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Large Scale Stochastic Dynamics

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Introduction by the Organisers

"Large Scale Stochastic Dynamics" is at the crossroad of probability theory and statistical physics. One central theme of statistical physics is the emergent behavior resulting from the interaction of many identical components, the paradigm being a fluid or a gas. On the atomistic scale they consist of a huge number of identical molecules. Their motion is governed by Newton's equations of classical mechanics (ignoring quantum effects). The emergent description, valid only for particular initial states and on a sufficiently coarse space-time scale, are the compressible Navier-Stokes equations of fluid dynamics. Roland Dobrushin (1929–1995) and Frank Spitzer (1926–1992) had the vision that in the context of stochastic dynamics with many identical components the issue of emergent behavior is both mathematically challenging and important in modelling applications. The latter judgement turned out to be more than true. Stochastic algorithms, such as kinetic Monte Carlo, importance sampling, Monte Carlo Markov chains, Glauber dynamics, and others, are daily practice. Their mathematical vision has evolved over the past twenty years into a rich, multifaceted research program. Our workshop is like a snap-shot of the current activities. A partial list of topics reads

- hydrodynamic limit for multi-component stochastic lattice gases
- steady state large deviations for driven systems
- random walks in random environments
- directed polymer in a random potential, spin glasses
- statics and dynamics of interfaces
- glassy dynamics

We had 49 participants from 11 countries, mostly probabilists, but also experts from partial differential equations and statistical physics. They all enjoyed tremendously the unique and stimulating atmosphere at the Mathematische Forschungsinstitut Oberwolfach and hope to return some day.

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Stefano Olla,
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Workshop: Large Scale Stochastic Dynamics

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Abstracts

Competition interfaces in last-passage percolation and second class particles

Pablo A. Ferrari
(joint work with Leandro P. R. Pimentel)

The particle configuration in the one-dimensional nearest neighbor totally asymmetric simple exclusion process can be mapped into the growth interface of a last-passage percolation model in $\mathbb{Z}^2$ (Rost, 1981).

The macroscopic behavior of the density profile of the exclusion process is governed by the Burgers equation (Benassi and Fouque 1987, Rezakhanlou 1991). This corresponds to the "shape theorem" in last-passage percolation (Rost 1981, Seppäläinen 1998). An important property of the exclusion process is that the so called second class particles (that follow roughly the behavior of a perturbation of the system) are asymptotically governed by the characteristics of the Burgers equation. When there is only one characteristic, the second class particle follows it (Ferrari 1992, 1994, Rezakhanlou 1995, Seppäläinen 2001); when there are infinitely many, the particle chooses one of them at random to follow (Ferrari and Kipnis 1995). These results hold when the initial distribution is a product measure with densities $\lambda \in (0,1]$, $\rho \in [0,1)$, to the left and right of the origin respectively. The existence of infinitely many characteristics occur at points where the solution of the Burgers equation is a rarefaction front. The rescaled position of the second class particle converges almost surely to a random variable uniformly distributed in the interval $[1-2\lambda, 1-2\rho]$ as time goes to infinity (Mountford and Guiol 2004).

Pimentel (2004) shows that the competition interface between two growing clusters in *first-passage* percolation, when suitable rescaled, converges almost surely to a random direction with a so far unknown distribution. Motivated by the similarity of these phenomena, we investigate the relation between the second class particle and the competition interface in *last-passage* percolation. We conclude that one object can be linearly mapped into the other (as processes) realization by realization. Indeed, the difference of the coordinates of the competition interface at time t is exactly the position of the second class particle at that time (Ferrari and Pimentel 2004). The map permits to describe the distribution of the angle of the competition interface for last-passage percolation in the random conic region corresponding to the particle configuration distributed with the product measure with densities λ and ρ to the left and right of the origin, respectively. The result is as follows: Call φ_n the position in $\mathbb{Z}^2$ of the nth point of the competition interface (this is a right-up path starting at $(1,1)$). It holds almost surely,

$$\lim_{n\to\infty} \frac{\varphi_n}{|\varphi_n|} = e^{i\theta} \tag{1}$$

where $\theta \in [0, 90^o]$ is given by

$$\tan\theta \;=\; \begin{cases} \frac{\lambda\rho}{(1-\lambda)(1-\rho)} & \text{if } \rho \geq \lambda \\ \left(\frac{U-1}{U+1}\right)^2 & \text{if } \rho < \lambda \end{cases} \tag{2}$$

and U is a random variable uniformly distributed in $[1-2\lambda, 1-2\rho]$. So that there is a deterministic/random phase transition in the asymptotic inclination of the boundaries of the conic region (corresponding to the particle densities), at the point $\lambda = \rho$. More details in Ferrari and Pimentel (2004a). A similar phenomena has been observed by Derrida and Dickman (1991) in first passage percolation.

On the other hand, the asymptotic behavior of the competition interface is related to the *geodesics*, random paths maximizing the passage time. We show that each semi-infinite geodesic has an asymptotic direction and that two semi-infinite geodesics with the same direction must coalesce. This has also been proven by James Martin (2004). The approach follows Newman (1995) who proved analogous statements for first-passage percolation (see also Licea and Newman 1996 and Howard and Newman 2001). As a consequence, we get a law of large numbers for the competition interface in the positive quadrant $(\mathbb{Z}^+)^2$; this corresponds to $\lambda = 1$ and $\rho = 0$. In this restricted case, the method is an alternative to the proof of Mountford and Guiol.

Acknowledgements

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Lyapunov Exponent for the Parabolic Anderson Model in $\mathbf{R^d}$

M.C. Cranston and T. Mountford

We consider the asymptotic almost sure behavior of the solution of the equation

$$u(t,x) \quad = \quad u_0(x) + \frac{\kappa}{2}\int_0^t \Delta u(s,x)ds + \int_0^t u(s,x)dW_x(s)$$

where $\{W_x : x \in \mathbf{R}^d\}$ is a field of Brownian motions. In fact, we establish existence of the Lyapunov exponent, $\lambda(\kappa) = \lim_{t\to\infty} \frac{1}{t}\log u(t,x)$. We also show that $c_1\kappa^{\frac{1}{3}}\lambda(\kappa) \le c_2\kappa^{\frac{1}{5}}$ as $\kappa \searrow 0$ under the assumption that the correlation function of the background field $\{W_x : x \in \mathbf{R}^d\}$ is C^β for $1 < \beta \le 2$.

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Universality in Sherrington-Kirkpatrick's Spin Glass Model

Philippe Carmona
(joint work with Yueyun Hu)

We show that the limiting free energy in Sherrington-Kirkpatrick's Spin Glass Model does not depend on the environment. It does not depend on the specific realization of environment, nor does it depend on the law of the centered environment, up to some normalization constant.

The physical system is an N-spin configuration $\sigma = (\sigma_1, \dots, \sigma_N) \in \{-1, 1\}^N$. Each configuration σ is given a Boltzmann weight $e^{\frac{\beta}{\sqrt{N}} H_N(\sigma) + h \sum_i \sigma_i}$ where $\beta = \frac{1}{T} > 0$ is the inverse of the temperature, h is the intensity of the magnetic interaction, $H_N(\sigma)$ is the random Hamiltonian

$$H_N(\sigma) = H_N(\sigma, \xi) = \sum_{1 \le i,j \le N} \xi_{ij} \sigma_i \sigma_j \,,$$

and $(\xi_{ij})_{1 \le i,j \le N}$ is an i.i.d family of random variables, admitting order three moments, which we normalize:

$$\mathbf{E}[\xi] = 0 \,, \quad \mathbf{E}\left[\xi^2\right] = 1 \,, \quad \mathbf{E}\left[|\xi|^3\right] < +\infty \,. \tag{1}$$

The object of interest is the random Gibbs measure

$$\langle f(\sigma) \rangle = \frac{1}{Z_N} 2^{-N} \sum_{\sigma} f(\sigma) e^{\frac{\beta}{\sqrt{N}} H_N(\sigma,\xi) + h \sum_i \sigma_i} \,,$$

and in particular the partition function

$$Z_N = Z_N(\beta, \xi) = 2^{-N} \sum_{\sigma} e^{\frac{\beta}{\sqrt{N}} H_N(\sigma,\xi) + h \sum_i \sigma_i} \,.$$

We shall denote by $g = (g_{ij})_{1 \le i,j \le N}$ an environment of i.i.d Gaussian standard random variables ($\mathcal{N}(0,1)$).

Recently, F. Guerra and F.L. Toninelli [1, 2] gave a rigorous proof, at the mathematical level, of the convergence of free energy to a deterministic limit, in a Gaussian environment,

$$\frac{1}{N} \log Z_N(\beta, g) \to \alpha_\infty(\beta) \quad \text{a.s. and in average.}$$

Talagrand [4] then proved that one can replace the Gaussian environment by a Bernoulli environment η_{ij}, $\mathbb{P}(\eta_{ij} = \pm 1) = \frac{1}{2}$, and obtain the *same limit*: $\alpha_\infty(\beta)$. We shall generalize this result.

Theorem 1. *Assume the environment ξ satisfies* (1)*. Then,*

$$\frac{1}{N} \log Z_N(\beta, \xi) \to \alpha_\infty(\beta) \quad \text{a.s. and in average.}$$

Furthermore, the averages $\alpha_N(\beta, \xi) \overset{def}{=} \frac{1}{N}\mathbf{E}[\log Z_N(\beta, \xi)]$ satisfy

$$|\alpha_N(\beta, \xi) - \alpha_N(\beta, g)| \le 9\mathbf{E}\Big[|\xi|^3\Big] \frac{\beta^3}{\sqrt{N}}\,.$$

Therefore the limiting free energy $\alpha_\infty(\beta)$ does not depend on the environment, hence the *Universality* in the title of this paper : this independence from the particular disorder was already clear to Sherrington and Kirkpatrick [3] although they had no mathematical proof of this fact (Guerra and Toninelli [2] provided a physical proof in the case the environment is symmetric with a finite fourth moment).

Notice eventually that $\alpha_\infty(\beta)$ can be determined in a Gaussian framework where Talagrand [5] recently proved that it is the solution of G. Parisi's variational formula.

The universality property can be mechanically extended to the ground states, that is the supremum of the families of random variables:

$$S_N(\xi) = \sup_\sigma \sum_{1 \le i,j \le N} \sigma_i \sigma_j \xi_{ij} = \sqrt{N} \lim_{\beta \to +\infty} \frac{1}{\beta} \log Z_N(\beta, \xi)\,.$$

F. Guerra and F.L. Toninelli [1, 2] proved that $N^{-3/2} S_N(g)$ converges as and in average to a deterministic limit e_∞. Here is the generalization :

Theorem 2. *Assume the environment ξ satisfies* (1)*. Then,*

$$N^{-3/2} S_N(\xi) \to e_\infty \quad \text{a.s. and in average.}$$

Furthermore, the averages satisfy, for a universal constant $C > 0$,

$$N^{-3/2} |\mathbf{E}[S_N(\xi)] - \mathbf{E}[S_N(g)]| \le C\Big(1 + \mathbf{E}\Big[|\xi|^3\Big]\Big) N^{-1/6}\,.$$

We end this introduction by observing that we do not need the random variables ξ_{ij} to share the same distribution. They only need to be independent, to satisfy (1) and such that $\sup_{ij} \mathbf{E}\left[|\xi_{ij}|^3\right] < +\infty$.

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Kawasaki dynamics at critical temperature

Cedric Bernardin

We are interested in the variance of the occupation time of a site for an interacting particle system known as Kawasaki dynamics. Formally, the dynamics is a Markov process whose state space (or configuration space) is $\Omega = \{0,1\}^{\mathbb{Z}^d}$. A configuration η describes the occupation of sites in the sense that $\eta(x) = 1$ if there is a particle on site x and $\eta(x) = 0$ otherwise. This interacting particle system $(\eta_t)_{t\geq 0}$ consists of particles performing random walks over the sites of $\mathbb{Z}^d$ with jump rates depending on the interaction with nearby particles and satisfying the exclusion rule: there is at most one particle by site. Consequently, a particle sitting on site x jumps to site y with rate $c_{x,y}(\eta)$ only if the site y is not occupied by an another particle (otherwise the jump is canceled). Consider a finite range and translation invariant ferromagnetic potential $(J_A)_{A\subset\mathbb{Z}^d}$ and an inverse temperature $\beta > 0$. The formal Hamiltonian is given by

$$H(\eta) = \sum_{A\subset\mathbb{Z}^d} J_A \eta^A$$

where for a finite subset A of $\mathbb{Z}^d$

$$\eta^A = \prod_{z\in A} \eta(z)$$

The lattice gas will be considered under a shift invariant Gibbs state μ associated to the potential $(J_A)_A$ and temperature β^{-1}. It means that μ is a probability on Ω satisfying the following DLR equations

$$\mu\left(\{\eta(x) = 1 \mid \eta_{\{x\}^c}\}\right) = \left(1 + \exp\left[\beta \sum_{x\in A} J_{A\setminus\{x\}} \eta^{A\setminus\{x\}}\right]\right)^{-1}$$

where $\eta_{\{x\}^c}$ is an arbitrary outside configuration on $\{x\}^c$. Every Gibbs measure is reversible for the dynamics. Since the density of particles is conserved, these measures are labeled by the density ρ of particles.
Let us fix the density ρ and the inverse temperature β and consider the gas in thermal equilibrium under the Gibbs measure $\mu_{\rho,\beta}$. We will often omit the index β (or ρ) when the temperature and density will be fixed. The expectation with respect to $\mu_{\rho,\beta}$ is denoted by $< \cdot >$. The quantity of interest is the density-density correlation function

$$u_t(x) =< \eta_t(x)\eta_0(0) > -\rho^2 \tag{1}$$

The Fourier transform of u_t, also known as structure function in the physical literature, is defined by

$$\hat{u}_t(k) = \sum_{x\in\mathbb{Z}^d} e^{2i\pi k\cdot x} u_t(x) \tag{2}$$

The static compressibility $\chi = \chi(\rho,\beta)$ is given by

$$\chi = \sum_{x\in\mathbb{Z}^d} u_0(x) = \hat{u}_0(0)$$

This quantity is well defined for $\beta < \beta_c$ where β_c is the inverse critical temperature defined as the minimal β for which χ diverges.

In the case of general Kawasaki dynamics, little is known about the density-density correlation function. Nevertheless, we know that time correlations cannot decay exponentially because of the conservation law (cf. [2], p. 176)

$$u_t(0) =< \eta_t(0)\eta_0(0) > -\rho^2 \geq ct^{-d/2}$$

In fact, we are not directly interested in the density-density correlation function but in the time t variance σ_t^2 of the occupation time of a site. This last quantity is related to the density-density correlation function by the following formula

$$\sigma_t^2 = \mathbb{E}_{\rho,\beta}\left[\int_0^t (\eta_s(0)-\rho)ds\right]^2 = 2\int_0^t (t-s)u_s(0)ds \tag{3}$$

where $\mathbb{E}_{\rho,\beta}$ denotes the expectation with respect to the law of the process $(\eta_t)_{t\geq 0}$ starting from $\mu_{\rho,\beta}$.
Here is our main theorem.

Theorem 1. *If $\beta < \beta_c$, we have the following lower bounds for the Laplace transform of the time t variance σ_t^2:*

$$\liminf_{\lambda\to 0} n(\lambda)\int_0^{+\infty} e^{-\lambda t}\sigma_t^2 dt \quad \geq \quad \begin{cases} C_1\chi(\rho)^{3/2} & \text{for } d=1 \\ C_2\chi(\rho)^2 & \text{for } d=2 \\ C_d\displaystyle\int_{k\in[0,1]^d} \frac{\hat{u}_0^2(k)}{\sum_{j=1}^d \sin^2(\pi k_j)}dk & \text{for } d\geq 3 \end{cases}$$

where C_d is a positive constant independent of ρ, β and the normalization function $n(\lambda)$ satisfies

$$n(\lambda) = \begin{cases} \lambda^{3/2} & \text{for } d = 1 \\ \lambda^2(-\log\lambda)^{-1} & \text{for } d = 2 \\ \lambda^2 & \text{for } d \geq 3 \end{cases}$$

In a Tauberian sense, this theorem means that

$$\begin{cases} \liminf\{\ t^{-3/2}\sigma_t^2\} > \ 0 & \text{for } d = 1 \\ \liminf\ \{(t\log t)^{-1}\sigma_t^2\} > 0 & \text{for } d = 2 \\ \liminf\ \{t^{-1}\sigma_t^2\} > 0 & \text{for } d \geq 3 \end{cases}$$

Remark that the normalization function is the same as the function given by Kipnis in [1] for the simple symmetric exclusion process, a simple example of Kawasaki dynamics.

More interesting than the normalization function is the dependence on the compressibility (and on the structure function for the dimension $d \geq 3$) of the lower bounds. These bounds are valid for all temperature greater than the critical temperature defined as the temperature for which the static compressibility $\chi(\rho, \beta)$ becomes infinite. In dimension $d = 1, 2$, they are clearly divergent as $\beta \to \beta_c$. The case of the dimension $d \geq 3$ is more intricate. Indeed, it is conjectured ([2], pp.209-210) that for $\beta = \beta_c$, the static structure function is of the following form

$$\hat{u}_0(k) \sim_{k\to 0} \|k\|^{-2+\eta} \tag{4}$$

with the critical exponent $\eta = 0.03$ for $d = 3$ and $\eta = 0$ for $d \geq 4$. Assuming this fact, we obtain that the lower bounds remain finite as β goes to β_c if and only if $d \geq 7$. In fact, at critical temperature, assuming that (4) reflects the real behavior of the static structure function, we are able to give some lower bounds for the Laplace transform of the time t variance of the occupation time. This is the contain of the following theorem.

Theorem 2. *Assume that at inverse critical temperature β_c, the critical static structure function is of the form*

$$\hat{u}_0(k) \sim_{k\to 0} \|k\|^{-2+\eta}\Phi(k)$$

where Φ is a bounded continuous function such that $\Phi(0) > 0$ and η is the critical static exponent given by $\eta = 1/4$ for $d = 2$, $\eta = 0.03$ for $d = 3$ and $\eta = 0$ for $d \geq 4$.

Then we have for sufficiently small λ,

$$\int_0^\infty dt e^{-\lambda t}\sigma_t^2 dt \;\geq\; \begin{cases} C_2\lambda^{-38/15} & \text{for } d=2 \\ C_3\lambda^{-(11-4\eta)/(4-\eta)} & \text{for } d=3 \\ C_4\lambda^{-5/2} & \text{for } d=4 \\ C_5\lambda^{-9/4} & \text{for } d=5 \\ C_6\lambda^{-2}|\log(\lambda)| & \text{for } d=6 \\ C_d\lambda^{-2} & \text{for } d\geq 7 \end{cases}$$

where C_d *is a positive constant.*

This theorem is established assuming only a static assumption concerning the structure function. A more general hypothesis is that at critical temperature the dynamical structure function is of the form:

$$\hat{u}_t(k) = \|k\|^{-2+\eta}\Psi(\|k\|^z t)$$

Here, Ψ is some function vanishing at infinity such that $\Psi(0)=1$. The parameter z is known as the dynamical critical exponent. [2] presents an heuristic argument to argue that $z = 4-\eta$. Assuming this conjectural value of the dynamical critical exponent, one can establish equalities in the theorem just above. Hence, our results are consistent with this conjectural value of z.

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On the Realizability of Point Processes with Specified One and Two Particle Densities

Joel L. Lebowitz

(joint work with O. Costin, T. Kuna and E. R. Speer)

The microscopic structure of macroscopic systems, such as fluids, is best described by the joint n-particle densities $\rho_n(\mathbf{r}_1, ...\mathbf{r}_n)$, where the $\mathbf{r}_1, ..., \mathbf{r}_n$ are position vectors in $\mathbb{R}^d$. The most important of these are the one particle density $\rho_1(\mathbf{r}_1)$ and the pair density $\rho_2(\mathbf{r}_1, \mathbf{r}_2)$. For spatially homogeneous systems, the only ones we shall consider here, $\rho_1(\mathbf{r}_1) = \rho$ and $\rho_2(\mathbf{r}_1, \mathbf{r_2}) = \rho^2 g(\mathbf{r}_1 - \mathbf{r}_2)$ with $g(\mathbf{r}) = g(-\mathbf{r})$; thus $\rho g(\mathbf{r})$ is the density of particles at a displacement $\mathbf{r}$ from the position of a specified particle. For pure fluid phases $g(\mathbf{r}) \to 1$ as $|\mathbf{r}| \to \infty$.

The theory of classical equilibrium fluids is based in large part on finding good approximations to $g(\mathbf{r})$. This leads to an interesting and important question: given a density $\rho > 0$ and a candidate $g(\mathbf{r})$, obtained via some approximate theory or just invented for capturing a certain behavior, do these arise from some actual distribution of particles in d-dimensional space? That is, does there exist a *point process*—a probability measure on sets of points in $\mathbb{R}^d$—with density ρ and

with pair density corresponding to the proposed $g(\mathbf{r})$? More generally, what are necessary and sufficient conditions on ρ and $g(\mathbf{r})$ for this to be true? For more background on the long history of this *realizability problem*, see [1] and references therein.

One may also consider the realizability problem on the lattice $\mathbb{Z}^d$. We assume the exclusion condition that each lattice site contains at most one particle. Now ρ is the probability that there is a particle on any fixed site and $\rho^2 g(\mathbf{r}_1 - \mathbf{r}_2)$ the probability that there are particles at $\mathbf{r}_1$ and $\mathbf{r}_2$; the exclusion condition corresponds formally to $g(\mathbf{0}) = 0$. In what follows we will frequently state results for the continuum case, making some comment on the translation to the lattice when that is more complicated than simply replacing integrals by sums.

There are some obvious conditions which ρ and $g(\mathbf{r})$ must satisfy [1]:

$$\rho > 0 \quad \text{and} \quad g(\mathbf{r}) \geq 0; \tag{1}$$

$$\hat{S}(\mathbf{k}) \equiv \rho + \rho^2 \int_{\mathbb{R}^d} e^{i\mathbf{k}\cdot\mathbf{r}} \, [g(\mathbf{r}) - 1] \, d\mathbf{r} \geq 0; \tag{2}$$

$$V_\Lambda \equiv \rho|\Lambda| + \rho^2 \iint_\Lambda [g(\mathbf{r}_1 - \mathbf{r}_2) - 1] \, d\mathbf{r}_1 d\mathbf{r}_2 \geq \theta(1-\theta). \tag{3}$$

In (3), V_Λ is the variance of the number N_Λ of particles in a region $\Lambda \subset \mathbb{R}^d$ and θ is the fractional part of the mean $\rho|\Lambda|$ of N_Λ: $\rho|\Lambda| = k + \theta$ with $k = 0, 1, \ldots$ and $0 \leq \theta < 1$. These conditions may, however, be only the tip of the iceberg. Necessary and sufficient conditions for realizability can be given in the form of an uncountable number of positivity conditions on ρ and $g(\mathbf{r})$ [2]: Given any functions $f_2(\mathbf{r}_1, \mathbf{r}_2)$ and $f_1(\mathbf{r})$ such that, for any n points $\mathbf{r}_1, \ldots, \mathbf{r}_n$, $\sum_{i \neq j} f_2(\mathbf{r}_i, \mathbf{r}_j) + \sum_i f_1(\mathbf{r}_i) + 1 \geq 0$, we must have

$$\rho^2 \iint_\Lambda g(\mathbf{r}_1, \mathbf{r}_2) f_2(\mathbf{r}_1, \mathbf{r}_2) d\mathbf{r}_1 d\mathbf{r}_2 + \rho \int_\Lambda f_1(\mathbf{r}) d\mathbf{r} + 1 \geq 0 \tag{4}$$

for all $\Lambda \subset \mathbb{R}^d$. Conditions (1)–(3) may be obtained from (4) by appropriate choices of f_1 and f_2, but clearly represent just a very small subset of the latter equations. Nevertheless, we must confess that we do not have at the present time any example which satisfies (1)–(3) and not (4) in $\mathbb{R}^d$. We do however have such an example on the lattice $\mathbb{Z}$, which we shall now describe. (The conditions for $\mathbb{Z}^d$ corresponding to (1)–(4) are obtained in the obvious way.)

Suppose that for $r \in \mathbb{Z}$ we define

$$g(r) = \begin{cases} 0, & \text{for } r = 0, \pm 1, \\ 1, & \text{for } |r| \geq 2. \end{cases} \tag{5}$$

This g describes a model with on-site and nearest neighbor exclusion and with no correlation, on the pair level, for sites separated by two or more lattice spacings. It is then easy to check that (1)–(3) are satisfied for for $\rho \leq 1/3$, but a simple argument shows that in fact there is a critical density $\rho_c < 1/3$ such that the process is not realizable for $\rho > \rho_c$ (a numerical calculation indicates that $\rho_c < 0.3287$). On the other hand it is easy to construct explicitly a realization of (5)

for $\rho \le 1/4$: Start with a Bernoulli measure on $\mathbb{Z}$ with density λ and then remove any particle from an occupied site x if and only if site $x+1$ is also occupied. This will yield a translation invariant state with density $\rho = \lambda(1-\lambda) \le 1/4$ and with $g(r)$ given by (5). Whether (5) is realizable for any $\rho > 1/4$ (i.e., whether or not $\rho_c > 1/4$) is a complete mystery to us at present. We can show, however, that the state constructed above corresponds to a *renewal* process, for which the sequence of interparticle distances is Markovian, and that such a renewal process with $g(\mathbf{r})$ given by (5) cannot exist for $\rho > 1/4$. (The corresponding result on $\mathbb{R}$ is discussed in [1], where it is shown that a renewal process exists with

$$g(r) = \begin{cases} 0, & \text{for } |r| \le 1, \\ 1, & \text{for } |r| > 1. \end{cases} \tag{6}$$

if and only if $\rho \le 1/e$. Here the maximum ρ consistent with (1)–(3) is 1/2.)

Let us now state a general theorem about realizability of a given ρ and $g(\mathbf{r})$, $\mathbf{r} \in \mathbb{R}^d$.

Theorem: *Let*

$$g(\mathbf{r}) = \begin{cases} 0, & |\mathbf{r}| \le D, \\ \exp(-\varphi(\mathbf{r})), & |\mathbf{r}| > D, \end{cases}$$

where for some $\Phi \ge 0$,

$$\sum_{i=1}^{n} \varphi(\mathbf{r}_i) \ge -2\Phi \quad \textit{whenever} \quad |\mathbf{r}_i - \mathbf{r}_j| \ge D, \quad 1 \le i < j \le n. \tag{7}$$

Then ρ *and* $g(\mathbf{r})$ *are realizable whenever*

$$\rho \le \left(e^{1+2\Phi} \int_{\mathbb{R}^d} |g(\mathbf{r}) - 1| \, d\mathbf{r} \right)^{-1}. \tag{8}$$

Note that condition (7) is satisfied for any $D \ge 0$ (and $\Phi = 0$) if $\varphi(\mathbf{r}) \ge 0$ for all r, and for $D > 0$ (i.e., with a hard core) if $\varphi(\mathbf{r})$ decays faster than $|\mathbf{r}|^{-(d+\epsilon)}$ for $|r| \to \infty$ [3]. A similar theorem holds in the lattice case $\mathbb{Z}^d$; the integral has to be replaced by a sum and the hard core condition is automatically satisfied because at most one particle can occupy any site. The theorem is a generalization of a result of R. V. Ambartzumian and H. S. Sukiasian (A-S) [4], who considered only the case $\varphi \ge 0$ ($g \le 1$). For the example (5) the theorem gives existence only for $\rho \le (3e)^{-1}$, so it is clearly not optimal. Readers with a statistical mechanics background will recognize the right hand side of (8) as the Ruelle-Penrose lower bound for the radius of convergence of the fugacity expansion for an equilibrium system with pair potential φ given by $g(\mathbf{r}) = e^{-\varphi(\mathbf{r})}$ [3].

The construction by A-S of the point process corresponding to ρ and $g(\mathbf{r})$, which we follow, does not in general yield a Gibbs measure. The existence of a Gibbs measure with a decaying pair potential which realizes a given ρ and $g(\mathbf{r})$ was proven by L. Koralov for lattice gases when ρ is sufficiently small and $\sum_{\mathbf{r}} |g(\mathbf{r}) - 1| < 2$ [5]. In general, if any measure realizes ρ and $g(\mathbf{r})$ then we can ask for the measure which maximizes the entropy subject to the constraint of the given ρ and g; this

should "formally" be a Gibbs measure with some pair potential [6]. There is no guarantee, however, that this potential would have any good decay property.

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Stochastic lattice gases with dynamical constraints and glassy dynamics

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(joint work with G.Biroli and D.S.Fisher)

I report results obtained in collaboration with G.Biroli and D.S.Fisher [9, 11, 10, 12] on some *kinetically constrained lattice gases* introduced in physical literature to study glassy dynamics (see for review [6]). These are stochastic lattice gases with hard core constraint and dynamics given by a continuous time Markov process which consists of a sequence of particle jumps. The rates are chosen in order to impose additional constraints (beyond hard core) to the allowed moves, namely the jump from x to y can have zero rate even if y is empty. This is an attempt to mimic the geometric constraints on the moves of a molecule in a dense liquid due to neighboring molecules. These constraints could indeed cooperatively produce a dramatic slowing down of dynamics at a finite density leading to liquid-glass transition, a phenomenon which still remains to be understood.

I will focus on a class of models (KA) which were introduced in [4]. Let $\Lambda \subset \mathbb{Z}^d$ and $m \in \{1, \dots, 2d-1\}$, a configuration is defined by giving for each $x \in \Lambda$ the occupation number $\eta_x \in \{0, 1\}$. The rate $c_{x,y}(\eta)$ for the jump from x to y is equal to one if: i) y is nearest neighbor to x ii) *and* y is empty iii) *and* no more than m nearest neighbors of x are occupied iv) *and* no more than m nearest neighbors of y different from x are occupied. If one or more of the above constraints is not satisfied the jump is not allowed, namely $c_{x,y}(\eta) = 0$. Note that for $m = 0$ we would recover the symmetric simple exclusion process (SSEP).

On a finite volume dynamics preserves the number of particles and the rates satisfy the detailed balance w.r.t. the uniform measure $\nu_{\Lambda,N}$ on the hyperplane

$\Omega_{\Lambda,N}$ with N particles. Therefore the generator is reversible w.r.t. $\nu_{\Lambda,N}$ and these are stationary measures. However, due to the degeneracy of rates, they are not the unique invariant measures. Indeed there exist configurations which do not evolve under dynamics and all measures concentrated on such configurations are invariant too. For instance, if $d = 2$ and $m = 1$, frozen configurations can be constructed by noticing that all the particles belonging to two fully occupied consecutive rows are forever blocked. Therefore the process is not ergodic on $\Omega_{\Lambda,N}$, which decomposes into disconnected irreducible components. On the other hand on infinite volume, i.e. for $\Lambda = \mathbb{Z}^d$, the rates satisfy detailed balance with respect to Bernoulli product measure μ_ρ at any $\rho \in [0, 1]$. A key question is whether the process is ergodic in $L_2(\mu_\rho)$, i.e. if zero is a simple eigenvalue of the generator. In this case spectral theorem would guarantee that time averages of local quantities converge in the long time limit to averages over the equilibrium ensemble μ_ρ. If the process is ergodic for $\rho \leq \rho_c$ and not ergodic for $\rho > \rho_c$ we say that and ergodic/non-ergodic transition occurs at ρ_c. From numerical simulations [4] it has been conjectured that such a transition occurs at $\rho_c \simeq 0.881$ for the model on the cubic lattice with $m = 3$. Analogous ergodicity breaking transitions have been found in different approximate theories for supercooled liquids (e.g. mode coupling theory and random first order scenario) and have been advocated as the cause of the slowing down of dynamics and the onset of liquid-glass transition.

Another relevant issue is the asymptotic displacement of a tagged particle. Indeed, for many supercooled liquids, a striking decrease of the mean square displacement of the tracer occurs near the glass transition [2]. For KA model, as long as the process is ergodic, a central limit theorem holds [3] and the position of the tagged particle in equilibrium converges, under diffusive space-time rescaling, to Brownian motion with a density dependent self-diffusion coefficient $D_S(\rho)$. This is expressed by a variational formula as in [7]. However, due to the degeneracy of rates, one cannot extend the strict positiveness for such formula proven for SSEP in [7]. Furthermore, from simulations it has been conjectured that a diffusive/sub–diffusive transition takes place at a critical density $\bar\rho$, with $D_S = 0$ for $\rho > \bar\rho$. Indeed, on the cubic lattice with $m = 3$, data were found to fit well with a transition at $\bar\rho \simeq 0.881$ and $D_S \propto (\rho - \bar\rho)^{3.1}$ for $\rho \to \bar\rho$ from below [4].

Our first result [9, 11, 12] is that, for any value of the spatial dimension d and the parameter m, an ergodic/non-ergodic transition does not occur at a non-trivial density $\rho \in (0, 1)$. More precisely, for $m \leq d-1$ the process on the infinite volume is not ergodic at any $\rho \in (0, 1]$, while for $m > d-1$ ergodicity holds at any $\rho \in [0, 1)$. The first statement can be easily established by noticing that, if $m \leq d - 1$, any fully occupied hypercube of particles can never be broken, therefore at any ρ there exists a finite fraction of frozen particles. In the following we will focus on the models with $m > d$, which are the interesting ones for the physical problem. In these cases, through the construction of allowed paths among configurations, one can identify an irreducible component which has unit probability w.r.t. any μ_ρ with $\rho < 1$. This, together with the product form of Bernoulli measure, allows to establish ergodicity. Therefore, on any finite lattice the process is not ergodic

and on the infinite lattice it is ergodic at any finite ρ. Furthermore, we derive a crossover size above which the configuration space of the finite system is dominated by a single ergodic component and below which finite size effects are important. More precisely, fix a density $\rho < 1$ and a finite cube $\Lambda \subset \mathbb{Z}^d$ of linear size ℓ and let M_ℓ be the irreducible component which is maximal w.r.t. μ_ρ. From the above result we have $\lim_{\ell\to\infty} \mu_\rho(M_\ell) = 1$. On the other hand, since at high density blocked configurations are more important, $\lim_{\rho\to 1} \mu_\rho(M_\ell) = 0$. The simultaneous limit $\ell \to \infty$ and $\rho \to 1$ depends on the relationship between ℓ and ρ: $\mu_\rho(M_\ell) \to 1$ ($\mu_\rho(M_\ell) \to 0$) for $\ell \to \infty$ faster (slower) than $\Xi(\rho)_{d,m}$. This crossover length scales as $\Xi(\rho)_{m,d} = \exp^{\circ s}[c(m,d)/(1-\rho)^{1/(m-d+1)}]$, with $\exp^{\circ s}$ the exponential iterated s times and $c(m,d)$ constants in ρ. Only in the two-dimensional case with $m = 2$ we determine the exact value of the constant, $c(2,2) = \pi^2/18$. The rapid divergence of Ξ when $\rho \to 1$ accounts for the apparent ergodic/non-ergodic transition detected in simulations.

We have then studied the asymptotic displacement of the tagged particle. For the two-dimensional case with $m = 2$ we prove that $D_S > 0$ at any $\rho \in (0,1)$ [10]. The key ingredient is the identification of particular clusters of vacancies which allow the displacement of the tagged particle when they are near it and that can move using the vacancies they typically find around. These regions, which play an analogous role as simple vacancies for SSEP, have a density dependent size and distance which are both finite at any finite ρ and diverge faster than any power law when $\rho \to 1$. This leads to a lower bound for D_S vanishing faster than any inverse power law. We conjecture that the same result holds in any dimension, namely a diffusive/non-diffusive transition does not occur at any density.[1] Furthermore, we expect that the mechanism we identify not only provides a lower bound but gives the correct scaling for D_S when $\rho \to 1$, that is $D_S \simeq 1/\Xi^d$ [9, 11, 12]. If typical relaxation times scale as $1/D_S$, this would correspond to a super-Arrhenius relaxation analogous to the one detected for supercooled liquids [2]. By performing numerical simulations for the two dimensional case with $m = 2$ we successfully checked that $\lim_{\rho\to 1}(1-\rho)\ln D_S(\rho) = 2c(2,2)$ while the validity of the predicted scaling for the three-dimensional case with $m = 3$ has been checked in [5].

Therefore, neither an ergodic/non-ergodic nor a diffusive/non-diffusive transition takes place for KA models [2] . However, at high density relaxation occurs only through the cooperative motion of special regions whose typical size and distance diverge very rapidly for $\rho \to 1$. This accounts for the dramatic slowing down and the glassy character of dynamics. Beyond the natural issue of establishing the connection (if any) of such a mechanism with the one occurring in real systems, several interesting questions remain open. A first issue concerns the macroscopic behaviour, i.e. the evolution of density profiles. For non–degenerate rates, the

[1] The lacking step in the proof is the extension to more than two dimensions of the diffusivity result for a particular problem of random walk on random environment. This, which we believe to hold on physical and numerical grounds, is needed to establish that the special regions diffuse.

[2] In relation with the physical problem, it is relevant to study the mean field version of the models obtained by considering the same dynamics on a Bethe lattice. In this case we find a different scenario: an ergodic/non-ergodic transition occurs at a finite density [9, 11, 12].

hydrodynamical limit [8] states that the coarse-grained density profile evolves under a diffusion equation $\partial_t \rho = \nabla (D(\rho) \nabla \rho)$. Although we conjecture that on large enough length and time scales hydrodynamics will be valid at any $\rho < 1$ for ergodic KA models, this seems to be hard to prove due to the degeneracy of rates. If the conjecture is correct, another issue related to both experiments and simulations [1] of glass-forming liquids, is the characteristic length and time scales beyond which hydrodynamic behavior sets in and how these increase with density. Other open questions are related to relaxation towards equilibrium. A natural quantity to consider is the dynamical structure factor $F(k,t)$, i.e. the Fourier transform of the density-density correlation function at non-zero wave vector. For the simple symmetric exclusion, $F(k,t)$ decays exponentially at long times [8]. For KA models numerical simulations suggest that relaxation could be slower than exponential (at least above some critical density). The decay is usually fitted by a *stretched exponential* $F(k,t) \sim \exp[-(t/\tau(k))^{\beta}]$, with $\beta < 1$ [5]. This is again reminiscent to the anomalous decay occurring at sufficiently low temperature in many glass-forming liquids [2] and understanding whether and why this occurs for kinetically constrained models could be very useful.

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Local Functional of Phase Separation Lines

Dmitry Ioffe

1. Introduction

For a large class of two-dimensional low temperature models of statistical mechanics in the phase co-existence regime the probabilistic structure of phase separation lines falls in the general framework of the thermodynamic formalism of sub-shifts over countable alphabets. In this way statistical weights of phase boundaries are reinterpreted in terms of actions of Ruelle type operators with Lipschitz continuous potentials and a local limit characterization of various observables of these phase boundaries follows from from a spectral analysis of the latter. Recent advances include a comprehensive implementation of such approach in the case of the nearest neighbour Ising model model at all sub-critical temperatures.

Below I shall give examples of results which could be obtained along these lines and try to stipulate a rather general features of the statistical weights on which the theory relies.

2. Ising model in a strip

A canonical example of phase separation lines is given in the framework of the nearest neighbour two dimensional Ising model below the critical temperature T_c. Let

$$\mathbf{S}^*_N = (1/2, 1/2) + [0, \dots, N-1] \times \mathbb{Z} \subset \mathbb{Z}^{2*}$$

be an infinite dual strip of width N. A spin configuration σ on $\mathbf{S}^*_N$ is an element of $\{-1, 1\}^{\mathbf{S}^*_N}$. Consider the so called Dobrushin's boundary conditions

$$\bar\sigma(y_1, y_2) \ = \mathbb{1}_{\{y_2>0\}} - \mathbb{1}_{\{y_2<0\}}.$$

Every $\sigma \in \{-1, 1\}^{\mathbf{S}^*_N}$ gives rise to the set of microscopic boundaries between regions occupied by spins of different signs. Using a "rounding of corners" procedure [DKS] contours can be represented as either open or closed self-avoiding curves in $\mathbb{R}^2$. In fact to each σ there corresponds precisely one open crossing contour $\gamma = \gamma[\sigma]$ with end points at 0 and $(N, 0)$

The contour γ models the $\vec e_1$-oriented microscopic interface between co-existing phases of the nearest neighbour Ising model at the inverse temperature $\beta > \beta_c$, where $\beta_c = 1/T_c$ is the phase transition threshold. The statistics $\mathbb{P}^{\pm}_{N,\beta}$ of γ is read from the Ising Gibbs distribution on $\{-1, +1\}^{\mathbf{S}^*_N}$ under Dobrushin boundary conditions $\bar\sigma$ defined above.

3. Large scale structure of γ

Our first result is an invariance principle for the interface γ under the diffusive scaling. Similar results were previously obtained at very low temperatures using the method of cluster expansions [DH, Ga, Hi]. Let γ_N be the scaling of γ by

$1/N$ in the horizontal (role of time) direction and by $1/\sqrt{N}$ in the vertical (role of space) direction.

Theorem 3.1. *For every $\beta > \beta_c$ the distribution of γ_N under $\mathbb{P}^{\pm}_{N,\beta}$ weakly converges on $C_0[0,1]$ to the distribution of $\sqrt{\kappa}_\beta B(\cdot)$, where $B(\cdot)$ is the standard Brownian bridge and κ_β is the curvature of the boundary β- equilibrium crystal shape in the horizontal direction.*

The theorem has been proved in [GI], actually for Ising interfaces stretched in arbitrary directions.

4. CLT for local functionals of γ

We continue to work with the Ising model on a strip with Dobrushin boundary conditions. Each crossing contour $\gamma = \gamma[\sigma]$ splits $\mathbf{S}^*_N$ into the upper (positive b.c.) part and the lower (negative b.c) part. Let $\chi_x(\sigma) = \pm 1$ be the indicator that xb lies in the $\pm$ part. A general approach to local anti-symmetric functionals of γ has been developed in [HIK]. Here I give two examples:

4.1. **Excess magnetization.** For $x \in \mathbf{S}^*_N$ let $< \sigma(x)|\gamma >_{N,\beta}$ be the conditional expectation given that γ is the crossing contour of σ. Let also $m^*_N(x)$ be the expectation of $\sigma(x)$ under the pure + boundary conditions on $\partial\mathbf{S}^*_N$. Consider the quantity,

$$\Delta_N(\sigma) = \sum_{x\in\mathbf{S}^*_N} \left(< \sigma(x)|\gamma >_{N,\beta} - \chi_x(\sigma) m^*_N(x)\right).$$

Theorem 4.1. *The random variable $\Delta_N/\sqrt{N}$ is asymptotically normal.*

4.2. **Glauber dynamics.** Let $\mathcal{L}$ be a generator of Glauber dynamics on $\{-1,1\}^{\mathbf{S}^*_N}$ reversible with respect to $\mathbb{P}^{\pm}_{N,\beta}$. Define the signed area under the (random) interface γ as $\mathfrak{A}_N(\gamma) = \mathfrak{A}_N(\sigma)$.

Theorem 4.2. *The random variable $\mathcal{L}\mathfrak{A}_N(\sigma)/\sqrt{N}$ is asymptotically normal.*

5. Properties of statistical weights

The probability distribution $\mathbb{P}^{\pm}_{N,\beta}$ is based on statistical weights $q_\beta(\gamma)$ on phase separation lines γ. A simplifying feature of the two dimensional Ising model is that these weights can be related by duality to the statistical weights which appear in the high temperature expansion [CIV1, CIV2, PV]. On the other hand, the Ornstein-Zernike theory developed in [CIV1] relies on the following four properties of $q_\beta(\cdot)$:

Strict exponential decay of the two-point function: There exists $C_1 < \infty$ such that, for all $x \in \mathbb{Z}^d \setminus \{0\}$,

$$\sum_{\lambda:\, 0\to x} q_\beta(\lambda) < C_1\, e^{-|x|/C_1}.$$

Finite energy condition: For any pair of compatible paths λ and η define the conditional weight

$$q(\lambda \,|\, \eta) = q(\lambda \amalg \eta)/q(\eta)$$

where $\lambda \amalg \eta$ denotes the concatenation of λ and η. Then there exists a universal finite constant $C_2 < \infty$ such that the conditional weights are controlled in terms of path sizes $|\lambda|$ as:

$$q(\lambda \,|\, \eta) \,<\, e^{-C_2|\lambda|}\,.$$

BK-type splitting property: There exists $C_3 < \infty$, such that, for all $x, y \in \mathbb{Z}^d \setminus \{0\}$ with $x \neq y$,

$$\sum_{\lambda:\, 0\to x\to y} q(\lambda) < C_3 \sum_{\lambda:\, 0\to x} q(\lambda) \sum_{\lambda:\, x\to y} q(\lambda)\,.$$

Exponential mixing : There exists $C_4 < \infty$ and $\theta \in (0,1)$ such that, for any four paths λ, η, γ_1 and γ_2, with $\lambda \amalg \eta \amalg \gamma_1$ and $\lambda \amalg \eta \amalg \gamma_2$ both admissible,

$$\frac{q(\lambda \,|\, \eta \amalg \gamma_1)}{q(\lambda \,|\, \eta \amalg \gamma_2)} < \exp\{C_4 \sum_{\substack{x\in\lambda \\ y\in\gamma_1\cup\gamma_2}} \theta^{|x-y|}\}\,.$$

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Current fluctuations in non-equilibrium diffusive systems

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(joint work with Bernard Derrida)

Understanding the fluctuations of the steady state current through a system in contact with two (or more) heat or particle reservoirs is one of the natural questions arising in non-equilibrium physics [6, 5, 11]. We report on the paper [4] where a functional for the current large deviations was proposed.

We consider a one dimensional diffusive open system of length N (with N large) in contact, at its two ends, with two reservoirs of particles at densities ρ_a and ρ_b. In the bulk, the system evolves under some conservative stochastic dynamics and, at the boundaries, particles are created or annihilated to match the densities of the reservoirs. Let Q_t be the integrated current up to time t, i.e. the number of particles which went through the system during time t. The large deviation functional G associated to Q_t can be written as

$$\lim_{t\to\infty}\lim_{N\to\infty}\frac{1}{tN}\log\left\langle \frac{Q_{tN^2}}{tN}\sim q\right\rangle = -G(q,\rho_a,\rho_b)\,,$$

where we used a diffusive scaling.

First of all, we would like to provide a heuristic derivation of the functional G by thermodynamical considerations. This is based on the assumption that G satisfies the following additivity principle. If we split a system of macroscopic unit length into two parts of lengths x and $1-x$, then we assume that

$$G(q,\rho_a,\rho_b) = \max_{\rho}\left\{\frac{G(qx,\rho_a,\rho)}{x}+\frac{G(q(1-x),\rho,\rho_b)}{1-x}\right\}. \tag{1}$$

The idea of factorizing a partition function as the product of partition functions in sub-domains may look innocuous from the perspective of equilibrium statistical mechanics, however such property is not true in general. In particular the density large deviation functional associated to the steady is not additive (see [7]). The assumption behind (1) is that conditioned on the current deviation q, the system remains essentially close to a fixed profile over a long period of time. This profile is chosen in order to facilitate the current deviations and is a solution of the variational principle (5).

The next step is to introduce the two macroscopic parameters from which the functional can be recovered. Suppose that for $\rho_a=\rho+\Delta\rho$ and $\rho_b=\rho$ with $\Delta\rho$ small, the diffusion coefficient $D(\rho)$ satisfies Fick's law

$$\frac{\langle Q_t\rangle}{t} = \frac{1}{N}D(\rho)\;\Delta\rho\,. \tag{2}$$

Suppose that for $\rho_a=\rho_b=\rho$ the variance of the current can expressed in terms of the conductivity $\sigma(\rho)$

$$\frac{\langle Q_t^2\rangle}{t}\sim\frac{1}{N}\sigma(\rho)\,. \tag{3}$$

If we keep dividing the system into smaller and smaller pieces and assume that for a piece of small (macroscopic) size Δx (i.e. of $N\Delta x$ sites), the behavior is essentially gaussian

$$\frac{1}{\Delta x} G(q\Delta x, \rho, \rho + \Delta\rho) \simeq - \frac{\left[q\Delta x + D(\rho)\,\Delta\rho\right]^2}{2\sigma(\rho)\Delta x}\,, \tag{4}$$

thus one finds a variational form for G

$$G(q, \rho_a, \rho_b) = \min_{\rho(x)} \left[\int_0^1 \frac{\left[q + D(\rho(x))\rho'(x)\right]^2}{2\sigma(\rho(x))}\, dx \right], \tag{5}$$

where the minimum is over all the functions $\rho(x)$ with boundary conditions $\rho(0) = \rho_a$ and $\rho(1) = \rho_b$.

For systems of purely classical interacting particles [11] in contact with two reservoirs the theory is, to our knowledge, less developed. However, one may wonder whether similar additivity principle pertains in this context and whether some universal features can be predicted by the formula (5). Notice also that the additivity principle can be used to study more complex diffusive networks including loops and that the expression (5) satisfies the Gallavotti-Cohen symmetry [8].

A mathematical justification of (5) can be achieved for some stochastic dynamics by using the formalism of hydrodynamic large deviations (see [10, 12]). In particular for the SSEP, the variational principle (5) can be recovered and the additivity principle justified a posteriori. In general only the space/time version of (5) is valid, i.e. that the exponential cost of observing a current profile $q(x,t)$ over a period $[0,T]$ and a density profile $\rho(x,t)$ such that $\partial_t \rho = -\nabla_x q$ is given by

$$G(q, \rho) = \int_0^T dt \int_0^1 dx\, \frac{\left[q(x,t) + D(\rho(x,t))\rho'(x,t)\right]^2}{2\sigma(\rho(x,t))}\,. \tag{6}$$

Observing a mean current $\hat{q} = \frac{1}{T}\int_0^T dt \int_0^1 dx\, q(x,t)$ amounts to integrate (6) over time and this cannot always be reduced to (5). This was understood by Bertini, De Sole, Gabrielli, Jona–Lasinio, Landim [3] who introduced a multi-dimensional extension of (6) and exhibit a counter-example of (5) for a specific choice of D and σ.

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The directed polymer in random environment is diffusive at weak disorder

Francis Comets
(joint work with Nobuo Yoshida)

In this popular model, the polymer is a long chain of size n which is stretches in one particular direction of $\mathbb{Z}^{d+1}$, and is modelled as a nearest neighbor path in $\mathbb{Z}^d$. The environment is given by independent identically distributed random variables $\eta(n,x)$, with all finite exponential moments. The polymer is attracted by large positive values of the environment, and repelled by large negative ones. All these considerations lead to the **polymer measure**:

$$\mu_n(d\omega) = Z_n^{-1} \exp\{\beta \sum_{t=1}^{n} \eta(t,\omega_t)\}\ P(d\omega)$$

with P the distribution of the simple random walk on the integer lattice $\mathbb{Z}^d$, $\beta \in [0,\infty)$ a "temperature inverse" prescribing how strongly the polymer path ω interacts with the medium, and Z_n the normalizing constant making μ_n a probability measure on the path space. Note the measure μ_n depends on n, β and on the environment η.

The issue is to study the asymptotics of the polymer ω as $n \to \infty$ under the measure μ_n, for typical realization of the environment; In particular, one is interested in the exponent $\xi \in [1/2, 1)$ such that

$$|\omega_n| \text{ is of order } n^{\xi}$$

as $n \to \infty$. The ground states (limit as $\beta \to \infty$) of the model is the oriented last passage percolation model, for which it is known, for $d = 1$ and very few specific cases for the distribution of η, that $\xi = 2/3$ ([7], [10]): the path is superdiffusive, in contrast with the simple random walk which is diffusive (corresponding to $\xi = 1/2$). Another quantity of interest is the exponent $\chi \in [0, 1/2]$ for the fluctuations of the normalizing constant, i.e. such that

$$\ln Z_n - a_n \text{ is of order } n^{\chi} \text{ for some constant } a_n$$

as $n \to \infty$. A number of predictions, conjectures and numerical estimates can be found in the physical literature [5], [8], on the values on such exponents, and relations between them. In particular, the scaling relation

$$\chi = 2\xi - 1$$

is believed to hold. In fact, for $d = 1$, $\beta = \infty$ and a few specific cases for the distribution of η, χ, it is proven that $\chi = 1/3$, i.e., the scaling relation holds in this case.

Weak and Strong disorder: Bolthausen [1] considered the positive martingale $Z_n/\mathbf{E}Z_n$ converges almost surely, and that a dychotomy takes place:

$$\lim_n Z_n/\mathbf{E}Z_n \quad \begin{cases} > 0 & \text{a.s.} \\ \text{or} & \\ = 0 & \text{a.s.} \end{cases}$$

A natural manner for measuring the disorder due to the random environment, is to call **weak disorder** the first case, and **strong disorder** the second one. Note that weak disorder can be defined as the region where $\chi = 0$ and $a_n = n \ln \mathbf{E}[\exp \beta\eta(t,x)]$. The terminology is justified, since the former case happens in large enough dimension for small β (including $\beta = 0$), and the latter case for large β and general unbounded environment. A more precise statement is given in the next theorem. But first, the reader should be warned that weak disorder is not equivalent to small β: Indeed, in dimension $d = 1$ or 2, strong disorder holds for all β non-zero ([2], [3]).

Theorem: ([6] and [1]) Assume $d \geq 3$ and β small enough so that

$$\frac{\mathbf{E}(\exp 2\beta\eta(t,x))}{(\mathbf{E}\exp \beta\eta(t,x))^2} \times P(\exists n > 0 : \omega_n = 0) < 1 \ .$$

Then, weak disorder holds and, for almost every realization of the environment, the rescaled path $\hat{\omega}^{(n)}$, $t \to \hat{\omega}_t^{(n)} := n^{-1/2}\omega_{nt}$, converges in law under μ_n to the Brownian motion with diffusion matrix $d^{-1}I_d$.

This result was quite a surprise for the scientific community who did not expect that diffusivity could take place!

The second moment method was used to derive the theorem. The assumption on β means that the above martingale is bounded in L^2, and it is far from being necessary. To improve on it, it was necessary to wait for fifteen year: In the following *criterium*, the crucial quantity is

$$I_n = \mu_{n-1}^{\otimes 2}(\omega_n = \tilde{\omega}_n) \ ,$$

i.e., the probability for two polymers ω and $\tilde{\omega}$ independently sampled from the polymer measure, to meet at time n.

Theorem: ([2] for the gaussian case, [3] for the general case). For non-zero β it holds

$$\Big\{ \lim_n Z_n/\mathbf{E}Z_n = 0 \Big\} = \Big\{ \sum_n I_n = \infty \Big\} \quad \text{almost surely.}$$

The result is obtained by writing the semi-martingale decomposition of $\ln Z_n/\mathbf{E}Z_n$, and studying separately the terms. This is a refined (conditional) second moment condition, and the criterium can also be used to obtain quantitative information on the polymer measure itself, on its concentration and localization.

Now, it is natural to conjecture that diffusive behavior takes place in the full weak disorder region. This indeed what we prove in this work in progress.

Theorem: Assume $d \geq 3$ and weak disorder. For all bounded continuous function F on the path space, we have

$$\lim_n \mu_n[F(\hat{\omega}^{(n)})] = \mathbf{E}F(B)$$

in probability. (B is the Brownian motion with diffusion matrix $d^{-1}I_d$.)

The statement shows that indeed the scaling relation holds in the full weak disorder region.

In the proof we introduce an infinite time horizon measure on the path space which is a natural limit of the sequence μ_n. This measure is a time inhomogeneous Markov chain which depends on the environment. We cannot prove the central limit theorem for this Markov chain directly. We need to average over the environment. In order to prove convergence in probability with respect to the environment, we use again a second moment method by introducing a second independent copy of the polymer before performing this average. All through, we make use of the convergence of the series $\sum I_n$ as the main technical ingredient.

To finish with, we mention an interesting open (so far) question. Does strong disorder imply super-diffusivity? For instance, what happens in the case of small dimension d and small β is not clear at all.

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Brownian Directed Polymers in Random Environment

Nobuo Yoshida
(joint work with Francis Comets)

We study the thermodynamics of a continuous model of directed polymers in random environment. The directed polymer in this model is a d-dimensional Brownian motion (up to finite time t) viewed under a Gibbs measure which is built up with a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}^d$ (=time $\times$ space). Here, the Poisson random measure plays the role of the random environment which is independent both in time and in space. We now give the definition of the random Gibbs measure which we call the *polymer measure*.

• *The Brownian motion:* Let $(\{\omega_t\}_{t\geq 0}, P)$ denote a d-dimensional standard Brownian motion. To be more specific, we let the measurable space $(\Omega, \mathcal{F})$ be $C(\mathbb{R}_+ \to \mathbb{R}^d)$ with the cylindrical σ-field, and P be the Wiener measure on $(\Omega, \mathcal{F})$ such that $P\{\omega_0 = 0\} = 1$.

• *The space-time Poisson random measure:* We let η denote the Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}^d$ with the unit intensity, defined on a probability space $(\mathcal{M}, \mathcal{G}, Q)$.

• *The polymer measure:* We let V_t denote a "tube"around the graph $\{(s, \omega_s)\}_{0<s\leq t}$ of the Brownian path,

$$V_t = V_t(\omega) = \{(s, x)\ ;\ s \in (0, t],\ x \in U(\omega_s)\}, \tag{1}$$

where $U(x) \subset \mathbb{R}^d$ is the closed ball with the unit volume, centered at $x \in \mathbb{R}^d$. For any $t > 0$, define a probability measure μ_t on the path space $(\Omega, \mathcal{F})$ by

$$\mu_t(d\omega) = \frac{\exp\left(\beta\eta(V_t)\right)}{Z_t}\ P(d\omega), \tag{2}$$

where $\beta \in \mathbb{R}$ is a parameter and

$$Z_t = P[\exp\left(\beta\eta(V_t)\right)]\ . \tag{3}$$

The model considered here, enables us to use stochastic calculus, with respect to both Brownian motion and Poisson process, leading to handy formulas for fluctuations analysis and qualitative properties of the phase diagram.
We discuss:

• The normalized partition function, its positivity in the limit which distinguishes the weak and strong disorder phases.

• The existence of quenched Lyapunov exponent, its positivity, and its agreement with the annealed Lyapunov exponent.

• An almost sure central limit theorem for the Brownian polymer for $d \geq 3$ and β smaller than some β_0.

• The longitudinal fluctuation of the free energy and some of its relations with the overlap between replicas and with the transversal fluctuation of the path. The results here includes an almost sure large deviation principle for the polymer measure. Our fluctuation results are interpreted as bounds on various exponents and provide a circumstantial evidence of super-diffusivity in dimension one.

• Relations to the Kardar-Parisi-Zhang equation.

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Diffusive behavior of isotropic diffusions in random environment

Ofer Zeitouni

(joint work with Alain-Sol Sznitman)

We present results concerning the asymptotic behavior of isotropic diffusions in random environment that are small perturbations of Brownian motion. When the space dimension is three or more we prove an invariance principle as well as transience. Our methods also apply to questions of homogenization in random media. The proof is based on an induction scheme that propagates from scale to scale a Hölder norm control on (truncated) transition probabilities, as well as a large deviations control on the strength of "traps".

Precise statements and a description of the main induction step are provided C.R. Acad. Sci. Paris, Ser I, v. 339 (2004), pp. 429–434.

Spontaneous symmetry breaking in a driven two-species lattice gas

Gunter M. Schütz

(joint work with R.D. Willmann and S. Großkinsky)

While single-species driven diffusive systems in one dimension are largely understood, two-species models show a variety of phenomena that are a matter of current research, such as phase separation and spontaneous symmetry breaking (see [1] for a recent review). The first such model that was shown to exhibit spontaneous symmetry breaking was a model with open boundaries that became known as the 'bridge model' [2, 3]. In this model, two species of particles move

in opposite directions. Although the dynamical rules are symmetric with respect to the two species, two phases with non-symmetrical steady states were found by Monte Carlo simulations and mean-field calculations. While the existence of one of the phases remains disputed [4, 5], a proof for the existence of the other one was given for the case of a vanishing boundary rate [6].

All symmetry breaking models considered so far evolve by random sequential update, corresponding to continuous time. We study a discrete time variation of the bridge model with parallel sublattice update. The dynamics in the bulk is deterministic, while stochastic events occur at the boundaries. Thus the complexity of the problem is reduced, which allows to elucidate the mechanism by which spontaneous symmetry breaking occurs in this model as well as to give a proof for the existence of a symmetry broken phase.

The model considered here is defined on a one-dimensional lattice of length L, where L is an even number. Sites are either empty (0) or occupied by a single particle of either species A or B. The dynamics is defined as a parallel sublattice update scheme in two half steps. In the first half-step particles are created and annihilated at the two boundary sites with probability α and β, respectively,

$$\begin{array}{llllll} |\,0\ldots & \xrightarrow{\alpha} & |\,A\ldots & |\,B\ldots & \xrightarrow{\beta} & |\,0\ldots \qquad \text{(left)} \\ \ldots 0\,| & \xrightarrow{\alpha} & \ldots B\,| & \ldots A\,| & \xrightarrow{\beta} & \ldots 0\,| \qquad \text{(right)}\ . \end{array}$$

Note that at each boundary site, either annihilation or creation can take place in a given time step, but not both. In the bulk, the following hopping processes occur deterministically between sites $2i$ and $2i+1$ with $0 < i < L/2$:

$$A0 \xrightarrow{1} 0A \qquad 0B \xrightarrow{1} B0 \qquad AB \xrightarrow{1} BA.$$

In the second half-step, these deterministic bulk hopping processes take place between sites $2i-1$ and $2i$ with $0 < i \leq L/2$. Note that the dynamics is symmetric with respect to the two particles species.

Phase diagram

The phase diagram of the model can be explored by Monte Carlo simulations. Two phases are found:

- If $\alpha < \beta$, the system exhibits a symmetric steady state. Here, the asymptotic bulk densities are $\rho_A(i) = 0$, $\rho_B(i) = \alpha\beta/(\alpha+\beta)$ if i is odd, and $\rho_A(i) = \alpha\beta/(\alpha+\beta)$, $\rho_B = 0$ if i is even.
- If $\alpha > \beta$, the system resides in the symmetry broken phase. Assume the A particles to be in the majority. Then, the bulk densities in the steady state are $\rho_B(i) = 0$ for all i, $\rho_A(i) = 1$ for i even and $\rho_A(i) = 1-\beta$ for i odd, so that the minority species is completely expelled from the system.

Thus, the dynamics of the majority species is as in the single species ASEP with parallel sublattice update. For this system, an analytical expression for the steady state density in a finite system is known [7]. The density profile of the majority species in the broken phase of the sublattice bridge model equals that of the high

density phase in the sublattice ASEP at the given parameters α and β. Particles of the expelled species only enter the system by fluctuations exponentially small in the system size L [8], so a typical symmetry broken configuration is stable against fluctuations.

To establish spontaneous symmetry breaking we additionally have to show that such a configuration can be reached from a symmetric initial condition within a time which is increasing only algebraically in the system size. Due to the deterministic bulk dynamics the state space of the process is not fully connected. But it is obvious that the empty lattice can be reached from any initial configuration as long as $\beta > 0$, so we take this as a symmetric initial condition. When one of the boundary probabilities is 0 or 1, the steady state can be given exactly, consistent with the above picture. But to avoid degeneracies in the following, we assume α, $\beta \in (0, 1)$ and $\alpha > \beta$.

Symmetry breaking dynamics

Starting from the empty lattice, A (B) particles are created at every time step with probability α at site 1 (L). Once injected, particles move deterministically with velocity 2 (-2). Therefore, at time $t = L/2$ the system is in a state where the density of A (B) particles is α (0) at all even sites and 0 (α) at all odd sites. In this situation both creation and annihilation of particles are possible. However, it turns out that the effect of creation of particles is negligible [8]: Whenever the bulk density of A-particles is above $\beta/2$, the deterministic hopping transports on average more A-particles towards site L than can be annihilated there. This leads to the formation of an A-particle jam at the right boundary, blocking the injection of B-particles, and vice versa a B-particle jam at the left boundary. In these jams, the only source of vacancies is annihilation at the boundaries with probability β. Therefore, in a jam, the density of A (B) particles at even (odd) sites is 1, while that at odd (even) sites is $1-\beta$. In each time step, the number of particles in each of the two jams reduces by one with probability β. Due to fluctuations, one of the jams, say the B-jam at the left boundary, is dissolved first. Then A-particles start to reenter at the left boundary. Let ΔN_1 be the number of entered A-particles until the A-jam on the right boundary is dissolved. At this point the system is again empty except for the ΔN_1 entered A-particles and some finite fluctuations, and the process of filling the lattice restarts. This establishes a cyclic behavior and we summarize the main steps of the next cycle.

1. There are ΔN_1 A-particles in the system.
2. The A-particles reach the right boundary and block the entrance of B's.

A-particles still enter with probability α and exit with probability $\beta < \alpha$, further increasing the majority of A particles.

3. The B-particles reach the left boundary, blocking also the entrance of A's.
4. The jam of B-particles is dissolved and A particles start to reenter.

Again, since $\alpha > \beta$ the majority of A-particles is increased and when the A-jam at the right boundary is dissolved, the cycle is finished and starts with initial

condition $\Delta N_2 > \Delta N_1$. The initial process from the empty lattice is a cycle with $\Delta N_0 = 0$, and of course $\Delta N_1 \in [-L/2, L/2]$ can also be negative due to symmetry. But conditioning on a typical outcome $\Delta N_1 = O(\sqrt{L})$ of the first cycle this fluctuation will be amplified after steps 2 and 4. This picture can be made rigorous and one can show that

$$\big\langle \Delta N_{k+1} - \Delta N_k \big| \Delta N_k \big\rangle = 2\big(\alpha/\beta - 1\big)\Delta N_k + O(1) \ .$$

Thus, for $\alpha < \beta$ the expected value of $\Delta N_{k+1} - \Delta N_k$ is proportional to $-\Delta N_k$ for $\alpha < \beta$. Fluctuations are strongly damped, leading to a symmetric phase. On the other hand for $\alpha > \beta$ the difference is proportional to $+\Delta N_k$ and thus fluctuations are amplified towards one of the boundaries of the domain $[-L/2, L/2]$. Once the boundary is reached, the amplification loop stops and one of the species will remain expelled from the system. The number of cycles to reach this symmetry broken phase from a symmetric initial condition is of order $\log L$ and the time for a cycle of order L, which yields an algebraic dependence.

We presented the sketch of a proof of spontaneous symmetry breaking for a version of the bridge model with deterministic bulk dynamics. The amplification loop sheds new light on the mechanism of symmetry breaking also for models with stochastic bulk dynamics, where rigorous statements are only available for degenerate boundary rates.

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Hyperbolic scaling problems: The method of compensated compactness

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(joint work with Bálint Tóth)

Historical remarks: A simple system with hyperbolic scaling is given by $\dot{r}_k = (1/2)(V'(r_{k+1}) - V'(r_{k-1}))$, where $V \in C^2(\mathbb{R})$ is convex at infinity, $r_k \in \mathbb{R}$ and $k \in \mathbb{Z}$. At a first glance $\rho_\epsilon(t,x) := r_k(t/\epsilon)$ if $|x - k\epsilon| < \epsilon/2$ solves $\partial_t \rho = \partial_x V'(\rho)$ as $\epsilon \to 0$. However, the first computer experiments performed by von Neumann and coworkers [11] demonstrated that this numerical procedure does not converge at all; see also Lax [9] for a rigorous treatment. In fact, the procedure should be stabilized by an additional elliptic term as $\dot{r}_k = (1/2)(V'(r_{k+1}) - V'(r_{k-1})) +$

$\sigma\,(r_{k+1}+r_{k-1}-2r_k)$, where the value of $\sigma>0$ may depend on initial data, see [8,13]. In general, hyperbolic equations and systems of conservation laws develop discontinuities (shocks), and a continuum of weak solutions emerges at the same time. The most fundamental hyperbolic scaling problems are certainly those of Hamiltonian systems of interacting particles; via physical arguments they result in a system of Euler type equations like the familiar p-system: $\partial_t u+\partial_x p(v)=0$, $\partial_t v-\partial_x u=0$. The mathematics of two-component hyperbolic systems is much more complex than that of a single equation, an existence theory goes back to R. DiPerna [2,3], while the problem of weak uniqueness has been solved much later, see A. Bressan [1]. Related results on the hydrodynamic limit (HDL) of microscopic Hamiltonian systems are all restricted to exactly solvable models.

A fairly general method for deriving HDL of interacting stochastic systems has been proposed by H.T. Yau [16], see also [12] on classical particle dynamics with weak conservative noise. Unfortunately, this relative entropy method needs smoothness of the macroscopic solution. Beyond shocks the coupling technique of Rezakhanlou [14] works in case of one-component attractive models; the proof relays on advanced methods of PDE theory. Here we are going to extend the Tartar - Murat - DiPerna [15,10,2,3] theory of compensated compactness to stochastic systems with two conservation laws. A first exposition of this method was presented in [4], some non-attractive one-component models are discussed in [5,7], HDL of the following two-component system is derived in [6].

Interacting exclusion processes: We consider particles with ±1 velocities on $\mathbb{Z}$ with full exclusion, thus $\omega_k=0,\pm1$ is the configuration at site $k\in\mathbb{Z}$. The microscopic evolution is generated by $\mathcal{L}=\mathcal{L}_0+\sigma\,\mathcal{S}$, the dynamics consists of independent exchanges at neighboring sites. The asymmetric component, $\mathcal{L}_0$ sends $(1,0)$ to $(0,1)$ and $(0,-1)$ to $(-1,0)$ at a unit rate, interaction means that $(1,-1)$ turns into $(-1,1)$ at rate two; any other action is banned. Finally, $\sigma=\sigma(\epsilon)>0$, and the symmetric $\mathcal{S}$ exchanges ω_k and ω_{k+1} at rate 1. The conserved quantities are chosen as $\eta_k:=1-\omega_k^2$ and $\xi_k:=-\omega_k$, then $\mathcal{L}_0\eta_k=\phi_{k+1}-\phi_k$, $\mathcal{L}_0\xi_k=\psi_{k+1}-\psi_k$,

$$\phi_k=\frac{1}{2}(\eta_k\xi_{k+1}+\eta_{k+1}\xi_k)+\frac{1}{2}(\eta_k-\eta_{k+1})\,,$$
$$\psi_k=\frac{1}{2}(\eta_k+\eta_{k+1}+2\xi_k\xi_{k+1}-2)+\frac{1}{2}(\eta_k\xi_{k+1}-\eta_{k+1}\xi_k)+\xi_k-\xi_{k+1}$$

are the fluxes. The symmetric component, $\mathcal{S}$ is acting as a discrete Laplacian; it plays the role of the elliptic stabilization what we need even for numerical procedures. Since all stationary states are superpositions of Bernoulli measures, the familiar Leroux system, $\partial_t\rho+\partial_x(\rho u)=0$, $\partial_t u+\partial_x(\rho+u^2)=0$ is expected as the result of HDL; ρ and u are the asymptotic densities of η and ξ, respectively.

Main result: For any $\epsilon>0$, $l=l(\epsilon)\in\mathbb{N}$ and space-time process ζ let

$$\hat{\zeta}_\epsilon(t,x):=\frac{1}{l}\sum_{k\in\mathbb{Z}}a\left(\frac{x-k\epsilon}{l\epsilon}\right)\,\zeta_k(t/\epsilon)$$

where $a \geq 0$ is a smooth density of compact support. For example, if $\zeta := (\eta, \xi)$ then $\hat{\zeta}_\epsilon$ denotes the empirical process, and $\hat{J}_\epsilon$ corresponds to the microscopic flux $J_k := (\phi_k, \psi_k)$. Suppose that $\sigma(\epsilon) \approx \log(1/\epsilon)\sqrt{1/\epsilon}$, $l(\epsilon) \approx \sqrt{(1/\epsilon)\log(1/\epsilon)}$, and let P_ϵ denote the distribution of $\hat{\zeta}_\epsilon$. We prove that P_ϵ is tight in the strong topology of $L^1_{\mathrm{loc}}(\mathbb{R}^2_+)$, and all limit points are concentrated on weak entropy solutions of the Leroux system. There is no result on uniqueness of the limit.

Compensated compactness: The first step of the proof is the usual one block replacement of $\hat{J}_\epsilon$ with $f(\hat{\zeta}_\epsilon)$, where $f(z)$, $z := (\rho, u)$ is just the flux of the Leroux system. This follows by LSI for $\mathcal{S}$. Next we introduce the Young measure $\Theta_\epsilon(dt, dx, dz) = dt\, dx\, \theta_\epsilon(t, x; dz)$ of $\hat{\zeta}_\epsilon$, its distribution is tight with respect to the weak topology of measures, and all limit distributions are concentrated on measure solutions of the Leroux sytem. To prove the Dirac property of the limit distributions, we consider Lax entropy pairs (S, F) and the related entropy production $X_\epsilon := \partial_t S(\hat{\zeta}_\epsilon) + \partial_x F(\hat{\zeta}_\epsilon)$. The crucial point of the proof is to establish the Murat decomposition $X_\epsilon = Y_\epsilon + Z_\epsilon$, where $X_\epsilon \to 0$ in probability in $H^{-1}(\mathbb{R}^2_+)$, while Z_ϵ is bounded in the space of measures; all that should be understood at the level of the distribution of Young measures. Moreover, convex entropy pairs satisfy the Lax entropy condition: $\limsup X_\epsilon \leq 0$ in probability, in the sense of distributions. Now we are in a position to conclude Tartar factorization: all limit distributions of the Young measure satisfy

$$\theta(t, x; S_1F_2 - S_2F_1) = \theta(t, x; S_1)\theta(t, x; F_2) - \theta(t, x; S_2)\theta(t, x; F_1)$$

a.e. with probability one for all couples (S_i, F_i) of entropy pairs. This probabilistic part of the argument is based on LSI and a general exponential moment bound. The final step of the proof is to show that, due to Tartar factorization, entropic measure solutions are weak entropy solutions in the Lax sense. Although there are general results of DiPerna on this issue, his conditions can not be verified in stochastic situations. However, in the case of the Leroux system we have two nice families of entropies, namely $S_a := \rho + au - a^2$ and $\bar{S}_a := |S_a|$ for $a \in \mathbb{R}$, so that an elementary calculation by Dafermos implies the final statement.

Concluding remarks: The probabilistic part of the argument above extends to several two-component models, but the last step is restricted to the Leroux system. Indeed, most physical systems of conservation laws have singular points where the conditions of strict hyperbolicity and genuine nonlinearity break down, therefore the general results of DiPerna and others do not apply. To exclude singularities from the phase space, a Conley - Chueh - Smoller type maximum principle would be needed for the Riemann invariants of the macroscopic equations. Unfortunately, this is not available for microscopic stochastic models, but there is a minor hope to prove it. In the case of one-component systems, the Lax entropy condition is sufficient for uniqueness of HDL, see [7]. Systems are much more difficult, in view of Bressan's [1] results, we ought to verify the Oleinik entropy condition for Riemann invariants; this is a formidable open problem.

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Perturbation of equilibria: a hydrodynamic limit

Bálint Tóth and Benedek Valkó

We present the results contained in [12]. We consider the pde

$$\begin{cases} \partial_t \rho + \partial_x \big(\rho u\big) = 0 \\ \partial_t u + \partial_x \big(\rho + \gamma u^2\big) = 0 \end{cases} \tag{1}$$

for $(t,x) \in [0,\infty) \times (-\infty,\infty)$ where $\rho = \rho(t,x) \in \mathbb{R}_+$, $u = u(t,x) \in \mathbb{R}$ are density, respectively, velocity field and $\gamma \in \mathbb{R}$ is a fixed parameter. For any fixed γ this is a *hyperbolic system of conservation laws* in the domain $(\rho,u) \in \mathbb{R}_+ \times \mathbb{R}$.

Phenomenologically, the pde describes a deposition/domain growth – or, in biological term: chemotaxis – mechanism: $\rho(t,x)$ is the density of population

performing the deposition and $u(t,x)$ is the negative gradient of the height of the deposition.

The pde (1) is invariant under the following scaling:

$$\widetilde{\rho}(t,x) := A^{2\beta}\rho(A^{1+\beta}t, Ax), \qquad \widetilde{u}(t,x) := A^{\beta}u(A^{1+\beta}t, Ax),$$

where $A > 0$ and $\beta \in \mathbb{R}$ are arbitrarily fixed. The choice $\beta = 0$ gives the straightforward hyperbolic scale invariance, valid for any system of conservation laws. More interesting is the $\beta = 1/2$ case. This is the natural scale invariance of the system, since the physical variables (density and velocity fields) change *covariantly* under this scaling. This is the (presumed, but never rigorously proved) asymptotic scale invariance of the Kardar-Parisi-Zhang deposition phenomena. The nontrivial scale invariance of the pde (1) suggests its *universality* in some sense. Our main result indeed states its validity in a very wide context.

The parameter γ of the pde (1) is of crucial importance: different values of γ lead to completely different behavior. Here are listed some particular cases which arose in the past in various contexts:
— The pde (1) with $\gamma = 0$ arose in the context of the 'true self-repelling motion' constructed by Tóth and Werner in [13]. For a survey of this case see also [14]. The same equation, with viscosity terms added, appear in mathematical biology under the name of (negative) chemotaxis equations (see e.g. [9], [8], [6])).
— Taking $\gamma = 1/2$ we get the 'shallow water equation'. See [1], [5]. This is the only value of the parameter γ when $m = \rho u$ is conserved and as a consequence the pde (1) can be interpreted as gas dynamics equation.
— With $\gamma = 1$ the pde is called 'Leroux's equation' which is of Temple class and for this reason much investigated. For many details about this equation see [10]. In the recent paper [3] Leroux's system has been derived as hydrodynamic limit under Eulerian scaling for a two-component lattice gas, going even beyond the appearance of shocks.

The goal of the paper [12] is to derive the two-by-two hyperbolic system of conservation laws (1) as decent hydrodynamic limit of some systems of interacting particles with two conserved quantities.

We consider one-dimensional, locally finite interacting particle systems with two conservation laws with periodic boundary conditions which under *Eulerian* hydrodynamic limit lead to two-by-two systems of conservation laws

$$\begin{cases} \partial_t \rho + \partial_x \Psi(\rho, u) = 0 \\ \partial_t u + \partial_x \Phi(\rho, u) = 0, \end{cases} \tag{2}$$

with $(t,x) \in [0,\infty) \times \mathbb{T}$, $(\rho, u) \in \mathcal{D} \subset \mathbb{R}^2$. Here $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the unit torus and $\mathcal{D}$ is a convex compact polygon in $\mathbb{R}^2$. The system is *typically* strictly hyperbolic in the interior of $\mathcal{D}$ with possible non-hyperbolic degeneracies on the boundary $\partial\mathcal{D}$ (see [11]). We consider the case of isolated singular (i.e. non hyperbolic) point on the interior of one of the edges of $\mathcal{D}$, call it $(\rho_0, u_0) = (0,0)$ and assume $\mathcal{D} \subset \{\rho \geq 0\}$ (otherwise we apply an appropriate linear transformation on the conserved quantities). We investigate the propagation of *small nonequilibrium*

perturbations of the steady state of the microscopic interacting particle system, corresponding to the densities (ρ_0, u_0) of the conserved quantities. We prove that for a very rich class of systems, under proper hydrodynamic limit the propagation of these small perturbations are *universally* driven by the system (1) on the unit torus, where the parameter $\gamma := \frac{1}{2}\Phi_{uu}(\rho_0, u_0)$ (with a proper choice of space and time scale) is the only trace of the microscopic structure. The proof is valid for the cases with $\gamma > 1$.

The proof essentially relies on H-T. Yau's relative entropy method (see e.g. [2], [4], [7], [15]) and thus, it is valid only in the regime of smooth solutions of the pde (1).

We should emphasize here the essential new ideas of the proof. Since we consider a *low density* limit, the distribution of particle numbers in blocks of mesoscopic size will have a *Poissonian* tail. The fluctuations of the other conserved quantity will be Gaussian, as usual. It follows that when controlling the fluctuations of the empirical block averages the usual large deviation approach would lead us to the disastrous estimate $\mathbf{E}\big(\exp\{\varepsilon\ GAU \cdot POI\}\big) = \infty$. It turns out that some very special cutoff must be applied. Since the large fluctuations which are cut off can not be estimated by robust methods (i.e. by applying entropy inequality), only some cancellation due to martingales can help. This is the reason why the cutoff function must be chosen in a very special way, in terms of a particular Lax entropy of the Euler equation (2). In this way the proof becomes an interesting mixture of probabilistic and pde arguments. The fine properties of the limiting pde, in particular the global behavior of Riemann invariants and some particular Lax entropies, play an essential role in the proof. The radical difference between the $\gamma \geq 1$ vs. $\gamma < 1$ cases, in particular applicability vs. non-applicability of the Lax-Chuey-Conley-Smoller maximum principle, manifests itself on the microscopic, probabilistic level.

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Euler hydrodynamics of one-dimensional attractive particle systems

Ellen Saada
(joint work with C. Bahadoran, H. Guiol, K. Ravishankar)

We consider attractive, irreducible, finite-range, asymmetric, conservative particle systems on $\mathbb{Z}$, with at most K particles per site ($K \in \mathbb{N}$), and without necessarily nearest-neighbor jumps or explicit invariant measures. We prove that their hydrodynamic limit under Euler time scaling exists, and is given by the entropy solution to some scalar conservation law with Lipschitz-continuous flux.

The simplest example of attractive system on $\mathbb{Z}$ is the asymmetric exclusion process, for which the hydrodynamic limit is well-known, and is given by the entropy solutions to the scalar conservation law

$$\partial_t u(t,x) + \partial_x G(u(t,x)) = 0 \tag{1}$$

where $G(u) = \gamma u(1-u)$ is the *macroscopic flux*, and γ is the mean drift of a particle.

The first approach to prove hydrodynamics for asymmetric exclusion was constructive, and developed among others by Andjel & Vares (1987), when the initial datum is a single step with density λ to the left and ρ to the right (the *Riemann problem* for (1)). Their proof combined the knowledge of the entropy solution to (1) in that case (when G is strictly convex or concave, the latter has a simple explicit form), and comparison (thanks to *attractiveness*) with explicit equilibrium systems; indeed, the extremal invariant and translation invariant measures of simple exclusion are Bernoulli product, of parameter the average particle density per site (see Liggett (1985)). The second more general approach, due to Rezakhanlou (1991) (see Kipnis & Landim (1999) for details), was based on entropy inequalities; it also needed attractiveness, and the existence of product invariant measures. It proved the hydrodynamic limit for the asymmetric exclusion process in any dimension, with an arbitrary initial datum in (1). In fact, both approaches apply to other attractive systems with explicit invariant measures, such as zero-range or misanthropes processes with ad hoc rates (provided the flux function is concave or convex when using the Andjel & Vares (1987) approach).

However, as soon as the rates do not satisfy some necessary algebraic conditions, the invariant measures for misanthropes process are unknown, even for the basic example of K-exclusion (see Seppäläinen (1999)). Our work enables us to treat those cases, without a further restriction to nearest neighbor interaction (see Rezakhanlou (2001)).

In Bahadoran et al. (2002), we studied processes with explicit product invariant measures. We relax this assumption in Bahadoran et al. (2004). Our approach relies on (i) a constructive proof of the hydrodynamic limit for Riemann initial profiles, and (ii) a general result which shows that the hydrodynamic limit for Riemann initial profiles implies the same for general initial profiles.

But we have first to give sense to a hydrodynamic equation of type (1), i.e. to define a flux function G for the processes we consider. Even if the invariant measures are unknown, we still have a one-parameter family of extremal invariant and translation invariant measures $\{\nu_\rho,\ \rho \in \mathcal{R}\}$, where $\mathcal{R}$ is a closed subset of $[0, K]$ containing 0 and K, and ρ represents the average particle density per site under the non-explicit measure ν_ρ. Then, for $\rho \in \mathcal{R}$, we define $G(\rho)$ 'as usual', by an equilibrium expectation of the microscopic flux; otherwise, $G(\rho)$ is defined by linear interpolation. The function G is thus Lipschitz-continuous, and the existence and uniqueness of an entropy solution to (1) for a Borel measurable initial datum can be deduced from Kružkov (1970) (see also Bressan (2000)).

Point (i) is a generalization of the constructive method of Andjel & Vares (1987). We relax the convexity assumption they need for the flux function by deriving a variational formula for the Riemann entropy solution, taking into account the restriction of densities to $\mathcal{R}$ (for an explicit construction of the Riemann entropy solution when $G \in \mathcal{C}^2(\mathbb{R})$, see Ballou (1970), or Godlewski & Raviart (1991)). We then solve the $\mathcal{R}$-valued Riemann problem.

Point (ii) is valid for finite-range attractive processes, and is inspired by the Glimm's scheme in the theory of hyperbolic conservation laws (see Glimm (1965), Serre (1999)). We use an approximation scheme, thanks to a 'finite propagation property' valid at microscopic and macroscopic levels, and a 'macroscopic stability property' at particles' level (that requires the finite-range assumption on the dynamics, see Bramson & Mountford (2002)). The former enables us to deal with a succession of non-interacting Riemann problems, while the latter implies that the hydrodynamic limit depends only on the density profile at time 0, and not on the underlying microscopic structure. We first prove hydrodynamics for $\mathcal{R}$-valued solutions, by taking an initial density profile a.e. $\mathcal{R}$-valued; then we extend the result to general profiles, and obtain hydrodynamics for arbitrary entropy solutions.

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Diffusive limit for a tagged particle in asymmetric zero-range

Sunder Sethuraman

The zero-range process is a collection of random walks on $\mathbb{Z}^d$ with transition probability $p(\cdot)$ which interact in the time domain. That is, if there are k walkers at a location i, then one of these "particles" jumps to j with rate $g(k)p(j-i)$. The function $g : \mathbb{N} \to \mathbb{R}_+$ from the non-negative integers to the non-negative reals which quantifies the interaction is called the process rate. The configuration space is $\Sigma = \mathbb{N}^{\mathbb{Z}^d}$, and the formal generator of the process is $(L\phi)(\eta) = \sum_{i,j\in\mathbb{Z}^d} g(\eta_i)p(j-i)(\phi(\eta^{i,j}) - \phi(\eta))$ where $\eta^{i,j}$ is the state obtained by moving a particle from i to j. Construction of the Markov process generated by L has been done in [1] under suitable conditions on g and p which include the case $g(0) = 0$, $g(k) > 0$ for $k \geq 1$, $\liminf g(k) > 0$ and g is Lipschitz, and also p is finite-range which we will assume throughout.

The equilibrium measures for this system are well known, and turn out to be in product form. Namely, there is a family of invariant measures P_ρ, each concentrating on a fixed density of particles $\rho \geq 0$. The measure $P_\rho = \prod_{i\in\mathbb{Z}^d} \mu$ where the common marginal $\mu(k) = (1/Z_\rho)(\alpha(\rho)^k/(g(k)\cdots g(1))$ for $k \geq 1$ and

$(1/Z_\rho)$ for $k = 0$ with normalization Z_ρ and $\alpha(\rho)$ chosen so that the mean number of particles $E_\mu[X] = \rho$.

The problem studied is the asymptotic motion of a distinguished, or tagged, particle in this system. Let $x(t)$ be the position of the tagged particle at time t. It turns out that the equilibrium measures for the process in the "reference-frame" of $x(t)$ are certain explicit "size-biased" versions of P_ρ denoted Q_ρ (cf. [4]).

When initial configurations are distributed in equilibrium Q_ρ, the law of large numbers has been proved $\lim_{t\to\infty} x(t)/t = (\alpha(\rho)/\rho)\sum ip(i)$ a.s. Q_ρ, as well as an invariance principle in Skorohod space when the jumps p are mean-zero, $\sum ip(i) = 0$, that is $\lim_{\lambda\to\infty}(x(\lambda\cdot)/\sqrt{\lambda} \Rightarrow B_\rho(\cdot)$ where B_ρ is Brownian motion with an explicit covariance depending on ρ and p [4], [5]. See also [3] for some "non-equilibrium" law of large numbers type results.

The results presented are diffusive variance bounds, under equilibrium Q_ρ, in one dimension $d = 1$ when the jumps have drift $\sum ip(i) \neq 0$. Namely,

$$0 \;<\; c_1 \;\leq\; \frac{\mathrm{Var}_{Q_\rho}(x(t))}{t} \;\leq\; c_2 \;<\; \infty$$

for constants $c_1 = c_1(\rho, p), c_2 = c_2(\rho, p)$, and also a corresponding invariance principle, under equilibrium Q_ρ, when the rate g is increasing, $g(k)/k$ is decreasing, and p is totally asymmetric nearest-neighbor–namely $p(1) = 1$,

$$\lim_{\lambda\to\infty} \frac{1}{\sqrt{\lambda}}[x(\lambda\cdot) - \alpha(\rho)/\rho] \;\Rightarrow\; \sigma(\rho)B(\cdot)$$

where B is standard Brownian motion and $\sigma^2(\rho)$ is the diffusion coefficient depending on ρ.

The methods are through some bounds on certain H_{-1} norms, and a coupling using the Newman-Wright theorem with respect to associated random variables [2].

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Large deviations from a macroscopic scaling limit for particle systems with Kac interaction and random potential

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(joint work with Enza Orlandi)

We consider a d-dimensional particles system interacting via a two-body Kac interaction and external random field given by independent random variables with translational invariant distribution.

Kac's potentials are two body interactions $J_\gamma(|x-y|)$ with range $\frac{1}{\gamma}$, and strength γ^d, where γ is a dimensionless parameter. They were introduced in [KUH], and then generalized in [LP], to present a rigorous derivation of the van der Waals theory of a gas-liquid phase transition. Since then there has been several papers on the subject. Recently many authors studied the equilibrium statistical properties of spin systems with Kac potential for γ small but finite. We mention only some of them [COP], [LMP]. In particular Random Field Kac models, in $d = 1$ and for γ small and fixed, have been recently studied in [COP1], [COPV]. Many authors studied also the time evolution of the macroscopic profile in particle systems interacting via long range Kac potential either in the case of conservative dynamics [LOP], [GL], [GLM], [MM], or in the case of non conservative dynamics [DOPT]. The formal Hamiltonian we consider is given by

$$H_\gamma^{\beta,\alpha}(\eta) = -\frac{\beta}{2}\sum_{x,y\in\mathbb{Z}^d} J_\gamma(x-y)\eta(x)\eta(y) - \sum_{x\in\mathbb{Z}^d}\alpha(x)\eta(x), \tag{1}$$

where β is a positive parameter and $\eta(x) \in \{0,1\}$, $\eta(x) = 1$ if there is a particle in x and $\eta(x) = 0$ means that the site is empty. The $\{\alpha(x)\ x \in \mathbb{Z}^d\}$ represent the external random field on the sites x. They are assumed to be independent random variables with translational invariant distribution, $\alpha(x) \in [-A, A]$ where $A > 0$.

Given the Hamiltonian (1) there is a standard way, see for example [Sp], to construct a dynamics which conserves the number of particles and for which the invariant measures are given by the one parameter family of Gibbs measures associated to (1). Performing a diffusive scaling limit, in [MOS] a law of large numbers when $d \geq 3$ was established for the density field, starting from a sequence of measures associated to some initial density profile ρ_0. The equation obtained for the density field is the following nonlocal, non linear partial differential equation

$$\frac{\partial\rho}{\partial t} = \frac{1}{2}\nabla\cdot\left(\sigma(\rho)\nabla\frac{\delta\mathcal{G}}{\delta\rho}\right), \qquad \rho(0,r) = \rho_0(r), \tag{2}$$

where the energy functional $\mathcal{G}(\rho)$ is of the form

$$\mathcal{G}(\rho) = \int dr g_0(\rho(r)) - \frac{\beta}{2}\int\int J(r-r')\rho(r)\rho(r')drdr', \tag{3}$$

g_0 is the (strictly convex) free energy density and $\sigma(\rho)$ is the conductivity, or mobility, of the system with only short range interaction, i.e. corresponding to $\beta \equiv 0$ in (1). It is linked, in a regime of linear response, to the diffusion matrix

$D(\rho)$, defined in [FM], via the Einstein relation $D(\rho) = \sigma(\rho)\chi(\rho)^{-1}$, see [Sp], where $\chi(\rho)$ is the static compressibility defined by

$$\chi(\rho) = \mathbb{E}\Big[\int \eta(0)^2 d\mu^{\alpha,\lambda_0(\rho)}(\eta) - \Big(\int \eta(0) d\mu^{\alpha,\lambda_0(\rho)}(\eta) \Big)^2 \Big],$$

$\mathbb{E}$ stands for expectation with respect to the disorder. The g_0 in (3) is the free energy density given by

$$g_0(\rho) = \rho\lambda_0(\rho) - p_0(\lambda_0(\rho))$$

where

$$p_0(\lambda) = \mathbb{E}\left[\log\left(1 + e^{\lambda+\alpha(0)} \right) \right],$$

and for any given $\rho \in [0,1]$, $\lambda_0(\rho)$ satisfies

$$\rho = \frac{dp_0}{d\lambda}(\lambda_0(\rho)) = \mathbb{E}\left[\frac{e^{\lambda_0(\rho)+\alpha(0)}}{1 + e^{\lambda_0(\rho)+\alpha(0)}} \right].$$

We show, in $d \geq 3$, the large deviations principle for the empirical random measures of the process briefly described above. The result holds almost surely with respect to the random field and the rate function, which depends on the statistical properties of the external random field, is lower semicontinuous and has compact level sets. The restriction on the dimension ($d \geq 3$) is a consequence of the fact that the law of large numbers has been proven in [MOS] only for $d \geq 3$. Results in all dimensions for a process without the Kac type of interaction, i.e. the one associated to the Hamiltonian (1) with $\beta = 0$, were announced by Quastel in [Q].

The proof of the large deviations principle for our dynamics follows the paper by J. Quastel, F. Rezakhanlou and S.R.S. Varadhan [QRV].

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Equilibrium Fluctuations for a System of Harmonic Oscillators with Conservative Noise

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(joint work with József Fritz and Stefano Olla)

During the last 15 years great progress has been made in the hydrodynamic limit theory of one component systems, whereas only a few results are available on two component models, see [3, 6, 4, 7]. These latter all concern hydrodynamic law of large numbers. Here we present an equilibrium fluctuation result for a two component system. Stochastic perturbations of mechanical systems is certainly an interesting field of hydrodynamic limit theory, see [3, 6, 5, 4, 2] for some examples. The model we discuss here is also of this kind.

We regard a linearly ordered system of harmonic oscillators with conservative noise. The oscillators are labeled by $k \in \mathbb{Z}$, the configuration space is denoted by $\Omega = (\mathbb{R} \times \mathbb{R})^{\mathbb{Z}}$, a typical configuration is of the form $\omega = (p_k, r_k)_{k\in\mathbb{Z}}$ where p_k denotes the velocity of the particle k and r_k stands for the distance between particle k and $k+1$. The dynamics is described by the following set of stochastic partial differential equations:

$$\begin{cases} dp_k(t) = (r_k - r_{k-1})dt - \gamma p_k dt + \sqrt{\gamma} p_{k+1} dW_k - \sqrt{\gamma} p_{k-1} dW_{k-1} \\ dr_k(t) = (p_{k+1} - p_k)dt \end{cases}$$

where $k \in \mathbb{Z}$ and $\{W_k\}_{k\in\mathbb{Z}}$ are independent Brownian motions and $\gamma > 0$ is the friction coefficient. This model appeared first in the thesis of C. Bernardin and it was proposed by his supervisor S. Olla.

For any $\beta > 0$ and $\rho \in \mathbb{R}$ the Gaussian product measures $\mu_{\beta,\rho}$ on Ω with marginal densities

$$\mu_{\beta,\rho}(dp_k, dr_k) = \frac{\beta}{2\pi} \exp\left\{-\frac{\beta}{2}(p_k^2 + (r_k - \rho)^2)\right\} dp_k dr_k$$

are invariant measures of the process. The formal generator of the system reads as

$$\mathcal{L} = \mathcal{A} + \mathcal{S}$$

where

$$\mathcal{A} = \sum_{k\in\mathbb{Z}} \{(p_{k+1} - p_k)\partial r_k + (r_k - r_{k-1})\partial p_k\}$$

$$\mathcal{S} = \frac{\gamma}{2} \sum_{k\in\mathbb{Z}} (p_{k+1}\partial p_k - p_k \partial p_{k+1})^2 .$$

Here $\mathcal{A}$ is the Liouville operator of a chain of interacting harmonic oscillators, $\mathcal{S}$ is the noise operator and acts only on velocities. It couples neighbouring velocities in such a way that the total energy of the system is conserved. Actually the model admits two conserved quantities: total interdistance (the sum of r_k -s) and total energy (the sum of H_k -s, where $H_k = \frac{1}{2}p_k^2 + \frac{1}{4}r_k^2 + \frac{1}{4}r_{k-1}^2$). Our aim is to study the equilibrium fluctuation of these two conserved quantities under diffusive scaling. We define the interdistance and energy fluctuation field as follows:

$$u_t^\varepsilon(\psi, \omega) = \sqrt{\varepsilon} \sum_{k\in\mathbb{Z}} \psi(\varepsilon k)(r_k(t/\varepsilon^2) - \rho)$$

$$e_t^\varepsilon(\varphi, \omega) = \sqrt{\varepsilon} \sum_{k\in\mathbb{Z}} \varphi(\varepsilon k)(H_k(t/\varepsilon^2) - \frac{1}{\beta} - \frac{\rho^2}{2})$$

where φ and ψ are smooth functions with compact support. We prove that $\xi_t^\varepsilon = (u_t^\varepsilon, e_t^\varepsilon)$ as a vector of two distribution valued processes converges in law to the solution $\xi_t = (u_t, e_t)$ of the following pair of stochastic partial differential equations of generalized Ornstein-Uhlenbeck type:

$$(1) \qquad \begin{cases} du = \frac{1}{\gamma}\Delta u + \frac{\sqrt{2}}{\sqrt{\gamma\beta}}\nabla j_1 \\ de = \frac{\gamma + \frac{1}{\gamma}}{2}\Delta e + \frac{\frac{1}{\gamma} - \gamma}{2}\Delta(\rho u) + \frac{\sqrt{2}\rho}{\sqrt{\gamma\beta}}\nabla j_1 + \frac{\sqrt{\gamma + \frac{1}{\gamma}}}{\beta}\nabla j_2 \end{cases}$$

where j_1, j_2 are independent δ-correlated space-time white-noises.

What makes this linear system interesting is the fact that one of the conserved quantities, namely the energy is not a linear function of the system, and the investigation of its fluctuation field is non trivial. While the derivation of the stochastic differential equation for u is straightforward, to obtain that for e one has to face two difficulties. Firstly we have to get rid of the singularity coming from the asymmetric part of the generator. Secondly we need to replace the microscopic current with linear functions of the conserved quantities.

For the sake of simplicity we suppose now that $\gamma = \beta = 1$, $\rho = 0$, and we introduce the shorthand notation $\nabla_1 a_k = a_{k+1} - a_k$, $\nabla_1^* a_k = a_{k-1} - a_k$, $\Delta_1 a_k = a_{k+1} + a_{k-1} - 2a_k$. Then the stochastic equation for H_k is of the following form:

$$dH_k = -\nabla_1^* J_k + \frac{1}{2}\Delta_1 p_k^2 + \nabla_1 p_{k-1} p_k dW_k$$

where $J_k = \frac{1}{2} r_k \left(p_k + p_{k+1}\right)$ is of non gradient type and this causes the singularity.

However using the fact that $\mathcal{S}J_k = -J_k$, J_k can be written as the sum of a gradient term and a remainder which adds up only to the martingale part:

$$J_k = \frac{1}{2}\nabla_1 \left(p_k^2 + r_{k-1} r_k\right) - \mathcal{L}J_k,$$

and this eliminates the singularity.

The next task is to replace p_k^2 and $r_{k-1} r_k$ with linear functions of the conserved quantities. Actually $r_{k-1} r_k$ will disappear in the limit of the fluctuation field while to treat p_k^2 we can again get use of the generator of the system, namely:

$$p_k^2 = \frac{p_k^2 + r_k^2}{2} + \frac{p_k p_{k+1} - p_k r_k}{2} - \frac{r_{k-1} r_k}{2} - \frac{\mathcal{L} p_k r_k}{2}.$$

Here $\mathcal{L} p_k r_k$ results in a vanishing martingale term, while being eigenfunctions of $\mathcal{S}$ an H_{-1} norm estimate shows that $p_k p_{k+1}$ and $p_k r_k$ will also disappear. The vanishing of $r_{k-1} r_k$ is shown again with the help of the generator $\mathcal{L}$ applied to a well chosen function, using H_{-1} norm estimates and some new ingredients. As its treatment is more involved we skip technical details here. The treatment of the martingale terms needs no special effort. The proof ends up with showing that the limit of the fluctuation fields is the unique solution of the martingale problem corresponding to the pair of equations (1) .

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Random walking shocks in interacting particle systems

Márton Balázs

It has been known for a while that many models from the field of stochastic interacting particle systems have deterministic partial differential equations as scaling limit, see e.g. the book of Kipnis and Landim [10], or Seppäläinen [11], Tóth and Valkó [12]. In the so-called *Eulerian scaling* this equation is usually a Burger-type one, developing shocks as entropy solutions from any decreasing initial data. The microscopic structure of these shocks is of great interest. Many results construct such structures from the viewpoint of the so-called *second class particle*, which is an object coming from *coupling* of these models. Some examples are De Masi, Kipnis, Presutti and Saada [5], Derrida, Lebowitz and Speer [6], Ferrari [7], and Ferrari, Fontes and Kohayakawa [8]. As this particle has a complex random motion in the system, it is not immediately clear how these structures look like from a fixed, non-moving position. On the other hand, this way of looking at the problem makes it difficult to deal with the case of multiple shocks. Ferrari, Fontes and Vares develop these methods and handle this case in [9].

Another work in this area is by Balázs [1], who introduces a simple product measure which shows the properties of a shock, and is stationary as seen by the second class particle. This result is valid for the so-called *exponential bricklayers' process*. Our aim here is to identify a structure corresponding to this distribution, but this time as seen from a fixed, non moving position. This way of looking at the problem makes it easy to handle multiple shocks as well, thus we can also investigate how these shocks interact with each other at the microscopic level.

We consider the class of *bricklayers' processes*, introduced in [1] and [2] based on ideas of B. Tóth. These models are slight generalizations of the misanthrope (see Cocozza-Thivent [4]) and also of the zero range processes. Each model from this class has a special one-parametered family of one dimensional discrete measures. If we fix that parameter and build the product of these measures for the sites, then we obtain a time-stationary distribution of the model. Hence we have a one-parametered family of equilibrium distributions; this parameter sets the *slope of the wall*, a quantity corresponding to particle density in particle systems.

We consider product measures with the same marginals as for the stationary distribution, except for that we allow the parameter to change from site to site. The main result states that distributions in this class evolve to linear combinations of distributions from the same class, but only in the special case of *exponential* bricklayers' process. The form of the linear coefficients allows us to interpret some of the situations as ordinary random walks of shocks having discontinuity of size one. We also obtain the nature of interaction between these shocks. It follows that a group formed by a number of such one-sized jump shocks is of stochastically bounded size in time, i.e. shocks of larger (integer-valued) discontinuities represented by such a group are sharp under any kind of hydrodynamic scaling.

Belitsky and Schütz [3] derive results similar to ours for the simple exclusion process with the use of a quantum algebra symmetry. Unfortunately, we were

not able to develop the quantum formalism for our locally infinite state space systems. However, there is a remarkable analogue between their work and the present settings: in [3] special relations are required between the shock densities and the particle jump rates. These relations are identical to those of Derrida, Lebowitz and Speer [6] allowing there an exact (not only asymptotic) product-description of the stationary distribution as seen from the second class particle. In the present setting we specially require integer values for the size of discontinuities of our shocks, in a very similar way as needed for the exact result of Balázs [1] from the viewpoint of the second class particle.

Some open questions

In the case of the simple exclusion process, Derrida et al. [6] use matrix product methods, Belitsky and Schütz [3] have a quantum matrix-algebra showing a special commutation symmetry. How can one introduce the corresponding formalism in our more general models? Where is the symmetry of type [3] for the exponential bricklayers' process?

We have two kind of shock-measures, one stationary as seen from the second class particle, and one performing ordinary random walk. How are these two related to each other? Is the latter ordinary random walk obtained by some integrated law of the second class particle w.r.t. the shock-measure? Can this integrated law be Markovian at all?

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Condensation in the Zero Range Process

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(joint work with G.M. Schütz and H. Spohn)

The zero range process (ZRP) is an interacting particle system without exclusion interaction and was originally introduced in [1]. The state space is $X = \mathbb{N}^\Lambda$ with $\mathbb{N} = \{0, 1, 2, \ldots\}$, where we take $\Lambda \subset \mathbb{Z}^d$ to be an arbitrary translation invariant lattice. The jump rate $g(\eta_x)$ of a particle at site $x \in \Lambda$ depends only on the occupation number $\eta_x \in \mathbb{N}$, and the leaving particle jumps to a new site $x + y$ according to an irreducible probability distribution $p(y)$ of finite range R. In general the process is non-reversible for asymmetric p but this does not affect results on the stationary measures.

There are rigorous results on the existence of the dynamics on general (infinite) lattices when the rate function $g : \mathbb{N} \to \mathbb{R}$ is sublinear [2]. In this case the generator

$$(\mathcal{L}f)(\boldsymbol{\eta}) = \sum_{x\in\Lambda} \sum_{y=-R}^{R} g(\eta_x)\, p(y) \big(f(\boldsymbol{\eta}^{x,x+y}) - f(\boldsymbol{\eta}) \big), \tag{1}$$

with Lipschitz continuous $f : X \to \mathbb{R}$ defines a Markov process on X. Since we consider translation invariant dynamics and lattices there are stationary product measures $\bar\nu_\phi$ with site independent marginals (see also [2])

$$\bar\nu_\phi^1(\eta_x) = W(\eta_x)\, \phi^{\eta_x} / Z(\phi) \ , \quad W(\eta_x) = \prod_{k=1}^{\eta_x} 1/g(k) \ , \tag{2}$$

where $\phi > 0$ is the fugacity and $Z(\phi) = \sum_{k\in\mathbb{N}} W(k)\, \phi^k$ the normalizing partition function. The measures are defined for all $\phi \in D_\phi = [0, \phi_c)$ where ϕ_c is the radius of convergence of Z. The particle density $R(\phi) = \langle \eta_x \rangle_{\bar\nu_\phi}$ is monotonic increasing in ϕ and invertible, with range $D_\rho = R(D_\phi) = [0, \rho_c)$. If g is a non-decreasing function then $\rho_c = \infty$ and there are product measures for every density.

In [3] it was found that for decreasing rates $g_{b,\gamma}(k) = \mathbb{1}_{k>0}\big(1 + b/k^\gamma\big)$ with either $\gamma \in (0, 1)$, $b > 0$ or $\gamma = 1$, $b > 2$ it is $D_\phi = [0, 1]$ and $D_\rho = [0, \rho_c]$. So there are no product measures beyond a finite critical density $\rho_c < \infty$. Fixing initial conditions with density $\rho > \rho_c$ the system is expected to phase separate into a homogeneous background with density ρ_c and a condensate with vanishing volume fraction where all excess particles accumulate. This condensation is of particular importance for two-species exclusion processes, where the ZRP with rates $g_{b,\gamma}$ can be used as an effective model for domain wall dynamics, providing a criterion for

phase separation [4]. On the other hand, as an (exactly solvable) example of a condensation transition this phenomenon is intriguing already on the level of the ZRP and there have been only non-rigorous studies so far.

Results

For general jump rates g on finite, translation invariant lattices Λ_L of size L there exist canonical stationary measures

$$\mu_{L,N}(\boldsymbol{\eta}) = \frac{1}{Z(L,N)} \prod_{x\in\Lambda_L} W(\eta_x)\, \delta\Big(\sum_{x\in\Lambda_L} \eta_x, N \Big) \;, \tag{3}$$

with fixed number of particles $N \in \mathbb{N}$ and canonical partition function $Z(L,N)$. We prove the above intuition on the condensation phenomenon by generalizing a standard result for $\rho \in D_\rho$ on the equivalence of canonical (3) and grand canonical ensemble (2) to densities $\rho > \rho_c$.

Theorem 1. *Define the extended inverse of* $R(\phi)$, $\bar\Phi(\rho) = \begin{cases} \Phi(\rho) & , \text{ for } \rho \in D_\rho \\ \phi_c & , \text{ for } \rho \notin D_\rho \end{cases}$. *Then for every* $\rho \in [0,\infty)$ *and bounded cylinder test functions*

$$\mu_{L,[\rho L]} \xrightarrow{w} \bar\nu_{\bar\Phi(\rho)} \;, \quad \text{for } L\to\infty \;. \tag{4}$$

For $\rho > \rho_c$ this proves the existence of the background phase with uniform product distribution $\bar\nu_{\phi_c}$. In general, the one-point marginal $\bar\nu^1_{\phi_c}$ decays subexponentially and the next theorem shows that for some special cases the condensed phase typically consists only of a single site if L is large.

Theorem 2. *Let* $\bar\nu^1_{\phi_c}(k) \simeq k^{-b}$ *have a monotonic decreasing power law tail with* $b > 2$ *and finite first moment* ρ_c. *Then for every* $\rho > \rho_c$ *the normalized maximum occupation number satisfies a weak law of large numbers*

$$\frac{1}{(\rho-\rho_c)L} \max_{x\in\Lambda_L} \eta(x) \overset{\mu_{L,[\rho L]}}{\longrightarrow} 1 \;, \quad \text{for } L\to\infty \;. \tag{5}$$

The proof uses large deviation estimates for subexponential distributions (see [5] and references therein), generalizing results of [6].

The first result can be generalized to ZRPs with two particle species, which have been introduced in [7, 8]. Such systems show a much richer phase diagram involving simultaneous condensation of both species [9]. A general rigorous analysis given in [10] shows that the fugacities $(\bar\Phi_1, \bar\Phi_2)(\rho_1,\rho_2)$ of the background phase are given by the unique maximizer of the thermodynamic entropy

$$S(\rho_1,\rho_2) = \sup_{(\phi_1,\phi_2)\in D_\phi} \Big(\rho_1 \log\phi_1 + \rho_2 \log\phi_2 - \log Z(\phi_1,\phi_2) \Big) \;. \tag{6}$$

Studies of the relaxation dynamics for uniform initial conditions [5, 10, 11] reveal an interesting coarsening phenomenon, which – in contrast to stationary results – depends on space dimension and reversibility of the process.

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Nonequilibrium CLT for a tagged particle in symmetric simple exclusion

Milton Jara

(joint work with Claudio Landim)

I report the first nonequilibrium central limit theorem for a tagged particle. Consider the one-dimensional nearest neighbor symmetric situation. In this context, as already observed by Arratia [1], the scaling changes dramatically since to displace the tagged particle from the origin to a site $N > 0$, all particles between the origin and N need to move to the right of N. This observation relates the asymptotic behavior of the tagged particle to the hydrodynamic behavior of the system. The correct scaling for the law of large numbers should therefore be X_{tN^2}/N and we expect $(X_{tN^2} - E[X_{tN^2}])/\sqrt{N}$ to converge to a Gaussian variable.

The central limit theorem in equilibrium was obtained by Rost and Vares [2] for a slightly different model. They proved that for each fixed $t > 0$, $X_{tN^2}/\sqrt{N}$ converges to a fractional Brownian motion W_t with variance given by $E[W_t^2] = \alpha t^{1/2}$. We extend their result to the nonequilibrium case.

The idea of the proof is to relate the position of the tagged particle to the well known hydrodynamic behavior of the symmetric exclusion process. Since particles

cannot jump over other particles, the position of the tagged particle is determined by the current over one bond and the density profile of particles. Therefore, a nonequilibrium central limit theorem for the position of the tagged particle follows from a joint central limit theorem for the current and the density profile. Since the current over a bond can itself, at least formally, be written as the difference between the mass at the right of the bond at time t and the mass at time 0, a central limit theorem for the position of the tagged particle should follow from a nonequilibrium central limit theorem for the density field.

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A microscopic interpretation of Stefan's melting and freezing problem

Glauco Valle

(joint work with Claudio Landim)

In this work we return to the classical Stefan's freezing on the ground model [3]. It could be described in the following way: Consider the real line occupied by a heat conducting material (heat is transmitted only by conduction). This material is initially almost everywhere characterized by a bounded and measurable temperature function $T : \mathbb{R} \to \mathbb{R}$. According to the temperature the material could be in one of two phases, a liquid phase for positive temperatures and a solid phase for negative temperatures. The temperature $T = 0$ is that of crystallization at which both phases may occur. Then it is required the determination of the temporal evolution of the temperature.

Lets consider this problem under more restrictive conditions. Suppose that at initial time the liquid phase fills the domain $0 < u < \infty$ at positive temperatures and the solid phase fills the domain $-\infty < u < 0$ at negative temperatures. We are able to determine a function $B = B(t)$ describing the time evolution of the boundary between the two phases and their temperature functions, respectively $\rho_1(t,u)$ and $\rho_2(t,u)$ for the solid and liquid phases. It is well known that these functions satisfy a Cauchy-Stefan Problem:

$$
(1) \qquad \begin{cases} \partial_t \rho_1 = a_1 \partial_{uu} \rho_1 \,;\ \partial_t \rho_2 = a_2 \partial_{uu} \rho_2 \\ \frac{dB(t)}{dt} = k\{a_1 \partial_u \rho_1(t, B(t)) - a_2 \partial_u \rho_2(t, B(t))\} \\ \rho_i(t, B(t)) = 0 \\ \rho_i(0, \cdot) = \rho_i^0(\cdot) \\ B(0) = 0\,, \end{cases}
$$

where $\rho_1^0 : \mathbb{R}_- \to \mathbb{R}_-$ and $\rho_2^0 : \mathbb{R}_+ \to \mathbb{R}_+$ are bounded measurable functions, $a_1 \geq 0$ and $a_2 > 0$ are the coefficients of heat conduction of the material with respect to the solid and liquid phases and $k > 0$ is a scaling factor for the temperature.

We give a simplified microscopic description of Stefan's Model Throughout Appropriated Interacting Particle Systems and scaling limit techniques. Such sort of description have already been studied previously by Chayes and Swindle [1] in the case of finite domains and coefficient $a_1 = 0$.

For the informal description of the microscopic model, consider the one dimensional lattice $\mathbb{Z}$ with each site being occupied by a molecular agglomerate of type 1 for the material in the solid state and of type 2 for the material in the liquid state. According to its internal energy, each agglomerate is classified by a heat unit of 0 or -1 for agglomerates of type 1 and 0 or 1 for agglomerates of type 2. A interaction between neighboring sites occurs independently in the following way: If the agglomerates are of the same type then their heat units are interchanged after a exponential time of mean a_1 for agglomerates of type 1 and after a exponential time of mean a_2 for agglomerates of type 2. If the agglomerates are of distinct type and the absolute value of their heat units are also distinct, the heat unit of absolute value 1 drops to 0 and the type of the agglomerate whose unit had absolute value 0 changes after a mean one exponential time. If the agglomerates are of distinct type and the absolute value of their heat units are equal to 1, both heat units drop to 0 after a mean one exponential time. Moreover, we start with configurations such that the agglomerates are of type 1 if they are at the left of the origin, otherwise they are of type 2. We also suppose that initially each site of $\mathbb{Z}$ (resp. $\mathbb{N}$) is occupied by an agglomerate of type 1 (resp. of type 2)

This model could be described as a coupling between two one dimensional nearest-neighbor simple symmetric exclusion processes in the semi-infinite lattice. To make this identification we simply put a particle at each site whose associated agglomerate has heat unit of absolute value 1 and say directly that a site is of type 1 or 2 according to the type of the associated agglomerate.

We show that this system has a hydrodynamical behavior under diffusive scaling whose hydrodynamical equation is a Cauchy-Stefan Problem of the type (1) with scaling factor $k = 1$, where the temperature is the macroscopic heat density profile. The general case with an arbitrary k can be obtained from the previous one rescaling the temperature by k^{-1}.

What makes this an interesting problem is the fact that the system considered is non-conservative and the usual methods to establish the Hydrodynamical behavior could not be applied without an adaptation which is particular to the model studied. We also refer to [4] and [2], where the hydrodynamics for other non-conservative one-dimensional systems is considered.

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Macroscopic current fluctuations in stochastic lattice gases

G. Jona-Lasinio
(joint work with L. Bertini, A. De Sole, D. Gabrielli, C. Landim)

The basic microscopic model is given by a stochastic lattice gas with a weak external field and particle reservoirs at the boundary. More precisely, let $\Lambda \subset \mathbb{R}^d$ be a smooth domain and set $\Lambda_N = N\Lambda \cap \mathbb{Z}^d$; we consider a Markov process on the state space X^{Λ_N}, where X is a subset of $\mathbb{N}$. The number of particles at the site $x \in \Lambda_N$ is denoted by $\eta_x \in X$ and the whole configuration by $\eta \in X^{\Lambda_N}$. The dynamics of the particles is described by a continuous time Markov process on the state space X^{Λ_N} with transition rates $c_{x,y}(\eta)$ from a configuration η to the configuration obtained from η by moving a particle from x to a neighbor site y. Similar rates $c_x^{\pm}$ describe the appearance or loss of a particle at the boundary site x. We assume the rates satisfy the *local detailed balance*, see [4, II.2.6]. The reservoirs are characterized by a chemical potential γ.

We introduce the empirical measure π^N corresponding to the density as follows. For each microscopic configuration $\eta \in X^{\Lambda_N}$ and each smooth function $G : \Lambda \to \mathbb{R}$, we set $\pi^N(G) = N^{-d} \sum_{x \in \Lambda_N} G(x/N)\eta_x$ Consider a sequence of initial configurations η^N such that $\pi^N(\eta^N)$ converges weakly to some density profile ρ_0. Under diffusive scaling, the empirical density at time t converges weakly, as $N \to \infty$, to $\rho = \rho(t, u)$ which is the solution of the hydrodynamic equation [3, 4]

$$\partial_t \rho = \nabla \cdot \Big[\frac{1}{2} D(\rho) \nabla \rho - \chi(\rho) \nabla V \Big]$$

with initial condition ρ_0 and boundary condition fixed by the reservoirs. Here D is the diffusion matrix, given by the Green–Kubo formula, see [4, II.2.2], χ is the conductivity, obtained by linear response theory, see [4, II.2.5], and ∇V the external field.

We now introduce the empirical current as follows. Denote by $\mathcal{N}_t^{x,y}$ the number of particles that jumped from x to y in the macroscopic time interval $[0, t]$. Here we adopt the convention that $\mathcal{N}_t^{x,y}$ represents the number of particles created at y due to the reservoir at x if $x \notin \Lambda_N$, $y \in \Lambda_N$ and that $\mathcal{N}_t^{x,y}$ represents the number of particles that left the system at x by jumping to y if $x \in \Lambda_N$, $y \notin \Lambda_N$. The difference $J_t^{x,y} = \mathcal{N}_t^{x,y} - \mathcal{N}_t^{y,x}$ represents the total current across the bond $\{x, y\}$ in the time interval $[0, t]$. Fix a macroscopic time T and denote by $\mathcal{J}^N$ the empirical measure on $[0, T] \times \Lambda$ associated to the current. For smooth vector fields $G = (G_1, \dots, G_d)$, the integral of G with respect to $\mathcal{J}^N$ is given by $\mathcal{J}^N(G) = N^{-(d+1)} \sum_{i=1}^d \sum_x \int_0^T G_i(t, x/N)\, dJ_t^{x,x+e_i}$, where e_i is the canonical basis and we sum over all x such that either $x \in \Lambda_N$ or $x + e_i \in \Lambda_N$. We normalized $\mathcal{J}^N$ so that it is finite as $N \to \infty$. Given a density profile ρ let us denote by $J(\rho) = -\frac{1}{2} D(\rho)\nabla\rho + \chi(\rho)\nabla V$ the current associated to ρ. If we consider an initial configuration η^N such that the empirical density $\pi^N(\eta^N)$ converges to some density profile ρ_0, then the empirical current $\mathcal{J}^N(t)$ converges, as $N \to \infty$, to $J(\rho(t))$, the current associated to the solution of the hydrodynamic equation.

We next discuss the large deviation properties of the empirical current. Fix a smooth vector field $j : [0,T] \times \Lambda \to \mathbb{R}^d$ and a sequence of configurations η^N whose empirical density converges to some profile ρ_0. Then, by the methods in [3, Ch. 10], it is possible to show that

$$\mathbb{P}^N_{\eta^N}\big(\mathcal{J}^N(t,u) \approx j(t,u)\big) \sim \exp\big\{ - N^d \mathcal{I}_{[0,T]}(j)\big\}$$

where the rate function is given by

$$\mathcal{I}_{[0,T]}(j) \;=\; \frac{1}{2}\int_0^T dt\, \big\langle [j - J(\rho)], \chi(\rho)^{-1}[j - J(\rho)]\big\rangle$$

in which $\rho = \rho(t,u)$ is obtained by solving the continuity equation $\partial_t \rho + \nabla \cdot j = 0$ with initial condition $\rho(0) = \rho_0$ and $\langle \cdot, \rangle$ is the inner product in $L_2(\Lambda, du)$. Of course there are compatibility conditions to be satisfied, for instance if we have chosen a j such that $\rho(t)$ becomes negative for some $t \in [0,T]$ then $\mathcal{I}_{[0,T]}(j) = +\infty$.

We want to study the fluctuations of the time average of the empirical current over a large time interval $[0,T]$; the corresponding probability can be obtained from the space–time large deviation principle. Fix some divergence free vector field $J = J(u)$ constant in time and denote by $\mathcal{A}_{T,J}$ the set of all currents j such that $T^{-1}\int_0^T dt\, j(t,u) = J(u)$. The condition of vanishing divergence on J is required by the local conservation of the number of particles. By the large deviation principle, for T large we have

$$\mathbb{P}^N_{\eta^N}\Big(\frac{1}{T}\int_0^T dt\, \mathcal{J}^N(t) \approx J\Big) \sim \exp\big\{ - N^d T \Phi(J)\big\}$$

where the logarithmic equivalence is understood by sending *first* $N \to \infty$ and *then* $T \to \infty$. The functional Φ is given by

$$\Phi(J) = \lim_{T\to\infty} \inf_{j \in \mathcal{A}_{T,J}} \frac{1}{T}\, \mathcal{I}_{[0,T]}(j)$$

By a standard sub–additivity argument it is indeed easy to show that the limit exists. One can also prove that Φ is a convex functional.

The next step is the study the variational problem on the right hand side of the previous equation. We derive an upper and a lower bound. Given $\rho = \rho(u)$ and $J = J(u)$, $\nabla \cdot J = 0$, let us introduce the functionals

$$\begin{aligned} \mathcal{U}(\rho, J) &= \frac{1}{2}\langle J - J(\rho), \chi(\rho)^{-1}[J - J(\rho)]\rangle \\ U(J) &= \inf_{\rho}\, \mathcal{U}(\rho, J) \end{aligned}$$

where the minimum is carried over all profiles ρ satisfying the boundary conditions. When J is constant, that is in the one–dimensional case, the functional U is the one introduced in [2].

From the convexity of $\Phi(J)$ it immediately follows that

$$\Phi(J) \le U^{**}(J)$$

where $U^{**}(J)$ is the convex envelope of U

To obtain the lower bound let us denote by $\tilde{\mathcal{U}}$ and $\tilde{U}$ the same functionals introduced previously, but now defined on the space of all currents without the conditions of vanishing divergence. Let also $\tilde{U}^{**}$ be the convex envelope of $\tilde{U}$.

Let $j \in \mathcal{A}_{T,J}$. By the convexity of $\tilde{U}^{**}$ in the set of all currents, we get

$$\begin{aligned} \frac{1}{T}\mathcal{I}_{[0,T]}(j) &= \frac{1}{T}\int_0^T dt\, \tilde{\mathcal{U}}(\rho(t), j(t)) \geq \frac{1}{T}\int_0^T dt\, \tilde{U}(j(t)) \\ &\geq \frac{1}{T}\int_0^T dt\, \tilde{U}^{**}(j(t)) \geq \tilde{U}^{**}(J) \end{aligned}$$

The upper and lower bounds are, in general, different. For a divergence free J we have $\tilde{U}(J) = U(J)$ but since the convex envelopes are considered in different spaces, we only have $\tilde{U}^{**}(J) \leq U^{**}(J)$.

To understand the physical meaning of the convex envelope, suppose $J = pJ_1 + (1-p)J_2$ and $U(J) > U^{**}(J) = pU(J_1)+(1-p)U(J_2)$ for some p, J_1, J_2. The values p, J_1, J_2 are determined by J and U. In addition we assume that $U^{**} = \tilde{U}^{**}$. If we condition on observing an average current J, the corresponding density profile is not determined, but rather we observe with probability p the profile $\hat{\rho}(J_1)$ and with probability $1-p$ the profile $\hat{\rho}(J_2)$. When U is not convex we have thus a situation in which the time averaged current J is realized with the coexistence of two dynamical regimes: we have a dynamical phase transition.

The derivation of the upper bound shows that, if U is not convex, our result differs from the one in [2]. On the other hand if $\tilde{U}^{**}(J) < U^{**}(J)$ it is possible that one can improve it by exploring currents with non vanishing divergence. In such a situation it is not clear to us if Φ can be directly related to U.

In the models where the diffusion coefficient $D(\rho)$ is constant and the mobility $\chi(\rho)$ is concave, for example in the symmetric simple exclusion where $\chi = \rho(1-\rho)$, it is not difficult to see that $\tilde{U}$ is convex. Therefore in these cases $\Phi = U$. In a forthcoming paper we shall discuss models where non convexities appear.

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Large deviation functional and fluctuations of density in the WASEP

Camille Enaud

(joint work with Bernard Derrida)

The main results presented in this talk are published in [1] and [3]. We consider the one dimensional weakly asymmetric simple exclusion process (WASEP) with open boundaries where the weak asymmetry scales like $q = 1 - \frac{\lambda}{L}$ with the system

size L. The system is in contact with two reservoirs at densities ρ_a and ρ_b which exchange particles with respectively the first and the last site of the system. The dynamics of this system is formally defined by the following generator $\mathbb{L}$ acting on given functions g of the system configuration $\eta = \{\eta_i\}_{i=1,\dots,L}$

$$\begin{aligned}
\mathbb{L}g(\eta) &= \sum_{i=1}^{L-1} \eta_i(1-\eta_{i+1})\big[g(\eta^{i,i+1}) - g(\eta)\big] \\
&+ \sum_{i=2}^{L} q\eta_i(1-\eta_{i-1})\big[g(\eta^{i-1,i}) - g(\eta)\big] \\
&+ \big[\rho_a(1-\eta_1) + q(1-\rho_a)\eta_1\big]\big[g(\eta^1) - g(\eta)\big] \\
&+ \big[q\rho_b(1-\eta_L) + (1-\rho_b)\eta_L\big]\big[g(\eta^L) - g(\eta)\big] \ ,
\end{aligned}$$

where $\eta^{i,i+1}$ is the configuration obtained from η by exchanging the occupation numbers of sites i and $i+1$ and η^i is obtained from η by changing the occupation number of site i.

Using an ansatz called the matrix method [4], we show [1] that the steady state probability of a given macroscopic profile $\{\rho\}$ can be written, in a certain range of boundary parameters, as a sum over weighted paths $\{y(x)\}$,

$$\mathrm{d}\mathbb{P}(\{\rho\}) \sim \mathcal{D}[\{\rho\}] \int_{\{y\}} e^{-LG(\{\rho\},\{y\})} \mathcal{D}[\{y\}] \tag{1}$$

where the sum is over all positive continuous functions $\{y(x), 0 \le x \le 1\}$, and the functional G is explicitly calculated in [1], equation (3.22). This allows us to derive the large deviation functional $\mathcal{F}$ of the WASEP,

$$\begin{aligned}
\mathcal{F}(\{\rho(x)\};\rho_a,\rho_b) = -K + \int_0^1 dx \Big\{ &\rho \log\frac{\rho}{F} + (1-\rho)\log\frac{1-\rho}{1-F} \\
&+ \log\left(F(1-F)\lambda - F'\right) + \frac{F'}{\lambda F(1-F)} \log\left(-\frac{F'}{F(1-F)\lambda - F'}\right)\Big\}
\end{aligned} \tag{2}$$

where the function $F(x)$ is the solution of the following differential equations

$$(F-\rho)F'^2 + F(1-F)F'' + \lambda F(1-F)(F-1+\rho)F' = 0 \tag{3}$$

with the boundary conditions

$$F(0) = \rho_a \qquad\qquad F(1) = \rho_b \tag{4}$$

This large deviation functional interpolates between two known results, the large deviation functional of the symmetric case (SSEP) (where $q = 1$) [5] and the one for the asymmetric case (ASEP) (where q is a constant smaller than 1) [6].

Another consequence of this "sum over paths" picture is the computation of the statistical properties of the fluctuations of density $\rho(x)$ around the average profile $\bar{\rho}(x)$ in the steady state [3]. We wrote them as a sum over two independent

Gaussian processes,

$$\sqrt{L}(\rho(x)-\bar\rho(x)) = \sqrt{\frac{\bar\rho(x)(1-\bar\rho(x))}{2}}B'(x) + \frac{1}{2}Y'(x)\ , \tag{5}$$

where $B(x)$ is a standard Brownian motion normalized such that

$$\left\langle [B(x)-B(x')]^2 \right\rangle = |x-x'| \tag{6}$$

while $Y(x)$ is a centered Gaussian process whose distribution is formally given by

(7)

$$\mathrm{d}\mathbb{Q}(\{Y\}) \propto \exp\left\{ -\int_0^1 \mathrm{d}x \left(\frac{-J\bar\rho'(x)Y(x)^2}{2\bar\rho(x)^2(1-\bar\rho(x))^2} + \frac{Y'(x)^2}{4\bar\rho(x)(1-\bar\rho(x))} \right) \right\} \mathcal{D}\big[\{Y\}\big]$$

where $\mathcal{D}\big[\{Y\}\big]$ is the standard Feynman measure, and J is related to the average current j by $J = \lim_{L\to\infty} Lj$.

From this, we deduce the 2-point correlation function: for $x < y$, we have

$$L\left(\left\langle \eta_{[Lx]}\eta_{[Ly]} \right\rangle - \left\langle \eta_{[Lx]} \right\rangle \left\langle \eta_{[Ly]} \right\rangle \right) \underset{L\to\infty}{\longrightarrow} \frac{J\bar\rho'(x)\bar\rho'(y)\int_0^x \frac{\mathrm{d}u}{\bar\rho'(u)} \int_y^1 \frac{\mathrm{d}v}{\bar\rho'(v)}}{\int_0^1 \frac{\mathrm{d}u}{\bar\rho'(u)}} \ . \tag{8}$$

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On a non-hierarchical version of the Generalized Random Energy Model

Erwin Bolthausen

(joint work with Nicola Kistler)

The **Random Energy Model** (REM for short) had been introduced by Derrida as a simple model exhibiting spin glass behavior. Consider independent centered Gaussian random variables $(H_N(\alpha))_{\alpha\in\Sigma_N}$, $\Sigma_N \overset{\text{def}}{=} \{1,\dots,2^N\}$, with variance N. The partition function is defined by $Z_N(\beta) \overset{\text{def}}{=} \mathrm{tr}_\alpha \exp[\beta H_N(\alpha)]$,where tr_α denotes averaging over α. The "quenched" free energy is defined by $F_N(\beta) \overset{\text{def}}{=} (1/N)\log Z_N(\beta)$, and its limiting value $f(\beta) \overset{\text{def}}{=} \lim_{N\to\infty}(1/N)\log Z_N(\beta)$. The

Gibbs measure on Σ_N is $\mathcal{G}_{N,\beta}(\sigma) \stackrel{\text{def}}{=} Z_N(\beta)^{-1} 2^{-N} \exp[\beta H_N(\alpha)]$. This is a *random* probability distribution on Σ_N. $f(\beta)$ can be determined by an application of the second moment method:

$$f(\beta) = \begin{cases} \beta^2/2 & \text{for } \beta \leq \beta_{\text{cr}} \stackrel{\text{def}}{=} \sqrt{2\log 2} \\ \sqrt{2\log 2}\beta - \log 2 & \text{for } \beta \geq \beta_{\text{cr}} \stackrel{\text{def}}{=} \sqrt{2\log 2} \end{cases}.$$

$\beta^2/2$ is of course just the "annealed free energy" $\lim_N (1/N) \log \mathbb{E} \operatorname{tr} \mathrm{e}^{\beta H}$.

More interesting is the description of the limiting Gibbs distribution. For a continuous function $f : \mathbb{R}^+ \to \mathbb{R}^+$, let $PPP(t \to f(t))$ stand for a Poisson point process on $\mathbb{R}^+$ with intensity measure f. If f is integrable at ∞, then a realization of the points has a maximal element, and we can order them downwards: $\eta_0 > \eta_1 > \eta_2 > \dots$. In case $\sum_i \eta_i < \infty$, we can "normalize" the point process by introducing the random points $\overline{\eta}_i \stackrel{\text{def}}{=} \eta_i / \sum_j \eta_j$. We denote this operation on point processes by $\mathcal{N}(\cdot)$, which then is a *random* probability distribution. With these settings, one proves that for $\beta > \sqrt{2\log 2}$, the point process $\sum_\alpha \delta_{\mathcal{G}_{N,\beta}(\alpha)}$ converges weakly as $N \to \infty$ to $\mathcal{N}\left(PPP\left(t \to x t^{-x-1}\right)\right)$, where $x(\beta) \stackrel{\text{def}}{=} \sqrt{2\log 2}/\beta$.

For $\beta \leq \sqrt{2\log 2}$, the limiting Gibbs distribution is in a sense trivial: No single configuration $\alpha \in \Sigma_N$ charges a positive mass in the limit, i.e. $\lim_{N\to\infty} \sup_\alpha \mathcal{G}_{N,\beta}(\alpha) = 0$, almost surely. For proofs of these facts, see [7].

The REM is too simple to shed much light on spin glasses, as there are no correlations between the energies. Derrida [5] introduced the **Generalized Random Energy Model** (GREM for short) which is a model with correlations. These correlations are tree-like organized. Consider a tree with 2^N leaves and K branching levels, with N large, where K stays fixed. We write the leaves as $\alpha = (\alpha_1, \dots, \alpha_K)$ where $\alpha_i \in \left\{1, \dots, 2^{N/K}\right\}$, and write again Σ_N for the collection of such α's. We attach centered Gaussian random variables with variance $\sigma_i^2 N$ to bonds of the tree at level i, where $\sum_i \sigma_i^2 = 1$, and then we sum these Gaussians along the tree to define the random energies for each leaf. Formally $H_N(\alpha) \stackrel{\text{def}}{=} \sum_{i=1}^K X^{(i)}_{\alpha_1,\dots,\alpha_i}$, where $X^{(i)}_{\alpha_1,\dots,\alpha_i}$ are independent Gaussians with variance $\sigma_i^2 N$. The $H_N(\alpha)$ are then evidently Gaussians with variance N, like in the REM case, but there are now correlations. Defining for $\alpha, \alpha' \in \Sigma_N$ the overlap

$$R(\alpha, \alpha') \stackrel{\text{def}}{=} \max\{i : (\alpha_1, \dots, \alpha_i) = (\alpha'_1, \dots, \alpha'_i)\}$$

one has $\mathbb{E} H_N(\alpha) H_N(\alpha') = N \sum_{i=1}^{R(\alpha,\alpha')} \sigma_i^2$. If $\sigma_1^2 > \sigma_2^2 > \dots > \sigma_K^2$, this model has K critical levels of "symmetry breaking" at $\beta_i^{\text{cr}} \stackrel{\text{def}}{=} \sqrt{2\log 2}/\sqrt{K}\sigma_i$. In the intervals $\left[\beta_{i-1}^{\text{cr}}, \beta_i^{\text{cr}}\right]$, the limiting free energy $f(\beta)$ is a quadratic polynomial, with curvature decreasing with i (see [5], [4]). Just below the first critical value, one still gets the annealed free energy $\beta^2/2$, and above the last, one has an affine function. I am not giving the exact expression which is not very revealing. More interesting is the behavior of the Gibbs measure. Let $x_i(\beta) \stackrel{\text{def}}{=} \sqrt{2\log 2}/\sqrt{K}\sigma_i\beta$. Surprisingly, if $\beta > \beta_K^{\text{cr}}$, the Gibbs distribution behaves as in the REM case: $\sum_\alpha \delta_{\mathcal{G}_N(\alpha)} \to \mathcal{N}\left(PPP\left(t \to x_K t^{-x_K-1}\right)\right)$. For $\beta \in \left(\beta_{i-1}^{\text{cr}}, \beta_i^{\text{cr}}\right)$, $2 \leq i \leq K$, the

statement has to be modified properly, as then $\lim_{N\to\infty} \sup_\alpha \mathcal{G}_N(\alpha) = 0$, again, but looking at appropriate margins, the same statement is true (with x_K replaced by x_{i-1}).

The GREM leads to an interesting clustering process which can be described in the following way: Order the random energies $(H_N(\alpha))$ downwards: $\eta_1^{(N)} > \eta_2^{(N)} > \eta_3^{(N)} > \dots$. Then determine at which level a pair $\left(\eta_i^{(N)}, \eta_j^{(N)}\right)$ has branched off. If one looks at the pairs (i,j) which have branched just at level $K-1$ (or later), this defines an equivalence relation, and therefore a partitioning on $\left\{1, 2, \dots, 2^N\right\}$. If one looks at the pairs which have branched at level $K-2$ or later, this defines a partitioning which is coarser than that before, and in this way one can go on, and gets a sequence of clusterings which get coarser and coarser. In the end, everything is clustered, of course. It turns out, that as $N \to \infty$, this sequence of clusterings converges in a sense to be made precise to a sequence of clusterings of $\mathbb{N}$. This sequence in the limit becomes independent of the Gibbs distribution. Furthermore, the limiting clusterings have a very explicit Markovian structure, a fact which had been found in [1]. For proofs of most of these facts, see [3].

A superficial generalization is the case where the branching number depends on the level, i.e. where $\#\alpha_i = 2^{\gamma_i N}$, $\sum_i \gamma_i = 1$. This leads after a trivial redefinition of the critical values to the same results.

The GREM plays in important rôle in spin glass theory. Originally invented as a simple model which exhibits replica symmetry breaking at various levels, it has become clear that more interesting models, like the celebrated one of Sherrington-Kirkpatrick, exhibit GREM-like behavior in the large N limit. Despite the spectacular recent progress in understanding the SK-model (see [6], [8]), many issues have not been clarified at all, the most prominent one being the so called *ultrametricity.* The GREM is of no use to investigate this, because it is ultrametrically organized from the start, so it offers *no* means to get an understanding why many systems are supposed to be ultrametric in the limit.

In [2] we have presented a simple, and natural, generalization of the GREM which has *no* built in ultrametric structure. We however show, that in the limit, the model is ultrametrically organized. We fix a number $n \in \mathbb{N}$, and consider the set $I = \{1, ..., n\}$, as well as a collection of positive real numbers $\{\sigma_J^2\}_{J\subset I}$ such that

$$\sum_{J\subset I} \sigma_J^2 = 1.$$

For convenience, we put $a_\emptyset \stackrel{\text{def}}{=} 0$. The relevant subset of I will be only the ones with positive a-value. For $A \subset I$ we set

$$\mathcal{P}_A \stackrel{\text{def}}{=} \{J \subset A : \sigma_J^2 > 0\},\ \mathcal{P} \stackrel{\text{def}}{=} \mathcal{P}_I.$$

For $n \in \mathbb{N}$, we set $\Sigma_N \stackrel{\text{def}}{=} \left\{1, \dots, 2^N\right\}$. We also fix positive real numbers γ_i, $i \in I$, satisfying

$$\sum_{i=1}^n \gamma_i = 1,$$

and write $\Sigma_N^i \stackrel{\text{def}}{=} \Sigma_{\gamma_i N}$, where for notational convenience we assume that $2^{\gamma_i N}$ is an integer. For $N \in \mathbb{N}$, we will label the "spin configurations" α as:

$$\alpha = (\alpha_1, ..., \alpha_n),\ \alpha_i \in \Sigma_N^i\ ,$$

i.e. we identify Σ_N with $\Sigma_N^1 \times \cdots \times \Sigma_N^n$. For $J \subset I$, $J = \{j_1, \ldots, j_k\}$, $j_1 < j_2 < \ldots < j_k$, we write $\Sigma_{N,J} \stackrel{\text{def}}{=} \prod_{s=1}^k \Sigma_N^{j_s}$, and for $\alpha \in \Sigma_N$, we write α_J for the projected configuration $(\alpha_j)_{j \in J} \in \Sigma_{N,J}$. Our spin glass Hamiltonian is defined as

$$H(\alpha) \stackrel{\text{def}}{=} \sum_{J \in \mathcal{P}} X^J_{\alpha_J} \tag{1}$$

where $X^J_{\alpha_J}$, $J \in \mathcal{P}$, $\alpha_J \in \Sigma_{N,J}$ are independent centered Gaussian random variables with variance $\sigma_J^2 N$. The $H(\alpha)$ are then Gaussian random variables with variance N. A special case is when $\mathcal{P} = \{I\}$, i.e. when only $\sigma_J^2 \neq 0$, in which case it has to be one. Then the $H(\alpha)$ are independent, i.e. one considers simply a set of 2^N independent Gaussian random variables with variance N. This is just the REM.

The GREM is a special case: It corresponds to the situation where the sets in $\mathcal{P}$ are "nested", meaning that $\mathcal{P}$ consists of an increasing sequence of subsets. Without loss of generality we may assume that in this case

$$\mathcal{P} = \{J_m : 1 \leq m \leq k\},\ J_m \stackrel{\text{def}}{=} \{1, \ldots, n_m\}, \tag{2}$$

where $1 \leq n_1 < n_2 < \ldots < n_k \leq n$.

Any of our models can be "coarse-grained" in many ways into a GREM. For that consider strictly increasing sequences of subsets of $I : \emptyset = A_0 \subset A_1 \subset \ldots \subset A_K = I$. We do not assume that the A_i are in $\mathcal{P}$. We call such a sequence a **chain** $\mathbf{T} = (A_0, A_1, \ldots, A_K)$. We attach weights $\hat{\sigma}^2_{A_j}$ to these sets by putting

$$\hat{\sigma}^2_{A_j} \stackrel{\text{def}}{=} \sum_{B \in \mathcal{P}_{A_j} \setminus \mathcal{P}_{A_{j-1}}} \sigma_B^2, \tag{3}$$

Evidently $\sum_{j=1}^K \hat{\sigma}^2_{A_j} = 1$, and if we assign random variables $H(\sigma, \mathbf{T})$, according to (1), we arrive after an irrelevant renumbering of I at a GREM of the form (2).

We define the partition functions $Z(\beta)$, and the free energies F_N, f in exactly the same way as before.

For any chain $\mathbf{T}$, we attach to our model a GREM $(H(\sigma, \mathbf{T}))_{\sigma \in \Sigma_N}$, as explained above, and then write $f(\mathbf{T}, \beta)$ for the corresponding limiting free energy. Our main result is that our generalization of the GREM does not lead to anything new in $N \to \infty$ limit, shedding (hopefully) some modest light on the "universality" of ultrametricity.

Theorem 1. *$f(\beta)$ is the free energy of a GREM. More precisely, there exists a chain $\mathbf{T}$ such that*

$$f(\beta) = f(\mathbf{T}, \beta)\ ,\ \beta \geq 0\,.$$

$f(\mathbf{T}, \beta)$ is minimal in the sense that

$$f(\beta) = \min_{\mathbf{S}} f(\mathbf{S}, \beta) \ ,$$

the minimum being taken over all chains $\mathbf{S}$.

There are also results about the limiting Gibbs distribution, and the overlap structure which are of the same nature, stating that the limiting Gibbs distribution is that of a REM, and the there is a limiting clustering, independent of the Gibbs weights. These results however need a kind of "irreducibility" assumption, and are in fact not true under the completely general assumptions given above. I am not going into details here.

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Hydrodynamics of asymmetric particle systems with open boundaries

Christophe Bahadoran

In this note we consider various generalizations of the one-dimensional asymmetric exclusion process on $\mathbb{Z}$. These systems are driven lattice gases, whose hydrodynamic limit is known to be governed by entropy solutions of scalar conservation laws of the form

$$\partial_t \rho(t,x) + \partial_x G(\rho(t,x)) = 0 \tag{1}$$

where $\rho(t,x) \in [0,1]$ is the density field, and $G(\rho)$ is the macroscopic flux. For instance, $G(\rho) = \rho(1-\rho)$ for the asymmetric exclusion process. See e.g. [6] or [10] for the theory of (1) and definitions of entropy solutions. Our purpose is 1) to describe the hydrodynamic behavior of these systems when they are restricted to some finite macroscopic region of size (say) 1 and coupled to reservoirs with prescribed densities at both ends; 2) to derive from this the stationary phase diagram, thus extending the result of [4] to more general systems and fluxes. We will now give some examples, describe the boundary dynamics and state the main results from [1].

1. Attractive systems with product invariant measures. A simple and interesting example is the exclusion process "with overtaking". We are given nonnegative coefficients β_j for $j \in \mathbb{Z} - \{0\}$, finitely many of them positive. A particle chooses right or left and moves to the first empty site in that direction, at rate $\beta_{\varepsilon j}$ if j particles are jumped over, with $\varepsilon = 1$ (resp. -1) if the jump is to the right (resp. left). We further assume that $\beta_{j+1} \le \beta_j$ for $j > 0$, $\beta_{j-1} \le \beta_j$ for $j < 0$, $\beta_1 + \beta_{-1} > 0$. The hydrodynamic limit follows from [9], with an explicit polynomial flux G vanishing at 0 and 1. Conversely, every polynomial flux vanishing at 0 and 1 can be generated by suitable parameters.

2. KLS-type models. Here a particle at site x chooses right (resp. left) with probability p (resp. $1-p$), and hops to the next site in this direction with rate $b(d^+, d^-)$, where d^+ (resp. d^-) is the number of vacant sites between x and the next particle in the chosen (resp. opposite) direction. We assume $b(n,m) = b(n \wedge r, m \wedge r)$ for some r (finite-range assumption), b nondecreasing (resp. nonincreasing) in its first (resp. second) argument, $b(0,.) = 0$. If b satisfies some algebraic relations (see [3]), the hydrodynamic limit (1) can be deduced from [9], with an explicit G. This model contains the following KLS model introduced in [7]: particles jump only to the right, with jump rates given (for $|\delta|, |\varepsilon| \le 1$) by

$$\begin{array}{lll} 0100 \to 0010 & \text{with rate} & 1+\delta \\ 1100 \to 1010 & \text{with rate} & 1+\varepsilon \\ 0101 \to 0011 & \text{with rate} & 1-\varepsilon \\ 1101 \to 1011 & \text{with rate} & 1-\delta \end{array}$$

with the additional condition $\varepsilon \ge |\delta|$. For large ε, the macroscopic flux exhibits two local maxima separated by a local minimum (see [8]).

The boundary dynamics. A configuration $\eta \in \{0,1\}^{\mathbb{Z}}$ on the infinite lattice is decomposed as follows: $\eta = (\eta_-, \eta_0, \eta_+)$, where η_- is the restriction to sites $x \le 0$, η_0 to sites $1, \dots, N$, and η_+ to sites $x > N$. For a single reservoir on the left with density ρ_l, the generator of the semi-infinite system on $\mathbb{N} - \{0\}$ is defined by

$$L^l_{\rho_l} f(\eta_0, \eta^+) = \mathbb{E}_{\nu_{\rho_l}}[Lf(\eta_-, \eta_0, \eta_+)|(\eta_0, \eta_+)] \tag{2}$$

where L denotes the generator of the infinite system on $\mathbb{Z}$, and $f(\eta_0, \eta_+)$ is a test function. Note that Lf depends also on η_-, because of the interaction between the reservoir and the bulk. Hence the conditional expectation in (2), which means that the boundary dynamics is a virtual extension of the bulk dynamics, where jump rates from or to the reservoir are averaged (given the bulk configuration) with respect to the virtual reservoir configuration η_-, forced to be in the steady state ν_{ρ_l}. Of course, in $L^l_{\rho_l}$, jumps from (resp. to) the reservoir are seen as particle creations (resp. annihilations) after averaging. Similarly one may define the generator of the semi-infinite system at sites $x \le N$ with a reservoir of density ρ_r at the right end:

$$L^r_{\rho_r} f(\eta_-, \eta_0) = \mathbb{E}_{\nu_{\rho_r}}[Lf(\eta_-, \eta_0, \eta_+)|(\eta_-, \eta_0)]$$

In the above examples, the resulting generator only modifies the dynamics within *finite range* of the boundary (where creations and/or annihilations may take place). This makes it possible to define the generator $L^N_{\rho_l,\rho_r}$ of the finite system on $\{1,\dots,N\}$ with reservoirs at both ends. In the hydrodynamic scaling limit, we let $N \to \infty$, and consider the system at times of order N (Euler time scaling).

The hydrodynamic limit. We should normally expect boundary conditions $\rho(t,0+) = \rho_l$, $\rho(t,1-) = \rho_r$ in the hydrodynamic equation (1). However, the results of [4] show that the bulk density profile in the stationary state need not connect smoothly to the reservoir densities. This can be understood thanks to the boundary conditions for (1) introduced in [2], and reformulated in [5]. We define a set $E_l(\rho_l)$ (resp. $E_r(\rho_r)$) of admissible left (resp. right) boundary values containing ρ_l (resp. ρ_r), but (generally) not reduced to it. We say that $\rho(t,x)$ satisfies the BLN boundary conditions, if it has traces at $x = 0$ and $x = 1$, with $\rho(t,0+) \in E_l(\rho_l)$ and $\rho(t,1-) \in E_r(\rho_r)$. The definition of $E_l(\rho_l)$ and $E_r(\rho_r)$ depends only on ρ_l, ρ_r and G (see [5]). The result of [2] is existence and uniqueness of of a weak entropy solution to (1) in $(0,1)$ that satisfies the BLN boundary conditions. Our first result is

Theorem 1. The hydrodynamic limit, under Euler time scaling, of the system with generator $L^N_{\rho_l,\rho_r}$, is given by the unique entropy solution to (1) with BLN boundary conditions.

From this result, we can deduce the hydrostatic profile as $N \to \infty$ of the unique invariant measure for the finite system. What we thus obtain is a rigorous proof for the above models of the formula introduced in [8]: define

$$\rho_{\mathrm{stat}}(\rho_l,\rho_r) := \begin{cases} \arg\inf_{\rho\in[\rho_l,\rho_r]} G(\rho) & \text{if} \quad \rho_l \le \rho_r \\ \arg\sup_{\rho\in[\rho_r,\rho_l]} G(\rho) & \text{if} \quad \rho_l \ge \rho_r \end{cases} \tag{3}$$

whenever the infimum or supremum is *uniquely* achieved. We prove that

Theorem 2. The limiting profile, as $N \to \infty$, of the unique invariant measure for $L^N_{\rho_l,\rho_r}$, is given by the flat density profile $\rho_{\mathrm{stat}}(x) = \rho_{\mathrm{stat}}(\rho_l,\rho_r)$ for every $x \in (0,1)$.

This includes the classical three-phase diagram (low density, high density, maximal current) derived in [4] for the asymmetric exclusion process, and the more complex seven-phase diagram evidenced in [8] in the presence of local minima. Theorem 2 can be understood from Theorem 1 as follows: when the formulas in (3) are uniquely defined, one can check that

$$E_l(\rho_l) \cap E_r(\rho_r) = \{\rho_{\mathrm{stat}}(\rho_l,\rho_r)\}$$

Thus, $\rho(t,x) = \rho_{\mathrm{stat}}(\rho_l,\rho_r)$ defines a stationary entropy solution of (1) with BLN boundary conditions, which can be proved to be unique.

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Hitting times for independent random walks on Z^d

Amine Asselah

(joint work with Pablo A.Ferrari)

We consider a system of asymmetric independent random walks on Z^d, denoted by $\{\eta_t, t \geq 0\}$, stationary under the product Poisson measure ν_ρ of marginal density $\rho > 0$. We fix a pattern A, an increasing local event, and denote by τ the hitting time of A. By using a Loss Network representation of our system, at small density, we obtain a coupling between the laws of η_t conditioned on $\{\tau > t, \eta_0 \sim \nu_\rho\}$ for all times t. When $d \geq 3$, this provides sharp estimates for the tail of τ, as well as bounds on the rate of convergence of the law of η_t conditioned on $\{\tau > t\}$ towards its limiting probability measure as t tends to infinity.

Our main result reads as follows. Assume that $d \geq 3$. There is $\rho_c > 0$ explicit such that for any $\rho < \rho_c$, there are $\lambda(\rho), \beta, M$ positive, such that for any $t \geq 0$

$$|e^{\lambda(\rho)t} P_{\nu_\rho}(\tau > t) - \frac{1}{\int u u^* d\nu_\rho}| \leq M \exp(-\beta t). \tag{1}$$

Here $\lambda(\rho)$ is the principal eigenvalue of the generator L with Dirichlet boundary on A, and u (resp. u^*) is the principal eigenfunction of L (resp. of the adjoint L^*) with Dirichlet boundary on A.

Also, we consider initial measures which are not product Poisson measures, but are "sandwiched" between two product Poisson measures.

Mutually catalytic branching: Continuum limit, Palm measures and multiscale analysis

Andreas Greven
(joint work with D. Dawson and I. Zähle)

The talk addresses the question of the qualitative properties of the spatial continuum limit for multitype population models with and without type interaction, close to critical dimension. This analysis highlights the qualitative change in behavior due to the type interaction which results in intermittency phenomena for the hot spots of the fluctuations in mutually catalytic models.

We consider systems of countably many linearly interacting (migration) multitype diffusions where each single multitype diffusion may also exhibit an interaction between the subpopulations of the different types at a site. The values of the components are (for the two-type case) taken in $(\mathbb{R}^+)^2$ and the components are indexed by the hierarchical group $\Omega_N = \bigoplus_{\mathbb{N}} Z^N$, $Z^N = \{0, 1, \ldots, N-1\}$, the latter with addition modulo N. Typical examples for the diffusive part of the evolution mechanism in each component are the mechanisms of two-type independent Feller branching diffusions, catalytic branching diffusions, mutually catalytic branching diffusions, two-type Anderson models and the Fisher-Wright diffusions.

In all these cases one can consider the question whether a suitable mass-time-space and if necessary an additional rate rescaling yields a spatial continuum limit with the space of sites being the group $\Omega_N^\infty = \bigoplus_{\mathbb{Z}} Z^N$, which is viewed as the continuum limit of Ω_N. The time-space scaling has in particular the property that the random walk on Ω_N underlying the migration term in all these systems converges to a Lévy jump process on Ω_N^∞. This procedure is analogous to passing from interacting systems indexed by $\mathbb{Z}^d$ to measure valued processes on $\mathbb{R}^d$.

For independent branching one obtains always nontrivial continuum limits without any rate rescaling, while for the case of other model classes, namely catalytic, mutually catalytic branching, this holds only for the critically recurrent case, other cases need an up- or down-scaling of the rate depending on the migration mechanism.For the Anderson-model and the Fisher-Wright case one obtains for critically recurrent and transient underlying random walks degeneration of the limits, namely one sees explosion of the fluctuations or a deterministic limit and only the strongly recurrent walks allow nontrivial limits for the Fisher-Wright model.

The key phenomenon is the following dichotomy for the qualitative properties of the continuum limit for branching models. In the case of strong recurrence of the underlying symmetrized migration, we obtain at fixed times for the states of the process densities (w.r.t. to the Haar measure on Ω_N^∞) of the two populations and also of the current reaction terms i.e. the derivative of the increasing process associated with the evaluations of the state of the process with test functions. The latter is relevant in case of catalytic and mutually catalytic models. On the other hand for critically recurrent and transient migration the reaction terms become singular w.r.t the Haar measure of the group Ω_N^∞. In the density case the task is

to describe the structure of the density field, in the singular case the geometric structure of the peaks of activity (hot spots).

For the continuum limit, one can now consider a further limit and let the parameter $N \to \infty$ and identify the limiting objects and study their properties. A simplification of the above procedure, introduced in [CDG03], is to consider the problems above in the hierarchical mean-field limit, that is to first let in a space-time-rate renormalized system tend $N \to \infty$ and then later pass to the continuum limit. Through this procedure we obtain a (possibly time-inhomogeneous) Markov chain which allows to describe local qualitative properties of the continuum limit and is called the small-scale characteristic. In order to justify the hierarchical mean-field limit, the important question is, do we really get the same result by interchanging the order of continuum limit and the limit $N \to \infty$, as is expected? This problem is first addressed for the two-type branching situation.

Next we apply the method of the hierarchical mean-field continuum limit systematically to branching and mutually catalytic branching. We analyze the local small-scale properties of the hierarchical mean-field continuum limit in more detail, namely we analyze the size of the regions of survival, respectively the monotype regions and furthermore the strength of reaction of or between the types (hot spots). More precisely their intensity and their spatial distribution. We introduce the size-biased laws of the small-scale characteristic as a key concept which in turn makes possible the use of the h-transform as a technical tool . Here the catalytic and mutually catalytic processes exhibit for the same migration mechanism much more peaked states than the classical branching case and the mutually catalytic case exhibits some entirely new features which can be described as local intermittency effects.

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Annealed deviations for random walk in random scenery

Wolfgang König

(joint work with N. Gantert and Z. Shi)

Let $(Z_n)_{n\in\mathbb{N}_0}$ be a d-dimensional *random walk in random scenery*, i.e., $Z_n = \sum_{k=0}^{n-1} Y(S_k)$ with $(S_k)_{k\in\mathbb{N}_0}$ a random walk in $\mathbb{Z}^d$ and $(Y(z))_{z\in\mathbb{Z}^d}$ an i.i.d. scenery, independent of the walk. The walker's steps are assumed to have mean zero and finite variance. For the purpose of this abstract, let S be simple random walk.

The random walk in random scenery has been introduced and analyzed in dimension $d \neq 2$ by H. Kesten and F. Spitzer [KS79] and by E. Bolthausen [B89] for $d = 2$. Under the assumption that $Y(0)$ has expectation zero and variance

$\sigma^2 \in (0,\infty)$, their results imply that

$$\frac{1}{n} Z_n \approx a_n^0 = \begin{cases} n^{-\frac{1}{4}} & \text{if } d=1, \\ (\frac{n}{\log n})^{-\frac{1}{2}} & \text{if } d=2, \\ n^{-\frac{1}{2}} & \text{if } d \geq 3. \end{cases} \tag{1}$$

More precisely, $\frac{1}{na_n^0} Z_n$ converges in distribution towards some non-degenerate random variable. In terms of the so-called *local times* of the walk,

$$\ell_n(z) = \sum_{k=0}^{n-1} 1_{\{S_k = z\}}, \qquad n \in \mathbb{N}, \;\; z \in \mathbb{Z}^d, \tag{2}$$

the random walk in random scenery may be identified as

$$Z_n = \sum_{z \in \mathbb{Z}^d} Y(z) \ell_n(z) = \langle Y, \ell_n \rangle. \tag{3}$$

It is the goal of the present work to identify the speed and the rate of the logarithmic decay of $\mathbb{P}(\frac{1}{n} Z_n > b_n)$ for various choices of sequences $(b_n)_n$ in $(0,\infty)$ satisfying $b_n \gg a_n^0$. Furthermore, we want to explain, at least on a heuristic level, the optimal joint strategy of the random walk and the scenery to meet the event $\{\frac{1}{n} Z_n > b_n\}$ in the cheapest way.

In order to demonstrate the main idea of the investigation, let us derive a lower bound for $\mathbb{P}(\frac{1}{n} Z_n > u)$ for $u > 0$ fixed. Our ansatz is to fix a scale function $1 \ll \alpha_n \ll n^{\frac{1}{d}}$ and to consider the event that the appropriately normalized rescaled local times and the rescaled scenery resemble fixed given shape functions $\psi^2, \varphi \colon \mathbb{R}^d \to [0,\infty)$, respectively, where we impose the conditions $\int \psi^2 = 1$ and $\langle \psi^2, \varphi \rangle \geq u$. This gives the lower bound

$$\begin{aligned} \mathbb{P}(\tfrac{1}{n} Z_n > u) &\geq \mathbb{P}\Big(\frac{\alpha_n^d}{n} \ell\big(\lfloor \cdot \, \alpha_n \rfloor\big) \approx \psi^2(\cdot), Y\big(\lfloor \cdot \, \alpha_n \rfloor\big) \approx \varphi(\cdot) \Big) \\ &\approx \exp\Big\{ -\frac{n}{\alpha_n^2} I(\psi^2) \Big\} \exp\Big\{ -\alpha_n^d J_H(\varphi) \Big\}, \end{aligned} \tag{4}$$

where the last line follows from well-known techniques in large-deviation theory, and the rate functions are

$$I(\psi^2) = \frac{1}{2} \|\nabla \psi\|_2^2 \qquad \text{and} \qquad J_H(\varphi) = \int \sup_{t>0} \Big(\varphi(x) t - H(t) \Big) \, \mathrm{d}x,$$

and $H(t) = \log \mathbb{E}(e^{tY(0)})$ is the cumulant generating function of the scenery. From (4) we already see that the choice $\alpha_n = n^{1/(d+2)}$ is the proper choice for which we can expect a non-trivial result. Indeed, A. Asselah and F. Castell [AC03] derived a theorem on the logarithmic asymptotics of $\mathbb{P}(\frac{1}{n} Z_n > u)$ for the case of *bounded* sceneries Y, which we do not formulate here.

However, we are interested in sceneries *unbounded* from above. In general, the random walk in random scenery with unbounded sceneries has interesting relations to self-intersection properties of the walk, which makes this subject particularly

interesting. As an example, if the scenery is standard Gaussian, then the distribution of Z_n given the random walk is equal to a centred Gaussian with variance equal to $\sum_{z\in\mathbb{Z}^d} \ell_n(z)^2$, the self-intersection number.

Our main result is the following.

Theorem. *Suppose that* $\log \mathbb{P}(Y(0) > r) \sim -Dr^q$ *as* $r \to \infty$, *for some* $D > 0$ *and* $q > \frac{d}{2}$. *Pick a sequence* $(b_n)_{n\in\mathbb{N}}$ *satisfying* $1 \ll b_n \ll n^{\frac{1}{q}}$. *Then*

$$\lim_{n\to\infty} n^{-\frac{d}{d+2}} b_n^{-\frac{2q}{d+2}} \log \mathbb{P}\Big(\frac{1}{n} Z_n > b_n\Big) = -K_{D,q}, \tag{5}$$

where

$$K_{D,q} \equiv \inf\Big\{\frac{1}{2}\|\nabla\psi\|_2^2 + D\|\psi^2\|_p^{-q} \colon \psi \in H^1(\mathbb{R}^d), \|\psi\|_2 = 1\Big\}, \tag{6}$$

(where $\frac{1}{p} + \frac{1}{q} = 1$*), and* $K_{D,q}$ *is positive.*

The constant $K_{D,q}$ is zero in the subcritical cases where $q < \frac{d}{2}$:

Proposition. (Positivity of $K_{D,q}$) *Fix* $d \in \mathbb{N}$ *and* $p, q > 1$ *satisfying* $\frac{1}{p} + \frac{1}{q} = 1$.

(i) *For any* $D > 0$,

$$K_{D,q} = (d+2)\Big(\frac{D}{2}\Big)^{\frac{2}{d+2}} \Big(\frac{\chi_{d,p}}{d}\Big)^{\frac{d}{d+2}}, \tag{7}$$

where

$$\chi_{d,p} = \inf\Big\{\frac{1}{2}\|\nabla\psi\|_2^2 \colon \psi \in H^1(\mathbb{R}^d) \colon \|\psi\|_2 = 1 = \|\psi\|_{2p}\Big\}. \tag{8}$$

(ii) *The constant* $\chi_{d,p}$ *is positive if and only if* $d \le \frac{2p}{p-1} = 2q$.

The constant $\chi_{d,p}$ is directly related to the well-known *Gagliardo-Niremberg* constant. In [We83], the existence of a minimizer in (8) has been established; indeed there is a minimizer which is rotationally symmetric, positive everywhere and infinitely often differentiable. Uniqueness of the minimizer has been proven in [MS81] for $d \in \{2, 3, 4\}$ for all $p \in (1, \frac{d}{d-2}]$ and for $d \in \{5, 6, 7\}$ for $p \in (1, \frac{8}{d})$.

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Wetting transition for effective gradient interface models

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The talk is in two parts. In the first part we review results on the wetting transition for effective interface with gradient interaction, in the second we look more closely at the $(1+1)$ case, and show (Brownian) scaling limits of the model.

The effective interface is a $(d+1)$ model with discrete basis and continuous height variables. Thus for $x \in \mathbb{Z}^d$, $\phi_x \in \mathbb{R}$ denotes the height of the interface above or below the site x. The interaction is of gradient type, i.e. depends only on the height differences of two neighboring sites with a strictly convex potential. A special case deals with quadratic potential which corresponds to a Gaussian also called harmonic crystal or (discrete) massless field. One of the difficulties of this model is the long range correlation due to the continuous symmetry. In particular, we have delocalization in lower lattice dimensions $d = 1, 2$ but localization in higher dimensions $d \geq 3$, cf. [8]

We focus on the interaction of the interface with its basis or wall: $\{x \in \mathbb{Z}^d : \phi_x = 0\}$. Two effects are considered: the repulsion and the weak pinning. The hard wall condition or repulsion forbids negative heights, this produces a delocalization of the interface, the so called entropic repulsion, cf. [1]. The weak pinning of reward $\epsilon > 0$ is concentrated at the wall and has the effect of localizing the interface in any dimensions, cf. [7]. Thus the two effects compete and we say that a wetting transition occurs at $\epsilon_0 > 0$, if the interface remains localized for strong pinning parameter $\epsilon > \epsilon_0$, respectively is delocalized for weak pinning $\epsilon \leq \epsilon_0$. Such a transition takes place in lower lattice dimensions, $d = 1, 2$, cf. [3], but not in higher dimensions $d \geq 3$, cf. [2].

In the second part we study the path properties for the δ-pinning wetting model in $(1+1)$-dimension, cf. [5]. In other terms, we study a random walk model with fairly general continuous increments conditioned to stay in the upper half plane and with a δ-measure reward for touching zero, that is the boundary of the forbidden region. Such a model displays a wetting transition, according to the size of the reward ϵ. Our focus is on getting a precise pathwise description of the system, in both the delocalized phase, that includes the critical case, and in the localized one. We then show that the properly rescaled interface converges in the delocalized phase to a Brownian excursion, respectively a Brownian bridge at criticality. This a generalization of the previous work of [6], which dealt with the special case of the simple random walk. The proofs are based on a careful analysis of the pinned sites and on excursion and regenerative set theory. Finally we present a dynamical version: the Landau-Ginsburg model on a wall with or without pinning, and discuss time-space scaling limit theorem to a partial stochastic differential equation with repulsion of the Nualard Pardoux type, cf. [9], [4], [10].

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Entropy dissipation estimates in a Zero–Range dynamics

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(joint work with Gustavo Posta)

We consider the time decay of relative entropy functionals for a class of zero range processes on the complete graph. The process is characterized by the jump rate function $c : \mathbb{N} \to \mathbb{R}_+$, $c(k)$ being the rate at which a vertex occupied by k particles expels one. The expelled particle is then relocated at another vertex chosen uniformly at random. We fix the number of vertices and the number of particles and consider the process with arbitrary initial distribution μ. Letting ν denote the reversible invariant measure and setting $f_t = \frac{\mathrm{d}\mu_t}{\mathrm{d}\nu}$ (μ_t being the distribution at time t of the process started with μ), the relative entropy functional $\mathrm{Ent}(f_t) = H(\mu_t|\nu)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Ent}(f_t) = -\mathcal{E}(f_t, \log f_t)\,, \tag{1}$$

where $\mathcal{E}(\cdot,\cdot)$ denotes the Dirichlet form associated to the process. The entropy dissipation estimate is a bound of the form

$$\mathrm{Ent}(f) \leq \gamma\,\mathcal{E}(f, \log f)\,, \tag{2}$$

where $\gamma > 0$ and the inequality holds for every nonnegative f. In view of (1) the above estimate implies exponential decay of $H(\mu_t|\nu)$ and γ can be used to estimate from above the mixing time of the process.

In the case where the jump rates c are uniformly increasing functions we show that the constant γ is finite uniformly in the number of vertices and the number of particles. The measure ν is globally log–concave in this case and the result is obtained by using a discrete version of the so–called Bakry–Emery criterium. In particular, we establish estimates of the form

$$\mathcal{E}(f, \log f) \leq \gamma\,\left[\mathcal{E}(-\mathcal{L}f, \log f) + \mathcal{E}(f, (-\mathcal{L}f)/f)\right]\,, \tag{3}$$

for all nonnegative f, where $\mathcal{L}$ denotes the generator of the dynamics and the term in brackets in the right hand side above is the second time–derivative of $\mathrm{Ent}(f_t)$ at time $t = 0^+$. We refer to [2] for the precise assumptions and statements.

We also study the standard case of possibly oscillating but roughly linearly increasing rates. Here the uniform entropy dissipation estimate cannot follow by a one–step computation as the one leading to (3). An important point is the observation that the invariant measure ν is such that its marginals at each vertex have bounded Radon Nikodym derivative wrt a log–concave probability measure. An estimate of the form (3) is therefore available on each coordinate and the final result is achieved by an adaptation of the iterative martingale approach.

In more than one way our results generalize and complement previous work in [4, 1] for the spectral gap problem and [3] for the logarithmic Sobolev inequality.

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Mott law as lower bound for a random walk in a random environment

Alessandra Faggionato

(joint work with H. Schulz–Baldes and D. Spehner)

Suppose given an infinite set of random points $\{x_j\} \subset \mathbb{R}^d$ distributed according to some stationary simple point process with a bounded mean density ρ. To each x_j is associated a random energy mark $E_j \in [-1, 1]$. All marks are drawn independently and identically according to a probability measure ν satisfying $\nu([-E, E]) \geq c_0 E^{1+\alpha}$ for some $\alpha \geq 0$ and $c_0 > 0$. Within each such *random environment* $\{x_j, E_j\}$, let us consider a *continuous-time random walk* over the points $\{x_j\}$ with energy dependent transition rates from x_j to x_k given by

$$c_{x_j,x_k}(E_j, E_k) \;=\; e^{-|x_j - x_k|}\; e^{-\beta(|E_j - E_k| + |E_j| + |E_k|)}\;, \tag{1}$$

where the positive parameter β is the inverse temperature. Our main result states that the random walk converges after appropriate space and time rescaling to a Brownian motion and that the associated diffusion coefficient $D(\beta)$ is bounded from below by

$$D(\beta) \;\geq\; c_1\, \beta^{-\frac{(\alpha+1)d}{\alpha+1+d}}\; \exp\left(-c_2\, \beta^{\frac{\alpha+1}{\alpha+1+d}}\right)\;, \tag{2}$$

where $d \geq 2$ is the dimension of space and c_1 and c_2 are some β-independent constants. The exponential factor on the r.h.s. is precisely as in Mott's law for the DC conductivity in disordered solids which is discussed below.

Based on the following heuristics due to Mott [Mot, SE], we expect that the power law in the exponential in (2) captures the good asymptotic behaviour of $\ln D(\beta)$ in the low temperature limit $\beta \uparrow \infty$ if $\nu([-E, E]) \sim c_0 E^{1+\alpha}$ as $E \to 0$. As β becomes larger, the rates (1) fluctuate widely with (x_j, x_k) because of the exponential energy factor. The low temperature limit effectively selects only jumps between points with energies in a small interval $[-E(\beta), E(\beta)]$ shrinking to zero as $\beta \to \infty$. Assuming that the diffusion coefficient is determined by those jumps with the largest rate, one can obtain directly the characteristic exponential factor on the right hand side of (2) by maximising these rates for a fixed temperature under the constraint that the mean density of points x_j with energies in $[-E(\beta), E(\beta)]$ is equal to $\rho\, \nu([-E(\beta), E(\beta)]) \sim c_0 \rho E(\beta)^{1+\alpha}$. As the characteristic mean distance $|x_j - x_k|$ between sites with optimal jump rates also varies heavily with the inverse temperature β, one speaks of a *variable range hopping regime*. A crucial (and physically reasonable, as discussed below) element of this argument is the independence of the energies E_j. The selection of the points $\{x_i\}$ with energies in the window $[-E(\beta), E(\beta)]$ then corresponds mathematically to a p-thinning with $p = \nu([-E(\beta), E(\beta)])$. It is then a well-known fact that an adequate rescaling of the p-thinning of a stationary point process converges in the limit $p \downarrow 0$ (corresponding to $\beta \uparrow \infty$) to a stationary Poisson point process (PPP) (*e.g.* [Kal, Theorem 16.19]). Hence one might call the stationary PPP the normal form of a model leading Mott's law, namely the exponential factor on the r.h.s. of (2) and we believe that proving the upper bound corresponding to (2) should therefore be most simple for the PPP. In dimension $d = 1$, a different behaviour of $D(\beta)$ is expected [LB] and this will not be considered here.

Our main motivation for studying the above model comes from its importance for *phonon-assisted hopping conduction* [SE] in disordered solids in which the Fermi level (set equal to 0 above) lies in a region of *strong Anderson localisation*. This means that, close to the Fermi level, the electron Hamiltonian has exponentially localised quantum eigenstates with localisation centres x_j and energy levels E_j. The DC conductivity of such materials would vanish if it were not for the lattice vibrations (phonons) at nonzero temperature. They induce transitions between the localised states, the rate of which can be calculated from first principle by means of the Fermi golden rule [MA, SE]. In the variable range hopping regime at low temperature, an adiabatic or rotating wave approximation can be used to treat quantum mechanically the electrons-phonon coupling [Spe]. Coherences between electronic eigenstates with different energies decay very rapidly under the resulting dissipative electronic dynamics and one can show that the hopping DC conductivity of the disordered solid coincides with the conductivity associated with a Markov jump process on the set of localisation centres $\{x_j\}$, hence justifying the use of a model of classical mechanics [BRSW]. Because Pauli blocking due

to Fermi statistics of the electrons has to be taken into account, this leads to a rather complicated exclusion process (*e.g.* [Qua, FM]). If, however, the blocking is treated in an *effective medium approximation*, one obtains a family of independent random walks with rates which in good first approximation are given by (1) in the limit $\beta \uparrow \infty$ [MA, AHL]. Let us discuss the remaining aspects of the model. The stationarity of the underlying simple point process $\{x_j\}$ simply reflects that the material is homogeneous, while the independence of the energy marks is compatible with Poisson level statistics, which is a general rough indicator for the localisation regime and has been proven to hold for an Anderson model [Min]. The exponent α allows to model a possible Coulomb pseudogap in the density of states [SE].

Having in mind the Einstein relation between the conductivity and the diffusion coefficient (which can be stated as a theorem for a number of models [Spo]), the lower bound (2) gives a lower bound on the hopping DC conductivity. In the above materials, the DC conductivity shows experimentally Mott's law, namely a characteristic low-temperature behaviour which is well approximated by the exponential factor in the r.h.s. of (2) with $\alpha = 0$ or $\alpha = d - 1$, as predicted by Mott [Mot] and Efros and Shklovskii [EF], respectively, based on the optimisation argument discussed above. A first convincing justification of this argument was given by Ambegoakar, Halperin and Langer [AHL], who first reduced the hopping model to a related random resistor network, in a manner similar to the work of Miller and Abrahms [MA], and then pointed out that the constant c_2 can be estimated using percolation theory [SE]. Our proof of the lower bound (2) is much inspired by this work, together with some classical results for random walks in random environment obtained by De Masi, Ferrari, Goldstein and Wick [DFGW].

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