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Optimal Constants in the Theory of Sobolev Spaces and PDEs

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Abstract. Recent research activities on sharp constants and optimal inequalities have shown their impact on a deeper understanding of geometric, analytical and other phenomena in the context of partial differential equations and mathematical physics. These intrinsic questions have applications not only to a-priori estimates or spectral theory but also to numerics, economics, optimization, etc.

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Introduction by the Organisers

The problem of finding sharp constants in geometric and functional inequalities is an issue of recognized importance in modern analysis. Prototypical instances of this kind of problems include isoperimetric inequalities, spectral estimates in mathematical physics, Sobolev and Hardy inequalities. Knowledge of the optimal form of the relevant inequalities and of the corresponding extremals not only provides deeper insight into the inequalities themselves, but is often crucial for applications to related partial differential equations and problems of the calculus of variations, as witnessed, for example, by the solution to the Yamabe problem.

Following classical results with their origins in the work of Steiner, Schwarz, Pólya, and Szegö, the proof of the isoperimetric inequality in the Euclidean space by De Giorgi was a breakthrough which paved the way to fundamental contributions to the study of optimal constants in the sixties and seventies of the last century by Federer, Maz'ya, Talenti, Aubin, Moser. Recent years have seen a renewed interest in investigations on these topics, which have benefited both from developments in classical methods such as symmetrizations and rearrangements, and also from new techniques, including scaling techniques and optimal mass transportation.

It was the aim of the proposed workshop to bring together mathematicians who work on various aspects of functional inequalities from spectral theory, shape optimization, probability, partial differential equations and related fields.

The workshop was attended by participants from various scientific communities including mathematical physics, nonlinear PDE, spectral theory, calculus of variations, optimal transportation and functional analysis. The common tie was the use of optimal estimates in different contexts. It came as a surprise even to the organizers how rich the theory of Sobolev and related inequalities is, and how varied the methods are. There were several contributions about the characterization of functions that provide sharp estimates, discussing qualitative properties such as symmetry or lack of symmetry, as well as quantitative aspects involving remainder terms in classical inequalities. To give an example, several talks dealt with inequalities like the Hardy-Sobolev inequality in the form

$$\int_\Omega |\nabla u|^p \, dx \geq C_0 \int_\Omega \frac{|u(x)|^p}{d(x,M)^p} \, dx + \left(\int_\Omega V(d(x,M))|u(x)|^q dx \right)^{\frac{p}{q}},$$

where M is a submanifold of the open set $\Omega \subset \mathbb{R}^n$ fulfilling suitable assumptions, C_0 is the optimal constant, and V is a one-dimensional function with an appropriate singularity at 0.

Some contributions were concerned with topics that at first sight appeared to be outside the central focus of the workshop, but then it turned out that they were indeed related and of great interest to the participants. Apart from the traditional presentations there were numerous informal discussion groups and an open problem session.

We made sure that all of the young participants could present their work. In particular, the three young postdocs that were supported as Oberwolfach Leibniz Graduate Students (OWLG) gave lectures on Monday and Tuesday. As it is often the case in Oberwolfach, there was a lot of interaction and a pleasant atmosphere, in which we shared information and brought each other up to the state of the art on particular questions. Of course the friendly and professional help from the Oberwolfach staff helped making the workshop a success.

Workshop: Optimal Constants in the Theory of Sobolev Spaces and PDEs

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Abstracts

Continuing an eikonal past a caustic

Giorgio Talenti

A principle (proposed by Felsen, Kravtsov, Ludwig and others) claims that any solution w to

$$\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2=n^2(x,y) \tag{1}$$

can be continued past the relevant caustic if a suitable complex-valued eikonal

$$u+iv$$

is called into play, i.e. the following system

$$\begin{cases} u_x^2+u_y^2-v_x^2-v_y^2 &= n^2(x,y) \\ u_xv_x+u_yv_y &= 0 \end{cases} \tag{2}$$

is solved under suitable initial conditions. In this talk, several properties of (2) are discussed.

On the distance between homotopy classes of S^m-valued maps

Itai Shafrir

It is well known that for $p \geq m$ the degree of maps in $W^{1,p}(S^m,S^m)$ is well defined and one has the following decomposition of this space as a disjoint union of homotopy classes, $W^{1,p}(S^m,S^m)=\bigcup_{d\in\mathbb{Z}}\mathcal{E}_d$. It is natural then to study the distance $\delta_p(d_1,d_2)$ between each pair of distinct homotopy classes $\mathcal{E}_{d_1}$ and $\mathcal{E}_{d_2}$, defined by

$$\delta_p^p(d_1,d_2)=\inf\Big\{\int_{S^m}|\nabla(u_1-u_2)|^p : u_1\in\mathcal{E}_{d_1}, u_2\in\mathcal{E}_{d_2}\Big\}\,.$$

In the one dimensional case ($m=1$) we find that the distance is given explicitly by the formula $\delta_p(d_1,d_2)=\frac{2^{1+1/p}|d_2-d_1|}{\pi^{1-1/p}}$.

In higher dimension, $m\geq 2$, it turns out that in the limiting case $p=m$ the distance between the homotopy classes is always zero. On the other hand, when $p>m$, for $d_1\neq d_2$, the distance is positive, but independent of d_1 and d_2, i.e., $\delta_p=c(m,p)$, where $c(m,p)$ is a positive number, that was computed explicitly before by Talenti (for $m=2$) and Cianchi (for any m) in the context of Sobolev-type inequalities on spheres.

We also studied the distance between homotopy classes for S^1-valued maps defined on a multiply connected, smooth and bounded domain D in $\mathbb{R}^2$. In this case, the space $H^1(D,S^1)$ can be written as a disjoint union of homotopy classes

$$H^1(D,S^1)=\bigcup_{\vec{d}\in\mathbb{Z}^n}\mathcal{E}_{\vec{d}},$$

where $\vec{d}=(d_1,\dots,d_n)$ is a vector of prescribed degrees on the boundaries of the n holes and $\mathcal{E}_{\vec{d}}$ consists of all the maps in $H^1(D,S^1)$ having these prescribed degrees. As above, we studied the distance $\delta_2(\vec{d_1},\vec{d_2})$ for $\vec{d_1}\neq\vec{d_2}$ and showed that

$$\delta_2^2(\mathcal{E}_{\vec{d}^{(1)}},\mathcal{E}_{\vec{d}^{(2)}})\geq\left(\frac{2}{\pi}\right)^2 I(\vec{d}^{(1)}-\vec{d}^{(2)})\,, \tag{1}$$

where

$$I(\vec{d})=\inf\{\int_D|\nabla u|^2\ :\ u\in\mathcal{E}_{\vec{d}}\}\,,$$

a quantity that was studied in the book [1]. We gave a necessary condition for an equality to hold in (1) (which includes the case of a strict inequality $\vec{d_2}>\vec{d_1}$, component-wise) but we also showed by an example that a strict equality may occur in (1).

Our report is based on a joint work with Jacob Rubinstein [3] and on a work in progress with Shay Levy [2].

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Extremal functions and symmetry breaking in Caffarelli-Kohn-Nirenberg inequalities

Jean Dolbeault

(joint work with Maria Esteban, Michael Loss, Gabriella Tarantello, Achilles Tertikas)

We consider the extremal functions for the interpolation inequalities introduced by Caffarelli, Kohn and Nirenberg in [1], that can be written as

$$\left(\int_{\mathbb{R}^d}\frac{|u|^p}{|x|^{bp}}\,dx\right)^{\frac{2}{p}}\leq \mathsf{C}_{\rm CKN}(\theta,a,b)\left(\int_{\mathbb{R}^d}\frac{|\nabla u|^2}{|x|^{2a}}\,dx\right)^{\theta}\left(\int_{\mathbb{R}^d}\frac{|u|^2}{|x|^{2(a+1)}}\,dx\right)^{1-\theta} \tag{1}$$

where u is a smooth function with compact support in $\mathbb{R}^d\setminus\{0\}$ and the parameters are in the range: $b\in(a+1/2,a+1]$ if $d=1$, $b\in(a,a+1]$ if $d=2$ and $b\in[a,a+1]$ if $d\geq 3$, $a\neq(d-2)/2=:a_c$, $p=\frac{2d}{d-2+2(b-a)}$ and $\theta\in[\vartheta(p,d),1]$ with $\vartheta(p,d):=d\,(p-2)/(2\,p)$.

We also consider *weighted logarithmic Hardy* inequalities, introduced in [4], which correspond to the limit $\theta=\gamma\,(p-2)$, $p\to 2_+$ and read as

$$\int_{\mathbb{R}^d}\frac{|u|^2}{|x|^{2(a+1)}}\,\log\left(|x|^{d-2-2a}\,|u|^2\right)\,dx\leq 2\gamma\,\log\left[\mathsf{C}_{\rm GLH}(\gamma,a)\int_{\mathbb{R}^d}\frac{|\nabla u|^2}{|x|^{2a}}\,dx\right] \tag{2}$$

for any smooth function u such that $\||x|^{-(a+1)}|u|\|_{L^2(\mathcal{C})}=1$, with compact support in $\mathbb{R}^d\setminus\{0\}$. The parameters are such that $a<a_c$, $\gamma\geq d/4$ and $\gamma>1/2$ if $d=2$.

Inequalities (1) and (2) can be extended to the larger space $\mathcal{D}^{1,2}_a(\mathbb{R}^d)$ obtained by completion with respect to the norm $u \mapsto \int_{\mathbb{R}^d} |x|^{-2a}\, |\nabla u|^2\, dx$. *Extremal* functions are such that the inequalities, written with their optimal constants, become equalities. We shall assume that $\mathsf{C}_{\rm CKN}(\theta,p,a)$ and $\mathsf{C}_{\rm GLH}(\gamma,a)$ are optimal, *i.e.* take their lowest possible value. By a Kelvin transformation (see [7]), the case $a > a_c$ can be reduced to the case $a < a_c$. For simplicity, we shall assume that $a < a_c$.

The case $\theta = 1$, $p \in [2, 2^*]$ and $d \ge 3$ has been widely discussed in the literature. Existence of extremal functions for (1) has been studied in various papers in case $\theta = 1$: see [2] and references therein for details. Radial symmetry of the extremal functions is an important issue, which has been established in a number of cases: see [3, 6, 7, 10, 11, 6]. Extremal functions are then entirely determined and the value of the optimal constants is known. On the other hand, *symmetry breaking*, which means that extremal functions are not radially symmetric, holds for

$$d \ge 2\ , \quad b < \frac{1}{2}\,(a-a_c)\Big[2 - \frac{d}{\sqrt{(a-a_c)^2 + d - 1}}\Big]\,, \tag{3}$$

as it has been established in [2, 7, 9]. Moreover, according to [6], a continuous curve $p \mapsto a(p)$ with values in the region $a < 0$, $b < a+1$ separates the symmetry breaking region from the region where radial symmetry holds.

The case $\theta < 1$ of Inequality (1) has been much less considered. Symmetry breaking has been established in [4] in a region which extends the one found in [7, 9]. If either $d = 1$ or $d \ge 2$ but for radial functions, existence of extremal functions for (1) has been proved in [4] for any $\theta > \vartheta(p,d)$. However, the best constant is not achieved if $\theta = \vartheta(p,d)$ and $d = 1$. Existence of extremal functions without symmetry assumption and some results of radial symmetry have also been obtained in [5, 8].

A symmetry breaking result for (2) has been established in [4] when

$$d \ge 2\ , \quad a < -1/2 \quad \text{and} \quad \gamma < \frac{1}{4} + \frac{(a-a_c)^2}{d-1}\ . \tag{4}$$

It is very convenient to reformulate Inequalities (1) and (2) in cylindrical variables as in [2]. By means of the Emden-Fowler transformation

$$t = \log|x| \in \mathbb{R}\ , \quad \omega = \frac{x}{|x|} \in \mathbb{S}^{d-1}\ , \quad y = (t,\omega)\ , \quad v(y) = |x|^{a_c - a}\, u(x)\ ,$$

Inequality (1) for u is equivalent to a Gagliardo-Nirenberg-Sobolev inequality on the cylinder $\mathcal{C} := \mathbb{R} \times \mathbb{S}^{d-1}$: for any $v \in H^1(\mathcal{C})$,

$$\left(\int_{\mathcal{C}} |v|^p\, dy\right)^{\frac{2}{p}} \le \mathsf{C}_{\rm CKN}(\theta,p,a) \left(\int_{\mathcal{C}} |\nabla v|^2\, dy + \Lambda \int_{\mathcal{C}} |v|^2\, dy\right)^{\theta} \left(\int_{\mathcal{C}} |v|^2\, dy\right)^{1-\theta}$$

with $\Lambda := (a_c - a)^2$. Similarly, with $w(y) = |x|^{a_c - a}\, u(x)$, (2) is equivalent to

$$\int_{\mathcal{C}} |w|^2 \log|w|^2\, dy \le 2\,\gamma\, \log\Big[\mathsf{C}_{\rm WLH}(\gamma,a)\Big(\int_{\mathcal{C}} |\nabla w|^2\, dy + \Lambda\Big)\Big]$$

for any $w \in H^1(\mathcal{C})$ such that $\|w\|_{L^2(\mathcal{C})} = 1$. We shall denote by $\mathsf{C}^*_{\rm CKN}(\theta,p,a)$ and $\mathsf{C}^*_{\rm WLH}(\gamma,a)$ the optimal constants for (1) and (2) respectively, when the set of functions is restricted to the radially symmetric ones. From [4], we know that

$$\begin{aligned} \mathsf{C}_{\mathrm{CKN}}(\theta,p,a) \geq \mathsf{C}^*_{\mathrm{CKN}}(\theta,p,a) = \mathsf{C}^*_{\mathrm{CKN}}(\theta,p,a_c-1)\,\Lambda^{\frac{p-2}{2p}-\theta} \\ \mathsf{C}_{\mathrm{WLH}}(\gamma,a) \geq \mathsf{C}^*_{\mathrm{WLH}}(\gamma,a) = \mathsf{C}^*_{\mathrm{WLH}}(\gamma,a_c-1)\,\Lambda^{-1+1/(4\gamma)} \end{aligned}$$

where $\Lambda = (a-a_c)^2$. Symmetry breaking means that the above inequalities are strict. Finding extremal functions amounts to minimize the functionals

$$\begin{aligned} \mathcal{E}[v] &:= \left(\|\nabla v\|^2_{L^2(\mathcal{C})} + \Lambda\,\|v\|^2_{L^2(\mathcal{C})}\right)^\theta \|v\|^{2(1-\theta)}_{L^2(\mathcal{C})} / \|v\|^2_{L^p(\mathcal{C})}\,, \\ \mathcal{F}[w] &:= \frac{\|\nabla w\|^2_{L^2(\mathcal{C})} + \Lambda\,\|w\|^2_{L^2(\mathcal{C})}}{\|w\|^2_{L^2(\mathcal{C})}}\,\exp\left[-\frac{1}{2\gamma}\int_{\mathcal{C}}\frac{|w|^2}{\|w\|^2_{L^2(\mathcal{C})}}\,\log\left(\frac{|w|^2}{\|w\|^2_{L^2(\mathcal{C})}}\right)dy\right]. \end{aligned}$$

Radial symmetry for (1) and (2) means that there are minimizers of $\mathcal{E}$ and $\mathcal{F}$ which depend only on t.

The method of [2, 9, 4] for proving symmetry breaking goes as follows. In case of Inequality (1), consider a symmetric minimizer v_* of $\mathcal{E}$, depending only on t. Up to a scaling and a multiplication by a constant, $v_*(t) = (\cosh t)^{-2/(p-2)}$ solves

$$(p-2)^2\,v'' - 4\,v + 2\,p\,|v|^{p-2}\,v = 0\,.$$

An expansion of $\mathcal{E}[v]$ at order two around v_* involves the operator $\mathcal{L} := -\Delta + \kappa\,{w_*}^{p-2} + \mu$ for some κ and μ which are explicit in terms of θ, p and d. Eigenfunctions are characterized in terms of Legendre's polynomials and spherical harmonic functions. The eigenspace of $\mathcal{L}$ corresponding to the lowest eigenfunction is generated by w_* (after a multiplication by a constant and a scaling). The eigenfunction $\lambda_{1,0}$ associated to the first non trivial spherical harmonic function is not radially symmetric. Condition (3) is determined by requiring that $\lambda_{1,0} < 0$, which implies that $\mathsf{C}_{\mathrm{CKN}}(\theta,p,a) > \mathsf{C}^*_{\mathrm{CKN}}(\theta,p,a)$. In case of Inequality (2), a similar analysis can be done. The radial minimizer is a Gaussian function in t and the operator $\mathcal{L}$ is the Schrödinger operator with harmonic potential.

Symmetry results in [6, 8] also involves some spectral analysis. By considering sequences $(v_n)_{n\in\mathbb{N}}$ of minimizers of $\mathcal{E}$ appropriately normalized by the condition $\|v_n\|^2_{L^p(\mathcal{C})} = 1$, one proves that $\|\nabla v_n\|^2_{L^2(\mathcal{C})}$ is bounded when either $b = b_n$ converges to $a+1$, or $a = a_n \to 0_-$ if $\theta = 1$, or $a = a_n \to a_{c\,-}$ if $\theta < 1$. Minimizers being solutions of an elliptic PDE, the convergence to a limit actually holds locally uniformly, which allows to write a linear equation for $D_\omega v_n$, where D_ω denotes an appropriate derivative with respect to ω. By spectral gap considerations, we conclude that $D_\omega v_n \equiv 0$ for n large enough: v_n depends only on t.

Using scaling properties, it has been proved in [6, 8] that there is a curve separating the region of symmetry for (3) from the region of symmetry breaking. The same property holds for (2). However, in both cases, no quantitative estimates are known about the position of the curve in the region $a < 0$. It is an open question to decide whether it coincides with the region defined by (3) and (4) or not.

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The shape of constrained minimizers of variational problems related to Sobolev type inequalities

Tobias Weth

(joint work with Pedro Girão)

We study the family of Poincaré-Sobolev type inequalities

$$(1) \qquad \left(\int_\Omega |u-u_\Omega|^q\,dx\right)^{\frac{p}{q}} \le C(p,q,\Omega)\int_\Omega |\nabla u|^p\,dx, \qquad u\in W^{1,p}(\Omega),$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $u_\Omega = \frac{1}{|\Omega|}\int_\Omega u\,dx$ is the average of u on Ω, $p>1$ and

$$1\le q\le \frac{pN}{N-p} \quad \text{if } N>p, \qquad 1<q<\infty \quad \text{if } N\le p.$$

Of interest are the best constants $C = C(p,q,\Omega)$ and extremal functions which give rise to equality in (1). It is easy to see that $C(p,q,\Omega)$ is the inverse of the number

$$(2) \qquad L_{p,q}(\Omega) = \inf\Big\{\int_\Omega |\nabla u|^p\,dx \;:\; u\in W^{1,p}(\Omega),\ \int_\Omega u = 0,\ \int_\Omega |u|^q = 1\Big\}.$$

Concerning the existence of minimizers for the minimization problem defined by (2), we quote the following result which follows by a standard compactness argument for subcritical q and is due to Demyanov and Nazarov in the critical case, see [2, Theorem 7.3].

Theorem 1. *There exists $\beta > 0$ such that the infimum in (2) is attained under each of the following assumptions.*

- $p \geq N$
- $p < N$ *and* $1 \leq q < \frac{pN}{N-p}$
- $p < \frac{N+1}{2} + \beta$ *and* $q = \frac{pN}{N-p}$

For the special case of a radial bounded domain, we now study the shape of minimizers for (2). Via spherical symmetrization, it is standard to see that, under the assumptions of Theorem 1, there exists a *foliated Schwarz symmetric* minimizer u of (2). By 'foliated Schwarz symmetric' we mean that u is axially symmetric with respect to an axis $\mathbb{R}e$, where $e \in \mathbb{R}^N$ is a unit vector, and u is nonincreasing in the angle $\theta := \arccos\big(\frac{x}{|x|} \cdot e\big)$. In fact, in the semilinear case $p = 2$ one can show that all minimizers must have this symmetry and further monotonicity properties. More precisely, we prove the following in [3].

Theorem 2. *Let $p = 2$, $q \geq 2$, and let u be a minimizer of (2). Then u is foliated Schwarz symmetric with respect to some unit vector e, i.e., $u = u(r, \theta)$ with $r = |x|$ and $\theta = \arccos\big(\frac{x}{|x|} \cdot e\big)$. Moreover:*

i) *u is strictly decreasing in $\theta \in (0, \pi)$.*
ii) *If q is sufficiently close to 2, then u is odd with respect to the reflection at the hyperplane $T(e) := \{x \in \mathbb{R}^N \,:\, x \cdot e = 0\}$.*

If Ω is the unit ball, then we have in addition:

iii) *$\partial_e u > 0$ on $\overline{\Omega} \setminus \{\pm e\}$. If τ is another unit vector in $\mathbb{R}^N$ orthogonal to e, then $\partial_\tau u$ has precisely four nodal domains. Here ∂_e and ∂_τ denote the directional derivatives in the direction of e and τ, respectively.*
iv) *If $N = 2$, the function u is not antisymmetric with respect to the reflection at $T(e)$ when q is sufficiently large.*

We point out that if Ω is a ball, properties i) and ii) imply that u takes its maximum and minimum precisely at the two antipodal points $\{\pm e\}$ on the boundary of Ω, and u has precisely two nodal domains. Moreover, in the case where u is odd with respect to the hyperplane $T(e)$, the four nodal domains of $\partial_\tau u$ considered in ii) are precisely the four quadrants in Ω cut off by the hyperplanes $T(e)$ and $T(\tau)$. This holds in particular for q close to 2. We also note that in case $q = 2$, minimizers of (2) are precisely the eigenfunctions of the Neumann-Laplacian on Ω corresponding to the first nonzero eigenvalue. These eigenfunctions are of the form $u(r, \theta) = g(r) \cos\theta$, and properties i)-iii) can be verified easily.

We briefly comment on the proof of Theorem 2. The most difficult parts are the strict inequality in i) and property ii), see [3, Section 5]. For both parts we need to carefully study the boundary values of the directional derivatives $\partial_e u$ and $\partial_\tau u$

for τ perpendicular to e. In a first step, we show that $\partial_e u$ is positive on $\partial\Omega \setminus \{\pm e\}$ and on the hyperplane $T(e)$ defined above. In a second step, we show that $\partial_e u$ can have at most two nodal domains. It then follows that $\partial_e u$ must be positive in one of the half balls cut off by the hyperplane $T(e)$. With this information, we then can conclude the proof of ii) by a moving plane argument. This is one of few examples where the moving plane method is applied to a problem with Neumann boundary conditions.

For general $p \neq 2$, we still conjecture that all minimizers of (2) are foliated Schwarz symmetric. However, the methods outlined above do not work, and up to now not much is known about the shape of minimizers. An application of a recent symmetry result by Maris [4], based on completely different methods, shows that for $N \geq 4$ minimizers u of (2) are radial with respect to a two-dimensional subspace V, so $u(x)$ only depends on the distance from x to V and the projection of x onto V. Moreover, by adjusting an approach which was introduced by Brock [1] in a different context, we can show that minimizers have a local type of symmetry. More precisely, the intersection of any superlevel set $u_c := \{x \in \Omega \,:\, u(x) \geq c\}$ with any sphere S contained in Ω (centered at zero) is a geodesic ball in S.

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Positivity preserving property for the hinged plate

Enea Parini

(joint work with Athanasios Stylianou)

A thin, vertically loaded, hinged plate can be modeled via the Kirchhoff-Love energy functional

$$I_\sigma(v) := \int_\Omega \left(\frac{(\Delta v)^2}{2} + (1-\sigma)(v_{xy}^2 - v_{xx}v_{yy}) - fv \right).$$

Here $\Omega \subset \mathbb{R}^2$ describes the shape of the plate, v represents the deflection when the load density $f \in L^2(\Omega)$ is applied and σ is the Poisson ratio of the plate. Although physical constraints impose that $-1 < \sigma < \frac{1}{2}$, we will consider the more general case $-1 < \sigma < 1$. A minimizer of the functional I_σ on the space $H_0(\Omega) := W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ is a weak solution of the equation

$$(1) \qquad \begin{cases} \Delta^2 v = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \\ \Delta v - (1-\sigma)\kappa v_n = 0 & \text{on } \partial\Omega, \end{cases}$$

where Δ^2 is the biharmonic operator, κ is the curvature of the boundary (with the convention that $\kappa \geq 0$ on convex parts) and v_n the exterior normal derivative. Since a maximum principle for fourth-order operators is not available, the following question is highly nontrivial:

$$\text{does } f \geq 0 \text{ imply } v \geq 0?$$

The answer to this question is in general negative, since it strongly depends on the geometry of the domain. In fact, there exist nonconvex domains where the solution to (1) is sign-changing for some positive f (see [1]).
For a convex, planar domain with boundary of class $C^{2,1}$, we are able to prove the following result.

Theorem 1 ([3], Theorem 3.1)**.** *Assume that $\Omega \subset \mathbb{R}^2$ is bounded and convex with $\partial\Omega \in \mathcal{C}^{2,1}$. Let $-1 < \sigma < 1$ and $f \in L^2(\Omega)$. Then the minimizer v_σ of I_σ is the unique weak solution in $H_0(\Omega)$ of (1). If moreover $f \geq 0$ and $f \not\equiv 0$, then there exists a positive constant c_f such that*

$$v_\sigma(x) > c_f d(x, \partial\Omega),$$

where $d(\cdot, \partial\Omega)$ is the distance to the boundary.

The proof follows easily from [2, Theorem 4.1], once one verifies that

$$0 \leq (1-\sigma)\kappa \leq \delta_{1,\kappa}\kappa,$$
$$(1-\sigma)\kappa \not\equiv \delta_{1,\kappa}\kappa,$$

where

$$\delta_{1,\kappa} := \inf_{v \in H_0(\Omega)} \frac{\int_\Omega (\Delta v)^2}{\int_\Omega \kappa v_n^2}.$$

Exploiting the property that

$$\int_\Omega (v_{xy}^2 - v_{xx}v_{yy}) = -\frac{1}{2}\int_{\partial\Omega} \kappa v_n^2$$

(see [3, Lemma 2.5]), one can easily prove that

$$\delta_{1,\kappa} \geq 2.$$

Remarking that $\kappa \geq 0$, $\kappa \not\equiv 0$, Theorem 1 follows. Observe that the bound $\delta_{1,\kappa} \geq 2$ is optimal, since for a disc one has $\delta_{1,\kappa} = 2$ (see [4]).
It remains open to verify whether the assumptions on the domain Ω are optimal. In fact, Theorem 4.1 in [2] requires the boundary of Ω to be of class $\mathcal{C}^2$, while the slightly stronger condition $\partial\Omega \in \mathcal{C}^{2,1}$ seems to be only a technical assumption and is used only in the preliminary lemma [3, Lemma 2.5]. Therefore it makes sense to wonder whether and how the result in Theorem 1 can be improved.

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The Buckling Problem and the Krein Laplacian

Mark S. Ashbaugh
(joint work with Fritz Gesztesy, Marius Mitrea, Roman Shterenberg, Gerald Teschl)

Recent developments on the buckling problem and the Krein Laplacian in which the author has been involved were discussed, including connections between these two problems, analysis of their spectral asymptotics, and inequalities for their eigenvalues. In particular, we note that the buckling problem is intimately related to the Krein Laplacian, and that, in fact, there is a unitary equivalence between the two problems if one considers the Krein Laplacian on the space orthogonal to its kernel (which turns out to be the space of harmonic functions on the domain considered). Old conjectures concerning the eigenvalues of the buckling problem were also discussed, including the Polya-Szegő conjecture for the first eigenvalue (which would be the Faber-Krahn result for this problem) and Payne's conjecture comparing the buckling eigenvalues to those of the Dirichlet Laplacian on the same domain.

The Krein Laplacian was first discussed by von Neumann around 1930 in the context of extensions of operators (see[6]), though perhaps without a full understanding of its significance. Later, in two long papers in 1947 [4, 5], Krein made an extensive investigation of it, presenting its most important properties and elucidating its place among the possible self-adjoint extensions of the Laplacian. It turns out that the Krein-von Neumann extension of the Laplacian (which we refer to briefly as the Krein Laplacian) is the minimal nonnegative self-adjoint extension of the Laplacian. This is to be contrasted with the fact that the Friedrichs extension is the maximal such extension. Because of this, the Krein-von Neumann extension is often referred to as the "soft" extension, while the Friedrichs extension is referred to as the "hard" extension. In fact, Krein's work dealt not just with the Laplacian, but with all densely defined, semi-bounded, symmetric operators.

Somewhat surprisingly, in light of the minimal property of the Krein Laplacian, its nonzero eigenvalues (which, due to the unitary equivalence, are also the eigenvalues of the buckling problem, multiplicities included) dominate the corresponding eigenvalues of the Dirichlet Laplacian. That is,

$$\lambda_{K,m}(\Omega) \geq \lambda_{D,m}(\Omega) \text{ for all } m \geq 1,$$

where $\lambda_{K,m}(\Omega)$ denotes the mth nonzero eigenvalue of the Krein Laplacian on Ω (counted with multiplicity), and similarly for the Dirichlet eigenvalues, a result which was well-known in the context of the buckling problem (via a simple Cauchy-Schwarz argument).

As an example of our results, we give the counting function asymptotics for the nonzero eigenvalues of the Krein Laplacian

$$N_{K,\Omega}(\lambda) = (2\pi)^{-n}\, v_n\, |\Omega|\, \lambda^{n/2} + O(\lambda^{(n-1/2)/2})$$

as $\lambda \to \infty$, where $v_n = \pi^{n/2}/\Gamma(n/2+1)$ denotes the volume of the ball of unit radius in $\mathbb{R}^n$ and where Ω is in the class of *quasi-convex domains.* To illustrate, this class includes (among other things)

1) all bounded (geometrically) convex domains;
2) all bounded Lipschitz domains satisfying a uniform exterior ball condition (no inward-pointing spikes);
3) all open sets which are the image of a domain as in 1) or 2) under a $C^{1,1}$-diffeomorphism; and
4) all bounded domains of class $C^{1,r}$ for some $r > 1/2$.

For the ball $B_n(R)$ of radius R in $\mathbb{R}^n$ we obtain the more detailed asymptotics

$$N_{K,B_n(R)}(\lambda) = (2\pi)^{-n}\, v_n^2\, R^n\, \lambda^{n/2} - (2\pi)^{-(n-1)}\, v_{n-1} \left[\frac{n}{4}\, v_n + v_{n-1}\right] R^{n-1}\, \lambda^{(n-1)/2} + O(\lambda^{(n-2)/2})$$

as $\lambda \to \infty$. This is to be contrasted with the counting function asymptotics of the Dirichlet Laplacian (=Friedrichs extension) which behaves as

$$N_{D,B_n(R)}(\lambda) = (2\pi)^{-n}\, v_n^2\, R^n\, \lambda^{n/2} - (2\pi)^{-(n-1)}\, v_{n-1}\, \frac{n}{4}\, v_n\, R^{n-1}\, \lambda^{(n-1)/2} + O(\lambda^{(n-2)/2}).$$

Note that the two counting functions agree at leading order, but that the counting function for the Krein Laplacian is less at the next order (in fact, the eigenvalue comparison mentioned earlier implies that for any domain Ω, $N_{K,\Omega}(\lambda) \leq N_{D,\Omega}(\lambda)$ for all λ). The proof relies on the inequalities

$$N_{K,B_{n-1}(R)}(\lambda) \leq N_{D,B_n(R)} - N_{K,B_n(R)}(\lambda) \leq N_{D,B_{n-1}(R)}(\lambda),$$

which can be proved based on the fact that both the Krein and Dirichlet problems for the ball admit solution via separation of variables leading to the determination of the nonzero eigenvalues in terms of Bessel function zeros (with multiplicities determined as the dimensions of the corresponding spaces of spherical harmonics).

Other results for eigenvalues for domains $\Omega \subset \mathbb{R}^n$ include (here "D" stands for "Dirichlet," "K" for "Krein")

$$\lambda_{D,2}(\Omega) \leq \lambda_{K,1}(\Omega),$$

$$\lambda_{K,2}(\Omega) \leq \frac{n^2+8n+20}{(n+2)^2}\, \lambda_{K,1}(\Omega),$$

$$\sum_{j=1}^{n} \lambda_{K,j+1}(\Omega) < (n+4)\, \lambda_{K,1}(\Omega) - \frac{4}{n+4}\, (\lambda_{K,2}(\Omega) - \lambda_{K,1}(\Omega)) \leq (n+4)\, \lambda_{K,1}(\Omega),$$

$$\sum_{j=1}^{m} (\lambda_{K,m+1}(\Omega) - \lambda_{K,j}(\Omega))^2 \leq \frac{4(n+2)}{n^2} \sum_{j=1}^{m} (\lambda_{K,m+1}(\Omega) - \lambda_{K,j}(\Omega))\, \lambda_{K,j}(\Omega)$$

for all $m \geq 1$, and

$$1 \leq \frac{\lambda_{K,1}(\Omega)}{\lambda_{D,1}(\Omega)} \leq 4,$$

the last holding for domains Ω such that the boundary $\partial\Omega$ is everywhere of nonnnegative mean curvature (and with all the other inequalities holding for general quasi-convex domains). These inequalities are the "Krein Laplacian formulations" of eigenvalue inequalities for the buckling problem associated with the names of (in order of appearance) Payne, Hile and Yeh, Ashbaugh, Cheng and Yang, and Payne (for the upper bound 4; the lower bound is elementary).

Much of the recent work presented in the talk represents joint work with Fritz Gesztesy, Marius Mitrea, Roman Shterenberg, and/or Gerald Teschl (see [2], [3]). In these papers we also present results for a "generalized buckling problem" gotten by replacing the Laplacian by a Schrödinger operator, $-\Delta + V$, where V is a nonnegative potential. Our initial motivation for these studies came from [1], but as we delved more deeply into the subject we learned of important related work by Grubb, Birman and Solomjak, Kozlov, and others. And on the buckling problem side of things, the key prior works are those of Payne and Payne, Pólya, and Weinberger (followed later by Hile and Yeh, Ashbaugh, and Cheng and Yang).

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Some applications of optimal constants for nonlinear PDEs on manifolds

Farid Madani

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. Denote by R_g, Δ_g the scalar curvature and the Laplacian of g respectively. Let us consider the following equation:

$$\Delta_g \psi + \frac{n-2}{4(n-1)} R_g \psi = \mu \psi^{\frac{n+2}{n-2}}$$

with $\mu \in \mathbb{R}$. Positive smooth solutions of this equation give solutions for the Yamabe problem [6]. The best constant in Sobolev inequalities was introduced by T. Aubin [1] and G. Talenti [5]. Probably, the principal motivation for which T. Aubin introduced it, is to solve the Yamabe problem. We show how the optimal constants help to find the principal assumption to solve this problem. We discuss the existence of solutions, when the metric g belongs to $W^{2,p}$ with $p > n/2$. This result is proven by the author [4]. Afterwards, we consider the equivariant Yamabe problem (in the presence of the isometry group), which generalizes the Yamabe problem and was introduced by E. Hebey and M. Vaugon [3]. We give the optimal constants in Sobolev inequalities for $G-$invariant functions, computed by Z. Faget [2], where G is a subgroup of the isometry group.

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Eigenvalue Problem for the 1-Laplace Operator

Zoja Milbers
(joint work with Friedemann Schuricht)

We consider the eigenvalue problem associated to the 1-Laplace operator, which formally has the form

$$\Delta_1 u := \operatorname{div}\left(\frac{Du}{|Du|}\right).$$

The problem is also of purely theoretical interest, since it can be obtained from the well-known eigenvalue problem for the p-Laplace operator

$$\Delta_p u := \operatorname{div}\left(|Du|^{p-2} Du\right)$$

by considering the limit $p \to 1$. Such problems are often studied by means of an associated variational problem. In the case $p \in (1, \infty)$ we consider the minimization of

$$E_p(u) := \int_\Omega |Du|^p \, dx \to \text{Min!}, \quad u \in W_0^{1,p}(\Omega) \tag{1}$$

under the constraint

$$G_p(u) := \int_\Omega |u|^p \, dx - 1 = 0 \tag{2}$$

for a suitable domain $\Omega \subset \mathbb{R}^n$. The eigenvalue equation for the p-Laplace operator is the corresponding Euler-Lagrange equation given by

$$-\operatorname{div}\left(|Du|^{p-2}Du\right) = \lambda |u|^{p-2}u \ \text{ on } \Omega, \quad u = 0 \ \text{ on } \partial\Omega\,, \tag{3}$$

which can be obtained for a minimizer of this problem by classical methods of the calculus of variations. In the case $p = 1$ it is reasonable to consider

$$E(u) := \int_\Omega d|Du| + \int_{\partial\Omega} |u|\, d\mathcal{H}^{n-1} \to \text{Min!}\,, \quad u \in BV(\Omega) \tag{4}$$

under the constraint

$$G(u) := \int_\Omega |u|\, dx - 1 = 0\,. \tag{5}$$

It is known that a minimizer of this variational problem exists in $BV(\Omega)$, but in contrast to the case $p \in (1, \infty)$ it is not necessarily unique and it might change sign in Ω. It is known that a suitable multiple of the characteristic function $u = \chi_C$ of the Cheeger set C of Ω is a minimizer of (4) under the constraint (5), where, roughly speaking, a Cheeger set is a subset of Ω which minimizes the quotient $|\partial D|/|D|$ among all sets $D \subset \Omega$.

The derivation of an Euler-Lagrange equation for a minimizer of (4), (5) is a difficult task, since both E and G are nonsmooth functionals. We obtain the Euler-Lagrange equation by a direct treatment of the variational problem (4), (5) with methods of nonsmooth analysis. For *any* measurable selection s of the set-valued sign function $\operatorname{Sgn}(u(x))$ there exists a vector field $z : \Omega \to B_1(0) \subset \mathbb{R}^n$, which depends on the choice of s, such that

$$-\operatorname{div} z(x) = \lambda\, s(x) \quad \text{on } \Omega\,. \tag{6}$$

The vector field z can be identified with $Du/|Du|$ if $|Du|$ is nonzero and is otherwise a suitable substitute for this expression. A measurable selection s replaces $u/|u|$ by some value in $[-1, 1]$ at points where u vanishes. Hence, it turns out that for minimizers $u \in BV(\Omega)$ of (4), (5) infinitely many Euler-Lagrange equations have to be satisfied in general. We call this equation with many s the multiple Euler-Lagrange equation.

Moreover, we study higher eigensolutions of the eigenvalue problem related to the 1-Laplace operator. By taking a closer look at the derivation of (6) we see that as a first condition we obtain equation (6) just for *one* measurable selection s. We call this equation with one s the single Euler-Lagrange equation. We define higher eigensolutions as critical points of E under the constraint (5) by means of the weak slope, which is defined for continuous and even some classes of lower semicontinuous functions. Then we use the nonsmooth critical point theory in order to obtain the existence of a sequence of critical points. Moreover, using a suitable nonsmooth version of the Lagrange multiplier theorem, we show that critical points have to satisfy the single Euler-Lagrange equation.

The situation is however not satisfactory, since it seems that not all of the functions satisfying the single Euler-Lagrange equation are critical points of (4), (5). One possibility to resolve this problem is to diminish the number of possible solutions by the derivation of a different necessary condition for critical points of (4), (5). We deduce a further necessary condition for critical points of (4), (5) by means of inner variations. For that purpose, we consider variations of the domain Ω instead of those of a minimizer u as in the derivation of the Euler-Lagrange equation. In the classical calculus of variations such an approach turns out to be useful if the minimizer is less regular than being in C^2, i. e. inner variations might produce an additional necessary condition which is different from the Euler-Lagrange equation. We adapt this method and use a suitable Lagrange multiplier rule which is applicable to our nonsmooth setting. It turns out that the new necessary condition is truly different from the single Euler-Lagrange equation, as we demonstrate on a model example. Moreover, in the scalar case $n = 1$ we are able to determine all eigenfunctions.

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Sharp quantitative isoperimetric inequalities in the plane

Vincenzo Ferone

(joint work with Angelo Alvino, Carlo Nitsch)

The classical isoperimetric inequality in the plane states that, among all the subsets of $\mathbb{R}^2$ of prescribed finite measure, the disk has the smallest perimeter, namely

$$P(E) \geq (4\pi|E|)^{\frac{1}{2}}, \qquad \text{with equality if and only if } E \text{ is a disk.}$$

Here $|E|$ and $P(E)$ denote, as usual, the measure and the perimeter of the set $E \subset \mathbb{R}^2$.

In [3, 4] Bonnesen introduced some remarkable inequalities which imply the isoperimetric one. For example, we recall that for bounded convex planar sets he proved that

$$P(E)^2 - 4\pi|E| \geq 4\pi d^2.$$

Here d is the thickness of the minimal annulus containing the boundary of E and we remark that the constant 4π and the exponent 2 on the right hand side are optimal. Later Bonnesen's work led to the study of a wider class of inequalities nowadays known as Bonnesen–style isoperimetric inequalities. Following Osserman [9] we say that a Bonnesen–style isoperimetric inequality can be written in the form

$$P(E)^2 - 4\pi|E| \geq F(E),$$

where the function F is nonnegative, vanishes only on the disks, and somehow measures how much E deviates from a disk. There are many different kinds of functions F satisfying these properties, and each one leads to a different refinement of the standard isoperimetric inequality.

We discuss the situations where the function F depends on the set E through the so–called *Fraenkel asymmetry index* or through the Hausdorff distance from a ball (see, e.g., [5, 7, 8, 6]). The *Fraenkel asymmetry index* $\alpha(E)$ is defined as

$$\alpha(E) = \min_{x\in\mathbb{R}^2} \frac{|E \setminus D_R(x)|}{|E|},$$

where $D_R(x)$ is the disk centered at x with $|D_R(x)| = |E|$. If $d_H(A, B)$ denotes the Hausdorff distance between the sets A, $B \subset \mathbb{R}^2$, the *Hausdorff asymmetry index* $\delta(E)$ is defined as the *translative Hausdorff distance* of E from a disk D_R having the same measure,

$$\delta(E) = \min_{x\in\mathbb{R}^2} d_H(E, D_R(x)).$$

In order to state the main results we need to define two classes of convex sets.

We say that a convex set S belongs to $\mathfrak{S}$ if it satisfies the following properties:

- S is symmetric with respect to two orthogonal axes (2-symmetric);
- S has a smooth C^1 boundary made of four circular arcs $\{a_i\}_{1\le i\le 4}$, two of which can possibly degenerate into parallel segments;
- $\alpha(S) = \frac{|S\setminus D|}{|D|}$, D being the disk having the same measure of S and centered at the intersection of the axes of symmetry of S;
- whenever a_i is a proper circular arc (for some $1 \le i \le 4$) then it does not cross ∂D, namely either $a_i \subset \overline{D}$ or $a_i \subset \mathbb{R}^2 \setminus D$.

A convex set Y belongs to $\mathcal{Y}$ if it is symmetric with respect to a straight line such that the part of it which stays on one side of the line coincides with a circular segment (the smallest part of a disk cut by a chord).

Theorem 1. *For every convex set $E \in \mathbb{R}^2$, the set $S \in \mathfrak{S}$, such that $\alpha(S) = \alpha(E)$ and $|S| = |E|$, satisfies the inequality*

$$P(S) \ge P(Y),$$

equality holding if and only if $E = S$, up to translations.

Theorem 2. *For every convex set $E \in \mathbb{R}^2$, the set $Y \in \mathcal{Y}$, such that $\delta(Y) = \delta(E)$ and $|Y| = |E|$, satisfies the inequality*

$$P(E) \ge P(Y),$$

equality holding if and only if $E = Y$, up to translations.

As a consequence of the above results we have, for example, that for a convex set $E \in \mathbb{R}^2$ the following inequalities with sharp constants hold true:

$$\frac{P(E) - 2\sqrt{\pi|E|}}{2\sqrt{\pi|E|}\alpha(E)^2} \ge \frac{\pi}{2(4-\pi)} - \frac{\pi^3(16-5\pi)(14-3\pi)}{24(4-\pi)^4(\pi-2)}\alpha(E)^2 + O\left(\alpha(E)^4\right),$$

$$P(E)^2 - 4\pi|E| \ge \delta(E)^2(4\pi^2 - O(\delta(E)).$$

The results discussed here are contained in [1, 2].

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On best constants of Hardy-Sobolev-Maz'ya inequalities

Achilles Tertikas

(joint work with Stathis Filippas and Jesper Tidblom)

For $n \geq 3$ we write $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$, with $1 \leq k \leq n$. We also introduce the codimension k affine subspace

$$S_k := \{x = (x_1, \dots x_k, \dots x_n) \in \mathbb{R}^n : \; x_1 = \dots = x_k = 0\}.$$

The Euclidean distance of a point $x \in \mathbb{R}^n$ from S_k is then given by

$$d(x) = d(x, S_k) = |\mathbf{X_k}|, \text{ where } \mathbf{X_k} := (x_1, \dots, x_k, 0, \dots, 0).$$

The classical Hardy inequality in $\mathbb{R}^n$ when distance is taken from S_k, reads

$$(1) \qquad \int_{\mathbb{R}^n} |\nabla u|^2 dx \geq \left(\frac{k-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{u^2}{|\mathbf{X_k}|^2} dx, \;\; u \in C_0^\infty(\mathbb{R}^n \setminus S_k),$$

where the constant $\frac{(k-2)^2}{4}$ is the optimal one.

Maz'ya, in his book, combined both inequalities when $1 \leq k \leq n-1$, establishing that for any $u \in C_0^\infty(\mathbb{R}^n \setminus S_k)$

$$(2) \;\; \int_{\mathbb{R}^n} |\nabla u|^2 dx \geq \left(\frac{k-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{u^2}{|\mathbf{X_k}|^2} dx + c_{k,Q} \left(\int_{\mathbb{R}^n} |\mathbf{X_k}|^{\frac{Q-2}{2}n - Q} \, |u|^Q dx \right)^{\frac{2}{Q}},$$

for $2 < Q \leq 2^* = \frac{2n}{n-2}$; cf. [8], Section 2.1.6/3. Concerning the best constant $c_{k,Q}$, it was shown in [10] that $c_{k,2^*} < S_n$ for $3 \leq k \leq n-1$, $n \geq 4$ or $k = 1$ and $n \geq 4$. Surprisingly, in the case $k = 1$ and $n = 3$ Benguria Frank and Loss [2] (see also Mancini and Sandeep [7]) established that $c_{1,6} = S_3 = 3(\pi/2)^{4/3}$! Maz'ya and

Shaposhnikova [9] have recently computed the best constant in the case $k = 1$ and $Q = \frac{2(n+1)}{n-1}$. These are the only cases where the best constant $c_{k,Q}$ is known.

In case $k = n$, that is, when distance is taken from the origin, inequality (2) fails. Brezis and Vazquez [3] considered a bounded domain containing the origin and improved the Hardy inequality by adding a subcritical Sobolev term. It turns out that in a bounded domain one can have the critical Sobolev exponent at the expense however of adding a logarithmic weight. More specifically let

$$X(t) = (1 - \ln t)^{-1},\ 0 < t < 1.$$

Then the Hardy-Sobolev inequality in the case of a bounded domain Ω containing the origin, reads:

(3)

$$\int_\Omega |\nabla u|^2 dx - \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} dx \geq C_n(\Omega) \left(\int_\Omega X^{\frac{2(n-1)}{n-2}} \left(\frac{|x|}{D}\right) |u|^{\frac{2n}{n-2}} dx\right)^{\frac{n-2}{n}},$$

where $u \in C_0^\infty(\Omega)$ and $D = \sup_{x\in\Omega} |x|$; cf [4]. The best constant in (3) was recently computed in [1] and is given by

$$C_n(\Omega) = (n-2)^{-\frac{2(n-1)}{n}} S_n.$$

It is worth noticing that in the case $n = 3$ once again one has $C_3(\Omega) = S_3 = 3(\pi/2)^{4/3}$!

We initially consider the case where distances are taken from different codimension subspaces $S_k \subset \mathbb{R}^n$, which are placed in the interior of the domain $\mathbb{R}^n$. We consider the cases $k = 3, \ldots, n$ since there is no positive Hardy constant in case $k = 2$ (cf (1)).

To state our next results we define

$$\beta_3 = -\alpha_3^2 + \frac{1}{4},\ \beta_m = -\alpha_m^2 + \left(\alpha_{m-1} - \frac{1}{2}\right)^2,\ m = 4, \ldots, n. \tag{4}$$

Theorem A (**Interior singularities**)
Suppose $n \geq 3$.
i) *Inequality*

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq \int_{\mathbb{R}^n} \left(\frac{\beta_3}{|\mathbf{X_3}|^2} + \ldots + \frac{\beta_n}{|\mathbf{X_n}|^2}\right) u^2 dx,$$

holds true for some real numbers $\beta_3, \beta_4 \ldots, \beta_n$ *and any* $u \in C_0^\infty(\mathbb{R}^n)$, *if and only if there exists nonpositive constants* $\alpha_3, \ldots, \alpha_n$, *such that the* $\beta_3, \ldots, \beta_n$ *are given by (4).*

ii) *Suppose that* $\alpha_3, \alpha_4 \ldots, \alpha_n$ *are nonpositive numbers and define* $\beta_3, \ldots, \beta_n$ *by the recursive relation (4). Then, if* $\alpha_n < 0$ *there exists a positive constant* C *such that for any* $u \in C_0^\infty(\mathbb{R}^n)$ *there holds*

(5)

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq \int_{\mathbb{R}^n} \left(\frac{\beta_3}{|\mathbf{X_3}|^2} + \ldots + \frac{\beta_n}{|\mathbf{X_n}|^2}\right) u^2 dx + C \left(\int_{\mathbb{R}^n} |\mathbf{X_2}|^{\frac{Q-2}{2}n-Q} |u|^Q dx\right)^{\frac{2}{Q}},$$

for any $2 < Q \leq \frac{2n}{n-2}$.

If $\alpha_n = 0$ *then there is no positive constant* C *such that (5) holds.*

Actually a much stronger interpolation inequality can be established using the stronger singular potential, $|x_1|^{\frac{Q-2}{2}n-Q}$. For this we have to restrict ourselves so that $|x_1|^{\frac{Q-2}{2}n-Q} \in L^1_{loc}(\mathbb{R}^n)$ and this amounts into supposing $\frac{2(n-1)}{n-2} < Q$. In particular we have

Theorem B (**Interior singularities**)
Let $\alpha_3, \alpha_4, \ldots, \alpha_n$*,* $n \geq 3$*, be arbitrary nonpositive real numbers and* $\beta_3, \ldots, \beta_n$ *are given by (4). Then, if* $\alpha_n < 0$ *there exists a positive constant* C *such that for any* $u \in C_0^\infty(\mathbb{R}^n)$ *there holds*

(6)

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq \int_{\mathbb{R}^n} \left(\frac{\beta_3}{|\mathbf{X_3}|^2} + \ldots + \frac{\beta_n}{|\mathbf{X_n}|^2} \right) u^2 dx + C \left(\int_{\mathbb{R}^n} |x_1|^{\frac{Q-2}{2}n-Q} |u|^Q dx \right)^{\frac{2}{Q}},$$

for any $\frac{2(n-1)}{n-2} < Q \leq \frac{2n}{n-2}$.

If $\alpha_n = 0$ *then there is no positive constant* C *such that (6) holds.*

If on the other hand the singularities are placed on the boundary, our results do not take a much different form. This time we define

$$\beta_1 = -\alpha_1^2 + \frac{1}{4}, \ \beta_m = -\alpha_m^2 + \left(\alpha_{m-1} - \frac{1}{2} \right)^2, \ m = 2, \ldots, n. \tag{7}$$

Theorem C (**Boundary singularities**)
Suppose $n \geq 3$.

i) *Inequality*

$$\int_{\mathbb{R}^n_+} |\nabla u|^2 dx \geq \int_{\mathbb{R}^n_+} \left(\frac{\beta_1}{|\mathbf{X_1}|^2} + \ldots + \frac{\beta_n}{|\mathbf{X_n}|^2} \right) u^2 dx,$$

holds true for some real numbers $\beta_1, \beta_2 \ldots, \beta_n$ *and any* $u \in C_0^\infty(\mathbb{R}^n_+)$*, if and only if there exists nonpositive constants* $\alpha_1, \ldots, \alpha_n$*, such that the* $\beta_1, \ldots, \beta_n$ *are given by (7).*

ii) *Suppose that* $\alpha_1, \alpha_2 \ldots, \alpha_n$ *are nonpositive numbers and define* $\beta_1, \ldots, \beta_n$ *by the recursive relation (7).*

Then, if $\alpha_n < 0$ *there exists a positive constant* C *such that for any* $u \in C_0^\infty(\mathbb{R}^n_+)$ *there holds*

(8)

$$\int_{\mathbb{R}^n_+} |\nabla u|^2 dx \geq \int_{\mathbb{R}^n_+} \left(\frac{\beta_1}{|\mathbf{X_1}|^2} + \ldots + \frac{\beta_n}{|\mathbf{X_n}|^2} \right) u^2 dx + C \left(\int_{\mathbb{R}^n_+} \mathbf{X_1}^{\frac{Q-2}{2}n-Q} |u|^Q dx \right)^{\frac{2}{Q}},$$

for any $2 < Q \leq \frac{2n}{n-2}$.

If $\alpha_n = 0$ *then there is no positive constant* C *such that (8) holds.*

For more information we refer to [5, 6].

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Role of fundamental solution in Hardy Sobolev inequalities

Adimurthi

Introduction. To begin with, recall the Hardy's inequalities in dimension one. Let $f \in L^2(0,\infty)$ and define the Hardy operator.

$$Tf(x) = \frac{1}{x}\int_0^x f(t)dt. \tag{1}$$

Then Hardy showed that T defines a bounded Linear Operator on $L^2(0,\infty)$ with

$$\int_0^\infty |f(x)|^2 dx - \frac{1}{4}\int_0^\infty |T(f)(x)|^2 dx \geq 0 \tag{2}$$

where equality holds if and only if $f = 0$. Hence $\frac{1}{4}$ is the best constant in (2) and is never achieved.

Now specialize (1) to $f = u'(x), u \in C_0^1(0,\infty)$ to obtain

$$H(u) = \int_0^\infty |u'(x)|^2 dx - \frac{1}{4}\int_0^\infty \left|\frac{u(x)}{x}\right|^2 dx \geq 0 \tag{3}$$

and this is called Hardy - Sobolev inequality.
From Hardy's result, $H(u) > 0$ for $u \neq 0$. Hence the question is "Is it possible to estimate H from below?"

In order to understand this, make the following transformation. Let $\alpha \in \mathbb{R}$ and $V = x^\alpha u(x)$. Then $V \in C_0^1(0,\infty)$ and $u = x^{-\alpha}V$. Taking the derivative to obtain

$$\begin{aligned} \frac{u'}{u} &= -\frac{\alpha}{x} + \frac{V'}{V} \\ {u'}^2 &= \frac{\alpha^2}{x^2}u^2 - \frac{2\alpha u^2 V'}{xV} + \frac{{V'}^2}{V^2}u^2 \\ &= \frac{\alpha^2}{x^2}u^2 - \alpha x^{-2\alpha-1}(V^2)' + V'^2 x^{-2\alpha}. \end{aligned}$$

Hence

$$\begin{aligned} \int_0^\infty x^{2\alpha+1}|u'(x)|^2 dx &= \alpha^2 \int_0^\infty \frac{u^2}{x^2} x^{2d+1} dx - \alpha \int_0^\infty (V^2)' dx + \int_0^\infty V'^2 x dx \\ &= \alpha^2 \int_0^\infty \frac{u^2}{x^2} x^{2\alpha+1} dx + \int_0^\infty V'^2 x dx, \end{aligned} \tag{4}$$

since $\int_0^\infty (V^2)' dx = V^2(\infty) - V^2(0) = 0$. Vanishing of this term is called "Magical cancellation" by Brezis - Vazquez [7].

Observe that if V is considered as a radial function in $\mathbb{R}^2$, then the last integral on the right hand side of (4) is the Dirichlet integral of V in $\mathbb{R}^2$. Therefore if $supp\, u \subset (0,R)$, then $supp\, V \subset (0,R)$. Let $\lambda_1(R)$ denote the first Dirichlet eigenvalue of $-\Delta$ in $B(0,R) \subset \mathbb{R}^2$, then

$$\int_0^R |V'(x)|^2 x dx \geq \lambda_1(R) \int_0^R |V(x)|^2 x dx = \lambda_1(R) \int_0^R |u|^2 x^{2\alpha+1} dx. \tag{5}$$

Consequences.

(a) Let $2\alpha + 1 = 0$, that is $\alpha = -\frac{1}{2}$, then (3) gives $H(u) \geq 0$ and $H(u) = 0$ if and only if $V \equiv 0$, hence $u \equiv 0$. This process $\frac{1}{4}$ is the best constant and is never achieved.

(b) Let $2\alpha + 1 = n - 1$ or $\alpha = \frac{n-2}{2}$, then (3) gives

$$H(u) = \int_0^\infty |u'(r)|^2 r^{n-1} dr - \left(\frac{n-2}{2}\right)^2 \int_0^\infty \frac{|u(r)|^2}{r^2} r^{n-1} dr \geq 0 \tag{6}$$

and $H(u) = 0$ if and only if $V \equiv 0$ and hence $u \equiv 0$. Hence $(\frac{n-2}{2})^2$ is the best constant in (5) and is never achieved. This inequality is the Hardy-Sobolev inequality in $n \geq 3$.
Furthermore if $u \in C_0^1(0,R)$, then consider u as a radial function in $\mathbb{R}^n$, then (5) and (6) imply for $B(R) = \{x \in \mathbb{R}^n; |x| < R\}, \lambda(R) = \lambda_1(\Omega)$,

$$H(u) = \int_\Omega |\nabla u|^2 dx - \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{|u|^2}{|x|^2} dx \geq \lambda_1(\Omega) \int_\Omega |u|^2 dx \tag{7}$$

By density argument, (7) holds for all u radial in $H_0^1(B(R))$. Furthermore equality holds in (7) if and only if $V = r^{\frac{(n-2)}{2}}u$ is the first Dirichlet eigenfunction in the ball of radius R in $\mathbb{R}^2$. Hence $V(0) \neq 0$. But $V(0) = 0$ which is a contradiction. This result was obtained in the seminal paper of Brezis - Vazquez [7] for an arbitrary bounded domain in $\mathbb{R}^n$ for $n \geq 3$. In view of the strict inequalities in (8) they raised the following question:

"Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, what is the best possible lower bound one can obtain for $H(u)$?".

This question has an affirmative answer by estimating H by infinite series involving logarithms. Basic idea in deriving this result is as follows:

Define $V_1(x) = (\log \frac{x}{R})^{\frac{-1}{2}} V(x) = (\log \frac{x}{R})^{\frac{-1}{2}} x^\alpha u(x)$

$$\frac{V'(x)}{V(x)} = \frac{1}{2}\frac{1}{x(\log \frac{x}{R})} + \frac{V_1'(x)}{V_1(x)}$$

$$x{V'}^2(x) = \frac{1}{4}\frac{x^{2\alpha+1}u^2}{x^2(\log \frac{x}{R})^2} + V_1^2 x \left(\log \frac{x}{R}\right) + \frac{1}{2}(V_1^2)'$$

and integrating to obtain

$$\int_0^\infty x{V'}^2 dx = \frac{1}{4}\int_0^\infty \frac{x^{2\alpha+1}u^2}{x^2\left(\log \frac{x}{R}\right)^2} + \int_0^\infty {V_1'}^2 x \left(\log \frac{x}{R}\right) \tag{8}$$

Combining (8) and (4) to obtain

$$\int_0^\infty x^{2\alpha+1}|u|^2 dx \geq \alpha^2 \int_0^\infty \frac{u^2}{x^2}x^{2\alpha+1}dx + \frac{1}{4}\int_0^\infty \frac{u^2 x^{2\alpha+1}}{x^2\left(\log \frac{x}{R}\right)^2}$$

and equality holds if and only if $u \equiv 0$. By the same argument this can be continued indefinitely by adding terms containing more logarithms (See [2], [8]).

Next we look for generalization of Hardy Sobolev inequalities on Manifolds. For the sake of simplicity, assume that $\Omega \subset \mathbb{R}^n$ be a domain and $L = -\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right)$ be an elliptic operator on Ω. For $u \in C^1(\Omega)$. define

$$|\nabla_L u|^2 = \sum a_{i_j}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} \tag{9}$$

Based on the same ideas as in the previous arguments we have the following generalized Hardy - Sobolev inequalities [1].

Theorem 1 (Generalized Hardy-Sobolev inequalities). *Let* $0 \in \Omega, E \in C^1(\Omega \backslash (0)) \cap L^1_{loc}(\Omega)$ *such that*

$$\begin{aligned} &LE = \delta_0 \quad in \quad \Omega \\ &E > 0 \quad in \quad \Omega \\ &E(0) = \infty \end{aligned} \tag{10}$$

then $\forall u \in C_0^1(\Omega)$, *we have*

$$\int\limits_\Omega |\nabla_L u|^2 \geq \frac{1}{4}\int\limits_\Omega \left|\frac{\nabla_L E}{E}\right|^2 u^2 \tag{11}$$

and equality holds if and only if $u \equiv 0$

Proof. Let $u \in C_0^1(\Omega)$ and define $V = E^{-\frac{1}{2}}u$. Then $V(0) = 0$ and $u = E^{\frac{1}{2}}V$. Then

$$\frac{\nabla_L u}{u} = \frac{1}{2}\frac{\nabla_L E}{E} + \frac{\nabla_L V}{V}$$

$$\begin{aligned} |\nabla_L u|^2 &= \frac{1}{4}\left|\frac{\nabla_L E}{E}\right|^2 u^2 + \frac{\langle \nabla_L E.\nabla_L V\rangle}{EV}u^2 + \frac{|\nabla_L v|^2}{V^2}u^2 \\ &= \frac{1}{4}\left|\frac{\nabla_L E}{E}\right|^2 u^2 + \frac{1}{2}\langle \nabla_L E.\nabla_L V^2\rangle + |\nabla_L V|^2 E \end{aligned}$$

Now $\int\limits_\Omega (LE)V^2 = V^2(0) = 0$, This is termed as the magical cancellation by Berzis and Vazquez. Hence (10) follows from this identity. □

As an application, we recollect the classical Hardy-Sobolev type inequalities in $\mathbb{R}^n (n \geq 2)$ as follows: let

$$E = \begin{cases} \frac{1}{|x|^{n-2}} & \text{if} \quad n \geq 3 \\ \log \frac{R}{|x|} & \text{if} \quad n = 2, \Omega \subset B(R), \end{cases}$$

then

$$\left|\frac{\nabla E}{E}\right|^2 = \begin{cases} \frac{(n-2)^2}{|x|^2} & \text{if} \quad n \geq 3 \\ \frac{1}{|x|^2(\log \frac{R}{|x|})^2} & \text{if} \quad n = 2. \end{cases}$$

Hence $\forall u \in C_0^1(\Omega)$,

$$\int\limits_\Omega |\nabla u|^2 \geq \begin{cases} (\frac{n-2}{2})^2 \int\limits_\Omega \frac{u^2}{|x|^2} & \text{if} \quad n \geq 3 \\ \frac{1}{4}\int\limits_\Omega \frac{u^2}{|x|^2(\log \frac{R}{|x|})^2} & \text{if} \quad n = 2. \end{cases}$$

Further Extensions.

(1) (11) extends to the L^p norm as well as for some degenerate operators (see [1]).
(2) (11) extends to Dirac operators (See [5]).

Dimension 2 Hardy - Sobolev inequalities.

Recall the Trudinger - Moser imbedding in $\mathbb{R}^2$. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, then $H_0^1(\Omega)$ embeds in an Orlicz space and in described by

$$\max_{\int\limits_\Omega |\nabla u|^2 \leq 1} \left\{ \int\limits_\Omega e^{4\pi u^2} dx \right\} < \infty \tag{12}$$

and 4π is the best constant. Associated to this we have the following (see [6]).

Theorem 2. *Let $0 \in \Omega$ and α, β are non negative real numbers. Then*

$$\sup_{\int_\Omega |\nabla u|^2 \leq 1} \int_\Omega \frac{e^{\alpha u^2}}{|x|^\beta} dx < \infty \tag{13}$$

if and only if $\frac{\alpha}{4\pi} + \frac{\beta}{2} \leq 1$.

Theorem 3. *Let $B = \{x \in \mathbb{R}^2, |x| < 1\}$ and equipped with the hyperbolic metric $\frac{dx}{(1-|x|^2)^2}$. Then $\forall u \in H_0^1(B)$,*

$$\sup_{\int_B |\nabla u|^2 \leq 1} \int_B \frac{(e^{4\pi u^2} - 1)}{(1 - |x|^2)^2} < \infty.$$

This is proved in ([9], [4]).

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Higher order Sobolev-Hardy inequalities with remainder terms

Filippo Gazzola

For $1 < p < \infty$ and $m \geq 1$ ($m \in \mathbb{N}$) consider the space $\mathcal{D}^{m,p}(\mathbb{R}^n)$ endowed with the norm

$$\|u\|_{m,p} := \begin{cases} \|\Delta^k u\|_p & \text{if } m = 2k, \\ \|\nabla(\Delta^k u)\|_p & \text{if } m = 2k+1. \end{cases}$$

For bounded domains Ω, this is also the norm in the space $W_0^{m,p}(\Omega)$ and in

$$W_\vartheta^{m,p}(\Omega) := \left\{ v \in W^{m,p}(\Omega);\ \Delta^j v_{|\partial\Omega} = 0 \text{ for } 0 \leq j < \frac{m}{2} \right\}$$

where restrictions on $\partial\Omega$ are intended in the sense of traces. When $1 \leq m < \frac{n}{p}$ put $p^*_m = \frac{np}{n-mp}$. It is well-known that there exist optimal constants $H = H(p,m,n) > 0$ and $S = S(p,m,n) > 0$ such that

$$\|u\|^p_{m,p} \geq H \int_{\mathbb{R}^n} \frac{|u|^p}{|x|^{mp}} \qquad \forall u \in \mathcal{D}^{m,p}(\mathbb{R}^n) \tag{1}$$

$$\|u\|^p_{m,p} \geq S \left(\int_{\mathbb{R}^n} |u|^{p^*_m} \right)^{p/p^*_m} \qquad \forall u \in \mathcal{D}^{m,p}(\mathbb{R}^n). \tag{2}$$

We refer to the classical works by Sobolev, Hardy, and Rellich, and to the more recent papers [2, 11, 13, 14] where these constants have been computed. Both inequalities (1) and (2) hold with the same constants H and S in the space $W^{m,p}_0(\Omega)$ in any bounded domain Ω. Although it is not explicitly assumed that $0 \in \Omega$, we have this case in mind. A natural question is then to wonder if the optimal embedding constants H and S remain the same in the larger space $W^{m,p}_\vartheta(\Omega)$. If $m \geq 2$ the proofs valid for the spaces $W^{m,p}_0(\Omega)$ cannot be carried on since there is no obvious extension of a function $u \in W^{m,p}_\vartheta(\Omega)$ to $\mathbb{R}^n$. Positive answers to this question were given in [10, 16] for (2) when $p = 2$. However, these proofs use the concentration-compactness method by Lions [11] which, in turn, extends functions from Ω to $\mathbb{R}^n$. Since it is not mentioned how these extensions can be obtained, the proof appears incomplete. Positive answers concerning H were obtained when $m = p = 2$ and $\Omega = B$ (the ball) in [3], and when $m = 2$ and $1 < p < \infty$ or $p = 2$ and $m \geq 2$ in general domains Ω in [6]. A complete positive answer to these questions was finally given in [8]:

THEOREM 1 *Let $n \in \mathbb{N}$, $p \in (1,\infty)$ and $m \in \mathbb{N}$ with $m < n/p$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in C^m$. Then,*

$$\|u\|^p_{m,p} \geq H \int_\Omega \frac{|u|^p}{|x|^{mp}} \quad \text{and} \quad \|u\|^p_{m,p} \geq S \left(\int_\Omega |u|^{p^*_m} \right)^{p/p^*_m} \qquad \forall u \in W^{m,p}_\vartheta(\Omega).$$

Theorem 1 becomes false for $p = 1$, see [1]. Moreover, the assumption $\partial\Omega \in C^m$ is necessary to have the full generality of the statement, see related work in [12] and also in [9]. We point out that the main idea for the proof of Theorem 1 was found by the authors during the Oberwolfach meeting *Topological and Variational Methods for Partial Differential Equations* held in may 2009. We conclude this first part with an open problem: *Which boundary conditions (other than Dirichlet and Navier) give the same constants for the embeddings in (1) and (2)?*

Since the inequalities in Theorem 1 are strict for $u \neq 0$, one is led to try to add **remainder terms**. There is a huge literature on inequalities with remainder terms. With no hope of being complete, let us just quickly mention some of the existing second order inequalities. These may be found in papers by Galaktionov (Diff. Int. Eq. 2006), Adimurthi-Grossi-Santra (JFA 2006), Tertikas-Zographopoulos (Adv. Math. 2006), Barbatis-Tertikas (J. Comp. Appl. Math. 2006), Barbatis (Math. Z. 2007), Ghoussoub-Moradifam (2009).

For general higher order spaces, we mention the paper Bartsch-Weth-Willem (Calc. Var. 2003), and that the following result is obtained in [5]:

THEOREM 2 *Let* $\Omega \subset \mathbb{R}^n$ *($n > 2m$) be a bounded domain. Let* q' *be the conjugate of* $q \in (1, \infty)$ *and consider*

$$|u|_{q,w} = \sup_{\substack{A \subseteq \Omega \\ |A|>0}} |A|^{-1/q'} \int_A |u|.$$

There exists $C = C(m, n, \Omega) > 0$ *such that*

$$\|u\|^2_{m,2} \geq S\|u\|^2_{2^*_m} + C|u|^2_{\frac{n}{n-2m},w} \quad \forall u \in H^m_0(\Omega).$$

The proof of this result heavily uses the Hilbert structure of the spaces $H^m_0(\Omega)$. We refer to [7] for a partial generalization of this result to the space $H^2_\vartheta(\Omega) = H^2 \cap H^1_0(\Omega)$ and to [4] to the non-Hilbertian framework $W^{1,p}_0(\Omega)$. However, a full generalization of this statement is still missing and we address the problem: *prove Theorem 2 in any space* $W^{m,p}_0$ *and* $W^{m,p}_\vartheta$.

The following somehow similar result concerning (1) was obtained in [6]:

THEOREM 3 *Let* $\Omega \subset \mathbb{R}^n$ *be a smooth bounded domain and let* $m \in \mathbb{N}$, $2m \leq n$. *There exist* $c_1, \dots, c_m$, *depending only on* m, n *and* $|\Omega|$, *such that*

$$\|u\|^2_{m,2} \geq H \int_\Omega \frac{u^2}{|x|^{2m}} + \sum_{\ell=0}^{m-1} c_\ell \int_\Omega \frac{u^2}{|x|^{2\ell}} \qquad \forall u \in H^m_\vartheta(\Omega).$$

More inequalities of this type are available, see again [6] and a nice improvement by Tertikas-Zographopoulos [15].

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Inversion positivity and the sharp Hardy–Littlewood–Sobolev inequality

Rupert L. Frank
(joint work with Elliott H. Lieb)

The Hardy–Littlewood–Sobolev inequality says that if $0 < \lambda < N$ and $f, g \in L^p(\mathbb{R}^N)$ with $p = 2N/(2N-\lambda)$, then

$$\left| I_\lambda[f,g] \right| \leq \mathcal{H}_{N,\lambda} \|f\|_p \|g\|_q \tag{1}$$

for some universal constant $\mathcal{H}_{N,\lambda}$. Here we abbreviated

$$I_\lambda[f,g] := \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\overline{f(x)}\; g(y)}{|x-y|^\lambda}\, dx\, dy\, .$$

We are interested in the sharp (that is, smallest possible) constant $\mathcal{H}_{N,\lambda}$ in (1). Our goal here is to sketch our new proof [3] of the following

Theorem 1. *Let $0 < \lambda < N$ if $N = 1, 2$ and $N - 2 \leq \lambda < N$ if $N \geq 3$. If $p = 2N/(2N-\lambda)$, then (1) holds with*

$$\mathcal{H}_{N,\lambda} = \pi^{\lambda/2} \frac{\Gamma((N-\lambda)/2)}{\Gamma(N-\lambda/2)} \left(\frac{\Gamma(N)}{\Gamma(N/2)} \right)^{1-\lambda/N} .$$

Equality holds if and only if

$$f(x) = \alpha \left(\beta + |x-\gamma|^2 \right)^{-(2N-\lambda)/2} \quad \text{and} \quad g(x) = \alpha' \left(\beta + |x-\gamma|^2 \right)^{-(2N-\lambda)/2} ,$$

for some $\alpha, \alpha' \in \mathbb{C}$, $\beta > 0$ and $\gamma \in \mathbb{R}^N$.

This theorem is originally due to Lieb [6], who can deal with the whole range of $\lambda \in (0, N)$. Another proof appeared in [2]. Both proofs in [6] and [2] rely on the non-linear technique of Schwarz symmetrization. In contrast, our proof relies on the *linear* notion of inversion positivity. A second ingredient of our proof is the geometric characterization of the optimizing functions $\alpha \left(\beta + |x-\gamma|^2 \right)^{-(2N-\lambda)/2}$, extending a result of Li and Zhu [5].

Inequality (1) is clearly invariant under translations and dilations. It is less obvious that it is actually invariant under the whole conformal group [6, 2]. This fact will also play a crucial role in our proof

We emphasize that since $|x|^{-N+2s}$ is a constant times the Green's function of $(-\Delta)^s$, inequality (1) is equivalent to the Sobolev inequality

$$\|(-\Delta)^{s/2}u\|_2^2 \geq \mathcal{S}_{N,s}\|u\|_q^2\,, \qquad q = 2N/(N-2s)\,, \tag{2}$$

for $0 < s < N/2$. Here 'equivalent' means that the sharp constant and the optimizer in one inequality determine the sharp constant and the optimizer in the other inequality. In passing we mention that we recently managed to extend the methods presented here to obtain the sharp logarithmic HLS inequality in dimensions $N = 1$ and 2. By duality and stereographic projection this is equivalent to Onofri's inequality (for $N = 2$) and the Lebedev–Milin inequality (for $N = 1$).

Our restriction $\lambda \geq N-2$ corresponds to the restriction $s \leq 1$ in (2), which covers the cases that appear most frequently in application. Since $\|(-\Delta)^{s/2}u\|_2^2 = \|\nabla u\|_2^2$ for $s = 1$, we provide a new proof of the standard Sobolev inequality [1, 9].

In the remainder of this note we sketch the key steps leading to the proof of Theorem 1.

Inversion positivity. Let $B = \{x \in \mathbb{R}^N :\ |x-a| < r\}$, $a \in \mathbb{R}^N$, $r > 0$, be a ball and denote by

$$\Theta_B(x) := \frac{r^2(x-a)}{|x-a|^2} + a$$

the inversion of a point x through the boundary of B. This map on $\mathbb{R}^N$ is lifted to an operator acting on functions f on $\mathbb{R}^N$ according to

$$(\Theta_B f)(x) := \left(\frac{r}{|x-a|}\right)^{2N-\lambda} f(\Theta_B(x))\,.$$

One easily finds that with $p = 2N/(2N-\lambda)$

$$I_\lambda[f] = I_\lambda[\Theta_B f] \qquad \text{and} \qquad \|f\|_p = \|\Theta_B f\|_p\,,$$

where we abbreviated $I_\lambda[f] := I_\lambda[f,f]$. Similarly, let $H = \{x \in \mathbb{R}^N :\ x\cdot e > t\}$, $e \in \mathbb{S}^{N-1}$, $t \in \mathbb{R}$, be a half-space and denote by $\Theta_H(x) := x + 2(t - x\cdot e)$ the reflection of a point x on the boundary of H. The corresponding operator is defined by $(\Theta_H f)(x) := f(\Theta_H(x))$ and it again satisfies $I_\lambda[f] = I_\lambda[\Theta_H f]$ and $\|f\|_p = \|\Theta_H f\|_p$. Our first ingredient in the proof of Theorem 1 is the following

Theorem 2 (Reflection and inversion positivity). *Let $0 < \lambda < N$ if $N = 1,2$, $N-2 \leq \lambda < N$ if $N \geq 3$, and let $B \subset \mathbb{R}^N$ be either a ball or a half-space. If $f \in L^{2N/(2N-\lambda)}(\mathbb{R}^N)$ and*

$$f^i(x) := \begin{cases} f(x) & \text{if } x \in B\,, \\ \Theta_B f(x) & \text{if } x \in \mathbb{R}^N \setminus B\,, \end{cases} \qquad f^o(x) := \begin{cases} \Theta_B f(x) & \text{if } x \in B\,, \\ f(x) & \text{if } x \in \mathbb{R}^N \setminus B\,, \end{cases}$$

then

$$\frac{1}{2}\left(I_\lambda[f^i] + I_\lambda[f^o]\right) \geq I_\lambda[f]\,.$$

If $\lambda > N-2$ then the inequality is strict unless $f = \Theta_B f$.

For half-spaces and $\lambda = N-2$ this theorem was long known to quantum field theorists [8]. The half-space case with $N-2 < \lambda < N$ was apparently first proved by Lopes and Mariş [7]. The case of balls seems to be new for all λ and we are grateful to E. Carlen for simplifying our original proof.

The Li–Zhu lemma. Our second ingredient in the proof of Theorem 1 is a generalization of a result by Li and Zhu [5].

Theorem 3 (**Characterization of inversion invariant measures**). *Let μ be a finite, non-negative measure on $\mathbb{R}^N$. Assume that*

(A) *for any $a \in \mathbb{R}^N$ there is an open ball B centered at a and for any $e \in \mathbb{S}^{N-1}$ there is an open half-space H with interior unit normal e such that*

$$\mu(\Theta_B^{-1}(A)) = \mu(\Theta_H^{-1}(A)) = \mu(A) \qquad \text{for any Borel set } A \subset \mathbb{R}^N .$$

Then μ is absolutely continuous with respect to Lebesgue measure and

$$d\mu(x) = \alpha \left(\beta + |x-y|^2\right)^{-N} dx$$

for some $\alpha \geq 0$, $\beta > 0$ and $y \in \mathbb{R}^N$.

For absolutely continuous measures $d\mu = v\,dx$ assumption (A) is equivalent to the fact that for any $a \in \mathbb{R}^N$ there is an $r_a > 0$ such that for any x

$$v(x) = \left(\frac{r_a}{|x-a|}\right)^{2N} v\left(\frac{r_a^2(x-a)}{|x-a|^2} + a\right),$$

and similarly for reflections.

We refer to [3] for how to deduce Theorem 1 from Theorems 2 and 3.

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On approximate differentiability of the maximal function

Piotr Hajłasz

Juha Kinnunen [3] proved that the Hardy-Littlewood maximal operator Mu is bounded in the Sobolev space $W^{1,p}(\mathbb{R}^n)$ when $p > 1$. More generally, if A is a sub-linear, translation invariant operator bounded in L^p, $p > 1$, then A is bounded in $W^{1,p}$. Since the maximal function is not bounded in L^1, there is no apparent reason to believe that the maximal function should be bounded in $W^{1,1}$. However, Tanaka [4], proved that the non-centered one dimensional maximal function of $u \in W^{1,1}(\mathbb{R})$ is locally in $W^{1,1}$ and has integrable derivative. This leads to a question: *Is the operator $u \mapsto |\nabla Mu|$ bounded from $W^{1,1}(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$?* This seems to be a very difficult problem. Even in dimension one it is not known if the Hardy-Littlewood maximal operator (i.e. the centered one) of a $W^{1,1}$ function belongs locally to $W^{1,1}$.

In the talk I discussed partial results from joint papers with Janni Onninen [2] and Jan Malý [1].

Theorem. ([2]) *Let $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$. If $Mu < \infty$ a.e. and $M|\nabla u| \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then $Mu \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ and $|\nabla Mu| \le 2M|\nabla u|$ a.e.*

This result allows to avoid reflexivity of L^p which is usually employed in the proof of Kinnunen's result.

Theorem. ([1]) *If $u \in L^1(\mathbb{R}^n)$ is approximately differentiable a.e., then Mu is approximately differentiable a.e.*

In particular $u \in W^{1,1}(\mathbb{R}^n)$ is approximately differentiable a.e. and hence the Hardy-Littlewood maximal function of $u \in W^{1,1}$ is approximately differentiable a.e. This is, however, much less than weak differentiability.

Another result concerns differentiability properties of the maximal function of an arbitrary function $u \in L^1(\mathbb{R}^n)$. Namely we have:

Theorem. ([1]) *Let $u \in L^1(\mathbb{R}^n)$. Then any open set $\Omega \subset \mathbb{R}^n$ contains a set of positive measure on which Mu is approximately differentiable. Equivalently, for any open set $\Omega \subset \mathbb{R}^n$ there is a C^1 function v such that the set $\Omega \cap \{Mu = v\}$ has positive measure.*

The proof involves potential theory of superharmonic functions.

In view of this result it is natural to inquire whether the maximal function of an integrable function is approximately differentiable a.e. The answer to this question turns out to be in the negative, a suitable example was constructed in [1].

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Critical Equations in $\mathbb{R}^2$

Manuel del Pino
(joint work with Monica Musso and Bernhard Ruf)

Let $\Omega \subset \mathbb{R}^2$ a smooth, bounded domain and $\lambda > 0$. We consider the problem

$$(P_\lambda) \qquad \Delta u + \lambda u\, e^{u^2} = 0, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

Solutions of Problem (P_λ) are critical points of the functional

$$J_\lambda(u_\lambda) = \int_\Omega |\nabla u|^2 - \lambda \int_\Omega e^{u^2}, \quad u \in H_0^1(\Omega).$$

In [1] it is proven that (P_λ) is solvable whenever $0 < \lambda < \lambda_1$. In [4] it is found that a solution u_λ with $J_\lambda(u_\lambda) \le C$ must satisfy that for some integer $k \ge 0$

$$J_\lambda(u_\lambda) \;=\; 4k\pi \;+\; o(1) \quad \text{as } \lambda \to 0.$$

In [3] we find solutions with this property. To state the result we introduce some notation.

Let $G(x,y)$ denote Green's function of the problem

$$-\Delta_x G(x,y) = 8\pi\delta_y(x) \quad \text{in } \Omega, \quad G(\cdot,y) = 0 \quad \text{on } \partial\Omega.$$

and $H(x,y) = 4\log\frac{1}{|x-y|} - G(x,y)$ its regular part.

For points $\xi = (\xi_1,\dots,\xi_k) \in \Omega^k$, $m = (m_1,\dots,m_k) \in \mathbb{R}_+^k$ we consider the functional

$$\varphi_k(\xi,m) = \sum_{j=1}^{k} 2m_j^2(2\log 8 - 2 + \log m_j^2) + m_j^2 H(\xi_j,\xi_j) - \sum_{i\neq j} m_i m_j\, G(\xi_j,\xi_j).$$

We say that φ_k has a *stable critical point situation* if for some region Λ compactly contained in its domain, any small $C^1(\bar\Lambda)$-perturbation of φ_k has a critical point in Λ.

- $\varphi_1(\xi,m)$ satisfies this property, with Λ a neighborhood of its minimum set.
- $\varphi_2(\xi,m)$ satisfies this property whenever Ω is not simply connected. We believe this is the case for any $k \ge 2$.

Theorem 1 ([3]). *Assume that $\varphi_k(\xi,m)$ has a stable critical point situation. Then there exists a solution a solution u_λ to (P_λ) that blows-up around k points ξ_j^λ as $\lambda \to 0$ and such that away from them, and for certain numbers m^λ it takes the form*

$$u_\lambda(x) \;=\; \sqrt{\lambda}\sum_{j=1}^{k} m_j^\lambda\; [\,G(x,\xi_j) \;+\; o(1)\,]$$

where $\nabla\varphi_k(\xi^\lambda, m^\lambda) \to 0$. *Furthermore*

$$J_\lambda(u_\lambda) \,=\, 4\pi k \,+\, \lambda \left[-|\Omega| \,+\, 8\pi\, \varphi_k(\xi^\lambda, m^\lambda) \,+\, o(1) \right] .$$

This result applies for $k = 1$, predicting bubbling near a minimizer of $H(\xi, \xi)$, and for $k = 2$ provided that Ω is not simply connected.

Existence of critical points in the Trudinger-Moser supercritical case. Let us consider the related problem of finding critical points in $H_0^1(\Omega)$ of the functional $\int_\Omega e^{u^2}$ subject to the constraint $\int_\Omega |\nabla u|^2 = \mu$. This amounts to solving the equation

$$(T_\mu) \qquad \Delta u \,+\, \mu\, \frac{u\, e^{u^2}}{\int_\Omega u^2\, e^{u^2}} = 0 \quad \text{in } \Omega,\ u = 0 \quad \text{in } \partial\Omega.$$

For $\mu \le 4\pi$ the supremum of this functional is finite, as the Trudinger-Moser inequality states [9, 6] while it is infinite for $\mu > 4\pi$. The supremum for $\mu = 4\pi$ is attained, as found in [2, 5].

In [7] it is found that this global maximum can be continued as a local maximum for a *supercritical range* of the form $4\pi < \mu < \mu_1$, and also that a second positive solution exists on a dense subset of $(4\pi, \mu_1)$. Existence of the second solution in the entire range has been recently obtained in [8]. Qualitative properties of this second solution are not known, nor existence of solutions for larger supercritical values of μ. We find the following result.

Theorem 2. *Assume that $k = 1$ or that $k = 2$ and that Ω is not simply connected. Then there exists a positive solution u_μ of Problem (T_μ) for $\mu \in (4k\pi, \mu_k)$ that blows up at exactly k points with a profile determined by a critical point of the functional φ_k, as in Theorem 1.*

We believe that this result should hold true for any $k \ge 1$ if Ω is not simply connected.

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A variational method for a class of parabolic PDEs

Wilfrid Gangbo

(joint work with Alessio Figalli, Türkay Yolcu)

Let $\mathbb{H}$ be a Hilbert space and $h : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ be such that $h(x, \cdot)$ is uniformly convex and grows superlinearly at infinity, uniformy in x. Suppose $U : \mathbb{H} \to \mathbb{R}$ is strictly convex and grows superlinearly at infinity. We assume that both h and U are smooth. If $\mathbb{H}$ is of infinite dimension, the initial value problem $\dot{x} = -\nabla_p h(x, -\nabla U(x)),\ x(0) = \bar{x}$ is not known to admit a solution. We study a class of parabolic equations on $\mathbb{R}^d$ (and so of infinite dimensional nature), analogous to the previous initial value problem and establish existence of solutions. First, we extend De Giorgi's interpolation method to parabolic equations which are not gradient flows but possess an entropy functional and an underlying Lagrangian. The new fact in the study is that not only the Lagrangian may depend on spatial variables, but it does not induce a metric. These interpolations reveal to be a powerful tool for proving convergence of a time discrete algorithm.
The specific system of PDEs we study is:

$$\partial_t \varrho_t + div(\varrho_t V_t) = 0, \quad \text{in} \qquad \mathcal{D}'((0,T) \times \mathbb{R}^d) \tag{1}$$

where

$$V_t := \nabla_p H\big(x, -\varrho_t^{-1} \nabla[P(\varrho_t)]\big) \qquad \varrho_t - \text{almost everywhere.}$$

Here, $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a Hamiltonian, convex in its second variables, $U \in C^1(0,\infty)$ is strictly convex and $P(s) = sU'(s) - U(s)$. Note that formally we have $\varrho_t^{-1}\nabla[P(\varrho_t)] = \nabla[U'(\varrho_t)]$, which explains the link with the ODE in Hilbert spaces. This report is based on a joint work with A. Figalli and T. Yolcu.

Sharp Hardy inequalities for fractional integrals on general domains

Michael Loss

(joint work with Craig Sloane)

Consider a domain $\Omega \subset \mathbb{R}^n$ with non-empty boundary. The following notion is taken from Davies [2]. Fix a direction $w \in \mathbb{S}^{n-1}$ and define

$$d_{w,\Omega}(x) = \min\{|t| \ : \ x + tw \notin \Omega\} \ .$$

Further, define the function

$$\delta_{w,\Omega}(x) = \sup\{|t| : x + tw \in \Omega\} \ ,$$

i.e., $\delta_{w,\Omega}(x)$ is the point in the intersection of the line $x+tw$ and Ω that is farthest away from x and set

$$\frac{1}{M_\alpha(x)^\alpha} := \frac{\int_{\mathbb{S}^{n-1}} dw \left[\frac{1}{d_{w,\Omega}(x)} + \frac{1}{\delta_{w,\Omega}(x)}\right]^\alpha}{\int_{\mathbb{S}^{n-1}} dw|w_n|^\alpha} .$$

The integral in the denominator can be easily computed to be

$$\int_{\mathbb{S}^{n-1}} dw|w_n|^\alpha = 2\pi^{\frac{n-1}{2}} \frac{\Gamma(\frac{1+\alpha}{2})}{\Gamma(\frac{n+\alpha}{2})}$$

These definitions are analogous to the one in [2] where all estimates are expressed in terms of

$$\frac{1}{m_2(x)^2} = \frac{\int_{\mathbb{S}^{n-1}} dw \frac{1}{d_{w,\Omega}(x)^2}}{|\mathbb{S}^{n-1}|/n} .$$

In case the domain Ω is convex, the quantity $M_\alpha(x)$ can be bounded in terms of $d_\Omega(x)$, the distance of x to $\partial\Omega$, and $D_\Omega(x)$, the *'width of Ω with respect to x'*. For convex domains with smooth boundary, this quantity is given by the width of the smallest slab that contains Ω and consists of two parallel hyper-planes one of which is tangent to $\partial\Omega$ at the point closest to x. This definition can be readily extended to general convex domains. It is not difficult to see that

$$\frac{1}{M_\alpha(x)^\alpha} \geq \left[\frac{1}{d_\Omega(x)} + \frac{1}{D_\Omega(x) - d_\Omega(x)}\right]^\alpha ,$$

The following theorem is proved in [4].

Theorem 1. *Let Ω be a domain with non-empty boundary and $1 < \alpha < 2$. For any $f \in C_c^\infty(\Omega)$*

$$\frac{1}{2}\int_{\Omega\times\Omega} \frac{|f(x)-f(y)|^2}{|x-y|^{n+\alpha}} dxdy \geq \kappa_{n,\alpha} \int_\Omega \frac{|f(x)|^2}{M_\alpha(x)^\alpha} dx . \tag{1}$$

In particular, if Ω is a convex region then for any $f \in C_c^\infty(\Omega)$

$$\frac{1}{2}\int_{\Omega\times\Omega} \frac{|f(x)-f(y)|^2}{|x-y|^{n+\alpha}} dxdy \geq \kappa_{n,\alpha} \int_\Omega |f(x)|^2 \left[\frac{1}{d_\Omega(x)} + \frac{1}{D_\Omega(x) - d_\Omega(x)}\right]^\alpha dx$$

The constant $\kappa_{n,\alpha}$ is given by

$$\kappa_{n,\alpha} = \pi^{\frac{n-1}{2}} \frac{\Gamma(\frac{1+\alpha}{2})}{\Gamma(\frac{n+\alpha}{2})} \frac{1}{\alpha} \left[\frac{2^{1-\alpha}}{\sqrt{\pi}} \Gamma(\frac{2-\alpha}{2})\Gamma(\frac{1+\alpha}{2}) - 1\right]$$

and is sharp.

The inequality

$$\frac{1}{2}\int_{\Omega\times\Omega} \frac{|f(x)-f(y)|^2}{|x-y|^{n+\alpha}} dxdy \geq \kappa_{n,\alpha} \int_\Omega \frac{|f(x)|^2}{d_\Omega(x)^\alpha} dx \tag{2}$$

has been shown previously by Bogdan and Dyda [1] for all $0 < \alpha < 2$ for the special case where Ω a half space. They conjectured that (2) holds for general convex domains.

The proof of (1) proceeds via a one dimensional inequality that is proved using certain invariance properties of the quadratic form

$$\frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}} \frac{|f(x)-f(y)|^2}{|x-y|^{1+\alpha}}dxdy$$

under fractional linear transformations and then reducing the general problem to one dimensions via the formula

$$\int_{\Omega\times\Omega} \frac{|f(x)-f(y)|^p}{|x-y|^{n+\alpha}}dxdy$$

$$= \frac{1}{2}\int_{\mathbb{S}^{n-1}} dw \int_{\{x:x\cdot w=0\}} d\mathcal{L}_w(x) \int_{x+sw\in\Omega} ds \int_{x+tw\in\Omega} dt \frac{|f(x+sw)-f(x+tw)|^p}{|s-t|^{1+\alpha}}$$

It was pointed out by Frank and Seiringer that the Theorem 1 can be extended, albeit in a weaker form, to a more general class of fractional integrals.

Theorem 2. *Let $1 < p < \infty$ and $1 < \alpha < p$. Then for any domain $\Omega \subset \mathbb{R}^n$ and any $f \in C_c^\infty(\Omega)$*

$$\int_{\Omega\times\Omega} \frac{|f(x)-f(y)|^p}{|x-y|^{n+\alpha}}dxdy \geq \mathcal{D}_{n,p,\alpha}\int_\Omega \frac{|f(x)|^p}{m_\alpha(x)^\alpha}dx$$

where

$$\frac{1}{m_\alpha(x)^\alpha} := \frac{\int_{\mathbb{S}^{n-1}} dw \frac{1}{d_{w,\Omega}(x)^\alpha}}{\int_{\mathbb{S}^{n-1}} dw|w_n|^\alpha} \ .$$

and

$$\mathcal{D}_{n,p,\alpha} = 2\pi^{\frac{n-1}{2}}\frac{\Gamma(\frac{1+\alpha}{2})}{\Gamma(\frac{n+\alpha}{2})}\int_0^1 \frac{|1-r^{\frac{\alpha-1}{p}}|^p}{(1-r)^{1+\alpha}}dr$$

is the sharp constant. In particular, for Ω convex

$$\int_{\Omega\times\Omega} \frac{|f(x)-f(y)|^p}{|x-y|^{n+\alpha}}dxdy \geq \mathcal{D}_{n,p,\alpha}\int_\Omega \frac{|f(x)|^p}{d_\Omega(x)^\alpha}dx \ .$$

The constant $\mathcal{D}_{n,p,s}$ has been computed before in [3] as the sharp constant for the Hardy inequality for the half-space.

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The isoperimetric inequality in the Gauss space

Nicola Fusco
(joint work with Andrea Cianchi, Francesco Maggi, Aldo Pratelli)

The Gauss measure is a probability measure on $\mathbb{R}^n$ defined by setting for any measurable set $E \subset \mathbb{R}^n$

$$\gamma_n(E) = \frac{1}{(2\pi)^{n/2}} \int_E e^{-\frac{|x|^2}{2}}\, dx\,.$$

If E is a set of locally fnite perimeter, the Gaussian perimeter of E is defined as

$$P_\gamma(E) = \frac{1}{(2\pi)^{n/2}} \int_{\partial^* E} e^{-\frac{|x|^2}{2}}\, d\mathcal{H}^{n-1}(x)\,,$$

where $\partial^* E$ stands for the essential boundary of E in the sense of De Giorgi and $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure. Clearly, both γ_n and P_γ are invariant by rotations around the origin. As in the Euclidean case, also the Gaussian perimeter can be characterized in a variational form. Namely, one has

$$P_\gamma(E) = \sup\Big\{\int_E \big(\mathrm{div}\varphi(x) - x\cdot\varphi\big)d\gamma_n \,:\, \varphi \in C^1_0(\mathbb{R}^n;\mathbb{R}^n),\ \|\varphi\|_\infty \le 1\Big\}\,.$$

It is well known that if E is a set such that $\gamma_n(E) = r \in (0,1)$, then

$$P_\gamma(E) \ge P_\gamma(H_{\nu,s})\,, \tag{1}$$

where $\nu \in \mathbb{S}^{n-1}$ and $H_{\nu,s}$ is the half-space $H_{\nu,s} = \{x : x\cdot\nu > s\}$ such that

$$r = \gamma_n(H_{\nu,s}) = \frac{1}{\sqrt{2\pi}} \int_s^\infty e^{-t^2/2}\, ds := \Phi(s)\,.$$

Using the function Φ, inequality (1) may be restated as

$$P_\gamma(E) \ge \frac{1}{\sqrt{2\pi}} e^{-[\Phi^{-1}(\gamma_n(E))]^2/2}\,.$$

The first proofs of the Gauss isoperimetric inequality (1) appeared in [6] and [1], followed later by different ones, both of geometric and probabilistic nature (see e.g. the references in [3]). However, only recently it was proved by Carlen and Kerce ([2]) that half-spaces are the only sets for which equality holds in (1). Their proof makes use of probabilistic arguments involving the Ornstein-Uhlenbeck semigroup. We present here a variational proof following the old idea of Steiner to deduce the isoperimetric inequality in the Euclidean case by a symmetrization argument. The analog in the Gauss space of the Steiner symmetrization is the so called Ehrhard symmetrization, first introduced in [4]. More precisely, in [3] the Gaussian isoperimetric inequality (1), together with the characterization of the equality cases, is quickly obtained by proving that the Gaussian perimeter strictly decreases under the Ehrhard symmetrization of a set E in a given direction $\nu \in \mathbb{S}^{n-1}$, unless the one dimensional sections of E parallel to ν are half-lines or lines.

By using Ehrhard symmetrization in [3] we prove also a quantitative version of inequality (1). In fact we show that the stronger inequality holds

$$P_\gamma(E) \geq P_\gamma(H_{\nu,s}) + \frac{\lambda^2(E)}{C^2(n,r)}\,, \tag{2}$$

where $\lambda(E)$ is the asymmetry index of the set E,

$$\lambda(E) = \min_{\nu\in\mathbb{S}^{n-1}} \big\{\gamma_n\big(E\triangle H_{\nu,s}\big) : \ \gamma(H_{\nu,s}) = \gamma_n(E) = r\big\}\,.$$

The quantitative inequality (2) can be also rewritten as

$$\lambda(E) \leq C(n,r)\sqrt{\delta(E)}\,,$$

where $\delta(E) = P_\gamma(E) - P_\gamma(H_{\nu,s})$ is the isoperimetric deficit of E.
Inequality (2) extends to the Gaussian context the quantitative (Euclidean) isoperimetric inequality proved in [5]

$$\Lambda^2(E) \leq C(n)\sqrt{\Delta(E)}\,,$$

where $\Lambda(E)$ is the Fraenkel asymmetry of E

$$\Lambda(E) = \min_{x\in\mathbb{R}^n}\Big\{\frac{|E\triangle B_r(x)|}{|E|} : |E| = |B_r|\Big\}$$

and $D(E)$ is the isoperimetric deficit

$$D(E) = \frac{P(E) - P(B_r)}{P(B_r)}\,,$$

$P(E)$ and $P(B_r)$ being the Euclidean perimeter of E and of a ball of radius r, respectively.

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On mean field equations of Liouville type over closed surfaces

Gabriella Tarantello

The study of selfdual vortices in gauge field theory relates, via Taubes approach, to the solvability of a certain class of mean field equations over a closed surface. Typically the 2-sphere S^2 to account (via the stereographic projection) to planar vortices with appropriate decay at infinity; or the flat 2-torus $\mathbb{T}^2$ to handle periodic configurations as they occur naturally in the physical applications.

To be more precise, let us focus on the following typified problem considered over the closed surface (M, g):

$$\Delta_g u + \lambda \left(\frac{ke^u}{\int_M ke^u dv_g} - \frac{1}{|M|} \right) = 4\bar{u} \sum_{j=1}^{N} \beta_j \left(\delta_{p_j} - \frac{1}{|M|} \right), \tag{1}$$

where Δ_g and dv_g denote respectively the Laplace-Beltrami operator and the volume element relative to the metric g, λ, $\beta_j \in (0, +\infty)$ and $p_j \in M$ for $j = 1, ..., N$; and k is a smooth (positive) function on M.

The interest for (1) has originated in several other contexts, including the "uniformization" theorem in differential geometry, when we extend the conformal class to include surfaces with "conical" singularities. On the other hand, from the point of view of selfdual vortex configurations, the point p_j $(j = 1, ..., N)$ plays a role of a *vortex point* and we shall need to obtain solutions with strong "localization" properties around p_j $(j = 1, ..., N)$. Such a "concentration" property can be attained only at a specific value of λ_* that depends on $\beta_1, ..., \beta_N$.

We show in some cases how to compute the Leray-Schauder degree of the Fredholm operator d_λ associated to (1), for λ in a neighbourhood of λ_*. By showing that d_λ has a "jump" as λ crosses λ_*, we prove the existence of "concentrated" solutions when λ approaches λ_*, as desired.

Open problem session

Nicola Fusco, Piotr Hajłasz, Bernd Kawohl, Pekka Koskela

Open problems presented by Pekka Koskela:

Let u be both harmonic and 3-harmonic, i.e. u satisfies $\Delta u = 0$ and $\Delta_3 u = 0$ on the unit ball in $\mathbb{R}^n$. **Prove** that then u is essentially trivial or that the Hessian D^2u of u vanishes everywhere in the unit ball.

For $n = 2$ this is known [1], but the proof uses function theoretic arguments. Instead of $\Delta_3 u = 0$ one could assume $\Delta_\infty u = 0$ or any $\Delta_p u = 0$ with $p \in (1, 2) \cup (2, \infty)$ without changing the problem. This question is related to quasiconformal mappings $F : B_1(0) \subset \mathbb{R}^3 \to \mathbb{R}^3$ which are homeomorphisms and harmonic and which satisfy the estimate $||DF(x)||^3 \leq KJ_K(x)$.

Let $E \subset [0, 1]$ be the ternary Cantor set. Then **prove or disprove** that for every Cantor set $E_2 \subset [0, 1]$ and for any $p > 2$ the set $E := E_1 \times E_2$ is removable for $W^{1,p}$, i.e. $W^{1,p}(\mathbb{R}^2 \setminus E) = W^{1,p}(\mathbb{R}^2)$.

It is known that the answer is positive if $p > p_0 > 2$, and p_0 is a known number.

Open problems presented by Bernd Kawohl:

The first eigenfunction u_1 of the p-Laplacian operator on a bounded domain Ω can be characterized as minimizing the Rayleigh quotient

$$R_p(v) = \frac{\int_\Omega |\nabla v|^p \; dx}{\int_\Omega |v|^p \; dx} \qquad \text{on } W_0^{1,p}(\Omega) \setminus \{0\}. \tag{1}$$

It is known to be unique up to scaling and it solves the associated Euler-equation $\Delta_p u + \lambda |u|^{p-2} u = 0$ in Ω under Dirichlet's boundary condition. In fact, any eigenfunction of only one sign must be the firsteigenfunction, see [4]. What about the second eigenfunction? It must change sign, and in particular in two dimensions it has a nodal line in Ω. For $p = 2$ nodal lines are well studied objects. When Ω is a disk and $p = 2$, the nodal line is a diameter, and when Ω is a square and $p = 2$, the nodal line can be a vertical, a horizontal or a diagonal.

For $p = \infty$ the second eigenfunction was investigated in [3] and for $p \to 1$ by Parini [5]. Their results as well as some numerical computations by J. Horák support the following conjectures:

a) For Ω a disk and any $p \in (1, \infty)$ the nodal line is a diameter.

b) For Ω a square and any $p \in (1, 2)$ the nodal line is vertical or horizontal, while for $p \in (2, \infty)$ it is diagonal.

In his numerical calculation J. Horák used the fact that second eigenfunctions can be characterized as a mountain pass connecting u_1 and $-u_1$, see [2].

Open problem presented by Nicola Fusco:

Let f be a function in $BV(\Omega)$ and $\Omega \subset \mathbb{R}^2$ a plane domain. If f_Ω denotes the average of f over Ω, then the following Poincaré type inequality

$$\left(\int_\Omega |f - f_\Omega|^2 \right)^{1/2} \leq c(\Omega) \int_\Omega |Df| \tag{2}$$

is known to have an optimal constant c_0, in the sense that $c(\Omega) \geq c_0 > 0$. In fact for any $f_0 \in C_0^\infty(\Omega)$ with $\int_\Omega f_0$ one finds $c_0 \geq ||f_0||_2 / \int |Df_0|$.

Show that the optimal constant among all domains of given area is attained for a disk. In fact, for a disk the optimal constant was explicitly calculated and it was shown that it is attained by a particular function in [6]. For general Ω with area $|\Omega| = 1$ Fusco and Pratelli could show that $c(\Omega)$ is realized by characteristic functions

$$c(\Omega) = \sup \left\{ \frac{\sqrt{t(1-t)}}{P(E;\Omega)}, \quad |E| = t, \; E \subset \Omega, \; 0 < t < 1 \right\}. \tag{3}$$

So if Ω is a disc, the optimal t is $1/2$, and after scaling, one can conclude that the conjecture would be true, if any plane domain of area π had a bisecting chord of length at most 2. However, the hope of proving this last statement has to be dismissed because of [7]. The so-called Auerbach triangle has area less than π and only bisecting chords of length 2.

Open problems presented by Piotr Hajłasz:

Extension domains.

We say that $\Omega \subset \mathbb{R}^n$ is a $W^{1,p}$-extension domain if there is a bounded linear operator $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ such that $Eu|_\Omega = u$ for $u \in W^{1,p}(\Omega)$.

Problem 1. *Suppose that Ω is a $W^{1,p}$-extension domain and $W^{1,q}$-extension domain for some $1 \le p < q \le \infty$. Does it follow that Ω is a $W^{1,r}$-extension domain for all $p < r < q$?*

All examples known to me have this property. Moreover the answer is in the positive when $p \ge n$, see [12]. The problem looks like an interpolation question, but the interpolation methods do not easily apply here. See also [9] for related results.

Sobolev homeomorphisms.

Let $f : \Omega_1 \to \Omega_2$ be a Sobolev homeomorphism between domains $\Omega_1, \Omega_2 \subset \mathbb{R}^n$, $f \in W^{1,p}$.

Problem 2. *Is it possible to construct a homeomorphism f, whose its Jacobian changes sign in the sense that $Jf > 0$ on a set of positive measure and $Jf < 0$ on a set of positive measure?*

The answer is "no" if $p > [n/2]$ (integer part of $n/2$), see [10]. The proof is based on the linking number and the method cannot be extended to smaller values of p, so a completely new idea is needed. On the other hand one can construct (P. Hajłasz, unpublished) an approximately differentiable homeomorphism with the Lusin property (sets of measure zero are mapped onto sets of measure zero) whose Jacobian changes sign.

Isoperimetric inequality.

To a vector function $u = (u_1, \ldots, u_n) : \mathbb{R}^n \to \mathbb{R}^n$, we associate the deformation tensor ε defined as the symmetric part of the gradient of u, i.e., $\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$, or in terms of components,

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

The well known Korn inequality implies that for $1 < p < n$ and $p^* = np/(n-p)$,

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le C\|\varepsilon(u)\|_{L^p(\mathbb{R}^n)}, \qquad u \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n). \tag{4}$$

Since the Korn inequality is not true for $p = 1$, one could suspect that inequality (4) should be false for $p = 1$. However, Strauss [13], proved a surprising result that (4) is actually true also for $p = 1$.

$$\left(\int_{\mathbb{R}^n} |u|^{n/(n-1)}\, dx\right)^{(n-1)/n} \le C \int_{\mathbb{R}^n} |\varepsilon(u)|\, dx. \tag{5}$$

It is well known that the classical Sobolev inequality for $p = 1$, with the best constant, is equivalent to the isoperimetric inequality.

Problem 3. *What is the best constant in (5)? Is there any isoperimetric interpretation of this inequality?*

The isoperimetric inequality is obtained form the Sobolev inequality applied to level sets of the function. However, in the case of the Strauss inequality, it is not clear what are the level sets, because the function is vector valued and the truncation is a problem in the vector valued case.

Convex functions.

Let u be a convex function on an open set in $\mathbb{R}^n$. It is known (but not well known) that for any $\varepsilon > 0$ there is $v \in C^2$ which coincides with u off a set of measure ε, see [8]. However, it is natural to ask:

Problem 4. *Is it possible to find such a function $v \in C^2$ that is also convex?*

This is a very natural question and a positive answer would be a very nice approximation result.

Subharmonic functions.

Let u be a subharmonic function defined in an open set $\Omega \subset \mathbb{R}^n$. Imomkulov [11], proved that for any $\varepsilon > 0$ there is a function $v \in C^2$ such that v coincides with u off a set of measure ε. This is a very natural, but rather unknown, result.

Problem 5. *Is it possible to find such a function $v \in C^2$ that is also subharmonic?*

This problem is somewhat related to Problem 4 and again a positive answer would give a very nice approximation result.

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