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Low Eigenvalues of Laplace and Schrödinger Operators

Organised by
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Abstract. This workshop brought together researchers interested in eigenvalue problems for Laplace and Schrödinger operators. The main topics of discussions and investigations covered Dirichlet and Neumann eigenvalue problems, inequalities for the spectral gap, isoperimertic problems and sharp Lieb–Thirring type inequalities. The focus included not only the analytic and geometric sides of the problems, but also related probabilistic and computational aspects.

Mathematics Subject Classification (2000): 35xx, 47xx, 81xx.

Introduction by the Organisers

The workshop *Low Eigenvalues of Laplace and Schrödinger Operators*, organised by Mark Ashbaugh, Columbia, Rafael Benguria, Santiago de Chile, Richard Laugesen, Urbana, and Timo Weidl, Stuttgart, took place at the MFO from February 8th until February 14th, 2009. The conference was attended by more than 25 international participants including experienced specialists as well as young researchers. During the workshop, 17 talks were delivered about new results concerning spectral theory and Low Eigenvalues of Laplace and Schrödinger Operators. Furthermore, open problems, recent developments and new strategies were presented and analysed in about five problem sessions and numerous vivid discussions. The most important topics of the talks as well as the discussions were

- Dirichlet and Neumann Eigenvalue Problems
- Isoperimetric Problems
- Gap Inequalities

- Geometrical Aspects
- Lieb–Thirring type Inequalities.

In the following we include the abstracts of the talks in chronological order. Since discussions of open problems have been a crucial part of the workshop we proceed with an extended abstract summarising the problems raised and investigated during the conference. This part of the report contains contributions from all paritcipants.

Organisers and participants are grateful to *Mathematisches Forschungsinstitut Oberwolfach* for facilitating this fruitful conference.

Workshop: Low Eigenvalues of Laplace and Schrödinger Operators

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Abstracts

Pólya's conjecture in the presence of a constant magnetic field

Rupert L. Frank
(joint work with Michael Loss, Timo Weidl)

Let $\Omega \subset \mathbb{R}^2$ be a domain of finite measure and let λ_n^Ω be the n-th eigenvalue (counting multiplicities) of the Dirichlet Laplacian on Ω. In the famous paper [7] Pólya has shown that if Ω is tiling, then $\lambda_n^\Omega \geq 4\pi|\Omega|^{-1}n$. This lower bound is sharp, since by Weyl's theorem $\lim_{n\to\infty} n^{-1}\lambda_n^\Omega = 4\pi|\Omega|^{-1}$. Pólya conjectured that his bound should be valid without the tiling assumption, but this is unproven. The best known lower bound for general domains, $\lambda_n^\Omega \geq 2\pi|\Omega|^{-1}n$, is due to Li and Yau [6] (see also [1]) and is off by a factor 1/2.

In this talk we discuss the analogue of Pólya's estimate in the presence of a constant magnetic field of strength $B > 0$. That is, we seek a lower bound for the n-th eigenvalue $\lambda_n^\Omega(B)$ of the operator $H^\Omega(B) = (-i\nabla - B\mathbf{A})^2$ in $L_2(\Omega)$ with Dirichlet boundary conditions. Here $\mathbf{A}(x) = \frac{1}{2}(-x_2, x_1)^T$. Our main result (see [4] and Theorem 1 below) is that

$$\lambda_n^\Omega(B) \geq 2\pi|\Omega|^{-1}n\,, \tag{1}$$

and that for any given Ω, the constant 2π can not be increased (if the inequality should be valid for all $B > 0$). Hence the analogue of Pólya's conjecture is *not* true in the presence of a constant magnetic field.

In fact, we prove a more general result for eigenvalue moments $\operatorname{tr}\left(H^\Omega(B) - \lambda\right)_-^\gamma = \sum_n \left(\lambda - \lambda_n^\Omega(B)\right)_+^\gamma$ with $\gamma \geq 0$. In the case $\gamma = 0$ this denotes the number of eigenvalues less than λ, and hence leads to (1).

Theorem 1. *Let $\gamma \geq 0$ and let $\Omega \subset \mathbb{R}^2$ be a domain of finite measure. Then*

$$\operatorname{tr}\left(H^\Omega(B) - \lambda\right)_-^\gamma \leq \rho_\gamma \, \frac{|\Omega| \; \lambda^{\gamma+1}}{4\pi(\gamma+1)} \qquad \text{for all } \lambda > 0 \text{ and } B > 0 \tag{2}$$

with the constant

$$\rho_\gamma = \begin{cases} 2 & \text{if } \gamma = 0\,, \\ 2\left(\gamma/(\gamma+1)\right)^\gamma & \text{if } 0 < \gamma < 1\,, \\ 1 & \text{if } \gamma \geq 1\,. \end{cases}$$

The constant ρ_γ is sharp in the following sense.

(a) *For any $0 \leq \gamma < 1$, any bounded domain $\Omega \subset \mathbb{R}^2$ and any $\epsilon > 0$ there exist $B > 0$ and $\lambda > 0$ such that*

$$\operatorname{tr}\left(H^\Omega(B) - \lambda\right)_-^\gamma \geq (1-\epsilon)\; \rho_\gamma \; \frac{|\Omega| \; \lambda^{\gamma+1}}{4\pi(\gamma+1)}\,.$$

(b) *For any* $\gamma \geq 1$, *any domain* $\Omega \subset \mathbb{R}^2$ *of finite measure, any* $B > 0$ *and any* $\epsilon > 0$ *there exists a* $\lambda > 0$ *such that*

$$\operatorname{tr}\left(H^\Omega(B) - \lambda\right)_-^\gamma \geq (1 - \epsilon)\, \frac{|\Omega|\ \lambda^{\gamma+1}}{4\pi(\gamma + 1)}\,.$$

The result for $\gamma \geq 1$ is (essentially) due to Erdős, Loss and Vougalter [2], while the inequality for $0 \leq \gamma < 1$ appeared in [4] (see also [3] for the sharpness assertion (a) as stated). Both papers contain results in dimensions $d \geq 3$ as well.

It is interesting to note that the sharpness of the constants in Theorem 1 has different origins for $\gamma \geq 1$ and for $0 \leq \gamma < 1$. For $\gamma \geq 1$ it is attained in the limit $\lambda \to \infty$ with B arbitrary but fixed, while for $0 \leq \gamma < 1$ it is attained in the limit $\lambda \to \infty$ and $B \to \infty$ with λ/B fixed at a certain, γ-dependent value.

In the context of semi-classical spectral estimates, Theorem 1 is the first example where the presence of a magnetic field has an influence on the constant in the inequality. Recall that both the sharp Lieb-Thirring inequality for $\gamma \geq 3/2$ and the sharp Berezin-Li-Yau inequality for $\gamma \geq 1$ remain valid with the same constant when an arbitrary, respectively a homogeneous magnetic field is added [5, 2]. This remarkable phenomenon can *not* be explained with the diamagnetic inequality, which only implies that exponential sums of the eigenvalues decrease as a magnetic field is added. While the lowest eigenvalue certainly goes up, higher eigenvalues may both increase or decrease as a magnetic field is added. Our main result (1) gives a sharp bound on the paramagnetic lowering of eigenvalues and states that the eigenvalues cannot decrease further than to half of the value that Pólya predicted. The problem of quantifying this 'failure of diamagnetism' in an abstract, operator-theoretic framework was addressed in [8, 3].

As a consequence of our result we see, in particular, that any attempt to prove Pólya's conjecture with a method which extends to constant magnetic fields must necessarily fail.

In [4] we also have shown that $\operatorname{tr}\left(H^\Omega(B) - \lambda\right)_-^\gamma$ can be estimated in terms of the density of states of the Landau Hamiltonian in $\mathbb{R}^2$. In contrast to (2) the right hand side in this estimate depends on B. More precisely, one has

Theorem 2. *Let* $\gamma \geq 0$ *and let* $\Omega \subset \mathbb{R}^2$ *be a domain of finite measure which is tiling. Then*

$$\operatorname{tr}\left(H^\Omega(B) - \lambda\right)_-^\gamma \leq |\Omega|\, \frac{B}{2\pi} \sum_{k=1}^{\infty} \left(B(2k-1) - \lambda\right)_-^\gamma \quad \text{for all } \lambda > 0 \text{ and } B > 0\,. \tag{3}$$

If $\gamma \geq 1$, *then this is true without the assumption that* Ω *is tiling.*

Inequality (3) for $\gamma \geq 1$ is *stronger* than (2). This follows from the mean value property of convex functions.

We conclude by remarking that an analogue of (3), with the inequality sign reversed, holds if Dirichlet boundary conditions are replaced by Neumann boundary conditions; see [3].

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Two-dimensional Berezin-Li-Yau inequalities with a correction term

Hynek Kovařík

(joint work with Semjon Vugalter and Timo Weidl)

1. Introduction

Let Ω be an open bounded set in $\mathbb{R}^d$ and let $-\Delta$ be the Dirichlet Laplacian on Ω. We denote by λ_j the non-decreasing sequence of eigenvalues of $-\Delta$. The main object of our interest in this paper is the lower bound

$$\sum_{j=1}^{k} \lambda_j \geq \frac{d\, C_d}{d+2}\, V^{-\frac{2}{d}}\, k^{\frac{d+2}{d}}, \qquad C_d = (2\pi)^2 \omega_d^{-2/d}, \tag{1}$$

where V stands for the volume of Ω and ω_d denotes the volume of the unit ball in $\mathbb{R}^d$. Inequality (1) was proved in [10], and is commonly known as the Li-Yau inequality. In [9] it was pointed out that (1) is in fact the Legendre transformation of an earlier result by Berezin, see [1]. Note also that the Li-Yau inequality yields an individual lower bound on λ_k in the form

$$\lambda_k \geq \frac{d\, C_d}{d+2}\, V^{-\frac{2}{d}}\, k^{\frac{2}{d}}\,. \tag{2}$$

For further estimates on λ_k see [14, 7, 8, 9].

It is important to compare the lower bound (1) with the asymptotical behavior of the sum on the left-hand side, which reads as follows:

$$\sum_{j=1}^{k} \lambda_j \;=\; \frac{d\, C_d}{d+2}\, V^{-\frac{2}{d}}\, k^{\frac{d+2}{d}} + \tilde{C}_d\, \frac{|\partial\Omega|}{V^{1+\frac{1}{d}}}\, k^{1+\frac{1}{d}} \;+ o\left(k^{1+\frac{1}{d}}\right) \quad \text{as } k \to \infty \tag{3}$$

with

$$\tilde{C}_d = \frac{\sqrt{\pi}\,\Gamma\left(2+\frac{d}{2}\right)^{1+\frac{1}{d}}}{(d+1)\,\Gamma\left(\frac{3}{2}+\frac{d}{2}\right)\Gamma(2)^{\frac{1}{d}}}\,.$$

The first term in the asymptotics (3) is due to Weyl, see [17]. The second term in (3) was established, under suitable conditions on Ω, in [4, 5, 13], see also [15, Chap. 1.6].

It follows from (3) that the constant in (1) cannot be improved. On the other hand, since the second asymptotical term is positive, it is natural to ask whether one might improve (1) by adding an additional positive term of a lower order in k to the right-hand side. The first step towards this goal was done by Melas, [12], who showed that the inequality

$$\sum_{j=1}^{k} \lambda_j \geq \frac{d\,C_d}{d+2}\, V^{-\frac{2}{d}}\, k^{\frac{d+2}{d}} + M_d\, \frac{V}{I}\, k, \qquad I = \min_{a\in\mathbb{R}^2} \int_\Omega |x-a|^2\, dx \tag{4}$$

holds true with a factor M_d which depends only on the dimension. However, the additional term in the Melas bound is not of the order $k^{1+1/d}$ predicted by the second term in (3). Moreover, the coefficient of the second term in (3) reflects explicitly the effect of the boundary of Ω, whereas such a dependence is not seen in the coefficient V/I of (4). Our aim is to improve (1) and (4) by adding a positive contribution which reflects the nature of the second term in the asymptotic (3).

To keep the presentation as short and stringed as possible, we have decided to restrict ourselves to the case $d = 2$. Let us finally mention that a similar effect of the boundary on the sum of the eigenvalues in the case of the discrete Laplace operator was already observed in [2], see also the later improvement in [3].

2. Main result

Following notation will be employed in the text. By $\Theta(\cdot) : \mathbb{R} \to \mathbb{R}$ we denote the Heaviside function defined by $\Theta(x) = 0$ if $x \leq 0$ and $\Theta(x) = 1$ if $x > 0$. Moreover, we impose the following condition of the domain Ω:

Assumption A. There exist C^2- smooth parts $\Gamma_j \subset \partial\Omega$ at the boundary of Ω. Let $j = 1, \dots, m$.

Our main result then reads as follows

Theorem 1. *Let Ω satisfy Assumption A. Then there exist natural numbers $k_1, \dots, k_m$ and constants c_1, c_2 (depending on the geometry of Γ_j), such that for any $k \in \mathbb{N}$ and any $\alpha \in [0,1]$ we have*

$$\sum_{j=1}^{k} \lambda_j \;\geq\; \frac{2\pi}{V}\, k^2 + \alpha\, c_2\, k^{\frac{3}{2}-\varepsilon(k)}\, V^{-3/2} \sum_{j=1}^{m} L(\Gamma_j)\,\Theta(k-k_j) + (1-\alpha)\,\frac{V}{32\,I}\, k, \tag{1}$$

where $L(\Gamma_j)$ denotes the length of Γ_j and

$$\varepsilon(k) = \frac{2}{\sqrt{\log_2(2\pi k/c_1)}} \tag{2}$$

2.1. Remarks.

Remark 1. The dependence of the constants k_j, c_1 and c_2 on the geometrical properties of Γ_j can be given explicitly, see [6] for details.

Remark 2. Note that the coefficient of the second term on the right hand side of (1) is very similar to the coefficient of the second term in the Weyl asymptotics (3). In particular, it reflects the expected effect of the boundary of Ω. On the other hand, this boundary term becomes visible only for k large enough. However, we would like to point out that the second term cannot be simply proportional to $\sum_j L(\Gamma_j)$. Indeed, one can make $\sum_j L(\Gamma_j)$ arbitrarily large by "folding" the boundary $\partial\Omega$ while keeping the eigenvalues λ_j with $j \leq k$ almost unchanged. This shows that the condition $k \geq k_j$ cannot be removed.

Remark 3. As for the constants in (1), notice that $\varepsilon(k) \ll 1$ for all k and that $\varepsilon(k) \to 0$ as $k \to \infty$. On the other hand, the values of k_j are in general very large. Nevertheless, the correction term on the right-hand side of (1) can be optimized according to the geometry of Ω by choosing the boundary segments Γ_j in an appropriate way.

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Universal Monotonicity for Eigenvalue Means

Joachim Stubbe

The fact that sharp phase space bounds for a Schrödinger operator can be derived from universal bounds on its eigenvalues has been discovered recently [1], [3], [6]. The goal of the present talk is to briefly review a few results starting from a joint work with Evans M. Harrell in 1997 [2] and to present a new proof of sharp Lieb-Thirring inequalities [6].

We consider the one-parameter family of Schrödinger operators

$$H(\alpha) = -\alpha\Delta + V(x), \quad \alpha > 0 \tag{1}$$

on $\Omega = \mathbb{R}^d$ or on a bounded domain $\Omega \subset \mathbb{R}^d$ with Dirichlet boundary conditions. The starting point of our analysis is the following trace formula for $H(\alpha)$ ([2],[6]):

Theorem 1. Suppose that $H(\alpha)$ given in (1) has a spectrum consisting of eigenvalues $E_k = E_k(\alpha)$ with associated eigenfunctions ϕ_k forming an orthonormal basis of the underlying Hilbert space $L^2(\mathbb{R}^d)$. Then for any function $f : \mathbb{R} \to \mathbb{R}$

$$d\sum_{E_j} f(E_j) + 2\alpha \sum_{E_j \neq E_k}\sum T_{jk}\frac{f(E_k) - f(E_j)}{E_k - E_j} = 0 \tag{2}$$

provided all sums are finite where

$$T_{jk} = T_{kj} = \left| \int_\Omega \phi_j \nabla \phi_k \, dx \right|^2 \tag{3}$$

denote the kinetic energy matrix elements.

The assumptions on the spectrum of $H(\alpha)$ can be relaxed to operators having also continuous spectrum [3] and recently we proved an algebraic projection operator version of (2) [4]. A standard choice is $f(E) = (z - E)_+^2$ and for any positive integer n and real z one gets ([2])

$$d\sum_{j=1}^n (z - E_j)^2 - 4\alpha(z - E_j)T_j = 4\alpha \sum_{j=1}^n \sum_{k=n+1}^n \frac{(z - E_j)(z - E_k)}{E_k - E_j} T_{jk} \tag{4}$$

with

$$T_j = \int_\Omega |\nabla \phi_j|^2 \, dx. \tag{5}$$

Example 1. For the Dirichlet Laplacian on a bounded domain one has $\alpha T_j = E_j$. The r.h.s of (4) is negative for $z \in [E_n, E_{n+1}]$ and by analyzing the quadratic function (in z) on the l.h.s. of (4) one easily obtains the following Weyl-sharp inequality ([2]):

$$\left(\frac{d+2}{d}\sum_{j=1}^{n} E_j\right)^2 - \frac{d+4}{d}\sum_{j=1}^{n} E_j^2 \geq 0. \tag{6}$$

In [1] Harrell and Hermi derived from (4) a differential inequality for the Riesz mean $R_2(z) = \sum(z - E_j)_+^2$:

$$\frac{d}{dz} z^{-2-d/2} R_2(z) \geq 0 \tag{7}$$

which implies the well known Berezin-Li-Yau bound for $R_2(z)$.

Example 2. Lieb-Thirring inequalities for $H(\alpha)$ are obtained from (4) using the fact that $T_j = \frac{d}{d\alpha} E_j(\alpha)$ by the Feynman-Hellmann theorem ([6]):

Theorem 2. Let $V(x)$ be a continuous function of compact support. For all $\sigma \geq 2$ the mapping

$$\alpha \mapsto \alpha^{\frac{d}{2}} \sum_{E_j(\alpha)<0} (-E_j(\alpha))^\sigma \tag{8}$$

is non increasing for all $\alpha > 0$. Consequently

$$\alpha^{\frac{d}{2}} \sum_{E_j(\alpha)<0} (-E_j(\alpha))^\sigma \leq L_{\sigma,d}^{cl} \int_{\mathbb{R}^d} (V_-(x))^{\sigma + \frac{d}{2}} \, dx \tag{9}$$

for all $\alpha > 0$ where

$$L_{\sigma,d}^{cl} = (4\pi)^{-\frac{d}{2}} \frac{\Gamma(\sigma+1)}{\Gamma(\sigma + \frac{d}{2} + 1)}. \tag{10}$$

The result also holds in the presence of magnetic fields or for matrix-valued Schrödinger operators.

Our result does not cover the full range of exponents $\geq \frac{3}{2}$ for which sharp Lieb-Thirring inequalities have been shown by Laptev and Weidl [5] via operator valued Schrödinger operators and an induction on the dimension. Indeed, explicit examples are known where the monotonicity property fails for all $\sigma < 2$. However, we expect that our method can be applied to situations where an induction on the dimension will not apply [4].

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Conformal Structure and the First Eigenvalue of the Laplacian

Nikolai Nadirashvili

Let (M,g) be a two-dimensional Riemannian manifold. In local coordinates (x_i, y_i), the metric writes $g = \sum g_{ij} dx_i dy_j$ and the Laplace-Beltrami operator has the form

$$\Delta_g = \frac{1}{|g|} \sum \frac{\partial}{\partial x_i}\Big(\sqrt{|g|} g^{ij} \frac{\partial}{\partial y^j}\Big)$$

where we have used the usual convention of repeated indexes and $g^{ij} = (g_{ij})^{-1}$, $|g| = \det(g_{ij})$.

We denote by $\lambda_1(g)$ the first non-zero eigenvalue of Δ_g and we have

$$\lambda_1(g) = \inf_{u \in E} R_{M,g}(u)$$

where $R_{M,g}(u)$ is the so-called Rayleigh quotient given by

$$R_{M,g}(u) = \frac{\int_M |\nabla u|^2 dA_g}{\int_M u^2 dA_g}$$

and the infimum is taken over the space

$$E = \left\{ u \in C^\infty(M), \quad \int_M u = 0 \right\}.$$

Due to the scaling property of the first eigenvalue under a metric change cg, it is natural to introduce a normalization for the metric and we denote by $\mathcal{A}(g)$ the set of all metrics on (M,g) satisfying $|A_g| = 1$. We then consider on M the class $[g]$ of metrics conformal to g in $\mathcal{A}(g)$, i.e.

$$[g] = \{ g' \in \mathcal{A}(g), \ \ g' \text{ conformal to } \ g \}.$$

We have the following definition of conformality.

Definition 1. *A metric g' is said to be conformal to g on the manifold M if there exists a map $\mu > 0$ from M into M such that*

$$g' = \mu g.$$

The following result was proved in the joint work of Y.Sire and the author.

Theorem 1. *Let (M, g) be as above. There exists a metric $\overline{g} \in [g]$ smooth outside a finite number of conical singularities such that*

(1) *The metric $\overline{g}$ extremalizes the first eigenvalue, i.e.*

$$\lambda_1(\overline{g}) = \sup_{g' \in [g]} \lambda_1(g').$$

(2) *If $U_1(\overline{g})$ is the eigenspace associated to $\lambda_1(\overline{g})$, there exists a family of eigenvectors $\{u_1, \cdots, u_\ell\} \subset U_1(\overline{g})$ such that the map*

$$\begin{cases} \phi : M \to \mathbb{S}^{\ell-1} \subset \mathbb{R}^\ell \\ x \to (u_1, \cdots, u_\ell) \end{cases}$$

is harmonic.

The theorem reduced to a careful analysis of a Schrödinger type operator. Indeed consider $g' \in [g]$, by conformal covariance, the equation $-\Delta_{g'} u = \lambda_1(g')u$ reduces to the following system

$$\begin{cases} -\Delta_g u = \lambda_1(g')\, \mu\, u, \text{ on } M \\ \int_M \mu\, dA_g = 1. \end{cases} \tag{1}$$

Maximization of Neumann and Steklov eigenvalues on planar domains

Iosif Polterovich

(joint work with Alexandre Girouard and Nikolai Nadirashvili)

Let Ω be a simply-connected bounded planar domain with Lipschitz boundary, and Δ be the Laplace operator. Consider the *Neumann* (1) and *Steklov* (2) eigenvalue problems on Ω:

$$-\Delta u = \mu u \text{ in } \Omega \text{ and } \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \tag{1}$$

$$\Delta u = 0 \text{ in } \Omega \text{ and } \frac{\partial u}{\partial n} = \sigma u \text{ on } \partial\Omega. \tag{2}$$

Both problems have discrete spectra (see [4, p. 7 and p. 113])

$$0 = \mu_0 < \mu_1(\Omega) \le \mu_2(\Omega) \le \mu_3(\Omega) \le \cdots \nearrow \infty,$$

$$0 = \sigma_0 < \sigma_1(\Omega) \le \sigma_2(\Omega) \le \sigma_3(\Omega) \le \cdots \nearrow \infty.$$

The eigenvalues $\mu_0 = 0$ and $\sigma_0 = 0$ are simple and correspond to constant eigenfunctions. The eigenvalues μ_k and σ_k satisfy the following variational characterizations:

$$\mu_k(\Omega) = \inf_{U_k} \sup_{0 \ne u \in U_k} \frac{\int_\Omega |\nabla u|^2\, dz}{\int_\Omega u^2\, dz}, \quad k = 1, 2, \ldots$$

$$\sigma_k(\Omega) = \inf_{E_k} \sup_{0 \ne u \in E_k} \frac{\int_\Omega |\nabla u|^2\, dz}{\int_{\partial\Omega} u^2\, ds}, \quad k = 1, 2, \ldots$$

The infima are taken over all k–dimensional subspaces U_k and E_k of the Sobolev space $H^1(\Omega)$ which are orthogonal to constants on Ω and $\partial\Omega$, respectively.

Both Neumann and Steklov eigenvalue problems describe the vibration of a free membrane. In the Neumann case the membrane is homogeneous, while in the Steklov case the whole mass of the membrane is uniformly distributed on $\partial\Omega$. In other words, the mass of the membrane Ω is equal to area(Ω) in the Neumann case, and to the perimeter $\mathrm{L}(\partial\Omega)$ in the Steklov case. We focus on the following

Question 1. How large can μ_k and σ_k be on a membrane of a given mass?

In 1954, this problem was solved by G. Szegő [9] for μ_1 (this result was later generalized by H. Weinberger [10] to arbitrary domains in any dimension) and by R. Weinstock [11] for σ_1. They showed that in both cases the maximum is attained if and only if the domain Ω is a disk.

In [1] we prove the following

Theorem 1. *(i) Let Ω be a simply-connected bounded planar domain with Lipschitz boundary. Then*

$$\mu_2(\Omega)\,\mathrm{area}(\Omega) < 2\,\mu_1(\mathbf{D})\,\pi \approx 6.78\pi.$$

(ii) There exists a family of simply-connected bounded Lipschitz domains $\Omega_\varepsilon \subset \mathbf{R}^2$, degenerating to the disjoint union of two identical disks as $\varepsilon \to 0+$, such that

$$\lim_{\varepsilon\to 0+} \mu_2(\Omega_\varepsilon)\,\mathrm{area}(\Omega_\varepsilon) = 2\,\mu_1(\mathbf{D})\,\pi.$$

Note that Theorem 1 implies the Pólya conjecture [8] for the second nonzero Neumann eigenvalue of a simply-connected planar domain:

$$\mu_2(\Omega)\,\mathrm{area}(\Omega) \le 8\pi.$$

The best previous upper bound on μ_2 was obtained in [7]:

$$\mu_2(\Omega)\,\mathrm{area}(\Omega) \le 16\pi.$$

In 1974, Hersch–Payne–Schiffer [5, p. 102] proved the following estimates for the products of Steklov eigenvalues:

$$\sigma_k(\Omega)\,\sigma_n(\Omega)\,\mathrm{L}(\partial\Omega)^2 \le \begin{cases} (k+n-1)^2\,\pi^2 & \text{if } k+n \text{ is odd,} \\ (k+n)^2\,\pi^2 & \text{if } k+n \text{ is even.} \end{cases}$$

In particular, for $n = k$ and $n = k+1$ we obtain

$$\sigma_k(\Omega)\,\mathrm{L}(\partial\Omega) \le 2\pi k, \quad k = 1, 2, \ldots,$$

$$\sigma_k(\Omega)\,\sigma_{k+1}(\Omega)\,\mathrm{L}(\partial\Omega)^2 \le 4\pi^2 k^2, \quad k = 1, 2, \ldots.$$

Note that for $n = k = 1$ one gets Weinstock's inequality. It is easy to check that the inequality is sharp for $k = 1$ and $n = 2$, with the equality attained on a disk. As was proved in [2], in fact a much stronger result holds:

Theorem 2. *There exists a family of simply-connected bounded Lipschitz domains $\Sigma_\varepsilon \subset \mathbb{R}^2$, degenerating to the disjoint union of k identical disks as $\varepsilon \to 0+$, such that*

$$\lim_{\varepsilon\to 0+} \sigma_k(\Sigma_\varepsilon)\,\mathrm{L}(\partial\Sigma_\varepsilon) = 2\pi k, \;\; k = 2, 3, \ldots$$

and

$$\lim_{\varepsilon\to 0+} \sigma_k(\Sigma_\varepsilon)\,\sigma_{k+1}(\Sigma_\varepsilon)\,\mathrm{L}(\partial\Sigma_\varepsilon)^2 = 4\pi^2 k^2, \;\; k = 2, 3, \ldots$$

In particular, the Hersch–Payne–Schiffer inequalities are sharp for all $n = k$ and $n = k + 1$, $k = 1, 2, \ldots$.

Let us note that one has to be careful in the choice of the family Σ_ε. In particular, joining the disks by vanishing thin channels (cf. [6]) works for the family Ω_ε in Theorem 1, but is not applicable in the Steklov case. As shown in [2], this construction leads to a "collapse" of the Steklov spectrum: $\lim_{\varepsilon\to 0+} \sigma_k(\Sigma_\varepsilon) = 0$ for all $k = 1, 2 \ldots$. One way to get around this difficulty is to pull the disks apart instead of joining them with channels.

Theorem 2 gives an almost complete answer to Question 1 for the Steklov eigenvalues σ_k. It remains to establish whether the Hersch–Payne–Schiffer inequalities are *strict* for all $n = k$, $k \geq 2$. We believe that this is true and prove it for $k = 2$ in [2].

Further discussion of maximization problems for Neumann and Steklov eigenvalues could be found in a recent survey [3].

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New isoperimetric inequalities for the Steklov problem

Antoine Henrot
(joint work with Gérard A. Philippin, Abdesselem Safoui)

In this talk, we are interested in the Steklov eigenvalue problem :

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial n} = pu & \text{on } \partial\Omega\,, \end{cases} \tag{1}$$

where Ω is a bounded Lipschitz open set in $\mathbb{R}^N$. The sequences of eigenvalues and corresponding eigenfunctions will be denoted $0 = p_0(\Omega) < p_1(\Omega) \le p_2(\Omega) \le \dots$ and $u_0(=\text{const.}), u_1, u_2, \dots$. If $\Omega = B_1$ is the unit ball, we have $p_1(\Omega) = p_2(\Omega) = \dots = p_N(\Omega) = 1$ and the associated eigenfunctions are the coordinates $u_k(X) = x_k$.

The classical isoperimetric inequality (analogous to Faber-Krahn inequality in the Dirichlet case or Szegö-Weinberger in the Neumann case) is due to Weinstock for $N = 2$ and Brock in the general case, see [1], [7], [3, section 7.3], and states:

$$p_1(\Omega) \le p_1(\Omega^*) \tag{2}$$

where Ω^* denotes the ball of same volume as Ω. Actually, Brock proves a stronger result, namely

$$\sum_{k=1}^{N} \frac{1}{p_k(\Omega)} \ge \sum_{k=1}^{N} \frac{1}{p_k(\Omega^*)}\,. \tag{3}$$

The proof relies on the classical Poincaré principle which is not so known, so it worth writing it here. Let $v_k(\not\equiv 0) \in H^1(\Omega),\ k = 1, ..., N$ be N linearly independent functions satisfying the conditions $\int_{\partial\Omega} v_k\, ds = 0$. Let us introduce the matrices $A := (a_{ij}), B := (b_{ij})$ defined by:

$$a_{ij} := \int_\Omega \nabla v_i \nabla v_j\, dx \quad \text{and} \quad b_{ij} := \int_{\partial\Omega} v_i v_j\, ds$$

and let us denote by $0 \le p'_1 \le p'_2 \le ... \le p'_N$ the N roots of the characteristic equation $\det|A - pB| = 0$ then Poincaré's variational principle (see e.g. [6]) asserts that

$$p_k(\Omega) \le p'_k\,, \quad k = 1, ..., N\,.$$

By means of a translation followed by an appropriate rotation, we can assume that $\int_{\partial\Omega} x_k\, ds = 0$ and $\int_{\partial\Omega} x_k x_j\, ds = 0$ for $k \ne j$.

The N functions defined as $v_k := x_k (I_k(\Omega))^{-1/2}$ (where $I_k(\Omega)$ is defined in (5) below) are admissible for the Poincaré principle and since the matrices A and B are diagonal here, it follows immediately that, for any domain Ω

$$p_k(\Omega) \le \frac{|\Omega|}{\int_{\partial\Omega} x_k^2\, ds} \tag{4}$$

with equality when Ω is a ball.

Therefore, we see that to obtain isoperimetric inequalities involving Steklov eigenvalues, we must look for similar inequalities for the moments of inertia

$\int_{\partial\Omega} x_k^2\, ds$. Let us introduce the two family of moments of inertia of the boundary and of the body itself:

$$I_k(\Omega) := \int_{\partial\Omega} {x_k}^2\, ds \quad \text{and} \quad J_k(\Omega) := \int_{\Omega} {x_k}^2\, dx\,. \tag{5}$$

Using a Theorem due to H. Knothe, see [5] (which is itself a consequence of the Prekopa-Leindler inequality), we prove in [4] the following inequality for **convex** domains:

$$I_k(\Omega)^{N+2} \geq (N+2)^{N+1}\omega_N J_k(\Omega)^{N+1} \tag{6}$$

where ω_N is the volume of the unit ball (with equality when Ω is ball). It allows us to generalize the Hersch-Payne-Schiffer inequality proved in [2] for $N = 2$ (and for general domains):

Theorem(Henrot-Philippin-Safoui 2008) For any convex domain Ω:

$$\prod_{k=1}^{N} p_k(\Omega) \leq \prod_{k=1}^{N} p_k(\Omega^*)\,. \tag{7}$$

Using the same technique, we can also generalize Brock's result (3) with any exponent $q > 1$:

$$\sum_{k=1}^{N} \frac{1}{p_k^q(\Omega)} \geq \sum_{k=1}^{N} \frac{1}{p_k^q(\Omega^*)}\,.$$

or prove similar isoperimetric inequalities involving symmetric expressions of the eigenvalues:

$$\sigma_j(p_1^{-1}(\Omega), p_2^{-1}(\Omega), \ldots, p_N^{-1}(\Omega)) \leq \sigma_j(p_1^{-1}(\Omega^*), p_2^{-1}(\Omega^*), \ldots, p_N^{-1}(\Omega^*))$$

where σ_j is the j-th elementary symmetric function (σ_1 is the sum and we get (3), σ_N is the product and we get (7)).

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Isoperimetric bounds for Neumann eigenvalues of triangles

Bartłomiej Siudeja
(joint work with Richard Laugesen)

Inequalities for eigenvalues of the Laplacian involving geometric quantities have been of interest for many years. Surprisingly, only a few of such inequalities have been obtained for Neumann boundary condition. The goal of the talk is to prove new bounds for Neumann eigenvalues of triangles.

The preprint on which the talk is based seems the first to study sharp isoperimetric inequalities for Neumann triangle eigenvalues. The Dirichlet case, on the other hand, has received considerable attention (see e. g. [2, 3, 5] and references therin).

One of the few known bounds for general domains was obtained by Szegő [6]

$$\mu_1 A \text{ is maximal precisely for disks,}$$

where μ_1 is the first nonzero eigenvalue of the Laplacian with Neumann boundary condition on a simply connected planar domain of area A. The analogous result in higher dimensions is due to Weinberger [7].

We prove a sharper result for triangular domains:

$$\mu_1 A \text{ is maximal when the triangle is equilateral.}$$

It is plausible that for polygons with four or more sides the maximizing polygon is regular. Such a result would complement Polýa-Szegő conjecture stating that the minimizer is regular in the Dirichlet case.

We also show three different stronger results. The first includes a stronger geometric functional, the second includes a stronger eigenvalue functional, and the last one is a trade-off between the two.

To strengthen the geometric functional, we write L for the perimeter and prove

$$\mu_1 L^2 \text{ is maximal for the equilateral triangle,}$$

which implies the result for $\mu_1 A$ above by invoking the triangular isoperimetric inequality. Even stronger result can be obtained using an isoperimetric excess obtained from the triangular isoperimetric inequality. This result provides two maximizers. In addition to equilateral triangles we get asymptotic equality for degenerate acute isosceles triangles.

Strengthening instead the eigenvalue functional, we show

$$\left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right)^{-1} A \quad \text{is maximal in the equilateral case.}$$

Note that disks are maximizers for general domains as has been proved by Szegő [6].

In order to further strengthen the eigenvalue functional we need to weaken the geometric functional. We show the arithmetic mean $(\mu_1 + \mu_2)/2$ of the first two non-zero eigenvalues is maximal for the equilateral triangle, with the scaling factor equal to the ratio of the square of the area over the sum of the squares of the side lengths.

In addition to such upper bounds, we prove a lower bound for μ_1 with diameter scaling. This bound saturates for the degenerate acute isosceles triangle. Thus we obtain a triangular sharpening of Payne and Weinberger's lower bound for arbitrary convex domains, which saturates for the degenerate rectangle [4].

We also show that the upper bound on $\mu_1 D^2$ for arbitrary convex domains due to Cheng [1] is asymptotically correct for degenerate obtuse isosceles triangles. Thus Cheng's result is best possible even in the restricted class of triangular domains.

Three kinds of trial function are employed to prove the upper bounds in this paper: for close-to-equilateral triangles we linearly transform the eigenfunctions of the equilateral triangle, for close-to-degenerate triangles we deform the eigenfunction of a circular sector, and for all other triangles we make do with linear or quadratic trial functions.

The lower bound on μ_1 under diameter normalization is proved by bisecting and stretching the triangle repeatedly, in order to approach the degenerate case. Payne and Weinberger's method of thinly slicing an arbitrary domain does not apply here, since the resulting slices would not be triangular. Our procedure relies on the proof of symmetry/antisymmetry of the first Neumann eigenfunction for isosceles triangles, which may be of independent interest.

In addition to their intrinsic interest, our results on triangles suggest new open problems for general domains, such as a possible strengthening of the Szegő–Weinberger upper bound by an isoperimetric excess term.

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Shape analysis of eigenvalues

Dorin Bucur

The generic problem we consider can be formally written as

$$\min_{|\Omega|=m} F(\lambda_1(\Omega), ..., \lambda_k(\Omega))$$

where $\Omega \subseteq \mathbb{R}^N$ is an open set and $|\Omega|$ is the Lebesgue measure. By

$$\lambda_1(\Omega) \leq \lambda_2(\Omega) \leq ... \leq \lambda_k(\Omega)$$

we denote the first k eigenvalues of the Laplacian with some boundary conditions that will be specified. Multiplicity is counted.

The questions we deal with are:

- prove the existence of a solution, i.e. an open set Ω;
- find qualitative properties of Ω: regularity, symmetry, convexity,...;
- perform numerical computations.

Here are some of the results that were presented in this talk.

Result 1. This is a joint work with P. Freitas. We prove in a purely variational way the Faber-Krahn inequality for the first eigenvalue of the Dirichlet-Laplacian. Precisely, we prove that the following problem

$$\min_{|\Omega|=m} \lambda_1(\Omega)$$

has the ball as unique solution.The method consists precisely in the following steps:

- prove the existence of a minmizing set;
- use a cut and reflect method in order to prove that is symmetric in all directions;
- deduce it is a union of annuli;
- select one annulus and use 1D analysis in order to prove that the ball is the minimizer.

Result 2. This is a joint work with A. Giacomini. We consider $\beta > 0$. An eigenvalue ν of the Laplacian with Robin boundary conditions on some *admissible* open set Ω, satisfies forammly

$$\begin{cases} -\Delta u = & \nu u \text{ in } \Omega \\ \frac{\partial u}{\partial n} + \beta u = & 0 \;\; \partial\Omega \end{cases}$$

The first one is given by the Rayleigh quotient

$$\nu_1(\Omega) = \min_{u \in H^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx + \beta \int_{\partial\Omega} |u|^2 d\mathcal{H}^{N-1}}{\int_\Omega u^2 dx}$$

Bossel (in $\mathbb{R}^2$, 1986) and Daners (in $\mathbb{R}^N$, 2006) proved that the ball minimizes ν_1 among Lipschitz sets of prescirbed measures.

In a joint work with A. Giacomini [2], we prove that a relaxed form of the Robin problem still has the ball as minimizer. This situation covers cracked domains, or more general non-smooth domains. Precisely

$$\nu_1(B) \leq \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2 dx + \beta \int_{J_u} ((u^+)^2 + (u^-)^2) d\mathcal{H}^{N-1}}{\int_{\mathbb{R}^N} u^2 dx}$$

for every measurable function u such that $u \geq 0$, $u^2 \in SBV(\mathbb{R}^N), |\{u \neq 0\} = m|$. Above, B stands for the ball of measure equal to m and J_u for the jump set of u^2.

In [1], we prove that among Lipschitz domains, the ball is the unique minimizer.

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On spectral minimal partitions on the sphere

Bernard Helffer
(joint work with T. Hoffmann-Ostenhof and S.Terracini)

Abstract *In continuation of works in collaboration with V. Bonnaillie-Noël, T. Hoffmann-Ostenhof, S. Terracini and G. Vial* [14, 12, 13, 2, 3], *we analyze the properties of spectral minimal partitions and focus in this paper our analysis on the case of the sphere. We prove that a minimal* 3*-partition for the sphere* $\mathbb{S}^2$ *should be up to rotation the so called* $\mathbf{Y}$*-partition. This question is connected to a celebrated conjecture of Bishop* [1] *in harmonic analysis.*

Let us consider the Laplacian $H(\Omega)$ on a bounded regular domain $\Omega \subset \mathbb{S}^2$ with Dirichlet boundary condition. We denote by $\lambda_j(\Omega)$ the increasing sequence of its eigenvalues and by u_j some associated orthonormal basis of eigenfunctions. We define for any function $u \in C_0^0(\overline{\Omega})$ the nodal set of u by $N(u) = \overline{\{x \in \Omega \mid u(x) = 0\}}$ and call the components of $\Omega \setminus N(u)$ the nodal domains of u.
We now introduce the notions of partition and minimal partition.
For $1 \leq k \in \mathbb{N}$, we call k**-partition** of Ω a family $\mathcal{D} = \{D_i\}_{i=1}^k$ of mutually disjoint sets such that

$$\cup_{i=1}^k D_i \subset \Omega \,, \tag{1}$$

and denote by $\mathfrak{O}_k$ the set of open connected partitions. For $\mathcal{D}$ in $\mathfrak{O}_k$, we introduce

$$\Lambda(\mathcal{D}) = \max_i \lambda(D_i) \text{ and } \mathfrak{L}_k = \inf_{\mathcal{D}\in\mathfrak{O}_k} \Lambda(\mathcal{D}).$$

and call $\mathcal{D} \in \mathfrak{O}_k$ minimal if $\mathfrak{L}_k = \Lambda(\mathcal{D})$.
If $k = 2$, it is rather well known (see [12] or [9]) that $\mathfrak{L}_2$ is the second eigenvalue and the associated minimal 2-partition is a **nodal partition**, i.e. a partition whose elements are the nodal domains of some eigenfunction. Applying this remark to the sphere, we get that the optimal 2-partition if the sphere is realized by two hemispheres.

More generally we can consider (see in [14]) for $p \in [1, +\infty[$

$$\Lambda^p(\mathcal{D}) = (\frac{1}{k}\sum_i \lambda(D_i)^p)^{\frac{1}{p}} \text{ and } \mathfrak{L}_{k,p}(\Omega) = \inf_{\mathcal{D}\in\mathfrak{O}_k} \Lambda^p(\mathcal{D})\,.$$

We write $\mathfrak{L}_{k,\infty}(\Omega) = \mathfrak{L}_k(\Omega)$ and recall the monotonicity property

$$\mathfrak{L}_{k,p}(\Omega) \leq \mathfrak{L}_{k,q}(\Omega) \text{ if } p \leq q\,.$$

The notion of p-minimal k-partition can be extended accordingly, by minimizing $\Lambda^p(\mathcal{D})$. Their existence and regularity (their boundary shares the properties of nodal sets except that now an odd number of lines can meet at a singular point) has been proved in [7, 8, 9] (see also [16, 5, 6] and references therein)

We describe as usual $\mathbb{S}^2$ in $\mathbb{R}^3_{x,y,z}$ by the spherical coordinates,$x = \cos\phi \sin\theta, y = \sin\phi \sin\theta, z = \cos\theta$ with $\phi \in [-\pi, \pi[$, $\theta \in]0, \pi[$, and we add the two poles "North" and "South", corresponding to the two points $(0, 0, 1)$ and $(0, 0, -1)$. Then a basic point is the introduction of the double covering $\mathbb{S}^2_{\mathcal{C}}$ of $\mathbb{S}^2 \setminus \{North, South\}$ which can be described by considering $\phi \in [-2\pi, 2\pi[$, $\theta \in]0, \pi[$.

We announce in our talk a proof of the following theorem :

Main Theorem: *Any minimal* 3*-partition of* $\mathbb{S}^2$ *is up to a fixed rotation obtained by the so called* **Y***-partition whose boundary is given by the intersection of* $\mathbb{S}^2$ *with the three half-planes defined respectively by* $\phi = 0, \frac{2\pi}{3}, \frac{-2\pi}{3}$ *. Hence*

$$\mathfrak{L}_3(\mathbb{S}^2) = \frac{15}{4}\,. \tag{2}$$

The value $\frac{15}{4}$ is the seventh eigenvalue of the lifted Laplacian on $\mathbb{S}^2_{\mathcal{C}}$ and corresponds to a spherical harmonic attached to the half integer $\ell = \frac{3}{2}$. The Y-partition is actually the projection on $\mathbb{S}^2$ of the nodal partition of this spherical harmonic defined on $\mathbb{S}^2_{\mathcal{C}}$. The proof consists in showing that any minimal 3-partition of the sphere cannot be nodal and can be lifted in a 6-partition of $\mathbb{S}^2_{\mathcal{C}}$. This involves a mixture between topological considerations (Euler's formula, Lyustenik-Shnirelman's theorem together with spectral considerations).

This theorem is immediately related (actually a consequence of) to a conjecture of Bishop (Conjecture 6) proposed in [1] stating that :

The minimal 3*-partition for* $\frac{1}{3}(\sum_{i=1}^{3} \lambda(D_i))$ *corresponds to the* **Y***-partition.*

A similar question was analyzed (with partial success) when looking in [13] at candidates of minimal 3-partitions of the unit disk $D(0, 1)$ in $\mathbb{R}^2$. The most natural candidate was indeed the Mercedes Star, which is the 3-partition given by three disjoint sectors with opening angle $2\pi/3$, i.e. $D_1 = \{x \in \Omega \mid \omega \in]0, 2\pi/3[\}$ and D_2, D_3 are obtained by rotating D_1 by $2\pi/3$, respectively by $4\pi/3$. Hence the Mercedes star in [13] is replaced here by the **Y**-partition in the Main Theorem. We observe that **Y**-partition can also be described the inverse image of the mercedes-star partition by the map $\mathbb{S}^2 \ni (x, y, z) \mapsto (x, y) \in D(0, 1)$.

We also present results based mainly on lower bounds obtained in [19] [11] of $\mathfrak{L}_k(\mathbb{S}^2)$ and apply this to the discussion for k large to the Hexagonal conjecture.
The best lower bound for $\mathfrak{L}_{2,1}(\mathbb{S}^2)$ is 2, which is optimal and corresponds to the case of two hemispheres [1].
The best lower bound for $\mathfrak{L}_{3,1}(\mathbb{S}^2)$ is at the moment around 3.39.
Another lower bound exploiting [11] is

$$\mathfrak{L}_{k,1}(\mathbb{S}^2) \geq \frac{1}{4} j_0^2 k - \frac{1}{8} j_0^2 - \frac{1}{4} \,. \tag{3}$$

Asymptotically, we obtain

$$\text{Area}(\mathbb{S}^2) \liminf_{k \to +\infty} \frac{\mathfrak{L}_{k,1}(\mathbb{S}^2)}{k} \geq \pi j_0^2 \,.$$

But πj_0^2 is the groundstate energy $\lambda(D^1)$ of the Laplacian on the disk D^1 in $\mathbb{R}^2$ of area 1. The Faber-Krahn Inequality gives for planar domains

$$|\Omega| \frac{\mathfrak{L}_{k,1}(\Omega)}{k} \geq \lambda(D^1) = \pi j_0^2 \,.$$

As for the case of plane domains, it is natural to conjecture (see for example [2, 5] but we first heared of this question from M. Van den Berg five years ago) that :

$$\lim_{k \to +\infty} \frac{\mathfrak{L}_k(\mathbb{S}^2)}{k} = \lim_{k \to +\infty} \frac{\mathfrak{L}_{k,1}(\mathbb{S}^2)}{k} = \lambda(Hexa^1) \,.$$

The first equality in the conjecture corresponds to the idea, which is well illustrated in the recent paper by Bourdin-Bucur-Oudet [4] that, asymptotically as $k \to +\infty$, a minimal k-partition for Λ^p will correspond to D_j's such that the $\lambda(D_j)$ are equal.

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The Random Displacement Model

Michael Loss
(joint work with Jeff Baker and Günter Stolz)

The random diplacement model describes the motion of a quantum particle interacting with a randomly deformed lattice. It is given by the Hamiltonian

$$H_\omega = -\Delta + V_\omega(x) \tag{1}$$

where the potential is of the form

$$V_\omega(x) = \sum_{i\in\mathbb{Z}^d} q(x-i-\omega_i) \tag{2}$$

The single site potential $q(x)$ is a reflection symmetric function with compact support in the unit cube centered at the origin and the variables ω_i are such that $q(x-i-\omega_i)$ maintains its support in the unit cube centered at the position $i \in \mathbb{Z}^d$. Thus, the variables ω_i measure the displacement of the single site potentials from the lattice points of $\mathbb{Z}^d$. If we assume that these variables are independent and identically distributed with respect to some probability measure, this model is ergodic with respect to shifts of $\mathbb{Z}^d$ and one can show [5] that there exists $\Sigma \subset \mathbb{R}$ such that

$$\sigma(H_\omega) = \Sigma \text{ for a.e. } \omega \ .$$

While this model is a natural candidate for describing conductivity properties of solids, or rather the absence thereof, it has been replaced by the Anderson model that is in many ways easier to treat and hence most of the research in random Schrödinger operators is devoted to the Anderson model. Few things are known about the random displacement model. It was proved in [4] that for independently distributed ω_i the one dimensional problem displays localization, i.e., the spectrum is dense point spectrum with exponentially localized eigenfunctions.

A much simpler but fundamental question is to determine the configuration for the single site potentials that minimizes the bottom of the spectrum of H_ω. For arbitrary dimensions, it was shown in [1] that the configuration for the minimizing single site potentials is given by squeezing them into the corners where 2^d cells of $\mathbb{Z}^d$ meet but keeping the support of the potential inside the cell. This determines the variables $\omega_i, i \in \mathbb{Z}^d$ and we denote this configuration by $\omega_{\min}$. This pattern is invariant by translation of *twice* the period of the underlying lattice $\mathbb{Z}^d$. Hence we know that

$$E_0 := \inf \Sigma = \inf \sigma(H_{\omega_{\min}}) \ .$$

The proof proceed by decoupling the various cells by imposing Neumann boundary conditions and then proving for this Neumann problem that the potential that minimizes the lowest eigenvalue wants to sit in a corner. Further, it was shown in [2] that if one places the Hamiltonian (1) in a box of size L^d with periodic boundary conditions then this minimizing configuration is unique for $d > 1$. The case $d = 1$ is very different, there is no uniqueness.

Concerning the Neumann problem, one may ask quite generally whether a potential must be placed on the boundary of a domain in order to minimize the energy. More precisely, consider a strictly convex open domain $D \subset \mathbb{R}^d$, bounded and with smooth boundary. Consider the family of Schrödinger operators $H := -\Delta + q(x-a)$ where $q(x) \in C_c^\infty(D)$ with Neumann conditions on the boundary of ∂D. Denote by G the set $\{a \in \mathbb{R}^d : \text{supp } q(x-a) \subset D\}$. The set G is open. Denote by $E_0(a)$ the lowest eigenvalue of H.

Theorem 1 (Strong minimum principle)**.** If there exists a point $a_0 \in G$ with $E_0(a_0) = \inf_G E_0(a)$, then the function $E_0(a)$ vanishes identically in G and for each $a \in G$ the eigenfunction is constant outside the support of the potential.

The proof [1] relies on second order perturbation theory expressing $\Delta E_0(a)$ in terms of the curvature matrix of the boundary ∂D. One should note that it is rather easy to construct potentials so that the lowest eigenvalue of the Neumann operator $-\Delta+q(x-a)$ vanishes identically as a function of a and the corresponding eigenfunction for each is constant outside the support of the potential. Thus one can move the potential around inside the domain without changing the energy.

Intuitively one expects localization to occur because there are very few eigenvalue degeneracies in the system that allow for the tunneling of the particle into far away regions, at least for energies close to the bottom of the spectrum. This is quantified by an estimate what is called Lifshitz tails. This together with the much harder Wegner estimate suffices to show localization by using a multiscale analysis. This method, which was pioneered in [7], has been developed further by many authors with some of the most significant progress in [6, 8, 3].

Assume that the distribution of the single site potential is concentrated with equal probability in the corners of each cell. One expects that the density of states, defined by

$$N(E) = \lim_{L\to\infty} \frac{1}{L^d} \mathbb{E}\left[\#\{\text{of eigenvalues of} H^P_{\omega,L}\} < E\right]$$

for $d \geq 2$ displays Lifshitz tail behavior, i.e.,

$$N(E) \approx e^{-c|E-E_0|^{-d/2}} \ , \tag{3}$$

as $E \to E_0$.

Here the Hamiltonian $H^P_{\omega,L}$ is the original Hamiltonian restricted to the cube of sidelength L with periodic boundary conditions and the expectation is with respect to the distribution of the random variables ω_i.

A natural and quite standard approach is to show by combining a large deviation argument with the uncertainty principle, that for most of the configurations of the potentials, the ground state energy of the Hamiltonian $H^P_{\omega,L}$ is not too low, i.e.,

$$E^P_{\omega,L} \geq E_0 + \frac{\text{const.}}{L^2} \ .$$

The difficulty with the displacement model is that the spectrum of the Hamiltonian H does not vary monotonically with the displacement of the potential. A similar difficulty occurs in the Anderson model in case the potential is not sign definite which was treated in [9]. Klopp and Nakamura, however, were able to adapt their method in [10] to the random displacement model. They obtain

$$N(E) \approx e^{-c|E-E_0|^{-1/2}} \ , \tag{4}$$

far from the desired result for dimensions larger than one. The exponent is a reminder that the method used is essentially one dimensional which is somewhat paradoxical because it was shown in [2] that for the one dimesional displacement model

$$N(E) \geq \frac{C}{log^2|E-E_0|} \ ;$$

thus, there are no Lifshitz tails in one dimension. This is strongly related to the fact that the minimizing configuration in one dimension is degenerate, in fact highly degenerate (see [2]). Thus in order to apply the method of [10], the fact that the minimizing configuration for the periodic Hamiltonian is unique for two and more dimensions has to be turned into a quantitative estimate.

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A Class of New Inequalities for the Eigenvalues of the Dirichlet Laplacian

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(joint work with Evans M. Harrell II)

The bulk of these results appear in the preprint [3] where various integral transform techniques were used to prove the equivalence of spectral inequalities hitherto proved by independent methods. We are concerned with bounds for different spectral functions of the eigenvalues $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$ of the fixed membrane problem

$$-\Delta u = \lambda\, u \quad in \quad \Omega \subset \mathbb{R}^d$$
$$u = 0 \quad on \quad \partial\Omega.$$

In [3] four key inequalities were shown to satisfy the following diagram

$$\begin{array}{ccc} \text{Yang} & \Leftrightarrow & \text{Harrell-Stubbe, } \rho \geq 2 \\ \Downarrow & & \Downarrow \\ \text{Kac} & \Leftrightarrow & \text{Berezin-Li-Yau, } \rho \geq 2 \end{array}$$

These statements are:

- The H. C. Yang inequality [6]

(1)
$$\sum_k (z - \lambda_k)_+^2 \leq \frac{4}{d} \sum_k (z - \lambda_k)_+ \, \lambda_k$$

- The Harrell-Stubbe inequality [6] [3] [4]

(2)
$$\sum_k (z - \lambda_k)_+^\rho \leq \frac{2\rho}{d} \sum_k (z - \lambda_k)_+^{\rho-1} \, \lambda_k$$

- The inequality of Kac [7]

(3)
$$Z(t) := \sum_{k=1}^{\infty} e^{-\lambda_k t} \leq \frac{|\Omega|}{(4\pi t)^{d/2}}$$

- The Berezin-Li-Yau inequality [9]

(4)
$$\sum_k (z - \lambda_k)_+^\rho \leq L_{\rho,d}^{cl} |\Omega| z^{\rho + d/2}$$

Certainly, (4) is known in this sharp form even when $1 \le \rho \le 2$ [9]. Here $z_+ = \max\{z, 0\}$ and

$$L^{cl}_{\rho,d} = \frac{\Gamma(\rho+1)}{(4\pi)^{d/2}\,\Gamma(\rho+1+d/2)}.$$

As already shown in [4], (2) is in fact equivalent to the monotonicity principle

$$\frac{R_\rho(z)}{z^{\rho+d/2}} \text{ is a nondecreasing function of } z \text{ when } \rho \ge 2.$$

Here $R_\rho(z) := \sum_k (z-\lambda_k)_+^\rho$ is the Riesz mean of order $\rho \ge 0$. This monotonicity principle, in addition to the semiclassical statement $\lim_{t\to 0+} t^{d/2} Z(t) = |\Omega|/(4\pi)^{d/2}$, are key ingredients paving the way from (3) to (4). Already from monotonicity and the statement

$$\prod_{k=1}^{m}\left(1+\frac{d}{2(\rho+k)}\right) = \frac{\Gamma(\rho+m+1+d/2)\,\Gamma(\rho+1)}{\Gamma(\rho+m+1)\,\Gamma(\rho+1+d/2)},$$

one can conclude for $z \ge z_0$, $\rho \ge 1$, and $m \ge 1$ that

$$R_\rho(z) \ge \frac{\Gamma(\rho+m+1+d/2)\,\Gamma(\rho+1)}{\Gamma(\rho+m+1)\,\Gamma(\rho+1+d/2)}\;\frac{R_{\rho+m}(z_0)}{z_0^{\rho+m+d/2}}\;z^{\rho+d/2},$$

which in light of the definition of $L^{cl}_{\rho,d}$ takes the form

$$\frac{R_\rho(z)}{L^{cl}_{\rho,d}\,z^{\rho+d/2}} \ge \frac{R_{\rho+m}(z_0)}{L^{cl}_{\rho+m,d}\,z_0^{\rho+m+d/2}}.$$

Fixing z_0 and sending $z \to \infty$ and exploiting the semiclassical statement $R_\rho(z) \sim L^{cl}_{\rho,d}\,|\Omega|\;z^{\rho+d/2}$ leads to the Berezin-Li-Yau inequality.

Sum rules in the spirit of Harrell-Stubbe play a key role in direct proofs of these inequalities and are central to the integral transform technique. Another sum rule attributed to Bethe [10] provides a new proof [8] [5] [2] to the lower bound universal statement: For $\rho \ge 1$

$$R_\rho(z) \ge H_d^{-1}\lambda_1^{-d/2}\;\frac{\Gamma(1+\rho)\Gamma(1+d/2)}{\Gamma(1+\rho+d/2)}\;(z-\lambda_1)_+^{\rho+d/2}.$$

Here

$$H_d = \frac{2\,d}{j^2_{d/2-1,1}\,J^2_{d/2}(j_{d/2-1,1})},$$

where $J_\alpha(x)$ denotes the Bessel function of order α and $j_{\alpha,p}$ is its $p-$th zero.

In fact, this statement is a particular case of the following general result valid for functions F and G related by the Weyl transform [1] [3]. For a nonnegative function f on $\mathbb{R}_+$ such that

$$\int_0^\infty f(t)\left(1+t^{-d/2}\right)\frac{dt}{t} < \infty$$

define

$$F(s) := \int_0^\infty e^{-st} f(t)\frac{dt}{t} \tag{5}$$

and let

$$G(s) := \mathcal{W}_{d/2}\{F(z)\}(s), \tag{6}$$

where

$$\mathcal{W}_\mu\{F(z)\}(s) := \frac{1}{\Gamma(\mu)} \int_s^\infty F(z)\,(z-s)^{\mu-1}\,dz$$

denotes the Weyl transform of order μ of the function $F(z)$. According to the Bateman project, $G(s) = \int_0^\infty \frac{e^{-st}}{t^{d/2}} f(t) \frac{dt}{t}$. Under such circumstances one has the following result.

Theorem: For $F(s)$ and $G(s)$ as defined above, and related by the Weyl transform,

$$\sum_{j=1}^\infty F(\lambda_j) \geq \frac{\Gamma(1+d/2)}{H_d}\,\lambda_1^{-d/2} G(\lambda_1).$$

It turns out that the correction introduced by Melas to the formula of Li-Yau leads to the following counterpart statement.

Theorem: For $F(s)$ and $G(s)$ as related by the Weyl transform, one has

$$\sum_{j=1}^\infty F(\lambda_j) \leq \frac{1}{(4\pi)^{d/2}}\,|\Omega|\, G\left(M_d \frac{|\Omega|}{I(\Omega)}\right).$$

Here M_d is a universal constant stemming from the work of Melas. This leads to the conjecture

$$\sum_{j=1}^\infty F(\lambda_j) \leq \frac{1}{(4\pi)^{d/2}}\,|\Omega|\, G(|\Omega|^{-2/d})$$

Here $\frac{1}{|\Omega|^{2/d}}$ replaces $M_d\,\frac{|\Omega|}{I(\Omega)}$. Such a conjecture would follow from the following adjustment of the Melas result: For $\rho \geq 1$

$$R_\rho(z) \leq L^{cl}_{\rho,d}|\Omega| \left(z - \frac{1}{|\Omega|^{2/d}}\right)_+^{\rho+\frac{d}{2}}.$$

A consequence of which is the following correction of the Kac inequality

$$Z(t) \leq \frac{|\Omega|}{(4\pi t)^{d/2}}\, e^{-\dfrac{t}{|\Omega|^{2/d}}}.$$

It seems that an inequality of the form

$$\left(\sum_{j=1}^k (z-\lambda_j)_+\right)^2 \leq \frac{4}{d} \sum_{j=1}^k (z-\lambda_j)_+^{1/2} \sum_{j=1}^k (z-\lambda_j)_+^{1/2}\lambda_j$$

is tractable [13] and might lead to an interesting statement for Riesz means of orders $\rho = 1/2, 1, 3/2$. It remains a question whether one can improve, at least for convex domains, a result of Van Den Berg [12] to the following statement

$$\left|Z(t) - \frac{|\Omega|}{(4\pi t)^{d/2}}\, e^{-t/|\Omega|^{2/d}}\right| \leq \phi(|\partial\Omega|, t, d)$$

where ϕ is an explicit function.

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A sharp Lieb-Thirring inequality with a remainder term

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(joint work with Timo Weidl)

Let $\Omega \subset \mathbb{R}^d$ be an open and bounded set and consider a potential $V: \ \Omega \to \mathbb{R}$, $V \geq 0$. Assume that a Schrödinger-Operator $H = -\Delta - V$ can be defined as a selfadjoint operator with Dirichlet boundary conditions, i. e. in the form sense on the form domain $H_0^1(\Omega)$, so that the negative spectrum of H consists of a finite number of negative eigenvalues $(-\lambda_k)_{k=1}^N$. These eigenvalues can be estimated with sharp Lieb-Thirring inequalities, see [6]. For $\sigma \geq 3/2$

$$\sum_k \lambda_k^\sigma \ \leq \ L_{\sigma,d}^{cl} \int_\Omega V^{\sigma+d/2} dx. \tag{1}$$

In the case of a constant potential $V \equiv \Lambda > 0$ on Ω it is known that the resulting semiclassical Berezin-Li-Yau inequalities, [2], can be improved by a negative remainder term, see [7], [8] and [5], and [3] for discrete operators. We improve the semiclassical inequalities (1) for general potentials $V \in L^{\sigma+d/2}(\Omega)$ by analysing the effect of the Dirichlet boundary condition on individual eigenvalues. Here we concentrate on the one dimensional case and use results from Sturm-Liouville Theory, see e.g. [9], to derive an improved Lieb-Thirring inequality with a remainder term.

Let $I \subset \mathbb{R}$ be an open interval of length $l < \infty$. For simplicity we first assume $I = (0, l)$ and $V \in C_0^\infty(I)$. Under these conditions define three types of Operators with different boundary conditions:

$$\begin{aligned} H_{\mathbb{R}} &= -\Delta - V \quad \text{with form domain } H^1(\mathbb{R}) \\ H_I^D &= -\Delta - V \quad \text{with form domain } H_0^1(I) \\ H_I^{(\alpha,\beta)} &= -\Delta - V \quad \text{for } 0 \le \alpha, \beta \le \frac{\pi}{2} \end{aligned}$$

through the closure of the quadratic form

$$h_I^{(\alpha,\beta)}[\varphi] = \int_I |\varphi'|^2 dx - \int_I V|\varphi|^2 dx + \cot\alpha\ |\varphi(0)|^2 + \cot\beta\ |\varphi(l)|^2$$

with form domain $H^1(I)$. For φ from the domain of $H_I^{(\alpha,\beta)}$ the boundary conditions

$$\varphi'(0) = \cot\alpha\ \varphi(0) \quad \text{and} \quad \varphi'(l) = -\cot\beta\ \varphi(l)$$

hold. Moreover define $\sigma_-(H_{\mathbb{R}}) = (-\mu_k)$, $\sigma_-\left(H_I^D\right) = (-\lambda_k)$ and $\sigma_-\left(H_I^{(\alpha,\beta)}\right) = (-\nu_k(\alpha,\beta))$. To establish a connections between these eigenvalues one can compare eigenfunctions of $H_{\mathbb{R}}$ and $H_I^{(\alpha,\beta)}$ in order to prove

Proposition 1. *Assume $\sigma_-\left(H_I^D\right) = (-\lambda_k)_{k=1}^N$. Then for the negative eigenvalues of $H_{\mathbb{R}}$ and $H_I^{(\alpha,\beta)}$ the identity*

$$-\mu_k = -\nu_k(\omega_k, \omega_k)$$

holds for all $k = 1, \ldots, N$ with $\omega_k = \operatorname{arccot}\sqrt{\mu_k}$.

To estimate the difference

$$\mu_k - \lambda_k = \nu_k(\omega_k, \omega_k) - \nu_k(0,0)$$

we use results from Sturm-Liouville-Theory. The eigenvalues $-\nu_k(\alpha,\beta)$ can be differentiated with respect to the boundary conditions depending on α and β and the derivative can be expressed with the help of functions $u(\nu,\alpha,t)$ and $\tilde{u}(\nu,\beta,t)$ defined by

$$\begin{aligned} -u'' - Vu &= -\nu\ u \text{ on } I \quad \text{and} \quad u(\nu,\alpha,0) = \sin\alpha \quad u'(\nu,\alpha,0) = \cos\alpha, \\ -\tilde{u}'' - V\tilde{u} &= -\nu\ \tilde{u} \text{ on } I \quad \text{and} \quad \tilde{u}(\nu,\beta,l) = \sin\beta \quad \tilde{u}'(\nu,\beta,l) = -\cos\beta. \end{aligned}$$

Proposition 2. *The identity*

$$\mu_k - \lambda_k = \int_0^{\omega_k} \| u(\nu_k(\alpha,\omega_k),\alpha) \|_{L^2(I)}^{-2}\ d\alpha + \int_0^{\omega_k} \| \tilde{u}(\nu_k(0,\beta),\beta) \|_{L^2(I)}^{-2}\ d\beta$$

holds for all $k \in \mathbb{N}$ where $-\lambda_k \in \sigma_-\left(H_I^D\right)$.

For $k = 1$, $\| u(\nu_1(\alpha, \omega_1), \alpha) \|_{L^2(I)}$ and $\| \tilde{u}(\nu_1(0, \beta), \beta) \|_{L^2(I)}$ can be estimated from below in terms of l and $\sqrt{\mu_1}$. Using these estimates, the variational principle and the inequality

$$\sqrt{\mu_1} \le \int V dt,$$

see [4], one can prove

Theorem 1. *For an open intervall $I \subset \mathbb{R}$ of length $l < \infty$ the estimates*

$$\sum_k \lambda_k^\sigma \le L_{\sigma,1}^{cl} \int_I V^{\sigma+1/2} dt - \left(\frac{2\left(\int V dt\right)^2}{\exp\left(l \int V dt\right) - 1} \right)^\sigma \quad \text{if} \quad l \int V dt \ge 2 \ln 3$$

$$\text{and} \quad \sigma_-(H) = \emptyset \quad \text{if} \quad l \int V dt < 2 \ln 3$$

hold for $\sigma \ge 3/2$ and all $V \in L^{\sigma+1/2}(I)$.

This result can be generalised to higher dimensions by applying operator valued Lieb-Thirring inequalities, [6], and an induction in the dimension argument similar to the one used e.g. in [6] and [8]. This process yields improved Lieb-Thirring inequalities with correction terms in higher dimensions.

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On some inequalities for eigenvalues of Schrödinger operators with complex-valued potentials

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(joint work with Oleg Safronov)

We consider the Schrödinger operator $H = -\Delta + V$ with a complex-valued potential V assuming that $\lim_{|x|\to\infty} V(x) = 0$ and would like to obtain some results on the distribution of eigenvalues of H in the complex plane.

The main result of [6] tells us that for any $t > 0$ the eigenvalues z_j of H lying outside the sector $\{z : \ |\Im z| < t\, \Re z\}$ satisfy the estimate

$$\sum |z_j|^\gamma \le C \int |V(x)|^{\gamma+d/2} dx, \qquad \gamma \ge 1,$$

where the constant C depends on t, γ and d.

A natural question that appears in relation to this result is what estimates are valid for the eigenvalues situated inside the conical sector $\{z : \ |\Im z| < t\Re z\}$, where the eigenvalues might be close to the positive half-line? The aim of this note is to present a number of statements that describe the rate of accumulation of eigenvalues to the set $\mathbb{R}_+ = [0, \infty)$.

Theorem 1. Let $\Re V \ge 0$ be a bounded function. Assume that $\Im V \in L^p(\mathbb{R}^d)$, where $p > d/2$ if $d \ge 2$ and $p \ge 1$ if $d = 1$. Then the eigenvalues λ_j of the operator $H = -\Delta + V$ satisfy the estimate

$$\sum_j \Big(\frac{\Im \lambda_j}{|\lambda_j + 1|^2 + 1} \Big)_+^p \le C \int_{\mathbb{R}^d} \Im V_+^p(x)\, dx, \tag{1}$$

where $\Im V_+$ is a positive part of $\Im V$. The constant C in this inequality can be computed explicitly:

$$C = (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{d\xi}{(\xi^2+1)^p}. \tag{2}$$

The right hand side of the estimate (1) does not contain the potential $\Re V$. This means that the conditions on $\Re V$ can be drastically relaxed. It is not the case when we try to obtain an estimate of the sum $\sum_j (\Im\lambda_j/(|\lambda_j+1|^2+1))_+^p$ for $p \le d/2$. A certain regularity of $\Re V$ is required in this case because of an essential reason.

Theorem 2. Let $\Re V \ge 0$ and $\Im V$ be two bounded real valued functions. Assume that $\Im V \in L^p(\mathbb{R}^d)$, where $p > d/4$ if $d \ge 4$ and $p \ge 1$ if $d \le 3$. Then the eigenvalues λ_j of the operator $H = -\Delta + V$ satisfy the estimate

$$\sum_j \Big(\frac{\Im \lambda_j}{|\lambda_j + 1|^2 + 1} \Big)_+^p \le (1 + ||V||_\infty)^{2p} C \int_{\mathbb{R}^d} \Im V_+^p(x)\, dx. \tag{3}$$

The constant C in this inequality can be computed explicitly:

$$C = (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{d\xi}{((\xi^2+1)^2+1)^p}. \tag{4}$$

One should mention, that the paper [6] in its turn was motivated by the question of E.B. Davies about an integral estimate for eigenvalues of H (see [1] and [4]). If $d = 1$ then all eigenvalues λ of H which do not belong to $\mathbb{R}_+$ satisfy

$$|\lambda| \le \frac{1}{4}\Big(\int |V(x)|dx\Big)^2.$$

Note that the constant in the latter inequality is sharp and is achieved for Dirac type potentials $e^{i\theta}\delta(x)$, $\theta \in [0, 2\pi]$. It is interesting that for the corresponding Dirichlet boundary value problem on $\mathbb{R}_+$ the constant in this inequality is not as good as $1/4$, see recent preprint [5].

The question is whether a similar integral estimate holds in dimension $d \ge 2$. By the word "similar", we mean an estimate by the L^p norm of the potential V with $p > d/2$. So, the problem can be formulated as a hypothesis in the following way:

Conjecture. Let $d \ge 2$ and let $\gamma > 0$ be given. There is a positive constant C such that

$$|\lambda|^\gamma \le C \int_{\mathbb{R}^d} |V(x)|^{d/2+\gamma} dx, \tag{5}$$

for every complex valued potential $V \in C_0^\infty$ and every eigenvalue $\lambda \notin \mathbb{R}_+$ of the operator $-\Delta + V$.

So far, we are able to prove only the following result related to this conjecture:

Theorem 3. Let V be a function from $L^p(\mathbb{R}^d)$, where $p \ge d/2$, if $d \ge 3$, $p > 1$, if $d = 2$, and $p \ge 1$, if $d = 1$. Then every eigenvalue λ of the operator $H = -\Delta + V$ with the property $\Re\lambda > 0$ satisfies the estimate

$$|\Im\lambda|^{p-1} \le |\lambda|^{d/2-1} C \int_{\mathbb{R}^d} |V|^p dx. \tag{6}$$

The constant C in this inequality depends only on d and p. Moreover, $C = 1/2$ for $p = d = 1$.

The relation (6) was established in [1] in the case $d = p = 1$. We prove it in higher dimensions and in dimension $d = 1$ for $p > 1$.

One can easily obtain the following elementary estimate although it is not quite the same as as (5).

Theorem 4. Let $d = 3$ and let $z = k^2 \notin \mathbb{R}_+$ be an eigenvalue of the operator $H = -\Delta + V$, $\Im k > 0$. Then there is a positive constant C depending only on $\gamma > 0$, such that

$$(\Im k)^{2\gamma} \le C \int_{\mathbb{R}^3} |V|^{3/2+\gamma} dx.$$

While we do not prove the conjecture directly, we find some interesting information about the location of eigenvalues of the operator $-\Delta + iV$ with a positive $V \ge 0$. In particular, in $d = 3$, we obtain that if $\int V dx$ is small and $\lambda \notin \mathbb{R}_+$ is an eigenvalue of $-\Delta + iV$, then $|\lambda|$ must be large. It might seem that eigenvalues do not exist at all for small values of $\int V dx$, however their presence in such cases can be easily established with the help of the following statement.

Proposition 3. *Let $d \geq 3$. Then there is a sequence of positive functions $V_n \geq 0$ such that the "largest modulus" eigenvalue $\lambda_n \notin \mathbb{R}_+$ of the operator $-\Delta + iV_n$ satisfies $|\lambda_n| \to \infty$ as $n \to \infty$, while* $\lim_{n\to\infty} \int V_n(x)dx = 0$.

Proof. If λ is an eigenvalue of $-\Delta + iV(x)$, then $n^2\lambda$ is an eigenvalue of $-\Delta + n^2 iV(nx)$. It remains to note that $\int n^2 V(nx)dx = Cn^{2-d}$. The proof of existence of a non-real eigenvalue of $-\Delta + iV(x)$ at least for one $V \geq 0$ is left to the reader. □

Remark. The proposition does not contradict Conjecture.

Note that our theorems imply also that the eigenvalues of $-\Delta + iV$ can not accumulate to zero in $d = 3$, if $V \geq 0$ is integrable.

One line proof of a Hardy inequality

Hardy's inequality for convex domains in $\mathbb{R}^d$ is usually given in terms of the distance to the boundary. Namely, let $\Omega \subset \mathbb{R}^d$ be a convex domain and let $\delta(x) = \operatorname{dist}(x, \partial\Omega)$ be the distance from $x \in \Omega$ to the boundary $\partial\Omega$. The following Hardy inequality is well known (see [2], [3])

$$\int_\Omega |\nabla u|^2 \, dx \geq \frac{1}{4} \int_\Omega \frac{|u(x)|^2}{\delta^2(x)} \, dx, \qquad u \in H_0^1(\Omega).$$

There is one line prove of this inequality based on two geometrical facts:

$$|\nabla \delta(x)| = 1$$

and if Ω is convex then

$$-\Delta \delta(x) > 0.$$

If now

$$Q = \nabla - \frac{1}{2} \frac{\nabla \delta(x)}{\delta(x)}$$

then we have

$$0 \leq Q * Q = -\Delta + \frac{\Delta \delta(x)}{\delta(x)} - \frac{1}{2} \frac{|\nabla \delta(x)|}{\delta^2(x)} + \frac{1}{4} \frac{|\nabla \delta(x)|}{\delta^2(x)}.$$

The proof is complete.

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Convex sets of constant width

Bernd Kawohl

A bounded convex set has constant width d iff any two parallel (and nonidentical) tangent planes to it have identical distance d from each other. Clearly balls have this property, but there are also other sets of constant width. This lecture was originally designed for a general audience as part of a series of lectures during the German "Year of Mathematics" 2008. It starts by presenting evidence from the Challenger desaster [11]. A lack of geometric insight was a serious contributing factor to this accident. Then the lecture treats two- (and later three-)dimensional sets of constant width and their occurence in daily life, for instance as shapes of coins [18]. These require in general less material than circular ones, because Barbier proved the following interesting isoperimetric property at the age of 21.

Theorem 1: (Barbier 1860) *All plane convex sets of constant width d have the same perimeter πd as the disc of diameter d.*

So by the classical isoperimetric inequality the disc has maximal area among all plane convex sets of given constant width. Another elegant and elementary proof that uses only the theorem of Pythagoras and polar coordinates was given by Littlewood in [19].

There are many plane convex sets of constant width. Their support function $p(\theta)$ necessarily satisfies the functional equation

$$p(\theta) + p(\theta + \pi) = d \qquad \text{on } [0, 2\pi], \tag{1}$$

and this equation has many solutions, for instance $p(\theta) = \frac{d}{2} + \varepsilon \sin(k\theta)$. It is only natural to ask for the shape of a coin that uses least material for a given width, and this question is answered by what is commonly called the Theorem of Blaschke and Lebesgue.

Theorem 2: *Among all plane convex sets of constant width d the Reuleaux-triangle minimizes area.*

The Reuleaux triangle is the intersection of three discs of radius d with centers at the corners of an equilateral triangle with sides of length d. Its beauty has inspired artists and architects as well as engineers. Very different proofs of this theorem were provided by Blaschke [4], Lebesgue [21, 22], Fujiwara [13], Eggleston [10], Besicovich [3], Ghandehari [14], Campi, Colesanti & Gronchi [7] and Harrell [15]. One can calculate the area A of a convex set in terms of its support function p and try to minimize A. This leads to the variational problem of minimizing the functional

$$A(p) := \int_0^{2\pi} \{p(\theta)^2 - p'(\theta)^2\}\, d\theta \quad \text{among } 2\pi\text{-periodic functions } p \tag{2}$$

under the nonlocal constant width constraint (1) and the convexity constraint

$$p''(\theta) + p(\theta) \geq 0 \quad \text{on } [0, 2\pi]. \tag{3}$$

Notice that (2) is a nonconvex minimization problem under nonstandard side constraints, and it cannot be attacked by direct methods in the calculus of variations.

Reuleaux triangles can be used to construct drills that drill square holes, see [23] for a video-clip or [27] for an instructive animation, or machines that transform a rotation into a sliding and stopping motion, see [26]. Reuleaux built and sold collections of small gears as instructional tools for students, and one of those collections has survived at Cornell University and was recently put on the web. Such movements were used in movie-projectors, see [28]. Reuleaux was an impressive scientist. I report on some of his achievements. He has only recently been compared to Leonardo da Vinci by Francis C. Moon, who discovered and saved the collection of Reuleaux's kinematic models at Cornell University [24, 26].

There are also threedimensional convex bodies of constant width. Photos of plaster models can be found in [16] or [12], other shapes in the website that comes with [6]. Incidentally, Stefan Cohn-Vossen was a postdoc of Courant, came to Cologne and gave his "Antrittsvorlesung"on convex surfaces on Feb 22, 1932. In April 1933 he was temporarily suspended from teaching because he was Jewish, in September 1933 he lost his job permanently. He emigrated to Moscow, where he died 1936 of pneumonia. It is truely admirable that Richard Courant, who was also driven out of this country, was later of instrumental help in supporting the Oberwolfach Institute. As in the twodimensional case, one ask if there is an analogue to Barbier's Theorem and one can try to maximize or minimize the volume of convex bodies of constant width.

Theorem: (Blaschke 1915) *Among all 3d convex bodies of given width d the ball maximizes volume and surface area, and the one that minimizes volume also minimizes surface area.*

The body of minimal volume or surface area is unknown, but there is a suspect.

Conjecture: (documented 1934 by Bonnesen & Fenchel) *The threedimensional convex bodies of constant width that minimize volume are exactly Meissner's bodies.*

Pictures of these bodies can be found at [25, 16, 12]. They are essentially constructed from modifications of a Reuleaux-tetrahedron, the intersection of four balls centered at the corners of a regular tetrahedron. There is, however an answer to the minimal volume problem if we look in the smaller class of rotational bodies,

Theorem: (Campi et al. 1996) *Among the class of rotational convex bodies of constant width d, the one that minimizes volume is the rotated Reuleaux triangle.*

Finally I mention two recent results that support the above conjecture. The first one shows how to construct an n-dimensional body of constant width from an $(n-1)$-dimensional one.

Theorem: (Lachand-Robert, Oudet 2006) *Suppose that $E_\pm$ denote the upper and lower half-plane in $\mathbb{R}^n$. Let $K_0 \subset E_+ \cap E_-$ be an $(n-1)$–dim. const. width body, and $Q \subset \mathbb{R}^n$ satisfy $K_0 \subset Q \subset E_- \bigcap_{x \in K_o} B(x, d)$. Set $K_+ := E_+ \cap \bigcap_{x \in Q} B(x, d)$*

and $K_- := E_- \bigcap_{x \in K_+} B(x,d)$. *Then* $K := K_+ \cup K_-$ *is an* n*-dimensional constant width body with* K_0 *as a cross section.*

If in this construction $n = 2$ and $Q = K_0 = (0, d)$, then K is the Reuleaux-triangle, and if $n = 3$ and $Q = K_0 =$ Reuleaux-triangle, then K is a Meissner body. Although this construction seems to be exhaustive only for $n = 2$, see [9], it can be used to randomly generate many threedimensional bodies of constant width. A student of mine, Martin Müller, has recently generated one million of those. None of them had smaller volume than the Meissner bodies.

And in a recent paper [2] Bayen, Lachand-Robert & Oudet derive a necessary condition that characterizes a 3d constant width body of (locally) minimal volume: If one squeezes such a body between two parallel planes, at one of the two points of tangency its surface is not smooth.

Clearly Meissner's bodies satisfy this condition, while a ball does not, and this supports the conjecture, but it does not prove it.

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Approximating low eigenvalues of the Laplacian: analysis, geometry and numerics

Pedro Freitas

We report on recent progress regarding the approximation of low eigenvalues of the Laplace operator on bounded domains in $n-$dimensional Euclidean space with Dirichlet boundary conditions. Although the general purpose is to be able to understand better the relationships between the geometry of the domain and the low eigenvalues, we divide our approach into (roughly) three categories as follows:

(1) asymptotic expansions
(2) bounds depending on geometric quantities
(3) more complex conjectured bounds supported by extensive numerical computations

Below we give some examples of results pertaining to each of these cases.

Asymptotic expansions. In this approach we expand eigenvalues in terms of a parameter ϵ measuring a scaling in one direction. In order to be able to write the coefficients of the asymptotic expansion explicitly, this is done around the (singular) *flattened* case which corresponds to vanishing ϵ – compare with a different approach in [6]. Furthermore, since we obtain all the coefficients of the unbounded terms, the approximation is actually good for ϵ not necessarily very small, and sometimes even up to the original domain, that is $\epsilon = 1$. As an example, we have the following expansion for the first eigenvalue of ellipses of radii 1 and ϵ.

$$\lambda_1(\epsilon) = \frac{\pi^2}{4\epsilon^2} + \frac{\pi}{2\epsilon} + \frac{3}{4} + \left(\frac{11}{8\pi} + \frac{\pi}{12}\right)\epsilon + \mathcal{O}(\epsilon^2), \quad \text{as } \epsilon \to 0.$$

This has a similar error (maximum of around 5%) as the corresponding expression derived by Joseph. However, we are not restricted by the fact that we might not know the eigenvalue of the original domain. An example of such a situation is given by the lemniscate,

$$\left(x_1^2 + x_2^2\right)^2 = x_1^2 - x_2^2.$$

for which we now obtain

$$\lambda_1(\epsilon) = \frac{2\pi^2}{\epsilon^2} + \frac{2\sqrt{3}\pi}{\epsilon} + \frac{97}{24} + \left(\frac{593}{64\sqrt{3}\pi} + \frac{\sqrt{3}\pi}{4}\right)\epsilon + \mathcal{O}(\epsilon^2), \quad \text{as } \epsilon \to 0.$$

The error made as ϵ approaches one is in fact smaller than in the case of the ellipses above. For details, see [3].

It is possible to follow a similar approach in the $n-$dimensional case, still scaling along one direction. The formulae involved are more complex, and we only obtained the first three terms [4]. These still control all the unbounded coefficients. For ellipsoids defined by

$$\mathcal{E} = \left\{(x_1, \ldots, x_d) \in \mathbb{R}^d : \left(\frac{x_1}{a_1}\right)^2 + \cdots + \left(\frac{x_d}{a_d}\right)^2 \leq 1\right\},$$

this yields

$$\lambda_1\left(\mathcal{E}_\epsilon\right) = \frac{\pi^2}{4a_d^2\epsilon^2} + \frac{\pi}{2a_d\epsilon}\sum_{i=1}^{d-1}\frac{1}{a_i} + \frac{1}{4}\left(3\sum_{i=1}^{d-1}\frac{1}{a_i^2} + \frac{1}{2}\sum_{i=1}^{d-1}\sum_{j=i+1}^{d-1}\frac{1}{a_i a_j}\right) + \mathcal{O}(\epsilon^{1/2})$$

as $\epsilon \to +0$.

Geometric bounds. We consider the problem of bounding the first Dirichlet eigenvalue of quadrilaterals. Our main result in this direction is given by the following theorem [5].

Theorem 1. *Let Q be a quadrilateral with side lengths ℓ_i, $i = 1, ..., 4$, and diagonals of lengths d_1 and d_2 forming an angle θ in $(0, \pi/2]$ between them, where it is assumed that d_2 is the length of the largest diagonal contained in Q. Then*

$$\frac{2\pi^2}{A} + \frac{\pi^2}{4A^2}(d_2\sin\theta - d_1)^2 \leq \lambda_1(Q) \leq \frac{\pi^2}{2A^2}\left(l_1^2 + l_2^2 + l_3^2 + l_4^2\right),$$

where A denotes the area of Q. Equality holds for squares and rectangles in the lower and upper bounds respectively.

The lower bound improves upon the Faber–Krahn-type bound of Polya and Szegö which states that among all quadrilaterals of the same area, the square minimizes the first Dirichlet eigenvalue. The upper bound generalizes a result established by Hersch in the case of parallelograms.

Numerics. By using the possibilities afforded not only by the increase in computer power but also by numerical methods which allow for the fast evaluation of low eigenvalues of the Dirichlet Laplacian on planar domains, we have analyzed numerically the dependence of the spectral gap of a domain K, $\gamma(K) := \lambda_2(K) - \lambda_1(K)$, on quantities such as the diameter, area and perimeter [1]. In particular, we considered the effect of the area in the long–standing gap conjecture. In its original form, as proposed by van den Berg [2], this reads as follows

Conjecture 1. *For any planar convex domain K we have*

$$\gamma(K) \geq \frac{3\pi^2}{d^2},$$

where d denotes the diameter of K.

Apart from providing extensive numerical results supporting the conjecture, our results also point in the direction that this is a particular case of the following conjecture which now takes the area into consideration.

Conjecture 2. *For any planar convex domain K we have*

$$\gamma(K) \geq \frac{12\pi^2\gamma(B_1)}{3\pi^3 d^2 + [4\gamma(B_1) - 3\pi^3]\sqrt{d^4 - \frac{16}{\pi^2}A^2}}$$

where B_1 is the ball of unit area. Equality holds if and only if K is a ball or asymptotically for infinite strips.

Note that this reduces to the gap conjecture when A vanishes.

We also examined in detail the dependence on the diameter for the gap of convex polygons and concluded that optimal isodiametric polygons seem to play a role here. Similar studies were carried out for the spectral quotient $\xi(K) = \lambda_2/\lambda_1$.

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An isoperimetric inequality for the free plate eigenvalue

L. Mercredi Chasman

Isoperimetric problems are about minimizing or maximizing a quantity subject to constraints. The classical isoperimetric inequality states that of all planar regions of the same perimeter, the disk has maximal area. Equivalently, of all regions of the same area, the disk minimizes perimeter. Many physical quantities satisfy isoperimetric-type inequalities.

Researchers have investigated and proved isoperimetric inequalities regarding frequencies of vibration in related situations. Lord Rayleigh conjectured, and Faber and Krahn proved, that of all membranes of the same area with constrained edges, a circular drum produces the lowest pitch. Kornhauser and Stakgold conjectured the opposite bound for a membrane with unconstrained edges; this result was proven by Szegő and Weinberger. In my talk, I proved an isoperimetric result for a free plate under tension with unconstrained edges: of all such plates having the same area, the disk has the highest fundamental pitch.

Mathematical formulation. Let Ω be a smoothly bounded region in $\mathbb{R}^d$, $d \geq 2$, and fix a parameter $\tau > 0$. The "plate" Rayleigh quotient is

$$Q[u] = \frac{\int_\Omega |D^2u|^2 + \tau |Du|^2 \, dx}{\int_\Omega |u|^2 \, dx}. \tag{1}$$

Here $|D^2u| = (\sum_{jk} u_{x_j x_k}^2)^{1/2}$ is the Hilbert-Schmidt norm of the Hessian matrix D^2u of u, and Du denotes the gradient vector.

Physically, when $d = 2$ the region Ω is the shape of a homogenous, isotropic plate. The parameter τ represents the ratio of lateral tension to flexural rigidity of the plate; for brevity we refer to τ as the tension parameter. Positive τ corresponds to a plate under tension, while taking τ negative would give us a plate under compression. The function u describes a transverse vibrational mode of the plate, and the Rayleigh quotient $Q[u]$ gives the bending energy of the plate.

From the Rayleigh quotient (1), one can derive the partial differential equation and boundary conditions governing the vibrational modes of a free plate. The critical points of (1) are the eigenstates for the plate satisfying the free boundary conditions and the critical values are the corresponding eigenvalues. For simplicity we give the boundary conditions taking $d = 2$. The equation is:

$$\Delta\Delta u - \tau \Delta u = \omega u, \tag{2}$$

where ω is the eigenvalue, with the natural (*i.e.*, unconstrained or "free") boundary conditions on $\partial\Omega$:

$$Mu := \frac{\partial^2 u}{\partial n^2} = 0 \tag{3}$$

$$Vu := \tau \frac{\partial u}{\partial n} - \frac{\partial}{\partial s}(n(D^2u)t) - \frac{\partial(\Delta u)}{\partial n} = 0 \tag{4}$$

Here n is the outward unit normal to the boundary, s the arclength, and t the unit tangent to the boundary.

The eigenvalue ω is the square of the frequency of vibration of the plate. The quantities appearing as boundary conditions have physical significance as well. The expression Mu is the *bending moment.* As the plate bends, one side compresses while the other expands, leading to a restoring moment which must vanish at an unconstrained edge.

The problem. Using coercivity, we can prove that the spectrum of the free plate under tension is discrete, consisting entirely of eigenvalues with finite multiplicity:

$$0 = \omega_0 < \omega_1 \leq \omega_2 \leq \cdots \to \infty.$$

We also have a complete L^2-orthonormal set of eigenfunctions $u_0 \equiv \text{const}$, u_1, u_2, and so forth.

We call u_1 the *fundamental mode* and the eigenvalue ω_1 the *fundamental tone*; the latter can be expressed using the Rayleigh-Ritz variational formula:

$$\omega_1(\Omega) = \min \left\{ Q[u] : u \in H^2(\Omega), \int_\Omega u \, dx = 0 \right\}.$$

In general, the kth eigenvalue is the minimum of $Q[u]$ over the space of all functions u L^2-orthogonal to the eigenfunctions u_0, u_1,..., u_{k-1}. Because u_0 is the constant function, the condition $u \perp u_0$ can be written $\int_\Omega u \, dx = 0$.

Let Ω^* denote the ball with the same volume as Ω. My main goal was to prove the following theorem.

Theorem 1. *For all smoothly bounded regions of a fixed volume, the fundamental tone of the free plate with a given positive tension is maximal for a ball. That is, if $\tau > 0$ then*

$$\omega_1(\Omega) \leq \omega_1(\Omega^*), \qquad \textit{with equality if and only if } \Omega \textit{ is a ball.} \tag{5}$$

In the limiting case $\tau = 0$, the first $d+1$ eigenvalues of Ω are trivial because $Q[u] = 0$ for all linear functions u. Thus we need the tension parameter τ to be positive to get a nontrivial conjecture.

The proof can be summarized as follows. Adapting Weinberger's approach for the membrane [1], we construct trial functions with radial part ρ matching the radial part of the fundamental mode of the ball. We use the trial function to bound the eigenvalue ω by a quotient of integrals over our region Ω, both of whose integrands are radial functions. These integrands will be shown to have a "partial monotonicity". The denominator's integrand is increasing and the numerator's integrand satisfies a decreasing partial monotonicity condition. From there we deduce the theorem.

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On computing the instability index of a non-selfadjoint differential operator associated with coating and rimming flows

Almut Burchard
(joint work with Marina Chugunova)

We examine the computation of the instability index for differential operators of the form

$$A[h] = -h'''' - \big(a(x)h\big)'' + \big(b(x)h\big)' - c(x)h\,, \tag{1}$$

acting on 2π-periodic functions. Such operators appear as linearizations of models for thin liquid films moving on the surface of a horizontal rotating cylinder [6, 3]. The flows are called **coating**, if the fluid is on the outside of the cylinder, and **rimming**, if the fluid is on the inside of a hollow cylinder. For instance, in [3], the linearized operator is given by

$$A[h](x) = -\frac{d}{dx}\left\{(1-\alpha_1\,\cos x)h + \alpha_2\,\sin x\frac{dh}{dx} + \alpha_3\left(\frac{dh}{dx} + \frac{d^3h}{dx^3}\right)\right\}$$

with periodic boundary conditions. The parameter α_1 is related to the gravitational drainage, α_2 is related to the hydrostatic pressure (in the lubrication approximation model this coefficient is very small), and the parameter α_3 describes surface tension effect. One would expect the flow to become unstable, if the fluid film is thick enough so that drops can form on the bottom of the cylinder (in case of a coating flow) or on its ceiling (in case of a rimming flow). In both cases, surface tension and higher rotation speeds should help to stabilize the fluid, but may also allow for more complicated steady states.

The *instability index* $\kappa(A)$ counts the number of eigenvalues of A (with multiplicity) that have positive real part. Our main result from [4] reduces the instability index of A to the instability index of its projection to a space of trigonometric polynomials, as follows. Define the operators A_x and A_y on a function $F(x,y)$ by acting with the single-variable operator A on $F(\cdot,y)$ and $F(x,\cdot)$, respectively. Suppose that $U(x,y)$ solves the partial differential equation

$$\big(A_x^* + A_y^*\big)U(x,y) = \delta_{y-x} \tag{2}$$

with periodic boundary conditions on $[0,2\pi]\times[0,2\pi]$, and let $U_0(x,y)$ be the solution in the special case where $a(x)=b(x)=0$ and $c(x)=1$. Direct computation shows that U_0 is piecewise smooth, with a jump in the third derivative across the line $x=y$. By elliptic regularity, the difference $U(x,y)-U_0(x,y)$ lies in the Sobolev space H^4. Although the integral kernel $U(x,y)$ just fails to lie in $H^{\frac{7}{2}}$, it defines a bounded linear operator from L^2 to H^4.

Proposition. *Let P_N denote the standard projection onto the space of trigonometric polynomials of order N, assume that U solves Eq. (2) on $[0,2\pi]\times[0,2\pi]$ with periodic boundary conditions, and set $M = 0.52\big(||a||_{H^1} + ||b||_{H^1} + ||c-1||_{H^1}\big)$. If*

$$N^2 > M(1+\sqrt{M+1})\big(1+||U(x,y)-U_0(x,y)||_{H^4}\big)\,,$$

then

$$\kappa(A) = \kappa\big(P_N A P_N\big)\,.$$

Remark. In the condition on N, the factor $1 + ||U(x,y) - U_0(x,y)||_{H^4}$ can be replaced with a smaller term, given by a suitable norm of U.

Eq. (2) is an instance of *Lyapunov's equation* $A^*U + UA = I$. Classical results state that it has a unique solution if the spectra of A and $-A^*$ are disjoint. Furthermore, $\kappa(A) = \kappa(U)$, and the positive and negative cones of U contain the invariant subspaces associated with the spectrum of A in the right and left complex half-planes, respectively [7, 5, 1, 2]. We argue that the quadratic form defined by the self-adjoint operator U is negative on high Fourier modes, because the fourth order term in A dominates the lower order derivatives. This implies that $\kappa(U) = \kappa(P_N U^{-1} P_N)$. The proof of the proposition is completed by estimating the off-diagonal terms in the Fourier representation for A and U.

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Problems from the Oberwolfach Workshop on Low Eigenvalues of Laplace and Schrödinger Operators

Contributed by all Participants

Below we present some open problems raised during the Oberwolfach Workshop on *Low Eigenvalues of Laplace and Schrödinger Operators*, February 9-13, 2009. The name of the participant who suggested the problem is stated in parentheses. We wish to thank Iosif Polterovich and Rupert L. Frank who collected the problmes in sections 1-5 and sections 8-10 and Rodrigo Bañuelos who contributed the problems in sections 6 and 7.

1. Dirichlet eigenvalues

Consider the Dirichlet boundary value problem on a bounded domain $\Omega \subset \mathbb{R}^n$:

$$\begin{cases} -\Delta u_j = \lambda_j u_j & \text{in } \Omega, \\ u_j = 0 & \text{on } \partial\Omega. \end{cases}$$

1.1. ***(Mark Ashbaugh)*** **Ratios of Dirichlet eigenvalues.**

(1) Payne–Pólya–Weinberger conjecture for ratios of consecutive eigenvalues. Prove that for arbitrary $\Omega \subset \mathbb{R}^n$

$$\lambda_{m+1}/\lambda_m(\Omega) \le \lambda_2/\lambda_1(\text{ball in } \mathbb{R}^n) \quad \text{for all } m. \tag{1}$$

The intuition for the conjectured bound is that one can approach as close as one wants to having equality in (1) by taking Ω to be m identical balls connected by thin tubes, and shrinking the widths of the tubes to 0. Assuming (1) is true, this argument would show sharpness of the bound. The harder part is to show the truth of (1). We also note that the right-hand side of (1) is explicitly given as the ratio of squares of Bessel function zeros $j^2_{n/2,1}/j^2_{n/2-1,1}$, where $j_{p,k}$ denotes the kth positive zero of the Bessel function J_p (our notation here is standard, following that of Abramowitz and Stegun [1]).

Inequality (1) is known for the beginning values of m, namely $m = 1, 2, 3$. These results were proved by Ashbaugh and Benguria [2, 3, 4, 5]. Indeed, proofs of the $m = 1$ case can be found in [2, 3], while a proof of the $m = 2$ case occurs in [4]. Later, in [5], it was shown that

$$\lambda_4/\lambda_2(\Omega) < \lambda_2/\lambda_1(\text{ball in } \mathbb{R}^n), \tag{2}$$

which immediately implies both the $m = 2$ and $m = 3$ cases of (1). One expects (1) to be strict for $m > 1$ (recall that a *domain* is a **connected** open set; if the connectedness hypothesis were removed, then cases with m identical disjoint balls would allow equality to occur all the way up).

Beyond what's been said above, all the cases $m \ge 4$ remain open. Indeed, it would be interesting to find any bound better than $1 + 4/n$ for those cases ($m \ge 4$). The bound $1 + 4/n$ was proved by Payne, Pólya, and Weinberger [9, 10] (done explicitly only for $n = 2$, but the proof generalizes straightforwardly to n dimensions). For $m \ge 4$ nothing better is known; for more on the literature of bounds for ratios of low eigenvalues, see [6, 7, 8].

(2) Prove that for arbitrary $\Omega \subset \mathbb{R}^n$,

$$\lambda_{2m}/\lambda_m(\Omega) \le \lambda_2/\lambda_1(\text{ball in } \mathbb{R}^n) \quad \text{for all } m. \tag{3}$$

This conjecture is motivated by (2) and is a considerable strengthening of the conjectured bound (1).

Much as above, one can see that, if true, (3) saturates at examples consisting of anywhere from m to $2m-1$ identical n–balls, connected by thin passages, in the limit as the widths of the passages are sent to 0. One expects (3) to be strict for $m > 1$. As already mentioned, the proof of the $m = 2$ case of (3) can be found in [5]. It relies on Courant's nodal domains result for the second eigenfunction (and, of course, the $m = 1$ case of (1)). The cases for $m = 3$ and above all remain open.

(3) Finally, we offer a conjectured inequality concerning the first n nontrivial ratios of eigenvalues λ_m to λ_1 for domains contained in $\mathbb{R}^n$, the $n = 2$ case of which was first stated and studied by Payne, Pólya, and Weinberger in [10].

Payne–Pólya–Weinberger conjecture for the sum of the first n eigenvalue ratios: Prove that for arbitrary $\Omega \subset \mathbb{R}^n$,

$$\frac{\lambda_2 + \lambda_3 + \cdots + \lambda_n + \lambda_{n+1}}{\lambda_1}(\Omega) \le \frac{\lambda_2 + \lambda_3 + \cdots + \lambda_n + \lambda_{n+1}}{\lambda_1}(\text{ball in } \mathbb{R}^n). \tag{4}$$

This inequality has been checked perturbatively for domains in $\mathbb{R}^2$ that are "nearly circular" and found to hold, much as can be done in the $m = 1$ case of (1). However, one finds that compared to the $m = 1$ case of (1), this bound is much tighter, essentially because as the domain breaks away from being circular (where $\lambda_2 = \lambda_3$) one finds λ_2/λ_1 decreases with a certain slope in the perturbation parameter, while λ_3/λ_1 increases according to the negative of that slope. Thus the sum of these ratios is stationary at first order, and one must go to second order to see that the sum is concave down, and hence that the desired bound holds for nearly circular domains. As in the two earlier problems, one can find explicit expressions for the right-hand side of (4) in terms of zeros of Bessel functions. Here the bound is $n\, j_{n/2,1}^2/j_{n/2-1,1}^2$, since for the n–ball all the eigenvalues λ_2 through λ_{n+1} are equal.

In general, inequality (4) is not known for any $n \ge 2$, though it has been proved for certain special cases where the domain has symmetry. See [6] for these more specialized results. Bounds of similar type to (4) have been proved, but the constant is always just slightly too large, assuming (4) is correct. An argument of Payne, Pólya, and Weinberger (given in [10] for $n = 2$) shows that $n + 4$ (i.e., $n(1 + 4/n)$) is an upper bound for the left-hand side of the inequality in Problem 3, with further refinements of this bound having been proved by various later authors.

See [6, 7, 8] for further work relating to these problems and references to some of the later literature.

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1.2. ***(Pedro Freitas, based on joint work with P. Antunes)*** **Upper and lower bounds for the first Dirichlet eigenvalue.** Payne and Weinberger proved in [2] that for a two–dimensional bounded simply connected domain Ω with area A and perimeter L we have

$$\lambda_1(\Omega) \le \frac{\pi j_{0,1}^2}{A}\left[1+\left(\frac{1}{J_1^2(j_{0,1})}-1\right)\left(\frac{L^2}{4\pi A}-1\right)\right].$$

Here J_1 is the Bessel function of the first kind of order 1, and $j_{0,1}$ is the first zero of the Bessel function of the first kind of order 0. This bound may be written explicitly in terms of the isoperimetric defect $L^2 - 4\pi A$ in the form

$$\lambda_1(\Omega) \le \frac{\pi j_{01}^2}{A} + C\frac{L^2-4\pi A}{A^2},$$

for some constant C. One may then pose the question of finding the optimal value of C for which the above inequality holds. Since the factor multiplying C vanishes when Ω is a ball, it should be possible to lower the constant until there is equality for another domain. Based on extensive numerical simulations, it was conjectured in [1] that this domain is the infinite strip (the equality being understood in an asymptotic sense), and that we should have

Conjecture. *For any planar simply connected domain* Ω *we have*

$$\lambda_1(\Omega) \le \frac{\pi j_{0,1}^2}{A} + \frac{\pi^2}{4}\frac{L^2 - 4\pi A}{A^2}.$$

Equality holds for disks and asymptotically for infinite strips.

Note that the coefficient in L^2/A^2 agrees with that in the following inequality due to Pólya [3]

$$\lambda_1(\Omega) \le \frac{\pi^2 L^2}{4A^2}.$$

It is possible to consider inequalities of the above type in the class of $n-$polygons, the difference being that then the conjecture should hold with $L^2 - 4\pi A$ replaced by $L^2 - \kappa_n A$, the isoperimetric defect for the corresponding class, and πj_{01}^2 by $\lambda_1(P_n^{reg})$, the first Dirichlet eigenvalue of the regular polygon of n sides and unit area.

Regarding lower bounds of this type, namely, bounds of the form

$$\frac{\pi j_{01}^2}{A} + K\frac{L^2 - 4\pi A}{A^2} \le \lambda_1(\Omega),$$

for some positive constant K, we conjecture that no such inequality exists for convex domains (the constant K multiplying the isoperimetric defect must vanish, yielding Faber–Krahn). However, if we again restrict ourselves to $n-$polygons $(n \ge 4)$, and consider

$$\frac{\lambda_1(P_n^{reg})}{A} + C_n\frac{L^2 - \kappa_n A}{A^2} \le \lambda_1(P_n),$$

the bound should hold when the constant C_n is such that it provides equality not only for the regular $n-$polygon (which is automatic since the isoperimetric deffect vanishes then) but also for the regular $(n-1)-$polygon – see [1] for the details. This is stronger than the Pölya–Szegő conjecture which states that of all $n-$polygons with the same area, the regular $n-$polygon yields the lowest first Dirichlet eigenvalue [4].

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1.3. Miscellaneous problems on Dirichlet eigenvalues.

(1) *(Antoine Henrot)* Prove that the disk minimizes the sum $\lambda_2 + \lambda_3$ of the second and the third eigenvalues of the Laplace-Dirichlet operator among domains of given area.

(2) *(Antoine Henrot)* This problem goes back to G. Pólya (see [4, pp. 50-51]). Prove that the regular n-gon minimizes the first eigenvalue of the Laplace–Dirichlet operator among polygons with n sides and given area. It was proved by Pólya for $n = 3, 4$ using Steiner symmetrization. Unfortunately, Steiner symmetrization does not preserve the class of n-gons for $n \geq 5$.

(3) *(Antoine Henrot)* Let Ω be a bounded, simply connected domain and B_0 a ball of small radius. Find the location of the ball in order to minimize or maximize the first eigenvalue of the doubly connected domain $\lambda_1(\Omega \setminus B_0)$ with Dirichlet boundary conditions. In particular, prove that the minimizing position is when B_0 touches the boundary of Ω (where?) and the maximizing one when B_0 is centered at some particular point of Ω (which one?). Such results have been obtained when Ω is a ball (several authors) and when Ω is a convex set with some symmetry properties [3].

(4) *(Dorin Bucur)* Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be a domain of finite volume V. For each $k \in \mathbb{N}$, prove that the isoperimetric problem

$$\min_{|\Omega|=V} \lambda_k(\Omega)$$

has at least one solution Ω^*.
Comments: For $k = 1, 2$ the previous problems are solved, being the well known Faber–Krahn inequalities. For $k = 3$, existence of a solution is proved in the family of quasi opens sets, while for $k \geq 4$ existence of a solution is conditioned by a mild regularity of the minimizers for λ_j, $j = 3, .., k-1$ (see [2]). It is conjectured that for $k = 3$ the solution is the ball in $\mathbb{R}^2, \mathbb{R}^3$ and three equal balls for $N \geq 4$ (see [4]).

(5) *(Dorin Bucur)* Prove that if Ω^* is a solution of the previous problem, then $\lambda_{k-1}(\Omega^*) = \lambda_k(\Omega^*)$. This conjecture has numerical evidence for several values of k (see [5]).

(6) *(Rafael Benguria)* Let $\Omega \subset \mathbb{R}^2$ be a convex domain, symmetric with respect to the origin, and let χ_Ω be its characteristic function. Consider the null set $\mathcal{N}(\Omega) = \{\xi \in \mathbb{R}^2 \,|\, \widehat{\chi}_\Omega(\xi) = 0\}$, where $\widehat{\chi}_\Omega$ denotes the Fourier transform of the characteristic function of the domain. Denote by $\kappa(\Omega)$ the distance in $\mathbb{R}^2$ from the set $\mathcal{N}(\Omega)$ to the origin. It is conjectured in [1] that $\kappa(\Omega) \leq \sqrt{\lambda_2(\Omega)}$. Note that one gets equality if Ω is a disk. A weaker estimate $\kappa(\Omega) \leq 2\sqrt{\lambda_1(\Omega)}$ was proved in [1].

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2. Neumann eigenvalues

Consider the Neumann boundary value problem on a bounded, Lipschitz domain $\Omega \subset \mathbb{R}^n$:

$$\begin{cases} -\Delta u_j = \mu_j u_j & \text{in } \Omega, \\ \partial u_j / \partial n = 0 & \text{on } \partial\Omega. \end{cases}$$

Here $\mu_0 = 0$ with corresponding eigenfunction that is constant. For planar domains, let A be the area of the domain, and L be the perimeter.

(1) *(Richard Laugesen)* Fix $k \geq 4$ and maximize $\mu_1 A$ over all k-gons Ω. Is the maximizer the regular k-gon? The triangular case $k = 3$ was proved recently by R. S. Laugesen and B. Siudeja [3]; they also considered harmonic, geometric and arithmetic means of the first two nonzero eigenvalues, μ_1 and μ_2, under various normalizations on the triangles.

Recall that for general domains in all dimensions, Szegő [6] and Weinberger [7] proved the disk/ball maximizes μ_1 subject to area/volume constraint, and that among simply connected domains in the plane, the disk maximizes the harmonic mean $2\,(\mu_1^{-1} + \mu_2^{-1})^{-1}$.

(2) *(Iosif Polterovich)* For planar domains, maximize $\sqrt{\mu_1 \mu_2}\, A$, the geometric mean of the first two positive eigenvalues, subject to area constraint. It is conjectured that $\sqrt{\mu_1 \mu_2}\, A \leq \pi \mu_1(D)$, where D is the unit disk, with the equality if and only if Ω is a disk [4]. Such a result would strengthen the Szegő's harmonic mean bound mentioned above. For simply–connected domains, the estimate on the geometric mean with an extra factor of $\sqrt{2}$ follows from [6] and [2]. Note that the equilateral triangle maximizes $\sqrt{\mu_1 \mu_2}\, A$ among all triangles [5].

(3) *(Richard Laugesen)* Maximize $\mu_1 L^2$ among convex plane domains. Are the maximizers the equilateral triangle and the square? Note that both these polygons yield the same value (is that a coincidence? or is there an underlying reason?), and that the disk yields a much lower value. The maximizer among triangles is equilateral, as is proved in the paper of Laugesen and Siudeja cited above.

(4) *(Dorin Bucur)* Assume that $(\Omega_n)_n \subset \mathbb{R}^N$ is a sequence of simply connected subsets of the unit ball B_1, and $\Omega^* \subset B_1$ such that
- $|\Omega_n| = m$ and $|\Omega^*| < m$;
- $\overline{B}_1 \setminus \Omega_n$ converges in the Hausdorff metric to $\overline{B}_1 \setminus \Omega^*$.

For every $k \in \mathbb{N}$ prove that

$$\limsup_{n\to\infty} \mu_k(\Omega_n) < \sup_{|\Omega|=m} \mu_k(\Omega).$$

This problem asserts that a sequence of domains converging in the Hausdorff metric and "losing measure" cannot be a maximizing sequence for the isoperimetric inequality associated to the k-th eigenvalue of the Neumann Laplacian (see [1]).

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3. Gap inequalities for triangles *(Richard Laugesen)*

Let Ω be a convex planar domain of diameter d. As above, $\lambda_1, \lambda_2, \dots$ are Dirichlet eigenvalues of Ω and $\mu_0 = 0, \mu_1, \dots$ are Neumann eigenvalues. Van den Berg's Gap Conjecture (1983) states that

$$(\lambda_2 - \lambda_1)d^2 \geq 3\pi^2.$$

The best known bound on the right hand side is π^2. Background information and many open problems related to the Gap Conjecture (including problems with convex potential) can be found in [1], see also section 10 of the current document.

Problem: minimize $(\lambda_2 - \lambda_1)d^2$ among triangles. The equilateral triangle is conjectured to be minimal; see [2]. Thus the Gap Conjecture for triangles differs from the case of general domains, where the minimizer is conjectured to be degenerate (a degenerate rectangular box).

The analogous Gap Problem for Neumann boundary conditions is to minimize $\mu_1 d^2$. For general (convex) domains the sharp lower bound of π^2 was found by L. E. Payne and H. F. Weinberger [4]. For triangles the minimizer is the degenerate acute isosceles triangle, as was recently proved in [3].

For Robin boundary conditions, the appropriate Gap Conjecture is discussed in [1]. Note that among triangles, the minimizer should presumably be isosceles, but whether that minimizer should be equilateral or degenerate acute isosceles is not yet clear. The answer might perhaps depend on the Robin parameter. Numerical studies are needed.

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4. Mixed eigenvalue problems *(Almut Burchard)*

Walter Craig's window placement problem: How should windows of a given size be positioned on the walls of a room to minimize long-term heat loss?

Let $\Omega \subset \mathbb{R}^n$ stand for the room, and $D \subset \partial\Omega$ the window, and $\partial\Omega \setminus D$ the insulating walls. Denote by Δ_D the Laplacian on Ω, with Dirichlet boundary conditions on D and Neumann boundary conditions on $\partial\Omega \setminus D$. The objective is to minimize the lowest eigenvalue $\lambda(D)$ of $-\Delta_D$ among all subsets $D \subset \partial\Omega$ of prescribed $(n-1)$–dimensional surface measure.

Conjectures: If Ω is smooth, then the minimizing window D should be connected and sufficiently regular so that the minimizing eigenfunction is continuous up to the boundary. Sufficiently small optimal windows should straddle a boundary point of maximal curvature. In the special case of a square, the optimal D should be an interval, centered either at a corner, or at the middle of a side (depending on the prescribed length of D).

Known results: The principal eigenvalue $\lambda(D)$ approaches its supremum (the first Dirichlet eigenvalue of Ω) if D is smeared out over $\partial\Omega$. Minimizing windows are known to exist for arbitrary Lipschitz domains. There are examples of non-convex domains where the minimizing D must be disconnected. If Ω is a ball, then the optimal D is a spherical cap. Partial results are known for the square.

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5. Isoperimetric inequalities as variational problems *(Nikolai Nadirashvili)*

From the general point of view, isoperimetric problems in mathematical physics are regular variational problems with constraints of special types. Here we discuss some questions motivated by this "unified" approach to isoperimetric inequalities. The central problems of the calculus of variations are existence, regularity and uniqueness of the minimizer (in the context of isoperimetric problems, the latter usually follows from certain symmetry properties). Typically, isoperimetric problems have the structure of a double variational problem: the physical quantity itself has a variational nature (like an eigenvalue of the Laplacian), and then one optimizes it under a certain constraint, say, the area of a domain. This naturally leads to overdetermined problems, e.g., for the first Dirichlet eigenvalue we are getting Serrin type conditions on the boundary. It is known from the general theory that the problems with the free boundary have smooth solutions with at most algebraic singularities. This motivates the following

Question 1. *Let M be a smooth complete 2-dimensional manifold. Does there exist a smooth domain which minimizes the first Dirichlet eigenvalue among all domains of the same area?*

On the space of plane domains of fixed area one can introduce a metric, taking as the distance between two domains the area of their symmetric difference. In this way, we can define local minimizers of the first Dirichlet eigenvalue on the space of plane domains.

Question 2. *Is a local minimizer of the first Dirichlet eigenvalue necessarily a smooth domain?*

The positive answer to Question 2 will imply the following generalization of the classical Faber–Krahn theorem: the disk is the unique local minimizer for the first Dirichlet eigenvalue. Let us remark that for smooth perturbations, the extremal property of the disk was proved in the original work of Rayleigh.

Notice that the existence of any minimizer in the problems of isoperimetric type is not clear at all. The difficulty is that the problem is not "coercive". By this we mean that the value of the first Dirichlet eigenvalue does not imply any regularity of the domain. Moreover, the set of domains of fixed area having the same first Dirichlet eigenvalue is not even closed under γ-convergence (see [2, p. 18] for the definition). Thus it is very interesting to find isoperimetric problems having "weak coercivity".

Conjecture 3. *Let M be a topological 2-sphere endowed with a Riemannian metric g. Suppose that (M,g) has unit area and let $\lambda_1(g)$ be the first nonzero eigenvalue of the Laplacian on (M,g). Then (M,g) is isometric to $(S^2, s\,g_0)$, where g_0 is the standard metric on the sphere S^2, and the conformal factor s satisfies*

$$||s||_{H^{-1}(S^2,g)} \leq C_{\lambda_1(g)}.$$

In other words, we conjecture that all conformal factors corresponding to metrics on S^2 of unit area and with the same first nonzero eigenvalue are uniformly bounded in the norm of the space *dual* to the Sobolev space $H^1(S^2, g)$ (hence we would get *weak* coercivity). Note that the conformal structure of the two–dimensional sphere is trivial: all Riemannian metrics are conformal to the standard round metric.

There are classical variational problems which admit singular solutions. Typically, such a solution has singularities of algebraic type. For minimizers of an isoperimetric problem it can be different. For example, let λ_1 be the first Dirichlet eigenvalue of a planar domain and α_1 be the first Robin eigenvalue (see [2, p. 107]) with a fixed positive Robin constant. Then the infimum of the ratio λ_1/α_1 over the set of all plane domains is equal to one. It is attained in the limit by a sequence of domains with increasingly oscillating boundaries [1]. This example motivates the following

Question 4. *Can an isoperimetric problem for eigenvalues have a fractal solution?*

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6. Four unknown constants (*Rodrigo Bañuelos*)

6.1. **Four inequalities.** Let $\Omega \subset \mathbb{R}^2$ be an arbitrary simply connected domain in the plane. We define $R_\Omega = \sup_{z\in\Omega} d_\Omega(z)$ (the inradius of the domain) where $d_\Omega(z)$ is the distance from z to the boundary of Ω. Let $\sigma_\Omega(z)$ be the density of the hyperbolic metric in Ω and let $\sigma_\Omega = \inf_{z\in\Omega} \sigma_\Omega(z)$. Finally, denote by λ_1 the lowest eigenvalue for the Dirichlet Laplacian in Ω and denote by τ_Ω the first exit time of Brownian motion from Ω. The following four inequalities hold.[1]

(1) There exists a positive constant C_1, independent of the domain, such that for all functions $u \in C_0^\infty(\Omega)$

$$\int_\Omega \frac{|u|^2}{d_\Omega^2} \leq C_1 \int_\Omega |\nabla u|^2. \tag{1}$$

This inequality is known as the "Hardy" inequality in the literature. It holds for domains which are more general than simply connected but does not hold for all domains, see [2]. The survey paper [11] contains a detailed account of this inequality as of around 1998. For some recent work, please see [1], [12], [13], [15], [17], [19] and references therein. In the setting of simply connected domains the inequality can be easily reduced to that of the unit disc or half–space with the aid of the Koebe $\frac{1}{4}$-theorem. In fact, the Koebe $\frac{1}{4}$-theorem proof gives the inequality with $C_1 = 16$, (see [2]).

(2) There exists a positive constant C_2, independent of the domain, such that

$$\frac{C_2}{R_\Omega^2} \leq \lambda_1 \leq \frac{j_0^2}{R_\Omega^2}. \tag{2}$$

The right hand side inequality is trivial by domain monotonicity of the eigenvalue–the larger the domain the smaller the eigenvalue. The constant j_0 is the smallest positive zero of the first Bessel function J_0. Of course,

[1] Many thanks to Ari Laptev and Tom Carroll for pointing out several recent papers related to these constants.

the right hand side inequality is sharp. The left hand side inequality follows from the variational characterization of the eigenvalue and the Hardy inequality (1). As above, the left hand side inequality holds for more general domains than just simply connected domains but not all. (Adding points to a domain has no affect on the eigenvalue but it can have a drastic affect on the inradius.) This inequality also has a long and interesting history, see [3] and [4].

(3) There exists a positive constant C_3, independent of the domain, such that

$$\frac{1}{2}R_\Omega^2 \le \sup_{z\in\Omega} E_z\left(\tau_\Omega\right) \le C_3 R_\Omega^2. \tag{3}$$

Here we use E_z to denote the expectation with respect to the Brownian motion starting at the point $z \in \Omega$. Again, the lower bound is trivial by domain monotonicity (the larger the domain the larger the lifetime). A necessary and sufficient condition (which includes all simply connected domains in $\mathbb{R}^2$) for a domain in $\mathbb{R}^d$ to have (3) is given in [8]. Again, since Brownian motion does not "see" points in two dimensions, the right hand side inequality cannot hold for all domains.

(4) There exist a positive constant C_4, independent of the domain, such that

$$\frac{C_4}{R_\Omega} \le \sigma_\Omega \le \frac{1}{R_\Omega}. \tag{4}$$

As above, the upper bound is obtained by domain monotonicity and the existence of the constant C_4 follows at once from the Koebe $\frac{1}{4}$-theorem since $\sigma_\Omega(z) = \frac{1}{|F'(0)|}$, where F is the conformal mapping from the unit disc onto the domain Ω with $F(0) = z$.

Problem 1. Identify the extremal constants C_1, C_2, C_3, C_4 in the above inequalities and the geometry of the "extremal" domains (whenever they exist).

6.2. **Convex domains.** In the case of convex domains, all constants are known:

(1) $C_1(convex) = 4$ which is the constant for the half space (or oven the one dimensional half-line). For a proof of this, see Davies [11]. There are also other sharper generalizations such as the one given in [1]. (Please also consult references given in [1] for more on these kind of extensions.) These results hold for convex domains in $\mathbb{R}^d$.
(2) $C_2(convex) = \pi^2/4$ and the extremal domain is an infinite strip. The same constant works also for any convex domain in $\mathbb{R}^d$. There are several proofs of this result including the original one given by J. Hersh in [14]. (See also [1] for a proof based on the Hardy inequality and other references.)
(3) $C_3(convex) = 1$ (see R. Sperb in [18]). Again, the extremal is given by an infinite strip (which reduces the problem to an interval). Here again, there

is a more general inequality which asserts that for any convex domain in $\mathbb{R}^d$ of inradius R_Ω,

(5) $$P_z\{\tau_\Omega > t\} \leq P_0\{\tau_{(-R_\Omega, R_\Omega)} > t\},$$

where $\tau_{(-R_\Omega, R_\Omega)}$ is the exit time from the interval $(-R_\Omega, R_\Omega)$ on the real line. (For this, see [6] and [7].) The inequality (5) together with the well-known classical characterization of the the eigenvalue as

$$-\lambda_1 = \lim_{t\to\infty} \frac{1}{t} \log P_z\{\tau_\Omega > t\}$$

gives a different proof that $C_2(convex) = \pi^2/4$. Again, the same results holds in all dimensions where the extremal domain is the infinite slab.

(4) $C_4(convex) = \pi/4$. This result was proved by Szegö in 1923 (see [3] for exact reference). Again, the extremal domain is the infinite strip.

6.3. **Arbitrary simply connected domains.** The following estimates for the optimal constants C_1, C_2, C_3, C_4 are known.

(6) $$4 \leq C_1 \leq 16$$

(7) $$0.6194 < C_2 < 2.095$$

(8) $$1.584 < C_3 < 3.228$$

(9) $$0.57088 < C_4 < 0.6563937$$

For the estimates for C_2 and C_3, and some history on these constants, we refer the reader to [3] and [9]. The paper [3] also contains some examples of simply connected domains which we conjecture are very close to the extremals for these four problems. The problem of determining the best constant C_4 (known as the Schlicht Bloch-Landau constant) has a long history in function theory. For the above estimates on C_4 we refer the reader to [16] and [10] and [9]. (The reference [10] contains many references to the literature on the Schlicht Bloch-Landau constant.) The upper estimate for C_3 follow from the lower estimate on C_4 and inequality (10) below. From the upper estimate on C_3 we get a lower estimate on C_2 using (11). The lower estimate for C_3 and upper estimate on C_2 follow from the example in [3], (see Theorems 2 and 3) and the calculations in [9]. For an approach using a Hardy-type inequality with σ_Ω replacing the distance function, see [5].

Theorem 1 ([3])**.** For any simply connected domain $\Omega \subset \mathbb{R}^2$, we have

(10) $$\frac{1}{2\sigma_\Omega^2} \leq \sup_{z\in\Omega} E_z(\tau_\Omega) \leq \frac{7\zeta(3)}{8\sigma_\Omega^2}$$

and

(11) $$\frac{2}{\sup_{z\in\Omega} E_z(\tau_\Omega)} \leq \lambda_\Omega \leq \frac{7\zeta(3) j_0^2}{8 \sup_{z\in\Omega} E_z(\tau_\Omega)},$$

where $7\zeta(3)/8 = \sum_{n=0}^{\infty} (2n+1)^{-3}$.

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7. The Isoperimetric Inequality for Riesz Capacities: A problem of P. Mattila

The Riesz kernels in $\mathbb{R}^d$, $d \geq 2$, are given by

$$K_\alpha(x,y) = \frac{\Gamma\left(\frac{d-\alpha}{2}\right)}{\Gamma(\frac{\alpha}{2})\pi^{d/2}2^{\alpha-1}} \frac{1}{|x-y|^{d-\alpha}},$$

for any $0 < \alpha < d$. Given a compact set $A \subset \mathbb{R}^d$, we define its α–Riesz capacity by

$$C_\alpha(A) = \left[\inf_\mu \int \int K_\alpha(x,y) d\mu(x) d\mu(y)\right]^{-1}$$

where the infimum is taken over all probability Borel measures supported in A. If $\alpha = 2$ and $d = 3$, this is the classic Newtonian capacity. Let $|A|$ denote the Lebesgue measure of the set A and let A^* be the ball centered at the origin and with same Lebesgue measure as A. Then the classical Pólya-Szegö inequality implies that

$$C_2(A) \geq C_2(A^*)$$

with equality if and only if A is ball.

Naturally one might ask if this inequality holds for all Riesz capacities of any order $0 < \alpha < d$. This problem was raised by P. Mattila in his paper "Orthogonal Projections, Riesz Capacities" (Indiana Univ. Math. J. 39, (1990),185-198). The case $0 < \alpha < 2$ was proved by D. Betsakos in "Symmetrization, symmetric stable processes, and Riesz capacities," Trans. Amer. Math. Soc., (2004), 735–755) using polarization inequalities for transition densities of killed symmetric stable processes and a well–known relationship between Green's functions and Riesz capacities. In "An Isoperimetric Inequality for Riesz Capacities," (Rocky Mountain J. Math. (2006), 675–682), P. J. Méndez-Hernández gives an alternative proof of Betsakos' result using the rearrangement techniques of Brascamp–Lieb–Luttinger and a characterization of capacities given by F. Spitzer in 1964 in terms of hitting times. (Spitzer's characterizations was originally proved for Brownian motion.) However, as it turns out, in 1983, in "The Isoperimetric Inequality for Isotropic Unimodal Lévy Processes," Z. Wahrscheinlichkeitstheorie (1983), 487–499, T. Watanabe had already proved a more general isoperimetric inequality for capacities of Lévy processes which implies, in particular, the result for $0 < \alpha \leq 2$. What remains open is the case of $2 < \alpha < d$. The probabilistic techniques do not apply to this case, at least not in the most "obvious" and direct way.

8. Pólya and Related Inequalities

Consider eigenvalues of the Dirichlet Laplacian on a bounded domain $\Omega \subset \mathbb{R}^n$:

$$\begin{cases} -\Delta u_j = E_j u_j & \text{in } \Omega, \\ u_j = 0 & \text{on } \partial\Omega. \end{cases}$$

Assume $n \geq 2$.

(1) *(Michael Loss)* The Pólya Conjecture claims that the Weyl asymptotic formula provides a lower bound:

$$E_j \geq (2\pi)^2 (n/|S^{n-1}||\Omega|)^{2/n} j^{2/n}, \qquad j = 1, 2, 3, \ldots.$$

The conjecture remains open even for $j = 3$.

The best partial result known is with a factor of $n/(n+2)$ (which is less than 1) on the righthand side, as one deduces by estimating $E_j \leq E_J$ in the Li-Yau result below.

(2) *(Timo Weidl)* Berezin proved in 1972 that

$$\sum_j (E - E_j)_+^\sigma \leq \frac{|\Omega|}{(2\pi)^n} \int_{\mathbb{R}^n} (E - |p|^2)_+^\sigma \, dp, \qquad \sigma \geq 1, \quad E > 0.$$

The cases $0 \leq \sigma < 1$ remain open. The Pólya conjecture is exactly the case $\sigma = 0$.

Note that the case $\sigma = 1$ implies the Li–Yau inequality

$$\sum_{j=1}^J E_j \geq \frac{n}{n+2} (2\pi)^2 (n/|S^{n-1}||\Omega|)^{2/n} J^{(n+2)/n}, \quad J = 1, 2, 3, \ldots .$$

The analogues of the Pólya and Li–Yau inequalities under Neumann boundary conditions are obtained simply by reversing the inequality in the Dirichlet cases above. The Pólya Conjecture remains open for Neumann boundary conditions, whereas the analogue of Li–Yau was proved by Pawel Kröger (1992). We do not know whether there exists a stronger Berezin–type result for the Neumann problem. For more information, see [8].

(3) *(Timo Weidl)* The Li–Yau result on Dirichlet eigenvalues extends to Hamiltonians with arbitrary magnetic field, as explained in the Lieb–Thirring section below. For more information and some progress see Item 10 below.

(4) *(Timo Weidl)* Can one strengthen the Li–Yau result by including a correction term, perhaps involving the surface area of the boundary? (There is a result by Melas of Li–Yau type with corrections involving *moments of inertia* rather than surface area, see [9]. Elliott Lieb says that this can be done for the discrete Laplacian on domains in a lattice, see[1].

February 2009: The result of Melas has been strengthened by inclusion of a correction term involving the surface area of the boundary, see [7]. An earlier improvement for $\gamma \geq 3/2$, involving a notion of effective boundary, is due to T. Weidl, [10]. For $\gamma < 3/2$ further improvements seem possible and desirable.

(5) *(Evans Harrell, Joachim Stubbe) Problems on eigenvalues of Schrödinger operators related to commutator methods*

Problem - corrections to Weyl type estimates. The trace identity of [3], [4] implies as shown in [2] that for $\sigma \geq 2$ the mapping

$$r_\sigma : E \mapsto E^{-\sigma - d/2} \left(\frac{|\Omega|}{(2\pi)^n} \int_{\mathbb{R}^n} (E - |p|^2)^\sigma - \sum_j (E - E_j)_+^\sigma \right) \tag{1}$$

is non increasing. According to Weyl's asymptotic formula $r_\sigma(E)$ tends to zero as E tends to infinity and therefore $r_\sigma(E) \geq 0$ which is the Berezin-Li-Yau-inequality. Can one strengthen this bound in the trace identity of

[3],[4] to obtain correction terms involving the surface area of the boundary? For the Laplacian with periodic boundary conditions a similar monotonicity property holds (see [5] for details). In this case the search for the correction term is related to the famous Gauss circle problem (or lattice point problem).

Problem - higher order operators, fractional powers of Laplacians etc. Prove monotonicity results like (1) for higher order operators (e.g. clamped plate problem) and fractional powers of Laplacians (see [6] for some results on $\sqrt{-\Delta}$) leading to Berezin-Li-Yau inequalities for these operators.

Problem - universal inequalities and Weyl type bounds. For $p > 0$ let

$$M_p(J) := \left(\frac{n+2p}{n} \frac{1}{J} \sum_{j=1}^{J} E_j^p \right)^{\frac{1}{p}} \tag{2}$$

and for $p = 0$ define

$$M_0(J) := e^{\frac{2}{n}} \left(\prod_{j=1}^{J} E_j \right)^{\frac{1}{J}}. \tag{3}$$

According the the Weyl asymptotic formula, for all $p \geq 0$,

$$M_p(J) \sim (2\pi)^2 (n/|S^{n-1}||\Omega|)^{2/n} J^{2/n}$$

as $J \to \infty$. In [3] it has been shown that

$$M_1^2(J) - M_2(J) \geq \frac{1}{4}(E_{J+1} - E_J)^2 (\geq 0)$$

and

$$M_1(J) - \sqrt{M_1^2(J) - M_2(J)} \leq E_J \leq E_{J+1} \leq M_1(J) + \sqrt{M_1^2(J) - M_2(J)}.$$

Both inequalities are sharp in the Weyl limit. For extensions to other $M_p(J)$ see [4]. For $p > 0$ find an upper bound of the form

$$M_p^{2p}(J) - M_{2p}^p(J) \leq C(p, \Omega) E_1^{2p} J^{2p\kappa}$$

with $\kappa < 2/n$.

Problem - universal inequalities. With the above notations does

$$E_J \leq M_p(J)$$

hold for all J and all $p \geq 0$?
Can one find Ω and J such that the inequality

$$M_1^2(J) - M_2(J) \geq \frac{1}{4}(E_{J+1} - E_J)^2$$

is saturated?

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9. Lieb–Thirring Inequalities

Write $E_1 < E_2 \le E_3 \le \cdots \le 0$ for the eigenvalues of $-\Delta - V$ on $L^2(\mathbb{R}^n)$, meaning

$$(-\Delta - V)u_j = E_j u_j.$$

The eigenfunctions u_j represent bound states with energies E_j. For simplicity we assume $V \ge 0$. Assume $n \ge 1$.

The Lieb–Thirring inequality can be written as

$$\sum_j |E_j|^\gamma \le L_{n,\gamma} \int_{\mathbb{R}^n} V^{\gamma+n/2}\, dx,$$

This inequality holds (with a constant $L_{n,\gamma}$ independent of V) iff the parameter γ satisfies $\gamma \ge 1/2$ if $n = 1$, $\gamma > 0$ if $n = 2$ and $\gamma \ge 0$ if $n \ge 3$. The case $\gamma = 0$ (counting eigenvalues) is the Cwikel–Lieb–Rozenblum Inequality (CLR).

In other words

$$\mathrm{Tr}\,(-\Delta - V)_-^\gamma \le \frac{C_{n,\gamma}}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (|p|^2 - V(x))_-^\gamma\, dpdx,$$

where

$$C_{n,\gamma} = \frac{L_{n,\gamma}}{L_{n,\gamma}^{\mathrm{cl}}} \qquad \text{and} \quad L_{n,\gamma}^{\mathrm{cl}} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (|p|^2 - 1)_-^\gamma\, dp.$$

The constant $L_{n,\gamma}^{\mathrm{cl}}$ is called the semiclassical Lieb–Thirring constant. Note that $C_{n,\gamma} \ge 1$ always, by the Weyl asymptotics, and that $C_{n,\gamma}$ is decreasing in γ for each fixed n, by the Aizenman–Lieb monotonicity result.

To start with, let us summarize some known results on the constants $C_{n,\gamma}$, along with conjectures about best (smallest) values of $C_{n,\gamma}$.

n	γ	Best known $C_{n,\gamma}$	Best constant?	status	last updated
1	$\frac{1}{2}$	2	2	known	
	$(\frac{1}{2}, \frac{3}{2})$	2*	$2\left(\frac{\gamma-1/2}{\gamma+1/2}\right)^{\gamma-1/2}$	conjectured	
	$[\frac{3}{2}, \infty)$	1	1	known	
2	$(0, \frac{1}{2})$	?			
	$[\frac{1}{2}, 1)$	3.64			Feb. 2009
	$[1, \frac{3}{2})$	1.82			Feb. 2009
	$[\frac{3}{2}, \infty)$	1	1	known	
$= 3$	$[0, \frac{1}{2})$	6.87	$8/\sqrt{3} \simeq 4.62$	conjectured	
	$[\frac{1}{2}, 1)$	3.64			Feb. 2009
	$[1, \frac{3}{2})$	1.82	1	conjectured	Feb. 2009
	$[\frac{3}{2}, \infty)$	1	1	known	
≥ 4	$[0, \frac{1}{2})$	10.34			Feb. 2009
	$[\frac{1}{2}, 1)$	3.64			Feb. 2009
	$[1, \frac{3}{2})$	1.82	1	conjectured	Feb. 2009
	$[\frac{3}{2}, \infty)$	1	1	known	

*better is known for $\gamma \in [1, \frac{3}{2})$, *e.g.* $C_{1,1} \leq \frac{\pi}{\sqrt{3}} \simeq 1.82$ via work of Eden–Foias.

Remark. References to the results in the table and to many of the questions below can be found in the lecture notes by Michael Loss and Timo Weidl, and in the survey paper by Dirk Hundertmark (which further states some better estimates on $C_{n,\gamma}$ for special values of n and γ).

For February 2009 updates see [6] and [13].

Now we state open problems on Lieb–Thirring inequalities.

(1) *(Richard Laugesen)* Must an optimal potential V exist, for those Lieb–Thirring inequalities in which the best constant is not known? In particular this question is open for $n = 1$ and $\frac{1}{2} < \gamma < \frac{3}{2}$.

A restricted version of the problem asks: within the class of potentials having m bound states (where $m \geq 1$ is given), does an optimal potential exist?

(2) *(Richard Laugesen)* If an optimal potential exists, then does it have just a single bound state? (In other words, does $-\Delta - V$ have just a single eigenvalue?) When $n = 1$ and $\frac{1}{2} < \gamma < \frac{3}{2}$, the natural conjecture is that the optimal potential is the one found by J. B. Keller when he determined the best constant in $|E_1|^\gamma \leq C \int_{\mathbb{R}} V^{\gamma+1/2}\, dx$, see [24].

This "single bound state" conjecture is due to Lieb and Thirring, 1976. In dimension $n = 1$, the conjecture is known to be true in the endpoint cases $\gamma = 1/2$ (in which case V is a delta function) and $\gamma = 3/2$ (in which case V is a transparent or reflectionless potential).

(3) *(Eric Carlen)* Does there exist a bound of the form $\sum_j |E_j|^\gamma \leq C|E_1|^\gamma$? Here the factor C could depend on n, γ, and on the integrability of a power of V sufficient to guarantee that the lefthand side is finite.

(4) *(Rafael Benguria)* The use of Korteweg–de Vries (KdV) integrable system methods when $n = 1, \gamma = 3/2$, suggests that one might similarly study Lieb–Thirring inequalities for the linear equation associated with the Benjamin–Ono equation (another integrable system). Tomas Ekholm, Rupert Frank and Dirk Hundertmark made progress during the Workshop already, by obtaining the analog of the Aizenman–Lieb "monotonicity toward best constants" result. The Lax pair for the Benjamin–Ono equation can be found for example in [1], see also [23]. D.J. Kaup and Y. Matsuno, *The inverse scattering transform for the Benjamin–Ono equation*, Studies in applied mathematics 101 (1998), 73–98.

(5) *(Rupert Frank)* The best constant when $n = 1, \gamma = 1$, is due to Eden–Foias, see [7]. More precisely, they proved a Sobolev inequality, which then gives a Lieb–Thirring inequality via the Legendre transform. So a question is: can one find a more direct proof of this Lieb–Thirring inequality?

Also, can one sharpen the Eden–Foias bound by including correction terms in their argument?

February 2009: An operator-valued version of the Eden–Foias bound has been proved by J. Dolbeault, A. Laptev, M. Loss, see [6]. By the 'lifting of dimension'-argument this result leads to the best known values for the constants in the Lieb–Thirring inequalities for $\gamma \geq 1$ if $n = 1$ and for $\gamma \geq 1/2$ if $n \geq 2$.

(6) *(Timo Weidl)* Can one find a way to directly estimate the sum of the eigenvalues, without going through the Birman–Schwinger transformation (which *counts* the eigenvalues rather than summing them)?

(7) *(Almut Burchard) The Ovals Problem.* Consider a smooth closed curve γ of length 2π in $\mathbb{R}^3$, and let $\kappa(s)$ be its curvature as a function of arclength. The curve determines the one-dimensional Schrödinger operator $H_C = -d^2/ds^2 + \kappa^2$ acting on 2π-periodic functions. This operator appears in the equation for the tension of a smooth, elastic, inextensible loop [4], and in connection with a Lieb–Thirring inequality in one dimension [3]; similar Schrödinger operators with quadratic curvature potentials have been studied in connection with quantum mechanics on narrow channels [11], Dirac operators on the sphere [15], and curvature-driven flows describing the motion of interfaces in reaction-diffusion equations [17].

A natural conjecture is that the principal eigenvalue $e(\gamma)$ is minimal when γ is a circle, where it takes the value 1. This question is open even for planar loops that enclose convex sets (*'ovals'*). It is known that the value $e(\gamma) = 1$ is attained for an entire family of planar curves whose curvature is given by $\kappa(s) = \big(\alpha^2 \cos^2 s + \alpha^{-2} \sin^2 s\big)^{-1}$. When $\alpha \to 0$, these curves collapse onto two straight line segments of length π joined at the ends. The inequality $e(\gamma) \geq 1$ has recently been shown for curves in some neighborhood of the family [4], and for curves satisfying additional geometric constraints [28]. The best universal lower bound on $e(\gamma)$ that is currently known is .6085 [28].

Several participants at the Workshop had worked on this problem previously (including Benguria, Loss, Burchard, Thomas, and Linde). All agreed that classical Calculus of Variations techniques may be exhausted at this point, and that rearrangement techniques seem to fail. Linde and Burchard claimed that minimizers can be shown to exist, and should be convex, but could conceivably contain one corner, or two corners joined by a straight line segment. Benguria pointed to the family of putative minimizers (which look like ellipses in polar coordinates) as evidence that the problem may have a hidden affine symmetry. Carlen, Mazzeo, and Benguria proposed to search for geometric flows that drive $e(\gamma)$ towards its minimum. The affine curvature flow [2] was mentioned as a promising candidate. Rapti and Lee proposed to analyze the Euler–Lagrange equation using ODE methods. Laugesen suggested applying the Birman–Schwinger transformation, after which the conjecture becomes that the largest eigenvalue of the operator $T = \kappa(d^2/ds^2 + \gamma)^{-1}\kappa$ is larger than 1, for each constant $0 < \gamma < 1$. Equivalently, take $\gamma = 1$ and try to show the largest eigenvalue of T is larger than 1, when T acts on functions ψ with $\kappa\psi$ orthogonal to $\sin s$ and $\cos s$. The hope is that a good choice of trial function (in the variational principle for the largest eigenvalue) might suffice to prove this conjecture.

(8) *(Timo Weidl)* For $n = 2, \gamma = 0$, can one prove a Cwikel–Lieb–Rozenblum Inequality that involves a logarithmic correction factor? Without some such correction factor, the inequality fails, since any nontrivial attractive potential has at least one bound state.

February 2009: This problem has been solved in [25].

(9) *(Timo Weidl)* Can one obtain improved Lieb–Thirring constants when working on a domain Ω rather than on all of $\mathbb{R}^n$? For example, can one obtain a boundary correction term?

(10) *(Timo Weidl) Magnetic Schrödinger operators on a domain.* Consider the Dirichlet Laplacian in a domain in $\mathbb{R}^n$. The technique of iteration-in-dimension gives sharp Lieb–Thirring constants for arbitrary magnetic fields for $\gamma \geq 3/2$ and any $n \geq 2$. (See the final part of [26].) For $1/2 \leq \gamma < 3/2$ one also gets estimates uniform in the magnetic field, but the constant is (probably) not sharp. With the same approach, the results

of D. Hundertmark, A. Laptev and T. Weidl, [19], carry over to magnetic operators; see the remark at the end of that paper.

The sharp Li–Yau bound (corresponding to $\gamma = 1$) has been proved by L. Erdös, M. Loss and V. Vougalter, [10], for constant magnetic fields. Does this bound hold true for *arbitrary* magnetic fields for $1 \le \gamma < 3/2$?

For $\gamma = 0$, does the Pólya conjecture hold true for tiling domains in the presence of magnetic fields?

February 2009: The answer to the latter question is negative for constant magnetic fields. Indeed, the sharp constant in the corresponding lower bound for $0 \le \gamma < 1$ was found in [14].

(11) *(Timo Weidl) Magnetic Schrödinger operators on* $\mathbb{R}^n$*.* Consider Lieb–Thirring bounds for magnetic Schrödinger operators on all of $\mathbb{R}^n$. In all cases where the sharp constant is known, either the magnetic field is not relevant (dimension $n = 1$) or the value of the constant is independent of the magnetic field ($\gamma \ge 3/2$ and $n \ge 2$ as above, where the sharp constant equals the classical constant).

Can the magnetic field change the optimal value of the Lieb–Thirring constant in the remaining cases? (February 2009: The magnetic field can change the optimal value at most by an explicit factor depending only on γ and d; see [12].

This question is rather speculative, because we do not know the sharp constants even in the non-magnetic case. But let us put forward the following more specific version:

Can one construct a counterexample to the Lieb–Thirring conjecture that the optimal constant is the classical one for $n = 3, \gamma = 1$, by using a suitable magnetic field?

(12) *(Eric Carlen) Generalization to manifolds.* Do there exist Lieb–Thirring inequalities on manifolds? As a basic first question, do the critical exponents ($\gamma = \frac{1}{2}$ when $n = 1$, and $\gamma = 0$ when $n = 2$) depend on the geometry?

Some references to get started here are [20] and [21]. A classic reference for applications to turbulence is [27].

February 2009: Intuition from recent results on continuous trees suggest that the critical exponents depend on both the local and global dimension of the manifold, see [9].

Sharp Lieb-Thirring inequalities on manifolds of bounded mean curvature and for periodic Schrödinger operators are proven in [18] when $\gamma \ge 2$. For applications to turbulence one should study the Stokes operator. For a Berezin-Li-Yau inequality on flat domains with Dirichlet boundary conditions see [22].

(13) *(Mark Ashbaugh) Reverse Lieb–Thirring Inequality.* For dimension $n = 1$, D. Damanik and C. Remling have proved a Reverse Lieb–Thirring Inequality in the subcritical range $0 < \gamma \le \frac{1}{2}$, see [5]. Sharp constants seem not

to be known. A Reverse Cwikel–Lieb–Rozenblum Inequality for the eigenvalue counting function for dimension $n = 2$ in the critical case $\gamma = 0$ has been proved by A. Grigor'yan, Yu. Netrusov, S.-T. Yau, [16].

(14) *(Rupert Frank) Powers of the Laplacian.* Can one prove a critical Lieb–Thirring inequality for arbitrary powers of the Laplacian? That is, one wants

$$\mathrm{tr}\ ((-\Delta)^s - V)_-^\gamma \le L_{\gamma,n} \int_{\mathbb{R}^n} V_+^{\gamma+n/2s}\, dx$$

for $\gamma = 1 - n/2s > 0$. Such an inequality is known for s a positive integer by work of Netrusov–Weidl.

Timo Weidl remarked that regardless of whether these operators have physical significance, the higher order situation can help shed light on what makes the second-order case work.

(15) *(Rupert Frank) Hardy–Lieb–Thirring Inequality.* Can one prove a Lieb–Thirring bound with a Hardy weight, on the half-line? That is, one wants

$$\mathrm{tr}\left(-\frac{d^2}{dr^2} - \frac{1}{4r^2} - V\right)_-^{\theta/2} \le C_\theta \int_0^\infty V(r) r^{1-\theta}\, dr$$

for $0 < \theta \le 1$. The inequality is known for $\theta = 1$ (Lieb–Thirring). For $\theta = 0$ it fails (although note that if it were true, it would resemble Bargmann's inequality).

February 2009: The inequality for all $0 < \theta \le 1$ has been proved in [8]. The sharp constant C_θ is not known, and there is not even a conjecture for it.

(16) *(Carlo Morpurgo) Cwikel–Lieb–Rozenblum bounds and heat kernel inequalities.*

Let Y be the Yamabe operator, or conformal Laplacian, on the euclidean "round" sphere (S^n, g). That is, $Y = \Delta_{S^n} + \frac{n}{2}\left(\frac{n}{2} - 1\right)$, where Δ_{S^n} denotes the Laplace–Beltrami operator on S^n.

Consider a positive smooth function W on S^n, normalized so that $\int_{S^n} W^{n/2}$ =volume of the round sphere. Define $Y_W = W^{-1/2} Y W^{-1/2}$, acting on $L^2(S^n, g)$.

Conjecture 1. For $n \ge 3$,

$$\max_{t>0} \left\{ t^{n/2} \mathrm{Tr}[e^{-tY_W}] \right\} \le \max_{t>0} \left\{ t^{n/2} \mathrm{Tr}[e^{-tY}] \right\}. \tag{1}$$

(Note that the eigenvalues of Y_W are the same as the eigenvalues of $W^{-(n+2)/4} Y W^{(n-2)/4}$ acting on $L^2(S^n, Wg)$, which is the natural Yamabe operator in the metric Wg.)

In other words we are looking for the best constant $C(W)$ in the inequality

$$\mathrm{Tr}[e^{-tY_W}] \le \frac{C(W)}{t^{n/2}}, \qquad t > 0, \tag{2}$$

and the conjecture states that this constant is attained precisely by the right side of (1), which is the best constant in (2) for $W \equiv 1$.

If Conjecture 1 is true then we can considerably improve the known CLR bounds, at least in low dimensions, noting that for a given positive potential V, the eigenvalues of the Birman–Schwinger operator $V^{-1/2}\Delta V^{-1/2}$ are the same as those of Y_W, with $W = (V \circ \pi)|J_\pi|^{2/n}$, π being the stereographic projection and J_π its Jacobian.

Conjecture 2. If $n \geq 4$ then the function $f_W(t) = t^{n/2}\mathrm{Tr}[e^{-tY_W}]$ is decreasing in t.

An asymptotic expansion $f_W(t) \sim a_0(W) + ta_1(W) + \dots$ holds as $t \to 0$, with $a_0(W) = (4\pi)^{-n/2}\int_{S^n} W^{n/2}$ and with $a_1(W)$ written explicitly in terms of the total curvature. Hence Conjecture 2 would imply (equality in) Conjecture 1 for $n \geq 4$, because Conjecture 1 normalizes the constant term $a_0(W)$ in the expansion.

It is known that $a_1(W)$ is negative for $n \geq 5$, zero for $n = 4$, and positive for $n = 3$, so that Conjecture 2 fails for small t when $n = 3$.

On the other hand, Conjecture 2 holds for large t and any $n \geq 3$, since the known sharp lower bound $\lambda_0(W) \geq \lambda_0(1) = \frac{n}{2}\left(\frac{n}{2} - 1\right)$ for the lowest eigenvalue of Y_W implies that $f_W(t)$ is decreasing when $t > \left(\frac{n}{2} - 1\right)^{-1}$.

Conjecture 2 is true if $W \equiv 1, n \geq 4$.

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10. Gap Inequalities

Consider eigenvalues of the Dirichlet Laplacian on a bounded convex domain $\Omega \subset \mathbb{R}^n$ with *convex* potential V:

$$\begin{cases} (-\Delta + V)u_j = \lambda_j u_j & \text{in } \Omega, \\ u_j = 0 & \text{on } \partial\Omega. \end{cases}$$

Assume $n \geq 1$. Notice the operator is written with $+V$, not $-V$ like in the previous section.

Van den Berg's Gap Conjecture is that

$$\lambda_2 - \lambda_1 \geq \frac{3\pi^2}{d^2}, \qquad d = \operatorname{diam}(\Omega),$$

with equality holding when $n = 1, V \equiv 0$. (In dimensions $n \geq 2$, the inequality should be strict, with equality holding only in the limit as the domain degenerates to an interval.)

In dimension $n = 1$ the conjecture has been completely proved by Richard Lavine (1994).

In dimensions $n \geq 2$, the best partial result says that $\lambda_2 - \lambda_1 \geq \pi^2/d^2$, which is missing the desired factor of 3 on the righthand side. The first proof of this result used P-function techniques based on the maximum principle. The second proof adapted the methods of Weinberger, who resolved the analogous Neumann gap problem long ago.

Now we state open problems, beginning with one dimension and then considering higher dimensional problems.

(1) *(Richard Lavine)* Can one expand the class of potentials for which the gap inequality holds, in one dimension? It is known for convex potentials, but also for single well potentials with a centered transition point. See the write-up by Mark Ashbaugh.

(2) *(Richard Lavine)* Normalize the eigenfunctions u_j in L^2 and define $\langle V \rangle_j = \int_\Omega V u_j^2 \, dx$. Are these means $\langle V \rangle_j$ an increasing sequence as j increases? The question is already interesting in one dimension.

(3) *(Richard Lavine)* Can one strengthen the gap inequality by adding to its righthand side a term that involves V? The question is already interesting in one dimension.

(4) *(Rodrigo Bañuelos)* Can Lavine's approach be extended to higher dimensions?

(5) *(Mark Ashbaugh)* In dimensions $n \geq 2$, one should try to understand whether genuine barriers exist to pushing the P-function techniques beyond the known π^2/d^2 bound. One seems to need to improve the log-concavity bound on the groundstate u_1 (due to Brascamp–Lieb). That is, instead of just discarding the Hessian of $\log u_1$ when it arises, on the grounds that it is ≤ 0, one seems to want to bound the Hessian strictly away from 0. Can this be achieved by the methods of Brascamp–Lieb, or of Korevaar?

(6) *(Antoine Henrot)* The Gap Conjecture is already very interesting in the case of vanishing potential $V \equiv 0$. A possible approach is as follows.
   (a) Prove the gap infimum $\inf_{\Omega \in \mathcal{O}}(\lambda_2 - \lambda_1)$ is not attained, when $\mathcal{O}$ is the class of convex domains with diameter 1.
   (b) Prove that minimizing sequences shrink to a segment of length 1.
   (c) Prove that the gap for a sequence of shrinking domains behaves like the gap of a one-dimensional Schrödinger operator with convex potential (semiclassical limit arguments).
   (d) Complete the proof using the results in the one dimensional case (Lavine's Theorem).

   It seems that points (b), (c) and (d) are OK. It remains to prove point (a)!

(7) *(Helmut Linde) Operator-valued potentials.* In order to prove the gap conjecture one could consider the Laplacian on a two-dimensional domain as being a one-dimensional operator with a matrix-valued potential. This

makes it possible to approach the problem via a sequence of simplified "toy models". For example, one can try to prove the gap conjecture first for very special classes of matrix-valued potentials, like potentials that have constant eigenvectors and whose eigenvalues are convex functions. Then one could gradually generalize this theorem to approach the "real" gap conjecture.

(8) *(Timo Weidl and Richard Laugesen) Magnetic Schrödinger operators.* For magnetic Schrödinger operators, the Gap Conjecture cannot hold as stated because the eigenvalue gap can be reduced to zero by the introduction of a magnetic field.

Can one still obtain a valid gap inequality by subtracting from the righthand side a term depending on the magnetic potential A?

(9) *(Rodrigo Bañuelos) Powers of the Laplacian.* Is the groundstate of $\sqrt{-\Delta}$ log-concave? See also the comments above on log-concavity of the groundstate of $-\Delta$.

(10) *(Rodrigo Bañuelos) Properties of the eigenfunction ratio.* The *Hot Spots* conjecture of Bernhard Kawohl says that the first nontrivial eigenfunction of the Neumann Laplacian attains its maximum and mimimum values on the boundary of the convex domain Ω. This has been proved only for some special classes of domains. The analogous conjecture for the Dirichlet Laplacian would be that the ratio u_2/u_1 attains its maximum and mimimum values on the boundary of Ω. Note u_2/u_1 satisfies Neumann boundary conditions (by explicit calculation, assuming the boundary is smooth) and satisfies a certain elliptic equation.

(11) *(Robert Smits) Robin boundary conditions.* Turn now from the Dirichlet boundary condition to the Robin condition $\partial u/\partial\nu = -\alpha u$ (for some given constant $\alpha > 0$, with ν denoting the outward normal). Is the gap $\lambda_2 - \lambda_1$ minimal when $V = 0$ and Ω degenerates to a segment having the same diameter as Ω?

In one dimension, is the gap minimal when $V = 0$ and Ω is a segment? Can Lavine's methods be adapted to Robin boundary conditions, in one dimension?

If one could prove the groundstate u_1 is log-concave, then existing methods could be adapted to imply $\lambda_2 - \lambda_1 \geq \pi^2/d^2$, like is already known for the Neumann and Dirichlet situations. Incidentally, the Rayleigh quotient for the gap can be shown (like in the Dirichlet case) to equal

$$\lambda_2 - \lambda_1 = \min_{\int_\Omega f u_1^2\,dx=0} \frac{\int_\Omega |\nabla f|^2 u_1^2\,dx}{\int_\Omega f^2 u_1^2\,dx},$$

with the potential entering implicitly through the dependence of u_1 on V.

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