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Homotopy Theory

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Abstract. Algebraic topology in general and homotopy theory in particular is in an exciting period of growth and transformation, driven in part by strong interactions with algebraic geometry, mathematical physics, and representation theory, but also driven by new approaches to our classical problems. This workshop was a forum to present and discuss the latest result and ideas in homotopy theory and the connections to other branches of mathematics. Central themes of the workshop were derived algebraic geometry, homotopical invariants for ring spectra such as topological Hochschild homology, interactions with modular representation theory, group actions on spaces and the closely-related study of the classifying spaces of groups.

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Introduction by the Organisers

Algebraic topology in general and homotopy theory in particular is in an exciting period of growth and transformation, driven in part by strong interactions with algebraic geometry, mathematical physics, and representation theory, but also driven by new approaches to our classical problems. There is also a human element to this change: there is a very strong group of relatively young researchers in the field. The purpose of the workshop was to present and discuss the latest result and ideas in homotopy theory and related fields. We made a special point to invite and feature some of the new voices.

One of the startling developments of the last few years has been the emergence of derived algebraic geometry, which may be defined as algebraic geometry with commutative rings replaced by some homotopy invariant analog, such as differential graded algebras, simplicial commutative rings, or E_∞-ring spectra. While quite historical, with origins in Serre's work on intersection theory, it is only since the work of Mike Hopkins and others that we've realized that some of the fundamental objects in algebraic geometry, such as the moduli stack of elliptic curves, are inherently derived in nature. In Tyler Lawson's talk, we heard the latest installment of this story. He reported on his work with Mark Behrens on derived Shimura varieties. Shimura varieties are a fundamental tool in number theory, central to the Harris-Taylor proof of the local Langlands conjecture, and the Behrens-Lawson work has the potential to make concrete a long-guessed-at connection between number theory and stable homotopy theory. The talks by Henn, Hill, and Naumann also worked out some of the interplay between number theory and homotopy theory.

In another link between homotopy theory and algebraic geometry, we have lately seen that classical invariants for rings can be generalized to great effect to ring spectra, an important example being (topological) Hochschild homology THH. In the talk by Vigleik Angeltveit, we heard about a computation of this invariant for the ring spectra associated to real- and complex-K-theory. Angeltveit and his coauthors have discovered new and very different-looking periodic phenomena. Gunnar Carlsson, motivated partly by questions from K-theory, discussed the iterations of THH and related constructions. There is a conjecture, due to John Rognes, that such iterations increase complexity in a very specific way.

The moduli spaces of surfaces, now for higher genus, also arise in other areas of homotopy theory as well. A few years ago, Madsen and Weiss gave a proof of the Mumford conjecture cohomology of the stable moduli space of Riemann surfaces; indeed, they did far more, as they calculated the homotopy type of this moduli space. The interest in these space has diverse origins, but currently they are central to our new understanding of various models for string theory and mirror symmetry in mathematical physics. This has created a great deal of interest in homotopy theoretic aspects of these moduli spaces. The talk by Elizabeth Hanbury explored some of these connections. These spaces are also part of current research on Gromov-Witten invariants; the talk by Stefan Bauer gave us a homotopy theoretic approach to dissecting the related Seiberg-Witten invariants.

The interactions between homotopy theory and modular representation theory are now well-established: the tools of stable homotopy theory are fundamental to representation theory, and results from representation theory feed back into our understanding of stable homotopy theory. We heard two talks in this areas, one from Henning Krause on the structure of sub-categories in the stable module category and another from Martin Langer on a subtle invariant of that same category inspired by a similar invariant from stable homotopy theory.

Another important point of contact between homotopy theory and algebra is in the theory of group actions on spaces and the closely-related study of the classifying spaces of groups. These ideas threaded their way through the talks by Adem, Grodal, Levi and Lück. The talk by Levi, for example, gave an overview of a long story about how one can recover the p-completion of a classifying space from simpler, homotopy theoretic data, and that of Grodal used related machinery to give a sophisticated representation theoretic extension of Smith theory.

A few talks told us something new about classical topics. For example, Jim McClure's talk applied rational homotopy theory in a novel way to the homotopy theory of the intersection pairing on a manifold – which normally relies on some differential or PL structure to get transversality. And the talk by Rekha Santhanam on units in ring spectra revisited a topic central to producing "twisted" versions of homology theories; twisted versions of K-theory have also played a role in the algebraic topology of mathematical physics, using the work of Freed-Hopkins-Teleman.

The participant list also reflected the changing nature of the field. Of the 52 registered participants, 17 (or about 30%) were young – that is, graduate students or in a post-doctoral position – and 9 (17%) were women. Of the 18 talks, 7 were from the younger participants. This emphasis was not completely accidental, but we had more proposed talks than slots and we emphasized quality of subject matter above other considerations when selecting speakers.

Workshop: Homotopy Theory

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Abstracts

Commuting elements and spaces of homomorphisms

Alejandro Adem
(joint work with Fred Cohen and Enrique Torres)

Let π denote a finitely generated discrete group and G a Lie group. The set of homomorphisms $\mathrm{Hom}(\pi, G)$ from π to G can be topologized in a natural way as a subspace of a finite product of copies of G, where the relations on π carve out what can be a variety with interesting singularities.

First we consider the question of path–connectivity for these spaces. In the case of $G = O(n)$ or $G = SO(n)$, the first two Stiefel–Whitney classes play an important role:

Theorem 1. *For a finitely generated discrete group π, the following two statements hold:*

(a) *There is a decomposition into non-empty, disjoint closed subsets (called w-sectors):*

$$\mathrm{Hom}(\pi, O(n)) \cong \bigsqcup_{w \in H^1(\pi, \mathbb{F}_2)} \mathrm{Hom}(\pi, O(n))_w,$$

where a w-sector corresponds to those homomorphisms with first Stiefel-Whitney class equal to w. In particular if $H^1(\pi, \mathbb{F}_2)$ is non-zero, then $\mathrm{Hom}(\pi, O(n))$ *is not path-connected. The sector for $w = 0$ is precisely* $\mathrm{Hom}(\pi, SO(n))$.

(b) *For n sufficiently large,* $\mathrm{Hom}(\pi, O(n))_w$ *has at least as many components as the cardinality of the image of $H^2(\pi/[\pi, \pi], \mathbb{F}_2) \to H^2(\pi, \mathbb{F}_2)$.*

This applies very well to fundamental groups of certain complements of complex hyperplane arrangements, such as the pure braid groups:

Corollary 2. *If π is the fundamental group of the complement of an arrangement of complex hyperplanes which has contractible universal cover, and if n is sufficiently large, then*

$$\#\pi_0(\mathrm{Hom}(\pi, O(n)) \geq |H^1(\pi, \mathbb{F}_2)||H^2(\pi, \mathbb{F}_2)|$$

Under certain conditions, the space of homomorphisms *must* be path-connected. This is the case when $\pi = \mathbb{Z}^n$ and G is a Lie group with path-connected maximal abelian subgroups. For certain classes of groups this can then be used to analyze the natural map $\mathrm{Hom}(\pi, G) \to [B\pi, BG]$, where $[B\pi, BG]$ denotes the based homotopy classes of maps between the classifying spaces $B\pi$ and BG.

Theorem 3. *Let π denote a homologically toroidal group*[1]*; then, if G is a Lie group with path-connected maximal abelian subgroups, the map*

$$B_0 : \pi_0(\mathrm{Hom}(\pi, G)) \to [B\pi, BG]$$

is trivial. In particular, if $\mathcal{M}$ is the complement of an arrangement of complex hyperplanes which is aspherical, then any complex unitary representation of $\pi = \pi_1(\mathcal{M})$ induces a trivial bundle over $B\pi$.

We specialize to the cohomology of the space of ordered commuting n-tuples of elements in a compact Lie group G. The main results here mostly apply to the case when $G = SU(2)$. The approach is to consider the complement, namely the space of ordered *non-commuting* n-tuples, which seems to be much more tractable. Duality is then used to compute the cohomology of the commuting n-tuples. For example, the complement of $\mathrm{Hom}(\mathbb{Z}^2, SU(2))$ in $SU(2) \times SU(2)$ is homotopy equivalent to $SO(3)$, and the cohomology of the commuting pairs can be easily computed from this. The case of commuting triples in $SU(2)^3$ requires more intricate calculations, which yield the following results:

Theorem 4.

$$H^i(\mathrm{Hom}(\mathbb{Z} \oplus \mathbb{Z}, SU(2)), \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ 0 & \text{if } i = 1 \\ \mathbb{Z} & \text{if } i = 2 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } i = 3 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } i = 4 \\ 0 & \text{if } i \geq 5 \end{cases}$$

$$H^i(\mathrm{Hom}(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}, SU(2)), \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ 0 & \text{if } i = 1 \\ \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} & \text{if } i = 2 \\ \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} & \text{if } i = 3 \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } i = 4 \\ \mathbb{Z} & \text{if } i = 5 \\ 0 & \text{if } i = 6 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } i = 7 \\ 0 & \text{if } i \geq 8 \end{cases}$$

Natural subspaces of $\mathrm{Hom}(\mathbb{Z}^n, G)$ arise from the so-called fat wedge filtration of the product G^n where the base-point of G is 1_G. Thus if F_jG^n is the subspace of G^n with at least j coordinates equal to 1_G, define subspaces of $\mathrm{Hom}(\mathbb{Z}^n, G)$ by the formula $S_n(j, G) = \mathrm{Hom}(\mathbb{Z}^n, G) \cap F_jG^n$, and write $S_n(G) = S_n(1, G)$.

A Lie group G is said to have *cofibrantly commuting elements* if the natural inclusions $I_j : S_n(j, G) \to S_n(j-1, G)$ are cofibrations for all n and j for which both spaces are non-empty. Many Lie groups satisfy this last property.

[1]A group π is said to be homologically toroidal if there exists a finite free product H of free abelian groups of finite rank and a homomorphism $H \to \pi$ which induces a split epimorphism in integral homology.

Theorem 5. *If G is a closed subgroup of $GL(n,\mathbb{C})$, then G has cofibrantly commuting elements.*

Theorem 6. *If G is a Lie group which has cofibrantly commuting elements, then there are homotopy equivalences*

$$\Sigma(\mathrm{Hom}(\mathbb{Z}^n,G)) \simeq \bigvee_{1\le k\le n} \Sigma\big(\bigvee^{\binom{n}{k}} \mathrm{Hom}(\mathbb{Z}^k,G)/S_k(G)\big).$$

The last two theorems imply the next corollary.

Corollary 7. *If G is a closed subgroup of $GL(n,\mathbb{C})$, then there are isomorphisms of graded abelian groups in cohomology (which may not preserve products)*

$$H^*(\mathrm{Hom}(\mathbb{Z}^n,G),\mathbb{Z}) \to H^*\big(\bigvee_{1\le k\le n} \bigvee^{\binom{n}{k}} \mathrm{Hom}(\mathbb{Z}^k,G)/S_k(G),\mathbb{Z}\big).$$

Applying this to the case of ordered commuting pairs in $SU(2)$, there is a homotopy equivalence which holds after a single suspension:

$$\mathrm{Hom}(\mathbb{Z}^2,SU(2)) \simeq SU(2) \bigvee SU(2) \bigvee (\mathbb{S}^6 - SO(3))$$

where $(\mathbb{S}^6 - SO(3))$ is the Spanier-Whitehead dual of $SO(3)$ in $\mathbb{S}^6$. These methods also yield the result that after one suspension, $\mathrm{Hom}(\mathbb{Z}^3,SU(2))$ is homotopy equivalent to

$$\bigvee^3 SU(2) \bigvee [\bigvee^3 (\mathbb{S}^6 - SO(3))] \bigvee [SU(2) \wedge (\mathbb{S}^6 - SO(3)].$$

A general decomposition for $\mathrm{Hom}(\mathbb{Z}^n,SU(2))$ is described, but the summands have not yet been identified as concretely in terms of Spanier-Whitehead duals.

In the last part of the lecture we describe the construction of simplicial spaces of homomorphisms using the descending central series of free groups. From this we obtain, for any topological group G, an interesting filtration

$$B(2,G) \subset B(3,G) \subset \cdots \subset B(q,G) \subset \cdots \subset BG$$

of BG, where $B(2,G)$ is assembled from the space of commuting n-tuples. A basic question is whether or not these spaces are $K(\pi,1)$'s when G is a finite group.

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Topological Hochschild homology of ku and ko

Vigleik Angeltveit
(joint work with Mike Hill and Tyler Lawson)

Topological Hochschild homology is a generalization of Hochschild homology to the context of structured ring spectra. In analogy with Hochschild homology, it helps classifying deformations and extensions of structured ring spectra.

In addition, the work of numerous authors on the cyclotomic trace now gives machinery allowing the computation of the algebraic K-theory of connective ring spectra R. The first necessary input to these computations is the topological Hochschild homology $THH(R)$.

We study the case where $R = \ell$, the Adams summand of the connective complex K-theory spectrum localized at a prime p, or $R = ko$, the connective real K-theory spectrum localized at $p = 2$. For simplicity we will concentrate on $THH(ku)$ and $THH(ko)$ localized at 2.

If p is odd, let $V(1) = S/(p, v_1)$ be the Smith-Toda complex. Then $V(1) \wedge \ell \simeq H\mathbb{F}_p$, and a calculation essentially due to McClure and Staffeldt [1] shows that

$$V(1)_* THH(\ell) \cong E(\lambda_1, \lambda_2) \otimes P(\mu),$$

an exterior algebra on two generators tensored with a polynomial algebra on one generator. Here $|\lambda_1| = 2p - 1$, $|\lambda_2| = 2p^2 - 1$ and $|\mu| = 2p^2$. This is the starting point of a calculation by Ausoni and Rognes [2] of $V(1)_* K(\ell_p)$ for $p \geq 5$. We can use THH with coefficients to circumvent (for the THH-calculations, not for the algebraic K-theory calculations) the nonexistence of $V(1)$ at $p = 2$ (and the lack of a ring structure on $V(1)$ at $p = 3$) by studying $THH(ku; H\mathbb{F}_2)$ instead. Regarding ku, $H\mathbb{Z}$, $ku/2 = k(1)$ and $H\mathbb{F}_2$ all as ku-bimodules, we get the following commutative diagram:

$$\begin{array}{ccccc} & & THH(ku) & & \\ & \swarrow & & \searrow & \\ THH(ku;H\mathbb{Z}_{(2)}) & & & & THH(ku;k(1)) \\ & \searrow & & \swarrow & \\ & & THH(ku;H\mathbb{F}_2) & & \end{array}$$

Each of the arrows in the above diagram gives a Bockstein spectral sequence going the other way. Thus we have the following spectral sequences:

$$THH_*(ku;H\mathbb{F}_2)[v_1] \implies THH_*\big(ku;k(1)\big); \tag{1}$$

$$THH_*(ku;H\mathbb{F}_2)[v_0] \implies THH_*(ku;H\mathbb{Z}_{(2)})^{\wedge}_2; \tag{2}$$

$$THH_*\big(ku;k(1)\big)[v_0] \implies THH_*(ku)^{\wedge}_p; \tag{3}$$

$$THH_*(ku;H\mathbb{Z}_{(2)})[v_1] \implies THH_*(ku). \tag{4}$$

We can understand the first two spectral sequences, and this gives us two spectral sequences which we can play against each other to understand $THH_*(ku)$. The calculation of Spectral Sequence (1) is due to McClure and Staffeldt [1] at odd primes, the argument for $p=2$ is formally identical. The idea is to use the $K(1)$-based Bökstedt spectral sequence to show that only the v_1-tower starting at the origin can survive, and then there is only one possible pattern of differentials. The behaviour of Spectral Sequence (2) follows from the differential $d_1(\mu)=v_0\lambda_2$ and an application of the "Leibniz rule" $d_{r+1}(x^p)=v_0x^{p-1}d_r(x)$.

We then have two spectral sequences converging to $THH_*(ku)$, and we can play them against each other. Gaps in Spectral Sequence (3) force certain differentials in Spectral Sequence (4), and from Spectral Sequence (3) we can also see that certain groups are cyclic, forcing certain hidden extensions in Spectral Sequence (4).

This gives some of the differentials in Spectral Sequence (4). To get the rest, we use topological Hochschild cohomology. There is a pairing

$$THH^m(ku)\otimes THH_n(ku)\to THH_{n-m}(ku),$$

and by finding elements in $THH^*(ku)$ that pair nicely with $THH_*(ku)$ we are able to "transport" the differentials we already found in Spectral Sequence (4) to get all of the differentials.

We end up with the following results:

Theorem 1. *The torsion free summand of $THH_*(ku)$ is $ku_*\oplus\Sigma^3F$, where F is the ku_*-module*

$$F=ku_*\left[\frac{v_1^{2^k+\cdots+2}}{2^k}\right]\subset ku_*\otimes\mathbb{Q}.$$

Theorem 2. *The torsion summand of the homotopy of $THH(ku)$, as a ku_*-module, is isomorphic to*

$$\bigoplus_{n\geq 0} \Sigma^{2^{n+3}+2} T_n,$$

where T_n is recursively built out of 2 copies of T_{n-1} and a v_1-tower of length $2^{n+2}-2$.

To facilitate understanding of the modules T_n, we have included a picture of the torsion starting in degrees 18 and 34 as Figure 1. These correspond to T_1 and T_2. In the picture, vertical and curved lines indicate multiplication by 2, while the sloped lines indicate multiplication by v_1.

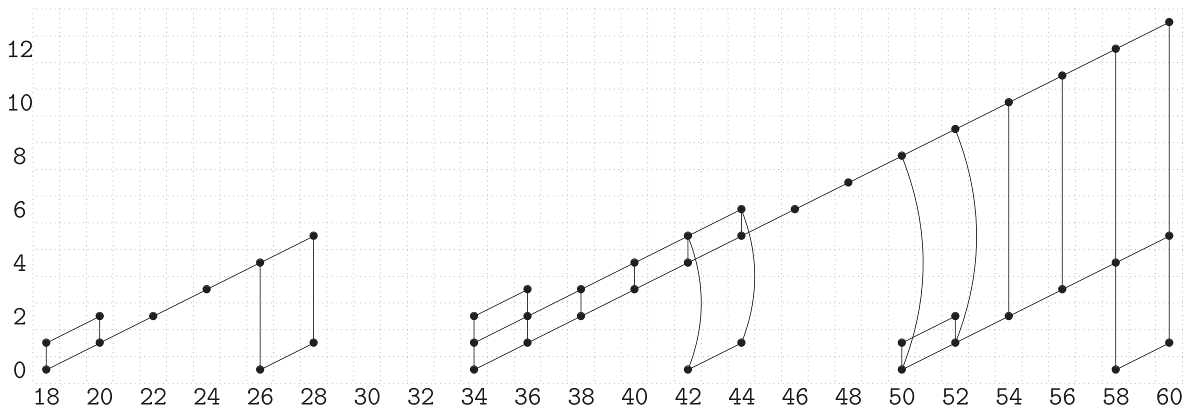

Figure 1. The torsion in degrees 18 through 60 for $p = 2$

The calculation of $THH_*(ko)$ is similar. We start with $THH_*(ko; H\mathbb{F}_2)$ and use the same combination of Bockstein spectral sequences to arrive at $THH_*(ko; ku)$. Then there is one last spectral sequence

$$THH_*(ko; ku)[\eta] \Longrightarrow THH_*(ko).$$

This spectral sequence has to collapse at the E_4 term at the latest, because $\eta^3 = 0$ in ko. The surprise is that on the complement of ko in $THH(ko)$, $\eta^2 = 0$.

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On refined Seiberg-Witten invariants

Stefan Bauer

Any closed and oriented differentiable 4-manifold X can be equipped with a spin^c-structure. Fixing a Riemannian metric, there is a refinement [3], [2] to the integer valued Seiberg-Witten invariant. This refined invariant is represented by the homotopy class of a stable, $U(1)$-equivariant map

$$[\mu] \in \{Th(\operatorname{ind} D), S^{H_+}\}^{U(1)}$$

from a Thom spectrum associated to the Dirac operator to the 1-point completion of the self-dual part $H_+ = H^2_+(X;\mathbb{R})$ of the second cohomology group of the given manifold X.

1. A naive definition

The group $Spin^c(4) \subset U(2)_+ \times U(2)_-$ consists of pairs (g_+, g_-) of unitary 2×2-matrices having the same determinant. Viewing the quaternions $\mathbb{H}$ as real multiples of special unitary 2×2-matrices, this complex spin group acts on the quaternions via

$$h \mapsto g_+ \, h \, g_-^*.$$

Now let P denote the $Spin^c(4)$-lift of the orthonormal frame bundle of X associated to the spin^c-structure. The spinor bundles

$$S^{\pm} \times_{pr_\pm} \mathbb{H}$$

are the complex rank-2 bundles associated to the representations obtained from the projections $pr_\pm : Spin^c(4) \to U(2)_\pm$. The bundle $\Lambda^2 T^* X$ of 2-forms decomposes as a sum $\Lambda^2_+ \oplus \Lambda^2_-$ of rank-3 real bundles. The bundle Λ^2_+ is obtained from P via the representation

$$Spin^c(4) \to SO(\operatorname{Im}\mathbb{H}), \quad (g_+, g_-) \mapsto (q \mapsto g_+ \, q \, g_+^*).$$

The Hopf map $\eta : \mathbb{H} \to \operatorname{Im}\mathbb{H}, \quad h \mapsto h \, i \, h^*$ globalizes to a fiber preserving map

$$\sigma = \mathrm{id}_P \times_{Spin^c(4)} \eta : S^+ \to \Lambda^2_+.$$

Let's assume for simplicity the vanishing of the first Betti number of X. Then the "naive" monopole map

$$\mu_n : \ker D_A \to H^2_+(X;\mathbb{R}), \quad \psi \mapsto \mathrm{pr}_{harm}(\sigma(\psi))$$

is defined using the Dirac operator D_A associated to a spin^c-connection A and maps harmonic spinors to harmonic self-dual 2-forms.

In the case of a K3-surface with its *spin*-structure, the bundles S^+ and Λ^2_+ are trivial. The kernel $\ker D$ of the Dirac operator and the space H_+ of self-dual harmonic 2-forms naturally identify with $\mathbb{H}$ and $\mathrm{I}m\,\mathbb{H}$ as the elements are constant sections. Thus, the naive monopole map identifies with the Hopf map. The refined invariant

$$[\mu] \in \{S^{\mathbb{H}}, S^{\operatorname{Im}\mathbb{H}}\}^{U(1)}$$

in this case is just the one-point completion of the naive monopole map.

The naive monopole map depends on the choice of a spin^c-connection. Combining all possible choices into one map results in the monopole map [3], which is a map between Fréchet manifolds and which admits a Fréchet group of symmetries, called gauge group. After suitable Sobolev completion, a theorem of A. Schwarz on compact perturbations of linear Fredholm maps can be applied to construct the refined invariant as an element in a suitable stable cohomotopy group.

2. Applications

Using the refined invariant, it is straightforward to deduce the following theorems on 4-manifolds:

Theorem 1. (Donaldson [5]**)** *Let X be an oriented 4-manifold with negative definite intersection form. Then the intersection form is diagonal.*

The number theoretic input to the proof is due to Elkies: A non-diagonal intersection form admits a characteristic element with square greater than the signature. The index of the Dirac operator can be computed from these data via the Atiyah-Singer index theorem. From the refined invariant, one gets a stable $U(1)$-equivariant map between spheres, which cannot exist by topological reasons: The argument uses the $K_{U(1)}$-theoretic mapping degree and in particular the fact that it is an element in the representation ring of $U(1)$ (and not of any localization thereof).

Theorem 2. (Furuta [6]**)** *Let X be a spin-manifold with negative signature. Then the second Betti number is bounded below by*

$$b_2(X) \geq 2 - \frac{10}{8} sign\, X.$$

A spin-structure allows for additional $Pin(2)$-symmetry of the monopole map instead of the $U(1)$-symmetry in the general case. The remaining argument uses the same kind of reasoning as the proof of Donaldson's theorem.

The ongoing quest for a Kodaira-classification of symplectic 4-manifolds is the background for the next theorem. It deals with the case of Kodaira-dimension 0 and implies that, at least from a cohomological level, all such symplectic manifolds are known.

Theorem 3. [1] *A closed, symplectic 4-manifold X with torsion first Chern class satisfies the inequality*

$$b_2^+(X) \leq 3.$$

For a proof, it suffices to consider the case of spin manifolds. A decisive input is due to Taubes [7]. According to his theorem, the absolute value of the integer valued Seiberg-Witten invariant of a symplectic 4-manifold is 1, as soon as $b_2^+(X) \geq 2$. This invariant appears as the degree of the refined invariant in Borel cohomology. The Atiyah-Singer index theorem specifies the spectra serving as source and target of the refined invariant. Using $Pin(2)$-equivariant obstruction theory, one shows that the relevant degree in Borel cohomology has to be even unless $b_2^+(X) \leq 3$.

3. Towards a QFT with values in spectra

The refined invariant, unlike the integer valued Seiberg-Witten invariant or Donaldson invariants, does not vanish in general for connected sums.

Theorem 4. [4] *The refined invariant of a connected sum is the smash product of the refined invariants of the summands.*

Ideally, one would like to view this theorem a special case of a more general setup: Given a compact 4-manifold X_i with boundary $Y = (-Y_{i-1}) \cup Y_i$ a disjoint union of an ,,in-going" and an ,,out-going" manifold, one would like to define a monopole map

$$\mu_{X_i} : M(X_i) \to N(Y_{i-1}, Y_i)$$

between equivariant spectra, together with gluing maps

$$V : M(X_i) \wedge M(X_{i+1}) \to M(X_i \cup_{Y_i} X_{i+1})$$

and composition maps

$$c : N(Y_{i-1}, Y_i) \wedge N(Y_i, Y_{i+1}) \to N(Y_{i-1}, Y_{i+1})$$

such that for $i = 0, 1$ the monopole maps are related by

$$\mu_{X_0 \cup_{Y_0} X_1} \circ V = c \circ \mu_{X_0} \wedge \mu_{X_1}$$

and such that the monopole maps recapture the refined invariants in case of empty boundaries.

Indeed, this scheme can be made to work in case the bounding manifolds have positive curvature, in this way generalizing the theorem above.

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Completion at A_∞-monads

Tilman Bauer
(joint work with A. Libman)

Let K be a commutative ring. Bousfield and Kan [1] construct a K-completion functor $\hat{K}\colon s\mathbf{Sets} \to s\mathbf{Sets}$ on simplicial sets with the following properties:

(1) $\hat{K}(X)$ is local with respect to homology with coefficients in K;
(2) a map $X \to Y$ is a K-homology equivalence if and only if the induced map $\hat{K}(X) \to \hat{K}(Y)$ is a homotopy equivalence.

The functor $\hat{K}$ shares these properties with the K-homology localization functor L_K; however, the canonical map $L_K(X) \to \hat{K}(X)$ is not always an equivalence. If it is an equivalence, the space X is called K-*good*. This is equivalent to $\hat{K}$ being idempotent, and to $X \to \hat{K}(X)$ being a K-homology isomorphism.

The completion $\hat{K}(X)$ is constructed by considering the monad $X \mapsto K(X)$ associating to a simplicial set the simplicial set underlying the free simplicial K-module on X. Iterating this monad gives a cosimplicial simplicial set $R_K(X)$ whose total space is defined to be $\hat{K}(X)$. In [1, I.5.6], it was claimed that the functor $\hat{K}$ was again a monad, but this claim was not proved and later retracted by the authors. It is a monad up to homotopy, as was proved in [5, 2]. However, if $\hat{K}$ was a monad on the space level, it would allow one to repeat the completion construction, replacing the monad K by $\hat{K}$, yielding a new functor $\hat{\hat{K}} \to \hat{K} \to K$. This new functor would also satisfy conditions (1) and (2), and it might be a better approximation to L_K than $\hat{K}$ was. Eventually, it could be hoped that there is a transfinite sequence of such functors, and if this sequence becomes constant beyond a certain stage, its homotopy limit would in fact be L_K, giving us an inverse limit construction of K-localization.

To construct the total space of a cosimplicial space, one can in fact restrict one's attention to the underlying diagram of cofaces; thus such a cofacial resolution can be obtained from any augmented functor $\mathrm{Id} \to K$, i.e., one does not really need the multiplication $K \circ K \to K$. It is obvious that if K is an augmented functor, then so is $\hat{K}$. Thus we can in fact construct a long localization tower. In the case of $K = \mathbb{F}_p$, this was studied in [4], where it was shown that for any space X, the tower converges to the localization. Whether this is also true for other monads is unknown, in particular in the important case of localization at a generalized homology theory K. In this setting, $X \mapsto \Omega^\infty(K \wedge \Sigma^\infty X)$ is a monad on topological spaces whose associated completion is the Bendersky-Thompson unstable K-completion [3].

However, the degeneracies can be valuable. For instance, they give rise to a ring structure on the Bousfield-Kan spectral sequence associated to the cosimplicial space $R_K(X)$ (along with higher multiplicative structure), which is usually indispensable in computations.

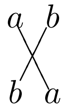
Theorem 1. *Let A be a (non-symmetric) A_∞-operad on* **Top***, and let K be a continuous A-monad on* **Top**. *Then the completion $\hat{K}$ is a continuous endofunctor on spaces which admits an $\hat{A}$-monad structure, where $\hat{A}$ is an A_∞-operad. Furthermore, there is an operad map $\hat{A} \to A$ such that the canonical map $\hat{K} \to K$ becomes a map of A_∞-monads.*

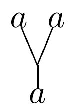
The statement involves the notion of an A_∞-monad, which is a monad up to coherent homotopies. In particular, if we choose $A = a$ to be the associative operad, the theorem tells us that $\hat{K}$ is an A_∞-monad for a suitable A_∞-operad. There is an A_∞-cobar construction for A_∞-monads which implies the existence of multiplicative structures in the Bousfield-Kan spectral sequence.

In the setting of simplicial sets, it seems unnatural to pass to the total space of the cosimplicial simplicial set $R_K(X)$, since the cosimplicial objects in any category carry a natural simplicial enrichment, the *external simplicial structure*. The functor R_K can be thought of as an endofunctor of cosimplicial objects by applying R_K degreewise (resulting in a bicosimplicial object) and taking the diagonal. Bousfield and Kan (using their famous "twist map") construct a natural transformation $R_K \circ R_K \to R_K$ in the case where K is the free K-module monad associated to a ring K, but it is not associative on the nose. We offer the following improvement on their construction:

Theorem 2. *There exists an explicit, combinatorially defined A_∞-operad A on simplicial sets such that for any ring K, $R_K \colon s\,\mathrm{Set} \to s\,\mathrm{Set}$ is an A-monad.*

Informally, k-simplices in the nth space of the operad A are braid-like diagrams with $n(k+1)$ incoming strands and $k+1$ outgoing strands, where two strands can

cross: 

or join: 

Furthermore, all strands are colored with a label in $\{0, \dots, k\}$ such that the labels of the incoming strands are $(0, 1, \dots, k, 0, 1, \dots, k, \dots, 0, 1, \dots, k)$ and the labels of the outgoing strands are $(0, 1, \dots, k)$, in this order from left to right. Only strands of the same color can join, and if two strands cross as in the first diagram, then $a > b$ must hold. Note that the crossing has no orientation, that is, none of the strands is considered passing above or below the other. Figure 1 shows an example of such a braid.

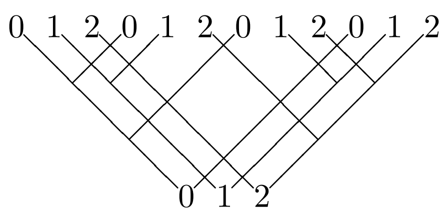
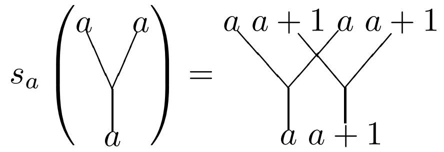


FIGURE 1. An example of an element of $(A_4)_2$

The sequence of $(A_n)_k$ forms a simplicial set, where the ith face map d_i is obtained by erasing the strands of color i, whereas the ith degeneracy map s_i duplicates the ith strand. It is obvious how this works on crossings; on joins, we define the resulting diagram to be



thus introducing one new crossing.

Theorem 2 leaves something to be desired: firstly, we do not know how to feed it a more general monad than the free K-module monad; secondly, it produces a monad for which it seems hopeless that it can be used as an input monad again, so that there is no real hope the completion process could be iterated in this setting.

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Iterated THH, homotopy fixed sets, and the chromatic tower

Gunnar Carlsson

(joint work with Morten Brun, Christopher Douglas, and Bjørn Dundas)

The chromatic point of view toward stable homotopy theory has been very productive over the last 20 years. It provides an organizing principle, and furthermore demonstrates that computations of stable homotopy groups can be usefully broken up into pieces which may be described in terms of the cohomology of certain algebraic groups over the p-adics. This philosophy is being carried out to make very explicit computations for the $K(2)$-local sphere by Goerss, Henn, and Mahowald (see [3] or [4]). It is clear from the complicated nature of the constructions involved that it would be very useful to have simpler and/or more geometric constructions of the cohomology theories involved (i.e. Morava K-theories, TMF) or the local spheres themselves. This is some of the motivation for the work of J. Lurie on "derived algebraic geometry" [6]. On the other hand, it has also been observed that if one views algebraic K-theory as a functor from the category of commutative S-algebras to itself, it is capable of increasing chromatic filtration. For example, it produces a p-local integral Eilenberg MacLane spectrum when applied to the field $\mathbb{F}_p$, and when applied to the p-complete integral Eilenberg-MacLane spectrum it produces the algebraic K-theory of the corresponding ring. Further, work of Ausoni and Rognes [1] shows that when applied to the ku-spectrum it produces a result which contains level 2 chromatic phenomena. This has motivated Rognes and others to make the "red shift" hypothesis, which asserts that this phenomenon is generic. On the other hand, we know from work of McCarthy [7] that relative K-theory of certain homomorphisms of commutative S-algebras is equivalent to certain topological cyclic homology spectra, written TC. The goal of this paper is to begin the exploration of the iteration of the functor TC, in the hope that one will see from it more explicit and geometric constructions of phenomena of higher chromatic filtration.

The construction of TC is obtained from the topological homology construction THH via a homotopy inverse limit construction involving certain symmetries and natural transformations of the THH-construction. These include an action by the circle group, as well as the power maps from the circle to itself. Iterating TC can therefore be obtained by studying the iterate of THH instead, and applying the symmetries and natural transformations in each factor. However, when one studies the iterate of THH, one finds that it admits more symmetries and natural transformations than just the individual symmetries arising in each individual factor. For example, in addition to the action of the torus group $\mathbb{T}^n$ on the n-fold iterate, one has the action of permutations on the factors, and actually any automorphism of $\mathbb{Z}_p^n$ will act. Further, any isogeny of $\mathbb{T}^n$ to itself will yield a natural transformation of the construction to itself. The rough observation is now that one obtains symmetries of the iterated construction which are related to matrix groups over the p-adic integers and numbers, and therefore the hope that in the future, one might obtain an understanding of the rings of operations of Morava K-theories via these constructions. In the present talk, I have outlined the construction of the required spectrum valued functors on the category of commutative S-algebras and some computations. The results are as follows.

- Attached to any commutative S-algebra A, there exists a functor from the category of finite sets to spectra which carries a finite set X to the smash product $\bigwedge_X A$, in such a way that when regarded as a spectrum with action by Σ_X, the symmetric group on X, it becomes an equivariant Σ_X-spectrum. Moreover, fixed point structure is describable in simple terms.
- We construct versions of the TF and TR constructions used in computations by Hesselholt and Madsen [5].
- We evaluate our version of TF explicitly for the sphere spectrum, obtaining a full description of the full mapping spectrum of $B\mathbb{T}^n$ into the p-adic sphere spectrum.
- We evaluate π_0 of our version of TR in terms of the equivariant Burnside ring of profinite groups defined by Dress and Siebeneicher in [2].

I also outlined various ideas about how to construct homotopy inverse limit constructions based on rings of matrices, which we expect to be useful in the study of this version of iterated TC.

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Algebraic models for finite G-spaces

Jesper Grodal

(joint work with Jeffrey H. Smith)

In my talk, based on [2], I explained how to assign certain finite algebraic models to G-spaces with finite mod p homology, for G a finite group. In general these models are too naïve to capture the mod p equivariant homotopy type, where by equivariant homotopy type we mean that two G-spaces are considered equivalent (called hG-equivalent) if they can be connected by a zig-zag of G-maps which are non-equivariant homotopy equivalences. We show, however, that the model does in fact capture the mod p equivariant homotopy type in the fundamental case when X has the mod p homology of a sphere – integral results can easily be obtained from the p-local results via the arithmetic square. Even when the model does not capture the homotopy type it still encodes important information about the group action, including all of classical Smith theory. We also show that for spheres the algebraic models are themselves determined by simple numerical information (Theorem 2), and that all models are realizable (Theorem 3). Our work also seems to point to that the models have independent algebraic interest. The models are given via the following theorem.

Theorem 1. *Let G be a finite group and consider the functor Φ : G-spaces $\to$ $\mathrm{Ch}(\mathrm{Vect}^{\mathbf{O}_p(G)^{\mathrm{op}}})$ which to a G-space X associates the functor on the opposite p-orbit category $\mathbf{O}_p(G)^{\mathrm{op}}$ given by*

$$G/P \mapsto \lim_n C_*(\mathrm{map}_G(EG \times G/P, P_n(\mathbb{F}_p)_n X); \mathbb{F}_p).$$

Then Φ sends a G-space with finite $\mathbb{F}_p$-homology to a perfect complex in $\mathrm{Vect}^{\mathbf{O}_p(G)^{\mathrm{op}}}$.

A perfect complex is a complex quasi-isomorphic to a finite chain complex of finitely generated projectives. Here $\mathrm{Ch}(\mathrm{Vect}^{\mathbf{O}_p(G)^{\mathrm{op}}})$ denotes chain complexes of functors from the opposite p-orbit category $\mathbf{O}_p(G)^{\mathrm{op}}$ to $\mathbb{F}_p$-vector spaces, C_* denotes singular chains reduced by setting $C_{-1} = \mathbb{F}_p$, map_G is the G-equivariant mapping space, P_n denotes the nth Postnikov truncation, and $(\mathbb{F}_p)_n X$ denotes the nth stage in the Bousfield-Kan tower of X converging to the Bousfield-Kan $\mathbb{F}_p$-completion of X. Under mild restrictions on X, the formula for $\Phi(X)(G/P)$ can be simplified to $C_*((X^\wedge_p)^{hP};\mathbb{F}_p)$, and if X is a genuine finite complex even to $C_*(X^P;\mathbb{F}_p)$, using the generalized Sullivan Conjecture (now a theorem by Miller and Lannes). The technical description in general is forced upon us by properties of Lannes' T-functor.

Note that, up to quasi-isomorphism, $\Phi(X)$ only depends on X up to hG-equivalence. Also note that by the existence of minimal resolutions, every perfect complex has a minimal model, a finite chain complexes of projectives, well-defined up to *isomorphism*. To get a feeling for Theorem 1, one may note that in the case where X is a point, the result implies that the constant functor in $\mathrm{Vect}^{\mathbf{O}_p(G)^{\mathrm{op}}}$ has finite projective dimension; a celebrated result first proved by Jackowski-McClure-Oliver [3].

The abelian category $\mathrm{Vect}^{\mathbf{O}_p(G)^{\mathrm{op}}}$ can be viewed as modules over the category algebra $\mathbb{F}_p\mathbf{O}_p(G)^{\mathrm{op}}$ of $\mathbf{O}_p(G)^{\mathrm{op}}$, i.e., the algebra with $\mathbb{F}_p$-basis consisting of the morphisms in $\mathbf{O}_p(G)^{\mathrm{op}}$ and multiplication given by composition. This is a finite dimensional algebra whose simple modules are described by the simple modules for $N_G(P)/P$, where P runs over the conjugacy classes of p-subgroups of G, and the projectives are given by left Kan extension. For instance for $G = C_p$ the orbit category looks like $G/G \to G/e$ so the simples are $k \to 0$ and $0 \to k$ with corresponding projectives $k \to k$ and $0 \to kC_p$, where we from now on let $k = \mathbb{F}_p$ for short. Examining the perfect complexes which can be built from them easily leads to the statements of classical Smith theory. For $G = \Sigma_3$ and $k = \mathbb{F}_2$ we get simples $k \Rrightarrow 0$, $0 \Rrightarrow k$, and $k \Rrightarrow \mathrm{St}$ (where $\Rrightarrow$ denotes 3 arrows, and St denotes the 2-dimensional simple module for Σ_3) with corresponding projectives $k \Rrightarrow k[G/C_2]$, $0 \Rrightarrow k[G/C_3]$, and $0 \Rrightarrow \mathrm{St}$. Notice that here the constant functor is not projective, but has a projective resolution with $k \Rrightarrow k[G/C_2]$ in degree zero and $0 \Rrightarrow \mathrm{St}$ in degree one.

The proof of Theorem 1 requires first developing a criterion for recognizing perfect complexes in terms of certain $kN_G(P)/P$-modules, and then using Lannes theory together with the theory of support varieties to verify that these conditions are met.

We say that a perfect complex $\mathbb{X}$ of $k\mathbf{O}_p(G)^{\mathrm{op}}$-modules is a $k\mathbf{O}_p(G)^{\mathrm{op}}$-sphere if $H(\mathbb{X}(G/e))$ is one-dimensional. An oriented dimension function consists of a functor $n : \mathbf{O}_p(G)^{\mathrm{op}} \to (\mathbb{Z}, \leq)$, together with an action of G on k. Every $k\mathbf{O}_p(G)^{\mathrm{op}}$-sphere has an associated oriented dimension function given by letting $n(G/P)$ be the (by Smith theory) unique n such that $H_n(\mathbb{X}(G/P)) \neq 0$, and taking as orientation the action of G on $H(\mathbb{X}(G/e))$. The next theorem gives a classification of $k\mathbf{O}_p(G)^{\mathrm{op}}$-spheres.

Theorem 2. *Let $\mathbb{X}$ and $\mathbb{Y}$ be $k\mathbf{O}_p(G)^{\mathrm{op}}$-spheres. Then $\mathbb{X}$ and $\mathbb{Y}$ are quasi-isomorphic if and only if $\mathbb{X}$ and $\mathbb{Y}$ have the same oriented dimension functions.*

Furthermore $[\mathbb{X},\mathbb{X}] \xrightarrow{\cong} k$ via $\phi \mapsto H(\phi)$, where brackets denote maps in the derived category.

The proof involves setting up the right obstruction theory, and then showing that the existence and uniqueness obstructions to extending a map vanish, once we have chosen the map on $\mathbb{X}(G/S)$ for a Sylow p-subgroup S. The vanishing result takes as one input extending properties of Steinberg complexes in [1] to the perfect complexes we are considering.

We conjecture that a given oriented dimension function is realizable by a $k\mathbf{O}_p(G)^{\mathrm{op}}$-sphere if and only if it satisfies the so-called Borel-Smith conditions (where only-if is easy). We have proved this for certain classes of groups including many groups of small order, and groups with normal Sylow p-subgroup (in particular p-groups).

The last theorem which we list here states that there is a 1-1-correspondence between p-complete G-spheres, up to homotopy, and $k\mathbf{O}_p(G)^{\mathrm{op}}$-spheres subject to obvious low-dimensional restrictions.

Theorem 3. *Let X and Y be G-spaces which homotopy equivalent to $\mathbb{F}_p$-complete spheres. Then X and Y are hG-equivalent if and only if $\Phi(X)$ and $\Phi(Y)$ are quasi-isomorphic.*

Furthermore, we have a 1-1-correspondence between $\mathbb{F}_p$-complete G-spheres up to hG-equivalence and $k\mathbf{O}_p(G)^{\mathrm{op}}$-spheres where $H(\mathbb{X}(G/S))$ has degree no less than -1 and the NS/S-action on $H(\mathbb{X}(G/S))$ is trivial if in degree -1 and factors through ± 1 if in degree 0, where S is a Sylow p-subgroup.

In addition to the results and methods of Theorem 1 and 2, the proof uses that the G-action on $\pi_*(X)$ has a filtration where the quotients are kG-modules of dimension less than p, which allows for passing between X and its abelianization $\mathbb{F}_pX$ via obstruction theory and tower arguments.

Note that Theorem 2 and 3 in particular shows that maps $[BG, B\operatorname{Aut}(S^n{}^\wedge_p)]$ are determined by the restriction to the normalizer of a Sylow p-subgroup in G. This shows that homotopical group actions on spheres are better behaved than homotopical representations $[BG, BU(n)^\wedge_p]$, where there are many exotic maps due to non-vanishing of certain higher limit obstructions, as first discovered by Jackowski-McClure-Oliver [3].

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Open-Closed Cobordism Categories

Elizabeth Hanbury

Perhaps the most fundamental cobordism category is the Riemann surface category, introduced by Segal and used by him to formalise field theories [4]. Its objects are the natural numbers - we think of $n \in \mathbb{N}$ as a disjoint union of n

oriented circles. A morphism from p to q is a Riemann surface with p incoming and q outgoing boundary components. Disjoint union gives rise to a symmetric monoidal structure on the category and Segal defined a conformal field theory to be a symmetric monoidal functor from the Riemann surface category to the category of vector spaces.

The analytic nature of the Riemann surface category makes it difficult to study and recently more topological models have been studied. Amongst these was Tillmann's 2-category model $\mathscr{S}$ in [2]. The objects in $\mathscr{S}$ are the natural numbers, the morphisms are oriented smooth surfaces, considered as cobordisms between circles, and the 2-morphisms are orientation-preserving diffeomorphisms between surfaces. For technical reasons, Tillmann also studied the subcategory $\mathscr{S}^b$ in which every component of a cobordism must have non-empty outgoing boundary. She proved that the classifying space $B\mathscr{S}^b$ has the structure of an infinite loop space and furthermore that $\Omega B\mathscr{S}^b$ is homotopy equivalent to $\mathbb{Z} \times B\Gamma_\infty^+$ where Γ_∞ is the stable mapping class group and $()^+$ denotes Quillen's plus construction. Thus she showed that $B\Gamma_\infty^+$ is an infinite loop space.

Another model which has been studied is the embedded surface category $\mathcal{C}_2$ of [3]. Its objects are closed 1-manifolds embedded in $\mathbb{R}^\infty$ and its morphisms are smooth cobordisms embedded in $\mathbb{R}^\infty \times [0, t]$ for some $t > 0$. In [3] the authors proved the following.

Theorem 1. *There is a homotopy equivalence* $\Omega B\mathcal{C}_2^b \simeq \Omega^\infty(\mathbb{C}P_{-1}^\infty)$.

Here $\mathbb{C}P_{-1}^\infty$ is the Thom spectrum of the complement of the canonical line bundle over $\mathbb{C}P^\infty$. Together with Tillmann's result this reproves Madsen-Weiss' generalisation of the Mumford conjecture i.e. we get that $\mathbb{Z} \times \Gamma_\infty^+$ is homotopy equivalent to $\Omega^\infty(\mathbb{C}P_{-1}^\infty)$.

There is an analogue of the Riemann surface category in which the objects are oriented 1-manifolds which may have boundary. The cobordisms between such objects are called open-closed cobordisms because of the links with open-closed string theory. If S_0, S_1 are oriented 1-dimensional manifolds, possibly with boundary, an open-closed cobordism between them is an oriented surface F with a decomposition $\partial F = \partial_{inF} \cup \partial_{out} F \cup \partial_f F$ such that

(i) There are identifications $\partial_{in} F = S_0$ and $\partial_{out} F = -S_1$, where $-S_1$ denotes the surface S_1 equipped with the opposite orientation.

(ii) $\partial_f F$, called the free boundary, is a cobordism from ∂S_0 to ∂S_1.

In [1] the authors studied a 2-category model $\mathscr{S}^{oc}$ of the open-closed cobordism category, analogous to Tillmann's 2-category $\mathscr{S}$. They calculated the homotopy type of $B\mathscr{S}^{oc}$ with the following theorem.

Theorem 2. *There is a homotopy equivalence* $\Omega B\mathscr{S}^{oc}_{\mathscr{B}} \simeq \Omega^\infty(\mathbb{C}P_{-1}^\infty) \times Q(\mathbb{C}P_+^\infty)$.

Here Y_+ denotes the space Y with a disjoint basepoint added and $Q(Y) = \operatorname{colim} \Omega^k \Sigma^k Y$.

This talk was about a variant of the open-closed cobordism category - a background space version denoted $\mathcal{C}_{X,N}$. Here X is a fixed space and N is a subspace of X. The objects in $\mathcal{C}_{X,N}$ are oriented 1-manifolds S embedded in $\mathbb{R}^\infty$

and equipped with a map $f : (S, \partial S) \to (X, N)$. The morphisms are oriented open-closed cobordisms F, embedded in $\mathbb{R}^\infty \times [0, t]$ for some $t > 0$, with a map $\phi : (F, \partial_f F) \to (X, N)$. The cobordisms should be embedded so that $\partial_{in} F \subset \mathbb{R}^\infty \times \{0\}$ and $\partial_{out} F \subset \mathbb{R}^\infty \times \{1\}$. For our techniques to work we are forced to make some technical modifications to the definition concerning the structure of the free boundary. For the same reasons we work with the subcategory $\mathcal{C}^b_{X,N}$ in which each component of a cobordism has non-empty outgoing boundary.

The category $\mathcal{C}_{X,N}$ is intended to be a model of open and closed strings moving in a background space X with the components of N as the set of D-branes. The main theorem presented was a calculation of its classifying space for particular cases of X, N.

Theorem 3. *Let X be simply-connected and $N \subseteq X$ discrete. Then there is a weak homotopy equivalence*

$$\Omega B\mathcal{C}^b_{X,N} \simeq \Omega^\infty(\mathbb{C}P^\infty_{-1} \wedge X_+) \times Q((\mathbb{C}P^\infty \times N)_+).$$

This is proved using the generalised group completion theorem, homological stability of mapping class groups and Pontryagin-Thom collapse.

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The rationalization of the homotopy of the $K(2)$-localization of the sphere at the prime 3

Hans-Werner Henn

(joint work with Paul Goerss, Nasko Karamanov, Mark Mahowald, and Charles Rezk)

1. Introduction

Let $K(n)$ be the n-th Morava K-theory at the prime p. The $K(n)$-local sphere can be studied via the Adams-Novikov spectral sequence which in this case can be identified with a descent spectral sequence

$$E_2^{s,t} \cong H^s(\mathbb{G}_n, (E_n)_t) \Longrightarrow \pi_{t-s}(L_{K(n)}S^0).$$

Here $\mathbb{G}_n$ denotes the automorphism group of the Honda formal group law F_n over $\mathbb{F}_{p^n}$ and E_n is the Landweber exact period ring spectrum with $(E_n)_0$ classifying deformations of F_n in the sense of Lubin and Tate.

Let us now focus on the case $p = 3$ and $n = 2$. In earlier work with Goerss, Mahowald and Rezk we constructed a resolution of the $K(2)$-local sphere at the prime 3 in terms of well understood homotopy fixed point spectra which are closely related to the spectrum TMF of topological modular forms. This filtration together with the calculation for the mod-3 Moore spectrum can be used to calculate the homotopy of the $K(2)$-local sphere after rationalization.

Theorem 1. *There are elements* $\zeta \in \pi_{-1}(L_{K(2)}S^0)$ *and* $e \in \pi_{-3}(L_{K(2)}S^0)$ *and there is an isomorphism of algebras*

$$\pi_*(L_{K(2)}S^0) \otimes \mathbb{Q} \cong \Lambda(\zeta, e) \otimes \mathbb{Q}.$$

The result is a byproduct of joint work with Mahowald and Karamanov, as well as of joint work with Goerss, Mahowald and Karamanov, in which we use the resolution mentioned above to calculate the homotopy of the $K(2)$-localization of the mod-3 Moore spectrum (with Mahowald and Karamanov) and finally (with Goerss, Mahowald and Karamanov - work in progress) to calculate the homotopy of the $K(2)$-local sphere. The resolution allows to break up Shimomura's earlier calculation of this homotopy into digestible pieces.

While our results do agree with Shimomura's earlier results in the case of the Moore spectrum, they disagree with the calculation by Shimomura and Wang for $L_{K(2)}S^0$ ([4]), in particular for the rational calculation Shimomura and Wang get an exterior algebra on ζ only. Our result is in agreement with the result predicted by the strong form of the chromatic splitting conjecture while theirs is not.

In the sequel we give an outline of the proof of the Theorem.

2. Background

We begin by recalling the Morava stabilizer group $\mathbb{G}_2$. Let

$$\mathcal{O}_2 = \mathbb{W}_{\mathbb{F}_9} < S > /(S^2 = 3, wS = Sw^\sigma).$$

Then $\mathbb{S}_2 := \mathcal{O}_2^\times$ is isomorphic to the group of automorphisms of the Honda formal group law (considered as a formal group law over $\mathbb{F}_9$) and $\mathbb{G}_2$ may be described as

$$\mathbb{G}_2 := \mathbb{S}_2 \rtimes Gal(\mathbb{F}_9 : \mathbb{F}_3)$$

with Galois action given by $\phi(a + bS) = a^\sigma + b^\sigma S$ with $a, b \in \mathbb{W}_{\mathbb{F}_9}$ if σ denotes a lift of Frobenius to the Witt vectors $\mathbb{W}_{\mathbb{F}_9}$.

The group $\mathbb{G}_2$ is a 3-adic analytic group in the sense of Lazard and such groups are of finite cohomological dimension unless they contain elements of order 3. However, $\mathbb{G}_2$ does contain elements of order 3. An explicit element of order 3 is given by

$$a = -\frac{1}{2}(1 + \omega S)$$

where ω is a fixed chosen primitive 8-th root of unity in $\mathbb{W}_{\mathbb{F}_9}$. This element prevents $\mathbb{S}_2$ and thus $\mathbb{G}_2$ to have finite cohomological dimension and thus the trivial module $\mathbb{Z}_3$ to admit a projective resolution of finite length. However, $\mathbb{G}_2$ still has virtual finite cohomological dimension and it admits a finite length resolution by permutation modules. In fact, such a resolution was constructed in [1] and uses the following two finite subgroups of $\mathbb{G}_2$:

- $G_{24} =< a, \omega^2, \omega\phi >\cong C_3 \rtimes Q_8$ (with ω^2 acting non-trivially and $\omega\phi$ acting trivially on C_3).
- $SD_{16} =< \omega, \phi >$, the semidihedral group of order 16.

The group $\mathbb{G}_2$ splits as a product $\mathbb{G}_2 \cong \mathbb{G}_2^1 \times \mathbb{Z}_3$ and the finite subgroups are automatically finite subgroups of $\mathbb{G}_2^1$. In order to construct a permutation resolution we can concentrate on the case of $\mathbb{G}_2^1$.

Theorem 2. [1] *There is an exact complex of $\mathbb{Z}_3[[\mathbb{G}_2^1]]$-modules of the following form*

$$0 \to C_3 \to C_2 \to C_1 \to C_0 \to \mathbb{Z}_3$$

with $C_0 = C_3 \cong \mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]]$ and $C_1 = C_2 \cong \mathbb{Z}_3[[\mathbb{G}_2^1]]\otimes_{\mathbb{Z}_3[SD_{16}]}\mathbb{Z}_3(\chi)$ where $\mathbb{Z}_3(\chi)$ is $\mathbb{Z}_3$ with ω and ϕ both acting by multiplication by -1.

Corollary 3. *Let M be a $\mathbb{Z}_3[[\mathbb{G}_2^1]]$-module. Then there is a first quadrant cohomological spectral sequence*

$$E_1^{s,t}(M) \cong Ext^t_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(C_s, M) \Longrightarrow H^{s+t}(\mathbb{G}_2^1, M)$$

with

$$E_1^{0,t}(M) = E_1^{3,t}(M) \cong H^t(G_{24}, M)$$

and

$$E_1^{1,t}(M) = E_1^{2,t}(M) \cong \begin{cases} Hom_{\mathbb{Z}_3[SD_{16}]}(\mathbb{Z}_3(\chi), M) & t = 0 \\ 0 & t > 0. \end{cases}$$

3. The spectral sequence in the case of $(E_2)_*/(3)$

In the sequel we let $M_* = (E_2)_*$. In the case of $M_*/(3)$ this spectral sequence has been completely worked out in joint work with Karamanov and Mahowald. There are two major difficulties:

• The action of $\mathbb{G}_2$ on $M_* = \mathbb{W}_{\mathbb{F}_9}[[u_1]][u^{\pm 1}]$ and on $M_*/(3) = (E_2)_*V(0)$ is very complicated. For example, no closed formula is known for the action of a. Here is an approximation which gets used in the calculation

$$a_*u \equiv \big(1 + (1+\omega^2)u_1 + (-1+\omega^2)u_1^3 + (1+\omega^2)u_1^5\big)u \mod (3, u_1^6)$$

However, the action of ω and of ϕ is well under control: ϕ is the identity on u and on u_1 and satisfies $\phi_*(wx) = w^\sigma\phi_*(x)$. The action of ω is $\mathbb{W}_{\mathbb{F}_9}$-linear and satisfies $\omega_*(u) = \omega u$, $\omega_*(u_1) = \omega^2 u_1$.

• The homomorphism $C_2 \to C_1$ in the permutation resolution is quite delicate. No concrete description was given in [1]. However, a sufficiently good approximation was described in the thesis of Karamanov [3].

The E_1-term of the spectral sequence can be described as follows: The spectral sequence for the Moore space is one of modules over $H^*(\mathbb{G}_2; (E_2)_*/(3))$. The following result describes it as a module over the subalgebra $\mathbb{F}_3[\beta, v_1]$ of

$H^*(\mathbb{G}_2;(E_2)_*/(3))$ where $\beta \in H^2(\mathbb{G}_2,(E_2)_{12}/(3))$ detects the image of the homotopy element $\beta_1 \in \pi_{10}S^0$ in $\pi_{10}(L_{K(2)}V(0))$ and v_1 detects the image of the homotopy element in $\pi_4 V(0)$ with the same name.

Theorem 4. *We have isomorphisms of $\mathbb{F}_3[\beta, v_1]$-modules (with β acting trivially on $E_1^{s,*,*}$ if $s = 1, 2$)*

$$E_1^{s,*,*} \cong \begin{cases} \mathbb{F}_3[[v_1^6\Delta^{-1}]][\Delta^{\pm 1}, v_1, \beta, \alpha, \widetilde{\alpha}]/(\alpha^2, \widetilde{\alpha}^2, v_1\alpha, v_1\widetilde{\alpha}, \alpha\widetilde{\alpha} + v_1\beta)e_s & s = 0, 3 \\ \omega^2 u^4 \mathbb{F}_3[[u_1^4]][u_1 u^{-2}, u^{\pm 8}]e_s & s = 1, 2. \end{cases}$$

We note that for $s = 0, 3$ this describes a suitable completion of the ring of mod-3-modular forms.

All differentials in this spectral sequence are v_1-linear. In particular, d_1 is completely described by continuity and the following formula.

Theorem 5. [2] *There are elements*

$$\Delta_k \in E_1^{0,0,24k}, \quad b_{2k+1} \in E_1^{1,0,16k+8}, \quad \overline{b}_{2k+1} \in E_1^{2,0,16k+8}, \quad \overline{\Delta}_k \in E_1^{3,0,24k}$$

for each $k \in \mathbb{Z}$ satisfying

$$\Delta_k \equiv \Delta^k e_0, \quad b_{2k+1} \equiv \omega^2 u^{-4(2k+1)} e_1, \quad \overline{b}_{2k+1} \equiv \omega^2 u^{-4(2k+1)} e_2, \quad \overline{\Delta}_k \equiv \Delta^k e_3$$

(where the congruences are modulo $v_1^6\Delta^{-1}$ resp. modulo $v_1^4 v_2^{-1}$) such that

$$d_1(\Delta_k) = \begin{cases} b_{2.(3m+1)+1} & k = 2m+1 \\ v_1^{4.3^n-2} b_{2.3^n(3m-1)+1} & k = 2m.3^n, n \geq 0, m \not\equiv 0 \ (3) \\ 0 & k = 0 \end{cases}$$

$$d_1(b_{2k+1}) = \begin{cases} \pm v_1^{6.3^n+2} \overline{b}_{3^{n+1}(6m+1)} & k = 3^{n+1}(3m+1), n \geq 0 \\ \pm v_1^{10.3^n+2} \overline{b}_{3^n(18m+11)} & k = 3^n(9m+8), n \geq 0 \\ 0 & \textit{else} \end{cases}$$

$$d_1(\overline{b}_{2k+1}) = \begin{cases} \pm v_1^2 \overline{\Delta}_{2m} & 2k+1 = 6m+1 \\ \pm v_1^{4.3^n} \overline{\Delta}_{3^n(6m+5)} & 2k+1 = 3^n(18m+17), n \geq 0 \\ (-1)^m v_1^{4.3^n} \overline{\Delta}_{3^n(6m+1)} & 2k+1 = 3^n(18m+5), n \geq 0 \\ 0 & \textit{else.} \end{cases}$$

The d_2-differential is non-trivial but can be derived from the following two principles:

- d_2 is $\mathbb{F}_3[v_1]$-linear.
- Whenever $\mathbb{F}_3[v_1]$-linearity and sparseness permit it, then $d_2(\Delta^k\alpha)$ and $d_2(\Delta^k\widetilde{\alpha})$ are non-trivial and uniquely determined up to sign.

All higher differentials are trivial.

4. The rational calculation

The calculation of the rational homotopy is an immediate consequence of the following result.

Theorem 6.
a) There are elements $\zeta \in H^1(\mathbb{G}_2, (E_2)_0)$ *and* $\epsilon_3 \in H^3(\mathbb{G}_2^1, (E_2)_0)$ *such that*

$$H^s(\mathbb{G}_2, (E_2)_0) \cong \Lambda(e_3, \zeta_2) \otimes \mathbb{Z}_3[\beta^2\Delta^{-1}]/(3\beta^2\Delta^{-1})$$

b) Let $t = 4.3^k m$ *with* $m \not\equiv 0$ *mod* (3). *Then* $3^{k+1}H^*(\mathbb{G}_2, (E_2)_t) = 0$.
c) In all other cases $H^s(\mathbb{G}_2, (E_2)_t) = 0$.

Part (b) and (c) are easy consequences of the product decomposition $\mathbb{G}_2 \cong \mathbb{G}_2^1 \times \mathbb{Z}_3$. Part (a) follows from Theorem 5 and a straightforward Bockstein spectral sequence argument. The crucial point is that Theorem 5 gives $E_2^{s,t,0} = 0$ if $s = 1, 2$.

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Recent Computational work on EO_n

Michael A. Hill

(joint work with Michael J. Hopkins and Douglas C. Ravenel)

The Hopkins-Miller theorem ensures that there is an action of the Morava stabilizer group S_n on the Lubin-Tate spectrum E_n by E_∞ ring maps. If n is divisible by $p-1$, then S_n has infinite cohomological dimension arising from p-torsion elements in the group. Since the $K(n)$-local sphere, $L_{K(n)}S^0$, is the homotopy fixed points of S_n acting on E_n [1], it is hoped that by restricting attention to finite subgroups (which carry the bulk of the higher cohomology), we can understand computationally the homotopy of $L_{K(n)}S^0$. Indeed, this was successfully done by Adams-Baird and Ravenel for $n = 1$ [2] and by Goerss-Henn-Mahowald-Rezk, and Behrens for $n = 2$, $p = 3$ [3, 4]. The algebraic approximation to $\pi_*L_{K(n)}S^0$ by a finite subgroup G can be made rigid by considering the homotopy fixed point spectra $EO_n(G) = E_n^{hG}$ of Hopkins and Miller.

The computations of the homotopy of EO_{p-1} by Hopkins and Miller allowed Nave to demonstrate quite strong results about the non-existence of Smith-Toda complexes [5]. This computation relied on an understanding of the homotopy of the Lubin-Tate spectrum E_{p-1} as an algebra over $\mathbb{Z}/p$ and was facilitated by a reinterpretation of this algebra using judiciously chosen invariant elements in the mod p homotopy. This talk focused on generalizations of this computation to higher heights divisible by $p-1$.

Using formal group machinery, Devinatz and Hopkins computed the action of S_n on the homotopy groups of E_n [6]. While complete, their description was difficult to apply to computations. Hopkins conjectured that there is a more natural collection of generators of E_{n*} for which the action of finite subgroups is especially simple. We begin by recalling that S_n is the group of units in the maximal order $\mathcal{O}_n$ of the division algebra D_n over $\mathbb{Q}_p$ of Hasse invariant $\frac{1}{n}$. The natural left action of S_n on $\mathcal{O}_n$ commutes with the right action of $\mathbb{Z}_{p^n}$, the Witt vectors for $\mathbb{F}_{p^n}$, and this makes $\mathcal{O}_n$ into a $\mathbb{Z}_{p^n}[S_n]$-module, the Dieudonné module M_n.

Conjecture 1 (Hopkins)**.** *If $G \subset S_n$ is a finite subgroup, then there is a G-equivariant isomorphism*

$$E_{n*} \cong S_{\mathbb{Z}_{p^n}}(M_n)[\Delta^{-1}]_I^\wedge,$$

where S denotes the symmetric algebra functor, M_n is placed in degree -2, Δ is a trivial representation corresponding to the multiplicative norm over the group on M_n, and I is an ideal in degree 0.

The conjecture is most important when $p-1$ divides n and p divides the order of G, as here an obstruction theory argument reduces the proof of this conjecture to verifying it for $\mathbb{Z}/p \subset G$. By using the theory of formal A-modules, Hopkins, Ravenel, and I have made significant headway in proving this conjecture.

Let $A = \mathbb{Z}_p[\zeta]$, where ζ is a p^{th} root of unity. There is an inclusion of A into $\mathcal{O}_n$, and this induces a formal A-module structure on F_n, the Honda formal group of height n. If we write $n = (p-1)f$, then as a formal A-module, F_n has height f, and there is a Lubin-Tate deformation theory of formal A-modules similar to that of formal groups, corepresented by a ring E^A_{f*}. Since A has a p^{th} root of unity, we can easily describe the action of $\mathbb{Z}/p$ on E^A_{f*}. Moreover, if we forget down to formal groups, then we get a natural $\mathbb{Z}/p$-equivariant map of corepresenting rings:

$$E_{(p-1)f*} \to E^A_{f*}.$$

This map is surjective, and it therefore produces a spectral sequence computing the cohomology of $\mathbb{Z}/p$ with coefficients in $E_{(p-1)f*}$ from the cohomology of $\mathbb{Z}/p$ with coefficients in E^A_{f*}. This spectral sequence has the advantage of having a well understood algebraic model, and by mirroring Devinatz and Hopkins original arguments, we have been able to show that through a large range, these spectral sequences coincide.

Assuming Hopkins' conjecture, we have also been able to describe the E_2 term of and compute the differentials in the homotopy fixed point spectral sequence of $\pi_* EO_{(p-1)f}(\mathbb{Z}/p)$. The differentials generalize those found by Hopkins and Miller in their original analysis of $\pi_* EO_{p-1}(\mathbb{Z}/p)$, and their construction is very similar.

Proposition 2. *As an algebra, the E_2 term of the homotopy fixed point spectral sequence for $EO_{(p-1)f}(\mathbb{Z}/p)$ is*

$$E(\alpha_1, \ldots, \alpha_f) \otimes P(\beta) \otimes P(\delta_1, \ldots, \delta_f^{\pm 1}) \oplus \mathit{Free},$$

where the bidegrees of the elements, written as $(t-s, s)$ are $|\alpha_i| = (-3, 1)$, $|\beta| = (-2, 0)$, and $|\delta_i| = -2p$.

The elements referred to as "Free" arise from free summands of E_{n*} and pair trivially with all elements of higher filtration. They also lie in the image of the transfer map from $EO_{(p-1)f}(\{1\})$, making them permanent cycles. The element β is the periodicity generator of $\mathbb{Z}/p$ cohomology, and the elements α_i, β, and δ_i are related by the power operation

$$\beta \mathcal{P}^0(\alpha_i) = \langle \alpha_i, \ldots, \alpha_i \rangle = \beta \delta_i.$$

Using formal group arguments lifted from the analogous story for E_f^A, we can relate the elements α_i to the elements $h_{i,0}$, appropriately translated by powers of δ_f, and the corresponding elements $b_{i,0}$ are similarly related to the classes labeled δ_i and β. These relations allow us to understand differentials that arise on norm classes in the homotopy fixed point spectral sequence.

Proposition 3. *The differentials are algebraically determined by the following properties.*

1. *There are differentials $d_{1+2(p^i-1)}(\delta_f^{p^{i-1}}) = \delta_f^{p^{i-1}} h_{i,0} \beta^{p^i - 1}$.*
2. *There are corresponding $d_{1+2(p-1)(p^i-1)}$ Toda style differentials truncating the β towers on δ_i.*
3. *The classes $\Delta_i = \delta_i / \delta_f$ are permanent cycles.*
4. *The class $\delta_f^{p^f}$ is a permanent cycle, and these describe all of the differentials.*

With the exception of the final statement, these results are all proved in essentially the same way: there exist universal examples for certain differentials in homotopy fixed point spectral sequences. Let g be a generator of $\mathbb{Z}/p$. If $u\colon S^k \to E_n$, then let

$$Nu = u \cdot gu \ldots g^{p-1} u \colon S^{pk} \to E_n.$$

Since S_n acts on E_n by E_∞ maps, this map is $\mathbb{Z}/p$-equivariant and therefore descends to homotopy fixed points. The spectrum $(S^{pk})^{h\mathbb{Z}/p}$ is the Spanier-Whitehead dual of a Thom spectrum, and the attaching maps of the top cell determine differentials on the class represented by Nu. The classes $\delta_f^{p^i}$ and the classes Δ_i are of the form Nu for appropriately chosen u, and this general argument produces the described differentials.

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Localising subcategories of the stable module category of a finite group

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(joint work with Dave Benson and Srikanth Iyengar)

Let G be a finite group and k a field of characteristic p. We write $\mathsf{StMod}\,kG$, respectively $\mathsf{stmod}\,kG$, for the stable module category whose objects are kG-modules, respectively finitely generated kG-modules, and whose morphisms are module homomorphisms modulo those that factor through a projective module. It is well known that this is a triangulated category. The thick subcategories of $\mathsf{stmod}\,kG$ were classified in [2]. More accurately, this was achieved when G is a finite p-group; for an arbitrary finite group only the tensor closed thick subcategories were classified. If G is a finite p-group then all thick subcategories are tensor closed.

Hopkins [5] and Neeman [8] classified the thick subcategories of perfect complexes over a commutative Noetherian ring. Neeman, in the same article, classified the localising subcategories of all complexes, thereby proving the algebraic version of the telescope conjecture in this context.

In this report we present the classification of the localising subcategories of the stable module category $\mathsf{StMod}\,kG$. Our main theorem is as follows.

Theorem. *Let G be a finite group and k a field of characteristic p. Then the tensor closed localising subcategories of $\mathsf{StMod}\,kG$ are in one to one correspondence with subsets of the set of non-maximal homogeneous prime ideals in $H^*(G,k)$. If S is such a subset then the corresponding localising subcategory is the full subcategory whose objects are the modules having their support contained in S.*

Let us discuss some of the essential ingredients of this work. First of all, we pass from $\mathsf{StMod}\,kG$ to the closely related category $\mathsf{K}(\mathsf{Inj}\,kG)$. Here, $\mathsf{K}(\mathsf{Inj}\,kG)$ denotes the category of complexes of injective kG-modules and homotopy classes of degree preserving morphisms of complexes. Note that injective and projective modules are the same in this context. There is the following useful recollement [7]

$$\mathsf{StMod}\,kG \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \mathsf{K}(\mathsf{Inj}\,kG) \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \mathsf{D}(\mathsf{Mod}\,kG)$$

which expresses the fact that the stable category $\mathsf{StMod}\,kG$ can be identified with the localising subcategory of $\mathsf{K}(\mathsf{Inj}\,kG)$ formed by all acyclic complexes and that the derived category $\mathsf{D}(\mathsf{Mod}\,kG)$ is equivalent to the corresponding quotient. In terms of classifying localising subcategories, $\mathsf{K}(\mathsf{Inj}\, kG)$ is obtained from $\mathsf{StMod}\, kG$ by adding the minimal localizing subcategory corresponding to the maximal homogeneous ideal of the cohomology ring $H^*(G, k)$.

We may think of $\mathsf{K}(\mathsf{Inj}\, kG)$ as an algebraic model for the category of modules over $C^*(BG; k)$, the differential graded algebra of cochains on the classifying space BG of G. This point of view is explained in [4]. In fact, we have an equivalence of triangulated categories between $\mathsf{K}(\mathsf{Inj}\, kG)$ and the derived category $\mathsf{D}(C^*(BG; k))$ in case G is a p-group.

Next we rely on the notion of support for objects in a triangulated category. This has been developed in recent joint work [3]. The support is defined relative to a ring of cohomological operators which in our context is the cohomology ring $H^*(G, k)$ of G. This ring is well known to be graded commutative and Noetherian.

The actual proof of the classification result is done in several steps. Following [6], it is sufficient to classify the minimal localising subcategories. We do this by reducing first to an elementary abelian subgroup of G and then to some appropriate Koszul complex. We view this as an exterior algebra, use then a refined Bernstein-Gelfand-Gelfand correspondence [1], and obtain a formal differential graded algebra which has commutative Noetherian cohomology. So we have reduced the classification problem to a case where we can apply a slight generalization of Neeman's original classification for commutative Noetherian rings [8].

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Adams filtration and infinite loop spaces

Nicholas J. Kuhn

1. Introduction

Let $\Sigma^\infty : \mathcal{T} \rightleftarrows \mathcal{S} : \Omega^\infty$ be the usual pair of adjoint functors between the category $\mathcal{T}$ of pointed topological spaces and the category $\mathcal{S}$ of spectra. Note that $\pi_*^S(Z) = \pi_*(\Omega^\infty \Sigma^\infty Z)$. A classic hard problem goes as follows: compute the image of the Hurewicz map

$$\pi_*^S(Z) \to H_*(\Omega^\infty \Sigma^\infty Z; \mathbb{Z}/2).$$

This image is not even known when $Z = S^0$, but conjecturally is very small: only Hopf invariant 1 elements, and Kervaire invariant 1 elements (if they exist) should be detected.

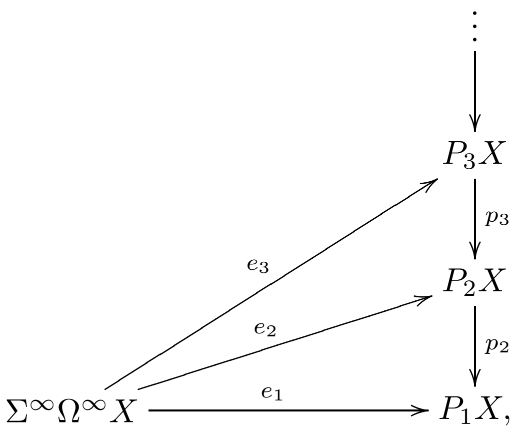
In intriguing work only partially published in the early 1980's, Jean Lannes and Saïd Zarati [4] studied these Hurewicz maps, and found a link between the Adams filtration of the domain, and the Dyer-Lashof algebra filtration of the range. In recent work, I have used Goodwillie tower methods to recover most of what they did. I can also greatly generalize the situation, studying the Hurewicz map

$$\pi_*(X) \to R_*(\Omega^\infty X),$$

where X is a spectrum, and R_* is the generalized homology theory associated to an E_∞ ring spectrum R.

2. The Goodwillie tower of $\Sigma^\infty \Omega^\infty X$

We use the model category of S–modules for our category of spectra. Given an S–module X, there is a tower under $\Sigma^\infty \Omega^\infty X$,



in which e_1 identifies with the evaluation map $\Sigma^\infty \Omega^\infty X \to X$, and the homotopy fiber of p_r is weakly equivalent to $D_r X = X_{h\Sigma_r}^{\wedge r}$. The tower strongly converges if X is 0–connected. (See [3] for more details and references.)

It is convenient to define $K_rX = \text{hofib}\{\Sigma^\infty\Omega^\infty X \to P_{r-1}X\}$, so that the K_rX form a decreasing filtration of $\Sigma^\infty\Omega^\infty X$:

$$\ldots \to K_3X \to K_2X \to K_1X = \Sigma^\infty\Omega^\infty X,$$

and the homotopy cofiber of $K_{r+1}X \to K_rX$ is D_rX.

For the rest of the paper, let X always be connective (i.e. -1-connected), and let R be a connective commutative S–module such that $\pi_0(S) \to \pi_0(R)$ is onto[1].

The canonical R–based Adams filtration of X is the filtration

$$\ldots \to X(2) \to X(1) \to X$$

defined by letting $X(1) = \text{hofib}\{X \to R \wedge X\}$, and, for $s > 1$, $X(s) = X(s-1)(1)$. Under our connectivity hypotheses, all the $X(s)$ will again be connective.

With this notation, our main theorem reads

Theorem 1. *There exist natural maps $R \wedge K_rX(s) \to R \wedge K_{2^sr}X$ making the diagram*

$$\begin{array}{ccc} R \wedge K_rX(s) & \longrightarrow & R \wedge K_{2^sr}X \\ \downarrow & & \downarrow \\ R \wedge \Sigma^\infty\Omega^\infty X(s) & \longrightarrow & R \wedge \Sigma^\infty\Omega^\infty X \end{array}$$

commute, and these maps are compatible as both r and s vary.

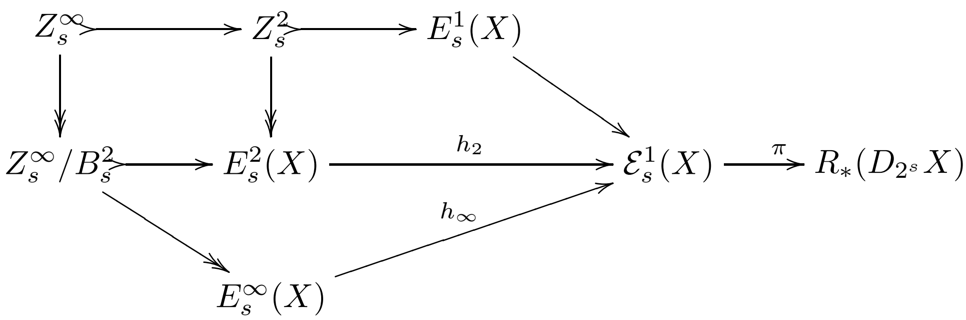
As an application, when $r = 1$, the theorem yields a map of spectral sequences

$$\{E_s^r(X)\} \to \{\mathcal{E}_s^r(X)\}$$

from the R-based Adams spectral sequence $\{E_s^r(X)\}$ to the 'exponential' tower spectral sequence converging to $R_*(\Omega^\infty X)$, with $\mathcal{E}_s^1(X) = R_*(K_{2^s}(X)/K_{2^{s+1}}(X))$.

Corollary 2. *Let $F_s^R\pi_*(X) = Im\{\pi_*(X(s)) \to \pi_*(X)\}$. The composite $F_s^R\pi_*(X) \hookrightarrow \pi_*(X) \to R_*(\Omega^\infty X) \to R_*(P_{2^s-1}X)$ is zero.*

When $R = H\mathbb{Z}/2$ and $X = \Sigma^\infty Z$, this recovers one of the main results of [4]. In this case, the Goodwillie tower splits and so the tower spectral sequence collapses at $\mathcal{E}^1$. With $B_s^r \subseteq Z_s^r \subseteq E_s^1(X)$ the evident cycles and boundaries indexed so that $Z_s^r/B_s^r = E_s^r(X)$, one thus gets an induced diagram



The commutation of the bottom diagram is the other main result in [4], and they go on to give an algebraic description of the middle composite $\pi \circ h_2$.

[1]One could also have the p-local generalizion of this.

3. Ideas in the proof of the main theorem

We use two theorems about the 'Topological André Quillen tower' construction in the category R–alg of non–unital commutative R–algebras.

Theorem 3. *For $I \in R$–alg, there exists a natural filtration in R–alg*

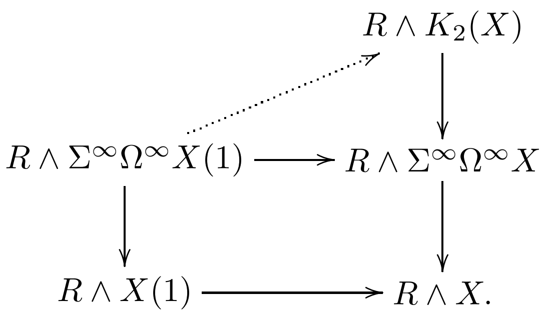
$$\ldots \to I^4 \to I^3 \to I^2 \to I^1 = I$$

such that
(i) I/I^2 is the Topological André Quillen R–module of I with trivial multiplication,
(ii) $I^r/I^{r+1} \simeq (I/I^2)^{\wedge^r_R}_{h\Sigma_r}$, also with trivial multiplication, and
(iii) there exists a natural map $\circ : (I^i)^j \to I^{ij}$ over I.

The first two properties are proved in [1, 5, 2], and lead quite easily to the next theorem. Property (iii) seems to be a new observation.

Theorem 4 ([1, 2]). *Let $I(X) = \mathrm{hofib}\{R \wedge \Sigma^\infty(\Omega^\infty X)_+ \to R\}$, viewed as an object in R–alg. If X is connective, then the $I^r(X)$ filtration of $I(X)$ agrees with the $R \wedge K_r(X)$ filtration of $R \wedge \Sigma^\infty\Omega^\infty X$.*

From these results, the main theorem follows quite easily. Firstly, it suffices to assume $s = 1$. The basic sequence $X(1) \to X \to R \wedge X$ leads to a commutative diagram in R–alg:



The dotted arrow exists because the bottom arrow is null, and this lift can be viewed as an R–alg map $I(X(1)) \to I^2(X)$. From this, and property (iii), one has, for all $r \geq 1$, a composite $I^r(X(1)) \to I^r(I^2(X)) \xrightarrow{\circ} I^{2r}(X)$, which can be interpreted as the desired map $R \wedge K_r(X(1)) \to R \wedge K_{2r}(X)$.

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On the notion of order in the stable module category

Martin Langer

Let G be a finite group and k be a field of prime characteristik $p > 0$. Denote by $\mathfrak{mod}$-kG the category of (right) kG-modules. In this category, the classes of injective and projective modules coincide; this allows us to form its stable category $\underline{\mathfrak{mod}}$-$kG$, whose objects are the same as in $\mathfrak{mod}$-kG, and the morphisms are given by morphisms in $\mathfrak{mod}$-kG modulo the subgroup of those morphisms factoring through a projective module. This is a triangulated category; denote the inverse of the shift functor by Ω. Then we have $\underline{\mathrm{Hom}}(\Omega^n X, Y) \cong \widehat{\mathrm{Ext}}^n_{kG}(X,Y)$; here $\widehat{\mathrm{Ext}}$ denotes Tate Ext-Groups. Suppose we are given a non-zero Tate cohomology class $[\zeta] \in \hat{H}^n(G) = \widehat{\mathrm{Ext}}^n(k,k)$ represented by some (surjective) map $\zeta : \Omega^n k \to k$. Denote by L_ζ the kernel of the map ζ; then we get an exact triangle

$$\cdots \to L_\zeta \to \Omega^n k \to k \to \Omega^{-1} L_\zeta =: k/\zeta \to \ldots$$

On k/ζ, we still have a multiplication by ζ; the following theorem will be the starting point of our discussion.

Theorem 1. (Carlson, [1]**)** *If p is odd and n is even, then multiplication by ζ on k/ζ vanishes; i.e.*

$$\Omega^n k \otimes k/\zeta \xrightarrow{\zeta \otimes id} k/\zeta$$

is stably zero.

If $p = 2$, then the previous theorem need not be true. For instance, one can take $G = \mathbb{Z}/2 \times \mathbb{Z}/2$; then for any non-zero $[\zeta] \in \hat{H}^{-2}(G)$, multiplication by ζ does not vanish on k/ζ. But why is the prime 2 special here?

In topology, we have a similar phenomenon. Suppose we are given a triangulated category $\mathcal{C}$, some object $X \in \mathcal{C}$ and a natural number m. On X, we have the 'multiplication by m', i.e. $m \cdot \mathrm{Id}_X : X \to X$; denote by X/m some choice of cone of this map. On this cone, we also have a multiplication by m. If $\mathcal{C}$ is the stable homotopy category, and p is a prime, then the mod-p Moore spectrum S/p (where S denotes the sphere spectrum) has a multiplication by p, which is zero if p is odd, but non-zero if $p = 2$.

Motivated by his proof of the Rigidity Theorem [3], Schwede introduced the notion of m-order (see [2]), which measures 'how zero multiplication by m on some object is'. More generally, the order can be defined for elements of the graded center of $\mathcal{C}$, i.e. natural transformations ζ from the identity functor to Σ^n: The ζ-order ζ-ord(X) of an object $X \in \mathcal{C}$ is an element of $\{0, 1, 2, \ldots, \infty\}$, defined inductively by the following condition:

ζ-ord$(X) \geq k$ if and only if

> for all objects K in $\mathcal{C}$ and all morphisms $f : K \to X$ there is an *extension* $\hat{f} : K/\zeta \to X$ such that for some (and hence any) cone $C_{\hat{f}}$ of $\hat{f}$, ζ-ord$(C_{\hat{f}}) \geq k - 1$.

Here, extension means that the following diagram commutes:

$$\begin{array}{ccccccc} \cdots \longrightarrow & \Sigma^{-n}K & \xrightarrow{\zeta} & K & \longrightarrow & K/\zeta & \longrightarrow \cdots \\ & & & \downarrow f & \swarrow \hat{f} & & \\ & & & Y & & & \end{array}$$

For instance, ζ-ord$(X) \geq 1$ if and only if $\zeta_X = 0$. In topology, we have the following results [2]:

(t1) Let $\mathcal{C}$ be a topological triangulated category and p be a prime number. For any object X of $\mathcal{C}$, the object X/p has p-order at least $p-2$.

(t2) In the stable homotopy category $\mathcal{SHC}$, the mod-p Moore spectrum S/p has p-order exactly $p-2$.

(t3) We can even go one step further. If the morphism $\alpha_1 \wedge X : S^{2p-3}X \to X$ is divisible by p, then X/p has p-order at least $p-1$. Here $\alpha_1 : \Sigma^{2p-3}S \to S$ is a generator of the p-torsion in the stable homotopy groups of spheres in the $(2p-3)$-stem.

In algebra, it is not very interesting to consider p-order, because in any algebraic triangulated category $\mathcal{C}$, for every object X and every integer n, the n-order of X/n is ∞ (see [2]).

Let us turn back to the case when $\mathcal{C} = \underline{\mathfrak{mod}}$-$kG$, the stable module category of kG-modules. Again, take a non-zero cohomology class $[\zeta] \in \widehat{\mathrm{Ext}}^n(k,k) = \hat{H}(G,k)$ of even degree n. This class is represented by some map $\zeta : \Omega^n k \to k$, which in turn induces an element (of degree n) in the center of $\mathcal{C}$, also denoted by ζ. Carlson's result (as stated above) can now be written as ζ-ord$(k/\zeta) \geq 1$ if $p \geq 3$. The following more general result explains to a certain amount why this is only true for 'large enough' primes p:

Theorem 2.

(a1) *For any object $X \in \underline{\mathfrak{mod}}$-$kG$, the ζ-order of X/ζ is at least $p-2$.*

(a3) *If $\beta P^{\frac{n}{2}-1}\zeta$ is a multiple of ζ in $\hat{H}^*(G)$, then the ζ-order of k/ζ is at least $p-1$. In the case $p=2$ the Steenrod Operation has to be replaced by Sq^{n-1}.*

Open questions include:

(a2) Given a prime p, are there examples of k, G and ζ such that the ζ-order of k/ζ is exactly $p-2$?

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Topological automorphic forms

Tyler Lawson
(joint work with Mark Behrens)

In a well known paper, Quillen described a relationship between stable homotopy theory and formal groups [8]. He showed that any cohomology theory E^* that has a multiplicative structure and an orientation for complex vector bundles carries a 1-dimensional formal group law. This formal group law describes the tensor product of the first Chern classes of two line bundles in terms of the first Chern classes of the line bundles themselves. Moreover, the complex cobordism ring MU^* is the representing object for formal group laws.

This structure has had wide-ranging applications to the understanding of stable homotopy theory. Roughly speaking, it becomes possible to approximate stable homotopy theory by the theory of sheaves on a moduli stack $\mathcal{M}_{FG}$ of 1-dimensional formal groups. One aspect of this connection is the Adams-Novikov spectral sequence. This spectral sequence can be interpreted as starting with the cohomology

$$H^*(\mathcal{M}_{FG}, \omega^*)$$

of the moduli stack, with coefficients in a graded structure sheaf ω^* of tensor powers of the cotangent sheaf, and converging to the stable homotopy groups of spheres. This can be regarded as a reflection of "faithfully flat descent" of modules for the map of ring objects $\mathbb{S} \to MU$ in the stable homotopy category, where $\mathbb{S}$ is the sphere spectrum.

The moduli of formal groups has a natural stratification by height that gives rise to the "chromatic filtration" in stable homotopy theory, and this filtration has proven to be effective for understanding periodic phenomena in stable homotopy groups. The chromatic filtration splits up the stable homotopy category into layers referred to as the $K(n)$-local categories. The $K(n)$-local categories contain Lubin-Tate spectra whose homotopy groups are the universal deformation rings of formal group laws over perfect fields of characteristic p, and the Hopkins-Miller theorem implies that these spectra admit actions by the automorphism groups of these formal group laws [9]. In particular, the Lubin-Tate spectrum E_n associated to the Honda height n formal group over $\mathbb{F}_{p^n}$ admits an action by the Morava stabilizer group $\mathbb{S}_n$.

Hopkins and Mahowald examined the relationship between the moduli of elliptic curves (which gives rise to a family of 1-dimensional formal groups) and the stable homotopy groups of spheres [7]. This line of research culminated in the construction, due to Goerss, Hopkins, and Miller, of a highly structured commutative ring spectrum of topological modular forms called TMF, whose homotopy groups detect several interesting phenomena in stable homotopy theory ([9], [5]). The homotopy groups of TMF are derived from rings of modular forms over $\mathbb{Z}$.

The spectrum TMF plays a prominent role in recent work of Goerss, Henn, Mahowald, and Rezk on the structure of the $K(2)$-local sphere [4]. They constructed a resolution of the $K(2)$-local sphere at the prime 3 in terms of fixed points of a Lubin-Tate spectrum E_2 by finite subgroups of the Morava stabilizer group, and these fixed point spectra can be expressed in terms of TMF. This gave a conceptual framework for computations of Shimomura and Wang [10], and helped to give an understanding of chromatic filtration 2.

Behrens gave a further reinterpretation of the constructions of Goerss-Henn-Mahowald-Rezk in terms of isogenies of elliptic curves [2]. He constructed a cosimplicial spectrum based on the category of supersingular elliptic curves with isogenies of degree a power of 2, and showed that the totalization Q of this spectrum is "half" of the resolution of the $K(2)$-local sphere. More precisely, there exists a cofiber sequence of spectra $DQ \to \mathbb{S}_{K(2)} \to Q$, where $\mathbb{S}_{K(2)}$ is the $K(2)$-local sphere and D denotes the $K(2)$-local Spanier-Whitehead dual. His result depended on the prime 2 being a topological generator of the group $\mathbb{Z}_3^\times$.

The construction of TMF was generalized by Behrens and the author in [3] to a collection of *topological automorphic forms* spectra. The construction of these spectra is based on a recent theorem of Lurie, which gives functorial constructions of structured ring spectra associated to moduli problems whose local geometry is governed by a 1-dimensional p-divisible group. There are various moduli of previously studied abelian varieties with this property, such as the Shimura varieties of type $\mathrm{U}(1, n-1)$ used by Harris and Taylor in [6] in their work on the local Langlands correspondence. These varieties parametrize high-dimensional abelian varieties with actions of division algebras that give rise to a 1-dimensional split summand of their p-divisible group, and hence of their formal group law. Lurie's machinery then constructs spectra whose homotopy groups are calculated from integral rings of automorphic forms, generalizing the spectrum TMF and giving rise to analogous structures at all chromatic levels.

Even in low dimensions there are several moduli of abelian varieties of interest to stable homotopy theory. For example, there are the various Shimura curves parametrizing "false elliptic curves", or abelian surfaces with an action by a maximal order in a quaternion algebra over $\mathbb{Q}$ with discriminant D, whose complex points are quotients of the upper half-plane by arithmetic Fuchsian groups. These give rise to global analogues of TMF whose homotopy groups are derived from rings of automorphic forms over $\mathbb{Z}[D^{-1}]$ for these arithmetic subgroups of $\mathrm{GL}_2(\mathbb{R})$. These rings can be rationally computed by standard techniques.

The initial input necessary for computational knowledge of these spectra is an understanding of which automorphic forms have integral lifts, making use of explicit analytic power series expansions of these functions over $\mathbb{C}$. These integral structures and congruences between them should be of independent interest.

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p-Local Compact Groups

Ran Levi
(joint work with Carles Broto and Bob Oliver)

The purpose of this project is to provide a very general context in which the p-local homotopy theory of classifying spaces of groups satisfying certain finiteness conditions can be studied. Here by p-local homotopy theory we mean the study of spaces which are invariant up to homotopy under the Bousfield-Kan p-completion functor [1], and maps between them.

The main motivation comes from compact Lie groups (in particular finite groups). If G is a compact Lie group whose group of components is a finite p-group, then the p-completed classifying space $BG^\wedge_p$ is homotopy equivalent to the classifying space of $G^\wedge_p$. In particular $BG^\wedge_p$ has a loop space whose mod-p homology is finite. Based on this observation, Dwyer and Wilkerson [4] defined a p-compact group to be a triple (X, BX, ϵ), where X is a loop space with a finite mod p homology algebra, BX is p-complete, and $\epsilon\colon X \to \Omega BX$ is a homotopy equivalence. With this simple definition, Dwyer-Wilkerson and others developed a spectacular theory which explained most of the homotopy theoretic properties of classifying spaces of compact Lie groups (with the restriction spelled out above), which until then were studied by more geometric means, by purely homotopy theoretic methods.

When the restriction on the group of components is dropped, the situation changes quite radically. If G is a finite group, the minimal normal subgroup of p-power index is denoted $O^p(G)$. If G is a compact Lie group with a group of components π, and the order of $O^p(\pi)$ is divisible by p, then $\Omega BG^\wedge_p$ is an infinite dimensional space. (This follows easily from [5].) This makes it impossible to extend the theory of p-compact groups to include all compact Lie groups, and in particular all finite groups.

The basic observation needed to fix this situation is that many p-local algebraic phenomena associated with a finite group G are controlled by the so-called p-fusion system of G. This is the category, whose objects are the p-subgroups of G, and whose morphisms are those homomorphism which are induced by conjugation in G. Since all Sylow p-subgroups in a finite group G are conjugate, the p-fusion system of G is equivalent to the full subcategory whose objects are the subgroups of a fixed Sylow p-subgroup.

The theory of p-local groups is based on this intuition. In the early 1990 Ll. Puig axiomatically defined the concept of a saturated fusion system over a finite p-group S, a concept modeled directly on the fusion systems of groups, and more generally fusion systems which arise in block theory. In [2] we used this concept to create and study p-local finite groups. In this lecture we introduce p-local compact groups, which is an extension of the notion to include objects such as p-completed classifying spaces of arbitrary compact Lie groups.

A discrete p-toral group is an extension of a group of the form $(\mathbb{Z}/p^\infty)^n$ by a finite p-group. A fusion system over a discrete p-toral group S is a category $\mathcal{F}$ whose objects are the subgroups of S, and whose morphisms are injective homomorphisms between them, which include all restrictions of inner automorphisms of S. Each morphism is required to be a composite of an isomorphism in $\mathcal{F}$ followed by an inclusion. A saturated fusion system $\mathcal{F}$ over S is required in addition to satisfy three extra axioms: a Sylow axiom, an Extension axiom, and a Continuity axiom. A centric linking system associated to $\mathcal{F}$ is, very roughly speaking, a category $\mathcal{L}$, whose objects are a certain collection of subgroups of S, called the $\mathcal{F}$-centric subgroups, and whose morphisms are "lifts" of the corresponding morphisms in $\mathcal{F}$, in such a way that the full subcategory of $\mathcal{F}$ whose objects are the $\mathcal{F}$-centric subgroups becomes a quotient category of $\mathcal{L}$. As in the finite case, the centric linking system, if it exists, contains just about enough information to equip the fusion system with a "classifying space". A p-local compact group is thus a triple $(S, \mathcal{F}, \mathcal{L})$, such that S is a discrete p-toral group, $\mathcal{F}$ is a saturated fusion system over S, and $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$. The classifying space of $(S, \mathcal{F}, \mathcal{L})$ is the p-completed nerve $|\mathcal{L}|^\wedge_p$.

Theorem 1. *Let BX be either the p-completed classifying space of a compact Lie group or the classifying space of a p-compact group. Then there exists a p-local compact group $(S, \mathcal{F}, \mathcal{L})$, unique up to equivalence, whose classifying space $|\mathcal{L}|^\wedge_p$ is homotopy equivalent to BX.*

The proof of the analogous theorem in the finite case is quite straight forward. The generalization presented here is much more laborious, and requires in particular the use of obstruction theory to obtain centric linking systems associated to saturated fusion systems. Reassuringly however, what one expects to be the natural examples of p-local compact groups, are indeed such examples.

Another family of examples arises from certain infinite discrete groups. A linear torsion group is a subgroup of $GL_n(k)$ for some field k, all of whose elements are of finite order. Natural examples of such groups are of course groups of Lie type over the algebraic closure of a finite field, and any of their subgroups. If $G \leq GL_n(k)$

is a linear torsion group, then the defining characteristic of G simply means the characteristic of k. Linear torsion groups G turn out to have "Sylow p-subgroups", i.e., discrete p-toral subgroups $S \leq G$, such that any homomorphism from any discrete p-toral group into G factors through S. It is rather straight forward to define the a fusion system $\mathcal{F}_S(G)$ associated to a linear torsion group, which can be shown to be saturated. In addition, since linear torsion groups are in particular discrete, the obvious definition of their centric linking systems $\mathcal{L}_S^c(G)$ turns out to work well within the theory. Thus linear torsion groups is a large and widely unexplored source of examples of p-local compact groups, as the following theorem states.

Theorem 2. *Fix a linear torsion group G, a prime p different from the defining characteristic of G, and a Sylow subgroup $S \leq G$. Then the triple*

$$(S, \mathcal{F}_S(G), \mathcal{L}_S^c(G))$$

is a p-local compact group, with classifying space homotopy equivalent to $BG_p^\wedge$.

The fundamental paper on the subject is [3]. In there we initiate the study of basic properties of p-local compact groups. We show in particular that the space of self homotopy equivalences of the classifying space of a p-local compact group can be studied by means of the centric linking system. This allows us to argue that each component of this space has only two nonvanishing homotopy groups in dimensions 1 and 2, and produce an algebraic description of the group of components, analogous to that we gave in the finite case. We also study spaces of maps from the classifying space of a discrete p-toral group into that of a p-local compact group, but more work is required in order to establish these results at the level of generality we achieved in the finite case. One big gap in our knowledge of the theory has to do with cohomology. In the finite case we have proven a "stable elements theorem". No such theorem is known to date in the theory of p-local compact groups.

Projects in Progress. An interesting problem in the subject is the construction and classification of certain maps between p-local compact groups. Fabien Junod has made substantial progress on the construction of unstable Adams operations on classifying spaces of p-local compact groups, under certain assumptions.

A related problem is the question whether every p-local compact group is in the appropriate sense a colimit of p-local finite groups. Many examples arising from compact Lie groups and p-compact groups suggest that this might be the case in greater generality. One remarkable example is the construction of the classifying space of the exotic 2-compact group $DI(4)$ as a colimit of classifying spaces of the exotic 2-local finite groups $\mathrm{Sol}(q)$ [6].

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The Burnside ring, equivariant cohomotopy and the Segal Conjecture for infinite groups

Wolfgang Lück

In this talk we want to extend some notions and results about equivariant homotopy theory from finite groups to infinite (discrete) groups. We firstly explain the main results of the paper [2]. We give various definitions of the Burnside ring for infinite groups. They are mutually different in general but agree for finite groups with the standard definition. The choice of the adequate definition depends on the aspect one is considering, e.g. universal equivariant Euler characteristics, completion theorems, induction properties and so on. We show

Theorem 1. *There is a construction of equivariant stable cohomotopy $\pi_?^*$ which defines an equivariant cohomology theory with multiplicative structure for finite proper equivariant CW-complexes. For every finite subgroup H of the group G the abelian groups $\pi_G^n(G/H)$ and π_H^n are isomorphic for every $n \in \mathbb{Z}$ and the rings $\pi_G^0(G/H)$ and $\pi_H^0 = A(H)$ are isomorphic.*

The main point in the construction is to use stabilization with G-vector bundles and not only with G-representations.

Let $\underline{E}G$ be the classifying space for proper G-actions. See [3] for more information about $\underline{E}G$. If there is a finite model for $\underline{E}G$, e.g., if G is word hyperbolic, a cocompact lattice in an almost connected Lie group, an arithmetic group, then we use the definition of the Burnside ring $A(G) := \pi_G^0(\underline{E}G)$. The main result of this talk is the following extension of the Segal Completion Theorem from finite groups, for which it was proved by Carlsson [1], to infinite groups.

Theorem 2. *Let G be a (discrete) group. Suppose that there is a finite G-CW-model for $\underline{E}G$. Let $\mathbb{I}_G$ be the augmentation ideal in the Burnside ring $A(G)$. Then there is an isomorphism of pro-$\mathbb{Z}$-modules*

$$\lambda_G^m(X)\colon \{\pi_G^m(X)/\mathbb{I}_G(L)^n \cdot \pi_G^m(X)\}_{n\geq 1} \xrightarrow{\cong} \{\pi_s^m\left((EG \times_G X)_{(n-1)}\right)\}_{n\geq 1}$$

In particular we obtain an isomorphism

$$\pi_s^m(EG \times_G X) \cong \pi_G^m(X)\widehat{_{\mathbb{I}_G}},$$

and an isomorphism

$$\pi_s^o(BG) \cong A(G)\widehat{_{\mathbb{I}_G}}.$$

Its topological K-Theory analogue has already been proved in [5] and [6]. We also explain the main result of [4].

Theorem 3. *Suppose that there is a cocompact G-CW-model for $\underline{E}G$. Denote by $K^*(BG)$ the topological (complex) K-theory of the classifying space BG. Then there is a $\mathbb{Q}$-isomorphism*

$$K^n(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \left(\prod_{i\in\mathbb{Z}} H^{2i+n}(BG;\mathbb{Q})\right) \times \prod_{p \text{ prime}} \prod_{(g)\in \operatorname{con}_p(G)} \left(\prod_{i\in\mathbb{Z}} H^{2i+n}(BC_G\langle g\rangle;\widehat{\mathbb{Q}_p})\right),$$

where $\operatorname{con}_p(G)$ is the set of conjugacy classes (g) of elements $g \in G$ of order p^d for some integer $d \geq 1$ and $C_G\langle g\rangle$ is the centralizer of the cyclic subgroup $\langle g\rangle$ generated by g.

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Rational homotopy theory of manifolds and stratified spaces

Jim McClure

(joint work with Greg Friedman and Scott Wilson)

Let X be a space with a PL structure (a family of compatible triangulations). The PL chain complex C_*X is the direct limit, over all compatible triangulations of X, of the simplicial chains of each triangulation.

Part I

Let M be a compact oriented PL manifold. The well-known intersection pairing in H_*M can be lifted to a partially defined chain-level operation whose domain is the subcomplex of $C_*M \otimes C_*M$ consisting of chains in general position with respect to the diagonal. It was shown in [3] that the inclusion of this subcomplex in $C_*M \otimes C_*M$ is a quasi-isomorphism; this is a strong "moving lemma" for PL chains. This result (and its analog for the iterated intersection pairing) implies that the chain-level intersection pairing induces a structure of Leinster partial commutative algebra on C_*M. Combining this with a result of Scott Wilson [4] shows that C_*M is canonically quasi-isomorphic to an E_∞ chain algebra, and that the quasi-isomorphism respects the partially defined algebra structure on C_*M.

A by-product of this work is the discovery that several formulas in the literature become more elegant if the usual Poincaré duality map from H^iM to $H_{m-i}M$ is given a sign $(-1)^{mi}$.

In more recent work, Scott Wilson and I have shown that the E_∞ structure obtained in this way is canonically quasi-isomorphic (via a chain-level version of Poincaré duality) to the usual E_∞ structure on the singular cochains S^*M. But Mandell has shown [2] that, for any space X, Sullivan's commutative DGA A^*X (and hence the minimal model of X) can be recovered from the E_∞ structure on S^*X, so combining everything that has been said shows that the minimal model of a compact oriented PL manifold M can be recovered from the chain-level intersection pairing on C_*M. This gives a strong connection between the rational homotopy type of M and the geometry of M. It would be interesting to know if the minimal model of M can be constructed directly from the intersection pairing on C_*M (without having to first convert to an E_∞ algebra).

Part II

The rest of the lecture was about the generalization of these ideas to intersection chains. So let X be a compact oriented PL pseudomanifold. For each triple of perversities $\bar{p}, \bar{q}, \bar{r}$ with $\bar{p} + \bar{q} \leq \bar{r}$ there is an intersection pairing

$$IH_i^{\bar{p}}X \otimes IH_j^{\bar{q}}X \to IH_{i+j-n}^{\bar{r}}X,$$

where n is the dimension of X. This pairing lifts to a partially-defined chain-level operation whose domain is a subcomplex of $IC_*^{\bar{p}}X \otimes IC_*^{\bar{q}}X$, and Greg Friedman has shown [1] that the inclusion of this subcomplex in $IC_*^{\bar{p}}X \otimes IC_*^{\bar{q}}X$ is a quasi-isomorphism, and similarly for the iterated intersection pairing.

The next step is to provide a suitable algebraic setting for this result, generalizing the Leinster partial commutative algebras and E_∞ algebras in Part I. First we give an abstract definition that encodes the algebraic structure actually present in the intersection pairing on IH_*X. By a *perverse chain complex* we mean a functor F from the poset of n-perversities to the category of chain complexes (the simplest example takes $\bar{p}$ to $IH_*^{\bar{p}}X$, with zero differential). The category of perverse chain complexes has a symmetric monoidal structure

$$(F \otimes G)(\bar{r}) = \bigoplus_{\bar{p}+\bar{q}\leq\bar{r}} F(\bar{p}) \otimes G(\bar{q}).$$

The intersection homology groups IH_*^*X, together with the intersection pairing, form a commutative monoid in this category. Greg Friedman's result shows that the intersection chain complexes IC_*^*X, together with the partially-defined chain-level intersection pairing, form a Leinster partial commutative monoid.

Next we observe that Wilson's result [4] generalizes to this situation to produce an E_∞ monoid quasi-isomorphic to IC^*_*X. Moreover, over the rationals one can "rectify" this E_∞ structure to a strictly commutative monoid. Comparing this with what has been shown in Part I, we would expect this commutative monoid to encode the "stratified rational homotopy type" of X. Making sense of this statement is work in progress.

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Arithmetically defined dense subgroups of Morava stabilizer groups

Niko Naumann

We reported on a potential application of topological automorphic forms [2] (see also T. Lawson's report in this volume) to the understanding of $K(n)$-local spheres. Let's begin with a brief review of the chromatic picture of stable homotopy, see [4] for a comprehensive account.

Let p be a prime and X a p-local finite spectrum. The chromatic convergence Theorem asserts that the homotopy type of X is recovered from a tower of localizations of X

$$X \xrightarrow{\sim} \mathrm{holim}(\ldots \to L_2X \to L_1X),$$

where L_n denotes localization with respect to Lubin-Tate theory E_n of height n, a complex orientable ring spectrum with

$$\pi_0(E_n) \simeq W(\overline{\mathbb{F}}_p)[[u_1, \ldots u_{n-1}]].$$

Furthermore, denoting by $K(n)$ Morava K-Theory, the canonical commutative diagram

$$\begin{array}{ccc} L_nX & \longrightarrow & L_{K(n)}X \\ \big\downarrow & & \big\downarrow \\ L_{n-1}X & \longrightarrow & L_{n-1}L_{K(n)}X \end{array}$$

is homotopy cartesian. This exhibits $L_{K(n)}X$ as a basic building block for the homotopy type of X and in the following we will concentrate on the fundamental special case that $X = S^0_{(p)}$ is the p-local sphere.

According to P. Goerss/M. Hopkins/H. Miller and E. Devinatz/M. Hopkins we have

$$L_{K(n)}S^0 \simeq E_n^{h\mathbb{G}_n},$$

where $\mathbb{G}_n$ is the extended Morava stabilizer group. This leads to a spectral sequence starting with continuous group-cohomology

$$E_2^{s,t} = H^s(\mathbb{G}_n, E_{n,t}) \Rightarrow \pi_{t-s} L_{K(n)} S^0.$$

This group-cohomology is hard to compute but the analogous problem for finite subgroups of $\mathbb{G}_n$ is much more amenable, see M. Hill's talk in this volume. This motivates the problem of constructing finite resolutions of $L_{K(n)}S^0$ by fixpoint-spectra of E_n with respect to finite subgroups of $\mathbb{G}_n$ ("higher real K-theories"). Substantial progress on this problem has been reported on by H.-W. Henn in this conference. Topological automorphic forms offer a different approach to the problem which has already seen great success in the case $n = 2, p = 3$ [1].

More details on the following can be found in [2]. One introduces

$$Q(l) := \mathrm{Tot}\begin{pmatrix} \text{a semi-cosimplicial } E_\infty\text{-ring spectrum} \\ \text{of length } n \text{ with constituent topological} \\ \text{automorphic forms of finite level} \end{pmatrix}.$$

The maps in this cosimplicial spectrum are nicely determined by l-power isogenies of abelian varieties. One can show that

$$Q(l) \simeq E_n^{h\Gamma}$$

for a dense subgroup $\Gamma \subseteq \mathbb{G}_n$ and thus hope that $Q(l)$ is closely related to $L_{K(n)}S^0$. Density of arithmetic subgroups such as $\Gamma \subseteq \mathbb{G}_n$ is the main concern of [3] and we gave a short account of this entirely arithmetic work.

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Units of Equivariant Ring Spectra

Rekha Santhanam

Motivation. Let R be an E_∞ ring spectrum. Then $\mathrm{GL}_1 R$ is defined to be the homotopy pullback of the following spaces

$$\begin{array}{ccc} \mathrm{GL}_1(R) & \longrightarrow & \Omega^\infty R \\ \downarrow & & \downarrow \\ \pi_0(R)^\times & \longrightarrow & \pi_0(R) \end{array}$$

May, Quinn and Ray [4] prove that $\mathrm{GL}_1(R)$ is in fact a connective spectrum. In particular, if K denoted the K-theory ring spectrum then $\mathrm{GL}_1(K)$ defines the twistings of twisted K theory. A recent result of Freed, Hopkins and Teleman relates twisted equivariant K-theory to Verlinde Algebra. As in the non-equivariant case, the twistings ought to arise homotopy theoritically from the units of equivariant spectra. We develop the framework defining units of equivariant spectra for finite group actions.

The Idea. We show that the category of E_∞-spaces is Quillen equivalent the category of Γ spaces (where special Γ-spaces are fibrant) with appropriate model categories. We can then reformulate the construction of the units of a ring spectrum. The zero space of an E_∞-ring spectrum R has two E_∞ structures. We consider the E_∞-structure corresponding to the product in R. The previously mentioned Quillen equivalence implies that there is a special Γ-space equivalent to this E_∞-space. Then given a special Γ-space, we can extract its units as a very-special Γ space. This describes the units of the E_∞-ring spectrum R as a very special Γ-space.

Let $\mathcal{T}$ and Spaces denote the category of pointed spaces.

Definition 1. Define the category Γ to have objects finited pointed sets and morphisms are pointed maps. Denote the n-pointed set $\mathbf{n} := \{0, 1, 2, \cdots, n\}$. Define a Γ-space to be a pointed functor from Γ to Spaces such that it maps $\mathbf{0}$ to the marked point in $\mathcal{T}$.

Given a pointed set $\mathbf{n}$, there exist n projection maps $\theta_i : \mathbf{n} \to \mathbf{1}$ which takes the ith element to 1 and everything else maps to 0. Then for any Γ-space X, we have a map $X(\mathbf{n}) \xrightarrow{\Pi X(\theta_i)} X(\mathbf{1})^n$. If this is a homotopy equivalence then X is said to be a special Γ-space. The map $\Delta : \mathbf{2} \to \mathbf{1}$ which maps both 1 and 2 to 1 induces a homotopy monoid structure on $X(\mathbf{1})$ if X is special.

In particular, this makes $\pi_0(X(\mathbf{1}))$ into a monoid. Further if $\pi_0(X(\mathbf{1}))$ is a group then X is called a very special Γ-space.

Segal [6] and Bousfield and Friedlander [1] show that very-special Γ spaces are equivalent to connective spectra.

Following the work of May and Thomason [5] we show that with appropriate model categories we get a zig-zag of Quillen equivalences between E_∞-spaces and Γ-spaces.

Define the category Π with objects being finite sets. Given $\mathbf{m}$ and $\mathbf{n}$ the morphism set $\Pi(\mathbf{m},\mathbf{n}) := \{\phi \in \Pi(\mathbf{m},\mathbf{n}) : \phi^{-1}(\mathbf{i})$ has at most one element if $i > 0\}$. As before, we can define special and very-special Π-spaces since Π contains all the θ_i's.

Definition 2. A category of operators is a category $\mathcal{G}$ whose object set is the set of finite pointed sets such that there exist functors $\Pi \to \mathcal{G} \to \Gamma$. The functor $\Pi \to \mathcal{G}$ is an inclusion functor.

Definition 3. Let $\mathcal{G}$ be a category of operators. Define a $\mathcal{G}$-space X to be a functor from $\mathcal{G}$ to Spaces such that $X(\mathbf{0})$ is the marked point.

We then have an adjoint pair of functors

$$\Pi[\mathcal{T}] \underset{\mathrm{R}}{\overset{\mathrm{L}}{\rightleftarrows}} \mathcal{T}$$

defined as $\mathrm{L}X = X(\mathbf{1})$ and $\mathrm{R}Y(\mathbf{n}) = Y^n$.

If $\mathcal{C}$ is a pointed operad, then we can construct a category of operators $\hat{\mathcal{C}}$ to be the category of operators with morphism sets

$$\hat{\mathcal{C}}(\mathbf{m},\mathbf{n}) := \coprod_{\phi\in\Gamma(\mathbf{m},\mathbf{n})} \prod_{1\le j\le n} \mathcal{C}(|\phi^{-1}(j)|).$$

In particular if we consider the operad $\mathcal{A}$ representing the algebra of monoids then the associated category of operators is the category Γ.

The category of operators $\hat{\mathcal{C}}$ defines a monad on Π-spaces. Note that if X is a $\mathcal{C}$-algebra then the Π-space $\mathrm{R}X$ is a $\hat{\mathcal{C}}$-algebra and the category of $\hat{\mathcal{C}}$-algebras is equivalent to the category of $\hat{\mathcal{C}}$-spaces. Denote the functor induced by R on $\mathcal{C}$-spaces by $\mathrm{R}_{\mathcal{C}}$.

Theorem 4. *There exists a functor $L_{\mathcal{C}} : \hat{\mathcal{C}}[\mathcal{T}] \to \mathcal{C}[\mathcal{T}]$ which is adjoint to $R_{\mathcal{C}}$ such that the following diagram of adjoint functors commutes:*

$$\begin{array}{ccc} \Pi[\mathcal{T}] & \underset{R}{\overset{L}{\rightleftarrows}} & \mathcal{T} \\ F_{\mathcal{C}} \downarrow\uparrow U_{\mathcal{C}} & & U \uparrow\downarrow F \\ \hat{\mathcal{C}}[\mathcal{T}] & \underset{R_{\mathcal{C}}}{\overset{L_{\mathcal{C}}}{\rightleftarrows}} & \mathcal{C}[\mathcal{T}] \end{array}$$

Here the functors $U_{\mathcal{C}}$ and U are forgetful functors and $F_{\mathcal{C}}$ and F are the free algebra functors.

Model categories. We consider the model category over Spaces where weak-equivalences are weak homotopy equivalences and fibrations are Serre fibrations.

The category of Π-spaces has a level model category structure [3] where a morphism $X \to Y$ is a weak equivalence (fibration) if $X(\mathbf{n}) \to Y(\mathbf{n})$ is a weak equivalence (Serre fibration) of spaces.

Then $\Pi[\mathcal{T}]$ is a left proper cellular model category. Consider the Π-spaces $\Pi^n(\mathbf{m}) = \Pi(\mathbf{m}, \mathbf{n})$. Note that $\mathrm{Map}_{\Pi[\mathcal{T}]}(\Pi^n, X) \cong X(\mathbf{n})$. The projection maps θ_i induce morphisms $\coprod \Pi^1 \to \Pi^n$, and localizing $\Pi[\mathcal{T}]$ with respect to these maps we get a new model category on $\Pi[\mathcal{T}]$. In this localized model category fibrant spaces are the special Π-spaces.

For a "nice" operad $\mathcal{C}$ we can show that $\mathcal{C}[\mathcal{T}]$ and $\hat{\mathcal{C}}[\mathcal{T}]$ form a model category as algebras over model categories $\mathcal{T}$ and $\Pi[\mathcal{T}]$. Under these model category structures we have the following theorem.

Theorem 5. *The adjoint pair*

$$\hat{\mathcal{C}}[\mathcal{T}] \underset{R_{\mathcal{C}}}{\overset{L_{\mathcal{C}}}{\rightleftarrows}} \mathcal{C}[\mathcal{T}]$$

is a Quillen equivalence.

Given any category of operators $\mathcal{G}$, the category $\mathcal{G}[\mathcal{T}]$ has a localized model category the same way as $\Pi[\mathcal{T}]$. In the case of $\hat{\mathcal{C}}[\mathcal{T}]$ the model category that it inherits as $\hat{\mathcal{C}}$ algebras over the localized model category on $\Pi[\mathcal{T}]$ is equivalent to that as a category of operators.

One can further show that if $\mathcal{G} \xrightarrow{v} \mathcal{H}$ is an equivalence of category of operators, then there exists adjoint pair of functors

$$\mathcal{G}[\mathcal{T}] \underset{v_*}{\overset{v^*}{\rightleftarrows}} \mathcal{C}[\mathcal{T}]$$

which is a Quillen equivalence with the localized model category structures.

If $\mathcal{C}$ is an E_∞ operad, then $\hat{\mathcal{C}} \to \hat{\mathcal{A}}$ is an equivalence, thus inducing a Quillen equivalence between $\hat{\mathcal{C}}[\mathcal{T}]$ and $\Gamma[\mathcal{T}]$. However from the previous theorem we know that $\hat{\mathcal{C}}[\mathcal{T}]$ is Quillen equivalent to $\Gamma[\mathcal{T}]$.

Thus we conclude that the category of E_∞-spaces is Quillen equivalent to the category of Γ-spaces.

Equivariant case. Given a finite group G, Shimakawa [7] defined Γ_G-spaces to model equivariant spectra. The category Γ_G is defined to be the category of finite pointed G-sets and maps are pointed G-maps. This is a G-enriched category.

A Γ_G-space is defined to be a G-functor from Γ_G to G-spaces. As before we can define a Γ_G-space X to be special if the map $X(S) \xrightarrow{\sim} \mathrm{Map}(S, X(\mathbf{1}))$ is a G-homotopy equivalence. Moreover he shows that the category of Γ_G-spaces is equivalent to the category of functors from Γ to G-spaces.

We define a category of equivariant operators to be a G-enriched category which has Π as a subcategory. Consider pointed operads in the category of G-spaces. Then using ideas similar to those in the non-equivariant case and above mentioned facts, we can show that the category of Γ_G-spaces is equivalent to the category of equivariant E_∞-spaces. This then allows us to define the units of equivariant spectra as before by constructing a very-special Γ-space out of special Γ-spaces extracting the units.

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