 > 0$ of the variable speed random walk is always non-degenerate, whereas $\sigma_C^2 \neq 0$ if and only if $\mathbb{E}[\mu_e] < \infty$. The main difference in the behavior of the constant speed versus variable speed random walk, is the possibility of traps for the constant speed random walk, which do not occur for the variable speed random walk.

The proof of the invariance principle is based on the construction of the corrector: χ so that

$$M_t = X_t - \chi(X_t)$$

is a martingale. Invariance principle for the martingale part $M_t^\epsilon = \epsilon M_{t/\epsilon^2}$ is rather standard, and the heat kernel estimates can be used in controlling the corrector.

In case of infinite mean $\mathbb{E}[\mu_e] = \infty$, the standard $L^2(\mathbb{P})$ calculus does not apply, instead we construct the corrector for the time discretized process. Here again both lower and upper heat kernel estimates play a crucial role.

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Passage from quantum to classical molecular dynamics

Jannis Giannoulis
(joint work with Gero Friesecke)

Within quantum mechanics an approximate description of the dynamics of M atoms is given via the wavefunction of their nuclei $\Psi_\epsilon(\cdot,t) \in \mathrm{L}^2(\mathbb{R}^d)$, $d = 3M$, at time $t \in \mathbb{R}$ governed by the time-dependent Schrödinger equation

$$\text{(SE)} \qquad \begin{cases} i\epsilon\partial_t \Psi_\epsilon(\cdot,t) = \big(-\frac{\epsilon^2}{2}\Delta + U\big)\Psi_\epsilon(\cdot,t) \quad \text{for } t \in \mathbb{R}, \\ \Psi_\epsilon(\cdot,0) = \Psi_\epsilon^0 \end{cases}$$

(in atomic units $m_e = |e| = \hbar = 1$ and for nuclei of equal mass), where $\epsilon^2 := m_e/m_n$ is the ratio of electronic to nucleonic mass in the atoms and $U : \mathbb{R}^d \to \mathbb{R}$ is the non-relativistic Born-Oppenheimer ground state potential energy surface, defined as follows. Let $Z_1, .., Z_M \in \mathbb{N}$ and $R_1, .., R_M \in \mathbb{R}^3$ denote the charges and positions of the nuclei, and let N denote the number of electrons in the system (usually $N = \sum_{\alpha=1}^M Z_\alpha$). Then for $x = (R_1, \ldots, R_M) \in \mathbb{R}^d$

$$(1)\ \ U = E_{e\ell} + V_{nn}, \quad E_{e\ell}(x) = \inf_\psi \langle \psi, H_{e\ell}(x)\psi\rangle, \quad V_{nn}(x) = \sum_{1\le\alpha<\beta\le M} \frac{Z_\alpha Z_\beta}{|R_\alpha - R_\beta|},$$

$$H_{e\ell}(x) = \sum_{i=1}^N \Big(-\frac{1}{2}\Delta_{r_i} - \sum_{\alpha=1}^M \frac{Z_\alpha}{|r_i - R_\alpha|}\Big) + \sum_{1\le i<j\le N} \frac{1}{|r_i - r_j|},$$

where the $r_i \in \mathbb{R}^3$ denote electronic coordinates and the infimum is taken over the usual subset of $\mathrm{L}^2((\mathbb{R}^3 \times \mathbb{Z}_2)^N; \mathbb{C})$ of normalized, antisymmetric electronic states belonging to the domain $\mathrm{H}^2((\mathbb{R}^3 \times \mathbb{Z}_2)^N; \mathbb{C})$ of $H_{e\ell}(x)$. The electronic part of the energy $E_{e\ell}$ can be shown to be bounded and globally Lipschitz. Hence, the potential (1) satisfies the standard Kato-type condition $U = U_b + U_s$ with

$$(2)\ \ U_b \in \mathrm{L}^\infty(\mathbb{R}^d), \quad U_s(x) = \sum_{1\le\alpha<\beta\le M} V_{\alpha\beta}(R_\alpha - R_\beta), \quad V_{\alpha\beta} \in \mathrm{L}^2(\mathbb{R}^3) + \mathrm{L}^\infty(\mathbb{R}^3).$$

For such potentials and any initial state $\Psi_\epsilon^0 \in \mathrm{H}^2(\mathbb{R}^d)$, it is known (cf. [2]) that (SE) has a unique solution $\Psi_\epsilon \in \mathrm{C}(\mathbb{R}; \mathrm{H}^2(\mathbb{R}^d)) \cap \mathrm{C}^1(\mathbb{R}; \mathrm{L}^2(\mathbb{R}^d))$ (with $\|\Psi_\epsilon(\cdot,t)\| = \|\Psi_\epsilon^0\|$ for all $t \in \mathbb{R}$; $\|\cdot\|$ the $\mathrm{L}^2(\mathbb{R}^d)$ norm), establishing quantum molecular dynamics.

Since ϵ^2 is very small in nature ($\sim 1/2000$ for hydrogen and even smaller for the other atoms), it is natural to raise the question whether some (and, if yes, what) sort of (approximate) description of molecular dynamics can be obtained in the limit $\epsilon \to 0$, when starting from the above framework of quantum molecular dynamics, which itself already constitutes an approximation to full Schrödinger dynamics for electrons and nuclei. More precisely, one asks for a limiting object W obtained from a sequence of solutions $\{\Psi_\epsilon\}$ to (SE) when $\epsilon \to 0$, and for an evolution equation describing the dynamics of W.

These questions are very well known. They are dealed with in the theory of semiclassical limits, where typically ϵ plays the rôle of Planck's constant $\hbar$. We

rely for our investigation on [3]. Based on ideas of E. Wigner [4], one applies on $\Psi_\epsilon(\cdot,t) \in \mathrm{L}^2(\mathbb{R}^d)$ the Wigner transformation

$$W_\epsilon(x,p,t) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \Psi_\epsilon(x+\frac{\epsilon y}{2},t)\overline{\Psi_\epsilon(x-\frac{\epsilon y}{2},t)} e^{-ip\cdot y}\,\mathrm{d}y. \tag{3}$$

Roughly speaking W_ϵ can be interpreted as a joint position and momentum density of the system, since they are given by

$$\int_{\mathbb{R}^d} W_\epsilon(x,p,t)\mathrm{d}p = |\Psi_\epsilon(x,t)|^2, \quad \int_{\mathbb{R}^d} W_\epsilon(x,p,t)\mathrm{d}x = \Big|\frac{1}{(2\pi\epsilon)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-i\frac{p}{\epsilon}\cdot x}\Psi_\epsilon(x,t)\mathrm{d}x\Big|^2,$$

respectively. Note however, that for general $\Psi_\epsilon(\cdot,t) \in \mathrm{L}^2(\mathbb{R}^d)$ the associated Wigner function $W_\epsilon(\cdot,t)$ is not necessarily nonnegative.

By Wigner transforming (SE) one obtains (cf. [1, Lemma 2.1]) the following formulation of quantum molecular dynamics, fully equivalent to (SE).

Lemma 1. *The Wigner transform* (3) *of any solution* $\Psi_\epsilon \in \mathrm{C}(\mathbb{R};\mathrm{H}^2(\mathbb{R}^d)) \cap \mathrm{C}^1(\mathbb{R};\mathrm{L}^2(\mathbb{R}^d))$ *to* (SE) *with* $U = U_b + U_s$ *as in* (2) *satisfies*

$$W_\epsilon \in \mathrm{C}^1(\mathbb{R};\mathrm{L}^\infty(\mathbb{R}^{2d})), \quad \frac{\partial}{\partial x_i}W_\epsilon, \frac{\partial^2}{\partial x_i \partial x_j}W_\epsilon, f_\epsilon \in \mathrm{C}(\mathbb{R};\mathrm{L}^\infty(\mathbb{R}^{2d})) \quad (i,j=1,\ldots,d)$$

and solves the Wigner equation

$$\text{(WE)}\quad \partial_t W_\epsilon = -\,p\cdot\nabla_x W_\epsilon \underbrace{-\frac{i}{(2\pi)^d}\int_{\mathbb{R}^d} \frac{U(x+\frac{\epsilon y}{2})-U(x-\frac{\epsilon y}{2})}{\epsilon}\Psi_\epsilon(x+\frac{\epsilon y}{2},t)\overline{\Psi_\epsilon(x-\frac{\epsilon y}{2},t)}e^{-ip\cdot y}\mathrm{d}y}_{=f_\epsilon(x,p,t)}.$$

From now on we focus on the specific energy potential (1). Due to the Lipschitz continuity of $E_{e\ell}$ and the fact that the singular set of V_{nn} is of measure zero, the difference quotient in f_ϵ satisfies $\frac{U(x+\frac{\epsilon y}{2})-U(x-\frac{\epsilon y}{2})}{\epsilon} \to \nabla U(x)\cdot y$ a.e. as $\epsilon \to 0$. Splitting f_ϵ into a term containing $\nabla U(x)\cdot y$ and a term containing the above difference quotient minus its limit, assuming that the latter tends to zero as $\epsilon \to 0$ and noticing that $ye^{-ip\cdot y} = i\nabla_p e^{-ip\cdot y}$, we obtain by passing formally to the limit $W_\epsilon \to W$ in (WE) the Liouville equation

$$\text{(LE)}\qquad \partial_t W = -p\cdot\nabla_x W + \nabla U(x)\cdot\nabla_p W. \tag{LE}$$

This is the transport equation for classical molecular dynamics in $\mathbb{R}^d \times \mathbb{R}^d$ with potential U: $\dot{x} = p$, $\dot{p} = -\nabla U(x)$. The object of our analysis [1] is exactly to justify rigorously the transition from (WE) to (LE).

Our work extends results obtained in [3] (which in the time-dependent case rely on smooth potentials U; for further related work cf. the references in [1, 3]). There it is shown that the sequence of Wigner functions $\{W_\epsilon\}$ to any bounded sequence $\{\Psi_\epsilon\} \subset \mathrm{L}^2(\mathbb{R}^d)$ contains a subsequence which converges weak* in $\mathcal{A}'$ to a nonnegative Radon measure of finite mass $W \in \mathcal{M}(\mathbb{R}^{2d})$ (Wigner measure), where

$$\mathcal{A} = \left\{\phi \in \mathrm{C}_0(\mathbb{R}^{2d}) \,|\, \|\phi\|_{\mathcal{A}} = \int_{\mathbb{R}^d} \sup_{x\in\mathbb{R}^d} \left|\int_{\mathbb{R}^d} e^{-ip\cdot y}\phi(x,p)\,\mathrm{d}p\right| \,\mathrm{d}y < \infty\right\}.$$

Moreover, it is shown that if $\{\Psi_\epsilon\}$ is normalized and the sequences of position and momentum densities are both tight (see below) then W is a probability measure.

Extending the above construction to one of time-dependent Wigner measures, and, more significantly, exploiting the specific Coulombic nature of the discontinuities of U, we obtain the following main result (see [1] for the complete proof).

Theorem 2. *Suppose $U : \mathbb{R}^d \to \mathbb{R}$ is the Born-Oppenheimer potential energy surface (1) of any molecule, or more generally $U = U_b + U_s$ with $U_b \in \mathrm{W}^{1,\infty}(\mathbb{R}^d)$,*

$$U_s(x) = \sum_{1\le\alpha<\beta\le M} \frac{C_{\alpha\beta}}{|R_\alpha - R_\beta|}, \quad C_{\alpha\beta} \ge 0, \quad x = (R_1, \dots, R_M) \in \mathbb{R}^d.$$

Let $\Psi^0_\epsilon \in \mathrm{H}^2(\mathbb{R}^d)$, $\|\Psi^0_\epsilon\| = 1$, $\|(-\frac{\epsilon^2}{2}\Delta + U)\Psi^0_\epsilon\| \le c \ \forall\ \epsilon > 0$, $\{|\Psi^0_\epsilon|^2\}$ tight, i.e.,

$$\lim_{R\to\infty} \limsup_{\epsilon\to 0} \int_{|x|>R} |\Psi^0_\epsilon(x)|^2 \,\mathrm{d}x = 0.$$

Let $\Psi_\epsilon \in \mathrm{C}(\mathbb{R}; \mathrm{H}^2(\mathbb{R}^d)) \cap \mathrm{C}^1(\mathbb{R}; \mathrm{L}^2(\mathbb{R}^d))$ be the corresponding solution to the time-dependent Schrödinger equation (SE) *and W_ϵ its Wigner transform* (3)*. Then:*

(i) For a subsequence, $W_\epsilon \rightharpoonup W$ in $\mathcal{D}'(\mathbb{R}^{2d+1})$.

(ii) $W \in \mathrm{C}_{weak*}(\mathbb{R}; \mathcal{M}(\mathbb{R}^{2d}))$, and $W(t)$ is a probability measure for all t, that is to say $W(t) \ge 0$ and $\int_{\mathbb{R}^{2d}} \mathrm{d}W(t) = 1$.

(iii) For all t we have $\nabla U_s \in \mathrm{L}^1(\,\mathrm{d}W(t))$, and $W(t)(\mathcal{S} \times \mathbb{R}^d) = 0$, where $\mathcal{S}$ is the singular set $\{x = (R_1, \dots, R_M) \in \mathbb{R}^{3M} \mid R_\alpha = R_\beta$ for some $\alpha \ne \beta$ with $C_{\alpha\beta} \ne 0\}$.

(iv) If $\Omega \subseteq \mathbb{R}^d$ is any open set such that $U_b \in \mathrm{C}^1(\Omega)$, then W is a global weak solution of the Liouville equation (LE) *in $\Omega \times \mathbb{R}^d \times \mathbb{R}$, that is to say*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^{2d}} \Big(\partial_t + p \cdot \nabla_x - \nabla U(x) \cdot \nabla_p\Big) \phi(x, p, t) \,\mathrm{d}W(t) \,\mathrm{d}t = 0$$

for all $\phi \in \mathrm{C}^\infty_0(\Omega \times \mathbb{R}^d \times \mathbb{R})$.

In this sense we extended previous results for smooth potentials by establishing the validity of the Liouville equation (LE) also for regions $\Omega \subset \mathbb{R}^d$ which may include Coulomb singularities of U. However, the question whether the above results remain valid also for regions which contain points of jump singularities of ∇U corresponding to eigenvalue crossings, as well as questions related to the uniqueness of the limiting Wigner measure W are addressed in work which is still in progress and which will hopefully lead to results of analogous analytical rigour.

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The parabolic Anderson model in catalytic random media

Jürgen Gärtner
(joint work with Frank den Hollander and Gregory Maillard)

We consider the parabolic Anderson problem

$$\begin{aligned} \partial_t u &= \kappa\Delta u + \gamma\xi u \qquad \text{on } \mathbb{R}_+ \times \mathbb{Z}^d, \\ u|_{t=0} &= 1 \qquad\qquad\quad \text{on } \mathbb{Z}^d \end{aligned}$$

in a space-time homogeneous random medium $\xi = \xi(t,x)$, where κ is a positive diffusion constant, Δ is the lattice Laplacian on $\mathbb{Z}^d$ and γ denotes a positive coupling constant. Our focus is on random media ξ for which the model describes a *diffusing reactant* u under the influence of a *moving catalyst* ξ.

We consider the following examples of catalytic media:

(a) *Independent random walks:*

$$\xi(t,\cdot) = \sum_k \delta_{Y_k(t)}(\cdot),$$

where (Y_k) is an infinite collection of random walks on $\mathbb{Z}^d$ with generator Δ in Poisson equilibrium with density $\varrho \in (0,\infty)$.

(b) *Simple symmetric exclusion dynamics* $\xi(t,\cdot)$ in Bernoulli equilibrium with density $\varrho \in (0,1)$.

(c) *Simple symmetric voter dynamics* $\xi(t,\cdot)$ with votes '0' and '1' and initial distribution being either the Bernoulli product measure or the ergodic invariant measure ($d \geq 3$) with density $\varrho \in (0,1)$.

We investigate the behavior of the annealed Lyapunov exponents

$$\lambda_p(\kappa) = \lim_{t\to\infty} \frac{1}{pt} \log \langle u(t,0)^p \rangle ,$$

i.e., the exponential growth rates as $t \to \infty$ of the successive moments of the solution u as a function of the moment $p \in \mathbb{N}$ and the diffusion constant $\kappa \in [0,\infty)$. Whereas the concrete behavior of $\lambda_p(\kappa)$ as $\kappa \to 0$ is model-dependent, our results indicate a *universal* behavior for $\kappa \to \infty$ in high dimensions.

Theorem 1 ([1])**.** *For the catalytic medium (a), $d \geq 3$ and $p\gamma G_d(0) < 1$,*

$$\lim_{\kappa\to\infty} \kappa(\lambda_p(\kappa) - \varrho) = \varrho\gamma^2 G_d(0) + 1_{d=3}[\varrho\gamma^2 p]^2 \mathcal{P}_3 \tag{1}$$

with $G_d(0) = \int_0^\infty p_t(0)\,dt$ the Green function at 0 of simple random walk (with generator Δ and transition kernel $p_t(x)$) and

$$\mathcal{P}_3 = \sup_{\|f\|_2=1} \left[\int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dy\, \frac{f^2(x) f^2(y)}{4\pi|x-y|} - \|\nabla f\|_2^2 \right]$$

the expression appearing in the polaron variational problem.

Hence, for $d \geq 3$, the asymptotics (1) is characterized by a *Green* term and, for $d = 3$, by an additional *polaron* term. In particular, for $d = 3$ and κ large, the solution u is *fully intermittent*, i.e., we have the *strict* inequality $\lambda_1(\kappa) < \lambda_2(\kappa) < \lambda_3(\kappa) < \dots$ For $d \geq 4$, the question about intermittency for large κ is still open. (For an explanation of intermittency see e.g. [5], Section 1.3.)

Although our approach for the exclusion dynamics (b) is totally different from that for the independent random walks (a), in both cases the results are essentially the same.

Theorem 2 ([2],[3]). *For the catalytic medium (b) and $d \geq 3$,*

$$\lim_{\kappa\to\infty} \kappa(\lambda_p(\kappa) - \varrho) = \varrho(1-\varrho)\gamma^2 G_d(0) + 1_{d=3}[\varrho(1-\varrho)\gamma^2 p]^2 \mathcal{P}_3.$$

In the proofs of both theorems we heavily use the reversibility of the catalytic medium $\xi(t,\cdot)$. Since the voter dynamics is non-reversible, these methods fail in the case (c). But the heuristics behind of the proofs ([2], Section 1.5) allows us to formulate the following conjecture.

Conjecture 3 ([4]). *For the catalytic medium (c), both initial distributions and $d \geq 5$,*

$$\lim_{\kappa\to\infty} \kappa(\lambda_p(\kappa) - \varrho) = \frac{\varrho(1-\varrho)\gamma^2}{G_d(0)} G_d^*(0) + 1_{d=5} \left[\frac{\varrho(1-\varrho)\gamma^2}{G_d(0)} p\right]^2 \mathcal{P}_5 \tag{2}$$

with $G_d^*(0) = \int_0^\infty t p_t(0)\, dt$ and

$$\mathcal{P}_5 = \sup_{\|f\|_2=1} \left[\int_{\mathbb{R}^5} dx \int_{\mathbb{R}^5} dy\, \frac{f^2(x) f^2(y)}{16\pi|x-y|} - \|\nabla f\|_2^2\right].$$

Again, the asymptotics (2) is given by a *Green*-like term. For $d = 5$, an additional *polaron*-like term appears.

For the behavior of the Lyapunov exponents $\lambda_p(\kappa)$ and intermittency of u for small κ and in low dimensions we refer to the papers listed below.

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Wegner estimate and level repulsion for Wigner matrices

László Erdős
(joint work with Benjamin Schlein and Horng-Tzer Yau)

Consider $N \times N$ random hermitian matrices $H = (h_{ij})$, $h_{ij} = \overline{h_{ji}}$. They form a *hermitian Wigner ensemble* if

$$h_{ij} = N^{-1/2} z_{ij}, \quad (i < j), \quad \text{and} \quad h_{ii} = N^{-1/2} x_{ii}, \tag{1}$$

where $z_{ij} \in \mathbb{C}$ $(i < j)$ and $x_{ii} \in \mathbb{R}$ are independent random variables with mean zero and we assume that z_{ij} $(i < j)$ all have a common distribution $\mathrm{d}\nu$. The diagonal elements, x_{ii}, also have a common distribution that may be different from $\mathrm{d}\nu$. Let $\mathbb{P}$ and $\mathbb{E}$ denote the probability and the expectation value w.r.t the joint distribution. We assume that there is a positive δ such that

$$\mathbb{E}\, e^{\delta |z_{ij}|^2} < \infty, \qquad \mathbb{E}\, e^{\delta x_{ii}^2} < \infty\,.$$

Moreover, we assume that either the distribution $\mathrm{d}\nu(z)$ depends only on $|z|$, or that the real an imaginary parts of z are independent. We set a normalization condition $\mathbb{E}\,|z_{ij}|^2 = 1$. Let $\mu_1 \le \mu_2 \le \ldots \le \mu_N$ denote the eigenvalues of H.

The normalization is chosen such that the bulk of the spectrum of H is $[-2, 2]$. In particular, the typical scaling between neighboring eigenvalues is of order $1/N$, $\mu_j - \mu_{j-1} \sim O(1/N)$. It is widely believed that the local eigenvalue statistics follows a universal pattern, namely the two-point correlation function is given by the Wigner-Dyson sine kernel, and higher order correlations are given by a determinant. In particular, neighboring eigenvalues repel each other. This has been justified in case of $\mathrm{d}\nu(z) = \exp(-|z|^2)$, i.e. for the Gaussian Unitary Ensemble (GUE) by Dyson. The same result was later extended by Pastur and Shcherbina [9] and Deift [2] to a big class of unitary invariant matrix ensembles, given by the law $\mathrm{d}\mathbb{P}(H) = \exp(-\mathrm{Tr}\, V(H))\mathrm{d}H$, where V satisfies certain growth and smoothness conditions. Johanssen [7] has established the sine-kernel for Wigner matrices if $\mathrm{d}\nu$ has a Gaussian component. All results about the sine-kernel have heavily relied on explicit formulas for the joint distribution of the eigenvalues. For example, the density function of the eigenvalues of the unitary ensemble $\mathrm{d}\mathbb{P}(H) = \exp(-\mathrm{Tr}\, V(H))\mathrm{d}H$ is given by

$$p(\mu_1, \mu_2, \ldots, \mu_N) = (\text{const.}) \prod_{i<j} (\mu_i - \mu_j)^2 \prod_j e^{-V(\mu_j)}. \tag{2}$$

Similar explicit formula is not available in case of the Wigner matrices. For general Wigner matrices it has been proven that the density of eigenvalues follows the Wigner semicircle law [11] and that the largest eigenvalues follow the Tracy-Widom distribution [10]. In this talk, based upon the work [5], we present several theorems towards the universality conjecture for Wigner matrices.

The Wigner semicircle law was originally proven macroscopically, more precisely it was shown that the empirical counting measure

$$\varrho_N(E) = \frac{1}{N}\sum_\alpha \delta(\mu_\alpha - E)$$

converges weakly, as $N \to \infty$, to the semicircle density $\varrho_{sc}(E) = \frac{1}{2\pi}\sqrt{4-E^2}$, $|E| \le 2$, in probability. Such result corresponds to a law of large numbers for the number of eigenvalues $\mathcal{N}(I)$ in an interval I of length $\eta = |I| = O(1)$, i.e. it involves the cumulative statistics of $O(N)$ eigenvalues. In our first result, we show that semicircle law holds on the shortest possible scales of order K/N, $K \gg 1$.

Theorem 1. *Fix $\kappa, \varepsilon > 0$ and let $E \in [-2+\kappa, 2-\kappa]$. There exists $K = K(\kappa,\varepsilon)$ and two constants C, c depending only on the distribution $\mathrm{d}\nu$ such that*

$$\mathbb{P}\left\{\left|\frac{\mathcal{N}[E-\frac{\eta}{2}, E+\frac{\eta}{2}]}{N\eta} - \varrho_{sc}(E)\right| \ge \varepsilon\right\} \le Ce^{-c\varepsilon\sqrt{N\eta}}$$

for all $N \ge N(\varepsilon)$ and $\eta \in (0,1]$ such that $N\eta \ge K$.

This result improves our previous works [3, 4] where we established semicircle law on scales of order $\eta \gg (\log N)^8/N$ and with a weaker estimate on the probability. Earlier related works [1, 6, 8] all work on scales at least $\eta \sim N^{-1/2}$ or larger. As a corollary, we obtain the complete delocalization of eigenvectors:

Corollary 2. *For any $\kappa > 0$ and $2 \le p < \infty$ there exist $M_0 = M_0(\kappa, p)$, $N_0 = N_0(\kappa, p)$ and $c = c(\kappa, p) > 0$ such that for any interval $I \subset [-2+\kappa, 2-\kappa]$ of length $|I| = 1/N$ we have*

$$\mathbb{P}\left\{\exists\ \mathbf{v} \text{ with } H\mathbf{v} = \mu\mathbf{v},\ \|\mathbf{v}\| = 1,\ \mu \in I \text{ and } \|\mathbf{v}\|_p \ge MN^{\frac{1}{p}-\frac{1}{2}}\right\} \le e^{-c\sqrt{M}} \tag{3}$$

for all $M \ge M_0$, all $N \ge N_0$.

Furthermore, for any $\kappa > 0$ there exist $M_0 = M_0(\kappa)$, $N_0 = N_0(\kappa)$ and $c = c(\kappa)$ such that

$$\mathbb{P}\left\{\exists\ \mathbf{v} \text{ with } H\mathbf{v} = \mu\mathbf{v},\ \|\mathbf{v}\| = 1,\ \mu \in I \text{ and } \|\mathbf{v}\|_\infty \ge \frac{M}{N^{1/2}}\right\} \le e^{-c\sqrt{M}} \tag{4}$$

for all $M \ge M_0(\log N)^2$, all $N \ge N_0$.

Our second result is an upper bound on the tail distribution of the eigenvalue gap. The obtained subexponential decay is probably not optimal, a Gaussian decay of the tail is expected.

Theorem 3. *Fix an energy $E \in [-2+\kappa, 2-\kappa]$. Denote by μ_α the largest eigenvalue below E and assume that $\alpha \le N-1$. Then there are positive constants C and c depending on κ such that*

$$\mathbb{P}\Big(\mu_{\alpha+1} - E \ge \frac{K}{N},\ \alpha \le N-1\Big) \le C\, e^{-c\sqrt{K}} \tag{5}$$

for any $N \ge 1$ and any $K \ge 0$.

On scales $\eta \leq 1/N$, the empirical density of states measure, $\varrho_N(E)$, is fluctuating. However its expectation, the averaged density of states, $\mathbb{E}\,\varrho_N(E)$, satisfies the Wegner estimate, which is our third main result:

Theorem 4. *Assume that the distribution* $\mathrm{d}\nu(z) = \mathrm{d}\nu(x,y)$, $z = x + iy$, *of the matrix elements are sufficiently smooth, i.e. its Fourier transform satisfies*

$$\widehat{\nu}(t,s) \leq \frac{C}{(1 + c(t^2+s^2))^5}. \tag{6}$$

Let $\kappa > 0$, *choose an energy* $|E| < 2 - \kappa$ *and consider an energy interval* $I = [E - \eta/2, E + \eta/2]$ *and set* $\varepsilon = N\eta$, *with* $\varepsilon > 0$. *Let* $\mathcal{N}_I$ *be the number of eigenvalues in* I *and assume* $N \geq 10$. *Then*

$$\mathbb{P}(\mathcal{N}_I \geq 1) \leq \mathbb{E}\,\mathcal{N}_I^2 \leq C\,\varepsilon \tag{7}$$

uniformly in N *and* E. *In particular,*

$$\sup_{I \subset [-2+\kappa, 2-\kappa]} \sup_{N \geq 10} \mathbb{E}\Big[\frac{\mathcal{N}_I}{N|I|}\Big] \leq C\,, \tag{8}$$

and therefore the averaged density of states, $\mathbb{E}\,\varrho_N(E)$, *is an absolutely continuous measure with a uniformly bounded density, i.e.*

$$\sup_{|E| \leq 2-\kappa} \sup_{N \geq 10} \mathbb{E}\,\varrho_N(E) \leq C \tag{9}$$

The constant C *in* (7), (8) *and* (9) *depends only on* κ *and on the distribution* $\mathrm{d}\nu$,

Finally, the following theorem establishes an upper bound on the level repulsion.

Theorem 5. *Let* $\kappa > 0$, *fix an energy* E *with* $|E| < 2 - \kappa$, *set* $\eta = \varepsilon/N$ *and let* $\mathcal{N}_\eta$ *be the number of eigenvalues in* $I_\eta = [E - \eta/2, E + \eta/2]$. *Fix* $k \in \mathbb{N}$, *and assume that* $\widehat{\nu}$ *satisfies a bound* (6) *with exponent* $k^2 + 5$ *instead of 5. Then, there exists a constant* $C > 0$, *depending on* k *and* κ, *such that*

$$\mathbb{P}(\mathcal{N}_\eta \geq k) \leq C\,\varepsilon^{k^2} \tag{10}$$

for all $\varepsilon > 0$ *and uniformly for all* $N \geq N_0(k)$.

The exponent k^2 is optimal for the GUE case, as it is a consequence of the appearance of the factor $\prod_{i<j}(\mu_i - \mu_j)^2$ in the joint density function (2).

The common starting point of all proofs is the following upper bound on the number of eigenvalues $\mathcal{N}_I$ in any interval $I = [E - \frac{\eta}{2}, E + \frac{\eta}{2}]$.

$$\mathcal{N}_I \leq C\eta\,\mathrm{Im} \sum_{k=1}^{N} \left[h_{kk} - z - \frac{1}{N} \sum_{\alpha=1}^{N-1} \frac{\xi_\alpha^{(k)}}{\lambda_\alpha^{(k)} - z} \right]^{-1} \qquad \xi_\alpha^{(k)} = |\mathbf{u}_\alpha^{(k)} \cdot \mathbf{a}^{(k)}|^2, \tag{11}$$

where $z = E + i\eta$. Here $\lambda_\alpha^{(k)}$ and $\mathbf{u}_\alpha^{(k)}$ denote the eigenvalues and eigenvectors of the $(N-1) \times (N-1)$ matrix $B^{(k)}$ obtained from H by removing the k-th row and the k-th column. The vector $\mathbf{a}^{(k)} \in \mathbb{C}^{N-1}$ is the k-th column of H after removing the diagonal element.

Using the estimate $\mathrm{Im}(a+bi)^{-1} \le (a^2+b^2)^{-1/2}$ on the right hand side of (11), we have

$$\mathcal{N}_I \le C\eta \sum_{k=1}^{N} \frac{1}{(a_k^2+b_k^2)^{1/2}} \tag{12}$$

with

$$a_k := \eta + \frac{1}{N}\sum_{\alpha=1}^{N-1} \frac{\eta \xi_\alpha^{(k)}}{(\lambda_\alpha^{(k)}-E)^2+\eta^2}, \qquad b_k := h_{kk} - E - \frac{1}{N}\sum_{\alpha=1}^{N-1} \frac{(\lambda_\alpha^{(k)}-E)\xi_\alpha^{(k)}}{(\lambda_\alpha^{(k)}-E)^2+\eta^2}\ ,$$

where a_k and b_k are the imaginary and real part, respectively, of the reciprocal of the summands in (11). Theorems 1 and 3 rely only on the imaginary part, i.e. b_k in (12) is neglected. In the proofs of Theorems 4 and 5, however, we make an essential use of b_k as well. Since typically $1/N \le c|\lambda_\alpha^{(k)} - E|$, we note that a_k^2 is much smaller than b_k^2 if $\eta \ll 1/N$ and this is the relevant regime for the Wegner estimate and for the level repulsion. Assuming a smoothness condition on the distribution $\mathrm{d}\nu$, the distribution of the variables $\xi_\alpha^{(k)}$ will also be smooth. Although $\xi_\alpha^{(k)}$ are not independent for different α's, they are sufficiently decorrelated so that the distribution of b_k inherits some smoothness which will make the expectation value $(a_k^2+b_k^2)^{-p/2}$ finite for certain $p>0$. This will give a bound on the p-th moment on $\mathcal{N}_I$ which will imply (7) and (10).

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Spectral measure of heavy tailed band and covariance random matrices

Amir Dembo

Consider the $N \times N$ symmetric matrix $\mathbf{A}_N^\sigma$ of entries $A_N^\sigma(i,j) = a_N^{-1}\sigma(\frac{i}{N}, \frac{j}{N})x_{ij}$ for $i \le j$, where $(x_{ij}, 1 \le i \le j < \infty)$ is an infinite array of i.i.d real variables, $\sigma : [0,1]^2 \to \mathbb{R}$ is a square integrable non-random, measurable function such that $\sigma(x,y) = \sigma(y,x)$ and

$$a_N := \inf\{u : \mathbb{P}[|x_{ij}| \ge u] \le \frac{1}{N}\}\,. \tag{1}$$

Such matrices are called "band matrices" in view of the possible choice of $\sigma(x,y) = \mathbf{1}_{|x-y|\le b}$. Let $(\lambda_1, \cdots, \lambda_N)$ denote the eigenvalues of $\mathbf{A}_N^\sigma$. In a joint work [1] with Serban Belinschi and Alice Guionnet we study the asymptotic behavior of the spectral measures $\hat{\mu}_{\mathbf{A}_N^\sigma} := \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i}$ as $N \to \infty$, in case of heavy tailed entries, where the common distribution of the absolute values of the x_{ij}'s is in the domain of attraction of an α-stable law, for $\alpha \in]0,2[$. That is, there exists a slowly varying function $L(\cdot)$ such that for any $u > 0$,

$$\mathbb{P}(|x_{ij}| \ge u) = L(u)u^{-\alpha}\,, \tag{2}$$

hence $a_N = L_0(N)N^{1/\alpha}$ (with $L_0(\cdot)$ a slowly varying function).

Predictions about the limiting spectral measure in case $\sigma(\cdot,\cdot) \equiv 1$ (the heavy tail analog of Wigner's theorem) have been made in [3] and rigorously verified in [2] (c.f. [2, Section 8]). We follow in [1] the approach of [2], which consists of proving the convergence of the resolvent, i.e. of the mean of the Cauchy-Stieltjes transform of the spectral measure, outside of the real line, by proving tightness and characterizing uniquely the possible limit points. In the latter task, for each $\alpha \in (0,2)$ the limiting spectral measure of $\mathbf{A}_N^\sigma$ is characterized in terms of the entire functions

$$g_\alpha(y) := \int_0^\infty t^{\frac{\alpha}{2}-1}e^{-t}\exp\{-t^{\frac{\alpha}{2}}y\}dt\,, \tag{3}$$

$$h_\alpha(y) := \int_0^\infty e^{-t}\exp\{-t^{\frac{\alpha}{2}}y\}dt = 1 - \frac{\alpha}{2}yg_\alpha(y)\,. \tag{4}$$

We define for any $\alpha \in (0,2)$ the usual branch of the power function $x \mapsto x^\alpha$, which is the analytic function on $\mathbb{C}\backslash\mathbb{R}^-$ such that $(i)^\alpha = e^{i\frac{\pi\alpha}{2}}$. This amounts to choosing $x^\alpha = r^\alpha e^{i\alpha\theta}$ when $x = re^{i\theta}$ with $\theta \in]-\pi,\pi[$. We also adopt the notation $x^{-\alpha}$ for $(x^{-1})^\alpha$. Recall [2, Theorem 1.4] that in case $\sigma(\cdot,\cdot) \equiv 1$, the limiting spectral measure μ_α for Wigner matrices with entries in the domain of attraction of an α-stable law has for $z \in \mathbb{C}^+ = \{z \in \mathbb{C} : \Im(z) > 0\}$, the Cauchy-Stieltjes transform

$$G_\alpha(z) := \int \frac{1}{z-x}d\mu_\alpha(x) = \frac{1}{z}h_\alpha(Y(z))\,, \tag{5}$$

where $Y(z)$ is the unique analytic on $\mathbb{C}^+$ solution of

$$z^\alpha Y(z) = C_\alpha g_\alpha(Y(z)) \tag{6}$$

tending to zero at infinity, and $C_\alpha := i^\alpha \Gamma(1-\frac{\alpha}{2})/\Gamma(\frac{\alpha}{2})$. In [2, Theorem 1.6] it is shown that μ_α has a smooth symmetric density ρ_α outside a compact set of capacity zero, and that $t^{\alpha+1}\rho_\alpha(t) \to \alpha/2$ as $t \to \infty$. Further analysis of $G_\alpha(z)$ as $\Im(z) \to 0$ yields [1, Proposition 1.1] which is as follows.

Proposition 1. *The unique analytic on $\mathbb{C}^+$ solution $Y(z)$ of (6) tending to zero at infinity takes values in the set $\mathcal{K}_\alpha := \{Re^{i\theta} : |\theta| \le \frac{\alpha\pi}{2},\, R \ge 0\}$ on which $g_\alpha(\cdot)$ is uniformly bounded. Its continuous extension to $\mathbb{R}\setminus\{0\}$ is analytic except possibly at a certain explicit finite set $\mathcal{D}_\alpha$. The symmetric uniformly bounded density of μ_α is*

$$\rho_\alpha(t) = \frac{\alpha|t|^{\alpha-1}}{2|C_\alpha|\pi}\Im\big(i^{-\alpha}Y(|t|)^2\big)\,, \tag{7}$$

continuous at $t \neq 0$, real-analytic outside $\mathcal{D}_\alpha$ and non-vanishing on any open interval.

The map $\alpha \mapsto \mu_\alpha$ is continuous on $(0,2)$ with respect to weak convergence of probability measures (see [2, Remark 1.5]), and as $\alpha \to 2$ the measures μ_α converge to the semi-circle law μ_2 (see [1, Lemma 5.2]).

We next state [1, Theorem 1.3] which provides the weak convergence of the spectral measures for $\mathbf{A}_N^\sigma$ and characterizes the Cauchy-Stieltjes transform of their limit, in case $\sigma \in \mathcal{C}_\star$, the set of piecewise constant functions $\sigma(x,y)$ such that for some finite q, some $0 = b_0 < b_1 < \cdots < b_q = 1$ and a $q \times q$ symmetric matrix of entries $\{\sigma_{rs}, 1 \le r,s \le q\}$,

$$\sigma(x,y) = \sigma_{rs} \qquad \text{for all} \quad (x,y) \in (b_{r-1},b_r] \times (b_{s-1},b_s]\,. \tag{8}$$

Theorem 2. *Fixing $\sigma \in \mathcal{C}_\star$, let $\Delta_r = b_r - b_{r-1}$ for $r = 1,\ldots,q$. With probability one, the sequence $\hat{\mu}_{\mathbf{A}_N^\sigma}$ converges weakly towards the non-random, symmetric probability measure μ^σ. The limiting measure has a continuous density ρ^σ on $\mathbb{R}\setminus\{0\}$ which is bounded off zero, and its Cauchy-Stieltjes transform is, for any $z \in \mathbb{C}^+$,*

$$G_{\alpha,\sigma}(z) := \int \frac{1}{z-x} d\mu^\sigma(x) = \frac{1}{z}\sum_{s=1}^q \Delta_s h_\alpha(Y_s(z))\,, \tag{9}$$

where $\underline{Y}(z) \equiv (Y_r(z), 1 \le r \le q)$ is the unique solution of

$$z^\alpha Y_r(z) = C_\alpha \sum_{s=1}^q |\sigma_{rs}|^\alpha \Delta_s g_\alpha(Y_s(z))\,, \tag{10}$$

composed of functions that are analytic on $z \in \mathbb{C}^+$ and tend to zero as $|z| \to \infty$. Moreover, $z^\alpha \underline{Y}(z)$ is uniformly bounded on $\mathbb{C}^+$ and for some $R = R(\sigma)$ finite, the mapping $\underline{Y}(z) \in (\mathcal{K}_\alpha)^q$ has an analytic extension through the subset (R,∞) where $\rho^\sigma(t) = -\frac{1}{\pi t}\sum_{s=1}^q \Delta_s \Im\big(h_\alpha(Y_s(t))\big)$ is real-analytic.

Theorem 1.7 of [1] extends many of the results of Theorem 2 to a large family $\mathcal{F}_\alpha$ of square integrable functions $\sigma(\cdot,\cdot)$. Further, it is shown there that $\mu^\sigma = \mu^{\widetilde{\sigma}}$,

i.e. σ and $\widetilde{\sigma}$ are equivalent, whenever $\sigma \in \mathcal{F}_\alpha$ and $\widetilde{\sigma} \in \mathcal{C}_\star$ with finite partition $0 = b_0 < b_1 < \cdots < b_q = 1$ are such that for any $1 \le r, s \le q$,

$$\int_{b_{s-1}}^{b_s} |\sigma(x,v)|^\alpha dv = |\widetilde{\sigma}_{rs}|^\alpha \qquad \text{for all} \quad x \in (b_{r-1}, b_r]\,.$$

For example, $\sigma(x,y) = \varphi(x-y)$ is in $\mathcal{F}_\alpha$ and equivalent to the constant $\widetilde{\sigma} = [\int_0^1 |\varphi(v)|^\alpha dv]^{1/\alpha}$ for any even, periodic function $\varphi : [-1,1] \to \mathbb{R}$ of period one and finitely many jump discontinuities. Hence, μ^σ then has the symmetric, uniformly bounded, continuous off zero, density $\widetilde{\sigma}^{-1}\rho_\alpha(t/\widetilde{\sigma})$.

Taking $\alpha = 2$ we have $C_2 = -1$ and $g_2(y) = h_2(y) = 1/(y+1)$ is well defined when $\Re(y) > -1$. Plugging the latter in the expressions of [1, Theorem 1.7], one finds Berezin's formula for the limit of $\hat{\mu}_{\mathbf{A}_N^\sigma}$ in case of entries with bounded variance (where we set $a_N = N^{-1/2}$). The latter limit (proved rigorously in [4]), is constant across each equivalence class of σ in $\mathcal{F}_2$. Similarly, using (5) and (6) with $\alpha = 2$ yields that $Y(z) = -\frac{1}{z}G_2(z)$ for the Cauchy-Stieltjes transform $G_2(z) = (z - \sqrt{z^2-4})/2$ of the semi-circle law μ_2.

Our next result, Theorem 1.10 of [1], is derived as a special interesting case of Theorem 2 with $q = 2$ and $\sigma_{rs} = \mathbf{1}_{|r-s|=1}$.

Theorem 3. *For a_N of (1) and $N \times M$ matrices $\mathbf{X}_{N,M}$ of i.i.d. heavy tailed entries x_{ij}, $1 \le i \le N$, $1 \le j \le M$, the law of which satisfies (2), consider the empirical covariance matrices $\mathbf{W}_{N,M} = a_{N+M}^{-2}\mathbf{X}_{N,M}\mathbf{X}_{N,M}^t$. If $N \to \infty$ and $\frac{M}{N} \to \gamma \in (0,1]$ then with probability one, the spectral measures $\hat{\mu}_{\mathbf{W}_{N,M}}$ converge to a non-random probability measure μ_α^γ. The probability measure μ_α^1 is absolutely continuous with the density*

$$\rho_\alpha^1(t) = 2^{1/\alpha} t^{-1/2} \rho_\alpha(2^{1/\alpha}\sqrt{t})$$

on $(0,\infty)$. Fixing $\gamma \in (0,1)$ let $(Y_1(z), Y_2(z))$ denote the unique analytic functions of $z \in \mathbb{C}^+$ tending to zero at infinity, such that

$$z^\alpha Y_1(z) = \frac{\gamma}{1+\gamma} C_\alpha g_\alpha(Y_2(z))\,, \qquad z^\alpha Y_2(z) = \frac{1}{1+\gamma} C_\alpha g_\alpha(Y_1(z))\,. \tag{11}$$

The functions $Y_1(z)$ and $Y_2(z)$ extend continuously to functions on $(0,\infty)$ that are analytic through (R,∞) for some finite $R = R_\alpha^\gamma$. The probability measure μ_α^γ then has an atom at zero of mass $1-\gamma$ and the continuous density

$$\rho_\alpha^\gamma(t) = -\frac{1}{\pi t}\Im\big(h_\alpha(Y_1(\sqrt{t}\,)\big)\,, \tag{12}$$

on $(0,\infty)$ which is real-analytic on (R,∞), bounded off zero, does not vanish in any neighborhood of zero and such that $t^{1+\alpha/2}\rho_\alpha^\gamma(t) \to \frac{\alpha\gamma}{2(1+\gamma)}$ as $t \to \infty$.

Note the contrast between the non-vanishing near zero density ρ_α^γ and the Pastur-Marchenko law μ_2^γ which vanishes throughout $[0, 1-\gamma]$ (c.f. [5]).

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Percolation and spectra on Cayley graphs

Ivan Veselić

We discuss geometric and spectral properties of certain types of homogeneous graphs. Homogeneity appears in two distinct variants: either one considers a single graph and in this case homogeneity means that the automorphism group of the graph is sufficiently rich. Or one considers a random family of graphs. In this context homogeneity means that the ensemble obeys an appropriate stationarity condition.

The results which are presented here are taken from the papers [12, 12, 7, 3, 4, 9, 2]. In particular they have been obtained in part jointly with Tonći Antunović, Daniel Lenz, and with Norbert Peyerimhoff, respectively.

§1 Geometric properties of quasi-transitive and percolation graphs

Denote the automorphism group of a graph $G = (E, V)$ by $\mathrm{Aut}(G)$. If the vertex set V decomposes under the action of $\mathrm{Aut}(G)$ into finitely many orbits the graph G is called quasi-transitive. If the degree of every vertex is finite G is said to be locally finite. From now on all graphs are assumed to be quasi-transitive and locally finite.

An (independent) site-percolation process on G is defined a follows. Let $\omega_v, v \in V$ be a collection of i.i.d. random variables with $\mathbb{P}(\omega_v = 1) = p = 1 - \mathbb{P}(\omega_v = 0)$. One defines a random sub-graph G_ω of as follows: if $\omega_v = 0$ one removes the vertex v together with all incident edges from the graph. All other vertices (and edges) are retained in the graph. The vector $(\omega_v)_{v\in V}$ will be denoted by ω and the set $\{(\omega_v)_{v\in V} | \forall \quad \in V : \omega_v \in \{0, 1\}\}$ of all possible realisations by Ω_{perc}. The natural probability measure associated to the percolation process is denoted by $\mathbb{P}_p$ and the associated expectation by $\mathbb{E}_p$. The connected component of G_ω containing a vertex v is denoted $C_v(\omega)$ and called cluster of v. The edge-percolation process is defined analogously by removing edges instead of vertices from G. All subsequent statements hold for both types of percolation processes. A fundamental observation of percolation theory is that there exists a critical value $p_c \in [0, 1]$ such that for all $p < p_c$ there is no infinite cluster in G_ω almost surely and for $p > p_c$ there is an infinite cluster in G_ω almost surely. Now we are in the position to formulate the main result of [3]:

Theorem 1. *Let G be a countable, quasi-transitive, locally finite graph. Consider independent site or edge-percolation on G. If $p < p_c$ then $\mathbb{E}_p\{|C_v|\} < \infty$.*

Here $|X|$ denotes the number of vertices in the graph X. Theorem 1 is a generalisation of earlier results obtained by Aizenman & Barsky and Menshikov in [10] and [1]. The proof in [3] uses the differential inequalities approach of [1]. Statements of this type are called *sharpness of the phase transition*. Note that no assumption on the volume growth of the graph is required in Theorem 1. In particular, the graph may be exponentially growing or even non-amenable.

§1 Integrated density of states for Hamiltonians on amenable graphs
Now we move on to discuss spectral properties of certain operators on the Hilbert space $\ell^2(V)$ over the vertex set of the graph G. In the following we assume that Γ is a sub-group of $\mathrm{Aut}(G)$ which acts freely and quasi-transitively on V, and ergodically by measure preserving transformations on a probability space $(\Omega, \mathbb{P})$.

In the sequel we will consider families of operators $(H_\omega)_{\omega\in\Omega}, H_\omega\colon \ell^2(V) \to \ell^2(V)$, which satisfy the following conditions for all $\omega \in \Omega$: H_ω is of finite hopping range (i.e. there is some $R \in (0,\infty)$ such that $H_\omega(x,y) = 0$ if $\mathrm{dist}(x,y) \geq R$), it is bounded (i.e. there is some $R \in (0,\infty)$ such that $|H_\omega(x,y)| \leq R$ for all $x, y \in V$, hermitian (i.e. $H_\omega(x,y) = \overline{H}_\omega(x,y)$ for all $x, y \in V$), and it is equivariant (i.e. $H_{\gamma\omega}(\gamma x, \gamma y) = H_\omega(x,y)$ for all $\gamma \in \mathrm{Aut}(G)$, and $x, y \in V$). Here we have used that H is determined by its matrix elements in the basis $\{\delta_x, x \in V\}$.

The next statement is taken from [7] and [12, 13] and concerns the non-randomness of the absolutely continuous, singular continuous, pure point, discrete and essential spectrum.

Theorem 2. *There exists $\Omega_0 \subset \Omega$ of probability one and subsets Σ, Σ_{ac}, Σ_{sc}, Σ_{pp}, Σ_{disc}, Σ_{ess}, Σ_{cmp} of $\mathbb{R}$ such that for all $\omega \in \Omega_0$*

$$\sigma(H_\omega) = \Sigma, \sigma_{ac}(H_\omega) = \Sigma_{ac}, \sigma_{sc}(H_\omega) = \Sigma_{sc}, \sigma_{pp}(H_\omega) = \Sigma_{pp},$$
$$\sigma_{disc}(H_\omega) = \Sigma_{disc}, \sigma_{ess}(H_\omega) = \Sigma_{ess}, \sigma_{cmp}(H_\omega) = \Sigma_{cmp}$$

Here $\sigma_{cmp}(H_\omega)$ denotes the set of eigenvalues of H_ω with compactly supported eigenfunctions.

Now we turn to the definition of the integrated density of states, also called spectral distribution function. Assume that there exists an increasing sequence of non-empty, finite sets $I_j \subset \Gamma, j \in \mathbb{N}$ such that $\lim_j \frac{|\partial I_j|}{|I_j|} = 0$ and that $\sup_j \frac{|I_{j+1} I_j^{-1}|}{|I_{j+1}|} < \infty$. Here ∂I denotes the outer vertex boundary of the set I. Such a sequence exists iff Γ is amenable. It is called a tempered Følner sequence, cf. [6]. Let $\mathcal{F}$ be a (finite) fundamental domain of Γ in G and define the sequence of sub-graphs by $\Lambda_j = \bigcup_{\gamma^{-1}\in I_j} \gamma\mathcal{F}$. Denote by $p_j\colon \ell^2(V) \to \ell^2(\Lambda_j)$ and $\iota_j\colon \ell^2(\Lambda_j) \to \ell^2(V)$ the natural projection and inclusion operators, by $H_{\omega,j} = p_j \circ H_\omega \circ \iota_j$ the finite volume restriction of H_ω, by $N_{\omega,j}(\lambda) = \mathrm{Tr}[\chi_{(-\infty,\lambda]}(H_{\omega,j})]/|I_j|$ the normalised eigenvalue counting function, and by $N(\lambda) = \mathbb{E}\{\mathrm{Tr}[\chi_{(-\infty,\lambda]}(H_\omega)]\}/|\mathcal{F}|$ the spectral distribution function of $(H_\omega)_\omega$.

Theorem 3. *There exists $\Omega_0 \subset \Omega$ of probability one such that for all $\omega \in \Omega_0$ the following uniform convergence statement holds:*

$$\lim_{j\to\infty} \sup_{\lambda\in\mathbb{R}} \big|N(\lambda) - N_{\omega,j}(\lambda)\big| = 0 \tag{1}$$

This is actually a special case of the main result of [9], which applies to a very general geometric setting. For various models partial results of this type have been obtained earlier by several authors, cf. the dicussion in [9]. Theorem 3 covers two important examples. The first one concerns periodic graphs and periodic operators. This situation occurs, if one specialises the setting of Theorem 3 to the case that Ω (and consequently Ω_0, too) consists of a single element. In this situation we will write for the sake of emphasis N_{per} for the integrated density of states. An operator of special interest is the adjacency operator A_G, which acts on $f \in \ell^2(V)$ according to $(A_G f)(x) = \sum_{y\sim x} f(y)$ where $y \sim x$ means that y is a vertex adjacent to x in the graph G.

The second example concerns percolation Hamiltonians. We explain the situation for egde-percolation, for site-percolation the definitions are similar. Given a percolation subgraph $G_\omega \subset G$ one has a new adjacency relation $\sim_\omega$ on the set of vertices V. Here $\omega \in \Omega = \Omega_{perc}$ is a realisation of the percolation process. The natural analoge of A is the percolation adjacency operator

$$(A_\omega f)(x) = \sum_{y\sim_\omega x} f(y), \quad f \in \ell^2(V), \quad x \in V.$$

For this operator a weaker version of the convergence statement in (1) (namely, pointwise convergence for all $\lambda \in \mathbb{R}$) has been proven earlier in [13].

§3 Low energy asymptotics of the spectral distribution of Laplacians

Let Γ be a discrete, finitely generated group, $S \subset \Gamma$ a finite, symmetric set of generators not containing the unit element of Γ, and $G = \mathrm{Cay}(\Gamma, S)$ the associated Cayley graph (which is obviously quasi-transitive and locally finite). Set $k = |S|$ (the degree of G). If Γ has polynomial growth, denote by d the growth exponent. This means that the volume of a ball with radius n divided by n^d tends to a constant. Let $\Delta_G = k\cdot \mathrm{Id} - A_G$ be the canonical Laplace operator on $\ell^2(G)$ (where A_G denotes the adjacency operator defined above) and N_{per} the integrated density of states associated to Δ_G. It follows from [8, 11] that $\lim_{\lambda\searrow 0} \frac{\log N_0(\lambda)}{\log \lambda} = \frac{d}{2}$ if Γ has polynomial growth and $\lim_{\lambda\searrow 0} \frac{\log N_0(\lambda)}{\log \lambda} = \infty$ if Γ has super-polynomial growth. Now we compare these asymptotics with the one of the Laplacian on percolation subgraphs of G. Let $p \in [0, p_c)$, $\Delta_\omega = k \cdot \mathrm{Id} - A_\omega$, and N_{rnd} the associated integrated density of states. (There are other natural types of Laplacians in this context, cf. the discussion in [5, 4].) The following three Theorems have been announced in [4] and proven in [2].

Theorem 4. *There is an $a_p \in (0, \infty)$ such that for all positive and sufficiently small λ we have*

$$N_{rnd}(\lambda) \le \exp\Big(-\frac{a_p}{2}\mathrm{Vol}\Big(\frac{1}{8\sqrt{2}k}\lambda^{-1/2} - 1\Big)\Big).$$

Here $\mathrm{Vol}\colon \mathbb{R} \to \mathbb{R}$ *is a monotone function such that* $\mathrm{Vol}(n)$ *equals the volume of the ball with radius* n *in* $Cay(\Gamma, S)$.
Moreover: If G *has superpolynomial growth then* $\lim_{E\searrow 0} \frac{\ln|\ln N_{rnd}(\lambda)|}{|\ln\lambda|} = \frac{d}{2}$ *and if* G *has super-superpolynomial growth then* $\lim_{E\searrow 0} \frac{\ln|\ln N_{rnd}(\lambda)|}{|\ln\lambda|} = \infty$.

Part of the proof of Theorem 4 is based on the strategy of [5]. Next we state a result which concerns an important class of Cayley graphs of exponential (in particular super-polynomial) growth.

Theorem 5. *Let* $\Gamma = \mathbb{Z}_m \wr \mathbb{Z}$ *be a Lamplighter group and* $Cay(\Gamma, S)$ *a Cayley-graph as above,* Δ_G *the Laplacian on* G *and* N_{per} *the associated integrated density of states, then*

$$\lim_{E\searrow 0} \frac{\ln|\ln N_{per}(\lambda)|}{|\ln\lambda|} = \frac{1}{2}.$$

The next result concerns sub-critical percolation on $\mathbb{Z}_m \wr \mathbb{Z}$.

Theorem 6. *Let* Γ, S *be as in the preceding Theorem,* $(G_\omega)_\omega$ *the ensemble of percolation subgraphs for* $p < p_c$, $(\Delta_\omega)_\omega$ *the associated percolation Laplacian with integrated density of states* N_{rnd}. *Then*

$$\lim_{E\searrow 0} \frac{\ln\ln|\ln N_{rnd}(\lambda)|}{|\ln\lambda|} = \frac{1}{2}$$

In [2] more precise estimates concerning the limits in Theorem 5 and 6 are given.

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Pinning of polymers and wetting of a substrate in two dimensions: the role of disorder

Giambattista Giacomin
(joint work with Hubert Lacoin and Fabio L. Toninelli)

The most basic, and most studied, model in the class we consider is based on a simple symmetric random walk $S = \{S_0, S_1, \ldots\}$ on $\mathbb{Z}$, *i.e.*, $S_0 = 0$ and $\{S_n - S_{n-1}\}_{n\in\mathbb{N}}$ is an IID sequence (with law $\mathbf{P}$) of random variables taking values ± 1 with probability $1/2$. It is convenient to take a directed walk viewpoint, that is to consider the process $\{(n, S_n)\}_{n=0,1,\ldots}$. This random walk is the *free model* and we want to understand what happens when the walk interacts with a substrate or with a defect line that provides *disordered* rewards/penalties each time the walk hits it. The interaction is introduced via the Hamiltonian

$$H_{N,\omega}(S) := -\sum_{n=1}^{N} (\beta\omega_n + h)\, \mathbf{1}_{S_n=0}, \tag{1}$$

where $N \in 2\mathbb{N}$ is the system size, h (homogeneous pinning potential) is a real number, $\omega := \{\omega_1, \omega_2, \ldots\}$ is a sequence of IID standard Gaussian random variables, $\beta \geq 0$ is the disorder strength and $\mathbb{E}$ denotes the average with respect to ω.

The Gibbs measure $\mathbf{P}_{N,\omega}$ for the pinning model is then defined as

$$\frac{\mathrm{d}\mathbf{P}_{N,\omega}}{\mathrm{d}\mathbf{P}}(S) = \frac{e^{-H_{N,\omega}(S)}}{Z_{N,\omega}}, \tag{2}$$

and of course $Z_{N,\omega} = \mathbf{E}[\exp(-H_{N,\omega}(S))]$, where $\mathbf{E}$ denotes expectation with respect to the simple random walk measure $\mathbf{P}$.

It is useful to introduce right away the (quenched) free energy

$$\mathrm{F}(\beta, h) := \lim_{N\to\infty} \mathbb{E}\frac{1}{N} \log Z_{N,\omega}. \tag{3}$$

The existence of the limit follows by a standard super-additivity argument. It is important to notice that $\mathrm{F}(\beta, h) \geq 0$, which follows simply by observing that $Z_{N,\omega} \geq \mathbf{P}(S_n \neq 0 \text{ for } n = 1, 2, \ldots, N) \overset{N\to\infty}{\sim} const.N^{-1/2}$. Convexity of $\mathrm{F}(\cdot,\cdot)$ as well as monotonicity of $\mathrm{F}(\beta,\cdot)$ are straightforward, so we set $h_c(\beta) := \sup\{h : \mathrm{F}(\beta, h) = 0\}$. It is not difficult to see that $h_c(\beta) \in [-\beta^2/2, 0]$ and $h_c(\beta) = 0$ if and only if $\beta = 0$.

The phenomena that one is watching out for are roughly summarized by the observation that if $h < h_c(\beta)$ then the contact fraction $N^{-1}\mathbb{E}\mathbf{E}_{N,\omega}\sum_{n=1}^{N}\mathbf{1}_{S_n=0}$ tends to zero as $N \to \infty$, while it stays positive if $h > h_c(\beta)$. This statement can be strongly refined (see [6]) but it already makes clear the type of transition undergone by this system: zero contact density below $h_c(\beta)$ (*delocalization*), and positive contact density above (*localization*).

Of course what we just stated is true also in absence of disorder ($\beta = 0$) and a remarkable fact of the homogeneous (*i.e.* non-disordered) model is that it is exactly solvable ([4, 6] and references therein). In particular, we know that $h_c(0) = 0$, *i.e.*, an arbitrarily small reward is necessary and sufficient for pinning, and that the free energy behaves quadratically close to criticality. If now we consider the *annealed measure* corresponding to (2), that is the model in which one replaces both $\exp(-H_{N,\omega}(S))$ and $Z_{N,\omega}$ by their averages with respect to $\mathbb{P}(\,\mathrm{d}\omega)$, one readily realizes that the annealed model is the homogeneous model ($\omega \equiv 0$) with h replaced by $h + \beta^2/2$. Therefore one finds that the *annealed critical point* $h_c^a(\beta)$ equals $-\beta^2/2$ for every β. Moreover the *annealed free energy* $\mathrm{F}^a(\beta, h)$ behaves, for $h \searrow h_c^a(\beta)$, like *const* $\times\, (h - h_c^a(\beta))^2$ (while it is of course zero for $h \le h_c^a(\beta)$).

Very natural questions are: does $h_c(\beta)$ differ from $h_c^a(\beta)$? Are quenched and annealed critical exponents different? These questions can be better understood if we considered a generalized context.

The random walk trajectories are not the fundamental mathematical object underlying the model: the discrete renewal process τ, describing the successive returns to the origin of the walk S, is. More precisely $\tau := \{\tau_j\}_{j=0,1,\ldots}$, defined by $\tau_0 = 0$ and $\tau_{j+1} := \inf\{n > \tau_j : S_{2n} = 0\}$, is a sequence with independent identically distributed increments and one can rewrite $Z_{N,\omega}$ in terms of τ in a straightforward way. The particularity of the simple random walk is that $K(n) := \mathbf{P}(\tau_{j+1} - \tau_j = n) \sim const.n^{-3/2}$. It is therefore very natural to assume that $K(\cdot)$ is a discrete probability density with regularly varying tail with exponent $1 + \alpha$, $\alpha \ge 0$. Or, more simply, that $K(n) \sim c/n^{1+\alpha}$ ($\alpha > 0$). For various applications in which models with $\alpha \ne 1/2$ appear we refer for example to [4, 6]. Here we are however more interested on the role of α in the questions that we have raised above.

Without surprises, the free energy $\mathrm{F}(\beta, h)$ is defined as in (3) and $\mathrm{F}(\beta, h) \ge 0$. Moreover $\mathrm{F}(\beta, h)$ enjoys precisely the same monotonicity and convexity properties as in the simple random walk model. Independently of α we still have $h_c(\beta) \in [h_c^a(\beta), 0]$, $h_c^a(\beta) = -\beta^2/2$. The α-dependence however definitely appears when we consider the critical behavior (of the annealed free energy, for now): if $\alpha \ne 1$

$$\mathrm{F}(\beta, h) \overset{h \searrow h_c^a(\beta)}{\sim} C_\alpha (h - h_c^a(\beta))^{\max(1,1/\alpha)}, \tag{4}$$

for some (explicit) $C_\alpha > 0$ (if $\alpha = 1$ there is a slowly varying correction [6, Ch. 2]). Therefore, playing on the value of α, all the range of critical exponents can be observed in this class of models.

So the annealed system is fully under control: what about the quenched system? Remarkably, one can find in the physical literature an argument, based on the

renormalization group, that leads to a criterion, called *Harris criterion* [10], aiming at deciding whether, when the disorder is *not too strong*, quenched and annealed critical behavior coincide or not. In the pinning model context, Harris' ideas have been developed [5, 3] and in this context the Harris criterion may be summed up by the following two points:

(I) *Irrelevant disorder regime*: if $\alpha < 1/2$ there exists $\beta_0 > 0$ such that, if $\beta \in (0, \beta_0)$, quenched and annealed critical points and critical behaviors coincide.

(R) *Relevant disorder regime*: if $\alpha > 1/2$ quenched and annealed critical points differ; moreover quenched and annealed free energies, at criticality, are not expected to behave in the same way.

The *rationale* behind the Harris approach is in principle rather simple: the point is understanding whether *renormalizing the system* suppresses the disorder, leading thus to scenario (I), or it enhances it, leading to scenario (R). By developing this idea, the authors of [5, 3] obtained disorder relevance if the free energy of the annealed system, as $h \searrow h_c^a(\beta)$, behaves like $(h - h_c^a(\beta))^{2-\nu}$ (possibly, times a slowly varying function) with $\nu > 0$. In the same framework, disorder is instead predicted to be irrelevant, for moderate values of β, if $\nu < 0$. And of course (4) tells us that the sign of ν is the sign of $\alpha - 1/2$.

These two scenarios are now fully rigorous: the irrelevant disorder regime has been identified in [1] and the relevant disorder regime has been identified in [9], where it is established that for $\alpha > 1/2$ the quenched free energy at criticality is smoother than the anneal one, and, for what concerns the shift of the critical point, in [2].

If $\nu = 0$, that is $\alpha = 1/2$, the Harris criterion does not yield a prediction (this case is named *marginal disorder case*, in the Harris sense) and a more delicate (model dependent) analysis is needed. This issue has been attacked in several papers, we mention here [5], standing for irrelevance of disorder at marginality, and [3], that stands for marginal relevance (predicting that $h_c(\beta) - h_c^a(\beta)$ behaves like $\exp(-const./\beta^2)$ for β small: an upper bound matching this claim can be found in [1]). We refer to [8] for more on the physical literature on this issue.

The purpose of the talk is to announce and to sketch some ideas of the proof of the following result [8]:

Theorem. *Disorder is relevant at marginality. Namely, when $\alpha = 1/2$ (that is $K(n) \overset{n\to\infty}{\sim} C/n^{3/2}$ for some $C > 0$) we have that $h_c(\beta) - h_c^a(\beta) > 0$ for every $\beta > 0$ and one can find $c > 0$ such that*

$$h_c(\beta) - h_c^a(\beta) \geq \exp(-c/\beta^4), \tag{5}$$

for $\beta \leq 1$.

The proof is based on a development of the *iterated fractional moment – change of measure* technique used in [7, 2]. In a nutshell: we have used a change of measure that introduces negative long range correlations in the disorder, combined with a coarse graining procedure close to the one exploited in [11].

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Directed polymer in dimension 1 + 1 and 1 + 2: Very strong disorder and bounds on the free energy

Hubert Lacoin

We present in this abstract some new results for a well-studied model of directed polymer in $\mathbb{N} \times \mathbb{Z}^d$ (for a survey on the model see [5]) for $d = 1$ and $d = 2$. This model has been introduced by Huse and Henley in 1985 to modelize impurity-induced domain-wall roughening in the 2D-Ising model [11] and has since extensively studied by numerous authors (e.g. [12, 2, 1, 14, 4, 6, 7, 3, 8]).

We give a brief definition of the model:
Let Γ_n the set of directed path γ of length n starting from $(0,0)$ in $\mathbb{N} \times \mathbb{Z}^d$. Given the realization of an i.i.d. random field $(\omega_{(i,z)})_{(i,z)\in\mathbb{N}\times\mathbb{Z}}$ with associated probability Q (in the sequel, expectation w.r.t. Q will also be denoted by Q), such that

$$Q\omega_{(1,0)} = 0, \quad Q\omega_{(1,0)}^2 = 1, \quad \lambda(\beta) := Q\exp(\beta\omega_{(1,0)}) < \infty \quad \text{for all } \beta > 0 \tag{1}$$

we associate to each path γ an energy given by the Hamiltonian

$$H_n(\gamma) = \sum_{i=1}^{n} \omega_{\gamma_i}. \tag{2}$$

Then we define $\mu_n^{(\beta)}$ the polymer measure at inverse temperature $\beta > 0$ given by

$$\mu_n^{(\beta)} := \frac{1}{Z_n}\exp(\beta H_n(\gamma)). \tag{3}$$

Where Z_n is the renormalization factor needed to make μ_n a probability:

$$\sum_{\gamma\in\Gamma_n} \exp(\beta H_n(\gamma)). \tag{4}$$

The limits

$$\mathrm{F}_q(\beta) = \lim_{n\to\infty} \frac{1}{n} \log Z_n = \lim_{n\to\infty} \frac{1}{n} Q \log Z_n. \tag{5}$$

are well defined (where the first one is an almost sure limit), and $\mathrm{F}_q(\beta)$ is refered to as the *quenched* free energy of the polymer model. Another notion of free energy for the model is the *annealed* free energy which can be defined

$$\mathrm{F}_a(\beta) = \lim_{n\to\infty} \frac{1}{n} \log Q Z_n. \tag{6}$$

Jensen inequality guaranties that

$$\mathrm{F}_q(\beta) \leq \mathrm{F}_a(\beta) \text{ for all } \beta. \tag{7}$$

In many respect, wether the inequality (7) is strict has a crucial effect on the asymptotic localization properties of the measure μ_n. For instance, if the two energy differs, two random path independently chosen with μ_n probability will tend to spend a positive fraction of the steps together. This situation is refered to has *very strong disorder* (for asymptotic properties of the polymer measure in very strong disorder regime see [4, 6, 3]).

It has been known for a long time that for $d \geq 3$ at high temperature, weak disorder holds and the polymer acts diffusively [12, 2, 1, 14]. A much more recent result due to Comets and Vargas proves that *very strong disorder* holds for all β in dimension 1 [8]. The following results makes improve this statement by estimating the difference

Theorem 1 (d=1)**.** *When $\omega_{i,z}$ are Gaussian random variables, there exist a constant c such that for all $\beta \leq 1$*

$$\frac{1}{c}\beta^4 \leq \mathrm{F}_a(\beta) - \mathrm{F}_q(\beta) \leq c\beta^4. \tag{8}$$

For general environment the statement remains true with rougher bounds

Theorem 2 (d=1)**.** *There exists a constant c such that for all $\beta \leq 1$*

$$\frac{1}{c}\beta^4 \leq \mathrm{F}_a(\beta) - \mathrm{F}_q(\beta) \leq c\beta^4(1 + (\log\beta)^2). \tag{9}$$

Moreover we can prove that *very strong disorder* holds for all temperature in dimension 2.

Theorem 3 (d=2). *When $\omega_{i,z}$ are Gaussian random variables, there exist a constant c such that for all $\beta \leq 1$ we have*

$$\exp(-c/\beta^4) \leq \mathrm{F}_a(\beta) - \mathrm{F}_q(\beta) \leq \exp(-1/(c\beta^2)). \tag{10}$$

The upper bound remains true for for arbitrary environment, but the proof of very strong disorder fails there for the moment.

The proof of these statement uses strongly the techniques used and sometimes developed for pinning models [13, 9, 10].

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Pinning and depinning of nearly flat martensitic interfaces in a heterogeneous environment

Patrick W. Dondl

(joint work with Kaushik Bhattacharya)

The motivation for this problem lies in the question of whether, and how, hysteresis can arise from a linear microscopic evolution law through the effect of heterogeneities, such as precipitates, that are always present in a realistic physical material. Hysteresis in the physical problems considered stems from a stick-slip

behavior of phase boundaries with respect to an applied force and this behavior is therefore often assumed in macroscopic models.

For the basic model we consider an elastic solid occupying a domain Ω with a bulk energy of the form

$$\mathscr{F}_{\text{elastic}} = \int_\Omega W(\nabla u, x), \tag{1}$$

where $u\colon \Omega \to \mathbb{R}^n$ is the displacement of the body. The domain is split into a subdomain E and its complement, separated by a phase boundary Γ—this represents the two phases the material can be in. There may also be inclusions $A = \bigcup_i A_i$ present in Ω. The elastic energy density $W(\nabla u, x)$ depends explicitly on the position x: its minimum, the transformation strain, is constant on the domains occupied by each phase or the inclusions and jumps across their respective boundaries.

Assuming smoothness of the involved quantities, one can calculate the rate of change of the energy in the system, depending on the normal velocity v_n of the phase boundary, to be

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{F}_{\text{elastic}} = -\int_\Gamma f\, v_n, \tag{2}$$

where f is the thermodynamic driving force. The goal is to analyze the free boundary problem arising from the kinetic assumption

$$v_n = f. \tag{3}$$

Our approach to an analysis uses an approximate elastic energy for a phase boundary with small slope. We assume that the phase boundary can be described by the graph of a smooth enough function g with periodic boundary conditions on a strip-like domain. We have derived [2], in the sense of Γ-convergence, a limiting functional for the self energy of a shallow-slope martensitic phase boundary, i.e., for $g = g_0 + \varepsilon g_1(x)$, g_0 constant, $g_1 \in C^1$, and $\varepsilon \to 0$. This limiting functional is of the type $\mathscr{F}_{\text{shallow}} = \int_{\Omega\setminus\{y=0\}} |\nabla u|^2$ if the jump in u equals g across the average height $y = 0$ of the interface; $\mathscr{F}_{\text{shallow}} = \infty$ otherwise. The Γ lim inf-inequality relies on the concentration of elastic energy around the interface if the limit of the test sequence does not admit the "correct" jump. The recovery sequence can be constructed by interpolation. The minimum of this energy, for a given g, can of course be easily found to be $\frac{1}{4}[g]^2_{H^{1/2}}$, i.e., the square of the $H^{1/2}$ seminorm of g.

The resulting viscous gradient flow problem of the limiting functional is given in Fourier space by the equation

$$\hat{g}_t(k) = -\,|k|\,\hat{g}(k) + \hat{\varphi}(k) + \hat{F}(k). \tag{4}$$

Here, the effect of the precipitates and that of an applied stress on the driving force are collected into the local periodic forcing $\varphi = \varphi(x, g(x))$ and a constant external load F.

We have proven existence of a solution to this problem, existence of a threshold force F^* up to which there is a stationary solution to the problem, and existence of time-space periodic solutions for $F > F^*$. The physical implication of this is

that the phase boundary is stuck up to a critical applied force and moves freely with a macroscopic average velocity thereafter.

In order to assert the existence of a threshold force and that of a time-space periodic solution, we use a fixed point method together with compact embeddings of fractional Sobolev spaces. The latter work follows [1], where the reaction-diffusion equation is considered with $|k|$ replaced by $|k|^2$. However, in our case, one cannot use elliptic or parabolic regularity.

Numerically, we found a power-law with exponent $1/2$ for the average velocity for a smooth periodic φ. In current work, we examine the power-law depinning transition of these interfaces in a periodic environment analytically. In experiments, however, different velocity exponents are often observed. The transition to these different pinning regimes, through the random nature of obstacles, is a challenging open problem.

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Optimal Transportation, Gradient Flows and Wasserstein Diffusion

Karl-Theodor Sturm

We present a brief introduction to recent progress in optimal transportation on manifolds and metric spaces. We recall the characterization of the heat equation on Riemannian manifolds M as the gradient flow for the relative entropy on the L^2-Wasserstein space of probability measures $\mathcal{P}(M)$, regarded as an infinite dimensional Riemannian manifold. Of particular interest are recent extensions to the heat flow on Finsler spaces, Heisenberg groups and Wiener spaces.
Convexity properties of the relative entropy $Ent(.|m)$ also play an important role in a powerful concept of generalized Ricci curvature bounds for metric measure spaces (M, d, m).

Moreover, we give a survey on recent results for the Wasserstein diffusion, a canonical reversible process $(\mu_t)_{t\geq 0}$ on the Wasserstein space $\mathcal{P}(\mathbb{R})$. This includes: particle approximation, logarithmic Sobolev inequaltiy, quasi-invariance of its invariant measure, the so-called entropic measure, $\mathbb{P}^\beta$ under push forwards $\mu \mapsto h_*\mu$ by means of smooth diffeomorphisms h of $\mathbb{R}$. We also indicate how to construct the entropic measure on multi-dimensional spaces, formally given as $\frac{1}{Z}\exp(-\beta \cdot Ent(.|m))\mathbb{P}^0(d\mu)$.

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Spatial random permutations with cycle weights

Volker Betz
(joint work with Daniel Ueltschi)

We investigate a model of spatial permutations which is motivated by its connection to the theory of Bose-Einstein condensation [5, 7, 6, 1]. We consider pairs $(\boldsymbol{x}, \pi)$ with $\boldsymbol{x} \in \Lambda^N$ (Λ is a cubic box in $\mathbb{R}^d$) and $\pi \in \mathcal{S}_N$ (the group of permutations of N elements). N is the number of "particles" of the system. The weight of $(\boldsymbol{x}, \pi)$ is given by the "Gibbs factor" $\mathrm{e}^{-H(\boldsymbol{x},\pi)}$ with Hamiltonian of the form

$$H(\boldsymbol{x}, \pi) = \sum_{i=1}^{N} \xi(x_i - x_{\pi(i)}) + \sum_{\ell \geq 1} \alpha_\ell r_\ell(\pi). \tag{1}$$

We always assume that ξ is a function $\mathbb{R}^d \to [0, \infty]$, with $\int \mathrm{e}^{-\xi} = 1$. The cycle parameters $\alpha_1, \alpha_2, \ldots$ are some fixed numbers, and $r_\ell(\pi)$ is the number of cycles of length ℓ in the permutation π. The most relevant choice for the function ξ is $\xi(x) = \gamma|x|^2$, $\gamma > 0$, which is related to the quantum Bose gas. We mainly consider the case where the weights α_ℓ decay at infinity faster than $1/\log \ell$, and where $\mathrm{e}^{-\xi}$ has positive Fourier transform. Intuitively, the Gibbs factor restricts the permutations so each jump is local, i.e. the distances $|x_i - x_{\pi(i)}|$ remain finite even for large systems.

Our main result on this model states that macroscopic cycles occur in the thermodynamic limit $N, |\Lambda| \to \infty$ when the density $\rho = N/|\Lambda|$ is larger than the *critical density* $\rho_{\mathrm{c}} \leq \infty$. We also give an explicit formula for ρ_{c}, cf. (4). When $\alpha_\ell = 0$ for all ℓ, we obtain the model of spatial random permutations that corresponds to the ideal Bose gas; in this case ρ_{c} is the well-known critical density for Bose-Einstein condensation for non-interacting particles.

Setting and main result. The state space is $\Omega_{\Lambda,N} = \Lambda^N \times \mathcal{S}_N$, with the Borel σ-algebra on Λ^N, and the discrete σ-algebra on $\mathcal{S}_N$. Write $\boldsymbol{x} = (x_1, \ldots, x_N)$. Our Hamiltonian is given by a slight modification of (1): define ξ_Λ through $\mathrm{e}^{-\xi_\Lambda(x)} = \sum_{y \in \mathbb{Z}^d} \mathrm{e}^{-\xi(x - Ly)}$, and put

$$H_\Lambda(\boldsymbol{x}, \pi) = \sum_{i=1}^{N} \xi_\Lambda(x_i - x_{\pi(i)}) + \sum_{\ell \geq 1} \alpha_\ell r_\ell(\pi). \tag{2}$$

The important point is that $\mathrm{e}^{-\xi_\Lambda}$ has a Λ-independent Fourier transform $\mathrm{e}^{-\varepsilon(k)}$ in finite volume. With the additional assumption that $\mathrm{e}^{-\varepsilon(k)}$ is positive, this enables us to relate our model with a probability model on Fourier modes. Note that $\int_\Lambda \mathrm{e}^{-\xi_\Lambda} = 1$, and that $H_\Lambda(\boldsymbol{x}, \pi) = H(\boldsymbol{x}, \pi)$ for large enough Λ if $\mathrm{e}^{-\xi}$ has compact support.

We introduce a probability measure on $\Omega_{\Lambda,N}$ such that a random variable $\theta : \Omega_{\Lambda,N} \to \mathbb{R}$ has expectation

$$E_{\Lambda,N}(\theta) = \frac{1}{Y(\Lambda,N)N!} \int_{\Lambda^N} \mathrm{d}\boldsymbol{x} \sum_{\pi\in\mathcal{S}_N} \theta(\boldsymbol{x},\pi)\, \mathrm{e}^{-H_\Lambda(\boldsymbol{x},\pi)}\,. \tag{3}$$

$Y(\Lambda,N) = \frac{1}{N!}\int_{\Lambda^N} \mathrm{d}\boldsymbol{x} \sum_{\pi\in\mathcal{S}_N} \mathrm{e}^{-H_\Lambda(\boldsymbol{x},\pi)}$ is the partition function. Put

$$\rho_{\mathrm{c}} = \sum_{n\geq 1} \mathrm{e}^{-\alpha_n} \int_{\mathbb{R}^d} \mathrm{e}^{-n\varepsilon(k)}\, \mathrm{d}k. \tag{4}$$

ρ_{c} is the critical density of the system. Precisely, define the finite volume free energy for density ρ through $q_\Lambda(\rho) = -\frac{1}{|\Lambda|}\log Y(\Lambda, |\Lambda|\rho)$. Then [2] there exists a convex function $q(\rho)$ such whenever $\rho_n \to \rho \geq 0$, we have $\lim_{n\to\infty} q_{\Lambda_n}(\rho_n) = q(\rho)$. q is an analytic function of ρ except at the critical density ρ_{c}. The non-analyticity of $q(\rho)$ at ρ_{c} is caused by the appearance of macroscopic cycles: let $\ell_i(\pi) = 1, 2, \ldots$ denote the length of the cycle of π that contains the index i. Let $V = L^d$, and $\boldsymbol{\varrho}_{m,n}(\pi) = \frac{1}{|\Lambda|}\#\{i = 1, 2, \cdots : m \leq \ell_i(\pi) \leq n\}$.

Theorem: *Assume* $\sum_{\ell\geq 1} \frac{|\alpha_\ell|}{\ell} < \infty$, $\mathrm{e}^{-\xi}$ *has Fourier transform* $\mathrm{e}^{-\varepsilon(k)} \geq 0$, *and* $\rho_{\mathrm{c}} < \infty$. *For any function* η *with* $\eta(V) \to \infty$ *and* $\eta(V)/V \to 0$ *as* $V \to \infty$, *and all* $s \geq 0$, *we have*

$$\lim_{V\to\infty} E_{\Lambda,\rho V}(\boldsymbol{\varrho}_{1,\eta(V)}) = \begin{cases} \rho & \text{if } \rho \leq \rho_{\mathrm{c}}; \\ \rho_{\mathrm{c}} & \text{if } \rho \geq \rho_{\mathrm{c}}; \end{cases} \qquad \textit{(microscopic cycles)}$$

$$\lim_{V\to\infty} E_{\Lambda,\rho V}(\boldsymbol{\varrho}_{\eta(V),V/\eta(V)}) = 0; \qquad \textit{(mesoscopic cycles)}$$

$$\lim_{V\to\infty} E_{\Lambda,\rho V}(\boldsymbol{\varrho}_{V/\eta(V),sV}) = \begin{cases} 0 & \text{if } \rho \leq \rho_{\mathrm{c}}; \\ s & \text{if } 0 \leq s \leq \rho - \rho_{\mathrm{c}}, \\ \rho - \rho_{\mathrm{c}} & \text{if } 0 \leq \rho - \rho_{\mathrm{c}} \leq s. \end{cases} \qquad \textit{(macroscopic cycles)}$$

When $\alpha_\ell = 0$ for all ℓ, we obtain the model of spatial random permutations that corresponds to the ideal Bose gas; in this case ρ_{c} is the well-known critical density for Bose-Einstein condensation for non-interacting particles. There the occurrence of macroscopic cycles has been understood in [7, 8]. The present setting with general functions ξ was considered in [1].

Main ideas of the proof. We express (3) as a model of random permutations on Fourier modes. Let $\Lambda^* = \frac{1}{L}\mathbb{Z}^d$. For $\boldsymbol{k} \in \Lambda^*$ We define

$$p_{\Lambda,N}(\boldsymbol{k},\pi) = \tfrac{1}{\widehat{Y}(\Lambda,N)N!}\, \mathrm{e}^{-\widehat{H}(\boldsymbol{k},\pi)} \prod_{i=1}^{N} \delta_{k_i,k_{\pi(i)}}, \tag{5}$$

with $\widehat{H}(\boldsymbol{k},\pi) = \sum_{i=1}^{N} \varepsilon(k_i) + \sum_{\ell\geq 1} \alpha_\ell r_\ell(\pi)$. This model offers an alternative representation to the model of spatial permutations, as far as the permutations are concerned: for any permutation π,

$$\int_{\Lambda^N} \mathrm{e}^{-H_\Lambda(\boldsymbol{x},\pi)}\, \mathrm{d}\boldsymbol{x} = \sum_{\boldsymbol{k}\in(\Lambda^*)^N} \mathrm{e}^{-\widehat{H}(\boldsymbol{k},\pi)} \prod_{i=1}^{N} \delta_{k_i,k_{\pi(i)}}.$$

In particular $\widehat{Y}(\Lambda,N) = Y(\Lambda,N)$, and $E_{\Lambda,N}(\Lambda^N \times \{\pi\}) = p_{\Lambda,N}(\Omega^*_{\Lambda,N},\pi)$ for all π. In order to separate the spatial component of the model from the permutations, we introduce a model of non-spatial permutations. For $\pi \in \mathcal{S}_n$, we put

$$p_n(\pi) = \frac{1}{h_n n!} \exp\Big\{-\sum_{\ell\geq 1} \alpha_\ell r_\ell(\pi)\Big\} \tag{6}$$

with normalization $h_n = \frac{1}{n!}\sum_{\pi\in\mathcal{S}_n} \mathrm{e}^{-\sum_\ell \alpha_\ell r_\ell(\pi)}$. $r_\ell(\pi)$ denotes the number of cycles of length ℓ in the permutation π. About this model, very detailed results can be obtained [2, 3]. In particular, put $N_{a,b} = \#\{i = 1, 2, \cdots : a \leq \ell_i(\pi) \leq b\}$. If $\sum_{\ell\geq 1} \frac{|\alpha_\ell|}{\ell} < \infty$, then

$$\lim_{n\to\infty} \tfrac{1}{n} E_n(N_{1,sn}) = s. \tag{7}$$

Next we introduce occupation numbers. Let $\mathcal{N}_\Lambda$ be the set of sequences $\boldsymbol{n} = (n_k)$ of integers indexed by $k \in \Lambda^*$, and $\mathcal{N}_{\Lambda,N} = \{\boldsymbol{n} \in \mathcal{N}_\Lambda : \sum_{k\in\Lambda^*} n_k = N\}$. To each $\boldsymbol{k} \in (\Lambda^*)^N$ corresponds an element $\boldsymbol{n} \in \mathcal{N}_{\Lambda,N}$, with n_k counting the number of indices i such that $k_i = k$. Thus we can view $\boldsymbol{n}$ as a subset of $(\Lambda^*)^N$. The probability (5) yields a probability on occupation numbers: summing over permutations and over compatible vectors $\boldsymbol{k}$, we have $p_{\Lambda,N}(\boldsymbol{n}) = \frac{1}{Y(\Lambda,N)} \prod_{k\in\Lambda^*} \mathrm{e}^{-n_k\varepsilon(k)}\, h_{n_k}$. Separation of the spatial and non-spatial aspects of the measure 3 is achieved by the identity

$$E_{\Lambda,N}(\boldsymbol{\varrho}_{a,b}) = \frac{1}{V} \sum_{\boldsymbol{n}\in\mathcal{N}_{\Lambda,N}} p_{\Lambda,N}(\boldsymbol{n}) \sum_{k\in\Lambda^*} E_{n_k}(N_{ab}). \tag{8}$$

In the light of (7) and (8), we can now focus on the quantity $p_{\Lambda,N}(\boldsymbol{n})$. By (7), macroscopic cycles appear if and only if at least one mode is macroscopically occupied, i.e. iff $p_{\Lambda,N}(n_k \geq sN) > 0$ uniformly in $N \in \mathbb{N}$ and Λ such that $N = \rho\Lambda$. It turns out that macroscopic occupation can occur only for $k = 0$ and that it occurs if and only if $\rho \geq \rho_{\mathrm{c}}$. The main step in proving this is a result that gives detailed information about the limiting distribution of the random variable n_0/V: putting $\rho_0 = \max(0, \rho - \rho_{\mathrm{c}})$, we have, for all $\lambda \geq 0$,

$$\lim_{V\to\infty} E_{\Lambda,\rho V}(\,\mathrm{e}^{\lambda n_0/V}\,) = \mathrm{e}^{\lambda\rho_0}\,.$$

The proof [2] is based on ideas of Buffet and Pulé [4] for the ideal Bose gas.

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Gibbs Measures for Delaunay Triangle Interactions: Existence and Variational Characterisation

Hans-Otto Georgii

(joint work with David Dereudre)

It is well-known that stationary renewal processes with reasonable spacing distribution can be characterised as Gibbs processes for an interaction between nearest-neighbour pairs of points [8, Section 6]. Here we consider an analogue in two dimensions, viz. Gibbsian point processes on $\mathbb{R}^2$ with an interaction depending on nearest-neighbour triples of points, where the nearest-neighbour triples are defined in terms of the Delaunay triangulation. Recall that the Delaunay triangulation is dual to the Voronoi tessellation, in the sense that two points are connected by a Delaunay edge if and only if their Voronoi cells have a common edge. Since the Voronoi cell of a point consists of the part of space that is closer to this point than to any other point, this means that the Delaunay graph defines a natural nearest-neighbour structure between the points. (Of course, the analogy with renewal processes does not reach too far because the independence of spacings under the Palm distribution, which is characteristic of one-dimensional renewal processes, is lost in two dimensions due to the geometric constraints.)

There is a principal difference between the Delaunay interactions considered here and the pair interactions that are common in Statistical Physics. Namely, suppose a particle x is added to a configuration ω. In the latter case, x is subject to some additional interaction with the particles in ω, but the interaction between the particles of ω is not affected by x. In the Delaunay case, however, the presence of x not only gives rise to some new tiles of the Delaunay triangulation, but also destroys some tiles that were present in the triangulation of ω. This so-called non-hereditary nature of the Delaunay triangulation makes it difficult to use a local characterisation of Gibbs measures in terms of their Campbell measures and Papangelou intensities. Such a local approach makes it necessary to restrict the

interaction to triangles with certain geometric constraints, as was done in [1, 2, 4, 5]. Instead, we propose here a global approach, which is based on translation invariance and the use of thermodynamic quantities such as pressure and free energy density.

To state the results more precisely, let Ω be the set of locally finite point configurations in $\mathbb{R}^2$, and for $\omega \in \Omega$ let $\mathsf{D}(\omega)$ be the associated set of Delaunay triangles, or tiles. (If, for example, four points lie on a circle with no point inside, we apply some rule to make the Delaunay triangulation unique.) The set $\mathsf{D}(\omega)$ is a subset of the set $\mathcal{T}$ of all triangles in $\mathbb{R}^2$. For each $\tau \in \mathcal{T}$ we write $R(\tau)$, $c(\tau)$ and $B(\tau)$ for the radius, centre and interior of its circumcircle, respectively. Next, let $\mathcal{P}_\Theta$ be the set of all shift-invariant probability measures P on Ω with finite intensity $\varrho(P)$, and let $P \in \mathcal{P}_\Theta$. The distribution of a typical Delaunay tile under P is described by the *tile intensity measure* μ_P. This is defined on the set $\mathcal{T}_0$ of centred tiles, and is characterised by the Palm formula

$$\int dx \int \mu_P(d\tau)\, f(x,\tau) = \int P(d\omega) \sum_{\tau \in \mathsf{D}(\omega)} f(c(\tau), \tau - c(\tau))\,,$$

which holds for every measurable function $f : \mathbb{R}^2 \times \mathcal{T}_0 \to [0,\infty]$. As a consequence of Euler's polyhedral formula, μ_P has total mass $\|\mu_P\| = 2\,\varrho(P)$.

We are interested in point processes with a point-interaction depending on the shape of Delaunay tiles. That is, we consider formal Hamiltonians of the form

$$H(\omega) = \sum_{\tau \in \mathsf{D}(\omega)} \varphi(\tau)\,,$$

where $\varphi : \mathcal{T} \to \mathbb{R}$ is a *bounded* measurable "tile-potential" that is invariant under space shifts. For example, one may take $\varphi(\tau) = \beta\,|c(\tau) - b(\tau)|/R(\tau)$, where $b(\tau)$ is the barycentre of τ, or $\varphi(\tau) = -\beta\,A(\tau)/R(\tau)^2$, where $A(\tau)$ the area of τ; $\beta > 0$ is a parameter. These potentials favour the equilaterality of tiles.

To be more specific, let $\Lambda \subset \mathbb{R}^d$ be any bounded Borel set. For every configuration $\omega \subset \Lambda^c$ and all configurations $\zeta \subset \Lambda$ let

$$H_{\Lambda|\omega}(\zeta) := \sum_{\tau \in \mathsf{D}(\zeta \cup \omega):\, B(\tau) \cap \Lambda \neq \emptyset} \varphi(\tau)$$

be the *Hamiltonian*, and

$$G_{\Lambda|\omega}(d\zeta) = e^{-H_{\Lambda|\omega}(\zeta)}\, \Pi_\Lambda(d\zeta)/Z_{\Lambda|\omega}$$

the *Gibbs distribution in* Λ *with boundary condition* ω; the normalisation factor $Z_{\Lambda|\omega}$ is called the partition function. A measure P on Ω is called a *Gibbs measure* for φ if $P(\,\cdot\,|\,\omega \text{ in } \Lambda^c) = G_{\Lambda|\omega}$ for almost all ω and all bounded $\Lambda \subset \mathbb{R}^2$. Here, Π_Λ is the standard Poisson point process on the configurations in Λ.

Next we introduce some thermodynamic quantities. For each $P \in \mathcal{P}_\Theta$, let

$$\Phi(P) := \int \varphi\, d\mu_P = \lim_{\Lambda \uparrow \mathbb{R}^2} \frac{1}{|\Lambda|} \int H_{\Lambda|\omega\setminus\Lambda}(\omega \cap \Lambda)\, P(d\omega)$$

be the *energy density* of P, and

$$I(P) := \lim_{\Lambda\uparrow\mathbb{R}^2} I(P_\Lambda;\Pi_\Lambda)/|\Lambda| \in [0,\infty]$$

be the *entropy density* of P; here we write $I(P_\Lambda;\Pi_\Lambda)$ for the relative entropy of P_Λ (the restriction of P to Λ) with respect to Π_Λ. The functional $I+\Phi$ is called the *free energy density*. It is well-known that I is affine and has compact sublevel sets in the usual weak topology on $\mathcal{P}_\Theta$. In fact, the latter even holds if $\mathcal{P}_\Theta$ is endowed with the weak* topology generated by the local functions that grow no faster than the particle number [7]. This topology turns out to be fine enough to make the energy density Φ continuous. This fact, which is crucial for the theorem below, relies on Euler's polyhedral formula, which implies that the number of tiles in a triangulation of any finite configuration of points is at most twice the number of points. It is this fact which makes it necessary to confine ourselves to two dimensions. It follows that, in this topology, $I+\Phi$ is lower semicontinuous with compact sublevel sets, and this implies that minimisers of $I+\Phi$ exist.

Theorem: *Each minimiser of $I+\Phi$ is a Gibbs measure, and vice versa. In particular, Gibbs measures exist.*

The proof has two ingredients. The first is the existence of the *pressure*

$$p(\varphi) = \lim_{|\Lambda|\to\infty} |\Lambda|^{-1} \log Z_{\Lambda|\mathrm{per}}\,,$$

where Λ runs through any unbounded sequence of squares and $Z_{\Lambda|\mathrm{per}}$ is the partition function with periodic boundary condition, which is obtained by viewing Λ as a torus; this fact follows directly from the large deviation results of [7]. The second ingredient is an adaptation of Preston's proof of the variational principle for the usual Gibbs measures [9]; here the technical problem is to control $R(\tau)$ for all Delaunay tiles τ that hit a given bounded set. See [6] for the details.

It is fairly straightforward to extend the preceding theorem to interactions involving a bounded number of adjacent Delaunay tiles. A more intriguing question is to allow for potentials with hard core; this is the object of current joint work with D. Dereudre and R. Drouilhet. Finally, we expect that this class of models is well-suited to establish the existence of a crystallisation transition, viz. the spontaneous emergence of a regular lattice structure. (A phase transition in a multi-species model with interspecies repulsion of Delaunay neighbours has been established in [3].)

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Convexity of the surface tension for non-convex potentials

Stefan Adams
(joint work with R. Kotecký and S. Müller)

Recently the study of gradients fields has attained a lot of attention because they are space-time analogy of Brownian motions, and are connected to the Schramm-Lowener evolution. The corresponding discrete versions arise in equilibrium statistical mechanics, e.g., as approximations of critical systems and as effective interface models. The latter models - seen as gradient fields - enable one to study effective descriptions of phase coexistence. Gradient fields have a continuous symmetry and coexistence of different phases breaks this symmetry. In the probabilistic setting gradient fields involve the study of strongly correlated random variables. One major problem has been open for several decades. What can be proved for the free energy and the Gibbs states for non-convex interactions of the microscopic subsystems (particles)?

We present in the talk the first break through for low temperature using Gaussian measures and renormalisation group techniques yielding an analysis in terms of dynamical systems. The nearest neighbor interaction potential W for the discrete gradients of the height fields $\varphi\colon \Lambda \to \mathbb{R}$, where $\Lambda \subset \mathbb{Z}^d$ is a finite subset, is given as a sum of the quadratic (Gaussian) part and a perturbation V, i.e.

$$W(\eta) = \frac{1}{2}\eta^2 - V(\eta).$$

Gradient fields are massless and hence their correlation functions are not integrable. Posing boundary conditions is crucial to define Gibbs distributions properly. We work on a torus with periodic boundary condition, where the discrete gradient vanishes in expectation. To implement the boundary condition given by a vector $u \in \mathbb{R}^d$ we use the method of [4]. To prove the strict convexity of the surface tension as a function of the tilt $u \in \mathbb{R}^d$ we need to compute the second derivative of the partition function. The partition function is now a Gaussian integral with an integrand which is non-convex in its exponent. As the gradients vanish in expectation due to the periodic boundary conditions on the torus Λ one can pull out the strictly convex part of the surface tension which is coming from the Gaussian measure. What remains is to prove that the contribution from the

remaining Gaussian integral

$$\mathcal{Z}_{N,\beta}(u) = \int \exp\big(-\beta \sum_{x\in\Lambda}\sum_{i=1}^{d} V\big(\frac{1}{\sqrt{\beta}}\nabla_i\varphi(x) - u_i\big)\big)\nu(\mathrm{d}\varphi), \tag{1}$$

where ν is a Gaussian measure independent on the tilt u and where β is the inverse temperature, does not spoil the strict convexity coming from the strict convex part of the potential. The finite volume surface tension (*free energy*) follows then as

$$f_{N,\beta}(u) = -\frac{1}{\beta|\Lambda|}\log Z_N + \tfrac{1}{2}|u|^2 - \frac{1}{\beta|\Lambda|}\log \mathcal{Z}_{N,\beta}(u). \tag{2}$$

To get the claim of the strict convexity we need to prove that the free energy $f_{N,\beta}(u)$ in (2) is strictly convex in u uniformly in $N \in \mathbb{N}$. The first term in (2) is a normalisation for ν and does not depend on u and the second is strictly convex and does not depend on N. Thus, we only need to prove that the derivatives of the third term are sufficiently small and do not destroy this strict convexity.

A natural way of dealing with the perturbation and evaluating $\log \mathcal{Z}_{N,\beta}(u)$ and its derivatives, would be to use some version of cluster expansion. Expanding the integrand

$$\prod_{x\in\Lambda}\Big(1 + \exp\big\{-\beta\sum_{i=1}^{d} V\big(\tfrac{1}{\sqrt{\beta}}\nabla_i\varphi(x) - u_i\big)\big\} - 1\Big),$$

in we can write

$$\mathcal{Z}_{N,\beta}(u) = \int \sum_{X\subset\Lambda} K(X,\varphi)\nu(\mathrm{d}\varphi).$$

However, here comes a difficulty: even though the function $K(X,\varphi)$ depends only on $\varphi(x)$ with x in the set X and its close neighbourhood and even if for a disjoint union $X = X_1 \cup X_2$ one has $K(X,\varphi) = K(X_1,\varphi)K(X_2,\varphi)$, the Gaussian measure $\nu(\mathrm{d}\varphi)$ with its slowly decaying correlations does not allow to separate the integral of $K(X,\varphi)$ into a product of integrals with the integrands $K(X_1,\varphi)$ and $K(X_2,\varphi)$. This is a non-locality that has to be overcome. The strategy is to perform the integration in steps corresponding to increasing scales, i.e. a rigorous renormalisation group analysis. We follow ideas from [3]. One basic important ingredient in this involved proof method is the finite range decomposition of the Gaussian measure ν. Here we follow [2] but need an extension to get better regularity estimates needed for our model. Basically we write the covariance operator of the given Gaussian measure as a finite sum of covariance operators which have finite ranges on given scales. Hence we can perform each single Gaussian integration, i.e. we can integrate out different fluctuating scales of the field. This roughly defines the renormalisation group map. To conclude we need fine estimates of the linearization of this mapping. These allow to prove that an underlying dynamical system (flow) reaches the trivial fix point at the origin. For strict convexity we need sufficiently large β (small temperature) and small tilts $u \in \mathbb{R}^d$ for certain small perturbations. Roughly speaking perturbations V are allowed which have compact support with some regularity (up to the eleventh derivative) as a function of the field and C^2dependence as a function of the tilt. Another example included

in our result are the potentials used in [1] for the zero tilt regime in dimension two.

We outline also the connection to the Cauchy-Born rule which states that the deformation on the atomistic level is locally given by an affine deformation at the boundary. The final aim of our study is to prove the following version of the Cauchy-Born rule: Is there for sufficiently small temperature and small affine deformations at the boundary a unique extremal gradient Gibbs measure? We hope to attack this problem in the near future as the strict convexity does not automatically imply this last statement conversely to other cases in statistical mechanics.

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Scaling limits for weakly pinned random walks with two large deviation minimizers

Tadahisa Funaki

(joint work with Tatsushi Otobe)

The scaling limits for d-dimensional random walks perturbed by an attractive force toward the origin are studied under the critical situation that the rate functional of the corresponding large deviation principle (LDP) admits two minimizers.

Let $\phi = \{\phi_i\}_{i=0}^N$ be the Markov chain on $\mathbb{R}^d$ with the transition probability density $p(x)$ starting at aN. Then, it is easy to see that the law of large numbers $h^N(t) := \frac{1}{N}\phi_{[Nt]} \to a + mt,\ t \in [0,1]$ holds, where $m = \int_{\mathbb{R}^d} xp(x)\,dx$ is the mean drift. We consider Markov chains modified by adding a pinning effect at 0 to ϕ (i.e., allowing occasional jumps to 0). Our goal is to study the scaling limit mentioned above for such modified chains, in particular when the corresponding LD rate functional admits two minimizers. The paper [1] discusses the case where $p(x)$ is mean-zero Gaussian, and we extend the results to the general case under the absence of the wall.

We will first define the weakly pinned Markov chains on $\mathbb{R}^d$ imposing the Dirichlet or free conditions for the arriving points at the final time $i = N$. Then, we will state our results on (1) LDP, (2) the free energies $\xi^{D,\varepsilon}$ and $\xi^{F,\varepsilon}$, (3) the minimizers of the LD rate functional, (4) the scaling limits under critical situation that two minimizers exist, and finally on (5) the critical exponents for the free energies.

The original motivation comes from the study of the (1+1)-dimensional interface model, or $(1+d)$-dimensional directed polymer model. The so-called Wulff shape

(for crystal) is characterized by a variational formula and our interest is to see what happens if the variational problem has non-unique solutions.

1. Markov chains with weak pinning

Let $\varepsilon \geq 0$ be a parameter representing the strength of pinning at 0. We define two probability measures $\mu_N^{D,\varepsilon}$ (called the Dirichlet case: $\phi_0 = aN$, $\phi_N = bN$, $a, b \in \mathbb{R}^d$) and $\mu_N^{F,\varepsilon}$ (called the free case: $\phi_0 = aN$) on $(\mathbb{R}^d)^{N+1}$, respectively, by

$$\mu_N^{D,\varepsilon}(d\phi) = \frac{\mathfrak{p}_N(\phi)}{Z_N^{a,b,\varepsilon}} \delta_{aN}(d\phi_0) \prod_{i=1}^{N-1} \big(\varepsilon\, \delta_0(d\phi_i) + d\phi_i\big) \delta_{bN}(d\phi_N),$$

$$\mu_N^{F,\varepsilon}(d\phi) = \frac{\mathfrak{p}_N(\phi)}{Z_N^{a,F,\varepsilon}} \delta_{aN}(d\phi_0) \prod_{i=1}^{N} \big(\varepsilon\, \delta_0(d\phi_i) + d\phi_i\big),$$

where $Z_N^{a,b,\varepsilon}$, $Z_N^{a,F,\varepsilon}$ are the normalizing constants and $\mathfrak{p}_N(\phi) = \prod_{i=1}^N p(\phi_i - \phi_{i-1})$.

Let $h^N = \{h^N(t), t \in [0,1]\} \in \mathcal{C} = C([0,1], \mathbb{R}^d)$ be the macroscopic path of the Markov chain defined as the polygonal approximation of $\{\frac{1}{N}\phi_{Nt}, t = 0, \frac{1}{N}, \ldots, \frac{N}{N}\}$. We assume that $\sup_{x\in\mathbb{R}^d} e^{\lambda\cdot x} p(x) < \infty$ for all $\lambda \in \mathbb{R}^d$ and the condition on the Legendre transform: $\Lambda^*(v) = \sup_{\lambda\in\mathbb{R}^d}\{\lambda \cdot v - \Lambda(\lambda)\} < \infty$ for all $v \in \mathbb{R}^d$, where $\Lambda(\lambda) = \log \int_{\mathbb{R}^d} e^{\lambda\cdot x} p(x) dx$.

2. Sample path LDP

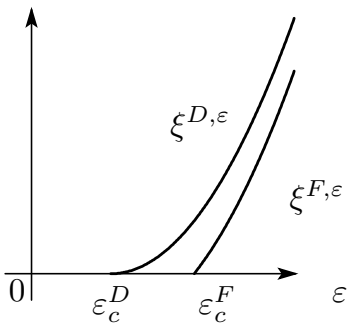
The LDP for $\mu_N = \mu_N^{D,\varepsilon}$ or $\mu_N^{F,\varepsilon}$, which is roughly stated as $\mu_N(h^N \sim h) \sim e^{-NI(h)}$, $h \in \mathcal{C}$, holds with the rate functional $I(h) = \Sigma(h) - \inf \Sigma$, where

$$\Sigma(h) = \int_0^1 \Lambda^*\big(\dot{h}(t)\big)\, dt - \xi^\varepsilon \Big| \{t \in [0,1]; h(t) = 0\} \Big|.$$

The case without pinning (i.e. $\xi^\varepsilon = 0$) was studied by Mogul'skii for the free case. Here, the pinning free energies $\xi^\varepsilon = \xi^{D,\varepsilon}$ or $\xi^{F,\varepsilon} \geq 0$ are defined respectively by

$$\xi^{D,\varepsilon} = \lim_{N\to\infty} \frac{1}{N} \log \frac{Z_N^{0,0,\varepsilon}}{Z_N^{0,0,0}},$$

$$\xi^{F,\varepsilon} = \lim_{N\to\infty} \frac{1}{N} \log \frac{Z_N^{0,F,\varepsilon}}{Z_N^{0,F,0}}.$$



$d \geq 3,\ m \neq 0$

We point out the following properties of the free energies:

(1) $^\exists$two critical values $0 \leq \varepsilon_c^D \leq \varepsilon_c^F$ s.t.
$\xi^{D,\varepsilon} > 0 \Leftrightarrow \varepsilon > \varepsilon_c^D$, $\quad \xi^{F,\varepsilon} > 0 \Leftrightarrow \varepsilon > \varepsilon_c^F$,

(2) $\varepsilon_c^D > 0$ $(d \geq 3)$, $\varepsilon_c^D = 0$ $(d = 1, 2)$,

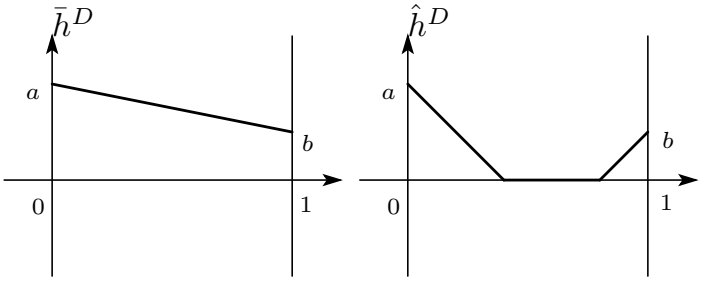
(3) $m = 0 \Rightarrow \varepsilon_c^D = \varepsilon_c^F, \xi^{D,\varepsilon} = \xi^{F,\varepsilon}$,
$m \neq 0 \Rightarrow \varepsilon_c^D < \varepsilon_c^F$, $\xi^{F,\varepsilon} < \xi^{D,\varepsilon}$ for $^\forall\varepsilon > \varepsilon_c^D$.

As an immediate consequence of the LDP, we see that the limits under μ^N are concentrated on the set of minimizers of the functional Σ.

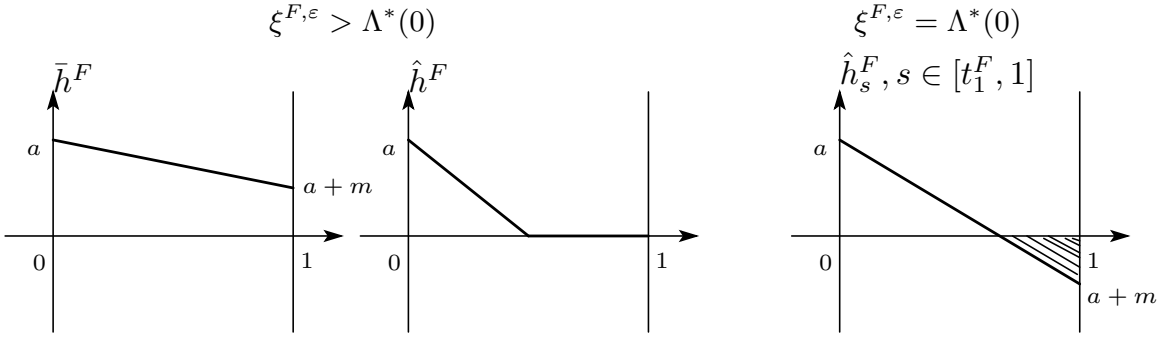
3. Minimizers of Σ

We assume $\xi^\varepsilon > 0$. Then, two (or more) possible minimizers $\bar{h}, \hat{h}$ are given as in the following figures.

Dirichlet case (i.e. $h(0) = a, h(1) = b$):



Free case (i.e. $h(0) = a$, $h(1)$ is free):



The Young's relation determining the times t_1^D or t_1^F when $\hat{h}^D$ or $\hat{h}^F$ first touch 0 is given by the formula: $-\frac{a}{t} \cdot \nabla\Lambda^*\big(-\frac{a}{t}\big) - \Lambda^*\big(-\frac{a}{t}\big) = \xi^\varepsilon - \Lambda^*(0)$, with $\xi^\varepsilon = \xi^{D,\varepsilon}$ or $\xi^{F,\varepsilon}$.

4. Main results

Scaling limits under the critical situation: $\Sigma(\bar{h}) = \Sigma(\hat{h})$ (we assume $\xi^{F,\varepsilon} > \Lambda^*(0)$ in the free case) are summarized in:

Theorem 1. (1) (Dirichlet case) The limits of h^N under $\mu_N^{D,\varepsilon}$ are $\hat{h}^D$ if $d = 1$, both $\bar{h}^D$ and $\hat{h}^D$ (coexistence) if $d = 2$, and $\bar{h}^D$ if $d \geq 3$.
(2) (Free case) The limits of h^N under $\mu_N^{F,\varepsilon}$ are both $\bar{h}^F$ and $\hat{h}^F$ (coexistence) if $d = 1$, and $\bar{h}^F$ if $d \geq 2$.

The central limit theorem for the first and the last hitting times of the weakly pinned Markov chains ϕ at 0 holds under a suitable scaling and conditioning (if necessary).

5. Critical exponents

Proposition 2. (1) (Dirichlet case) As $\varepsilon \downarrow \varepsilon_c$,

$$\xi^{D,\varepsilon} \sim \begin{cases} C(\varepsilon - \varepsilon_c^D)^2, & d = 1, 3, \\ e^{-2\pi\sqrt{\det Q}/\varepsilon}, & d = 2, \\ C(\varepsilon - \varepsilon_c^D)/\log(\varepsilon - \varepsilon_c^D), & d = 4, \\ C(\varepsilon - \varepsilon_c^D), & d \geq 5, \end{cases}$$

where C are different positive constants depending on d.
(2) (Free case) (i) If $m = 0$, $\xi^{F,\varepsilon} = \xi^{D,\varepsilon}$.
(ii) If $m \neq 0$, we have $\xi^{F,\varepsilon} \sim C(\varepsilon - \varepsilon_c^F)$ as $\varepsilon \downarrow \varepsilon_c^F$ for all $d \geq 1$.

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Perturbed Allen–Cahn equation and forced mean curvature flow

Matthias Röger
(joint work with L. Mugnai)

We consider deterministic perturbations of the Allen–Cahn equation of the form

$$\begin{aligned} \varepsilon\partial_t u_\varepsilon &= \varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}W'(u_\varepsilon) + g_\varepsilon \quad \text{in } \Omega_T, \\ u_\varepsilon(0,\cdot) &= u_\varepsilon^0 \quad \text{in } \Omega, \\ \nabla u_\varepsilon \cdot \nu_\Omega &= 0 \quad \text{on } (0,T)\times\partial\Omega, \end{aligned} \tag{1}$$

where the spatial domain Ω is given by an open bounded set in $\mathbb{R}^n$ with Lipschitz boundary, $(0,T)$ is a fixed time intervall, $\Omega_T := (0,T)\times\Omega$, and W is the standard quartic double-well potential $W(r) = \frac{1}{4}(1-r^2)^2$. For $g_\varepsilon = 0$ equation (1) reduces to the standard Allen–Cahn equation and solutions converge to an evolution of phase boundaries by mean curvature flow [1, 2, 4]. Our goal is to prove that solutions of the perturbed equation converge to an evolution by *forced mean curvature flow.*

Theorem 1. *Let $n = 2, 3$ and let $(u_\varepsilon)_{\varepsilon>0}$, $(g_\varepsilon)_{\varepsilon>0}$, and $(u_\varepsilon^0)_{\varepsilon>0}$ be given such that (1) is satisfied and such that for all $\varepsilon > 0$*

$$\int_0^T \int_\Omega \frac{1}{\varepsilon} g_\varepsilon(t,x)^2 \, dxdt \leq \Lambda, \quad \int_\Omega \Big(\frac{\varepsilon}{2}|\nabla u_\varepsilon^0|^2 + \frac{1}{\varepsilon}W(u_\varepsilon^0)\Big)\, dx \leq \Lambda_0 \tag{2}$$

Then there exists a subsequence $\varepsilon \to 0$ and a phase indicator function $u \in BV(\Omega_T) \cap L^\infty(0,T;BV(\Omega;\{-1,1\})$ such that $u_\varepsilon \to u$ in $L^1(\Omega_T)$ and such that $\Gamma_t := \partial^*\{u(t,\cdot) = 1\}$ is a generalized solution of

$$v = H + g. \tag{3}$$

Here v describes the velocity vector of the evolution, $H(t,\cdot)$ denotes the mean curvature vector of Γ_t, and g is an appropriate limit of $-g_\varepsilon \nabla u_\varepsilon$.

To define a generalized solution of (3) we use the concept of L^2-*flows*, which describes an evolution of integral varifolds with weak mean curvature and generalized velocity and which complements the evolution of phase boundaries Γ_t. This formulation and the technique used to derive Theorem 1 emerges from our work [8] on the Allen–Cahn action functional

$$\mathcal{S}_\varepsilon(u) := \int_0^T \int_\Omega \Big(\sqrt{\varepsilon}\partial_t u + \frac{1}{\sqrt{\varepsilon}}\big(-\varepsilon\Delta u + \frac{1}{\varepsilon}W'(u)\big)\Big)^2 dx\,dt.$$

The assumption of the main Theorem yields the estimate $\mathcal{S}_\varepsilon(u_\varepsilon) \le \Lambda$. In [8] we obtain the compactness of uniformly action-bounded sequences, propose a relaxed formulation of the *reduced action functional*, which was introduced by Kohn, Otto, M. Westdickenberg, and Vanden-Eijnden [5] as the sharp interface limit of $\mathcal{S}_\varepsilon$, and prove a lower estimate.

In the following we sketch the line of arguments that we use derive Theorem 1. To obtain good compactness properties it is important to consider not only the limit of phase fields u_ε but also the limit of the *diffuse surface energy measures*

$$d\mu_\varepsilon^t := \Big(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2(t,\cdot) + \frac{1}{\varepsilon}W(u_\varepsilon(t,\cdot))\Big)\,dx, \quad d\mu_\varepsilon := \Big(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\Big)\,dxdt$$

and the diffuse surface energy at time $t \in (0,T)$,

$$E_\varepsilon(u_\varepsilon(t,\cdot)) = \int_\Omega \Big(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2(t,\cdot) + \frac{1}{\varepsilon}W(u_\varepsilon(t,\cdot))\Big)\,dx.$$

Furthermore we write $w_\varepsilon := -\Delta u_\varepsilon + \frac{1}{\varepsilon}W'(u_\varepsilon)$. A first important observation is that the action functional $\mathcal{S}_\varepsilon(u_\varepsilon)$ controls the diffuse surface energy,

$$\sup_{t\in(0,T)} E_\varepsilon(u_\varepsilon(t_0,\cdot)) + \int_0^T \int_\Omega \Big(\frac{\varepsilon}{2}(\partial_t u_\varepsilon)^2 + \frac{1}{2\varepsilon}w_\varepsilon^2\Big)\,dxdt \le \Lambda + \Lambda_0.$$

Another key estimate is the control of the time derivative of the energy measures.

Proposition 1. *Choose an arbitrary $\psi \in C_c^1(\Omega_T)$. Then*

$$2\partial_t\mu_\varepsilon^t(\psi) = \int_\Omega \big(\sqrt{\varepsilon}\partial_t u_\varepsilon + \frac{1}{\sqrt{\varepsilon}}w_\varepsilon\big)^2\psi - \int_\Omega \big(\varepsilon(\partial_t u_\varepsilon)^2 + \frac{1}{\varepsilon}w_\varepsilon^2\big)\psi - 2\int_\Omega \varepsilon\nabla\psi\partial_t u_\varepsilon\nabla u_\varepsilon.$$

Therefore $\partial_t\mu_\varepsilon^t(\psi)$ is uniformly bounded in $L^1(0,T)$.

By an application of the results of Modica–Mortola [7, 6] and the compactness of uniformly bounded Radon measures we now obtain the convergence of phase fields and energy measures.

Proposition 2. *There exists a subsequence $\varepsilon \to 0$ and a phase inidicator function $u \in BV(\Omega_T,\{-1,1\}) \cap L^\infty(0,T;BV(\Omega))$ such that*

$$u_\varepsilon \to u \quad \text{in } L^1(\Omega_T), \qquad u_\varepsilon(t,\cdot) \to u(t,\cdot) \quad \text{in } L^1(\Omega_T).$$

Moreover, for all but countably many $t \in [0,T]$ there exist Radon measures μ^t on Ω such that

$$\mu_\varepsilon^t \rightharpoonup \mu^t, \quad d\mu_\varepsilon \rightharpoonup d\mu^t\, dt \qquad \text{as Radon measures,}$$
$$\mu^t \geq \frac{\sigma}{2}|\nabla u(t,\cdot)|, \quad \sigma := \int_{-1}^{1} \sqrt{2W(s)}\, ds,$$

and for all $\psi \in C^1(\overline{\Omega})$ the mapping $t \mapsto \mu^t(\psi)$ is of bounded variation in $(0,T)$.

By an application of the compactness statement for diffuse energy measures with uniformly bounded diffuse Willmore energy [9] we obtain that the measures μ^t are much better than just being Radon measures: They are geometric objects with generalized tangential planes and generalized mean curvature vector.

Proposition 3. *For almost all $t \in (0,T)$*

$$\frac{1}{\sigma}\mu^t \text{ is an integral } (n-1)\text{-varifold},$$
$$\mu^t \text{ has weak mean curvature } H(t,.) \in L^2(\mu).$$

Moreover we have equipartition of energy *in the sense that*

$$\Big|\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 - \frac{1}{\varepsilon}W(u_\varepsilon)\Big| \rightharpoonup 0 \quad \text{in } L^1(\Omega_T).$$

Finally for all $\eta \in C_c^0(\Omega, \mathbb{R}^n)$

$$\lim_{\varepsilon\to 0} \int_{\Omega_T} \eta \cdot w_\varepsilon \nabla u_\varepsilon = \int_{\Omega_T} \eta \cdot H\, d\mu.$$

The next key step is that the evolution $(\mu^t)_{t\in(0,T)}$ has a generalized velocity.

Proposition 4. *There exists $v \in L^2(\mu, \mathbb{R}^n)$, such that for all $\eta \in C_c^1(\Omega_T)$*

$$\Big|\int_0^T \int_\Omega \big(\partial_t \eta + \nabla\eta \cdot v\big)\, d\mu^t dt\Big| \leq C\|\eta\|_\infty. \tag{4}$$

For all $\eta \in C_c^0(\Omega_T, \mathbb{R}^n)$

$$\lim_{\varepsilon\to 0} \int_{\Omega_T} -\eta \cdot \varepsilon \partial_t u_\varepsilon \nabla u_\varepsilon = \int_{\Omega_T} \eta \cdot v\, d\mu.$$

Remark 1. *In the case of a smooth evolution of smooth hypersurfaces* (4) *uniquely determines the velocity vector v of the evolution. In our case we obtain v as the sharp interface limit of the diffuce velocities*

$$v_\varepsilon := -\frac{\partial_t u_\varepsilon}{|\nabla u_\varepsilon|}\frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|},$$

more precisely we prove that (μ, v) is the measure–function pair limit in the sense of Hutchinson [3] *of $(\varepsilon|\nabla u_\varepsilon|^2 \mathcal{L}^{n+1}, v_\varepsilon)$.*

Finally we show the convergence of the forcing terms and pass (1) to the limit.

Proposition 5. *There exists $g \in L^2(\mu, \mathbb{R}^n)$, such that for all $\eta \in C_c^0(\Omega_T, \mathbb{R}^n)$*

$$\lim_{\varepsilon \to 0} \int_{\Omega_T} -\eta \cdot g_\varepsilon u_\varepsilon \nabla u_\varepsilon \, = \, \int_{\Omega_T} \eta \cdot g \, d\mu.$$

If we choose an arbitrary test function $\eta \in C_c^0(\Omega, \mathbb{R}^n)$ and multiply (1) by $\eta \cdot \nabla u_\varepsilon$ we can pass to the limit $\varepsilon \to 0$ and obtain that

$$\int_{\Omega_T} \eta \cdot v \, d\mu \, = \, \int_{\Omega_T} \eta \cdot (H + g) \, d\mu,$$

which yields that μ-almost everywhere $v = H + g$ holds.

Remark 2. *We can also show that v, H, g are properties of the evolutions of phase boundaries Γ_t and we therefore obtain that $(\Gamma_t)_{t \in (0,T)}$ itself evolves (in a generalized formulation) by forced mean curvature flow.*

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Dynamics and self-similarity in min-driven clustering

Barbara Niethammer

(joint work with G. Menon and R.L. Pego)

We consider a mean-field model for a family of clustering processes which are given by a simple rule in an infinite number of clusters: At each step a random integer $k \geq 1$ is chosen with probability p_k, and the smallest cluster merges with k randomly chosen clusters. The mean-field assumption is that all these random

variables are independent. The only assumptions we impose on the probabilities are that

$$p_k \geq 0, \quad \sum_{k=1}^{\infty} p_k = 1, \quad \sum_{k=1}^{\infty} k p_k < \infty. \tag{1}$$

The evolution of the number density under this process is described by the following rate equation for cluster size density $f(t,x)$, obtained by summing over all loss and gain events:

$$\partial_t f(t,x) = f(t,l)\dot{l} \sum_{k=1}^{\infty} p_k \left(\rho_t^{\star k}(x-l) - k\rho_t(x)\right), \quad x > l(t). \tag{2}$$

Here $\rho_t(x) := f(t,x)/\int_0^\infty f(t,x)\,dx$ denotes a probability density and we denote by $\rho_t^{\star k}$ the k-fold self-convolution.

An important feature of the model (2) is its invariance under reparametrization in time and a careful choice of the time scale is key to the analysis. In the first mathematical study of a variant of (2) in [1], the authors imposed the relation $f(t,l)\dot{l} = 1$ such that the number of coalescence events per unit time is constant. In a more recent paper, Gallay and Mielke [2] parametrized time by the minimum size, that is $l(t) = t$. This leads to an elegant solution procedure that was used to prove some basic results on well-posedness and the approach to self-similarity. In [3] we choose yet another time scale. We parametrize time inversely to the total number of domains, so that

$$t = N(t)^{-1}. \tag{3}$$

With this time scale we can obtain existence and uniqueness for measure-valued solutions, necessary and sufficient conditions for convergence to self-similar form and a characterization of eternal solutions (cf. [3], Section 7, for details on the latter).

To any solution of (2) we associate a probability measure F_t with distribution function written

$$F_t(x) = \frac{1}{N(t)} \int_0^x f(t,y)\,dy = \int_0^x \rho_t(y)\,dy. \tag{4}$$

For any probability distribution F on $[0,\infty)$, we call $l = \inf\{x|F(x) > 0\}$ the *min* of F. (This is short for "minimum size," regarding F as a probability distribution for size.) We prove that the initial-value problem for an appropriate weak form of (2) is well-posed for initial probability measures F_{t_0} with positive min. That is, (2) with (3) determines a continuous dynamical system on the space of probability measures with positive min, equipped with the weak topology.

Our main interest is in self-similar long-time behavior within the model (2). It has been found in [2] that there is a one-parameter family of self-similar solutions of the form

$$F_t(x) = F^{(\theta)}\left(\frac{x}{l^{(\theta)}(t)}\right), \quad l^{(\theta)}(t) = t^{1/\theta}, \quad \theta \in (0,1], \quad t > 0. \tag{5}$$

Here $F^{(\theta)}$ is a probability distribution with density $\rho^{(\theta)}$ supported on $[1,\infty)$. The density $\rho^{(\theta)}$ is known explicitly only through its Laplace transform. The only density with finite mass is $\rho^{(1)}$ in the case that

$$\sum_{k=1}^{\infty} p_k k \log k < \infty\,. \tag{6}$$

The distributions $F^{(\theta)}$ for $0<\theta<1$ have heavy tails, with $\rho^{(\theta)}(x) \sim c_\theta x^{-(1+\theta)}$ as $x \to \infty$.

As in many other coarsening and coagulation models the domains of attraction of the self-similar solutions are determined by the tails of the initial size distribution. For the case that only a finite number of the p_k are nonzero, Gallay and Mielke showed that all densities with finite mass are attracted to the self-similar solution with $\theta = 1$. Moreover, for $0 < \theta \le 1$ they showed that if the initial data ρ_{t_0} is sufficiently close to $\rho^{(\theta)}$ in a suitable weighted norm then the rescaled probability density $t\rho_t(tx)$ approaches $\rho^{(\theta)}$ with a rate of convergence determined by the weighted norm.

In our main result we establish conditions that are both necessary and sufficient to answer a more general question about *arbitrary* scaling limits. We characterize the set of all non-degenerate limits under a general rescaling of the form $F_t(\lambda(t)x)$ where $\lambda(t)$ is a measurable, positive function such that $\lim_{t\to\infty}\lambda(t) = \infty$. A limit is non-degenerate if it is suitable data for the initial-value problem. That is, non-degenerate limits are probability distributions with a positive min.

Theorem *Let $t_0 > 0$, let F_{t_0} be an arbitrary probability measure on $(0,\infty)$ with positive min, and let F_t $(t \ge t_0)$ be the associated measure-valued solution of (2).*

(i) Suppose there is a measurable, positive function $\lambda(t) \to \infty$ as $t \to \infty$ and a probability measure F_* with positive min, such that

$$\lim_{t\to\infty} F_t(\lambda(t)x) = F_*(x) \tag{7}$$

at all points of continuity of F_*. Then there exists $\theta \in (0,1]$ and a function L slowly varying at infinity such that the initial data F_{t_0} satisfies

$$\int_0^x y F_{t_0}(dy) \sim x^{1-\theta} L(x), \quad \text{as} \quad x \to \infty. \tag{8}$$

Moreover, the min $l(t)$ of F_t and the rescaling $\lambda(t)$ satisfy

$$\lambda(t) l_* \sim l(t) \sim t^{1/\theta}\tilde{L}(t), \quad \text{as} \quad t \to \infty, \tag{9}$$

where $l_* > 0$ is the min of F_* and $\tilde{L}$ is slowly varying at infinity, related to L by an explicit formula.

(ii) Conversely, assume there exists $\theta \in (0,1]$ and a function L slowly varying at infinity such that the initial data satisfies (8). Then $l(t)$ satisfies (9), and

$$\lim_{t\to\infty} F_t(l(t)x) = F^{(\theta)}(x), \quad x \in (0,\infty). \tag{10}$$

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Correlations of phase transformations in polycrystals

Alexei Novikov

(joint work with Leonid Berlyand and Oscar Bruno)

We investigate the statistics of the transformation-strains that arise in random martensitic polycrystals as boundary conditions cause its component crystallites to undergo solid-to-solid (martensitic) phase transitions. Martensitic transformations are shape-deforming phase transitions that can be induced in certain alloys as a result of changes in the imposed strains, stresses or temperatures. These transitions occur when a crystalline solid transforms between its parent phase (austenite) and any of a number of variants of the product phase (martensite). We focus on a setting that, while sufficiently simple to allow for a complete analytical treatment, provides significant insights on the problem: we study laminated polycrystals that consist of sequences of n of grains of rectangular cross-section—of base $1/n$ and height $L = L(n)$, so that a complete polycrystal is an infinite parallelepiped with rectangular cross-section of base 1 and height L. The goal of this work is to provide a rigorous probabilistic theory for the misfit statistics in such polycrystals and, in particular, to provide a rationale for the approximations implicit in the numerical algorithms [2, 4] for polycrytalline phase transitions in two- and three-dimensional space.

The microstructure in a laminated polycrystal is described by a sequence of the orientation angles θ_i, $i = 1, \ldots, n$: θ_i represents the orientation of the two-dimensional lattice structure in the i-th grain. We assume θ_i is a sequence of n independent identically distributed (i.i.d.) random variables. The transformation in the i-th grain gives rise to a strain-tensor, the transformation strain ε_i^T, $(i = 1, \ldots, n)$, which is constant and it takes one of three *admissible* values: no deformation (the original square lattice remains square), or deformation into one of two rectangular crystalline lattices parallel to the original square lattice. The phase transition in the polycrystal gives rise to a sequence of transformation strains ε_i^T, $i = 1, \ldots, n$ obtained by the minimization of the elastic *misfit energy* among all admissible configurations.

The main results of this work characterize the probability distribution of the random variables ε_i^T that results as minimizers of the overall elastic energy for a given i.i.d. distribution of the angle sequence θ_i. Such results are provided in three

different cases according to whether the grains are 1) infinitely long ($L = \infty$); 2) of finite but large height ($L = L \gg 1$); and 3) short height ($L = l_0/(2n)$, $l_0 \ll 1$). In case 1) our treatment applies to arbitrary i.i.d. probability measures ρ defining the distribution of angles, in cases 2) and 3), in turn, we restrict consideration to i.i.d. distribution of angles with Bernoulli probability measures ρ. Our main results can be briefly described as follows:

(1) *Infinitely long grains.* For an arbitrary i.i.d. distribution of angles θ_i, $i = 1, 2, \dots, n$, under certain technical assumptions, in the limit $n \to \infty$, the transformation strains ε_i^T, $i = 1, 2 \dots, n$ are also i.i.d. with probability measure μ, where the measure μ is the minimizer of a certain functional. In particular, in the case of infinitely long grains there are no correlations between transformation strains of any two grains.

(2) *Long finite grains.* $L \gg 1$. If θ_i, $i = 1, \dots, n$ are Bernoulli random variables, then in the limit $n \to \infty$, ε_i^T, $i = 1, 2 \dots, n$ have long-range but no short-range correlations.

(3) *Short grains.* $L = l_0/(2n)$, $l_0 \ll 1$. If θ_i, $i = 1, \dots, n$ are Bernoulli random variables, then in the limit $n \to \infty$, ε_i^T, $i = 1, 2 \dots, n$ have short-range but no long-range correlations.

The results 2 and 3 can be explained as follows. The cornerstone of our study is the maximization of an integral energy functional of the form $\int K_L(x-t)f(x)f(t)dxdt$. Its integral kernel $K_L(x)$ decays on different length scales for long and short grains. For long grains it decays on the length scale of the composite (on $O(1)$ scale), while for short grains it decays on the length scale of a grain (on $O(1/n)$ scale). Maximization with respect to this integral kernel leads to long-range and short-range correlations for long and short grains, respectively. Formally, correlations arise because grains that undergo the stress-free transformation tend to "group together" on the scale of the decay of the integral kernel. We justify this heuristic idea in case of long grains by applying a randomized version of the Riesz rearrangement inequality. In the case of short grains we show the transforming grains group together—by applying an isoperimetric inequality. A more detailed account of this work could be found in [1].

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Resistance analysis of infinite networks

Erin P. J. Pearse
(joint work with Palle E. T. Jorgensen)

An operator-algebraic approach to the analysis of resistance networks. In [6], we introduce and develop new methods for the analysis of finite-energy functions on infinite resistance networks. Such a network is a connected simple graph G whose edges have been weighted by a "conductance" function c which assigns nonnegative values to edges ($c_{xy} > 0$ means that an edge connects x to y, denoted $x \sim y$). The network is considered as a metric space under the resistance metric R, which is defined in terms of the energy form

$$\mathcal{E}(u,v) := \sum_{x\sim y} c_{xy}(u(x)-u(y))(v(x)-v(y)), \quad \text{for} \quad u,v : G \to \mathbb{C},$$

where the sum is taken over all edges in the graph. The effective resistance distance between vertices x and y is often phrased as inverse capacitance via

$$R(x,y)^{-1} := \min\{\mathcal{E}(u) : u(x) = 0, u(y) = 1\},$$

where $\mathcal{E}(u) = \mathcal{E}(u,u)$. We discuss several formulations of this metric and describe how it is better suited to understanding network topology than shortest-path (combinatorial) distance. In particular, we show how $R(x,y)$ is related to random walks and hence the asymptotic structure of infinite networks. Different types of limits on networks (e.g., *shorting* and *truncation*, discussed below) are expressed naturally in terms of boundary conditions and hence yield to operator-analytic inspection. We explain how these approaches are explained by Hilbert space geometry. *New approaches to resistance analysis.* It is known from [4] that transience of the random walk on a network is equivalent to the existence of a finite-energy solution to a certain Dirichlet problem (i.e., a *monopole*), and hence equivalent to "unit current flow to infinity". (These terms are discussed below.) We take the novel approach of using the resistance metric R on the graph G as the foundation for an algebraic investigation: the initial step is the natural embedding $x \mapsto v_x$ of the metric space (G,R) into a Hilbert space (the "energy space" $\mathcal{H}_\mathcal{E}$), following von Neumann's method.

The resulting Hilbert space $\mathcal{H}_\mathcal{E}$ consists of *equivalence classes* of functions on the vertices, possibly including *nonconstant harmonic functions of finite energy.* It is known that $\ell^2(G)$ contains no such objects [6, 7], so we instead study the geometry of $\mathcal{H}_\mathcal{E}$ via the energy inner product $\langle\cdot,\cdot\rangle_\mathcal{E} = \mathcal{E}(\cdot,\cdot)$.

The action of the network Laplace operator is defined for $u : G \to \mathbb{C}$ by

$$\Delta u(x) := \sum_{y\sim x} c_{xy}(u(x)-u(y)).$$

The energy space decomposes as $\mathcal{H}_\mathcal{E} = [\mathcal{F}in] \oplus \mathcal{H}arm$, where $\mathcal{F}in = [\mathrm{span}\{\delta_x\}]_\mathcal{E}$ is the energy closure of the space of functions of finite support and $\mathcal{H}arm = \{u \in \mathcal{H}_\mathcal{E} : \Delta u \equiv 0\}$ is the space of nontrivial harmonic functions [3, 6]. Thus the usual onb candidate $\{\delta_x\}_{x\in G}$ is not always dense in $\mathcal{H}_\mathcal{E}$, where δ_x denotes the

characteristic function of $\{x\}$. Hence, we use the naturally dense set of ***dipoles***. The dipole v_x is the image of the vertex x under the embedding described above, and is a *reproducing kernel* for $\mathcal{H}_\mathcal{E}$, i.e., $\langle u, v_x\rangle = u(x) - u(o)$, for any $u \in \mathcal{H}_\mathcal{E}$.

A ***monopole*** *at* x is a function $w \in \mathcal{H}_\mathcal{E}$ with $\Delta w = \delta_x$. It is an important fact that dipoles always have finite energy [6], but there are only monopoles of finite energy when the network has a transient random walk [2, 3].

The energy has a corresponding decomposition as a discrete Gauss-Green formula

$$\mathcal{E}(u,v) = \sum_{x\in G} u(x)\Delta v(x) + \sum_{\mathrm{bd}\, G} u\tfrac{\partial v}{\partial \mathbb{n}} \tag{1}$$

where $\frac{\partial v}{\partial \mathbb{n}}$ denotes a discrete analogue of the normal derivative of v (defined just below.) If $v \in \mathcal{H}_\mathcal{E}$ is written (uniquely) as $v = f + h$ with $f \in \mathcal{F}in$ and $h \in \mathcal{H}arm$, then $\mathcal{E}(v) = \sum_{x\in G} f(x)\Delta f(x) + \sum_{\mathrm{bd}\, G} h\frac{\partial h}{\partial \mathbb{n}}$, so that the energy of the harmonic part of v comes from the boundary, and the energy of the finite part of v comes from the vertices. From the discrete Gauss-Green formula one obtains a boundary representation for $\mathcal{H}arm$:

$$h(x) = \sum_{\mathrm{bd}\, G} v_x \tfrac{\partial h}{\partial \mathbb{n}} + h(o).$$

Both the discrete Gauss-Green formula and the boundary representation are valid for the energy space of any network; there are no other hypotheses.

From these considerations, it follows that $\mathcal{F}in$ is dense in $\mathcal{H}_\mathcal{E}$ if and only if $\mathcal{H}arm = 0$ if and only if the boundary term in the discrete Gauss-Green formula vanishes identically. This leads to the main question here:

Question 1. ***What is*** $\mathrm{bd}\, G$***?***

The formula (1) holds in the following sense: let $\{G_n\}$ be an *exhaustion* of G; that is, a nested sequence of finite subnetworks whose union is G. Define the *boundary of a subnetwork* G_n to be

$$\mathrm{bd}\, G_n := \{x \in G_n : \exists z \in G \setminus G_n \text{ with } z \sim x\},$$

and for $x \in \mathrm{bd}\, G_n$, define the *normal derivative* to be

$$\tfrac{\partial v}{\partial \mathbb{n}}(x) := \sum_{\{y\sim x\,:\, y\in G_n\}} c_{xy}(v(x) - v(y)),$$

Then the boundary term is defined as the limit of sums taken over the boundaries of finite subnetworks:

$$\sum_{\mathrm{bd}\, G} u\tfrac{\partial v}{\partial \mathbb{n}} := \lim_{n\to\infty} \sum_{x\in \mathrm{bd}\, G_n} u(x)\tfrac{\partial v}{\partial \mathbb{n}}(x).$$

Obviously, one expects this quantity to be more properly described as an integral, so Question 1 is really asking: with respect to what measure, and over what space, should this integral be taken? We offer a couple of preliminary responses.

Previous notions of boundary. The boundary of an infinite network appearing in [6] is a relative of the Martin boundary (for certain Markov processes) cf. [8, 9], but for finite-energy harmonic functions instead of nonnegative harmonic functions.

A Gel'fand triple for $\operatorname{bd} G$. The goal is to obtain a natural probability measure on the boundary. One can construct a *Gel'fand triple* (also called a *rigged Hilbert space*) $S \subseteq \mathcal{H}_{\mathcal{E}} \subseteq S'$, where S is a suitable dense subspace of $\mathcal{H}_{\mathcal{E}}$ defined in terms of the domain of Δ, and S' is the dual of S. The duality between S and S' allows extension of the inner product on $\mathcal{H}_{\mathcal{E}}$ to a pairing of S and S'.

The construction is extended by applying Schoenberg's theorem to the resistance metric and Minlos' theorem to the resulting positive definite function, to obtain a Gaussian probability measure $\mathbb{P}$ on S' for which

$$\mathbb{E}_\xi(e^{\mathrm{i}\langle u,\xi\rangle_\mathcal{E}}) = e^{-\frac{1}{2}\|u\|_\mathcal{E}^2} \quad \text{and} \quad \mathbb{E}_\xi(\langle u,\xi\rangle_\mathcal{E}^2) = \|u\|_\mathcal{E}^2, \quad \text{for } \xi \in S'.$$

When applied to $\mathcal{H} = \mathcal{H}_{\mathcal{E}}$, the construction yields a Fourier-type duality for the energy space of an arbitrary graph, and a concrete representation of $\mathcal{H}_{\mathcal{E}}$ as an L^2 measure space $\mathcal{H}_{\mathcal{E}} \cong L^2(S', \mathbb{P})$. This is not trivial: while every separable Hilbert space is isomorphic to $\ell^2(X, d\mu)$, it is *not* the case that $\mathcal{H}_{\mathcal{E}}$ is isomorphic to $\ell^2(G, d\mu)$ for any μ.

Additionally, for $e_x(\xi) := e^{\mathrm{i}\langle v_x,\xi\rangle_\mathcal{E}}$, one has

$$\mathbb{E}_\xi(\overline{e_x} e_y) = \int_{S'} \overline{e_x(\xi)} e_y(\xi)\, d\mathbb{P} = e^{-\frac{1}{2}R(x,y)},$$

and the boundary of G may be studied as $\operatorname{bd} G = S'/\operatorname{dom}(\Delta)$. As hoped, this gives a new way of understanding the boundary representation for $\mathcal{H}arm$:

$$h(x) = \int_{S'} \tilde{v}_x(\xi) \tfrac{\partial \overline{h}}{\partial \mathrm{n}}(\xi)\, d\mathbb{P}(\xi) + h(o).$$

The motivation for our investigation originated in a series of papers by Bob Powers, in which the boundedness of $R(x, y)$ on the integer lattice $\mathbb{Z}^3$ is used to explain features of the Heisenberg model for ferromagnetism. We explain the key role of the resistance metric in this setting, and how it explains long-range correlation of spins between particles.

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Exponential Decay Estimates for Dispersion Management Solitons

Dirk Hundertmark
(joint work with M. B. Erdoğan and Young-Ran Lee)

The one-dimensional non-linear Schrödinger equation (NLS) with periodically varying dispersion coefficient

$$iu_t + d(t)u_{xx} + |u|^2 u = 0. \tag{1}$$

describes the amplitude of a signal transmitted via amplitude modulation of a carrier wave through a fiber-optical cable where the dispersion is varied periodically along the fiber, see, e.g., [11, 13, 15]. In (1) t corresponds to the distance along the fiber, x denotes the (retarded) time, and $d(t)$ is the dispersion along the waveguide, which, for practical purposes, one can assume to be piecewise constant.

The dispersion management technique uses alternating sections of constant but (nearly) opposite dispersion, introducing a rapidly varying dispersion $d(t)$ with small average dispersion which leads to well-localized stable soliton-like pulses changing periodically along the fiber. It has been enormously fruitful (see, *e.g.*, [1, 2, 3, 7, 14, 15]).

Writing $d(t) = d_0(t) + d_{av}$, the envelope equation takes the form

$$iu_t + d_0(t)u_{xx} + d_{\mathrm{av}}u_{xx} + |u|^2 u = 0. \tag{2}$$

Here $d_0(t)$ is the mean zero part which we assume to be piecewise constant, and d_{av} the average dispersion over one period. Since the full equation (2) is very hard to study, Gabitov and Turitsyn suggested to separate the free motion given by the solution of $iu_t + d_0(t)u_{xx} = 0$ in (2), and to average over one period, [2, 3]. This yields the following (Gabitov-Turitsyn) equation for the "averaged" solution v

$$iv_t + d_{\mathrm{av}}v_{xx} + Q(v, v, v) = 0, \quad \text{where} \tag{3}$$

$$Q(v_1, v_2, v_3) := \int_0^1 T_r^{-1}\Big(T_r v_1 \overline{T_r v_2} T_r v_3\Big)\, dr, \tag{4}$$

and $T_r = e^{ir\partial_x^2}$. In some sense, v is the slowly varying part of the solution u of (1) and the varying dispersion is interpreted as a fast background oscillation, justifying formally the above averaging procedure. This is similar to Kapitza's treatment

of the unstable pendulum, see [8]. This averaging procedure yielding (3) is well-supported by numerical and theoretical studies, see, for example, [1, 14, 15], and was rigorously justified in [16] in the limit of large local dispersion $d_0(t)$.

One can find stationary solutions of (3) by making the ansatz $v(t,x) = e^{i\omega t} f(x)$, yielding the time independent equation

$$-\omega f = -d_{\rm av} f_{xx} - Q(f,f,f) \tag{5}$$

describing stationary soliton-like solutions, the so-called dispersion managed solitons. Despite the enormous interest in dispersion managed solitons, there are few rigorous results available. One reason for this is that it is a nonlinear and nonlocal equation. Existence and smoothness of solutions of (5) had first been rigorously established in [16] for positive average dispersion $d_{\rm av} > 0$. In the case $d_{\rm av} = 0$, the existence was obtained in [6], also see [5] for a simplified proof. Smoothness in the case $d_{av} = 0$ was established in [12].

Of course one has to be a bit careful what one means by a solution of (5). We focus on weak solutions, meaning $f \in H^1$ in the case $d_{av} > 0$, or $f \in L^2$ in the case $d_{av} = 0$, such that

$$-\omega\langle g, f\rangle = d_{av}\langle g', f'\rangle - \langle g, Q(f,f,f)\rangle. \tag{6}$$

for all $g \in H^1$. Here $\langle g, f\rangle = \int_{\mathbb{R}} \overline{g(x)} f(x)\, dx$ is the usual scalar product on $L^2(\mathbb{R})$. By a formal calculation, using the unicity of T_r in L^2, we have

$$\langle g, Q(f,f,f)\rangle = \mathcal{Q}(g,f,f,f),$$

where

$$\mathcal{Q}(f_1,f_2,f_3,f_4) = \int_0^1 \int_{\mathbb{R}} \overline{T_t f_1(x)} T_t f_2(x) \overline{T_t f_3(x)} T_t f_4(x) dx dt. \tag{7}$$

Thus the weak formulation of (5) is given by

$$-\omega\langle g, f\rangle = d_{av}\langle g', f'\rangle - \mathcal{Q}(g,f,f,f) \tag{8}$$

for all $g \in H^1(\mathbb{R})$ if $d_{av} > 0$, respectively all $g \in L^2(\mathbb{R})$ if $d_{av} = 0$. Due to Strichartz inequality, the functional $\mathcal{Q}(f_1,f_2,f_3,f_4)$ is well-defined for $f_j \in L^2(\mathbb{R})$, see [16, 4].

The decay of (weak) solutions was first addressed by Lushnikov in [9]. He gave convincing but non-rigorous arguments that any (weak) solution f of (5) for $d_{av} = 0$ satisfies

$$f(x) \sim |x| \cos(a_0 x^2 + a_1 x + a_2) e^{-b|x|} \quad \text{as } x \to \infty \tag{9}$$

for some suitable choice of real constants a_j and $b > 0$, see also [10]. In particular, he predicted that f and $\widehat{f}$ decay exponentially at infinity. For $d_{av} = 0$, the first rigorous x-space decay bounds were established in [4], where it was shown that both f and $\widehat{f}$ decay faster than any polynomial in the case $d_{\rm av} = 0$. In particular, any weak solution is a Schwartz function.

Our main result confirms Lusnikov's exponential decay prediction:

Theorem 1. *Let $f \in L^2$ be a weak solution of (5) with $d_{av} = 0$. Then there exists $\mu > 0$ such that*

$$|f(x)| \lesssim e^{-\mu|x|}, \quad |\widehat{f}(\xi)| \lesssim e^{-\mu|\xi|},$$

where $\widehat{f}$ is the Fourier transform of f.

We have the following immediate

Corollary 2. *Let $f \in L^2$ be a weak solution of (5) with $d_{av} = 0$. Then both f and $\widehat{f}$ are analytic in a strip containing the real line.*

The difficulty in obtaining any decay for weak solutions comes from the fact that the kernel of the free Schrödinger evolution has no point-wise decay. Instead one has to use the multi-linear structure of $\mathcal{Q}$ and the oscillation of the kernel. In particular, our result follows from exponentially weighted multi-linear estimates for $\mathcal{Q}$. These are refinements of the x-space Strichartz estimates which were developed in [4] to prove, in conjunction with well-known Fourier space Strichartz estimates, that any weak solution $f \in L^2(\mathbb{R})$ of (5) for $d_{\rm av} = 0$ is already a Schwartz function.

We would also like to stress that our proof of Theorem 1 uses only the new exponentially weighted multi-linear estimates for $\mathcal{Q}$ together with $f \in L^2$ for any weak solution and as such does not require any of the previous smoothness or decay results.

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Hydrodynamic limit for the interface model with general potentials

Takao Nishikawa
(joint work with Jean-Dominique Deuschel and Yvon Vignard)

1. Model and settings

Let us consider the Ginzburg-Landau $\nabla\phi$ interface model, that is, the dynamics governed by the following stochastic differential equations (SDEs)

$$d\phi_t(x) = -\frac{\partial H}{\partial \phi(x)}(\phi_t)\,dt + \sqrt{2}dw_t(x), \quad x \in \mathbb{Z}^d, \tag{1}$$

where Hamiltonian H is defined by

$$H(\phi) = \sum_{x,y\in\mathbb{Z}^d;\,|x-y|=1} V(\phi(x)-\phi(y)), \tag{2}$$

and $\{w_t(x);\, x \in \mathbb{Z}^d\}$ is the family of independent standard Brownian motions. Similarly to above, we introduce the time evolution on the periodic lattice $\Gamma_N \equiv (\mathbb{Z}/N\mathbb{Z})^d$ with the periodic boundary condition, and the time evolution on the discretized microscopic domain D_N corresponding to a bounded domain D with a Lipschitz boundary, with Dirichlet boundary conditions. More precisely saying, we define D_N by

$$D_N = \{x \in \mathbb{Z}^d;\, B(x/N, 5/N) \subset D\},$$

where $B(\alpha, l)$ represents the hypercube in $\mathbb{R}^d$ with center α and side length l. Here, we consider the boundary condition described by

$$\phi_t(x) = N^{d+1}\int_{B(x/N,1/N)} f(\theta)\,d\theta, \quad x \in \mathbb{Z}^d \setminus D_N,\, t \geq 0 \tag{3}$$

with $f \in C_0^2(\mathbb{R}^d)$. We note that the function f describes the macroscopic boundary condition and the identity (3) describes the microscopic one.

We regard the time evolution (1) as the model describing the motion of microscopic interfaces and introduce the macroscopic height variable h^N as follows:

$$h^N(t,\theta) = \sum_{x\in\mathbb{Z}^d} N^{-1}\phi_{N^2t}(x)1_{B(x/N,1/N)}(\theta), \quad \theta \in \mathbb{T}^d \equiv [0,1)^d,$$

for the dynamics with the periodic lattice Γ_N, we also introduce similarly to above for the dynamics on the domain D_N by replacing $\mathbb{T}^d$ by $\mathbb{R}^d$. Note that we adopt the diffusive scaling.

If the potential V in (2) is strictly convex, it is known that the anisotropic mean curvature motion is derived as the limit of h^N, see [3] in the case with the periodic boundary condition, and [4] in the case with the Dirichlet boundary condition.

Recently, the analysis of the system with non-convex potentials is highly developed. Especially, in the high temperature regime, that is, if nonconvex perturbation is small, the surface tension remains strictly convex, see [2] and [1]. Moreover, in this regime, ergodic Gibbs measure of the gradient field is unique when specifying a tilt, see [1]. We shall discuss the hydrodynamic scaling limit in this regime, that is, we assume the conditions stated below for the potential V in (2):

(1) V can be decomposed into two parts:

$$V(x) = V_0(x) + g(x), \quad x \in \mathbb{R},$$

where $V_0 \in C^2(\mathbb{R})$ is symmetric and strictly convex in the following sense: there exist constants $c_-, c_+ > 0$ such that

$$c_- \le V_0''(x) \le c_+, \quad x \in \mathbb{R},$$

and $g \in C^2(\mathbb{R})$ is also symmetric and satisfying

$$\sup_{x \in \mathbb{R}} (|g'(x)| + |g''(x)|) < \infty$$

(2) For each $u \in \mathbb{R}^d$, there exists a unique ergodic Gibbs measure for gradient field with tilt u.

(3) The surface tension (free energy) $\sigma = \sigma(u)$ is strictly convex in $u \in \mathbb{R}^d$.

2. Main results

In the case with the periodic boundary condition, it is possible to derive the anisotropic mean curvature motion.

Theorem 1. *We assume that*

$$\sup_{N \ge 1} E\left[\|h^N(0)\|^2_{H^1(\mathbb{T}^d)}\right] < \infty$$

and that there exists $h_0 \in H^1(\mathbb{T}^d)$ such that

$$\lim_{N \to \infty} E\|h^N(0) - h_0\|^2_{L^2(\mathbb{T}^d)} = 0.$$

We then have

$$\lim_{N \to \infty} E\|h^N(t) - h(t)\|^2_{L^2(\mathbb{T}^d)} = 0,$$

where h is the weak solution of nonlinear PDE

$$\frac{\partial h}{\partial t} = \mathrm{div} \nabla \sigma(\nabla h) \tag{4}$$

on $\mathbb{T}^d$ with initial data h_0.

In the case with the Dirichlet boundary condition, we can derive the equation though we require an additional assumption $d \le 2$.

Theorem 2. *Assume $d \le 2$. We assume that*

$$\sup_{N\ge 1} E\left[\|h^N(0)\|^2_{H^1(D)}\right] < \infty$$

and

$$\lim_{N\to\infty} E\|h^N(0) - h_0\|^2_{L^2(D)} = 0$$

for some $h_0 \in C^2(D)$ such that $h_0 - f \in C_0^2(D)$. We then have

$$\lim_{N\to\infty} E\|h^N(t) - h(t)\|^2_{L^2(D)} = 0,$$

where h is the weak solution of nonlinear PDE (4) with initial data h_0 and Dirichlet boundary condition given by f.

The proof for Theorems 1 and 2 is based on H^{-1}-method used in [3] and [4]. The main difficulty is the characterization of equilibrium states of the gradient dynamics, which is defined by $\eta_t(b) = \nabla\phi_t(b)$ for $b \in (\mathbb{Z}^d)^*$, where $(\mathbb{Z}^d)^*$ is the set of all directed bonds $b = (x, y)$, $x, y \in \mathbb{Z}^d, |x - y| = 1$ and ∇ is the discrete gradient operator defined by

$$\nabla\psi(b) = \psi(x) - \psi(y), \quad \psi \in \mathbb{R}^{\mathbb{Z}^d}, b = (x, y) \in (\mathbb{Z}^d)^*.$$

We denote the state space of η_t by $\mathscr{X}$, that is,

$$\mathscr{X} = \left\{\nabla\phi \in \mathbb{R}^{(\mathbb{Z}^d)^*};\, \phi \in \mathbb{R}^{\mathbb{Z}^d}\right\}.$$

In the case with periodic boundary condition, since the entropy method does work, the characterization of Gibbs measures on $\mathscr{X}$ corresponding to η_t is enough for this aim, and therefore it is not so difficult to obtain Theorem 1. But, it is difficult to apply the entropy method when the Dirichlet boundary condition is imposed, because the space-time average is not absolutely continuous with respect to finite volume Gibbs measure and thus we cannot use the relative entropy in the proof of local equilibrium. However, once we completely characterize the family of stationary measures, it is known that the calculation of relative entropy is not needed in the proof of local equilibrium, see [4]. The key to handle stationary measures is the following theorem, which implies that the problem of the characterization of stationary measures can be reduced to that for Gibbs measures. Note that the condition "$d \le 2$" in Theorem 2 is coming from here.

Theorem 3. *We assume $d \le 2$. Let a measure μ on $\mathscr{X}$ be tempered in the following sense:*

$$\sup_{b\in(\mathbb{Z}^d)^*} E^\mu[\eta(b)^2] < \infty.$$

If μ is a stationary measure of η_t, μ is then a Gibbs measure.

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A stochastic particle approximation for the Ricci flow on surfaces

Robert Philipowski

The evolution of a Riemannian metric $g = g_t$ on a connected d-dimensional closed manifold M under the (normalized) Ricci flow is described by the partial differential equation

$$\frac{\partial g}{\partial t} = \frac{2}{d}\bar{R}g - 2\,\mathrm{Ric}\,. \tag{1}$$

Here Ric denotes the Ricci curvature, R the scalar curvature and $\bar{R} := \frac{1}{\mathrm{vol}(M)}\int_M R$ the average scalar curvature (all quantities taken with respect to $g = g_t$).

From now on we concentrate on the two-dimensional case (i.e. M is a surface). In this case $\mathrm{Ric} = \frac{1}{2}Rg$, and therefore (1) is equivalent to

$$\frac{\partial g}{\partial t} = \left(\bar{R} - R\right) g. \tag{2}$$

Hamilton [2] and Chow [1] proved the following result:

Proposition 1. *Let g_0 be any Riemannian metric on M.*

1. *The Ricci flow equation has a unique solution $(g_t)_{t\geq 0}$ with initial data g_0.*
2. *As $t \to \infty$, g_t converges uniformly in any C^k-norm to a smooth metric g_∞ of constant curvature.*

Remark 2. *From (2) it is obvious that the conformal class of g is conserved under the two-dimensional Ricci flow.*

Remark 3. *The volume of M is conserved under the Ricci flow (this holds in all dimensions). Without loss of generality we may therefore assume that* $\mathrm{vol}(M) = 1$.

Thanks to Remark 2 it is possible to represent g_t in the form

$$g_t = u(t,\cdot)\tilde{g} \qquad \forall t \geq 0,$$

where $\tilde{g}$ is a fixed reference metric on M and u a smooth function on $\mathbb{R}_+ \times M$ (one can for instance take $\tilde{g} = g_0$, so that in this case $u_0 := u(0,\cdot) = 1$). Then u is the solution of the following PDE:

$$\frac{\partial u}{\partial t} = \Delta_{\tilde{g}}(\log u) - R_{\tilde{g}} + 4\pi\chi(M)u, \tag{3}$$

where $\Delta_{\tilde{g}}$ and $R_{\tilde{g}}$ are the Laplace-Beltrami operator and the scalar curvature with respect to $\tilde{g}$, and $\chi(M)$ is the Euler characteristic of M. From now on all geometric and analytic quantities on M are taken with respect to the reference metric $\tilde{g}$.

We now present a stochastic particle approximation for equation (3) and hence the Ricci flow. For this purpose it is essential that the metric $\tilde{g}$ has non-positive curvature.

Assumption 1. $R_{\tilde{g}} \leq 0$ *everywhere on* M.

The particle system lives on a set S_k of k points of M which is supposed to be approximately uniform in M in the sense that the discrete measure $\mu_k := \frac{1}{k}\sum_{x\in S_k}\delta_x$ converges weakly to the Riemannian volume measure on M as $k \to \infty$. It is well known that this weak convergence is equivalent to the property that

$$D_k := \sup_{f\in BL_1}\left|\frac{1}{k}\sum_{x\in S_k} f(x) - \int_M f(y)dy\right| \to 0 \qquad \text{as } k\to\infty, \tag{4}$$

where BL_1 is the set of all Lipschitz continuous functions on M with $\|f\|_{BL} := |f|_{lip} + \|f\|_\infty \leq 1$.

Remark 4. *One possible choice is* $S_k := \{\xi_1, \ldots, \xi_k\}$ *with independent and uniformly (with respect to the Riemannian volume measure) distributed* M*-valued random variables* $\xi_1, \ldots, \xi_k$. *In this case the weak convergence of* μ_k *to the Riemannian volume measure and hence* (4) *holds almost surely (Glivenko-Cantelli theorem).*

The particle system consists of N particles moving as follows in discrete time steps of length $\tau > 0$ on S_k: Suppose that at time $n\tau$ (i.e. after n time steps) there are exactly $m \geq 1$ particles at a site $x \in S_k$. Then each of these particles, independently of the past and of the behaviour of all other particles, behaves as follows: With probability

$$\pi(x,y) := \frac{\tau}{k}\left[\frac{p_r(x,y)}{r}\varphi(km/N) - R(y)\right] \tag{5}$$

it jumps to a point $y \neq x$, otherwise it stays at x. Here $r > 0$ is a parameter and $p_r(x,y)$ is the heat kernel on M, i.e. the fundamental solution of the heat equation $\frac{\partial p}{\partial t} = \Delta p$. Moreover

$$\varphi(u) := \begin{cases} \frac{\log u - \log u_{min}}{u} & \text{if } u \geq u_{min} \\ 0 & \text{if } u < u_{min}, \end{cases}$$

where $u_{min} := \inf\{u(t,x) \mid t \geq 0, x \in M\}$. ($u_{min}$ is strictly positive as a consequence of Proposition 1.) Note that $\pi(x,y)$ is always nonnegative because of our assumption $R \leq 0$. If τ is small enough, then $\sum_{y\neq x}\pi(x,y) \leq 1$, so that the jump rule (5) is well-defined.

In order to study the macroscopic behaviour of the particle system we consider the empirical measure defined as

$$\overline{X}^N_{\tau,r,k}(n\tau, x) := \frac{1}{N}\sum_{i=1}^{N} 1_{X^i_n}(x),$$

where X^i_n denotes the position of the i-th particle after n time steps. We extend this definition from the discrete set $\{n\tau | n \in \mathbb{N}_0\}$ to arbitrary values of $t \in \mathbb{R}_+$ by piecewise linear interpolation. Then we have the following result (Theorem 1 in [3]):

Theorem 5. *There are constants $A_1, A_2, A_3 < \infty$ (depending on $(M, \tilde{g})$, R and u_0) such that for all $T \geq 0$, all $N, k \in \mathbb{N}$, all $r \in (0, 1]$ and all $\tau > 0$:*

$$\begin{aligned} & E\left[\sup_{0\leq t\leq T}\sup_{f\in BL_1}\left|\int_M f(y)u(t,y)dy - \sum_{x\in S_k} f(x)\overline{X}^N_{\tau,r,k}(t,x)\right|\right] \\ \leq\ & E\left[\frac{1}{k}\sum_{x\in S_k}|u_0(x) - k\overline{X}_0(x)|\right] + T\left[A_1\tau + A_2 r + A_3 r^{-5/2}D_k + \frac{k}{2\tau\sqrt{N}}\right]. \end{aligned}$$

This means that the empirical measure of the particle system converges locally uniformly in time to the measure with density u as N and k tend to ∞ and τ and r tend to 0, provided that the initial configuration is chosen appropriately and the constraints $r^{-5/2}D_k \to 0$ and $\frac{k}{\tau\sqrt{N}} \to 0$ are respected.

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Ageing in the parabolic Anderson model

Marcel Ortgiese

(joint work with Peter Mörters and Nadia Sidorova)

Ageing is a phenomenon that can be observed in many physical systems that are out of equilibrium. Vaguely speaking, if starting at time t we observe for how long the system stays in the same state, then if the answer depends on t, the system ages. Mathematical treatments have focused mainly on the dynamical properties of spin glasses and related trap models, see [1] for an overview. Also in [2], the occurrence of ageing is investigated for different models of interacting diffusion processes. In our project we show that ageing occurs in the parabolic Anderson model.

The parabolic Anderson model is given by the heat equation on the lattice $\mathbb{Z}^d$ with a random potential, i.e. we consider the solution $u : [0,\infty) \times \mathbb{Z}^d \to [0,\infty)$ of the Cauchy problem

$$\begin{aligned} \tfrac{\partial}{\partial t}u(t,z) &= \Delta u(t,z) + \xi(z)u(t,z)\,, \qquad (t,z) \in (0,\infty) \times \mathbb{Z}^d\,, \\ u(0,z) &= \mathbf{1}_0(z)\,, \qquad z \in \mathbb{Z}^d\,. \end{aligned}$$

Here Δ is the discrete Laplacian

$$\Delta f(x) = \sum_{y \in \mathbb{Z}^d : y \sim x} (f(y) - f(x))\,,$$

where $y \sim x$ means that y is a nearest-neighbour of site x. The potential $\xi = (\xi(z) : z \in \mathbb{Z}^d)$ is a collection of independent, identically distributed random variables, which we assume to be Pareto-distributed for some $\alpha > d$, i.e.

$$\mathrm{Prob}\{\xi(z) \le x\} = 1 - x^{-\alpha}\,, \quad \text{for } x \ge 1\,.$$

As our main result we show that the ageing phenomenon occurs in this model in the following form.

Theorem 1. *Let $0 < \varepsilon < \frac{1}{2}$. For any $c > 0$,*

$$\lim_{t\to\infty} \mathrm{Prob}\Big\{ \sup_{z\in\mathbb{Z}^d} \sup_{t\le s\le t(1+c)} \Big| \frac{u(t,z)}{\sum_{x\in\mathbb{Z}^d} u(t,x)} - \frac{u(s,z)}{\sum_{x\in\mathbb{Z}^d} u(s,x)} \Big| < \varepsilon \Big\} = I(c)\,,$$

for a constant $I(c) \in (0,1)$ given explicitly by

$$I(c) = \frac{1}{B(\alpha - d + 1, d)} \int_0^1 v^{\alpha-d}(1-v)^{d-1}\phi_c(v)\mathrm{d}v\,,$$

where the weight $\phi_c(v)$ is defined as

$$\phi_c(v) = \Big(1 - \tilde{B}(v, \alpha-d, d) + (1+c)^\alpha (\frac{c}{v} + 1)^{d-\alpha} \tilde{B}\Big(\frac{v+c}{1+c}, \alpha - d, d\Big)\Big)^{-1}\,.$$

Here B is the Beta function and $\tilde{B}$ the incomplete Beta function

$$\tilde{B}(x,a,b) = \frac{1}{B(a,b)} \int_0^x v^{a-1}(1-v)^{b-1}\mathrm{d}t\,.$$

Note that as one would expect, if $c \to \infty$, then $I(c) \to 0$ and if $c \to 0$, then $I(c) \to 1$.

Our proofs rely on the techniques developed in [4], where the authors show that at most times the solution u is localized in one point. After a certain time another point becomes more attractive and the solution relocalizes. Now, if $Z_t = \mathrm{argmax}\{u(t,z) : z \in \mathbb{Z}^d\}$ denotes the point where most of the mass is localized, then it is shown that as $t \to \infty$

$$\frac{u(t,Z_t)}{U(t)} \to 1 \quad \text{in probability, and} \quad \frac{u(t,Z_t) + u(t,Z_t^{(2)})}{U(t)} \to 1 \quad \text{almost surely}\,,$$

where $Z_t^{(2)}$ is an auxiliary process, which is only important for the relative short time intervals when u relocalizes from one point to another. Using the point process technique developed in [3], we can show that for any $c > 0$,

$$\text{Prob}\{Z_t = Z_{t(1+c)}\} \to I(c) \quad \text{as } t \to \infty\,,$$

from which we can then deduce Theorem ((1)).

If we define the "remaining lifetime" function R as

$$R(t) = \sup\{s \geq 0 \,:\, Z_t = Z_{t+s}\}\,,$$

then we have shown that $R(t)/t$ converges weakly to a random variable with law given by $1 - I$. In addition to these results about the weak convergence, we can also describe the almost sure asymptotics of the function R. Let $(\tau_n)_{n\geq 0}$ denote the jump times of the process $(Z_t)_{t\geq t_0}$, where t_0 can be chosen such that $(Z_t)_{t\geq t_0}$ never returns to the same point in $\mathbb{Z}^d$. Then the function $(R(t), t \geq t_0)$ is of a very simple structure, see also Figure 1. At time τ_n, R jumps from zero to $\tau_{n+1} - \tau_n$, then it decreases linearly with slope -1 during the interval $[\tau_n, \tau_{n+1})$. Therefore, in order to describe the almost sure asymptotics of R it suffices to describe the asymptotics of $\{\tau_n\}$.

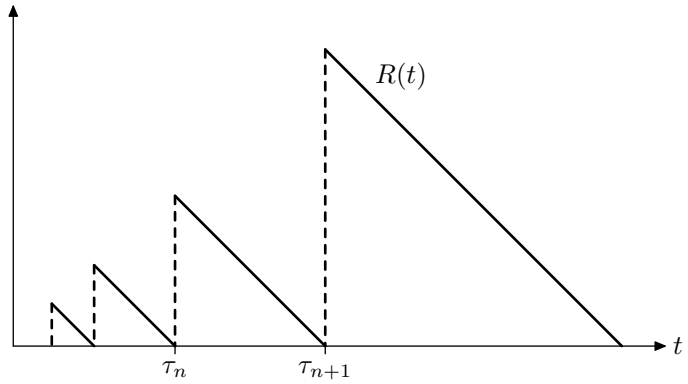

FIGURE 1. A schematic representation of the remaining lifetime function R.

Theorem 2. *For all n sufficiently large, for $\beta, \beta' \gg 1$,*

$$(\log \tau_n)^{-\beta} \leq \frac{R(\tau_n)}{\tau_n} \leq (\log \tau_n)^{\beta'}\,.$$

As it is known that one sees qualitatively different localization behaviour depending on the tail of the distribution of the potential, it is a natural question to ask at what point the ageing behaviour in the form of Theorem 1 changes qualitatively as the tails of the potential get lighter. Since our proofs rely on the precise estimates developed in [4] that only work for polynomial tails, further research that might reveal a more complete picture requires the development of more general techniques.

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Reporter: Nadia Sidorova

Participants

Dr. Stefan Adams
Mathematics Institute
University of Warwick
Zeeman Building
GB-Coventry CV4 7AL

Dr. Volker Betz
Mathematics Institute
University of Warwick
Gibbet Hill Road
GB-Coventry CV4 7AL

Prof. Dr. Erwin Bolthausen
Institut für Mathematik
Universität Zürich
Winterthurerstr. 190
CH-8057 Zürich

Prof. Dr. Anton Bovier
Institut für Angewandte Mathematik
Universität Bonn
Wegelerstr. 6
53115 Bonn

Dr. Andrea Collevecchio
Dipartimento di Matematica
Universita degli Studi di Venezia
I-30123 Venezia

Prof. Dr. Amir Dembo
Department of Mathematics
Stanford University
Stanford , CA 94305
USA

Prof. Dr. Jean Dominique Deuschel
Institut für Mathematik
Sekr. MA 7-4
Technische Universität Berlin
Straße des 17. Juni 136
10623 Berlin

Dr. Nicolas Dirr
Department of Mathematical Sciences
University of Bath
Claverton Down
GB-Bath BA2 7AY

Dr. Patrick W. Dondl
Max-Planck-Institut für Mathematik
in den Naturwissenschaften
Inselstr. 22 - 26
04103 Leipzig

Prof. Dr. Laszlo Erdös
Mathematisches Institut
Ludwig-Maximilians-Universität
München
Theresienstr. 39
80333 München

Prof. Dr. Ryoki Fukushima
Department of Mathematics
Faculty of Science
Kyoto University
Kyoto 606-8502
JAPAN

Prof. Dr. Tadahisa Funaki
Graduate School of
Mathematical Sciences
University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo 153-8914
JAPAN

Prof. Dr. Jürgen Gärtner
Institut für Mathematik
Fakultät II; Sekr. MA 7-5
Technische Universität Berlin
Straße des 17. Juni 136
10623 Berlin

Prof. Dr. Hans-Otto Georgii
Mathematisches Institut
Ludwig-Maximilians-Universität
München
Theresienstr. 39
80333 München

Dr. Giambattista Giacomin
U.F.R. de Mathematiques
Case 7012
Universite de Paris 7
2, Place Jussieu
F-75251 Paris Cedex 05

Dr. Johannes Giannoulis
Zentrum für Mathematik
Technische Universität München
Boltzmannstr. 3
85748 Garching bei München

Prof. Dr. Dirk Hundertmark
Dept. of Mathematics, University of
Illinois at Urbana-Champaign
273 Altgeld Hall MC-382
1409 West Green Street
Urbana , IL 61801-2975
USA

Prof. Dr. Palle E.T. Jorgensen
Dept. of Mathematics
University of Iowa
Iowa City , IA 52242-1466
USA

Prof. Dr. Wolfgang D. König
Mathematisches Institut
Universität Leipzig
Postfach 100920
04009 Leipzig

Hubert Lacoin
Laboratoire de Probabilites et
Modeles Aleatoires
Universite Paris VII
175 rue du Chevaleret
F-75013 Paris Cedex

Dr. Frederic Legoll
LAMI-ENPC
6 et 8 Avenue Blaise Pascal
Cite Descartes - Champs sur Marne
F-77455 Marne la Vallee Cedex 2

Prof. Dr. Stephan Luckhaus
Mathematisches Institut
Universität Leipzig
Johannisgasse 26
04103 Leipzig

Prof. Dr. Peter Mörters
Department of Mathematical Sciences
University of Bath
Claverton Down
GB-Bath BA2 7AY

Prof. Dr. Barbara Niethammer
Mathematical Institute
Oxford University
24-29 St. Giles
GB-Oxford OX1 3LB

Dr. Takao Nishikawa
Department of Mathematics
College of Science and Technology
Nihon University
1-8-14 Kanda-Surugadai, Chiyoda-ku
Tokyo 101-8308
JAPAN

Prof. Dr. Alexei Novikov
Department of Mathematics
Pennsylvania State University
University Park , PA 16802
USA

Prof. Dr. Stefano Olla
CEREMADE
Universite Paris Dauphine
Place du Marechal de Lattre de
Tassigny
F-75775 Paris Cedex 16

Dr. Marcel Ortgiese
Department of Mathematical Sciences
University of Bath
Claverton Down
GB-Bath BA2 7AY

Dr. Erin Pearse
Dept. of Mathematics
University of Iowa
Iowa City , IA 52242-1466
USA

Dr. Robert Philipowski
Institut für Angewandte Mathematik
Universität Bonn
Wegelerstr. 6
53115 Bonn

Dr. Matthias Röger
Max-Planck-Institut für Mathematik
in den Naturwissenschaften
Inselstr. 22 - 26
04103 Leipzig

Prof. Dr. Manfred Salmhofer
Institut für theoretische Physik
Universität Heidelberg
Philosophenweg 19
69120 Heidelberg

Prof. Dr. Michael Scheutzow
Institut für Mathematik
Fakultät II; Sekr. MA 7-5
Technische Universität Berlin
Straße des 17. Juni 136
10623 Berlin

Tom Schmitz
Max-Planck-Institut für Mathematik
in den Naturwissenschaften
Inselstr. 22 - 26
04103 Leipzig

Dr. Nadia Sidorova
Department of Mathematics
University College London
Gower Street
GB-London WC1E 6BT

Prof. Dr. Karl-Theodor Sturm
Universität Bonn
Institut für Angewandte Mathematik
Poppelsdorfer Allee 82/1
53115 Bonn

Prof. Dr. Juan J. L. Velazquez
Institute Ciencias Matematicas (ICMAT)
Universidad Complutense
Plaza de Ciencias 3
E-28040 Madrid

PD Dr. Ivan Veselic
Fakultät für Mathematik
TU Chemnitz
Reichenhainer Str. 41
09126 Chemnitz

Hendrik Weber
Universität Bonn
Institut für Angewandte Mathematik
Poppelsdorfer Allee 82/1
53115 Bonn

Dr. Johannes Zimmer
Department of Mathematical Sciences
University of Bath
GB-Bath BA2 7AY