Mathematisches Forschungsinstitut Oberwolfach

Report No. 32/2019

DOI: 10.4171/OWR/2019/32

Mathematical Theory of Water Waves

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14 July – 20 July 2019

Abstract. Water waves, that is waves on the surface of a fluid (or the interface between different fluids) are omnipresent phenomena. However, as Feynman wrote in his lecture, *water waves that are easily seen by everyone, and which are usually used as an example of waves in elementary courses, are the worst possible example; they have all the complications that waves can have*. These complications make mathematical investigations particularly challenging and the physics particularly rich. Indeed, expertise gained in modelling, mathematical analysis and numerical simulation of water waves can be expected to lead to progress in issues of high societal impact (renewable energies in marine environments, vorticity generation and wave breaking, macro-vortices and coastal erosion, ocean shipping and near-shore navigation, tsunamis and hurricane-generated waves, floating airports, ice-sea interactions, ferrofluids in high-technology applications, . . .). The workshop was mostly devoted to rigorous mathematical theory for the exact hydrodynamic equations; numerical simulations, modelling and experimental issues were included insofar as they had an evident synergy effect.

Mathematics Subject Classification (2010): 76B15.

Introduction by the Organizers

From a mathematical viewpoint, the water-wave problem poses surprisingly deep and subtle challenges for rigorous analysis and numerical simulation. Although the governing equations are widely accepted, a rigorous theory of their solutions is extremely complex due not only to the fact that the water-wave problem is a classical free-boundary problem, but also because the boundary conditions (and, in some cases, the equations) are strongly nonlinear. The last thirty years have seen rapid progress in analysis, modelling and numerical simulation of surface waves. On the one hand we now have rigorous results on well-posedness, long-time existence, appearance of singularities, existence of small-amplitude three-dimensional waves, and the role of vorticity; on the other hand numerical simulations of large-amplitude and three-dimensional waves have been made possible by theoretical and technological advances. These developments are evidenced by a number of recent high-profile workshops, conferences and special research semesters together with minisymposia at the major applied mathematics conferences, an explosion of new talent at the PhD and postdoctoral level in these areas, and the founding of a new interdisciplinary scientific journal on water waves.

In view of the continuing vigorous interest in the mathematical theory of water waves it appeared timely to convene a further workshop at Oberwolfach (previous workshops were held in 2001, 2006 and 2015). Its aims were to review the state of the art and stimulate research in major open problems in the following themes.

- Water waves with vorticity and application to wave-current interactions;
- (Boundary) initial-value problems;
- Coherent structures;
- Waves in domains with complex geometry and wave-structure interaction.

Significant new results in these areas were reported and are summarised in the extended abstracts below. The workshop was attended by twenty-six participants from twelve countries; there was a good mix of senior and junior researchers. Twenty-two lectures were held in a friendly and informal atmosphere and many collaborative discussions took place.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1641185, “US Junior Oberwolfach Fellows”. Moreover, the MFO and the workshop organizers would like to thank the Simons Foundation for supporting Professor Vera Hur in the “Simons Visiting Professors” programme at the MFO.

Workshop: Mathematical Theory of Water Waves

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Abstracts

Low regularity solutions for gravity water waves

Albert Ai

The gravity water wave equations are a system of partial differential equations which govern the evolution of the interface between a vacuum and an incompressible, irrotational fluid, in the presence of gravity. We are concerned with the well-posedness of the Cauchy problem for the gravity water wave equations. In particular, we are concerned with the situation where the initial data has low regularity, corresponding to surface waves which are not necessarily smooth.

To formulate the problem and equations, we establish the following notation. Let Ω denote a time dependent fluid domain contained in a fixed domain $\mathcal{O}$, located between a free surface and a fixed bottom:

$$\Omega = \{(t, x, y) \in [0, 1] \times \mathcal{O} \ ; \ y < \eta(t, x)\},$$

where $\mathcal{O} \subseteq \mathbb{R}^d \times \mathbb{R}$ is a given connected open set, with $x \in \mathbb{R}^d$ representing the horizontal spatial coordinates and $y \in \mathbb{R}$ representing the vertical spatial coordinate. We also assume the free surface

$$\Sigma = \{(t, x, y) \in [0, 1] \times \mathbb{R}^d \times \mathbb{R} : y = \eta(t, x)\}$$

is separated from the fixed bottom $\Gamma = \partial\Omega\backslash\Sigma$ by a curved strip of depth $h > 0$:

$$\{(x, y) \in \mathbb{R}^d \times \mathbb{R} : \eta(t, x) - h < y < \eta(t, x)\} \subseteq \mathcal{O}.$$

We consider an incompressible, irrotational fluid flow. In this setting the fluid velocity field v may be given by $\nabla_{x,y}\phi$ where $\phi : \Omega \to \mathbb{R}$ is a harmonic velocity potential,

$$\Delta_{x,y}\phi = 0.$$

In our setting of a constant downward gravitational force and vanishing surface tension, the water waves system is then given by

$$\begin{cases} \partial_t\phi + \dfrac{1}{2}|\nabla_{x,y}\phi|^2 + P + gy = 0 & \text{in } \Omega, \\ \partial_t\eta = \partial_y\phi - \nabla_x\eta \cdot \nabla_x\phi & \text{on } \Sigma, \\ P = 0 & \text{on } \Sigma, \\ \partial_\nu\phi = 0 & \text{on } \Gamma, \end{cases} \tag{1}$$

where $g > 0$ is the constant acceleration due to gravity, ν is the normal to Γ, and P is the pressure, recoverable from the other unknowns by solving an elliptic equation. Here, the first equation is the Euler equation with vertical gravitational force, the second is the kinematic condition requiring that fluid particles at the interface remain at the interface, the third indicates no surface tension, and the fourth indicates a solid bottom.

A substantial literature regarding the well-posedness of the Cauchy problem for the system (1) has been produced. We refer the reader to [ABZ14a], [ABZ14b], [Lan13] for a more complete history and references. In our direction, the well-posedness of (1) was first established for initial data in certain high-regularity Sobolev spaces by Wu [Wu97], [Wu99].

Alazard-Burq-Zuily [ABZ14a] later improved the H^s well-posedness of Wu to a lower regularity threshold corresponding to Lipschitz fluid velocity fields (precisely, to Sobolev spaces H^s with $s > \frac{d}{2} + 1$). This was further sharpened in one surface dimension to velocity fields with only BMO derivatives by Hunter-Ifrim-Tataru [HIT16]. These results were shown primarily by establishing estimates exhibiting the conservation of energy.

On the other hand, it has been shown that for dispersive equations, one can lower well-posedness regularity thresholds below that which is attainable by energy conservation alone. This was first realized for the nonlinear wave equation via dispersive estimates known as Strichartz estimates, in the works of Bahouri-Chemin [BC99b], [BC99a], Tataru [Tat00], [Tat01], [Tat02], Klainerman-Rodnianski [KR03], and Smith-Tataru [ST05]. Strichartz estimates have similarly been studied for the water wave equations with surface tension; see [CHS10], [ABZ11], [dPN15], [dPN16], [Ngu17].

This low regularity Strichartz paradigm was first applied toward gravity water waves by Alazard-Burq-Zuily in [ABZ14b]. The argument proceeds by first establishing a formulation of the water waves system

$$(\partial_t + T_V \cdot \nabla + iT_\gamma)u \approx 0 \tag{2}$$

where $T_V \cdot \nabla$ and T_γ denote variable coefficient differential operators. Then by proving a Strichartz estimate for (2), one can obtain improvements to the well-posedness theory over results obtained by using conservation of energy alone.

However, the low regularity of the variable coefficients V and γ impose difficulties which appear to force losses in the Strichartz estimates established in [ABZ14b], and thus losses in the well-posedness theory. In fact, it is likely that at the regularity of the coefficients V and γ in (2) required by our low regularity problem, it is impossible to prove Strichartz estimates without loss. For instance, in the context of the wave equation with Lipschitz metric, counterexamples to sharp Strichartz estimates were provided by Smith-Sogge [SS94] and Smith-Tataru [ST02].

Thus, to further improve the well-posedness threshold obtained in [ABZ14b], we need to invoke properties of V and γ other than their regularities, by recalling that they arise from solutions to the water waves system. We implement this by observing an integration structure that reveals hidden regularity in a change of variables from Eulerian to Lagrangian coordinates. This approach applies to all dimensions with no additional difficulty, but still turns out to be insufficient for achieving sharp Strichartz estimates. In the case of one surface dimension, we use a more refined argument, based on both an integration structure, and additional local smoothing effects enjoyed by dispersive equations. Here we are able to prove sharp Strichartz estimates, implying the largest improvement to the well-posedness threshold that can be attained using the Strichartz paradigm.

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Existence theory in the Wiener algebra: Vortex sheets, Boussinesq equations, and other problems

David M. Ambrose

We discuss a framework for proving existence of small solutions for nonlinear evolution problems. This framework, introduced by Duchon and Robert for the vortex sheet problem [9], uses function spaces based on the Wiener algebra. The method starts from a fixed point formulation based on Duhamel formulas, and then uses a contraction mapping argument to prove existence of the fixed point. The method applies to problems which either are elliptic in space-time or are parabolic. For parabolic problems, the method gives existence of solutions, while for elliptic problems the method yields not only existence of solutions but also ill-posedness of the initial value problem. The vortex sheet, as studied by Duchon and Robert, is an example of the use of the method for an elliptic problem.

We discuss application of this method to some water wave model equations, namely the *abcd*-systems of Bona, Chen, and Saut [7], [8]. For some values of the parameters a, b, c, and d, the model systems are linearly ill-posed. For such values of the parameters, we adapt the Duchon-Robert proof to give existence of solutions for the corresponding *abcd*-system and we use these solutions to conclude lack of continuous dependence of the solutions upon initial data; this is joint work with Jerry Bona and Timur Milgrom [5].

The Duchon-Robert method also applies to parabolic problems, and we have used it to study an epitaxial growth model and, in joint work with Anna Mazzucato, the two-dimensional Kuramoto-Sivashinsky equation [3], [6]. We also have used it to study mean field games, for which the equations are a coupled system of a forward parabolic equation and a backward parabolic equation [1], [2].

Both the vortex sheet and the Boussinesq problems can be written in the following way:

$$u_t - Av = (F(u,v))_x, \qquad v_t - Au = (G(u,v))_x,$$

where A is an elliptic operator. The terms F and G are the nonlinearities, so the problem linearizes to the system of equations $u_t - Av = 0$ and $v_t - Au = 0$; this can be simplified as $u_{tt} - A^2u = 0$, which is elliptic in space-time. For elliptic problems we expect to specify half the data so we now proceed by specifying $u(0,\cdot) = u_0$. On the time interval $[0,\infty)$, we have a forward-backward Duhamel formula,

$$u = Su_0 - \frac{1}{2}I_0(F+G) + \frac{1}{2}I^+(F+G) + \frac{1}{2}I^-(F+G), \tag{1}$$

$$v = -Su_0 + \frac{1}{2}I_0(F+G) - \frac{1}{2}I^+(F-G) + \frac{1}{2}I^-(F+G). \tag{2}$$

Here, the operator S is the semigroup for the linear operator, $S(t) = e^{-At}$. The operators I^+ and I^- are Duhamel integrals, forward and backward in time, respectively, defined by

$$(I^+h)(t,\cdot) = \int_0^t e^{-(t-s)A} h_x(s,\cdot)\, ds, \qquad (I^-h)(t,\cdot) = \int_t^\infty e^{-(s-t)A} h_x(s,\cdot)\, ds.$$

The operator I_0 is given by $I_0 = I^-(0)$. A derivation of the forward-backward Duhamel formula (1), (2) can be found in [11] or [4].

Of the Bona-Chen-Saut *abcd*-systems, we draw particular attention to the Kaup system [10], which is a completely integrable model:

$$\eta_t + w_x + (w\eta)_x + \frac{1}{3}w_{xxx} = 0, \tag{3}$$

$$w_t + \eta_x + ww_x = 0. \tag{4}$$

Here η represents the position of the water wave surface while w is a horizontal velocity. We study this problem on spatial domain $[0,1]$, with periodic boundary conditions.

We change variables from η to v, where $v = \Theta H\eta$, with H being the periodic Hilbert transform. The operator Θ is described by its Fourier symbol,

$$\hat{\Theta}(k) = \left(\frac{4\pi^2}{3}k^2 - 1\right)^{-1/2}.$$

Under this change of variables, the Kaup system becomes

$$v_t - Aw = \partial_x\left(\Theta H\left(wH\Theta^{-1}v\right)\right),$$

$$w_t - Av = \partial_x\left(-\frac{1}{2}w^2\right),$$

where the operator A is given in terms of its Fourier symbol as

$$\hat{A}(k) = (2\pi|k|)\left(\frac{4\pi^2}{3}k^2 - 1\right)^{1/2}.$$

Now we introduce functions spaces. Given $\rho \geq 0$, we let B_ρ be the set of functions f for which the norm

$$\|f\|_{B_\rho} = \sum_{k\in\mathbb{Z}} e^{\rho|k|}|\hat{f}(k)|$$

is finite. For $\rho = 0$, this is exactly the periodic Wiener algebra. We also introduce space-time versions, given $\alpha > 0$ and $j \in \mathbb{N}$. The norm for $\mathcal{B}^j_\alpha$ is

$$\|f\|_{\mathcal{B}^j_\alpha} = \sum_{k\in\mathbb{Z}} \sup_{t\in[0,\infty)} \left((1+|k|^j)e^{\alpha t|k|}|\hat{f}(t,k)|\right).$$

Note that for $j \in \mathbb{N}$, a function f is in $\mathcal{B}^j_\alpha$ if and only if f and its first j spatial derivatives are all in $\mathcal{B}^0_\alpha$. Note that all of these spaces are Banach algebras. It can be proved that for sufficiently small values of α, the operators I^+ and I^- defined above are bounded from $\mathcal{B}^j_\alpha$ to $\mathcal{B}^{j+1}_\alpha$.

We have the following existence theorem:

Theorem 1. *Let α satisfy $\alpha \in \left(0, 2\pi\sqrt{\frac{4\pi^2}{3} - 1}\right)$. There is an $r_0 = r_0(\alpha) > 0$ and a constant $C = C(\alpha)$ such that if $0 < r < r_0$, then for $w_0 \in B_0$ with $\|w_0\|_{B_0} \leq r$, there exists $(\eta, w) \in \mathcal{B}^0_\alpha \times \mathcal{B}^1_\alpha$ that solves the system (3), (4) with $w(0,\cdot) = w_0$ and*

$$\|(\eta, w)\|_{\mathcal{B}^0_\alpha\times\mathcal{B}^1_\alpha} \leq Cr.$$

Theorem 1 is proved by the contraction mapping theorem; the quadratic nature of the nonlinear terms in (3), (4), the algebra property for the function spaces, and the boundedness of the operators I^+ and I^- combine to provide the necessary contracting property. Notice that the initial data is only in B_0; thus it need not be smooth. However, at time t the solutions are in $B_{\alpha t}$, and are thus analytic with radius of analyticity at least αt. This provides a proof of ill-posedness of the initial value problem in Sobolev spaces upon reversing time, as small smooth solutions may blow up arbitrarily quickly. We have the following ill-posedness result.

Theorem 2. *The Kaup system is ill-posed in Sobolev spaces. More precisely, for any $s_1 \in \mathbb{N}$ and $s_2 \in \mathbb{N}$, there is a sequence $\{(\eta_0^n, w_0^n)\}_{n\in\mathbb{N}}$ of initial data in $H^{s_1}(\mathbb{T}) \times H^{s_2}(\mathbb{T})$ and positive times $\{t_n\}_{n\in\mathbb{N}}$, both of which tend to zero in their respective norms, such that*

$$\lim_{t\uparrow t_n} \left\|(\eta_n(\cdot,t), w_n(\cdot,t))\right\|_{H^{s_1}(\mathbb{T})\times H^{s_2}(\mathbb{T})} = +\infty.$$

While we have stated the results for the specific case of the Kaup system, the *abcd*-systems are treated more generally in [5].

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Variational principles for water waves with time-dependent vorticity

Thomas J. Bridges

Extending Luke's variational principle to water waves with fully space-time dependent vorticity is a challenging problem. This talk will review recent success in this direction. Four cases will be considered. First Luke's variational principle (VP) is reformulated using Hamilton's principle, where the boundary conditions are imposed and are not natural boundary conditions. Secondly Luke's VP is re-formulated in terms of the stream function. This VP is successful and has an interesting connection with the GKK conservation law, but generates an irrotational flowfield. Thirdly constrained (Euler-Poincaré) variations are used with the stream function and this VP is successful in generating the fully inviscid vortical Euler equations with a free surface, but is quite complicated. The fourth VP is based on the Hodge decomposition where both a velocity potential and a stream function are used, with the stream function determined by the vorticity field only.

1. Luke's VP and Hamilton's principle

Consider the water wave problem in the $(x, y)-$plane in standard form with constant unit density and free surface at $y = h(x, t)$. Luke's variational principle [4], reformulated as Hamilton's principle (kinetic minus potential energy), is just

$$\delta \int_{t_1}^{t_2} \int_{x_1}^{x_2} L \, \mathrm{d}x\mathrm{d}t = 0 \,,$$

with

$$L = \int_0^h \left(\tfrac{1}{2}(u^2 + v^2) - gy \right) \mathrm{d}y + \int_0^h \lambda(u_x + v_y) \, \mathrm{d}y + \Lambda(h_t + uh_x - v)\Big|^{y=h} . \qquad (1)$$

Here λ and Λ are Lagrange multipliers. Although formulated differently, this VP generates the same equations as Luke's VP [4].

The boundary conditions on solid boundary conditions are not natural (do not emerge from the VP). They are Dirichlet conditions and so have to be imposed

$$v(x, 0, t) = u(x_1, y, t) = u(x_2, y, t) = 0 \,.$$

With these constraints, and the imposition of fixed endpoint conditions on the variation of L, the full governing equations for water waves are recovered with λ a velocity potential and Λ the velocity potential evaluated at the free surface. Even though the pressure does not appear explicitly the correct free surface boundary condition still emerges from $\delta L/\delta h = 0$.

2. Hamilton's principle with stream function

Now parameterize the incompressibility condition by introducing a stream function

$$u = \psi_y \quad \text{and} \quad v = -\psi_x \,. \tag{2}$$

Substitute into (1)

$$L = \int_0^h \left(\tfrac{1}{2}(\psi_x^2 + \psi_y^2) - gy\right) \mathrm{d}y + \Lambda(h_t + \Psi_x) + \mu\psi_x\big|_{y=0} \,. \tag{3}$$

The first two terms are kinetic minus potential energy. The Lagrange multiplier Λ is associated with the kinematic free surface boundary condition with $\Psi(x,t) := \psi(x, h(x,t), t)$. The fourth term shows an alternative to specified boundary conditions: they can be added as constraints with μ as a Lagrange multiplier.

Note that irrotationality is not assumed. However the variational principle generates irrotational flow only [2]. Otherwise, the VP generates the correct equations and boundary conditions. A novelty is that

$$\frac{\delta L}{\delta h} = K_t + \left(\psi_y K - \tfrac{1}{2}(\psi_x^2 + \psi_y^2) + gh\right) \,, \tag{4}$$

where $K = h_x\psi_y - \psi_x$. Setting the expression on the right-hand side of (4) to zero is the GKK conservation law [3] in terms of the stream function. The GKK conservation law serves as the pressure boundary condition at the free surface even for flow with vorticity.

3. Hamilton's principle with constrained variations

In the variational principle generated by the Lagrangian (3) the stream function is treated as a free variation. Hence, when a term like

$$\int_0^h \Delta\psi \, \delta\psi \, \mathrm{d}y = 0 \,, \tag{5}$$

appears in the first variation of L, this equation can only be satisfied if ψ is harmonic, generating an irrotational flow.

On the other hand, one can argue from first principles that the stream function variation is not free. The general stream function variations are of the form,

$$\delta\psi = w_t + \{w, \psi\} \,,$$

where $\{w,\psi\}$ is the Poisson bracket of functions, $\{w,\psi\} = w_x\psi_y - w_y\psi_x$, and here it is $w(x,y,t)$ that is the free variation. A derivation and justification of this formula from first principles is in [1]. Substituting the constrained variation into (5) then gives

$$0 = \int_0^h \Delta\psi \,(w_t + \psi_y w_x - \psi_x w_y) \, \mathrm{d}y = \int_0^h -\frac{D}{Dt}(\Delta\psi) w \, \mathrm{d}y + \textsf{boundary terms} \,, \tag{6}$$

where D/Dt is the usual convective derivative. Setting the latter term to zero gives the vorticity equation in the interior. All the boundary conditions are also recovered [1], giving a VP for water waves with general space-time varying vorticity field.

On the other hand, constrained variations are a bit more difficult to work with. For example they are difficult to implement numerically.

4. Towards a VP with a Hodge decomposition

The Hodge decomposition includes both a velocity potential and a stream function. It combines the best features of irrotational flow, represented by a velocity potential, with all the vorticity concentrated in the stream function. The Hodge decomposition of the velocity field in two space dimensions is

$$u = \phi_x + \psi_y \quad \text{and} \quad v = \phi_y - \psi_x \,. \tag{1}$$

The problem is over-determined so the strategy is to set $\psi = 0$ on all boundaries. In this case ψ is determined by solving

$$\Delta\psi = -\omega \,, \quad \psi\big|_{\partial\mathcal{D}} = 0 \quad \text{with} \quad \frac{D\omega}{Dt} = 0 \,,$$

where $\partial\mathcal{D}$ represents all solid and moving boundaries. With the zero Dirichlet boundary conditions on the stream function, the problem is well defined and all the vorticity is concentrated in ψ. This VP is useful when the irrotational flow is dominant and vorticity is treated as a perturbation. The strategy is to substitute (1) into the Lagrangian

$$L = \int_0^h \left[\tfrac{1}{2} \left(\phi_x + \psi_y\right)^2 + \tfrac{1}{2} \left(\phi_y - \psi_x\right)^2 - gy \right] \mathrm{d}y + \mathsf{Constraints} \,. \tag{2}$$

But what are the appropriate constraints? Subtleties appear. Pursuit of this strategy is currently work in progress.

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Traveling heteroclinic waves in a Frenkel-Kontorova chain

Boris Buffoni
(joint work with Hartmut Schwetlick and Johannes Zimmer)

The problem. The talk was on a recent paper [2] about moving kinks in the so-called Frenkel-Kontorova chain. They are obtained by a perturbation of an explicit large kink connecting two periodic trains in a special case.

In 1938, Frenkel and Kontorova [6] studied an infinite chain of nonlinear oscillators linearly coupled to their nearest neighbors, described by the equation

$$\ddot{v}_j(t) = \gamma\left[(v_{j+1}(t) - v_j(t)) - (v_j(t) - v_{j-1}(t))\right] - g'(v_j(t)).$$

This is Newton's equation of motion for atom $j \in \mathbb{Z}$ with mass 1, $\gamma > 0$ is the elastic modulus of the elastics springs and g' is periodic (the force of the on-site potential g).

Traveling waves are of the form $v_j(t) = u(j - ct)$ with some traveling profile u. Setting $x := j - ct$ with c being the wave speed, this leads to

$$\text{(FK-STAT)} \qquad c^2 u''(x) - \gamma \Delta_D u(x) + g'(u(x)) = 0,$$

where Δ_D is the discrete Laplacian

$$(\Delta_D u)(x) := u(x+1) - 2u(x) + u(x-1).$$

In the paper [6], $g'(u) = \sin(2\pi u)$ (in suitable units). Equation (FK-STAT) is a differential-difference equation that is non-local and nonlinear. Since the publication of [6], there have been numerous papers on infinite chains of oscillators, see, e.g., (in chronological order) [15, 5, 1, 3, 4, 7, 10, 12, 13, 9, 14, 16, 17, 19, 11, 20, 21] for related contributions.

Let us also mention that heteroclinic connections between periodic solutions is a classical subject that, in addition to differential-difference equations, also appears in PDEs; see, e.g., [8] for the Swift-Hohenberg equation. We have not tried to sum up the many examples and approaches that have been dealt with. However, we acknowledge that we learned from these ideas.

Our contribution. We show the existence of heteroclinic subsonic solutions by a perturbation argument from the case of g having piecewise quadratic wells (which results in non-smooth forces). The perturbation parameter will be noted by $\epsilon > 0$.

Firstly, periodic (not heteroclinic) wave trains are studied in anharmonic (but near harmonic) wells. This is obtained by a "global" center manifold description, following the ideas of the local analysis of Iooss and Kirchgässner [10].

We then construct a one-parameter family $w_\beta = w_{\beta,\epsilon}$, $\beta \in [-1, 1]$, of approximate asymptotically correct (as $x \to \pm\infty$) heteroclinic solutions. When $\epsilon = \beta = 0$, $w_{0,0}$ is the known heteroclinic solution in [11] corresponding to the piecewise harmonic potential g.

Finally, an exact solution for $\epsilon > 0$ is obtained from the approximate solutions via a topological fixed point argument. A key property is that the family w_β satisfies a transversality condition with respect to β.

The governing equation. We choose $\gamma = 1$ and place two neighboring minima of g at ± 1. Moreover g is supposed even, and solutions u are sought among odd functions. In traveling wave coordinate, we restrict ourselves to moving profiles such that all atoms in the left half-line are in one well and all atoms in the right half-line are in the other well. Moreover g will have two wells only, rather than being periodic. However, the obtained solutions for a two-well potential are also solutions for the same equation with a periodic potential.

The equation for moving kinks is

$$\text{(MAIN)} \qquad c^2 u''(x) - \Delta_D u(x) + \alpha u(x) - \alpha \psi'(u(x)) = 0,$$

where $\alpha > 0$ and $\psi' = \psi'_\epsilon$ is a perturbation of the sign function as $\epsilon \to 0+$.

For $\psi' = \operatorname{sgn}(u)$, the on-site potential is $\frac{\alpha}{2} \min\{(u+1)^2, (u-1)^2\}$, a primitive of the force $\alpha u - \alpha \operatorname{sgn}(u)$.

The operator L given by

$$u \to Lu := c^2 u'' - \Delta_D u + \alpha u$$

is written in Fourier space as

$$-c^2 k^2 + 2(1 - \cos k) + \alpha =: D(k),$$

where D is the dispersion function. Let α be given by

$$\alpha := c^2 \left(\frac{\pi}{2}\right)^2 - 2 > 0.$$

Then trivially $k_0 := \frac{\pi}{2}$ is one root of D and $-k_0$ is another. Furthermore, for $c = 1$, $D'(k)$ vanishes only at $k = 0$. Thus, if $c \leq 1$ is sufficiently close to 1, then D vanishes exactly at k_0 and $-k_0$.

Main Theorem. *For small $\epsilon \in (0, 1/2)$, the on-site potential ψ_ϵ is assumed to be an even function $\psi = \psi_\epsilon \in C^\infty(\mathbb{R}, \mathbb{R})$ satisfying the following conditions. Let*

$$|\psi''_\epsilon(u)| \leq 2\epsilon^{-1} \text{ for } |u| < \epsilon,$$

and, for $|u| \geq \epsilon$,

$$|\psi'_\epsilon(u) - \operatorname{sgn}(u)| < C\,\epsilon,$$

$$|\psi''_\epsilon(u)| < C\,\epsilon,\ |\psi'''_\epsilon(u)| < C\,\epsilon,\ |\psi^{(4)}_\epsilon(u)| < C\,\epsilon,\ |\psi^{(5)}_\epsilon(u)| < C\,\epsilon.$$

It is also assumed that $\psi'_\epsilon(u) - u$ vanishes at $u = 1$ and $u = -1$.
Let $k_0 := \frac{\pi}{2}$ and $\alpha := c^2 \left(\frac{\pi}{2}\right)^2 - 2$.

If $\epsilon > 0$ is small enough, then there exists a range of velocities $c \leq 1$ close to 1 such that for these velocities, there exists an odd heteroclinic solution $u = u_\epsilon$ to (MAIN) *such that u converges to a positive periodic solution as $x \to +\infty$ and to a negative periodic solution as $x \to -\infty$.*

The asymptotic state near $-\infty$ is in one well while the state near $+\infty$ is in the other.

Our result covers cases of (FK-STAT) with g' anharmonic, periodic and C^∞. As our argument is perturbative in nature, it is not clear whether the original choice of Frenkel and Kontorova, $g'(u) = \sin(2\pi u)$ can be dealt with.

Some of the references above study a modified model, with an added force. There are also extensions to higher space dimensions, for example Srolovitz and Lomdahl [18]. Could the above theorem be extended to this case as well?

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Improved model for long internal gravity waves

Didier Clamond
(joint work with Denys Dutykh and André Galligo)

We consider two-dimensional internal gravity waves in an idealised situation where two liquid layers are bounded from below and above by rigid impermeable horizontal surfaces (i.e., the so-called rigid lid approximation). Additionally, the fluids are assumed to be perfect and the waves are long compared to both layer thicknesses. Serre–Green–Naghdi type models [1] can be derived for internal waves in [2], in particular as the Euler–Lagrange equations for the Lagrangian density

$$\mathcal{L} \;=\; \mathcal{K} \;-\; \mathcal{V} \;+\; \rho_1 \{ h_{1t} + [h_1 \bar{u}_1]_x \} \phi_1 \;+\; \rho_2 \{ h_{2t} + [h_2 \bar{u}_2]_x \} \phi_2, \tag{1}$$

where ρ_j are the densities, ϕ_j are Lagrange multipliers, $\bar{u}_j$ are the horizontal velocities averaged over the j-th layer and where

$$\mathcal{K} \;=\; \rho_1 \left(\frac{h_1 \bar{u}_1^2}{2} + \frac{h_1^3 \bar{u}_{1x}^2}{6} \right) + \rho_2 \left(\frac{h_2 \bar{u}_2^2}{2} + \frac{h_2^3 \bar{u}_{2x}^2}{6} \right), \tag{2}$$

$$\mathcal{V} \;=\; \tfrac{1}{2} (\rho_1 - \rho_2)\, g\, h_1^2 \;+\; \tfrac{1}{2} \rho_2\, g\, D^2, \tag{3}$$

are, respectively, the kinetic and potential energies ($\mathcal{V}$ is measured from the bed $y = -d_1$; see [3] and Figure 1 for details). The equations derived from (1) are

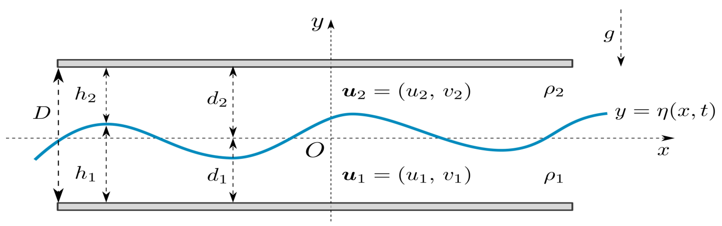

Figure 1. *Definition sketch.*

fully nonlinear and weakly dispersive. In order to improve the dispersive properties without increasing the order of the derivatives involved in the equations, we proceed as for surface waves [4] introducing the modified Lagrangian density

$$\mathcal{L}' \;=\; \mathcal{L} + \tfrac{1}{12} \left(\beta_1 h_1^3 - \beta_2 h_2^3 \right) [\rho_1 (\bar{u}_{1t} + \bar{u}_1 \bar{u}_{1x}) - \rho_2 (\bar{u}_{2t} + \bar{u}_2 \bar{u}_{2x}) + (\rho_1 - \rho_2) g h_{1x}]_x\,, \tag{4}$$

where the β_j are parameters at our disposal. It can be easily checked that $\mathcal{L}'$ is asymptotically consistent with $\mathcal{L}$, i.e., $\mathcal{L}' - \mathcal{L}$ is of the same order than the leading neglected term in the expansion of $\mathcal{L}$ with respect of a shallowness parameter (see [4] for the case of surface waves).

A suitable choice for the parameters β_j is obtained consider infinitesimal traveling waves in presence of a mean sheared flow, i.e. the flow in the j-th layer is close to the mean velocity U_j. Therefore, the dependent variables are sought in the form

$$h_1(x,t) \;=\; d_1 \;+\; \eta(x,t), \qquad \bar{u}_j(x,t) \;=\; U_j \;+\; \mu_j(x,t), \tag{5}$$

where μ_j and η are small quantities. Moreover, looking for traveling waves propagating at constant speed c, solutions are sought in the form $\eta = a\cos k(x-ct)$ and $\mu_j = A_j \cos k(x-ct)$, so we obtain the linear dispersion relation

$$(6) \qquad (\rho_1-\rho_2)\,g \;=\; \sum_{j=1}^{2} \frac{\rho_j}{d_j}\left[1 + \frac{2\,(kd_j)^2}{6 + 3\,\beta_1\,(kd_1)^2 + 3\,\beta_2\,(kd_2)^2}\right](U_j-c)^2.$$

to be compared with the (exact linear) one for arbitrary thicknesses

$$(7) \qquad (\rho_1-\rho_2)\,g \;=\; (\rho_1/d_1)\,(U_1-c)^2\,kd_1\coth(kd_1) \;+\; (\rho_2/d_2)\,(U_2-c)^2\,kd_2\coth(kd_2).$$

The β_j are chosen such that the discriminants of (6) and (7) match up to k^4 in their Taylor expansion for c around $k=0$, hence

$$(8) \qquad \beta_1\,d_1^2 \;+\; \beta_2\,d_2^2 \;=\; \frac{2}{15}\,\frac{\rho_1 d_1^3+\rho_2 d_2^3}{\rho_1 d_1+\rho_2 d_2} \;+\; \frac{2}{15}\,\frac{(\rho_2 d_1+\rho_1 d_2)\,d_1 d_2}{\rho_1 d_1+\rho_2 d_2}\,\frac{V}{V+G},$$

$$G \;\stackrel{\text{def}}{=}\; (\rho_1-\rho_2)\,g\,d_1\,d_2\,(\rho_1 d_1+\rho_2 d_2)\,(\rho_2 d_1+\rho_1 d_2)\,,$$

$$V \;\stackrel{\text{def}}{=}\; \rho_1\,\rho_2\,(\rho_2 d_1-\rho_1 d_2)\,(d_1-d_2)\,(d_1+d_2)\,(U_1-U_2)^2\,.$$

Kelvin–Helmholtz instabilities are found when the discriminant of (6) is negative (then c is complex), i.e., if (assuming $g>0$ and $\rho_1>\rho_2$)

$$(9) \qquad \frac{(U_1-U_2)^2}{(\rho_1-\rho_2)\,g} \;>\; \frac{d_1}{\rho_1\,K_1} \;+\; \frac{d_2}{\rho_2\,K_2}, \qquad K_j \;\stackrel{\text{def}}{=}\; 1 \;+\; \frac{2\,(kd_j)^2}{6 + 3\,\beta_1\,(kd_1)^2 + 3\,\beta_2\,(kd_2)^2}.$$

If $\beta_1=\beta_2=0$ and $U_1 \neq U_2$, there always exists a wavenumber k_s for which the inequality (9) is satisfied for all $k>k_s$, i.e., the Kelvin–Helmholtz instability always occurs. If $\beta_j>0$ the inequality (9) is satisfied only if $|U_1-U_2|$ is large enough, i.e., the Kelvin–Helmholtz instability is completely eliminated for small $|U_1-U_2|$ and, for sufficiently large $|U_1-U_2|$, the instability threshold k_s appears at rather low value that is consistent with the long wave assumption.

For travelling waves of permanent form the flow is steady in the frame of reference moving with the wave and the equations can be reduced to a single ordinary differential equation of the form

$$(\,\mathrm{d}h_1\,/\,\mathrm{d}x\,)^2 \;=\; P_8(h_1)\,/\,Q_8(h_1),$$

where P_8 and Q_8 are algebraic polynomials of degree eight (too complicated to be reported here). Using the phase plane analysis together with algebraic methods described in [5], we can construct singular (weak) solutions as the slug (or plug) flow depicted in Figure 2. It should be noted that such solutions exist only if $\beta_j \neq 0$.

In summary, using a variational principle and modifying the Lagrangian in an asymptotically consistent way, we derive a model with free parameters. The equations of motion are conservative. The free parameters can be chosen to improve the dispersion properties without increasing the model complexity. A first byproduct of this approach is that the Kelvin–Helmholtz instability is reduced,

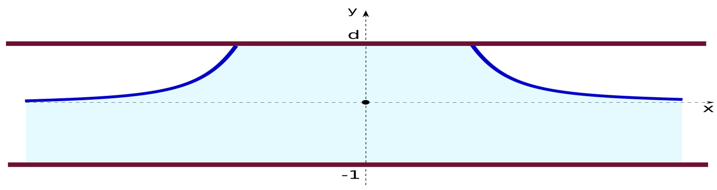

FIGURE 2. *Slug wave.*

and completely removed for small mean shear currents. A second byproduct is the existence a weak solutions that are not present in the original (Serre-like) model.

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Smooth stationary water waves with exponentially localized vorticity

Mats Ehrnström
(joint work with Chongchun Zeng and Samuel Walsh)

We prove the existence of two-dimensional smooth, finite energy, travelling waves with spatially highly localized vorticity. They are modeled as (stationary) solutions to the incompressible Euler equation

(1) $$\partial_t v + (v \cdot \nabla)v + \nabla p + g e_2 = 0,$$

on the fluid domain

(2) $$\Omega = \left\{(x_1, x_2) \in \mathbb{R}^2\colon\ -1 < x_2 < 1 + \eta(t, x_1)\right\}.$$

Here, $v = v(t, \cdot)\colon \Omega(t) \to \mathbb{R}^2$ is the velocity, $p = p(t, \cdot)\colon \Omega(t) \to \mathbb{R}$ is the pressure, $g > 0$ is the gravitational acceleration, and $e_2 = (0, 1)$. The asymptotic depth is normalized to be 2. The velocity field v can then be represented through a stream function via the relation

$$v = \nabla^{\perp}\Psi = (-\Psi_{x_2}, \Psi_{x_1}),$$

where the kinematic boundary conditions imply that Ψ is a constant along each component of $\partial\Omega$; to achieve finite-energy waves, we take

(3) $$\Psi|_{\partial\Omega} = 0.$$

The dynamic condition can be expressed as the well-known Bernoulli equation

$$\tfrac{1}{2}|\nabla\Psi|^2 + gx_2 + \alpha^2\kappa = g \qquad \text{on } x_2 = 1 + \eta(x_1), \tag{4}$$

where $\alpha > 0$ measures surface tension and κ is the signed curvature.

A central object of interest for this paper is the *vorticity* $\omega = \nabla \times v = \Delta\Psi$. We shall construct large families of solitary stationary water waves with a *smooth and highly localized vorticity and a finite energy*: in a perturbed disk around the origin the vorticity is large and negative, and outside it is positive and exponentially decaying. We call this a *vortex spike*. To this aim, we search for Ψ as a solution to

$$\Delta\Psi = \frac{1}{\delta^2}\gamma(\Psi) \qquad \text{on } \Omega, \tag{5}$$

where γ is the vorticity function, for which there exists a radial, exponentially decreasing *ground state* U, which is a solution to

$$\Delta U = \gamma(U) \qquad \text{in } \mathbb{R}^2. \tag{6}$$

Our assumptions are:

(A) The vorticity function $\gamma \in C^{k_0}(\mathbb{R},\mathbb{R})$, $k_0 \geq 2$, satisfies $\gamma(0) = 0$, $\gamma'(0) = 1$, and (6) has a radial solution $U \in C^{k_0+2}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$.

(B) The kernel of $-\Delta+\gamma'(U)\colon H^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is equal to $\operatorname{span}\{\partial_{x_1}U, \partial_{x_2}U\}$.

Prototypical functions fulfilling Assumptions (A) and (B) are $\gamma(t) = t - t^p$, for integers $p \geq 2$ and all $t \geq 0$, but many others will do as well. Classical results for dimension $n = 2$ may be found in for example [1, 4]. Under the above assumptions, our main theorem is as follows.

Theorem. *For any γ as in Assumptions* (A) *and* (B)*, there exists $\delta_0 > 0$ such that, for each $\delta \in (0, \delta_0)$, there is a finite energy solution*

$$(\Psi, \eta) \in \big(H_0^1(\Omega) \cap H^{k_0}(\Omega)\big) \times H^{k_0}(\mathbb{R})$$

to the stationary water wave problem (3), (4), *and* (5)*. Both Ψ and η are even in x_1. Moreover, there exists a constant $C > 0$, independent of δ but depending on γ, such that for each $\delta \in (0, \delta_0)$ there exists τ with $|\tau| \leq C\delta^{-\frac{1}{2}}e^{-\frac{2}{3\delta}}$ satisfying*

$$|\Psi - \Psi_0|_{H^{k_0}(\Omega)} \leq C\delta^{1-2k_0}e^{-\frac{2}{\delta}}, \tag{7}$$

where

$$\Psi_0(x) = U\left(\frac{x_1}{\delta}, \frac{x_2 - \tau}{\delta}\right) - U\left(\frac{x_1}{\delta}, \frac{2 - x_2 - \tau}{\delta}\right) - U\left(\frac{x_1}{\delta}, \frac{-2 - x_2 - \tau}{\delta}\right),$$

and

$$|\eta|_{H^{k_0}(\mathbb{R})} \leq C\delta^{1-k_0}e^{-\frac{2}{\delta}}, \quad |\eta - \eta_0|_{H^{k_0}(\mathbb{R})} \leq C\delta^{\frac{3}{4}-2k_0}e^{-\frac{3}{\delta}}, \tag{8}$$

with

$$\eta_0 = -\frac{1}{\alpha\sqrt{g}\delta^2}e^{-\frac{\sqrt{g}}{\alpha}|\cdot|} * \left(\left(\partial_{x_2}U(\frac{\cdot}{\delta}, \frac{1}{\delta})\right)^2\right).$$

A few remarks: one can prove that the kinetic energy is $O(1)$,

$$|v|_{L^2(\Omega)} = |\nabla\Psi|_{L^2(\Omega)} = |\nabla U|_{L^2(\mathbb{R}^2)} + o(e^{-\frac{1}{2\delta}}),$$

while the corresponding vorticity is spiked in the sense that

$$\omega = \frac{1}{\delta^2}\gamma\left(U(\frac{\cdot}{\delta})\right) + o(e^{-\frac{1}{2\delta}}), |\omega|_{L^\infty(\Omega)} = O\left(\tfrac{1}{\delta^2}\right), \quad |\omega|_{L^1(\Omega)} = |\Delta U|_{L^1(\mathbb{R}^2)} + o(e^{-\frac{1}{2\delta}}).$$

On the other hand, the total vorticity is exponentially small in $0 < \delta \ll 1$:

$$\int_\Omega \omega\,\mathrm{d}x = \int_\Omega \Delta\Psi\,\mathrm{d}x = \int_{\partial\Omega} N\cdot\nabla\Psi\,\mathrm{d}x = o(e^{-\frac{1}{2\delta}}).$$

Therefore, as a measure, $\omega\,\mathrm{d}x$ converges weakly to 0 as $\delta \searrow 0$. However, the vorticity has a rich spatial structure in a domain on the scale of $O(\delta)$ where its point-wise value is $O(\frac{1}{\delta^2})$. Moreover, as ω is $O(1)$ in $L^1(\Omega)$, these waves exhibit a highly localized but strong rotational vector field with kinetic energy of order $O(1)$.

The first rigorous construction of traveling capillary-gravity waves with localized vorticity in infinite depth is due to Shatah, Walsh, and Zeng [7]. In that paper, two classes of compactly supported vorticity were studied: solitary and periodic waves with a submerged point vortex, and solitary waves with a vortex patch. Our construction is a combination of ideas from that paper and the theory of spike and spike-layer solutions to singular perturbations of semi-linear elliptic PDE. These equations typically have the form

$$\delta^2\Delta u = u - u^p \qquad \text{in } D, \tag{9}$$

where $D \subset \mathbb{R}^n$ is a smooth bounded domain, $p > 1$, and Dirichlet or Neumann conditions are prescribed on ∂D. Beginning in the late 80s, versions of (9) were investigated intensively, see, for example, [5, 6]. To the best of our knowledge, ours is the first work exploring singularly perturbed elliptic equations in the hydrodynamical context. The method bears certain similarities to Li and Nirenberg's treatment of (9) in [5], although the water wave problem presents substantial new difficulties.

The capillary-gravity waves in the current work can be viewed, for $0 < \delta \ll 1$, as smoothed vortex patches or as the limit, as the period tends to infinity, of steady periodic waves with critical-layers, see, for example, [2]. We do not perturb from a shear flow, but singularly from the ground state U which has fixed, positive energy. In this respect, the families constructed in the above theorem represent a new kind of water waves.

While we focus on stationary capillary-gravity waves in the current paper, our result immediately furnishes families of *traveling* capillary-gravity waves with exponentially localized vorticity. On the other hand, smooth *finite*-energy waves with a non-zero wavespeed are unlikely to exist. We also do *not* expect smooth spatially localized stationary waves to exist in infinite depth unless the free surface is overturned. With some modifications, the approach of the current paper should also apply when the bed has nontrivial topography.

Acknowledgements

The submission [3] was finalized while ME and SW attended the workshop *Mathematical Theory of Water Waves* at Oberwolfach, July 2019.

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Peaked periodic waves in the reduced Ostrovsky equation: instability and uniqueness

Anna Geyer

(joint work with Dmitry Pelinovsky)

The reduced Ostrovsky equation is a model for small-amplitude long waves in a rotating fluid [4, 7, 8] and can be written in the form

$$u_t + uu_x = \partial_x^{-1} u, \tag{1}$$

which is locally well-posed in $\dot{H}^s_{\text{per}}$ with $s > 3/2$, see [9]. Peaked periodic waves of this equation are known to exist since the original work of Ostrovsky [7]:

$$U_*(z) := \frac{3z^2 - \pi^2}{18}, \quad z \in [-\pi, \pi], \tag{2}$$

continued periodically beyond $[-\pi, \pi]$.

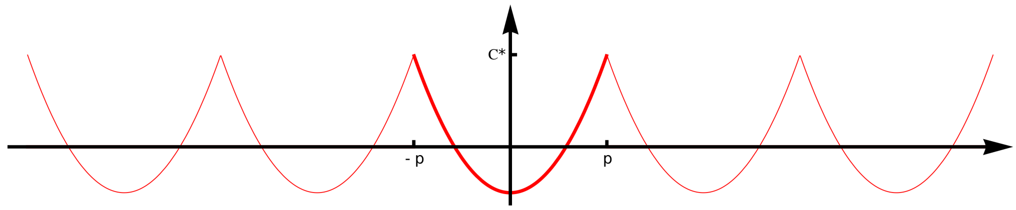

Figure 1. The peaked periodic wave defined in (2).

In this talk I will present recent results in which we answer in particular the long standing open question whether these solutions are stable.

The peaked periodic waves are, informally speaking, located at the boundary between global and breaking solutions in the reduced Ostrovsky equation: If the initial datum u_0 is smooth, it was shown that global solutions of (1) exist if $m_0(x) := 1 - 3u_0''(x) > 0$ for every x and wave breaking occurs if $m_0(x)$ is sign-indefinite [3, 5]. Substituting U_* for u_0 yields $m_0(x) = 0$ almost everywhere except at the peaks. It might therefore be natural to expect that the peaked periodic waves are unstable in the time evolution of the reduced Ostrovsky equations.

We first prove linear instability of the peaked periodic waves using semi-group theory and energy estimates, see [2]. Moreover, we show that the peaked wave (2) is the unique peaked travelling wave solution of the reduced Ostrovsky equation in the space of L^2 periodic functions with zero mean and a single minimum per period. We show that the solution is Lipschitz continuous and exists in $\dot{H}^s_{\text{per}}$ with $s < 3/2$. Moreover, we are able to prove that the reduced Ostrovsky equation does not admit any Hölder continuous solutions. This means in particular that the formally constructed cusped waves are not solutions to the reduced Ostrovsky equation.

We then show that the peaked wave is also spectrally unstable [1] and discover an unusual instability phenomenon: the spectrum of the linearized operator at the peaked wave completely covers a closed vertical strip of the complex plane.

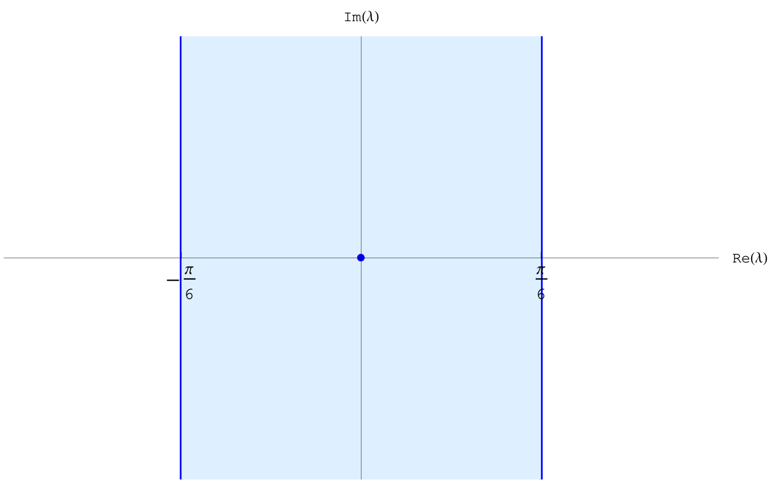

Figure 2. The spectrum of the linearized operator at the peaked periodic wave U_* given by (2) completely covers a closed vertical strip in the complex plane with zero being the only eigenvalue. This shows that the peaked wave is spectrally unstable with respect to co-periodic perturbations.

In order to prove the spectral instability of the peaked periodic waves, we proceed as follows. We first show that the point spectrum of the linearized operator A consists of only the zero eigenvalue. We then observe that A is the sum of the linearization A_0 of the quasi-linear part of the equation and a non-local term, which we may view as a compact perturbation K. The truncated spectral problem for A_0 is then transformed to a problem on the line by a change coordinates. This facilitates the explicit computation of the spectrum of A_0. Finally, we justify the truncation of the linearized operator to its differential part by verifying the assumptions of an abstract result. Using Floquet-Bloch theory, we can even extend the instability result to subharmonic and localized perturbations by showing that the spectrum remains invariant when changing the Floquet exponent.

The proof of nonlinear instability of the peaked periodic waves is still open. One of the main obstacles for nonlinear stability analysis is the lack of well-posedness results for initial data in $\dot{H}^s_{\rm per}$ with $s < \frac{3}{2}$, which would include the peaked periodic waves U_* given by (2). Another obstacle is the discrepancy between the domain of the linearized operator $A = \partial_z L = \partial_z(c_* - U_*^p) + \partial_z^{-1}$ in $\dot{L}^2_{\rm per}$ and the Sobolev space $\dot{H}^1_{\rm per}$: while the former allows finite jumps of perturbations at the peaks, the latter requires continuity of perturbations across the peaks.

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Prediction of the free-surface elevation for rotational water waves using pressure measurements

David Henry

(joint work with Alan Compelli and Gareth Thomas)

The work presented here provides a systematic analysis of the role that the dynamic pressure distribution plays in the kinematics of rotational flows. The main focus is in establishing new formulae which determine how the dynamic pressure at the bed (or more generally, any fixed-depth beneath the wave trough) prescribes the surface profile. This is a theoretically fascinating exercise in itself, however the primary motivation behind this work is provided by applications. Directly measuring the surface of water waves is extremely difficult, and costly, particularly in the ocean. A commonly employed alternative is to calculate the free-surface profile of water waves by way of a pressure transfer function. This recovers the free-surface elevation using measurements from submerged pressure transducers, which are most conveniently located on the sea-bed.

The key to the success of this approach is in the derivation of a suitable candidate for the pressure transfer function— an issue which is the subject of a large body of experimental and theoretical research in the irrotational setting. To this point there has been little, if any, progress in this direction for water waves with vorticity. However, flows with vorticity (rotational) are highly physically relevant, for instance being vital in the modelling of wave-current interactions. This is particularly pertinent in the context of predicting the surface profile by way of pressure measurements since the pressure sensors are frequently located on the sea-bed, a region where currents are ubiquitous (accounting for sediment transport, for instance).

In the presented work we address this issue, considering the streamfunction Ψ, dynamic pressure p, and vorticity Ω in the form:

$$\begin{aligned} \Psi(\theta,z) &= \sum_{n=0}^{\infty} \psi_n(z)\cos n\theta \\ p(\theta,z) &= \sum_{n=0}^{\infty} p_n(z)\cos n\theta, \\ \Omega\,(\theta,z) &= \sum_{n=0}^{\infty} \Omega_n(z)\cos n\theta. \end{aligned} \tag{1}$$

Here $\theta = x - ct$ is the 'phase variable', where the $x-$axis represents the horizontal direction and $c > 0$ is the wave phasespeed. We consider a novel formulation of the Euler equations for travelling rotational waves which relates the dynamic pressure in the underlying fluid motion to the streamfunction; upon substitution of the functions in (1) we obtain the following relation:

$$\begin{aligned} \frac{1}{\rho}\sum_{n=0}^{\infty} p_n(z)\cos n\theta &= -A_0 - \frac{1}{2}\left(\psi_0'(z)\right)^2 + \int^z \psi_0'(z)\Omega_0(z)dz \\ &- \frac{1}{4}\sum_{m=1}^{\infty}\left[\left(\psi_m'(z)\right)^2 + \left(mk\psi_m(z)\right)^2 - 2\int^z \psi_m'(z)\Omega_m(z)dz\right] \\ &+ \sum_{n=1}^{\infty}\left\{\frac{\omega}{k}\psi_n'(z) - \frac{1}{2}\left[\psi_n'(z)\psi_0'(z) - 2\psi_n(z)\Omega_0(z)\right]\right\}\cos n\theta \\ &-\frac{1}{4}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left[\psi_n'(z)\psi_m'(z) - nmk^2\psi_n(z)\psi_m(z) - \frac{2n}{m+n}\psi_n(z)\Omega_m(z)\right]\cos(n+m)\theta \\ &-\frac{1}{4}\sum_{n=0}^{\infty}\sum_{\substack{m=1\\ m\neq n}}^{\infty}\left[\psi_n'(z)\psi_m'(z) + nmk^2\psi_n(z)\psi_m(z) + \frac{2n}{m-n}\psi_n(z)\Omega_m(z)\right]\cos(m-n)\theta. \end{aligned} \tag{2}$$

Matching harmonic components on each side of (2), one can derive explicit information on the pressure—streamfunction relation for linear, and nonlinear, water waves.

At the linear wave level, the first order pressure-streamfunction relation describing wave-field kinematics takes the form

$$\frac{p_1(z)}{\rho} = \left(\frac{\omega}{k} - U(z)\right)\psi_1'(z) + U'(z)\psi_1(z). \tag{3}$$

Here $U(z)$ is the (wave independent) mean-current The first-order component of the streamfunction, $\psi_1(z)$, is determined by the equation

$$\psi_1''(z) - \left(k^2 - \frac{kU''(z)}{\omega - kU(z)}\right)\psi_1(z) = 0. \tag{4}$$

This is the Rayleigh equation of hydrodynamic stability theory, or the inviscid form of the Orr-Sommerfeld equation. For regular waves over a horizontal bed with mean water depth h, the appropriate boundary conditions are

$$\begin{aligned} &(c - U(0))^2\psi_1'(0) + [(c - U(0))\,U'(0) - g]\,\psi_1(0) = 0, && \\ &\psi_1(0) = a\,(c - U(0)) && \text{on } z = 0, \\ &\psi_1(-h) = 0 && \text{on } z = -h, \end{aligned}$$

The Rayleigh equation (4) can only be solved analytically for a few simple current profiles: for non-zero currents these are the constant current $U(z) = U_c$ and the linear profile $U(z) = U_s + \Omega z$. It is worth noting that many profiles can be approximated in applications by a number of linear components, with appropriate matching conditions applied at the interfaces. If $U(z)$ is arbitrary a numerical solution of (4) is required; the imposed bottom boundary conditions for the dynamic pressure ensure an initial-value problem that should not cause numerical difficulties.

In order to derive a pressure transfer function for rotational waves we work with (3). Taking the standard *ansatz* for the linear free-surface, $\eta(\theta) = a\cos\theta$ where the constant a is the (to-be-determined) wave-amplitude, we observe that $p_1(z)$ must satisfy $p_1(0) = \rho g a$ at the linearised surface. The aim is to measure the dynamic bed-pressure $p_b = p_1(-h)$ and use this to determine the surface amplitude a. Since $\psi_1(z)$ is linearly proportional to a we can write $\psi_1(z) = a\chi_1(z)$ and the equation (4) is transformed to a boundary value problem for χ which is now independent of the wave amplitude a. It follows that if p_b is the known quantity, then the surface-recovery from pressure measurement formula is given by

$$a = \frac{p_b}{\rho\,[c - U(-h)]\,\chi_1'(-h)}.$$

It is shown that we are not restricted in our approach to working at the flat bed in taking pressure measurements; we obtain similar (if more complicated) formulae relating the surface profile to dynamic pressure measurements for arbitrary depths. Also, the formula above reduces to the classical pressure transfer function in the irrotational setting. It is also shown in the talk that implementing the moderate current approximation (MCA) leads to even more tractable, and easily computed, formulae. These formulae are verified numerically for a range of rotational wave motions with constant vorticity.

Stokes waves in a constant vorticity flow

Vera Mikyoung Hur
(joint work with Sergey A. Dyachenko)

In the 1800s, Stokes [10, 11] made many contributions about periodic waves at the surface of an incompressible inviscid fluid in two dimensions, under the influence of gravity, propagating a long distance at a practically constant velocity without change of form. For instance, he observed that the crests become sharper and the troughs flatter as the amplitude increases and that the so-called wave of greatest height exhibits a 120° peaking at the crest. About a century later, Amick, Fraenkel and Toland [1], and Plotnikov [8] proved that such an extreme wave exists. In an irrotational flow of infinite depth, notable recent advances were based on a formulation of the problem as a nonlinear pseudo-differential equation:

$$c^2\mathcal{H}y' - gy - g(y\mathcal{H}y' + \mathcal{H}(yy')) = 0 \tag{1}$$

due to Babenko [2] and others. Here $\mathcal{H}$ denotes the Hilbert transform, defined by

$$\mathcal{H}e^{iku} = -i\,\mathrm{sgn}(k)e^{iku} \quad \text{for } k \in \mathbb{Z},$$

and $(u + \mathcal{H}y(u), y(u))$, $u \in [-\pi, \pi]$, represents the fluid surface; c means the speed of wave propagation and g the constant of gravitational acceleration. The prime means the ordinary differentiation in the u variable.

The irrotational flow assumption is well justified in some circumstances. But rotational effects are significant in many others. Constant vorticity is of particular interest because it greatly simplifies the mathematics. Moreover, it is representative of a wide range of physical scenarios. In the 1980s, Simmen and Saffman [9], Teles da Silva and Peregrine [12], and others used a boundary integral method and numerically computed Stokes waves in a constant vorticity flow. They found a 'fold' in the wave speed versus amplitude plane for some values of the vorticity and overhanging profiles, among others.

Recently, Constantin, Strauss and Varvaruca [3] employed conformal mapping techniques, modified (1) and supplemented it with a scalar constraint, to allow constant vorticity and finite depth, and they established a global bifurcation result. Moreover, they conjectured that at the boundary of the connected solution curve, one reaches: either an *extreme wave*, which exhibits a peaking at the crest and whose profile is single valued or overhanging, or a *touching wave*, whose profile contacts with itself at the trough line, enclosing a bubble of air.

Dyachenko and the author [5, 6] took matters further to eliminate the Bernoulli constant from the equation in [3] and, hence, the scalar constraint. The result is

$$c^2\mathcal{H}y' - (g+\omega c)y - g(y\mathcal{H}y' + \mathcal{H}(yy')) - \frac{1}{2}\omega^2(y^2 + \mathcal{H}(y^2y') + y^2\mathcal{H}y' - 2y\mathcal{H}(yy')) = 0, \tag{2}$$

where ω means the value of constant vorticity. The associated linearized operator is self-adjoint, whence they [5, 6] solved (2) efficiently using the Newton-conjugate gradient method.

For zero vorticity, Longuet-Higgins and Fox [7] and others predicted that the wave speed experiences infinitely many oscillations whereas the wave amplitude increases monotonically toward the extreme wave. For negative constant vorticity, the crests become sharper and lower [9, 12]. For a large value of positive constant vorticity, on the other hand, we find that the amplitude increases, decreases and increases during the continuation of numerical solution. Namely, a fold appears in the wave speed versus amplitude plane. The fold becomes larger as the value of the vorticity increases. For a larger value of the vorticity, a *gap* of 'inadmissible' solutions appears in the wave speed versus amplitude plane, bounded by two touching waves. The gap becomes larger as the value of the vorticity increases. More folds and gaps follow as the value of the vorticity increases even further. By the way, the numerical method of [9, 12] and others diverges in a gap, whence it is incapable of finding higher gaps.

We find that the amplitude increases monotonically past all folds, whereas the wave speed experiences oscillations. Moreover, overhanging profiles disappear as the amplitude increases past all folds, and the crests become sharper, like for zero vorticity. Therefore, we claim that an extreme wave appears ultimately, which exhibits a $120°$ peaking at the crest and whose profile is single valued.

Furthermore, we find that touching waves at the beginnings of the lowest gaps tend to the limiting Crapper wave [4] as the value of positive constant vorticity increases indefinitely — a striking and surprising link between rotational and capillary effects — while the profile encloses a circular bubble of fluid in rigid body rotation at the ends of the gaps. Touching waves at the beginnings of the second gaps tend to the circular vortex wave on top of the limiting Crapper wave in the infinite vorticity limit, and circular vortex wave on top of itself at the ends of the gaps. Touching waves at the boundaries of higher gaps accommodate more circular bubbles of fluids.

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Long time dynamics for two dimensional water waves

Mihaela Ifrim
(joint work with Daniel Tataru)

The aim of this talk was to provide an overview of recent work of the author, joint with Daniel Tataru, and in part with several other coauthors as listed in the references, on the long time dynamics for two dimensional water waves.

The water wave equations describe the motion of the free surface of a fluid moving under the action of various physical forces. The fluid motion is governed by the incompressible Euler equations

$$\rho(\partial_t + v \cdot \nabla) + \nabla p = -g\mathbf{j}, \qquad \nabla \cdot v = 0$$

in the fluid domain Ω_t, where v is the velocity, p is the pressure, and $g \geq 0$ is the gravity. To these one adds a boundary condition (impermeability) on the rigid bottom, as well as two boundary conditions on the free surface Γ_t, the water-air interface:

$$\begin{cases} \partial_t + v \cdot \nabla \text{ is tangent to } \bigcup \Gamma_t & \text{(kinematic)} \\ p = -2\sigma \mathbf{H} \quad \text{on } \Gamma_t & \text{(dynamic).} \end{cases}$$

Here $\mathbf{H}$ is the mean curvature of the free boundary, and $\sigma \geq 0$ is the surface tension.

Under the additional condition that the flow is irrotational $\nabla \times v = 0$, one can represent the velocity using a velocity potential ϕ, namely $v = \nabla\phi$. Here ϕ is a harmonic function in Ω_t which is uniquely determined by its values ψ on the top Γ_t. Because of this, it has been known for a long time that [14] the fluid motion can be reduced to evolution equations for the free boundary. A similar reduction can be made if the vorticity is constant instead of zero.

To write these equations one needs to make a choice of coordinates. The two traditional choices are the Eulerian coordinates, where the frame is fixed, with flat geometry, but the particles are moving, respectively the Lagrangian coordinates, where the particles are fixed but the frame is moving and curved. In two dimensional water waves there is a third choice, that of holomorphic (conformal) coordinates [12, 13, 3], where both particles and the frame move but in a conformally flat geometry.

In Eulerian coordinates, for instance, the water wave equations can be seen as a system in (η, ψ) in one space dimension, where η describes the elevation of the free surface and ψ is the trace of the velocity potential on the free surface. The equations have the form

$$\begin{cases} \partial_t \eta - G(\eta)\psi = 0 \\ \partial_t \psi + g\eta - \sigma \mathbf{H}(\eta) + \dfrac{1}{2}|\nabla\psi|^2 - \dfrac{1}{2}\dfrac{(\nabla\eta\nabla\psi + G(\eta)\psi)^2}{1+|\nabla\eta|^2} = 0, \end{cases}$$

where $G(\eta)$is the Dirichlet to Neumann operator associated to the fluid domain, which makes the evolution fully nonlinear and nonlocal. Instead we prefer to work in holomorphic coordinates, where we model the fluid domain as a half-plane (in the infinite depth case) or as a strip (in the finite depth case) via a holomorphic change of coordinates $Z = 1 + W$ as in the Riemann mapping theorem. The velocity potential can be viewed as the real part of a complex velocity potential Q, and the equations have the following form

$$\begin{cases} W_t + F(1 + W_\alpha) = 0 \\ Q_t + FQ_\alpha - g\mathcal{T}_h[W] + P_h\left[\dfrac{|Q_\alpha|^2}{J}\right] = 0, \end{cases} \tag{1}$$

where the Hilbert/Tilbert transform $\mathcal{T}_h$ and the projector P_h depend on the depth h and

$$J = |1 + W_\alpha|^2, \qquad F = P_h\left[\frac{Q_\alpha - \bar{Q}_\alpha}{J}\right].$$

This is an evolution for (W, Q) in what we, by a slight abuse, call the space of holomorphic functions, i.e. functions on the top which admit a natural holomorphic extension in the full half-space or strip, with imaginary part vanishing on the bottom.

In this set-up we have considered four problems so far:

(G): Gravity waves in deep water.
- infinite bottom, gravity, no surface tension .

(T): Capillary waves in deep water
- infinite bottom, surface tension, no gravity (short waves) .

(V): Constant vorticity gravity waves in deep water
- infinite bottom, no surface tension, gravity, constant vorticity .

(B): Gravity waves in shallow water
- finite bottom, no surface tension, gravity .

These problems have a number of common features, such as the fully nonlinear structure, the nonlocality, the dispersive character, and the non-resonant structure for bilinear wave interactions. There are also key differences which are due to the different dispersion relations, and in particular the different low/high frequency regimes. For these four problems we have investigated a range of questions, which we briefly enumerate:

- Low regularity local well-posedness for (G),(T),(V),(B) in [3, 1, 9, 2].
- Cubic lifespan bounds for small data for (G),(T),(V),(B) in [3, 1, 9, 2].
- Global solutions for small localized data for (G),(T), see [3, 4, 1].
- Cubic NLS approximation for (G), see [6].
- Absence of solitary waves for (G),(T), see [10].
- Morawetz estimates for (G), see [11].

The above global solutions for (G), (T) exhibit dispersive decay as well as some form of modified scattering. This is no longer possible for (B) and (V), which admit small solitons arising from the KdV, respectively the Benjamin-Ono approximation at low frequency. As a first step in their study we have considered the similar problems for KdV and Benjamin-Ono, proving results as follows:

- Dispersive solutions for KdV with small localized data on an (optimal) quartic time scale [8].
- Dispersive solutions for Benjamin-Ono with small localized data on an (optimal) almost global time scale [7].

Two of the ideas which have emerged from the above sequence of works and which have found a broader range of applications are as follows:

(1) **The modified energy method,** as a quasilinear alternative to the classical normal form method; the idea here [15], [3] is that in quasilinear problems it is better to modify energy functionals rather than transform the equation.

(2) **Testing by wave packets,** useful in capturing the asymptotic equation in the study of modified scattering for global dispersive solutions, [5], [4]. The use of (generalized) wave packets for this purpose balances better the linear and nonlinear errors in the asymptotic equation.

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A mathematical analysis of the Isobe–Kakinuma model for water waves

Tatsuo Iguchi

We consider the motion of a water filled in $(n+1)$-dimensional Euclidean space together with the motion of the water surface. We assume that the water surface and the bottom are represented as $z = \eta(x,t)$ and $z = -h + b(x)$, respectively. As was shown by J. C. Luke [9], the water wave problem has also a variational structure. His Lagrangian density is of the form

$$\mathcal{L}(\Phi, \eta) = \int_{-h+b(x)}^{\eta(x,t)} \left(\partial_t \Phi(x,z,t) + \frac{1}{2} |\nabla_X \Phi(x,z,t)|^2 + gz \right) \mathrm{d}z, \tag{1}$$

where Φ is the velocity potential of the water. M. Isobe [3, 4] and T. Kakinuma [5, 6, 7] approximated the velocity potential Φ in Luke's Lagrangian by

$$\Phi^{\mathrm{app}}(x,z,t) = \sum_{i=0}^{N} \Psi_i(z;b)\phi_i(x,t),$$

where $\{\Psi_i\}$ is an appropriate function system in the vertical coordinate z and may depend on the bottom topography b and $(\phi_0, \phi_1, \ldots, \phi_N)$ are unknown variables, and derived an approximate Lagrangian density $\mathcal{L}^{\mathrm{app}}(\phi_0, \phi_1, \ldots, \phi_N, \eta) = \mathcal{L}(\Phi^{\mathrm{app}}, \eta)$. The Isobe–Kakinuma model is the corresponding Euler–Lagrange equation for the approximated Lagrangian. We have to choose the function system $\{\Psi_i\}$ carefully in order that the Isobe–Kakinuma model would be a good approximation for the water wave problem. Here, we adopt the approximation

$$\Phi^{\mathrm{app}}(x,z,t) = \sum_{i=0}^{N} (z + h - b(x))^{p_i} \phi_i(x,t), \tag{2}$$

where $p_0, p_1, \dots, p_N$ are nonnegative integers satisfying $0 = p_0 < p_1 < \cdots < p_N$. Then, the corresponding Isobe–Kakinuma model has the form

$$
(3) \quad \begin{cases}
H^{p_i}\partial_t\eta + \displaystyle\sum_{j=0}^{N}\Big\{\nabla\cdot\Big(\frac{1}{p_i+p_j+1}H^{p_i+p_j+1}\nabla\phi_j - \frac{p_j}{p_i+p_j}H^{p_i+p_j}\phi_j\nabla b\Big) \\
\quad + \dfrac{p_i}{p_i+p_j}H^{p_i+p_j}\nabla b\cdot\nabla\phi_j - \dfrac{p_ip_j}{p_i+p_j-1}H^{p_i+p_j-1}(1+|\nabla b|^2)\phi_j\Big\} = 0 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad i = 0,1,\dots,N, \\
\displaystyle\sum_{j=0}^{N}H^{p_j}\partial_t\phi_j + g\eta \\
\quad + \dfrac{1}{2}\left\{\left|\displaystyle\sum_{j=0}^{N}(H^{p_j}\nabla\phi_j - p_jH^{p_j-1}\phi_j\nabla b)\right|^2 + \left(\displaystyle\sum_{j=0}^{N}p_jH^{p_j-1}\phi_j\right)^2\right\} = 0,
\end{cases}
$$

where $H(x,t) = h + \eta(x,t) - b(x)$ is the depth of the water. Here and in what follows we use the notational convention $0/0 = 0$.

The hypersurface $t = 0$ in the space-time $\mathbf{R}^n \times \mathbf{R}$ is characteristic for the Isobe–Kakinuma model (3), so that the initial value problem to (3) is not solvable in general. In fact, if the problem has a solution $(\eta, \phi_0, \dots, \phi_N)$, then by eliminating the time derivative $\partial_t\eta$ from the equations we see that the solution has to satisfy the relation

$$
\begin{aligned}
&H^{p_i}\sum_{j=0}^{N}\nabla\cdot\Big(\frac{1}{p_j+1}H^{p_j+1}\nabla\phi_j - \frac{p_j}{p_j}H^{p_j}\phi_j\nabla b\Big) \\
(4) \quad &= \sum_{j=0}^{N}\Big\{\nabla\cdot\Big(\frac{1}{p_i+p_j+1}H^{p_i+p_j+1}\nabla\phi_j - \frac{p_j}{p_i+p_j}H^{p_i+p_j}\phi_j\nabla b\Big) \\
&\qquad + \frac{p_i}{p_i+p_j}H^{p_i+p_j}\nabla b\cdot\nabla\phi_j - \frac{p_ip_j}{p_i+p_j-1}H^{p_i+p_j-1}(1+|\nabla b|^2)\phi_j\Big\}
\end{aligned}
$$

for $i = 1, \dots, N$. Therefore, as a necessary condition the initial date and the bottom topography have to satisfy the relation (4) for the existence of the solution.

Theorem 1 ([8]). *Under appropriate assumptions on the initial data and the bottom topography, the initial value problem to the Isobe–Kakinuma model* (3) *is well-posed locally in time.*

Now, we introduce a new unknown ϕ by

$$
(5) \qquad \phi = \Phi^{\mathrm{app}}|_{z=\eta} = \sum_{i=0}^{N}H^{p_i}\phi_i,
$$

which corresponds to Zakharov's canonical variable. In fact, (η, ϕ) are canonical variables for the Isobe–Kakinuma model.

Proposition 1 ([8]). *Under appropriate assumptions on the water surface and the bottom, once the canonical variables* (η, ϕ) *are given,* (4)–(5) *determine uniquely* $(\phi_0, \phi_1, \ldots, \phi_N)$.

The Isobe–Kakinuma model has a conserved energy

$$E^{\mathrm{IK}} = \int_{\Omega(t)} \frac{1}{2} |\nabla_X \Phi^{\mathrm{app}}(X,t)|^2 \mathrm{d}X + \frac{1}{2} g \int_{\mathbf{R}^n} \eta(x,t)^2 \mathrm{d}x,$$

which can be written explicitly in terms of $(\eta, \phi_0, \ldots, \phi_N)$. Thanks of Proposition 1, this energy function can be written implicitly in terms of the canonical variables (η, ϕ), which will be denoted by $\mathcal{H}^{\mathrm{IK}}$.

Theorem 2 ([1]). *The Isobe–Kakinuma model* (3) *is equivalent to Hamilton's canonical form*

$$\partial_t \eta = \frac{\delta \mathcal{H}^{\mathrm{IK}}}{\delta \phi}, \quad \partial_t \phi = -\frac{\delta \mathcal{H}^{\mathrm{IK}}}{\delta \eta}.$$

In the following, according to the bottom topography, we restrict the choice of the approximation (2) as

$$\Phi^{\mathrm{app}}(x,z,t) = \begin{cases} \displaystyle\sum_{i=0}^{N} (z+h)^{2i} \phi_i(x,t) & \text{in the case of the flat bottom,} \\ \displaystyle\sum_{i=0}^{2N} (z+h-b(x))^{i} \phi_i(x,t) & \text{in the case of the variable bottom.} \end{cases}$$

Even in the case of the flat bottom, we can adopt the later choice of the above approximation. However, it turns out that the terms of odd degree do not play any important role in such a case so that the former choice economizes the computational resources in the numerical computations. We introduce a nondimensional parameter δ as a ratio of the mean depth h to the typical wavelength λ, that is, $\delta = h/\lambda$ which measures the shallowness of the water.

Theorem 3 ([2]). *Let* η^{WW} *and* η^{IK} *be solutions to the water wave problem and to the Isobe–Kakinuma model in a nondimensional form, respectively. Under appropriate assumptions on the initial data, we have*

$$|\eta^{\mathrm{WW}}(x,t) - \eta^{\mathrm{IK}}(x,t)| \lesssim \delta^{4N+2}$$

for some time interval independent of small δ.

It is well-known that the solutions to the shallow water equations and the Green–Naghdi equations approximate the solution to the water wave problem with errors of order $O(\delta^2)$ and $O(\delta^4)$, respectively. Therefore, the Isobe–Kakinuma model is a much more precise approximate model than the well-known models in the strongly nonlinear shallow water regime.

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The complex dynamics of Faraday pilot waves: a hydrodynamic quantum analogue

Paul A. Milewski

A millimetric droplet may bounce periodically on the surface of a vertically vibrating bath of the same fluid; the thin air layer separating the droplet from the bath during impact prevents coalescence. Each impact excites a wave-field consisting primarily of temporally decaying (due to viscosity) Faraday waves, whose longevity depends on the reduced acceleration $\Gamma = A\omega_0^2/g$, where A is the shaking amplitude, ω_0 is the frequency, and g is the gravitational acceleration. As Γ increases, the bouncing may destabilize to horizontal "walking" across the bath, whereby the droplet is propelled at each impact by the slope of its associated Faraday wave field generated at previous impacts. The decay time of the Faraday waves increases with Γ for $\Gamma < \Gamma_F$, where the Faraday threshold Γ_F is the critical vibrational acceleration at which Faraday waves would arise spontaneously (i.e. the Faraday instability) in the absence of a droplet. This decay time results in a "path-memory" of previous impacts, where the "memory" timescale is inversely proportional to the proximity of the Faraday threshold Γ_F. The resulting non-Markovian dynamics are similar in many respects to the pilot-wave dynamics envisaged by de Broglie as a physical framework for understanding quantum mechanics.

We develop a hydrodynamic pilot-wave model, formulated from first principles, that exhibits the behaviour observed in the laboratory experiments. In describing the wave field, our approach is a linear quasi-potential flow description inspired by Benjamin & Ursell (1954), with viscous damping being incorporated in the manner outlined by Lamb (1932) and Dias et. al. (2008). We use a variety of models to describe the droplet-bath interaction, depending on the level of detail needed for particular experimental comparisons. Three such "impact" models are discussed, in increasing complexity:

- A Periodic, instantaneous and localised impact model that allows for an elegant reduction of the full hydrodynamic coupled system to period map, enabling mathematical analysis of the dynamics (see Durey et. al. (2018));
- A continuous time "nonlinear spring" rebound model of Molacek & Bush (2013) which allows for richer "exotic" periodic behaviour and non-periodic motion (see Milewski et. al. (2015));
- A detailed and parameter-free "kinematic-match" solid-sphere impact model which captures the hydrodynamic details of individual impacts (see Galeano-Rios et. al. (2017)).

In simulations, the first model above is appropriate for elucidating certain statistical long-time behaviour, whereas the third model captures details of the drop impact previously inaccessible to computations.

We discuss the results of these three models and also show that in confined states and near the Faraday threshold, a mean hydrodynamic pilot wave field arises from the stationary distribution of droplet positions. We prove a relation between the droplet position distributions and this mean wave-field, rationalising aspects of the dynamics at high memory (Durey et. al. (2018)).

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Solitary wave solutions of a Whitham-Boussinesq system

Dag Nilsson
(joint work with Evgueni Dinvay)

We consider the Whitham–Boussinesq system

$$\eta_t = -v_x - \mathrm{i}\tanh(\mathrm{D})(\eta v), \tag{1}$$

$$v_t = -\mathrm{i}\tanh(\mathrm{D})\eta - \mathrm{i}\tanh(\mathrm{D})\left(\frac{v^2}{2}\right), \tag{2}$$

with $\mathrm{D} = -\mathrm{i}\partial_x$ and $\mathcal{F}(\tanh(\mathrm{D})f)(\xi) = \tanh(\xi)\widehat{f}(\xi)$, where $\mathcal{F}$ is the Fourier transform

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}} f(x)\mathrm{e}^{-\mathrm{i}x\xi}\,\mathrm{d}x.$$

This system was introduced recently in [1] as a full-dispersion model for the two-dimensional water wave problem for an inviscid incompressible flow.

We have shown that (1)-(2) possesses solitary wave solutions, that is, travelling wave solutions that decay to zero as $|x| \to \infty$. This existence result is obtained using a variational approach together with Lion's method of concentration compactness [2].

In order to obtain solitary wave solutions of (1)-(2), we first make the travelling wave ansatz:

$$\eta(x,t) = \eta(x+ct),\ v(x,t) = v(x+ct).$$

The system (1)-(2) can then be written as

$$Kv + \eta v + cK\eta = 0,$$

$$\eta + \frac{v^2}{2} + cKv = 0,$$

where

$$K = \frac{D}{\tanh(D)}.$$

After making the change of variables $v = cK^{-\frac{1}{2}}u$ we reduce this system to a scalar equation in u:

(3)

$$\frac{1}{c^2}u - K^{-1/2}\left(\frac{(K^{-1/2}u)^3}{2}\right) - K^{-1/2}(K^{\frac{1}{2}}uK^{-1/2}u) - K^{1/2}\left(\frac{(K^{-1/2}u)^2}{2}\right) - Ku = 0.$$

Equation (3) has a variational structure, indeed, it can be written as

$$\mathrm{d}\mathcal{E}(u) + \lambda\mathrm{d}\mathcal{Q}(u) = 0,$$

where

$$\mathcal{E}(u) = \frac{1}{2}\int_{\mathbb{R}} uKu + K^{1/2}u(K^{-1/2}u)^2 + \frac{(K^{-1/2}u)^4}{4}\,\mathrm{d}x,$$

$$\mathcal{Q}(u) = \frac{1}{2}\int_{\mathbb{R}} u^2\,\mathrm{d}x.$$

and $\lambda = -1/c^2$. Hence, in order to find solutions of (3) we can consider the constrained minimization problem

$$\inf_{u\in U_q} \mathcal{E}(u) \quad \text{with} \quad U_q = \left\{ u \in H^{\frac{1}{2}}(\mathbb{R}) \,:\, \mathcal{Q}(u) = q \right\}.$$

At this stage we put our problem in a more general framework by allowing for a more general class of Fourier multipliers, instead of the specific Fourier multiplier K, and thus a more general constrained minimization problem.

Definition 1 (Admissible Fourier multipliers)**.** *Let operator L be a Fourier multiplier, with symbol m, i.e.*

$$\mathcal{F}(Lf)(\xi) = m(\xi)\widehat{f}(\xi).$$

We say that L is admissible if m is even, $m(0) > 0$ and for some $s' > 1$ and $s > 1/2$ the symbol satisfies the following restrictions.

(i) The function $\xi \mapsto \frac{m(\xi)}{\langle\xi\rangle^s}$ is uniformly continuous, and

$$m(\xi) - m(0) \simeq |\xi|^{s'} \text{ for } |\xi| \leqslant 1,$$
$$m(\xi) - m(0) \simeq |\xi|^{s} \text{ for } |\xi| > 1.$$

(ii) For each $\varepsilon > 0$ the kernel of operator $L^{-1/2}$ satisfies

$$\mathcal{F}^{-1}\left(m^{-1/2}\right) \in L^2(\mathbb{R} \setminus (-\varepsilon, \varepsilon)).$$

There exists $p \in (1,2) \cap [2/(s+1), 2)$ such that

$$\mathcal{F}^{-1}\left(m^{-1/2}\right) \in L^p(-1,1).$$

We have the corresponding functional

$$\mathcal{E}(u) = \frac{1}{2}\int_{\mathbb{R}} \left(L^{1/2}u + \frac{1}{2}(L^{-1/2}u)^2 \right)^2 \,\mathrm{d}x$$

defined on $H^{s/2}(\mathbb{R})$. Our main goal is then to obtain a solution of the minimization problem

$$\inf_{u\in U_q} \mathcal{E}(u) \quad \text{with} \quad U_q = \left\{ u \in H^{s/2}(\mathbb{R}) \,:\, \mathcal{Q}(u) = q \right\}. \tag{4}$$

Our main result establish that there does indeed exist solutions of (4).

Theorem 2. *Let D_q be the set of minimizers of $\mathcal{E}$ over U_q. There exists $q_0 > 0$ such that for each $q \in (0, q_0)$, the set D_q is nonempty and $\|u\|^2_{H^{\frac{s}{2}}} \lesssim q$ uniformly for $u \in D_q$. Each element of D_q is a solution of the Euler–Lagrange equation*

$$\lambda u + L^{-1/2}\left(\frac{(L^{-1/2}u)^3}{2}\right) + L^{-1/2}(L^{1/2}uL^{-1/2}u) + L^{1/2}\left(\frac{(L^{-1/2}u)^2}{2}\right) + Lu = 0. \tag{5}$$

The Lagrange multiplier λ *satisfies*

$$\frac{m(0)}{2} < -\lambda < m(0) - Dq^{\beta},$$

where $\beta = \frac{s'}{2s'-1}$ *and* D *is a positive constant.*

As previously mentioned, the main tool for proving Theorem 2 is Lion's concentration compactness theorem [2]. In the application of this theorem one needs to show that the 'dichotomy' and 'vanishing' scenarios cannot occur, and then use the 'concentration' scenario to obtain a convergent subsequence of a minimizing sequence. In general the difficulty in the application of this theorem lies in excluding 'dichotomy'. This is true in our case as well, but in addition it turns out that excluding 'vanishing' is also a nontrivial task. This issue stems from the fact that the Fourier multiplier operator L appears in the nonlinearity in (5) and also that we are working in low regularity Sobolev spaces ($H^{\frac{s}{2}}(\mathbb{R}),\ s > 1/2$). To resolve this issue we impose integrability conditions on the kernel in Definition 1 (ii). With these extra assumptions we are able to deduce that 'vanishing' cannot occur.

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(Yet another) Reformulation of the water-wave problem: Asymptotic models and conservation laws

Katie Oliveras

We consider a nonlocal formulation of the water-wave problem for a free surface with an irrotational flow, and show how the problem can be reduced to a single equation for the interface. The formulation is also extended to constant vorticity and interfacial flows of different density fluids. Finally, we show how this formulation can be used to systematically derive Olver's conservation laws not only for an irrotational fluid, but for constant vorticity and interfaces. Finally, we conjecture an extension to generalized vorticity, potentially leading to new conservation laws.

Failure of modulation equations

Guido Schneider

Amplitude, modulation, or envelope equations, such as the KdV or the NLS equation play a big role for the qualitative understanding of the water wave problem and other dispersive wave systems in spatially unbounded domains. It has been shown that these equations make correct predictions about the dynamics of the water wave problem, cf. [2, 15, 16, 6] for the KdV approximation and [17, 5, 10, 7] for the NLS approximation.

Only for a few examples [11, 12] it has been known that amplitude equations can fail to make correct predictions. Therefore, besides proving approximation results, we started to investigate the failure of modulation equations more systematically in the last years, cf. [14, 1, 9]. It turned out that the question of validity of modulation equations in many situations is really subtle. It can fail for Sobolev initial conditions, but can make correct predictions for analytic conditions, cf. [11, 3]. It can fail for periodic boundary conditions, but can make correct predictions on the whole real line, cf. [4].

There are essentially two possibilities how a modulation equation can fail, namely failure by linear instability **(1)** and failure by nonlinear dynamics **(2)**. Examples for **(1)** are the failure of a number of modulation equations for the approximate description of the dynamics near unstable dispersive periodic waves [13, §7.6] and the non-validity of a number of amplitude equations for the description of modulations of periodic waves at the Eckhaus boundary [8, §8.4] in dissipative systems. Examples for **(2)** are the failure of the Newell-Whitehead equation for pattern forming systems via quadratic transverse instabilities [11, §4], the failure of the NLS approximation for the water wave problem with suitably chosen small surface tension and periodic boundary conditions [14] via unstable resonances, and the failure of the NLS approximation for a modified Zakharov system [1, §4].

In this talk we focus on the failure of the N wave interaction (NWI) approximation for a special model problem which is a first example for **(2)**, without imposing periodic boundary conditions on the original system, cf. [9]. The main ingredients of our construction are a periodic arrangement of resonant wave numbers and a finite speed of propagation in the original system. The construction goes in two steps. First, we use the periodically arranged quadratic resonances to give a new simplified proof for the failure of the NWI approximation in case of suitably chosen periodic boundary conditions. Secondly, we use the result of the first step and the finite speed of propagation of the original system to give, to our knowledge, the first rigorous proof that an amplitude system fails in the description of the original system without imposing periodic boundary conditions on the original system.

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Existence of two-hump surface waves in water of finite depth

Shu-Ming Sun
(joint work with Shengfu Deng)

The talk concerns the existence of two-hump surface waves with small oscillatory tails at infinity on a layer of water with finite depth. The fluid of constant density is assumed to be incompressible and inviscid, such as water, and bounded above by a free surface and below by a horizontal rigid bottom. The flow is irrotational and the wave on the free surface is traveling with a constant speed under gravity and surface tension forces. The existence of single-hump (solitary) waves traveling on the surface has a long history since the observation of John Scott Russell [12]. When the traveling speed was near a critical value, the Korteweg-de Vries (KdV) equation [3, 8] was derived as a first-order approximate model equation for the exact governing equations (also called Euler equations) and the solitary-wave solutions were also found for the KdV equation, which explained the Scott Russell's observation.

Since the KdV equation is the first-order model equation, to establish the validity of the solitary-wave solutions of the KdV equation, we need to show whether the solutions of the KdV equation are approximations of some solutions of the Euler equations. The existence of solitary-wave solution for the Euler equations, whose approximation is the solitary-wave solution of the KdV equation, was proved for zero surface tension case by Lavrentév [9], Friedrichs and Hyers [5], and Beale [2]. For nonzero surface tension case, Hunter and Vanden-Broeck [6] numerically found that for large surface tension, the Euler equations have a solitary-wave solution decaying to a uniform flow at infinity and for small surface tension, the Euler equations have a solitary-wave solution, which has small non-decaying oscillations at infinity, called a generalized solitary-wave solution. Amick and Kirchgässner [1] and Sachs [11] proved the existence of solitary waves for large surface tension case and Beale [2], Sun [13], Iooss and Kirchgässner [7], and Lombardi [10] obtained the existence of generalized solitary-wave solutions for small surface tension case. Moreover, it was shown that if surface tension is small and near its critical value, there is no solitary-wave solution of elevation which decays to zero at infinity [14] (more detailed discussions can be also found in [15]).

The possibility of the existence of multi-hump waves for the Euler equations under the assumptions given above remained elusive. For zero surface tension case, Craig and Sternberg [4] showed that the Euler equations have only symmetric single-hump waves for the super-critical steady state case, which rules out the existence of multi-hump solutions for the Euler equations. The talk gave a recent development on the existence of multi-hump solutions and presented the proof of the existence of two-hump solutions of the Euler equations for the small surface tension case, based on the generalized solitary-wave solutions obtained before. Basic idea to derive two-hump solutions is that from a generalized solitary-wave solution that must be symmetric, we intentionally break the symmetry of the solution so that the amplitudes of the oscillations at plus or minus infinity are different, with the difference as a new parameter. If the symmetry is broken and we assume that the solution is bounded and oscillatory at infinity in the positive direction, then such solution may become unbounded at the negative infinity. However, we know that if the amplitude difference of the oscillations at positive and negative infinity is zero, the solution must be bounded. Therefore, by adjusting the difference very carefully, we can show that there exists a point $-x_0$ near the negative infinity with $x_0 > 0$ such that all the odd derivatives of the solution at $-x_0$ are zero. Therefore, by the translation and reflection invariance properties of the Euler equations, the bounded solution obtained in $[-x_0, +\infty)$ can be extended symmetrically to $(-\infty, -x_0]$, which gives a solution of the Euler equations in $(-\infty, +\infty)$. Since the solution already has a hump at zero, by the symmetry about $-x_0$, there is another hump at $-2x_0$, which implies that the solution has two humps. The similar idea may be applied to construct 2^m-hump solutions for the Euler equations.

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(No) Solitary waves in deep water in two dimensions

Daniel Tataru

(joint work with Mihaela Ifrim)

The aim of this talk was to describe the author's recent results, joint with Mihaela Ifrim, concerning non-existence of solitary waves in infinite depth water.

The water wave equations describe the motion of the free surface of an incompressible, irrotational fluid moving under the action of various physical forces. The fluid motion is governed by the incompressible Euler equations

$$\rho(\partial_t + v \cdot \nabla) + \nabla p = -g\mathbf{j}, \qquad \nabla \cdot v = 0$$

in the fluid domain Ω_t, where v is the velocity, p is the pressure, and $g \geq 0$ is the gravity. These equations are coupled with two boundary conditions on the free surface Γ_t which represents the water-air interface:

$$\begin{cases} \partial_t + v \cdot \nabla \text{ is tangent to } \bigcup \Gamma_t & \text{(kinematic)} \\ p = -2\sigma \mathbf{H} \quad \text{on } \Gamma_t & \text{(dynamic).} \end{cases}$$

Here $\mathbf{H}$ is the mean curvature of the boundary, and $\sigma \geq 0$ is the surface tension.

For irrotational flows $\nabla \times v = 0$, one can represent the velocity using a velocity potential ϕ, namely $v = \nabla\phi$. Here ϕ is a harmonic function in Ω_t which is uniquely determined by its values ψ on the top. This allows one (see for example [22]) to reduce the fluid motion to evolution equations for the free boundary Γ_t. In Eulerian coordinates, e.g., the water wave equations can be seen as a system in (η, ψ), where η is the elevation of the free surface and ψ is the trace of the velocity potential on the free surface:

$$\begin{cases} \partial_t \eta - G(\eta)\psi = 0 \\ \partial_t \psi + g\eta - \sigma \mathbf{H}(\eta) + \frac{1}{2}|\nabla\psi|^2 - \frac{1}{2}\frac{(\nabla\eta\nabla\psi + G(\eta)\psi)^2}{1 + |\nabla\eta|^2} = 0. \end{cases}$$

Here $G(\eta)$ is the Dirichlet to Neumann operator associated to the fluid domain. This can be viewed a nonlinear and nonlocal pseudodifferential operator.

A solitary wave is a solution to the water wave system which maintains a fixed profile, and moves with constant velocity,

$$(\eta, \psi)(x, t) = (\eta, \psi)(x - ct, 0).$$

Here one also assumes some qualitative averaged decay for η_x at infinity, in order to distinguish solitary waves from periodic traveling waves. The constant c is the horizontal speed of the solitary wave.

The existence and stability of solitary waves is a fundamental question in the study of water waves. Solitary waves have been discovered in many of these settings in both two and higher dimensions, beginning with the results in finite depth in [10], [5] and [2], followed by the bifurcation result in [16]. This famously includes for instance the Stokes waves of greatest height, which have an angular crest; these were conjectured in [18] and proved to exist in [19], [1]. Within this family of problems, one of the most difficult cases has turned out to be that of deep water.

For water waves in deep water, solitary waves have been recently proved to exist provided that both gravity and surface tension are present, see [15, 6, 7]. However, in the seemingly simpler cases where exactly one of these forces is active, the problem has remained largely open, and only partial nonexistence results are known [8, 17, 12].

Our present work is devoted to the two dimensional case; in brief, our results assert that no pure gravity or pure capillary solitary waves exist. We begin our discussion with pure gravity waves, where our main result is as follows:

Theorem 1. *The two dimensional gravity wave equation in deep water admits no solitary waves* (η, ψ) *with critical regularity* $\eta_x \in \dot{B}^{\frac{1}{2}}_{2,1}, \nabla\phi_{|y=\eta} \in \dot{H}^1$.

This result settles the two dimensional case of a longstanding open problem, where only partial results were known before. In [8] it was shown that there are no solitary waves with either positive elevation ($\eta \geq 0$) or negative elevation ($\eta \leq 0$) in both two and three dimensions, and more recently, that there are no two dimensional solitary waves with at least $|x|^{-1-\epsilon}$ decay at infinity, see [17, 12]. The expansion at infinity was discussed in greater detail in [20].

For convenience the result was stated above in Eulerian coordinates, where it is implicitly assumed that the free surface is a graph. We do not use Eulerian coordinates or make such an assumption in the proof of the theorem.

The a-priori regularity that is assumed in the theorem is completely natural, and is dictated by the scaling of the problem. Indeed, below this regularity it is largely meaningless to consider the dynamic evolution. This regularity does imply the averaged decay of η_x to zero at infinity, but little more. Indeed, these assumptions allow for functions η which have nearly linear growth at infinity.

The a-priori regularity above excludes angular crests. However, we also have an alternate version of the result which also excludes solitary waves with finitely many angular crests. We refer the reader to the full article for more details.

Our second result applies to the case of pure capillary waves in two dimensions, where not even partial results were previously available:

Theorem 2. *The two dimensional capillary wave equations in deep water admits no solitary waves* (η, ψ) *with critical regularity* $\eta_x \in \dot{B}^{\frac{1}{2}}_{2,1}$, $\nabla\phi_{|y=\eta} \in L^2$.

Our approach for both of these problems relies on the use of the holomorphic formulation of the equations, which uses conformal coordinates within the fluid domain. This approach has been initially used precisely in the study of periodic traveling waves in work of Babenko [4, 3], and later for solitary waves in [16, 12] and many other works. It has also been implemented in the study of the dynamical problem for gravity waves in work of Wu [21], Dyachenko-Kuznetsov-Spector-Zakharov [9], and more recently by the authors and Hunter [11], as well as for capillary waves by the authors in [13]. The equivalence between the solitary wave equation and the Babenko equation has also been explored in [12].

In the present work we derive a set of equations for steady gravity waves in holomorphic coordinates, which are easily seen to imply the Babenko equations for gravity waves; however, the converse is far less obvious. Similarly, we also produce a related set of equations for capillary waves.

Once the problem is properly formulated in the conformal setting, our proof of both results relies on a positive commutator argument which is somewhat reminiscent of results on absence of embedded eigenvalues in elliptic problems. Matters are more difficult here because of the nonlocality, so a limiting argument is required, with Coifman-Meyer style commutator estimates.

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Doubly periodic steady waves on Beltrami flows

Erik Wahlén
(joint work with Evgeniy Lokharu and Douglas Seth)

The last 15 years have seen a great deal of progress on two-dimensional steady water waves with vorticity, but the three-dimensional problem remains largely unexplored. The main reason for this is undoubtedly the lack of a general reduction to an elliptic free-boundary problem, as offered by the stream function formulation for two-dimensional rotational flows or by the velocity potential formulation for irrotational flows. In my talk I reported on recent progress in the special case of Beltrami flows, that is, flows whose vorticity vector is proportional to the velocity vector. This is another case which can be formulated as an elliptic free-boundary problem, but as a system rather than a scalar equation. In a moving frame of reference, the problem takes the form

$$\nabla \times \boldsymbol{u} = \alpha \boldsymbol{u} \qquad \text{in } \Omega^\eta, \tag{1a}$$

$$\nabla \cdot \boldsymbol{u} = 0 \qquad \text{in } \Omega^\eta, \tag{1b}$$

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \qquad \text{on } \partial\Omega^\eta, \tag{1c}$$

$$\frac{1}{2}|\boldsymbol{u}|^2 + g\eta - \sigma\nabla \cdot \left(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right) = Q \qquad \text{on } z = \eta, \tag{1d}$$

where α is a constant, g is the gravitational constant of acceleration, σ the coefficient of surface tension and Q the Bernoulli constant. The fluid domain is given by $\Omega^\eta = \{(\boldsymbol{x}', z) \in \mathbb{R}^2 \times \mathbb{R} : -d < z < \eta(\boldsymbol{x}')\}$, where $\boldsymbol{x}' = (x, y)$, $d > 0$ is the undisturbed water depth and the surface elevation η is an unknown function.

There are two important special classes of solutions. Laminar flows ($\eta \equiv 0$, no dependence on the horizontal coordinates) are given by

$$\boldsymbol{U}[c_1, c_2] = c_1\boldsymbol{U}^{(1)}(z) + c_2\boldsymbol{U}^{(2)}(z), \quad c_1, c_2 \in \mathbb{R}, \text{ where}$$

$$\boldsymbol{U}^{(1)}(z) = (\cos(\alpha z), -\sin(\alpha z), 0), \quad \boldsymbol{U}^{(2)}(z) = (\sin(\alpha z), \cos(\alpha z), 0).$$

and $Q(c_1, c_2) = \frac{1}{2}(c_1^2 + c_2^2)$. Physically this means that the velocity field is constant at each vertical position but its direction varies with height (see Figure 1, left). The other special solutions are so-called $2^1/_2$-dimensional flows, meaning that the velocity field is independent of one of the horizontal coordinates, although the corresponding velocity component doesn't vanish. Suppose for simplicity that the solution is independent of the horizontal coordinate y. Then it has the form

$$\boldsymbol{u} = (-\psi_z(x, z), \alpha\psi(x, z) + \beta, \psi_x(x, z)), \quad \eta = \eta(x),$$

with $\psi_{xx} + \psi_{zz} + \alpha^2\psi + \alpha\beta = 0$. Thus, any $2^1/_2$-dimensional Beltrami field corresponds to a two-dimensional velocity field with affine vorticity function.

In the recent preprint [5] we present an existence theory for genuinely three-dimensional solutions of problem (1). These are doubly periodic small-amplitude solutions bifurcating from a laminar flow $\boldsymbol{U}[c_1^\star, c_2^\star]$ for appropriate values $\boldsymbol{c} = (c_1^\star, c_2^\star)$ of the bifurcation parameters. The solutions are periodic with respect to a

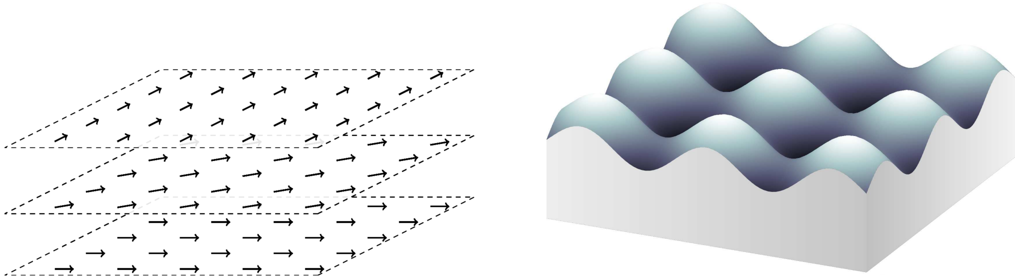

Figure 1. Left: A laminar flow. Right: A doubly periodic wave.

given lattice $\Lambda = \{m_1\boldsymbol{\lambda}_1 + m_2\boldsymbol{\lambda}_2 \colon m_1, m_2 \in \mathbb{Z}\} \subset \mathbb{R}^2$ (see Figure 1, right) and satisfy the symmetry conditions $\eta(-\boldsymbol{x}') = \eta(\boldsymbol{x}')$, $\boldsymbol{u}(-\boldsymbol{x}', z) = (\boldsymbol{u}'(\boldsymbol{x}', z), -u_3(\boldsymbol{x}', z))$. Note that the lattice is not required to be symmetric, that is, we allow $|\boldsymbol{\lambda}_1| \neq |\boldsymbol{\lambda}_2|$. To single out a two-parameter family of solutions, we require that $Q = Q(c_1, c_2)$ and impose the additional integral conditions

$$\int_{\Omega_{00}^{\eta}} u_j \,\mathrm{d}V = \int_{\Omega_{00}^{\eta}} U_j[c_1, c_2] \,\mathrm{d}V \qquad \text{for } j = 1, 2, \tag{1e}$$

where Ω_{00}^{η} denotes one periodic cell of the fluid domain. The appropriate values of the bifurcation parameters $\boldsymbol{c}$ are found by solving the 'dispersion equation'

$$\rho(\boldsymbol{c}, \boldsymbol{k}) = g + \sigma|\boldsymbol{k}|^2 - \frac{(\boldsymbol{c}\cdot\boldsymbol{k})^2}{|\boldsymbol{k}|^2}\kappa(|\boldsymbol{k}|) + \alpha\frac{(\boldsymbol{c}\cdot\boldsymbol{k})(\boldsymbol{c}\cdot\boldsymbol{k}_\perp)}{|\boldsymbol{k}|^2} = 0,$$

where

$$\kappa(|\boldsymbol{k}|) = \begin{cases} \sqrt{\alpha^2 - |\boldsymbol{k}|^2}\cot(\sqrt{\alpha^2 - |\boldsymbol{k}|^2}d) & \text{if } |\alpha| > |\boldsymbol{k}|, \\ \sqrt{|\boldsymbol{k}|^2 - \alpha^2}\coth(\sqrt{|\boldsymbol{k}|^2 - \alpha^2}d) & \text{if } |\alpha| < |\boldsymbol{k}|, \end{cases}$$

and $\boldsymbol{k}_\perp = (-l, k)$, in which $\boldsymbol{k} = (k, l)$ denotes the wave vector living in the dual lattice $\Lambda' = \{n_1\boldsymbol{k}_1 + n_2\boldsymbol{k}_2 \colon n_1, n_2 \in \mathbb{Z},\ \boldsymbol{k}_i\cdot\boldsymbol{\lambda}_j = 2\pi\delta_{ij},\ i, j = 1, 2\}$. The bifurcation condition is that the dispersion equation has precisely four solutions in the dual lattice, and for simplicity we assume that these are given by $\pm\boldsymbol{k}_1$ and $\pm\boldsymbol{k}_2$ (note that $\rho(\boldsymbol{c}, -\boldsymbol{k}) = \rho(\boldsymbol{c}, \boldsymbol{k})$). This means that the kernel of the linearised problem is two-dimensional when taking the symmetries into account. In addition, we require the non-resonance condition $\sqrt{\alpha^2 - |\boldsymbol{k}|^2} \notin \frac{\pi}{d}\mathbb{Z}_+$ for all $\boldsymbol{k} \in \Lambda'$, as well as the transversality condition $\nabla_{\boldsymbol{c}}\rho(\boldsymbol{c}^\star, \boldsymbol{k}_1) \nparallel \nabla_{\boldsymbol{c}}\rho(\boldsymbol{c}^\star, \boldsymbol{k}_2)$.

Our main result is the following.

Theorem 1. *Let $\alpha \in \mathbb{R}$, $\sigma > 0$ and the depth $d > 0$ be given. Furthermore, let Λ' be the dual lattice generated by the linearly independent vectors $\boldsymbol{k}_1, \boldsymbol{k}_2 \in \Lambda'$. Assume that the non-resonance condition holds, that within the lattice Λ', the dispersion equation with $\boldsymbol{c} = \boldsymbol{c}^\star$ has exactly four roots $\pm\boldsymbol{k}_1$ and $\pm\boldsymbol{k}_2$, and that the transversality condition holds. Then there exists an $\epsilon > 0$ such that for any $\boldsymbol{t} = (t_1, t_2) \in B_\epsilon(0; \mathbb{R}^2)$ there is a solution $(\boldsymbol{u}, \eta) \in C^{1,\gamma}_{per,e}(\Omega^\eta; \mathbb{R}^3) \times C^{2,\gamma}_{per,e}(\mathbb{R}^2)$ of problem (1) with $\boldsymbol{c} = \boldsymbol{c}^\star + \mathcal{O}(|\boldsymbol{t}|^2)$, $Q = Q(\boldsymbol{c})$ and*

$$\eta(\boldsymbol{x}') = t_1\cos(\boldsymbol{k}_1\cdot\boldsymbol{x}') + t_2\cos(\boldsymbol{k}_2\cdot\boldsymbol{x}') + \mathcal{O}(|\boldsymbol{t}|^2), \quad |\boldsymbol{t}| < \epsilon.$$

Here $C^{k,\gamma}$ denotes the class of k times Hölder continuously differentiable functions, and the subscripts 'per' and 'e' stand for periodicity and symmetry, respectively. We also show that near the bifurcation point, there are no other solutions in this class except for laminar flows and 2½-dimensional solutions.

In the paper we give examples of when the bifurcation conditions are satisfied. Assume e.g. that α, $\kappa(|\boldsymbol{k}_1|)$ and $\kappa(|\boldsymbol{k}_2|)$ are all positive. Then if

$$0 \leq \arctan\left(\frac{\kappa(|\boldsymbol{k}_1|)}{|\alpha|}\right) - \arctan\left(\frac{\kappa(|\boldsymbol{k}_2|)}{|\alpha|}\right) < \theta < \pi,$$

where θ is the angle between $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$, there exists a $\boldsymbol{c}^\star$ such that $\rho(\boldsymbol{c}^\star, \boldsymbol{k}_1) = \rho(\boldsymbol{c}^\star, \boldsymbol{k}_2) = 0$ and the transversality condition is satisfied. Generically, one expects there to be no other solutions (except for $-\boldsymbol{k}_1, -\boldsymbol{k}_2$). In the special case $|\boldsymbol{k}_1| = |\boldsymbol{k}_2|$ we show rigorously that for σ outside a countable set of forbidden values, this is indeed the case. The non-resonance condition can be satisfied by choosing the angle θ appropriately. We can also handle the irrotational case $\alpha = 0$, which has previously been investigated by Reeder & Shinbrot [6], Craig & Nicholls [3], Groves & Mielke [1] and Groves & Haragus [2] among others. It is important, however, that $\sigma > 0$; otherwise one expects to encounter small divisor issues as in the irrotational case (see e.g. Iooss & Plotnikov [4]). It would be interesting to treat this case as well, although it's much more challenging.

The proof involves reducing the problem to a single scalar equation for η and using a multi-parameter local bifurcation argument based on Lyapunov-Schmidt reduction. Note that when $\alpha = 0$ there is an additional symmetry which allows one to use a one-dimensional bifurcation approach in the symmetric case (treated e.g. by Reeder & Shinbrot and Groves & Mielke). This additional symmetry is lacking when $\alpha \neq 0$, but the multi-parameter approach allows us to bypass this difficulty.

Acknowledgment. This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement no 678698).

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Orbital stability and instability of fractional KdV solitary waves

Samuel Walsh
(joint work with Kristoffer Varholm and Erik Wahlén)

We consider nonlinear dispersive PDEs of the form

$$\partial_t u = \partial_x(|\partial_x|^\alpha u - u^p), \tag{1}$$

where $u = u(t,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the unknown, α describes the strength of the dispersion, and $p > 1$ represents a "generic" power law nonlinearity.

This family of problems encapsulates a number of important hydrodynamical models, including the famous Korteweg–de Vries ($\alpha = 2$, $p = 2$) and Benjamin–Ono equations ($\alpha = 1$, $p = 2$). Recently, there has been increased interest in the "fractional" regime that corresponds to values of $\alpha < 1$. While these do not come directly from fluid equations, they can be thought of as dispersive corrections to the inviscid Burgers equation.

Heuristically, at least, one expects that stronger nonlinearity leads to blow-up due to steepening or self-focusing. On the other hand, dispersion causes different frequency components of $u(t)$ to separate in physical space, driving the solution to spread and decay (in L^∞_x, say) as $t \to \infty$. A fundamental problem is to quantify how the interplay between nonlinearity and dispersion relates to the long-time dynamics of the system.

For example, this balancing is responsible for the existence of solitary waves solutions to (1), and there is an extensive literature devoted to studying their stability properties. Notably, Bona, Souganidis, and Strauss [2] obtained a general stability/instability criteria that applied for $\alpha \in [1, 2]$. In this talk, we present a new, more direct proof of this seminal result. At the same time, we expand its scope to the fractional case.

The starting point for both our work and the original paper [2] is to rewrite the PDE (1) as a Hamiltonian system. Formally, $|\partial_x|^\alpha u - u^p$ is the L^2 gradient of the *energy* functional

$$E(u) := \frac{1}{2}\int_{\mathbb{R}} (|\partial_x|^{\frac{\alpha}{2}} u)^2 \, dx - \frac{1}{p+1}\int_{\mathbb{R}} u^{p+1} \, dx.$$

This suggests that the natural *energy space* for the problem is $\mathbb{X} := H^{\frac{\alpha}{2}}(\mathbb{R})$. To make this rigorous, we need E to be smooth with domain $\mathbb{X}$. For that reason, we restrict to dispersion strengths $\alpha \in (1/3, 2]$ and power laws

$$p \in \mathbb{N} \cap \begin{cases} (1, \frac{1+\alpha}{1-\alpha}) & \alpha \in (1/3, 1), \\ (1, \infty) & \alpha \in [1, 2]. \end{cases}$$

By Sobolev embedding, when α and p are as above, we have $\mathbb{X} \hookrightarrow L^{p+1}(\mathbb{R})$ and that $E \in C^\infty(\mathbb{X}; \mathbb{R})$.

Then the dispersive PDE admits the Hamiltonian description

$$\partial_t u = JE'(u), \tag{2}$$

where the Poisson map $J = \partial_x$ and the above equation is satisfied in the sense of distributions.

To discuss stability/instability, we require that the Cauchy problem be at least locally well-posed in time. Unfortunately, this is not currently known for all α and p in the admissible range. We therefore introduce a smoother space

$$\mathbb{W} := \begin{cases} \mathbb{X} & \text{if } \frac{\alpha}{2} > s_0(\alpha,p), \\ H^{s_0+}(\mathbb{R}) & \text{if } \frac{\alpha}{2} \leq s_0(\alpha,p). \end{cases}$$

where $s_0 = s_0(\alpha, p)$ is defined as the minimum regularity index such that (2) is locally well-posed in H^{s_0+}.

At present, it is known that the problem is *globally* well-posed in $\mathbb{X}$ for all $\alpha > 6/7$ for $p = 2$; see [5]. This is conjectured to hold for all $\alpha > 1/2$, which corresponds to the L^2 subcritical case. However, for fKdV with $\alpha \in (1/3, 6/7)$, our functional analytic setup will lead to a *conditional* stability or instability result. Roughly speaking, a solitary wave is conditionally orbitally stable if, for any initial data u_0 sufficiently close in $\mathbb{X}$, the corresponding solution $u(t)$ remains close in $\mathbb{X}$ to a spatial translate of the steady wave for as long as it exists and is uniformly bounded in $\mathbb{W}$. Conditional orbital instability is defined analogously.

It is very important to note here that while J is injective, it is *not* surjective. This unfortunate fact means that one cannot directly use the Grillkais–Shatah–Strauss (GSS) method [3] to infer stability/instability. In its place, Bona, Souganidis, and Strauss made use of an additional conserved quantity, the mass $\int u\,dx$. While this works quite nicely for $\alpha \in [1, 2]$, it requires an additional argument relating to the spatial decay rates of solutions with initial data near a solitary wave. These estimates actually *fail* for the fractional regime, so something new is required.

Our argument relies on a relaxed version of GSS that can treat Hamiltonian systems like (2) for which the Poisson map has dense range; see, [6]. Indeed, it is quite straightforward to see that the range of ∂_x is dense in the appropriate space.

With this general theory in hand, it is fairly simple to prove the following theorem characterizing both conditional stability and instability.

Theorem 1. *If $p < 2\alpha + 1$, then solitary waves solutions of* (1) *are conditionally orbitally stable when $\frac{\alpha}{2} \leq s_0(\alpha, p)$, and orbitally stable when $\frac{\alpha}{2} > s_0(\alpha, p)$. If $p > 2\alpha + 1$, the waves are orbitally unstable.*

As mentioned above, for the range $\alpha \in [1, 2]$, this recovers completely the classical Bona, Souganidis, and Strauss result [2]. Earlier works by Angulo Pava [1] and Linares, Pilod, and Saut [4] treat $\alpha \in (1/2, 1)$ and $p = 2$. Because our new argument does not rely on tail estimates, it applies just as immediately to the fractional regime $\alpha \in (1/3, 1)$. In particular, this gives the first proof that solitary waves are conditionally orbitally unstable instability when $\alpha \in (1/3, 1/2)$.

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Waves near resonance: from high speed train to moving loads on very large floating structures

Zhan Wang

(joint work with Paul Milewski and Jean-Marc Vanden-Broeck)

Hydroelasticity, a name adapted from aeroelasticity, is concerned with the motion and distortion of deformable bodies responding to hydrodynamic excitations, and the associated reactions on the motion of the environmental fluid. Hydroelastic waves enjoy wide usages in marine structures and sea transport. Modern applications of hydroelastic waves abound: very large floating structures usable as fully functional airport runways (e.g. Mega-Float project in Japan); large fast merchant ships and container vessels which are relatively more flexible; flexible risers to transport hydrocarbon (mainly refers to oil) from the seabed to shore or offshore facilities; safe use of lake and ocean ice for roadways and landing strips.

There has been a renewed interest in hydroelasticity problems dealing with the interaction between moving fluids and deformable sheets. The response of the floating elastic sheet to moving loads on it is sometimes surprising, since localized, stable, large responses were observed as the plane moving steadily on an ice cover with speed in a certain range. The linear theory shows that there exists a critical speed $c_{\min}$ due to competition between gravity and elastic bending. The critical speed occurs at a finite wavenumber where the group velocity equals the phase velocity. It follows physically that if a moving load is travelling with $c_{\min}$, then energy will not be able to propagate relative to the load and will therefore accumulate. This phenomenon is very much like vibrations of fast train track which can be described by an external forcing moving on a beam resting on an elastic foundation (i.e. Pasternak-type foundations). Although the linear theory identifies the critical speed and provides some physical explanation, it fails to depict the real wave phenomenon near resonance since it predicts unlimited growth of wave amplitude owing to the non-dispersive nature.

In order to understand certain measured effects of large responses when forces move slightly below the critical speed, the nonlinear factors need to be considered.

Using a weakly nonlinear normal-form analysis of the free and forced hydroelastic wave problem around $c_{\min}$, Părău & Dias (2002) found that there is a critical depth H_c above which there are no free solitary waves bifurcating from a uniform stream since the envelope was governed by a defocussing nonlinear Schrödinger equation (NLS). While for depths shallower than H_c, their analysis shows that there are wavepacket solitary waves which are qualitatively similar to the experimental measurements carried out at Lake Saroma in Hokkaido (Takizawa 1988). However, people did observe similar locally confined travelling-wave solutions in deep water (experiments at McMurdo Sound in Antarctica, Squire *et al.* 1998), which cannot be explained by the weakly nonlinear theory.

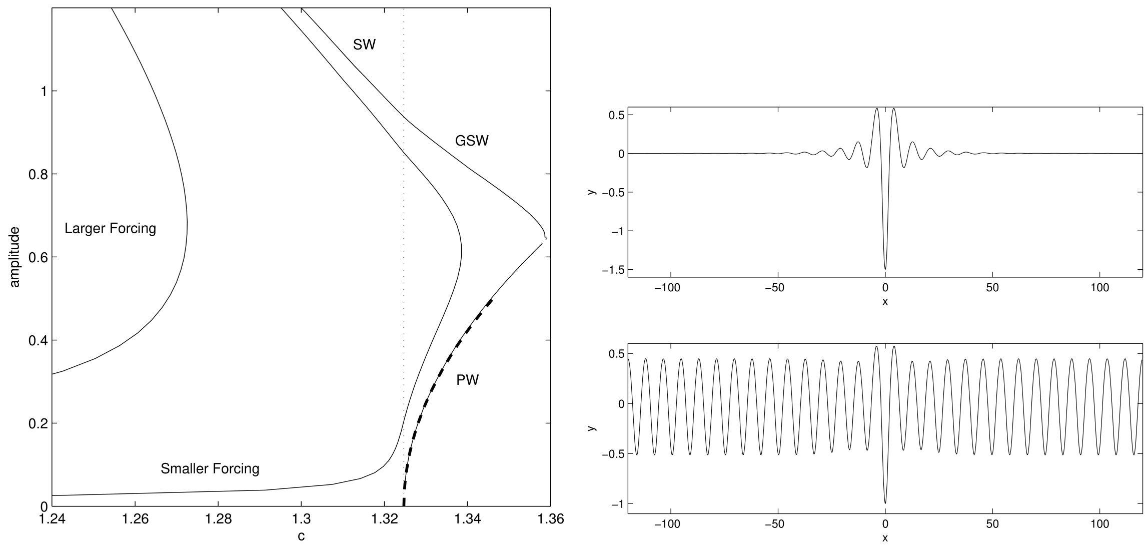

FIGURE 1. (Left) Travelling wave solution branches near the critical speed $c_{\min}$ (which is shown by the vertical line). Branches of forced solutions are shown by two different forcing amplitudes, $a = 0.2$ and $a = 0.02$. The uppermost curve is a branch of unforced solitary waves for $c < c_{\min}$ (labelled SW) and generalized solitary waves for $c > c_{\min}$ (labelled GSW). The branches originating at $c_{\min}$ are periodic Stokes solutions (labelled PW) and they are compared to the NLS prediction (thick dashed curve). (Right) Examples of unforced solitary waves and generalized solitary waves.

Even though small amplitude solitary waves are not predicted to exist by standard perturbation analyses, we show that solitary waves do occur in deep water in the full Euler equations, but they are a new type in that they occur along a branch of generalized solitary waves that itself bifurcates from periodic waves of finite amplitude. The bifurcation of deep water hydroelastic waves near the critical speed is rich with Stokes, solitary, generalized solitary, and dark solitary waves (see Figure 1). When the problem is forced by a moving load, we find that, for small forcing, steady responses are possible at all subcritical speeds, but for larger loads there is a transcritical range of forcing speeds for which there are no steady solutions. If the problem is forced at a speed in this range, large unsteady responses are obtained, and that when the forcing is released, a solitary wave is generated. These solitary waves appear stable, and can coexist within a sea of small-amplitude waves, which may result in the breakup of ice. It is worth noting that this new type of bifurcation mechanism was also found in other fluid dynamical systems later on, for example, interfacial capillary-gravity waves between two immiscible fluids as the density ratio is above a critical value and capillary-gravity waves propagating on a conducting fluid under a strong normal electric field.

Overall, in deep water the hydroelastic wave problem is very different from capillary-gravity waves. Some open problems on the theoretical side of this topic are mentioned in the end, including global well-posedness of the initial value problem for 2D and 3D hydroelastic waves, existence of 2D and 3D hydroelastic solitary waves in deep water, and linear and nonlinear transverse instabilities of hydroelastic plane solitary waves. The defocusing NLS deprives us of a powerful tool to prove the existence of solitary waves in the primitive Euler equations, and therefore new techniques need to be developed.

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Rotational bores with critical layers

Miles H. Wheeler

(joint work with Robin Ming Chen and Samuel Walsh)

Consider an infinite two-dimensional channel containing two superposed fluids: a denser lower fluid which is irrotational and a lighter upper fluid with constant vorticity. In this talk we rigorously construct front-type solutions or bores in this system, i.e. solutions where the interface between the fluids travels with constant speed c and has distinct asymptotic heights as $x \to \pm\infty$, say h and h'. In future work we will use global bifurcation techniques to construct large-amplitude solutions, but for the moment we restrict ourselves to the small-amplitude regime $|h - h'| \ll 1$. Some of the solutions we construct have critical layers and the striking "half cat's eye" streamline patterns shown in Figure 1.

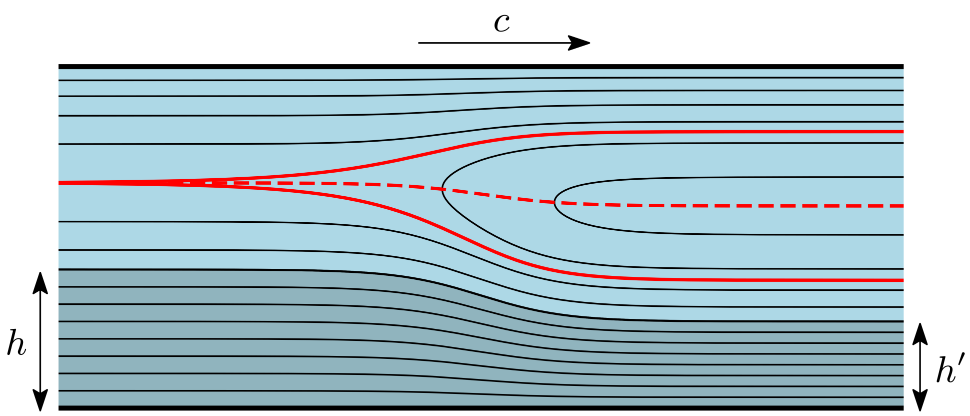

FIGURE 1. Streamlines in a bore with a critical layer. The fluid particles move from right to left below the dashed red line, and from left to right above it. Between the two red lines, the streamlines extend only to $x = +\infty$; otherwise they tend to both $x = +\infty$ and $x = -\infty$.

Standard arguments using the conservation of mass, momentum, and energy yield a set of two "conjugate flow" equations which constrain the parameters h, h', c. In our case these are highly nonlinear polynomial equations which cannot be solved explicitly except in special cases. One such limiting case is when both layers are irrotational. The parameters h' and c are then uniquely determined by the ratio between $0 < \rho < 1$ between the upper and lower densities [1, 6, 8]. Another limiting case — which does not appear to have been considered before — is when the two densities are equal so that $\rho = 1$. In general, we study the equations perturbatively for $|h - h'| \ll 1$.

To obtain both the existence of the solutions and the qualitative picture in Figure 1, we use a novel center manifold reduction theorem which reduces the full PDE to a second-order ODE for the vertical displacement η of the interface. Inspired by [4], this reduction is "without a phase space" in the sense that we never explicitly reformulate our PDE as an evolution equation. It is also specialized to a class of quasilinear elliptic problems that are amenable to the global bifurcation arguments we have in mind. Most of the heavy lifting in the proof is outsourced to [2]; in principle we could also have used the more classical spatial dynamics theory [7, 5]. We find our reduction theorem to be particularly convenient for calculations, and in [3] also apply it to problems from biology and elasticity.

Because the bores we construct are less symmetric than solitary waves ("non-reversible" in the spatial dynamics terminology), they are more difficult to construct. Indeed, in addition to the usual parameter c we must also vary the upstream depth ratio h and take advantage of a conserved quantity called the flow force — this eventually boils down to having a good local understanding of the conjugate flow equations discussed above.

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Some recent results on capillary-gravity water waves and water waves with point vortices

Sijue Wu

(joint work with Siddhant Agrawal and Qingtang Su)

In this talk, I presented some of the recent work of my Ph.D students Siddhant Agrawal and Qingtang Su. Agrawal's work concerns the capillary-gravity water waves, and Su's work is on the long time behavior of water waves with point vortices.

The n-dimensional full water wave problem is described by the following set of equations defined in moving domains $\Omega(t)$:

$$\begin{cases} \mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} = \mathbf{g} - \nabla P \qquad \text{on } \Omega(t),\ t\geq 0, \\ \operatorname{div}\mathbf{v} = 0 \qquad \text{on } \Omega(t),\ t\geq 0, \\ P = -\sigma\kappa, \qquad \text{on } \Sigma(t) \\ (1,\mathbf{v}) \text{ is tangent to the free surface } (t,\Sigma(t)), \end{cases} \tag{1}$$

where $\mathbf{v}$ is the fluid velocity, P is the fluid pressure; $\Omega(t)$ is the fluid domain at time t, $\Sigma(t) = \partial\Omega(t)$ is the free interface. $\mathbf{g}$ indicates the gravity, $\sigma \geq 0$ is the surface tension coefficient, and κ is the mean curvature of the interface $\Sigma(t)$. A solution is irrotational if in addition to (1), the vorticity

$$\operatorname{curl}\mathbf{v} = 0, \qquad \text{on } \Omega(t). \tag{2}$$

Both Agrawal and Su's work concern the motion of the two-dimensional water waves.

Agrawal studied the 2-dimensional irrotational capillary-gravity water waves in a regime that allows for angled crested interfaces, extending the results in [5] to capillary-gravity water waves. In [5], we constructed an energy functional $\mathcal{E}(t)$, which allows for interfaces with angled crests, and showed that the gravity water wave equation (i.e. $\sigma = 0$) is locally well-posed in the class where $\mathcal{E}(t) < \infty$. In [1], Siddhant studies the case where the surface tension coefficient $\sigma > 0$. Using Riemann mapping, Siddhant obtains a precise formula for the important quantity $-\frac{\partial P}{\partial \mathbf{n}}$, which shows that $-\frac{\partial P}{\partial \mathbf{n}}$ is the sum of two terms: the first is a non-negative term associated with the pure gravity water waves ($\sigma = 0$), the second term is (nonlinearly) proportional to $\sigma|D|\kappa$, which shows that the Taylor sign condition can fail if σ is large. He then constructs an energy functional $\mathcal{E}_\sigma(t)$, with the properties that $\mathcal{E}_\sigma(t) \geq \mathcal{E}(t) > 0$ for all $\sigma > 0$ and $\lim_{\sigma\to 0} \mathcal{E}_\sigma = \mathcal{E}$ and proves an a-priori estimate, and shows, without any assumptions on the sign of the quantity $-\frac{\partial P}{\partial \mathbf{n}}$, that for any initial data satisfying $\mathcal{E}_\sigma(0) < \infty$, there is a unique solution in the class where $\mathcal{E}_\sigma(t) < \infty$ for the capillary gravity water wave equation (1)-(2), on a time interval $[0, T^*]$, with $T^* > 0$ depending only on $\mathcal{E}_\sigma(0)$. His results advances earlier works in several ways: while the time of existence of the solution is $T \leq C\|\kappa\|_{L^\infty}^{-1}$ in earlier works, $T^* = O(1)$ in Siddhant's result; moreover for any given $\sigma_1 > 0$, T^* depends only on σ_1 for all $0 \leq \sigma \leq \sigma_1$. Furthermore, Siddhant shows that the solution of the gravity water wave equation ($\sigma = 0$) in the class where $\mathcal{E}(t) < \infty$ can be obtained as a zero surface tension limit of the corresponding solutions for the capillary gravity water wave equation (1)-(2).

Qingtang Su studied the long-time behavior of the 2-dimensional water wave equation with point vortices [4]. (i.e. $\sigma = 0$, but curl $\mathbf{v} \neq 0$, in $\Omega(t)$). He shows that for certain initially symmetric vortex pairs, if the motion of the water wave is also symmetric initially, then despite the interaction with the interface, the vortex pair will keep moving downward with a positive speed. For this initial configuration, Qingtang shows that for initial data of size ϵ, the solution of the water wave equation (1) with point vortices exist on a time interval of length $O(\epsilon^{-2})$, with interface remaining small and smooth. Qingtang also shows that for this symmetric vortex pair, if the initial data is sufficiently localized, then the solution exist for all time. Qingtang Su's result is the first concerning the long time behavior of the water waves with non-zero compactly supported vorticity.

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