Mathematisches Forschungsinstitut Oberwolfach

Report No. 34/2005

Dynamical System Methods in Fluid Dynamics

Organised by
Jerrold E. Marsden (Pasadena)
Jürgen Scheurle (München)

July 31st – August 6th, 2005

Abstract. The workshop was organized around the infusion of new techniques from dynamical systems, geometric methods, multiscale analysis, scientific computation, and control theory into traditional methods in fluid mechanics. It was well attended with about 45 participants with broad geographic representation from all continents. There was an excellent blend of senior researchers, students, postdocs and junior faculty.

Mathematics Subject Classification (2000): 37-xx, 58-xx, 76-xx.

Introduction by the Organisers

The workshop was organized around the following classical as well new and emerging techniques. Traditional methods in fluid mechanics continue to be invigorated with the infusion of ideas from dynamical systems, geometric methods, multiscale analysis, new developments in scientific computation, and control theory. The specific themes of the workshop were as follows:

(1) ***Dynamical Systems, Geometric, Analytical, and PDE Methods.*** The meeting presented recent advances in these basic tools that are of importance to fundamental investigations in fluid mechanics, with an emphasis on those tools that are relevant to one of the following topics.

(2) ***Mixing in Geophysical Flows.*** Recent investigations using finite time Lagragnian coherent structures and invariant manifolds have proven useful in specific investigations involving mixing and transport in oceanographic flows and in flows in bays and estuaries.

(3) ***Control.*** There are interesting investigations underway involving optimization and applications to, for example, coordinated control of groups of underwater vehicles. The dynamics and control of these vehicles as well as how their motion interacts with ambient (highly time dependent) currents in the oceans are of importance for emerging studies on sensing

and the gathering of oceanographic data. Other work on control theory, such as locomotion of articulated bodies in fluids, as well as control of separation and cavity oscillations was also represented at the meeting.

(4) ***Computational Methods.*** Recent discrete formulations of fluids have led to algorithms that have, for example, an exact discrete Kelvin theorem. The meeting expored the developments of these algorithms and their application to geophysical and other flows in which vorticity (or potential vorticity) is important. Several groups worldwide have been investigating these aspects.

(5) ***Interface Problems.*** There is much current interest in interface problems in fluid mechanics, such as two and three phase flows and the ocean-atmosphere interface. Recent mathematical methods in this area include level-set methods and phase-field methods.

(6) ***Symmetry Methods.*** Techniques such as the Karhunen-Loève decomposition, self-similarity, pattern formation, stability and bifurcation, can greatly benefit from the use of symmetry methods, including computational speed-up.

(7) ***Averaged Euler and Navier Stokes Equations.*** These models for fluids have all the structure of standard fluid mechanics and have computational properties similar to LES (large eddy simulation). The conference presented progress in this area with emphasis on turbulence calculations and wall bounded flow, both for incompressible and compressible flows.

(8) ***Vortex Dynamics.*** Vortex methods, vortex filaments, investigated for their Hamiltonian structure and as a computational tool continues to be an exciting and active area for both geophysical flows and turbulence as well as for its intrinsic mathematical interest.

Structure of the Meeting. The meeting had a balance of senior researchers, postdoctoral fellows and graduate students. Consistent with the general approach advocated by Oberwolfach, we selected about 20–25 people to give a lecture at the meeting. They suggested students, postdoctoral fellows and junior faculty all of whom participated through two poster sessions.

Posters. The poster session was one of the most interesting aspects of the meeting. The organizers decided to award prizes for the best poster and an ad hoc committee (consisting of Friedlander, Haller, Reich, Marsden and Scheurle) was formed to choose the four best posters. This number matched the number of gifts that were available. The winners were (in alphabetical order):

- Francois Gay-Balmaz, Lausanne, *Sphere actions on chains of Hilbert manifolds and applications to fluid dynamics*.
  Prize: a copy of *A Mathematical Introduction to Fluid Mechanics* by Alexandre Chorin and Jerrold Marsden.
- Juan Meli-Huber, Princeton, *Motion planning for an articulated body in a perfect fluid*
  Prize: a copy of *Chaos near resonance* by George Haller.

- Mustafa Sabri Kilic, MIT *Reduced Navier-Stokes equations near a flow boundary*
  Prize: a copy of *Simulating Hamiltonian Dynamics* by Ben Leimkuhler and Sebastian Reich
- Amit Surana, MIT *Kinematic theory of separation in 3D fluid flows*
  Prize: a copy of *The N-Vortex Problem* by Paul Newton.

Impressions, Connections, Insights. The mixture of participants, from those who were very applied and those who came from a more of a dynamical systems background, but all with a strong mathematical dedication, was very fruitful.

Dynamical systems methods have now started to be utilized successfully for the study of complex real fluid systems, from controlled man-made experiments to multi-scale and multivariate natural dynamics, from oceanic motions to flows around jellyfish. Dynamical system approaches are then applied to complex data, models or combinations thereoff. There is a wide range of research opportunities, including the possibilty for new theoretical and methodological progress.

Amongst the specific new connections and insights that were gained, the power of dynamical systems ideas, such as invariant manifolds or Lyapunov exponents, seemed to be quite impressive as well as useful in the context of metereology as well as oceanography. The use of dynamical systems ideas in control of fluids (separation and cavity oscillations, for example) was also quite interesting for a number of the participants.

Workshop: Dynamical System Methods in Fluid Dynamics

Table of Contents

Abstracts

Spatio-temporal patterns in equivariant evolution equations

Wolf-Jürgen Beyn
(joint work with Vera Thümmler)

In this contribution we consider general evolution equations of the form

$$u_t = F(u), \quad u(0) = u_0, \tag{1}$$

where $u(t) \in X$(a Banach space), u_t denotes the time derivative and

$$F : Y \subset X \mapsto X, \quad Y \subset X \text{ (a dense subspace)},$$

is a vector field with an equivariance property

$$F(a(\gamma)u) = a(\gamma)F(u) \quad \forall\, u \in Y, \gamma \in G.$$

Here G denotes a Lie group of dimension s, generally noncompact, that acts on X via a homomorphism $a : G \mapsto GL(X)$ satisfying $a(\gamma)Y \subset Y$ for all $\gamma \in G$.

Typical examples are parabolic or hyperbolic systems on unbounded domains such as the quintic Ginzburg Landau equation, cf.[4],[6]

$$u_t = \Delta u + f(u),\ f(u) = u(\mu + \beta|u|^2 + \gamma|u|^4), u(x,t) \in \mathbb{C}, x \in \mathbb{R}^d, t \geq 0. \tag{2}$$

The group in this case is $G = SE(d) \times S^1$ with the Euclidean group $SE(d)$ in $\mathbb{R}^d$ and the action given by

$$[a(\gamma)u](x) = e^{-i\theta}u(b^{-1} \circ x),\ x \in \mathbb{R}^d, \quad \gamma = (b, \theta) \in SE(d) \times S^1.$$

Our aim is to present and analyze the so called *freezing method* developed in [2] and to discuss extensions of this method as well as some open problems. The method reformulates equation (1) as a differential algebraic evolution equation which is subsequently discretized and solved numerically. The approach allows to set up a time-dependent moving coordinate system in which certain spatio-temporal patterns actually become stationary. Patterns treated this way include e.g. traveling and rotating waves, rotating spiral waves and spinning solitons. A similar idea was put forward, first in the context of Karhunen-Loéve expansion in [8], and later in [7] for more general equivariances that allow scalings of time.

The basic idea is to split the solution $u(t)$ of eqn. (1) in the form

$$u(t) = a(\gamma(t))v(t), \tag{3}$$

where both $\gamma(t) \in G$ and $v(t) \in Y$ are unknown. These extra degrees of freedom are compensated for by $s = \dim(G)$ algebraic constraints that are called *phase conditions* or *pinning conditions*. These are chosen to minimize the variation of the function $v(t)$ as much as possible, a particular useful feature when adaptive meshes in space are needed. The decomposition (3) also underlies many theorems for equivariant evolution equations and their bifurcations (cf. [3],[5],[9]) but seems not to have been used systematically for numerical purposes.

We introduce the Lie algebra $\mathcal{A} = T_eG$ of G, the derivative $d\gamma(e) : \mathcal{A} \mapsto T_\gamma G$ of the left multiplication $g \mapsto \gamma \circ g$ at the unit element $g = e$ and for each $v \in Y$ the derivative $S(v) = d[a(e)v] : \mathcal{A} \mapsto X$ of the map $\gamma \in G \mapsto a(\gamma)v \in X$ at $\gamma = e$. Inserting the ansatz (3) into (1) and using equivariance leads to the following differential algebraic equation for the variables $\gamma(t) \in G, \mu(t) \in \mathcal{A}, v(t) \in Y$

$$\begin{array}{ll} v_t = F(v) - S(v)\mu, & v(0) = u_0 \\ \gamma_t = d\gamma(e)\mu, & \gamma(0) = e \\ 0 = \psi(v, \mu). & \end{array} \tag{4}$$

Here, the phase condition is defined by a mapping $\psi : Y \times \mathcal{A} \mapsto \mathcal{A}^*$. A choice for ψ that works particularly well in practice, is obtained from minimizing $\langle v_t, v_t\rangle = |v_t|^2$ where $\langle\cdot,\cdot\rangle$ is some continuous inner product on X. Using the first equation in (4) this leads to a least squares problem in μ with normal equations

$$0 = \psi_{\min}(v, \mu)\varphi := \langle F(v) - S(v)\mu, S(v)\varphi\rangle \quad \forall \varphi \in \mathcal{A}. \tag{5}$$

For any v with trivial stabilizer $\mathrm{stab}(v) = \{\gamma \in G : a(\gamma)v = v\}$ this has a unique solution μ in which case (4) turns out to be a DAE of index 1. For the PDE example (2) with $d = 1$ eqn. (4) leads to a *Partial Differential Algebraic Equation* (PDAE) for $v(x, t) \in \mathbb{C}, x \in \mathbb{R}, b(t), \theta(t) \in \mathbb{R}, t \geq 0$

$$\begin{array}{ll} v_t = v_{xx} + f(v) + \mu_1 v_x + \mu_2 iv, & v(\cdot, 0) = u_0 \\ b_t = \mu_1,\ \theta_t = \mu_2, & b(0) = \theta(0) = 0 \\ 0 = \langle v_t, v_x\rangle,\ 0 = \langle v_t, iv\rangle. & \end{array} \tag{6}$$

The following results and problems are discussed in the talk:

(1) *Applications to rotating and traveling waves in 1D and spiral waves in 2D:* The PDAE's arising from (4), such as (6), are first truncated to a finite domain and supplied with boundary conditions (e.g. Dirichlet or Neumann). Then the resulting initial boundary value problems with algebraic constraints can be solved by any suitable discretization method. For the case $d = 1$ we use implicit Euler in time and finite differences in space and for $d = 2$ we employ half-explicit Euler (implicit only in the algebraic variables μ) and finite differences/ elements in space. The examples clearly show that solutions that converge to a traveling wave $\bar{u}(x, t) = \bar{v}(x-\bar{\mu}t)$ or to a traveling and rotating wave $\bar{u}(x, t) = e^{i\bar{\theta}t}\bar{v}(x - \bar{\mu}t)$ (cf. (2) and [6]) can be decomposed into a function $v(\cdot, t)$ that converges to the spatial profile $\bar{v}(\cdot)$, and into algebraic variables $\mu(t)$ that converge to the translational and rotational speeds. For pictures and movies we refer to [2]. While it is easy to freeze the spinning solitons of (2) from [4], somewhat more care has to be taken of the rigidly rotating spiral waves in Barkley's system [1]. In this case the PDAE resulting from (4) becomes a mixed hyperbolic-parabolic system (cf. the convection term v_x appearing in (6)) that has to be solved by a suitable upwind/downwind switching scheme. For details of the method and results see [2].

(3) *Relative equilibria and their stability:*
Relative equilibria are solutions $\bar{u}(t) = a(\bar{\gamma}(t))\bar{v}$ of eqn. (1) that stay in the group orbit of a single element $\bar{v}$. If the stabilizer of $\bar{v}$ is trivial then one finds that there exists some $\bar{\mu} \in \mathcal{A}$, such that $(\bar{v}, \bar{\mu})$ is a steady state of $v_t = F(v) + S(v)\mu$ and

satisfies the second equation in (4). Choosing a suitable phase function ψ one can then compute branches and bifurcation points of relative equilibria by applying to the nonlinear system $0 = F(v) - S(v)\mu,\ \psi(v,\mu) = 0$ a standard bifurcation package. Moreover, it is known for several cases that spectral assumptions on the linearized operator $Pv = DF(\bar{v})v - S(v)\bar{\mu}$ (e.g. 0 is an s-fold eigenvalue of P and the remaining spectrum lies strictly in the left half plane) imply asymptotic stability of $\bar{u}$ with asymptotic phase for the equation (1) (see e.g. [3] for the ODE case). One further expects that this leads to asymptotic stability (in the classical sense of Liapunov) of the pair $(\bar{v},\bar{\mu})$ for the PDAE (4). In [10] this is proved for the traveling wave case ($G = \mathbb{R}, d = 1$) and a generalization to Lie groups of dimension ≥ 2 for $d = 1$ is indicated. Moreover, for this case the thesis [10] provides a complete stability analysis for discretizations of (4) on a finite interval $[x_-, x_+]$ with step-size $h = (x_+ - x_-)/N$ and arbitrary linear two-point boundary conditions. Under appropriate assumptions on the boundary operator (that turn out to be rather sharp) asymptotic exponential estimates are derived that are uniform in $x_\pm$ and h. For spiral waves in dimension ≥ 2 nonlinear stability theorems seem neither to be known for parabolic sytems of type (2) nor for the PDAE formulation.

(3)*Extensions:*

If eqn. (1) has a Hamiltonian structure then one can show that the first equation of the frozen system (4) retains this property provided that the phase condition $\psi(v,\mu) = 0$ can be solved for $\mu = \mu(v)$. As an application we consider the sine-Gordon equation $u_{tt} = u_{xx} + \sin(u)$ with initial values near a traveling wave and we apply a symplectic scheme to the Hamiltonian system corresponding to the PDAE. Numerical experiments for the (viscous) Burgers equation show that the method even applies to viscous and hyperbolic conservation laws. A stability theory for the corresponding PDAE has not yet been developed.

References

[1] D. Barkley, *Euclidean symmetry and the dynamics of rotating spiral waves*, Phys. Rev. Lett., **72** (1994),164–167.

[2] W.-J. Beyn and V. Thümmler, *Freezing solutions of equivariant evolution equations*, SIADS **3** (2004) 85-116.

[3] P. Chossat and R. Lauterbach, *Methods in Equivariant Bifurcations and Dynamical Systems*, World Scientific (2000).

[4] L.-C. Crasovan, B. A. Malomed, and D. Mihalache, *Spinning solitons in cubic-quintic nonlinear media*, Pramana J. Phys., **57** (2001), 1041–1059.

[5] B. Fiedler, B. Sandstede, A. Scheel, and C. Wulff, *Bifurcation from relative equilibria of noncompact group actions: Skew products, meanders, and drifts*, Doc. Math., **1** (1996),479–505.

[6] P. Marcq, H. Chaté and R. Conte, *Exact solutions of the one-dimensional quintic complex Ginzburg-Landau equation*, Physica D, **73** (1994),305-317.

[7] C. W. Rowley, I. G. Kevrekidis, J. E. Marsden, and K. Lust, *Reduction and reconstruction for self-similar dynamical systems*, Nonlinearity, **16** (2003),1257–1275.

[8] C. W. Rowley and J. E. Marsden, *Reconstruction equations and the Karhunen–Loève expansion for systems with symmetry*, Phys. D, **142** (2000),1–19.

[9] B. Sandstede, A. Scheel, and C. Wulff, *Bifurcations and dynamics of spiral waves*, J. Nonlinear Sci., **9** (1999), 439–478.
[10] V. Thümmler, *Numerical analysis of the method of freezing traveling waves*, PhD Thesis, Bielefeld University (2005), available via www.math.uni-bielefeld.de/~beyn/AG_Numerik/preprints.

Issues of integrability in some classical problems of the dynamics of rigid bodies and vortices in an ideal fluid

Alexey Borisov
(joint work with Ivan Mamaev and Alexander Kilin)

In this talk we discuss several classical problems concerning the dynamics of rigid bodies and vortices in an ideal fluid.

The first one is the problem of the motion of two circular cylinders with circulations which are equal in magnitude and opposite in sign (so that whole circulation is zero). A similar system of two moving spheres in a fluid was studied by C. Bjerknes [1], N. Joukowski [2], G. Lamb [3] and others; they found certain unusual effects in the behaviour of the bodies. Here we present most rigorous qualitative analysis of the system and reveal some new effects in the behaviour of the cylinders. We also show that in the general case, this system is not integrable.

The second problem is the motion of n bodies in an ideal fluid on the plane, when circulations around the bodies are assumed to be arbitrary and distances between bodies are much more then the bodies' sizes (so called mass vortices). The dynamics of two mass vortices is studied in detail.

The last problem is about the dynamics of an arbitrary cylinder interacting with point vortices. For the most general case (when vortices and circulations around the cylinders are arbitrary) the equations of motion are obtained and their Hamiltonian form is indicated. It was shown numerically that the system of an elliptic cylinder and one vortex is not integrable.

References

[1] Bjerknes C. Vorläufige Mittheilungen über die Kräfte, die entstehen, wenn kugelformige Körper, indem sie Dilatations und Contractions — Schwingungen ausführen, in einer incompressible Flussigkeit sich bewegen, Repert für rein und angew. Math., 1876, p. 264.
[2] Joukowski N. E. Generalization of the Bjerknes problem of the hydrodynamic forces acting on pulsating and oscillating bodies immersed in a mass of liquid. Collected Works. V. 2., GITTL Moscow-Leningrad. 1949, p. 670-688.
[3] Lamb H. Hydrodynamics, Ed. 6-th, N.Y. Dover publ., 1945.

On the approximation of transport phenomena

Michael Dellnitz

Over the last years so-called *set oriented* numerical methods have been developed in the context of the numerical treatment of dynamical systems, see [1, 2]. The basic idea is to cover the objects of interest – for instance *invariant manifolds* or *invariant measures* – by outer approximations which are created via adaptive multilevel subdivision techniques. These schemes allow for an extremely memory and time efficient discretization of the phase space and have the flexibility to be applied to several problem types. An overview about these set oriented methods can be found in [3].

In this talk we show that set oriented techniques can particularly be useful for the approximation of *transport processes* which play an important role in many real world phenomena. We mainly focus on two related applications: first we analyze the transport of asteroids in the solar system – this work is particularly motivated by the explanation of the existence of the asteroid belt between Mars and Jupiter. Secondly we show how to analyze transport phenomena in ocean dynamics. Here the related mathematical models depend explicitly on time and this makes the numerical treatment inherently more difficult. However, following [5] we demonstrate the strength of an appropriate set oriented approach by a study of transport in Monterey Bay which is based on real data.

In addition we illustrate how to make use of these set oriented numerical techniques for the solution of *multiobjective optimization problems*. In these problems several objective functions have to be optimized at the same time. For instance, for a perfect economical production plan one wants to simultaneously *minimize cost* and *maximize quality*. As indicated by this example the different objectives typically contradict each other and therefore certainly do not have identical optima. Thus, the question arises how to approximate the "optimal compromises" which, in mathematical terms, define the so-called *Pareto set*. In order to make our set oriented numerical methods applicable we first construct a dynamical system which possesses the Pareto set as an attractor. In a second step we develop appropriate step size strategies. The corresponding techniques are applied to the optimization of an active suspension system for cars, see [4].

References

[1] M. Dellnitz and A. Hohmann, *A subdivision algorithm for the computation of unstable manifolds and global attractors*, Numerische Mathematik **75** (1997), 293–317.

[2] M. Dellnitz and O. Junge, *On the approximation of complicated dynamical behavior*, SIAM Journal on Numerical Analysis **36**(2) (1999), 491–515.

[3] M. Dellnitz and O. Junge, *Set oriented numerical methods for dynamical systems*, B. Fiedler, G. Iooss and N. Kopell (eds.): Handbook of Dynamical Systems II: Towards Applications, World Scientific (2002), 221–264.

[4] M. Dellnitz, O. Schütze and T. Hestermeyer, *Covering Pareto sets by multilevel subdivision techniques*, Journal of Optimization, Theory and Applications, **124**(1) (2005), 113–136.

[5] K. Padberg, *Numerical analysis of transport in dynamical systems*, PhD Thesis, University of Paderborn (2005).

Nonlinear instability for the Navier-Stokes equations

Susan Friedlander
(joint work with Roman Shvydkoy and Nataša Pavlović)

For PDE's there is no general theorem that linear instability implies nonlinear instability. Furthermore, the issue of stability/instability for PDE's is very sensitive to the function space in which growth is measured. For the Navier-Stokes equations Yudovich [3] proved in the function space $L^q(\Omega)$, with $q \geq$ the spatial dimension n, that linear instability implies nonlinear instability. In our talk we present a more transparent proof of this result using a bootstrap argument. This new method of proof extends the result to L^p, $p > 1$, which has the advantage of including the physically interesting case of the L^2 energy norm.

Let $u_0(x) \in C^\infty$ be a steady state velocity satisfying the incompressible Navier-Stokes equations. We write the evolution equations in perturbation form as

$$\frac{\partial v}{\partial t} = -(u_0 \cdot \nabla)v - (v \cdot \nabla)u_0 - \nabla \cdot (v \otimes v) - \nabla p + \nu \Delta v, \tag{1}$$

$$\nabla \cdot v = 0, \tag{2}$$

$$v(t = 0) = v_0. \tag{3}$$

Applying the Leray projector P onto the space of divergence-free functions we write these equations in operator form as

$$\frac{\partial v}{\partial t} = Av + N(v, v), \tag{4}$$

where A is a relatively compact perturbation of the Stokes operator $\nu P\Delta$, and $N(v, v) = P\nabla \cdot (v \otimes v)$. The operator A generates a strongly continuous semigroup in every Sobolev space $W^{s,p}$.

We utilize the following definition of Lyapunov (nonlinear) stability/instability for the Navier-Stokes equations:

Definition. Let $q \geq n$, and $p > 1$. An equilibrium u_0 is called nonlinearly stable if, no matter how small $\rho > 0$, there exists $\delta > 0$ so that $v_0 \in L^q$ and $\|v_0\|_p < \delta$ implies the following

(i) there exists a global in time solution to (4) such that $v(t) \in C([0, \infty); L^q)$
(ii) $\|v(t)\|_p < \rho$ for a.e. $t \in [0, \infty)$.

An equilibrium u_0 that is not stable in the above sense is called nonlinearly unstable.

The concept of existence that we employ is that of "mild" solutions of Kato-Fujita [2]. The solution to (4) is represented as an integral via Duhamel's formula

$$v(t) = e^{tA}v_0 + \int_0^t e^{(t-s)A} N(v, v)(s) ds. \tag{5}$$

It is well known that there is local in time existence for $v(t)$ in L^q when $q \geq n$.

The bootstrap proof of L^p, $p > 1$, nonlinear instability given in Friedlander et al [1] is valid in $\mathbb{T}^n$, a smooth bounded domain $\Omega \subset \mathbb{R}^n$, and in $\mathbb{R}^n$. The proof is

easiest in a finite domain but can be adapted to $\mathbb{R}^n$. The unstable eigenfunction ϕ for the operator A with eigenvalue l of maximal real part is used as the initial condition, i.e. we take $v_0 = \epsilon\phi$. The first term on the LHS of (5) is then $\epsilon\phi e^{\lambda t}$. The idea of the proof of nonlinear instability is to show that the second term on the RHS of (5) grows at most like the square of the norm of $v(t)$ as long as the latter is bounded by a constant multiple of $\epsilon e^{\lambda t}$. The L^q-metric in which such control is possible has to satisfy $q > n$. We use L^q as an auxiliary space, while the final instability result is proved in L^p.

The instability result can be generalized to all the equations of motion that are augmented versions of the incompressible Navier-Stokes equations, e.g. the dissipative MHD equations and the rotating stratified dissipative equations of geostrophical fluid dynamics.

References

[1] Susan Friedlander, Nataša Pavlović and Roman Shvydkoy, *Nonlinear instability for the Navier-Stokes equations*, submitted to Comm. Math. Phys.

[2] Tosio Kato and Hiroshi Fujita, *On the nonstationary Navier-Stokes system*, Rend. Sem. Mat. Univ. Padova **32** (1962), 243–260.

[3] V. I. Yudovich, *The linearization method in hydrodynamical stability theory*, Translations of Mathematical Monographs, vol. 74, American Mathematical Society, Providence, RI, 1989.

On a normal form for one-dimensional excitable media

Georg A. Gottwald

(joint work with Lorenz Kramer)

We present a generic normal form for one-dimensional excitable media. The normal form is constructed around the well-known generic saddle-node bifurcation present in excitable media for isolated pulses. In the case of wave trains or pulses in a ring of length L with velocity c_0, this saddle-node will be disturbed and depends on the wavelength. The interaction with the preceding pulse (or more accurately with its inhibitor) modifies the bifurcation behaviour. The normal form reads

$$\partial_t X = -\mu - gX^2 - \beta(\gamma + X(t-\tau) + \gamma_1 X(t)) \,, \tag{1}$$

where $\beta = \beta_0 \exp(-\kappa\tau)$. Here X measures for example the deviation of the maximal amplitude of the pulse with respect to the saddle-node of the isolated pulse. The delay time $\tau = L/c_0$ is the temporal distance of two consecutive pulses, and κ is the decay rate of the inhibitor. The last term models the influence of the inhibitor of the preceding pulse.

Besides the saddle-node of the isolated pulse ($\beta \to 0$) and the shifted saddle-node of a wave train of finite wave length L, Equation (1) exhibits three new bifurcation scenarios. In particular, a Hopf bifurcation which may coalesce in a Bogdanov-Takens point with the saddle-node, and an inhomogeneous pitchfork bifurcation in which every second pulse dies. These bifurcations have so far not been observed

and we can verify the predictions of our normal form in a modified Barkley model [1] and the Fitzhugh-Nagumo model [2].

We determine the parameters of the normal form by fitting to numerical data obtained by simulating a particular excitable medium, the Barkley model. We test the predictions against numerical simulations of partial differential equation models of excitable media. The quantitative agreement and the predictions are striking.

Moreover, we presented a non-perturbative approach to study bifurcations in excitable media [3]. It is based on the observation that close to the saddle-node the pulse shape is approximately a bell-shaped function. Employing a test function approximation that optimises the two free parameters of a bell-shaped function, i.e. its amplitude and its width, we find the actual bifurcation point and determine the pulse shape for close-to-critical pulses at excitabilities near the bifurcation. This method which makes explicit use of the bell-shaped character at the bifurcation point has also been successfully applied to other reaction diffusion systems such as bistable and autocatalytic systems. It was also successful in describing solution behaviour of reaction diffusion systems far away from the bifurcation [4, 5]. We show that this method is successful in describing retracting fingers in two-dimensional excitable media. Our method may be used to determine the parameters of the normal form (1) directly from the PDE.

References

[1] D. Barkley, Physica **49D**, (1991), 61.
[2] R. Fitzhugh, Biophys. J. **1**, (1961), 445; J. Nagumo, S. Arimoto and S. Yoshizawa, Proc. IRE **50**, (1962), 2061.
[3] G. Gottwald and L. Kramer, Chaos **14**, (2004) 855.
[4] S. Menon and G. Gottwald, Phys. Rev. E **71**, (2005) 066201.
[5] S. Cox and G. Gottwald, submitted.

Fluid flow separation and invariant manifolds

George Haller

In his landmark 1904 paper on boundary layers, L. Prandtl derived a criterion for flow separation in steady two-dimensional incompressible flows [1]. The criterion states that fluid particles separate from a no-slip boundary at a point where the wall-normal derivative of the wall-tangential velocity is zero. This zero-skin-friction principle has become the most broadly used indicator of separation, even though numerical work in the 1970's by Sears and Tellionis ([2],[3]) and others showed that the principle fails for unsteady flows.

Despite all available ad hoc criteria, experiments and simulations, separation in unsteady fluid flows has remained an unsolved problem. As a notable advance, dynamical systems theory has been invoked to describe separation in mathematical

terms: the ejection of the fluid from the boundary is governed by a nonhyperbolic unstable manifold, as observed by Shariff, Pulliam, and Ottino [4] and Yuster and Hackborn [5]. Still, the technical difficulties in finding such manifolds analytically in aperiodic flows seemed insurmountable.

In this talk, I describe a recent solution to the above problem, i.e., an analytic criterion that predicts the location of flow separation in general mass-conserving two-dimensional fluid flows [6]. The criterion identifies the separation point–the point of attachment of a time-dependent nonhyperbolic unstable manifold to the wall–as the location where an appropriately weighted backward-time integral of the wall-nomal velocity derivative remains bounded. The weight function in this integral is the squared reciprocal of the time-dependent fluid density.

I also show numerical and experimental results confirming the analytic criterion, as well as applications of the new separation criterion to flow control [7]. Finally, I describe a refinement of the theory for two-dimensional flows with a steady asymptotic mean [8], and very recent extensions of the theory to three-dimensional flows [9].

References

[1] L. Prandtl, *Über Flüssigkeitsbewegung bei sehr kleiner Reibung*, Verh. III, Int. Math. Kongr., Heidelberg (1904), 484–491.

[2] W. R. Sears and D.P. Tellionis, *Unsteady boundary layer separation*, in Recent Research on Unsteady Boundary Layers (ed. E. Eichelbrenner) (1971), 404–447.

[3] W. R. Sears and D.P. Tellionis, *Boundary-layer separation in unsteady flow.* SIAM J. Appl. Math. **28** (1975) 215–235.

[4] K. Shariff, T. H. Pulliam, T. H., and J.M. Ottino, *A dynamical systems analysis of kinematics in the time-periodic wake of a circular cylinder*, Lect. Appl. Math. **28** (1991) 613–646.

[5] T. Yuster and W.W. Hackborn, *On invariant manifolds attached to oscillating boundaries of Stokes flows*, Chaos **7** (1997), 769–776.

[6] G. Haller, *Exact theory of unsteady separation for two-dimensional flows*, J. Fluid. Mech. **512** (2004), 257-311.

[7] M.R. Alam, W. Liu and G. Haller, *Closed-Loop Separation Control: An Analytic Approach.* submitted to Phys. Fluids. (2005)

[8] M.S. Kilic, G. Haller and A. Neishtadt, *Unsteady flow separation by the method of averaging*, Phys. Fluids **17** (2005) 067104.

[9] A. Surana, O. Grunberg and G. Haller, *Exact theory of three-dimensional flow separation. Part I. Steady separation*, submitted to J. Fluid. Mech. (2005).

Global optimal control of perturbed systems

Oliver Junge

(joint work with Lars Grüne)

We propose a new numerical method for the computation of the optimal value function of perturbed control systems and associated globally stabilizing optimal feedback controllers. The method is based on a set oriented discretization of state space in combination with a new algorithm for the computation of shortest paths

in weighted directed hypergraphs. Using the concept of a multivalued game, we prove convergence of the scheme as the discretization parameter goes to zero.

The method is based on a novel approach to the solution of general nonlinear stabilization problems presented in [4]. The approach relies on a division of state space into boxes that constitute the nodes of a directed weighted graph, where the weights are constructed from the given cost function. On this graph, standard graph theoretic algorithms for computing shortest paths can directly be applied, yielding an approximate value function which is piecewise constant on the state space. At the same time, for every node in the graph, these algorithms compute the successor node on a shortest path, yielding approximate optimal pseudo-trajectories of the original system. Hence, this method combines a simple and hierarchically implementable discretization technique with efficient graph theoretic algorithms. For the problem of feedback stabilization the solution from [4], however, is not directly applicable, because the resulting pseudo-trajectories would have to be postprocessed in order to obtain true solutions of the system.

In [3] it was subsequently shown that the approximate optimal value function can in fact be used in order to construct a stabilizing feedback controller. Based on concepts from dynamic programming [1] and Lyapunov based approximate stability analysis [5], a statement about its optimality properties was given and a local a posteriori error estimate derived that enables an adaptive construction of the division of state space. However, due to the fact that the approximate optimal value function is not continuous, the constructed feedback law is in general not robust with respect to perturbations of the system.

In this talk, we show how to incorporate arbitrary perturbations into the framework sketched above. These perturbations can be either inherently contained in the underlying model, describing, e.g., external disturbances or the effect of unmodelled dynamics, or they could be added on top of the original model to account, e.g., for discretization errors.

The goal is to construct a feedback which is robust in the sense that on a certain subset of state space it stabilizes the system regardless on how the perturbation acts. Conceptually, this problem leads to a dynamic game, where the controls and the perturbations are associated to two "players" that try to minimize and to maximize a given cost functional, respectively. We show how the discretization of state space in a natural way leads to a multivalued dynamic game and give a result on the convergence of the associated value function when the images of the inclusion shrink to the original single-valued map. From this multivalued game we derive a directed weighted hypergraph that gives a finite state model of the original game. We formulate an adapted version of Dijsktra's algorithm in order to compute the associated approximate value function and state a result on the convergence when the box-diameter of the state space division goes to zero.

A particular advantage of our technique is that we are able to keep track of the effects of discontinuities in the approximated optimal value function as induced, e.g., by state space constraints. This allows us to prove not only L^∞ convergence in regions of continuity but also L^1 convergence in the whole domain of the optimal

value function, provided that the optimal value function is continuous with respect to small changes in the state space constraints [2].

References

[1] L. Grüne. An adaptive grid scheme for the discrete Hamilton-Jacobi-Bellman equation. *Numer. Math.*, 75(3):319–337, 1997.
[2] L. Grüne and O. Junge. Global optimal control of perturbed systems. Submitted, 2005.
[3] L. Grüne and O. Junge. A set oriented approach to optimal feedback stabilization. *Systems Control Lett.*, 54(2):169–180, 2005.
[4] O. Junge and H. M. Osinga. A set oriented approach to global optimal control. *ESAIM Control Optim. Calc. Var.*, 10(2):259–270 (electronic), 2004.
[5] D. Nešić and Andrew R. Teel. A framework for stabilization of nonlinear sampled-data systems based on their approximate discrete-time models. *IEEE Trans. Automat. Control*, 49(7):1103–1122, 2004.

Multiscale models for meteorological applications

Rupert Klein

The earth's atmosphere is of overwhelming complexity due to a rich interplay between a large number of phenomena interacting on very diverse length and time scales. There are mathematical equation systems which, in principle, provide a comprehensive description of this system. Yet, exact or accurate approximate solutions to these equations covering the full range of complexities they allow for are not available. As a consequence, one of the central themes of theoretical meteorology is the development of simplified model equations that are amenable to analysis and computational approximate solution, while still faithfully representing an important subset of the observed phenomena associated with specific length and time scales.

It is common agreement that the conservation laws for three-dimensional compressible flow of a mixture of ideal gases with concentration-dependent specific heat capacities, supplemented with a number of source terms, represent a reasonable starting point for such derivations. Yet, most derivations in the literature of reduced model equations for specific atmospheric flow phenomena use already simplified sets of equations as their basis. Examples are the "incompressible Boussinesq approximation" for boundary layer phenomena, or the "Hydrostatic Primitive Equations" for models of very large scale flows, [1, 2]. The decision which model to adopt as the foundation for further derivations is generally based on physical arguments. The same is true for the choice of the particular length and time scales that are to be addressed by a simplified set of equations. Thus, one arrives at the desired simplified model equations through a combination of physical reasoning and mathematical derivations that is hard to digest for the mathematically trained but meteorologically untrained.

This contribution has summarized recent efforts by the author, [3, 4], to provide a unified mathematical modelling framework that satisfies the following requirements. (i) All derivations are based on the full three-dimensional compressible flow

equations. (ii) Specific length and time scalings are introduced within the general framework of multiple scales asymptotics based on a single asymptotic small parameter, ϵ. (iii) The multitude of small or large characteristic numbers found after systematic non-dimensionalization of the governing equations is related to ϵ via a series of distinguished limits.

This approach allows one to recover a large number of well-known "single-scale models", i.e., models covering a single time, a single horizontal, and a single vertical scale, by suitable specializations of the general multiple scales ansatz. Remarkably, one and the same distinguished limit among the various non-dimensional parameters or the system is sufficient to this end. This is an unexpected result which reveals a certain inherent mathematical consistency among all the models that can be reproduced in this way.

The recovery of well-known models may be considered as a "validation" of the suggested mathematical approach. Its potential comes to full fruition with more recent derivations of true multi-scale models. Thus, Majda & Klein, [5], suggest an entire hierarchy of "systematic multiscale models for the tropics" involving synoptic-planetary interactions; Klein et al., [6], demonstrate how turbulent boundary layer models can be incorporated in the same framework; Klein & Majda, [7], provide multiscale models for moist atmospheric processes which explicitly address the multiple time scales associated with condensation-evaporation of cloud water, with the autoconversion of cloud droplets into raindrops, and with the collection of cloud droplets by falling precipitation.

References

[1] A. E. Gill, *Atmosphere-Ocean Dynamics*, volume 30 of *Intl. Geophysics Series*, Academic Press, San Diego, (1982)

[2] J. Pedlosky, *Geophysical Fluid Dynamics*, Springer, 2nd edition, (1987)

[3] Klein R., *Asymptotic Analyses for Atmospheric Flows and the Construction of Asymptotically Adaptive Numerical Methods*, ZAMM, **80**, (2000), 765–777

[4] Klein R., *An Applied Mathematical View of Theoretical Meteorology*, in: Applied Mathematics Entering the 21st century; Invited talks from the ICIAM 2003 Congress. SIAM Proceedings in Applied Mathematics, **116**, (2004)

[5] Majda A.J., Klein R., *Systematic Multi-Scale Models for the Tropics*, J. Atmosph. Sci., **60**, (2003), 393–408

[6] Klein R., Mikusky E., Owinoh A., *Multiple Scales Asymptotics for Atmospheric Flows*, in: 4th European Conference of Mathematics, Stockholm, Sweden, 2004, Ari Laptev (ed.); European Mathematical Society Publishing House, (2005), 201–220

[7] Klein R., Majda A.J., *Systematic Multi-Scale Models for the Tropics*, Theor. & Comp. Fluid Dyn., submitted, August (2005)

Dynamics and Lagrangian Coherent Structures in the Ocean and their Uncertainty

Pierre F.J. Lermusiaux
(joint work with Francois Lekien)

The observation, computation and study of "Lagrangian Coherent Structures" (LCS) in turbulent geophysical flows have been active areas of research in fluid mechanics for the last 30 years. Growing evidence for the existence of LCSs in geophysical flows (e.g., eddies, oscillating jets, chaotic mixing) and other fluid flows (e.g., separation profile at the surface of an airfoil, entrainment and detrainment by a vortex) generates an increasing interest for the extraction and understanding of these structures as well as their properties.

In parallel, realistic ocean modeling with dense data assimilation has developed in the past decades and is now able to provide accurate nowcasts and predictions of ocean flow fields to study coherent structures. Robust numerical methods and sufficiently fast hardware are now available to compute real-time forecasts of oceanographic states and render associated coherent structures. It is therefore natural to expect the direct predictions of LCSs based on these advanced models.

The impact of uncertainties on the coherent structures is becoming an increasingly important question for practical applications. The transfer of these uncertainties from the ocean state to the LCSs is an unexplored but intriguing scientific problem. These two questions are the motivation and focus of this presentation.

Using the classic formalism of continuous-discrete estimation [1], the spatially discretized dynamics of the ocean state vector $\mathbf{x}$ and observations are described by

$$d\mathbf{x} = \mathcal{M}(\mathbf{x}, t) + d\boldsymbol{\eta} \tag{1a}$$

$$\mathbf{y}_k^o = \mathcal{H}(\mathbf{x}_k, t_k) + \epsilon_k \tag{1b}$$

where $\mathcal{M}$ and $\mathcal{H}$ are the model and measurement model operator, respectively. The stochastic forcings $d\boldsymbol{\eta}$ and ϵ_k are Wiener/Brownian motion processes, $\boldsymbol{\eta} \sim \mathcal{N}(0, \mathbf{Q}(t))$, and white Gaussian sequences, $\epsilon_k \sim \mathcal{N}(0, \mathbf{R}_k)$, respectively. In other words, $\mathcal{E}\{d\boldsymbol{\eta}(t) d\boldsymbol{\eta}^T(t)\} \doteq \mathbf{Q}(t)\, dt$. The initial conditions are also uncertain and $\mathbf{x}(t_0)$ is random with a prior PDF, $p(\mathbf{x}(t_0))$, i.e. $\mathbf{x}(t_0) = \widehat{\mathbf{x}}_0 + \mathbf{n}(0)$ with $\mathbf{n}(0)$ random. Of course, vectors and operators in Eqs. (1a-b) are multivariate which impacts the PDFs: e.g. their moments are also multivariate.

The estimation problem at time t consists of combining all available information on $\mathbf{x}(t)$, the dynamics and data (Eqs. 1a-b), their prior distributions and the initial conditions $p(\mathbf{x}(t_0))$. Defining the set of all observations prior to time t by $\mathbf{y}_{t-}$, the conditional PDF of $\mathbf{x}(t)$, $p(\mathbf{x}, t \,|\, \mathbf{y}_{t-})$, contains all of this information and is the solution for the prediction to time t. For the filtering problem at t_k, it is $p(\mathbf{x}, t_k \,|\, \mathbf{y}_0^o, ..., \mathbf{y}_k^o)$. Under classic hypotheses of differentiability and continuity, $p(\mathbf{x}, t \,|\, \mathbf{y}_{t-})$ is governed between observations by the Fokker-Planck equation or Kolmogorov's forward equation (Eq. 2a). At measurement times t_k, one can simply apply Bayes' rule and use the assumed white property of ϵ_k to obtain the update

Eq. 2b.

(2a)

$$\frac{\partial p(\mathbf{x},t\,|\,\mathbf{y}_{t-})}{\partial t} = -\sum_{i=1}^{n}\frac{\partial\,(\,p(\mathbf{x},t\,|\,\mathbf{y}_{t-})\mathcal{M}_i(\mathbf{x},t)\,)}{\partial \mathbf{x}_i} + \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2\,(\,p(\mathbf{x},t\,|\,\mathbf{y}_{t-})\mathbf{Q}_{ij}\,)}{\partial \mathbf{x}_i \partial \mathbf{x}_j}$$

(2b) $$p(\mathbf{x},t_k\,|\,\mathbf{y}_0^o,...,\mathbf{y}_k^o) = \frac{p(\mathbf{y}_k^o\,|\,\mathbf{x})\;p(\mathbf{x},t_k\,|\,\mathbf{y}_0^o,...,\mathbf{y}_{k-1}^o)}{\int\;p(\mathbf{y}_k^o\,|\,\chi)\;p(\chi,t_k\,|\,\mathbf{y}_0^o,...,\mathbf{y}_{k-1}^o)\;d\chi}$$

Equations for governing the moments, modes, etc of the PDF can be obtained from Eqs. 2a-b. When data are assumed to be continuous in time, Eqs. 2a-b are replaced by the Kushner equation if PDFs are retained or by the Zakai equation if a non-normalized form is employed Both explicitly depend on data value increments.

Approximations of these equations were solved using the Error Subspace Statistical Estimation (ESSE, [2]) for the estimation of uncertainties associated to LCSs in Monterey Bay. The Harvard Ocean Prediction System (HOPS) and ESSE provide ocean modeling, data assimilation and uncertainty estimates for the flow fields. These estimates are input to MANGEN [5, 3, 4] to generate the corresponding uncertainties attached to the LCSs in the region. The HOPS-ESSE-MANGEN combination leads to a useful nonlinear scheme for the estimation of oceanic LCSs and their uncertainties via multivariate data assimilation.

The transfer of uncertainties from ensembles of ocean fields to ensembles of coherent structures is studied for three specific regimes in the Monterey Bay area: two upwelling events and one relaxation event. It is shown that such estimates can discriminate the least robust LCS and identify highly certain structures. The Lagrangian uncertainty varies strongly from one regime to the other. However, numerical studies reveal that the more intense DLE ridges are usually more certain.

Future work includes the investigation of higher momenta of the LCS distribution as well as a larger range of oceanographic regime. In addition, LCS and uncertainties in coupled acoustic and biological systems are of major interest for practical applications.

Acknowledgments. We thank J.E. Marsden and J. Scheurle for the invitation to a great workshop. We are grateful to S.C. Shadden, J.E. Marsden and N.E. Leonard for enlightening discussions, as well as the Office of Naval Research for their continuous support.

References

[1] Jazwinski, A.H., 1970. *Stochastic Processes and Filtering Theory*, Academic Press.

[2] Lermusiaux, P.F.J. Uncertainty Estimation and Prediction for the Interdisciplinary Ocean. Invited manuscript, Special Issue. J. Comp. Phys., Submitted.

[3] Lekien F., Coulliette C., Mariano A.J., Ryan E.H., Shay, L.K., Haller, G. and Marsden J., Pollution Release Tied to Invariant Manifolds: A Case Study for the Coast of Florida, *Physica D*, **210** (1–2), 1–20, 2005.

[4] S.C. Shadden, F. Lekien, J.E. Marsden, Definition and Properties of Lagrangian Coherent Structures: Mixing and Transport in Two-Dimensional Aperiodic Flows, Physica D, (submitted), 2005.

[5] `http://www.mangen.info`

Plane Kolmogorov flows and bifurcation without parameters

Stefan Liebscher
(joint work with Andrei Afendikov and Bernold Fiedler)

We are interested in the Kolmogorov problem of a viscous incompressible fluid flow under external forcing

$$\begin{aligned} \partial_t u &= \Delta u - (u\cdot\nabla)u - \nabla p + G\cdot \tbinom{f(x_2)}{0} \\ 0 &= \nabla\cdot u \end{aligned} \tag{1}$$

in a plane canal with periodic boundary conditions

$$\{(x_1,x_2)\mid x_1\in\mathbb{R},\ x_2\in\mathbb{R}/2\pi\mathbb{Z}\},\qquad u(t,x_1,x_2) = u(t,x_1,x_2+2\pi). \tag{2}$$

The forcing originally suggested by Kolmogorov is $f(x_2)=\sin x_2$. We consider generalized odd forcing functions with zero mean value.

The primary stationary x_1-independent profile

$$u_*(x_1,x_2) = \tbinom{U(x_2)}{0},\qquad p_*(x_1,x_2) \equiv \text{constant}, \tag{3}$$

given by

$$U''(x_2)+Gf(x_2) = 0,\qquad \langle U\rangle := \tfrac{1}{2\pi}\textstyle\int U(x_2)\,\mathrm{d}x_2 = 0, \tag{4}$$

becomes unstable for increasing Grashof number G. However, this instability is caused by long-wavelength perturbations. Thus, prior approaches by Meshalkin, Sinai, and Iudovich [1, 2] imposing an artificial periodicity in the originally unbounded x_1-direction are not suited to describe the fluid profiles near the critical instability.

Our approach utilizes Kirchgässner reduction to obtain a spatial dynamical system on a 6-dimensional center manifold, see for example [3]. The dynamics is generated by translations of stationary profiles in the unbounded spatial x_1-direction. Although the elliptic stationary problem (1) yields an ill-posed initial-value problem with respect to cross-sectional profiles evolving in x_1-direction, we still obtain a flow on a manifold tangential to the critical eigenspace of the primary profile u_* at $G=G_0$. This reduced system inherits the three conserved quantities of (1)

$$J = \big(\,\langle u_1\rangle, \langle u_1^2+p\rangle, \langle u_1u_2-\partial_{x_1}u_2\rangle\,\big) \tag{5}$$

and a 3-parameter family of equilibria given by the stationary x_1-homogeneous profiles of (1):

$$u_{*,\beta}(x_1,x_2) = \tbinom{\beta_1+U_{\beta_2}(x_2)}{\beta_2},\qquad p_{*,\beta}(x_1,x_2) \equiv \beta_3, \tag{6}$$

with

$$U''_{\beta_2}(x_2)-\beta_2U'_{\beta_2}(x_2)+Gf(x_2) = 0,\qquad \langle U_{\beta_2}\rangle = 0. \tag{7}$$

In the critical level set $J=(0,0,0)$, a line of equilibria remains. This line of equilibria is neither induced by symmetries, nor by first integrals. The failure of normal hyperbolicity of the line of equilibria gives rise to "bifurcation without parameters" as introduced in [4, 5, 6]

Fixing $G \gtrapprox G_0$, the rescaled reduced spatial-dynamics system reads

$$\dddot{y} + \dot{y} - 3y^2\dot{y} \;=\; a y\ddot{y} + b\dot{y}^2 + \text{ small terms.} \tag{8}$$

Here, the coordinates $(\ddot{y}, \dot{y}, y)$ correspond to variations of the cross sectional profile $U(\cdot)$ of the primary solution (3). The line of equilibria is given by $\{\dot{y} = \ddot{y} = 0\}$.

There are two additional "time"-reversibilities of (8),

$$\begin{array}{ll} S_1: & (\ddot{y}, \dot{y}, y) \mapsto (-\ddot{y}, \dot{y}, -y), \\ S_2: & (\ddot{y}, \dot{y}, y) \mapsto (\ddot{y}, -\dot{y}, y), \qquad (a = b = 0 \text{ only}), \end{array} \tag{9}$$

inherited from the symmetries

$$\begin{array}{ll} S_1: & x_1 \mapsto -x_1, \quad x_2 \mapsto -x_2, \\ S_2: & x_1 \mapsto -x_1, \quad x_2 \mapsto x_2 + \pi \end{array} \tag{10}$$

of (1). The original Kolmogorov forcing, $f(x_2) = \sin x_2$, satisfies both symmetries, whereas generalizations, e.g. $f(x_2) = \sin x_2 + \varepsilon \sin 2x_2$, may break in particular the second symmetry S_2 with 2-dimensional fixed-point space. We will discuss the generalized problem (with symmetry S_1 only) as a small symmetry-breaking perturbation of the original Kolmogorov case (with both symmetries).

Let us start with the fully symmetric case, $a = b = 0$. Truncation of higher-order terms of (8) yields

$$\dddot{y} + \dot{y} - 3y^2\dot{y} \;=\; 0. \tag{11}$$

Integrating (11) once, we obtain the integrable Hamiltonian system

$$\ddot{y} + y - y^3 \;=\; \Theta \tag{12}$$

with energy

$$H \;=\; \tfrac{1}{2}\dot{y}^2 - \tfrac{1}{4}y^4 + \tfrac{1}{2}y^2 - \Theta y \;=\; -\ddot{y}y + \tfrac{1}{2}\dot{y}^2 + \tfrac{3}{4}y^4 - \tfrac{1}{2}y^2 \tag{13}$$

on any fiber of constant Θ.

The set of all nontrivial bounded trajectories, i.e. the trajectories corresponding to inhomogeneous stationary bounded profiles of the fluid flow, is given by a bubble of periodic orbits around the centers $\{|y| < \sqrt{3}/3, \dot{y} = 0, \ddot{y} = 0\}$. Its boundary is provided by the line of equilibria and the homoclinic orbits to the saddles $\{\sqrt{3}/3 < |y| < 1, \dot{y} = 0, \ddot{y} = 0\}$. The two sets of homoclinic orbits meet at the pair of heteroclinics to the saddles $y = \pm 1$ in the fiber $\Theta = 0$. This periodic bubble persists under small perturbations respecting both reversibilities S_1, S_2. The main reason is the intersection of all periodic orbits with the 2-dimensional fixed-point plane $\{\dot{y} = 0\}$ of the reversibility S_2.

We continue with the generalized case of only one reversibility S_1, with one-dimensional fix space given by the $\dot{y}$-axis. However, we restrict the problem to a small symmetry breaking perturbation of the fully symmetric case, e.g. by a small perturbation of the original Kolmogorov forcing,

$$\dddot{y} + \dot{y} - 3y^2\dot{y} \;=\; \varepsilon a y\ddot{y} + \varepsilon b\dot{y}^2 + \text{ small terms}, \qquad 0 < \varepsilon \ll 1. \tag{14}$$

Higher-order terms have only to respect the reversibility S_1, in general.

We switch to coordinates Θ, H, y instead of $\ddot{y}, \dot{y}, y$ and obtain

$$(15) \qquad \begin{aligned} \dot{\Theta} &= \varepsilon \left(a(\Theta - y + y^3)y + 2b(H - \tfrac{1}{2}y^2 + \tfrac{1}{4}y^4 + \Theta y)\right) + \text{ small terms}, \\ \dot{H} &= -y\dot{\Theta}. \end{aligned}$$

For small ε, the variables Θ and H change slowly, of order $\mathcal{O}(\varepsilon)$, whereas the variable y evolves at a time scale $\mathcal{O}(1)$. In fact, the y-$\dot{y}^+$ halfplane defines a Poincaré section to all periodic orbits of (11). The corresponding Poincaré return map of the perturbed system (14) can be interpreted as some first-order time discretization, with time step ε, of the averaged vector field

$$(16) \qquad \dot{\Theta} = \varepsilon \oint a\ddot{y}y + b\dot{y}^2 \, \mathrm{d}\tau, \qquad \dot{H} = \varepsilon \oint -a\ddot{y}y^2 - b\dot{y}^2 y \, \mathrm{d}\tau.$$

At the boundary of the periodic bubble, the averaging procedure amounts to a Melnikov calculation: the integrals can be expressed in terms of elliptic integrals and prove transversality of the splitting of the homoclinic boundary. Discussion of further transverse intersections of center-stable and center-unstable manifolds of the saddle equilibria and the influence of the remaining small terms yields a description of the set of bounded solutions.

In particular, we find a complicated set of saddle-saddle connections that correspond to multi-bump profiles in the fluid-flow problem. Additionally, there exists a continuum of focus-focus and saddle-focus heteroclinic orbits corresponding to stationary profiles with oscillatory tails.

Several important problems are still open in this context. The stability of the profiles with respect to the p.d.e. problem (1) is not understood yet. Only small symmetry-breaking perturbations have been investigated so far. Large perturbations cannot be discussed by the averaging approach used here, since the periodic bubble will not cover all bounded solutions anymore.

References

[1] L. Meshalkin and J. Sinai. Investigation of the stability of the stationary solution of the problem of the motion of some plane incompressible fluid flow (Russian). *Prikl. Math. Mech.*, **25**:1140–1143, 1961.

[2] V. Iudovich. On the instability of parallel flows of a viscous incompressible fluid with respect to spatially periodic perturbations (Russian). In *Numerical methods of solving the problems of Mathematical Physics*. Nauka, Moscow, 1966.

[3] A. Afendikov and A. Mielke. Multi-pulse solutions to the Navier-Stokes problem between parallel plates. *Angew. Math. Phys.*, **51**:79–101, 2001.

[4] B. Fiedler, S. Liebscher, and J.C. Alexander. Generic Hopf bifurcation from lines of equilibria without parameters: I. Theory. *J. Diff. Eq.*, **167**:16–35, 2000.

[5] B. Fiedler and S. Liebscher. Takens-Bogdanov bifurcations without parameters, and oscillatory shock profiles. In H. Broer, B. Krauskopf, and G. Vegter, editors, *Global Analysis of Dynamical Systems, Festschrift dedicated to Floris Takens for his 60th birthday*, pages 211–259. IOP, Bristol, 2001.

[6] B. Fiedler and S. Liebscher. Bifurcations without parameters: Some ODE and PDE examples. In T.-T. Li et al., editor, *International Congress of Mathematicians, Vol. III: Invited Lectures*, pages 305–316. Higher Education Press, Beijing, 2002.

On Three Posters

Jerrold E. Marsden

This talk gave a survey of three of the posters and a few other things. The first poster by Shawn Shadden concerned LCS or Lagrangian Coherent Structures, a notion that, while it has a complex history, is largely due to George Haller (see also Haller's extended abstract). The talk showed some movies that demonstrate some of the key properties of LCS, namely that it divides even unsteady flows into regions in which particles of fluid have different fates, such as recirculation zones in airfoils and in oceanic flows, such as in Monterey Bay. The idea of the computations is to use software like Mangen (see [8]) to compute FTLE (finite time Liapunov exponents), which is a time dependent field that gives the maximum expansion rate of nearby particles. Ridges in this field are the LCS. Some discussion of the theory that gives a precise estimate on the flux across LCS (and hence the extent to which LCS are transported as curves, by the flow) were given; the theory is given in [13]. Additional applications to pollution release in Florida (see [9] as well as the swimming of Jellyfish (see [12]) were given as well.

The general relation of this work to the computation of AIS (Almost Invariant Sets) was also mentioned—roughly, the LCS are boundaries of the AIS. See [3] and [10].

The second poster discussed was that of Eva Kanso on the swimming of an articulated body (a fish) in a perfect fluid (see [5]). There are two main issues here; first of all, the issue of whether or not a fish can swim at all in potential flow—movies were shown demonstrating that it can and can do so fairly efficiently. This already is an interesting achievement, because many previous attempts at this problem (mainly by Montgomery, Kelly and Radford) used either approximate models or perturbation techniques that limited the analysis to relatively small shape changes. Here the method is to use a combination of geometric mechanics (to separate the equations into the shape dynamics and the locomotion dynamics) and direct numerical simulation, without making any analytical approximations. The second thing discussed is the use of DMOC (discrete mechanics and optimal control) from [7] to find *optimal swimming gaits*. This was done in [4]. This is interesting in part, because nature presumably optimizes shape changes to give the best locomotion—of course in reality it may be a multiobjective optimization problem.

The third poster (based on [6]) discusses the phenomenon of dissipation induced instabilities and in particular the example of the baroclinic instability in atmospheric and ocean dynamics using a specific two-layer quasigeostrophic β-plane PDE model that was introduced by Phillips in 1951. It was suspected, based on formal eigenvalue arguments (see [11] and [2]) that this system had its origin of instability in a dissipative mechanism, but this had not been proven rigorously before.

The notion of dissipation induced instabilities is well-known in the finite dimensional case, going back to Thomson and Tait over a century ago and studied by

many people since then, including Chetayev and Merkin, although the name was coined only recently (see [1]). However, the extension to the infinite dimensional case is much more subtle and this is the main point of the poster. Besides detailed analytical estimates for the PDE's involved, the tools that were useful in carrying this out for the baroclinic instability include Arnold's nonlinear stability method as well as the work of Yudovich on the linearized stability instability problem (see also the extended abstract of Friedlander and Shvydkoy).

References

[1] Bloch, A. M., P. S. Krishnaprasad, J. E. Marsden, and T. S. Ratiu [1994], Dissipation induced instabilities, *Ann. Inst. H. Poincaré, Analyse Nonlineéaire* **11**, 37–90.

[2] M. G. Clerc and J. E. Marsden [2001], Dissipation-induced instabilities in an optical cavity laser: A mechanical analog near the 1:1 resonance. *Phys. Rev. E*, **64**, 067603.

[3] Dellnitz, M., O. Junge, W. S. Koon, F. Lekien, M. W. Lo, J. E. Marsden, K. Padberg, R. Preis, S. Ross, and B. Thiere [2005], Transport in dynamical astronomy and multibody problems, *Intern. J. of Bifurcation and Chaos*, **15**, 699–727.

[4] Kanso, E. and J. E. Marsden [2005], Optimal motion of an articulated body in a perfect fluid, *Proc. CDC (to appear)*.

[5] Kanso, E., J. E. Marsden, C. W. Rowley, and J. Melli-Huber [2005], Locomotion of articulated bodies in a perfect fluid, *J. Nonlinear Science* **15**, 255–289.

[6] Kreshetnikov, R. and J. E. Marsden [2005], *Dissipation-induced instability phenomena in infinite-dimensional systems*. Submitted.

[7] Junge, O., J. E. Marsden, and S. Ober-Blöbaum [2005], Discrete mechanics and optimal control, *Proc. IFAC Conf, Prague, June, 2005*.

[8] Lekien, F. [2003], Time-Dependent Dynamical Systems and Geophysical Flows, *Thesis, California Institute of Technology*.

[9] Lekien, F., C. Coulliette, A. J. Mariano, E. H. Ryan, L. K. Shay, G. Haller, and J. Marsden [2005], Pollution release tied to invariant manifolds: A case study for the coast of Florida, *Physica D*. (to appear).

[10] Padberg, K. [2005], *Numerical Analysis of Chaotic Transport in Dynamical Systems*, PhD thesis, University of Paderborn.

[11] R. A. Romea [1977], The effects of friction and β on finite-amplitude baroclinic waves. *J. Atmospheric Sci.*, **34**, 1689–1695.

[12] Shadden, S. C., J. O. Dabiri, and J. E. Marsden [2005], Lagrangian analysis of entrained and detrained fluid in vortex rings, *in preparation*.

[13] Shadden, S. C., F. Lekien, and J. E. Marsden [2005], Definition and Properties of Lagrangian Coherent Structures: Mixing and Transport in Two-Dimensional Aperiodic Flows, *Physica D (submitted)*.

Isotropic LANS-α Equations for Anisotropic Turbulent Flow Simulations

Kamran Mohseni

Turbulent flows play an important role in many areas of atmospheric and oceanic flows as well as engineering fluid mechanics. Accurate simulation of a turbulent flow requires that the energetics of the large scale energy containing eddies, dissipative small scales, and inter-scale interactions to be accounted for. While the direct numerical simulation (DNS) of most geophysical flows seems unlikely

in near future, turbulence modeling could provide qualitative and in some cases quantitative measures for many applications. Large Eddy Simulations (LES) and the Reynolds Averaged Navier-Stokes Equations (RANS) are among the numerical techniques to reduce the computational intensity of turbulent calculations. In LES, the dynamics of the large turbulence length scales are simulated accurately and the small scales are modeled. On the other hand, RANS models are obtained by time averaging the Navier-Stokes equations. In this case most of the unsteadiness is averaged out. Consequently, the time mean quantities are calculated while the faster scale dynamics are modeled.

More recently, Holm, Marsden and their coworkers [4] introduced a Lagrangian averaging technique for the mean motion of ideal incompressible flows. Unlike the traditional averaging or filtering approach used for both RANS and LES, where the Navier-Stokes equations are averaged or spatially filtered, the Lagrangian averaging approach is based on averaging at the level of the variational principle. In the isotropic Lagrangian Averaged Euler-α (LAE-α) equations, fluctuations smaller than a specified scale α are averaged at the level of the flow maps. Mean fluid dynamics are derived by applying an averaging procedure to the action principle of the Euler equations. Both the Euler and the Navier-Stokes equations can be derived in this manner. The usual Reynolds Averaged Navier-Stokes (RANS) or LES equations are then obtained through the subsequent application of either a temporal or spatial average. The critical difference with the Lagrangian averaging procedure is that the Lagrangian (kinetic energy minus potential energy) is averaged *prior to the application of Hamilton principle and a closure assumption is applied at this stage.* This procedure results in either the Lagrangian averaged Euler Equations (LAE-α) or the Lagrangian averaged Navier-Stokes Equations (LANS-α), depending on whether or not a random walk component is added in order to produce a true molecular diffusion term. Since the Hamilton principle is applied after the Lagrangian averaging is performed, all the geometrical properties (e.g., invariants) of the inviscid dynamics are retained even in the presence of the model terms which arise from the closure assumption [4]. For instance, LAE equations posses a Kelvin circulation theorem. Thus it is potentially possible to model the transfer of energy to the unresolved scales without an incorrect attenuation of quantities such as resolved circulation. This is an important distinction for many engineering and geophysical flows where the accurate prediction of circulation is highly desirable.

However, most geophysical flows of interest are often anisotropic. For example, due to rapid damping of turbulent fluctuations in the vicinity of a wall, the application of the isotropic LANS-α equations with a constant α is not appropriate for long term calculations. In order to capture the correct behavior in such systems the parameter α must be spatially or/and temporally varied in the direction of anisotropy [2], i.e., wall normal direction. There has been some attempt (with limited success) in order to remedy this problem. There are at least two approaches to anisotropy in the LANS-α equations:

(i) To derive a set of *anisotropic* LANS-α equations. See alternative derivations in [3, 5].

(ii) Use the isotropic LANS-α equations, but with a variable α to compensate for the anisotropy.

At this point much more work must be done on the anisotropic LANS-α equations before they can be applied to practical problems. The second approach listed above is what will be explored in this study.

In this talk a *dynamic* procedure for the Lagrangian Averaged Navier-Stokes-α (LANS-α) equations is developed where the variation in the parameter α in the direction of anisotropy is determined in a self-consistent way from data contained in the simulation itself. The dynamic model is initially tested in forced and decaying isotropic turbulent flows where α is constant in space but it is allowed to vary in time. In order to evaluate the applicability of the dynamic LANS-α model in anisotropic turbulence, *a priori* test of the dynamic LANS-α in channel flows is performed at various Taylor Reynolds numbers between 180 and 550 based on the wall friction velocity to find the variation of α in the wall-normal direction. It is found that in the wall region the parameter α rapidly increases away from the wall and saturates to an almost constant value in the outer region. An appropriate scaling for α is also identified. As a result, the isotropic LANS-α equations can now be easily used in anisotropic wall bounded flows with a universally damped α. Current numerical experiments exhibit a promising application of the isotropic LANS-α model for anisotropic flows in complex geometries. For more references and details see Zhao and Mohseni [7, 6] and Bhat et.al., [1].

References

[1] H. S. Bhat, R. C. Fetecau, J. E. Marsden, K. Mohseni, and M. West. Lagrangian averaging for compressible fluids. *SIAM Journal on Multiscale Modeling and Simulation*, 3(4):818–837, 2005.

[2] S.Y. Chen, C. Foias, D.D. Holm, E. Olson, E.S. Titi, and S. Wynne. Camassa-Holm equations as a closure model for turbulent channel and pipe flow. *Phys. Rev. Lett.*, 81:5338–5341, 1998.

[3] D.D. Holm. Fluctuation effects on 3D Lagrangian mean and Eulerian mean fluid motion. *Physica D*, 133:215–269, 1999.

[4] D.D. Holm, J.E. Marsden, and T.S. Ratiu. Euler-Poincaré models of ideal fluids with nonlinear dispersion. *Phys. Rev. Lett.*, 349:4173–4177, 1998.

[5] J.E. Marsden and S. Shkoller. The anisotropic Lagrangian averaged Euler and Navier-Stokes equations. *Arch. Rational Mech. Anal.*, 166:27–46, 2002.

[6] H. Zhao and K. Mohseni. Anisotropic turbulent flow simulations using the Lagrangian Averaged Navier-Stokes-α equations. AIAA paper 2005-5352, American Institute of Aeronautics and Astronautics, *35^th AIAA Fluid Dynamics Conference and Exhibit*, Toronto, Canada, June 6-9 2005.

[7] H. Zhao and K. Mohseni. A dynamic model for the Lagrangian averaged Navier-Stokes-α equations. *Phys. Fluids*, 17(7), 2005.

Dynamical systems models of the atmosphere

Paul K. Newton, University of Southern California

We examine a simplified model of the atmosphere based on the two-dimensional spherical shell model for incompressible, inviscid fluids. We discretize the vorticity form of the Euler equations on the unit sphere using a collection of N point vortices [4]. First we describe a one-way coupled model for rotation [7] in which the point vortices are advected by the solid-body velocity field, but the solid-body field is not affected by the point vortices. The dynamical system is given by [7]

$$\dot{\mathbf{x}}_\alpha = \frac{1}{4\pi} \sum_{\beta=1;\beta\neq\alpha}^{N} \Gamma_\beta \frac{\mathbf{x}_\beta \times \mathbf{x}_\alpha}{(1 - \mathbf{x}_\alpha \cdot \mathbf{x}_\beta)} + \Omega \hat{e}_z \times \mathbf{x}_\alpha \tag{1}$$

for $\alpha = 1, ..., N$ and $\mathbf{x}_\alpha \in \mathbb{R}^3, \quad \|\mathbf{x}_\alpha\| = 1$. Here, $\mathbf{\Omega}$ is the solid-body rotational frequency around the north pole $\hat{\mathbf{e}}_\mathbf{z}$.

The *center of vorticity* vector $\mathbf{J}$ (also known as the momentum map) defined as

$$\mathbf{J} = \sum_{\alpha=1}^{N} \Gamma_\alpha \mathbf{x}_\alpha = \left(\sum_{\alpha=1}^{N} \Gamma_\alpha x_\alpha, \sum_{\alpha=1}^{N} \Gamma_\alpha y_\alpha, \sum_{\alpha=1}^{N} \Gamma_\alpha z_\alpha \right) = (J_x, J_y, J_z) \tag{2}$$

plays a central role in our discussion. We can consider its evolution equation by multiplying (1) by Γ_α and summing over α

$$\begin{aligned} \sum_{\alpha=1}^{N} \Gamma_\alpha \dot{\mathbf{x}}_\alpha \quad &= \textstyle\sum_{\alpha=1}^{N} \sum_{\beta=1\neq\alpha}^{N} \Gamma_\alpha \Gamma_\beta \frac{\mathbf{x}_\beta \times \mathbf{x}_\alpha}{(1-\mathbf{x}_\alpha \cdot \mathbf{x}_\beta)} + \sum_{\alpha=1}^{N} \Omega \hat{\mathbf{e}}_z \times \Gamma_\alpha \mathbf{x}_\alpha \\ &= \Omega \hat{\mathbf{e}}_z \times \textstyle\sum_{\alpha=1}^{N} \Gamma_\alpha \mathbf{x}_\alpha. \end{aligned} \tag{3}$$

$\mathbf{J}$ then satisfies

$$\dot{\mathbf{J}} = \Omega \hat{\mathbf{e}}_z \times \mathbf{J}, \tag{4}$$

whose solution is given by

$$\mathbf{J}(t) = \mathbf{M}_\Omega \mathbf{J}(0) \tag{5}$$

where $\mathbf{M}_\Omega$ is the solid-body rotation matrix

$$\mathbf{M}_\Omega = \begin{pmatrix} \cos\Omega t & -\sin\Omega t & 0 \\ \sin\Omega t & \cos\Omega t & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6}$$

From this, we conclude that the length of $\mathbf{J}$ is constant since

$$\|\mathbf{J}\|^2 =< \mathbf{J}, \mathbf{J} >=< \mathbf{M}_\Omega \mathbf{J}(0), \mathbf{M}_\Omega \mathbf{J}(0) >=< \mathbf{M}_\Omega^T \mathbf{M}_\Omega \mathbf{J}(0), \mathbf{J}(0) >= \|\mathbf{J}(0)\|^2. \tag{7}$$

The components satisfy

$$J_x^2 + J_y^2 = C_1 = \text{const.} \tag{8}$$

$$J_z = C_2 = \text{const.} \tag{9}$$

Our first result based on (1) is

Theorem 1: *Solutions on the rotating sphere,* $\mathbf{x}_\alpha(t)$*, are mapped to solutions on the non-rotating aligned sphere,* $\mathbf{z}_\alpha(t)$ *via the unitary operator*

$$L^J_\Omega(t) \equiv \mathbf{M}_\Omega(t){\mathbf{M}_z}^{-1}{\mathbf{M}_y}^{-1}, \text{ i.e.}$$

$$\begin{aligned}
\mathbf{x}_\alpha(t) &= L^J_\Omega(t)\mathbf{z}_\alpha(t) \\
\mathbf{M}_z &= \begin{pmatrix} \cos\gamma_z & -\sin\gamma_z & 0 \\ \sin\gamma_z & \cos\gamma_z & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
\mathbf{M}_y &= \begin{pmatrix} \cos\gamma_y & 0 & \sin\gamma_y \\ 0 & 1 & 0 \\ -\sin\gamma_y & 0 & \cos\gamma_y \end{pmatrix}
\end{aligned}$$

Here, the angles γ_y and γ_z are the angles between the $\mathbf{J}$ vector and the xz and xy planes, respectively and the dynamical trajectory $\mathbf{z}_\alpha(t)$ evolves on the non-rotating sphere in which the $\mathbf{J}$ vector is aligned with the north pole. See [7] for details. From this, along with results from [3] and the properties of $L^J_\Omega(t)$ which yield:

$$\begin{aligned}
\|\mathbf{x}_\alpha - \mathbf{x}_\beta\|^2 &= < \mathbf{x}_\alpha - \mathbf{x}_\beta, \mathbf{x}_\alpha - \mathbf{x}_\beta > \\
&= 2(1 - < \mathbf{x}_\alpha, \mathbf{x}_\beta >) \\
&= 2(1 - < \mathbf{M}_\Omega(t)\mathbf{M}_z^{-1}\mathbf{M}_y^{-1}\mathbf{z}_\alpha, \mathbf{M}_\Omega(t)\mathbf{M}_z^{-1}\mathbf{M}_y^{-1}\mathbf{z}_\beta >) \\
&= 2(1 - < \mathbf{M}_y\mathbf{M}_z\mathbf{M}_\Omega(t)^{-1}\mathbf{M}_\Omega(t)\mathbf{M}_z^{-1}\mathbf{M}_y^{-1}\mathbf{z}_\alpha, \mathbf{z}_\beta >) \\
&= 2(1 - < \mathbf{z}_\alpha, \mathbf{z}_\beta >) = \|\mathbf{z}_\alpha - \mathbf{z}_\beta\|^2
\end{aligned}$$

it follows that:

Corollary: *The 3-vortex problem on the rotating sphere is integrable for all vortex strengths. The 4-vortex problem is integrable if* $\mathbf{J} = 0$.

We then discuss a new set of *dipole coordinates* on the sphere [8] which can be viewed as normal coordinates for a system of N interacting dipoles. The coordinates are based on the centers of vorticity and centroids of each of the interacting dipoles. In these coordinates, the system can be viewed as a billiard system on the sphere, where the leading term governing the motion of each dipole is geodesic flow, but there are long range interactions which couple the dipoles with each other. Properties of this new coordinate system along with some canonical dipole scattering events are described, as are rings of point vortices and Platonic solid configurations [1] on the rotating sphere.

Towards the end of the talk, we compare some of the evolutions in the one-way coupled model with corresponding ones in a fully coupled two-way model [5] in which the background vorticity is discretized into strips, with constant vorticity in each strip constructed so that initially the strips give rise to solid-body rotation. The fully coupled system is an infinite dimensional dynamical system as the contours between adjoining strips must be discretized and their evolution must be tracked along with the point vortices. In this fully coupled system, we show that

the **J** vector is no longer conserved and the orientation tends to drift towards the north pole.

We finish with a discussion of the Anarctic polar vortex splitting event of September 2002 in which the polar vortex in the Southern hemisphere underwent a topological bifurcation from a center to a figure eight pattern (i.e elliptic point to hyperbolic point). Non-integrable models of this event and the resulting particle transport and mixing on the full sphere are described in a poster of S. Ross [6].

References

[1] H. Aref, P. K. Newton, M. Stremler, T. Tokeida, D. Vainchtein, *Vortex crystals*, Advances in Appl. Mech., **39**, (2003), 1–79.

[2] R. Kidambi, P. K. Newton, *Streamline topologies for integrable vortex motion on a sphere*, Physica D, **140**, (2000), 95–125.

[3] R. Kidambi, P. K. Newton, *Motion of three vortices on a sphere*, Physica D, **116**, (1998), 143–175.

[4] P. K. Newton, **The N-Vortex Problem: Analytical Techniques**, *Appl. Math. Sci. Vol. 145*, Springer-Verlag, (2001).

[5] P. K. Newton, T. Sakejo, *Vortex ring evolution on the rotating sphere, in progress.*

[6] P. K. Newton, S. Ross, *Chaotic advection in the restricted four vortex problem on the sphere*, University of Southern California, preprint (2005).

[7] P.K. Newton, H. Shokraneh, *The N-vortex problem on a rotating sphere: I. Multi-frequency configurations*, University of Southern California, preprint (2004).

[8] P. K. Newton, H. Shokraneh, *Vortex dipole coordinates on the sphere, in press*, in **Vortex Dominated Flows**, World Scientific (2005).

Model reduction via degenerate variational asymptotics: convergence and long-time behavior

Marcel Oliver

(joint work with Georg Gottwald)

1. Introduction

We study a new method for constructing reduced models for the slow motion of multi-scale mechanical and fluid-dynamical systems with gyroscopic or magnetic terms in the limit of vanishing inertia. In the context of geophysical fluid mechanics, this limit corresponds to the well-studied semigeostrophic scaling limit where Rossby number and Burger number tend to zero at the same rate.

The main advantage of going the variational route is the persistence of proper analogs of the parent conserved quantities under model reduction. In the geophysical context, the important conserved quantities are energy and potential vorticity, and we derive reduced equations of motion that can be cast in terms of potential vorticity advection and non-standard potential vorticity inversion.

The main difference to earlier work on variational asymptotics, particularly that of Salmon [5, 6] is that, as a first step, we perform a near-identity change of coordinates, which is expanded jointly with the Lagrangian of the system. Model

reduction is achieved by imposing that the transformation must be such that the expanded Lagrangian, when truncated to the desired formal order of accuracy, becomes affine, i.e. linear in the velocities. By allowing very general near-identity changes of coordinates, we introduce degrees of freedom into the construction which turn out to be crucial for maintaining well-posedness and regularity of the "slow" equation in the PDE context.

We apply this method in two cases. As an illustration, we use a finite dimensional toy model for which we can rigorously prove that solutions to the reduced system will converge to solutions to the parent equations if the latter are initialized with "balanced" data. For the rotating shallow water equations, we demonstrate that the variational method yields, to first order, a new model which possesses a third order PV inversion law—the potential vorticity is advected by a velocity field which is smoother than the velocity field for any of the standard nearly geostrophic models (Hoskins' semigeostrophy or Salmon's L_1 dynamics).

2. Illustration of the method

We begin by considering a toy situation, namely the equations for an anharmonic oscillator in the plane with an external magnetic field. We keep the charge of the particle fixed while letting the mass go to zero. This limit is analogous of the semigeostrophic limit for shallow water which we discuss later. The Lagrangian for this system is

$$L_\varepsilon = \frac{\varepsilon}{2}\,|\dot{q}_\varepsilon|^2 - \frac{1}{2}\,\dot{q}_\varepsilon^T J q_\varepsilon - V(q_\varepsilon) \tag{1}$$

where $q\colon \mathbb{R} \to \mathbb{R}^2$ and

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \tag{2}$$

The corresponding equations of motion are

$$\varepsilon\,\ddot{q}_\varepsilon - J\dot{q}_\varepsilon + \nabla V(q_\varepsilon) = 0\,. \tag{3}$$

Step 1: Near identity transformation. Starting from the full Lagrangian, where we denote physical coordinates by q_ε and new coordinates by q, we introduce a near-identity change of coordinates between them via

$$q_\varepsilon = q + \varepsilon\,q' + \tfrac{1}{2}\,\varepsilon^2\,q'' + \dots\,. \tag{4}$$

Note that neither q_ε nor q are assumed to be the solutions of any equation at this point; they are simply the arguments of the Lagrangian as a functional on the full configuration space. As such, we can choose the coefficients q', q'', etc. freely.

This transformation is plugged into the Lagrangian, which we then expand in powers of ε. For our toy model, we obtain

$$L_\varepsilon = L_0 + \varepsilon\,L_1 + \tfrac{1}{2}\,\varepsilon^2\,L_2 + \dots\,, \tag{5}$$

with

$$L_0 = -V(q) - \tfrac{1}{2}\,\dot{q}^T J q\,, \qquad \text{and} \qquad L_1 = \tfrac{1}{2}\,|\dot{q}|^2 - \nabla V(q)\cdot q' - \dot{q}^T J q'\,. \tag{6}$$

Step 2: Degeneracy condition. We now choose q', q'', etc. such that the n-th order Lagrangian is affine, i.e. linear in the velocities:

$$L_n = F_n(q) \cdot \dot{q} + V_n(q) \tag{7}$$

For our model problem, this can be achieved by choosing

$$q' = -\tfrac{1}{2}\, J\dot{q} + f(q) \tag{8}$$

where, in particular, we take the one-parameter family of transformations

$$q' = -\tfrac{1}{2}\, J\dot{q} + \mu\, \nabla V(q)\,. \tag{9}$$

The resulting slow equations of motion are easily derived, and take the form

$$\left[1 + \varepsilon\,(\tfrac{1}{2} + \mu)\,\Delta V\right] J\dot{q} = \nabla V + 2\varepsilon\mu\, \nabla\nabla V\, \nabla V\,. \tag{10}$$

Step 3: Dirac constraint. The Dirac constraint which defines the slow manifold in the full phase space of the parent dynamics is then given by

$$p_n = \frac{\partial L_n}{\partial \dot{q}} = F_n(q)\,. \tag{11}$$

In [1, 2], we prove the following convergence result.

Theorem. *Let q solve the reduced equation including terms of order $n-1$,*

$$\dot{q} = F(q) \tag{12}$$

Let q_ε solve the full nonlinear parent dynamics with $\dot{q}_\varepsilon(0) = F(q_\varepsilon(0))$. Then

$$\|q(t) - q_\varepsilon(t)\| \le c\,\varepsilon^n \tag{13}$$

for $t = O(1)$. Moreover,

$$\|q(t) - q_\varepsilon(t)\| \le c\,\varepsilon^{n-k} \tag{14}$$

for $t = O(\varepsilon^{-k})$.

3. Variational asymptotics for shallow water

The same formal procedure applies to Hamiltonian PDEs with magnetic or gyroscopic terms. A detailed discussion in the context of the rotating shallow water equations is given in [7]. The important difference with respect to the finite dimensional setting is that well-posedness and regularity of the reduced models now crucially depend on the choice of the free parameter λ, and on similar free parameters when computing higher orders of the expansion.

For example, at first order for the semigeostrophically scaled shallow water equations, we obtain a one-parameter family of models that includes Salmon's L_1 and LSG equations, where the latter model is ill posed, as well as a new equation which reads

$$\left[1 - \tfrac{1}{2}\,\varepsilon\,(h\,\Delta + 2\,\boldsymbol{\nabla} h \cdot \boldsymbol{\nabla})\right] \boldsymbol{u} = \boldsymbol{\nabla}^\perp h\,, \tag{15}$$

where u denotes the fluid velocity and h the fluid height function in non-standard coordinates, and the evolution is determined by advection of the potential vorticity $q = (1 + \frac{1}{2}\,\varepsilon\,\Delta h)/h$, or

$$(q - \tfrac{1}{2}\,\varepsilon\,\Delta)h = 1\,. \tag{16}$$

The remarkable consequence is that, for this model, potential vorticity inversion "gains" three derivatives, the maximum possible for first order models of this type. This case resembles the regularity type of the two dimensional Lagrangian averaged Euler equations [4, 3]. Although these equations are, in principle, equally difficult to solve as the L_1 equations, we expect that the built-in non-dissipative smoothing will make the new model numerically much better behaved.

References

[1] G. Gottwald and M. Oliver, *Averaged dynamics via degenerate variational asymptotics*, in preparation.

[2] G. Gottwald, M. Oliver, and N. Tecu, *Long-time accuracy for approximate slow manifolds in a finite dimensional model of balance*, preprint.

[3] D.D. Holm, *Fluctuation effects on 3D Lagrangian mean and Eulerian mean fluid motion*, Phys. D **133** (1999), 215–269.

[4] D.D. Holm, J.E. Marsden, and T.S. Ratiu, *Euler-Poincaré equations and semidirect products with applications to continuum theories*, Adv. in Math. **137** (1998), 1–81.

[5] R. Salmon, *Practical use of Hamilton's principle*, J. Fluid Mech. **132** (1983), 431–444.

[6] R. Salmon, New equations for nearly geostrophic flow, *J. Fluid Mech.* **153** (1985), 461–477.

[7] M. Oliver, *Variational asymptotics for shallow water near geostrophy: A transformational approach*, J. Fluid Mech., to appear.

Lie-Poisson structure for the α-Euler equations

Tudor S. Ratiu

(joint work with François Gay-Balmaz)

The role of Hamiltonian structures for evolutionary conservative equations in mathematical physics is well established. In the finite dimensional case classical symplectic and Poisson geometry and their Lagrangian counterparts form the framework in which the dynamics is formulated. When dealing with infinite dimensional systems one is confronted with serious technical and conceptual difficulties: the formulation uses a weak symplectic form, or the time evolution is not smooth in the function spaces that are natural to the problem. If the system is linear, this corresponds to the fact that the right hand side of the evolutionary equation is given by an unbounded operator. There is very little general theory dealing with the natural questions that arise when working with Hamiltonian PDEs. The first systematic attempt at such a devleopment can be found in [6] and more recently, motivated by questions regarding coherent states quantization, in [12].

Arnold [2] has given a formal Hamiltonian formulation of the Euler equations for an incompressible homogeneous perfect fluid (see also [3], [4], [9]). Ebin and Marsden [7] have shown that, in appropriate Sobolev spaces, the Euler equations

are the spatial representation of a L^2 geodesic spray whose dynamics is that of such a fluid in material representation. This geodesic spray is smooth. Vasyklevych and Marsden [16] have given a Hamiltonian formulation of the Euler equations by carefully analyzing the function spaces on which Poisson brackets are defined and carrying out a non-smooth Lie-Poisson reduction. It is remarkable that the passage from the previous analytically rigorous Lagrangian formulation to this Hamiltonian picture is nontrivial, mainly due to the fact that the flow is not C^1 from the Sobolev space of the initial conditions to itself.

In this talk it will be shown that the program outlined in [16] holds for the α-Euler equation (see [8]). It has been proved in [11], [13], [14] that these equations, either on boundaryless manifolds or with Dirichlet boundary conditions, are the spatial representation of a smooth geodesic spray of a H^1-like weak Riemannian metric on appropriate diffeomorphism groups. These equations are related to the Camassa-Holm equation (see [5]) for which this program can also be carried out.

Let M be a smooth compact oriented Riemannian manifold with smooth boundary ∂M. Let $s > \frac{1}{2}\dim M + 1$ and $\mathcal{D}^s$ be the group of Sobolev class H^s diffeomorphisms on M. We study three boundary conditions:

- the *Dirichlet diffeomorphism group*: $\mathcal{D}^s_D := \{\eta \in \mathcal{D}^s \mid \eta_{|\partial M} = id_{\partial M}\}$
- the *Navier diffeomorphism group*: $\mathcal{D}^s_N := \{\eta \in \mathcal{D}^s \mid (T\eta_{|\partial M} \circ n)^{tan} = 0 \text{ on } \partial M\}$, where $(\cdot)^{tan}$ is the tangential part to the boundary of a vector in $TM|\partial M$.
- the *mixed diffeomorphism group*: $\mathcal{D}^s_{mix} := \{\eta \in \mathcal{D}^s \mid \eta \text{ leaves } \Gamma_i \text{ invariant}, \eta_{|\Gamma_1} = id_{|\Gamma_1}, (T\eta_{|\Gamma_2} \circ n)^{tan} = 0 \text{ on } \Gamma_2\}$, where Γ_1 and Γ_2 are two disjoint subsets of ∂M such that $\partial M = \Gamma_1 \cup \Gamma_2$ and $\overline{\Gamma}_1 = \partial M \setminus \Gamma_2$; furthermore, we assume that for all $m \in \Gamma_i$ we can find a local chart U of M at m such that $\overline{U} \cap \partial M \subset \Gamma_i$.

$\mathcal{D}^s_D, \mathcal{D}^s_N$ and $\mathcal{D}^s_{mix}$ are smooth Hilbert submanifolds and subgroups of $\mathcal{D}^s$. The tangent spaces at e are

$$\mathcal{V}^s_D := T_{id_M}\mathcal{D}^s_D = \{u \in \mathfrak{X}^s_\parallel \mid u_{|\partial M} = 0\},$$

$$\mathcal{V}^s_N := T_{id_M}\mathcal{D}^s_N = \{u \in \mathfrak{X}^s_\parallel \mid (\nabla_n u_{|\partial M})^{tan} + S_n(u) = 0 \text{ on } \partial M\},$$

$$\mathcal{V}^s_{mix} = T_{id_M}\mathcal{D}^s_{mix} = \{u \in \mathfrak{X}^s_\parallel \mid (\nabla_n u_{|\Gamma_1})^{tan} + S_n(u) = 0 \text{ on } \Gamma_1,\ u_{|\Gamma_2} = 0\},$$

where $S_n : T\partial M \to T\partial M$ is the Weingarten map $S_n(u) := -\nabla_u n$.

Denote by $\mathcal{D}^s_\mu$ the topological subgroup of $\mathcal{D}^s$ formed by volume preserving diffeomorphisms, where μ is the Riemannian volume form on M. $\mathcal{D}^s_\mu$ is a smooth Hilbert submanifold of $\mathcal{D}^s$. $\mathcal{D}^s_{\mu,D}, \mathcal{D}^s_{\mu,N}, \mathcal{D}^s_{\mu,mix}$ are the corresponding diffeomorphism groups with the volume-preserving constraint imposed. They are smooth Hilbert submanifolds and topological subgroups of $\mathcal{D}^s_\mu$ and $\mathcal{D}^s$. Their tangent spaces at the identity are $\mathcal{V}^s_{\mu,D}, \mathcal{V}^s_{\mu,N}, \mathcal{V}^s_{\mu,mix}$, which correspond to the previous spaces with the divergence zero condition imposed. We shall work first with the Dirichlet diffeomorphism group and address the other two later.

In what follows we shall use the following notations: $\mathrm{Def}(u) := \frac{\nabla u + (\nabla u)^t}{2}$ is the deformation tensor, $(\nabla u)^t$ is the transpose of the $(1,1)$-tensor ∇u relative to the metric g, $\overline{g}$ is the inner product on $(1,1)$-tensors (in coordinates, if R, S are $(1,1)$-tensors then $\overline{g}(R,S) = g_{ik}g^{j\ell}R^i_j S^k_\ell = \mathrm{Tr}(R^t \cdot S)$), $\flat : \mathfrak{X} \to \Omega^1$ is defined by

$u^\flat := g(u,\cdot)$ forall $u \in \mathfrak{X}$, $\sharp := \flat^{-1} : \Omega^1 \to \mathfrak{X}$, $\mathrm{R}(u,v) := \nabla_u\nabla_v - \nabla_v\nabla_u - \nabla_{[u,v]}$ is the curvature operator, $\mathrm{Ricci}(u,v) := \mathrm{Tr}(w \longmapsto \mathrm{R}(w,u)v)$ is the Ricci tensor, $\mathrm{Ric} : \mathfrak{X} \to \mathfrak{X}$: $g(\mathrm{Ric}(u),v) := \mathrm{Ricci}(u,v)$ is the Ricci operator, δ is the codifferntial associated to g, $\Delta u = -[(d\delta + \delta d)u^\flat]^\sharp$ is the Hodge Laplacian on vector fields, $\Delta_r := \Delta + 2\,\mathrm{Ric}$ is the Ricci Laplacian, and $\mathcal{L} := \Delta_r + \operatorname{grad}\operatorname{div}$.

For an arbitrary constant $\alpha > 0$, introduce , on $\mathfrak{X}^{C^1}$ the inner product

$$\langle u,v\rangle_1 := \int_M \big(g(x)(u(x),v(x)) + 2\alpha^2\overline{g}(x)(\mathrm{Def}(u)(x),\mathrm{Def}(v)(x))\big)\mu(x),$$

for all $u, v \in \mathfrak{X}^1$. A right invariant weak Riemannian metric on $\mathcal{D}^s_{\mu,mix}$ is defined by $\mathcal{G}^1(\eta)(u_\eta,v_\eta) := \langle u_\eta\circ\eta^{-1}, v_\eta\circ\eta^{-1}\rangle_1$ for $u_\eta, v_\eta \in T_\eta\mathcal{D}^s_{\mu,mix}$

The analysis begins with the following well-known result. *Let $\eta(t) \in \mathcal{D}^s_{\mu,D}$ be a curve in $\mathcal{D}^s_{\mu,D}$ and let $u(t) := TR_{\eta(t)^{-1}}(\dot\eta(t)) = \dot\eta(t)\circ\eta(t)^{-1} \in \mathcal{V}^s_{\mu,D}$. Then the following properties are equivalent :*
(1) *$\eta(t)$ is a geodesic of $(\mathcal{D}^s_{\mu,D},\mathcal{G}^1)$*
(2) *$u(t)$ is a solution of the α-Euler equations*

$$(1-\alpha^2\Delta_r)\partial_t u(t) + \nabla_{u(t)}[(1-\alpha^2\Delta_r)u(t)] - \alpha^2\nabla u(t)^t\cdot\Delta_r u(t) = -\operatorname{grad} p(t)$$

(3) *$u(t)$ is a solution of $\partial_t u(t) + \mathcal{P}_e\left(\nabla_{u(t)}u(t) + \mathcal{F}^\alpha(u(t))\right) = 0$, where $\mathcal{F}^\alpha := \mathcal{U}^\alpha + \mathcal{R}^\alpha : \mathcal{V}^s_{\mu,D} \longrightarrow \mathcal{V}^s_D$, for $\mathcal{U}^\alpha(u) := (1-\alpha^2\mathcal{L})^{-1}\alpha^2\,\mathrm{Div}(\nabla u\cdot\nabla u^t + \nabla u\cdot\nabla u - \nabla u^t\cdot\nabla u), \mathcal{R}^\alpha(u) := (1-\alpha^2\mathcal{L})^{-1}\alpha^2\Big(\mathrm{Tr}\left(\nabla_\cdot(\mathrm{R}(\cdot,u)u) + \mathrm{R}(\cdot,u)\nabla_\cdot u + \mathrm{R}(u,\nabla_\cdot u)\cdot\right) - (\nabla_u\,\mathrm{Ric})u - \nabla u^t\cdot\mathrm{Ric}(u)\Big)$*

The geodesic spray $\mathcal{S}^1(u_\eta) = T\overline{\mathcal{P}}\big(S\circ u_\eta - \mathrm{Ver}_{u_\eta}(\overline{\mathcal{F}}^\alpha(u_\eta))\big)$ of $(\mathcal{D}^s_{\mu,D},\mathcal{G}^1)$ is C^∞, where $\overline{\mathcal{F}}^\alpha(u_\eta) := \mathcal{F}^\alpha(u_\eta\circ\eta^{-1})\circ\eta$, S is the geodesic spray of (M,g), and $\mathrm{Ver}_{u_\eta}(v_\eta) \in T_{u_\eta}(T\mathcal{D}^s_{\mu,D})$ is the vertical lift of $v_\eta \in T_\eta\mathcal{D}^s_{\mu,D}$ at $u_\eta \in T_\eta\mathcal{D}^s_{\mu,D}$.

With this background we turn to the Hamiltonian formulation of the α-Euler equations. If $F : T\mathcal{D}^s_{\mu,D} \to \mathbb{R}$ of class C^1, define the **horizontal derivative** $\frac{\partial F}{\partial\eta} : T\mathcal{D}^s_{\mu,D} \to T^*\mathcal{D}^s_{\mu,D}$ of F by

$$\left\langle \frac{\partial F}{\partial\eta}(u_\eta), v_\eta \right\rangle := \left.\frac{d}{dt}\right|_{t=0} F(\gamma(t)),$$

where $\langle\,,\rangle$ is the duality paring and $\gamma(t) \subset T\mathcal{D}^s_{\mu,D}$ is a smooth path defined in a neighborhood of zero, with base point denoted by $\eta(t) \subset \mathcal{D}^s_{\mu,D}$, satisfying the following conditions: $\gamma(0) = u_\eta$, $\dot\eta(0) = v_\eta$, and γ is parallel, that is, its covariant derivative of the $\mathcal{G}^1$ Levi-Civita connection vanishes.

The **vertical derivative** $\frac{\partial F}{\partial u} : T\mathcal{D}^s_{\mu,D} \to T^*\mathcal{D}^s_{\mu,D}$ is the fiber derivative:

$$\left\langle \frac{\partial F}{\partial u}(u_\eta), v_\eta \right\rangle := \left.\frac{d}{dt}\right|_{t=0} F(u_\eta + tv_\eta).$$

These derivatives induce **horizontal** and **vertical functional derivatives** $\frac{\delta F}{\delta \eta}, \frac{\delta F}{\delta u} : T\mathcal{D}^s_{\mu,D} \to T\mathcal{D}^s_{\mu,D}$ relative to the weak Riemannian metric $\mathcal{G}^1$ by

$$\mathcal{G}^1(\eta)\left(\frac{\delta F}{\delta \eta}(u_\eta), v_\eta\right) = \left\langle \frac{\partial F}{\partial \eta}(u_\eta), v_\eta \right\rangle, \ \mathcal{G}^1(\eta)\left(\frac{\delta F}{\delta u}(u_\eta), v_\eta\right) = \left\langle \frac{\partial F}{\partial u}(u_\eta), v_\eta \right\rangle$$

for any $u_\eta, v_\eta \in T\mathcal{D}^s_{\mu,D}$. Since $\mathcal{G}^1$ is weak, the existence of the functional derivatives is not guaranteed but if they exist, they are unique.

For $k \geq 1$ and $r, t > \frac{1}{2}\dim M + 1$ define :

$$C^k_r(T\mathcal{D}^t_{\mu,D}) := \left\{ F \in C^k(T\mathcal{D}^t_{\mu,D}) \Big| \exists \, \frac{\delta F}{\delta \eta}, \frac{\delta F}{\delta u} : T\mathcal{D}^t_{\mu,D} \longrightarrow T\mathcal{D}^r_{\mu,D} \right\}.$$

The Poisson bracket of $F, G \in C^k_r(T\mathcal{D}^t_{\mu,D})$ is defined by

$$\{F, G\}^1(u_\eta) = \mathcal{G}^1(\eta)\left(\frac{\delta F}{\delta \eta}(u_\eta), \frac{\delta G}{\delta u}(u_\eta)\right) - \mathcal{G}^1(\eta)\left(\frac{\delta F}{\delta u}(u_\eta), \frac{\delta G}{\delta \eta}(u_\eta)\right).$$

As in the case of Euler equation we have the following result. Define $\pi_R : T\mathcal{D}^s_{\mu,D} \longrightarrow \mathcal{V}^s_{\mu,D}$ *by* $\pi_R(u_\eta) := u_\eta \circ \eta^{-1}$. *Let* F_t *be the flow of the smooth geodesic spray* $\mathcal{S}^1$ *of the* $\mathcal{G}^1$*-metric and* $\widetilde{F}_t := \pi_R \circ F_t$. *Then* $\widetilde{F}_t$ *is the flow of the* α*-Euler equation and* $\pi_R \circ F_t = \widetilde{F}_t \circ \pi_R$. We need more precise function spaces to describe the Poisson brackets.

(1) For $k, t \geq 1$ and $r \geq s > \frac{1}{2}\dim M + 1$ define:

$$C^k_{r,t}(\mathcal{V}^s_{\mu,D}) := \{f \in C^k(\mathcal{V}^s_{\mu,D}) | \exists \, \delta f : \mathcal{V}^r_{\mu,D} \longrightarrow \mathcal{V}^t_{\mu,D}\}, \quad C^k_t(\mathcal{V}^s_{\mu,D}) := C^k_{s,t}(\mathcal{V}^s_{\mu,D}),$$

where δf is the **functional derivative** of f with respect to $\langle \ , \ \rangle_1$:

$$\langle \delta f(u), v\rangle_1 = Df(u)(v), \quad \forall \, u, v \in \mathcal{V}^r_{\mu,D}.$$

(2) For $k \geq 0$, $r \geq s > \frac{1}{2}\dim M + 1$, and $t \geq 1$ define:

$$\mathcal{K}^k_{r,t}(\mathcal{V}^s_{\mu,D}) := \{f \in C^{k+1}_{r,t}(\mathcal{V}^s_{\mu,D}) | \delta f \in C^k(\mathcal{V}^r_{\mu,D}, \mathcal{V}^t_{\mu,D})\}, \quad \mathcal{K}^k(\mathcal{V}^s_{\mu,D}) := \mathcal{K}^k_{s,s}(\mathcal{V}^s_{\mu,D}).$$

(3) Let $k \geq 1$, $r \geq s > \frac{1}{2}\dim M + 1$, and $t > \frac{1}{2}\dim M + 1$. The **Poisson bracket** on $C^k_{r,t}(\mathcal{V}^s_{\mu,D})$ is defined by: $\{f, g\}^1_+(u) := \langle u, [\delta g(u), \delta f(u)]\rangle_1$, for all $u \in \mathcal{V}^r_{\mu,D}$. For $s, t > \frac{1}{2}\dim M + 1$, $r \geq s$, and $k \geq 1$ the following hold:

(1) $\{ \ , \ \}^1_+$ *is* $\mathbb{R}$*-bilinear and anti-symmetric on* $C^k_{r,t}(\mathcal{V}^s_{\mu,D}) \times C^k_{r,t}(\mathcal{V}^s_{\mu,D})$.

(2) $\{ \ , \ \}^1_+$ *is a derivation in each factor:*

$$\{fg, h\}^1_+ = \{f, h\}^1_+ g + f\{g, h\}^1_+, \forall \, f, g, h \in C^k_{r,t}(\mathcal{V}^s_{\mu,D}).$$

(3) *If* $s > \frac{1}{2}\dim M + 2$, $\{ \ , \ \}^1_+$ *satisfies the Jacobi identity, that is, for all* $f, g, h \in \mathcal{K}^k(\mathcal{V}^s_{\mu,D})$ *and* $u \in \mathcal{V}^{s+1}_{\mu,D}$ *we have:*

$$\{f, \{g, h\}^1_+\}^1_+(u) + \{g, \{h, f\}^1_+\}^1_+(u) + \{h, \{f, g\}^1_+\}^1_+(u) = 0.$$

To show that $\pi_R, F_t, \tilde{F}_t$ are Poisson maps one needs to show that the vertical and horizontal functional derivatives exist. For $f \in C^k(\mathcal{V}^s_{\mu,D})$ denote $f_R := f \circ \pi_R \in C^k(T\mathcal{D}^{s+k}_{\mu,D})$. Let $k \geq 1$ and $r > \frac{1}{2}\dim M + 1$ such that $s + k \geq r$. If $f \in C^k_r(\mathcal{V}^s_{\mu,D})$

then one shows that the vertical functional derivative of f_R with respect to $\mathcal{G}^1$ exists. Similarly, one shows that if $k \geq 1$ and $r > \frac{1}{2}\dim M + 2$ such that $s + k \geq r$, $f \in C^k_r(\mathcal{V}^s_{\mu,D})$, then the horizontal functional derivative of f_R with respect to $\mathcal{G}^1$ exists. Then one proves the following results:

(1) *Let $k \geq 1$ and $r > \frac{1}{2}\dim M + 2$ such that $s + k \geq r$. Let $f \in C^k_r(\mathcal{V}^s_{\mu,D})$. Then $f_R := f \circ \pi_R$ is in $C^k_r(T\mathcal{D}^{s+k}_{\mu,D})$.*

(2) π_R is a Poisson map: *$k \geq 1$, $r > \frac{1}{2}\dim M + 2$ such that $s + k \geq r$.*

$$\{f \circ \pi_R, g \circ \pi_R\}^1(u_\eta) = \left(\{f, g\}^1_+ \circ \pi_R\right)(u_\eta),\ f, g \in C^k_r(\mathcal{V}^s_{\mu,D}),\ u_\eta \in T\mathcal{D}^{s+k}_{\mu,D}.$$

(3) F_t is a Poisson map: *Let F_t be the flow of $\mathcal{S}^1$, $t_1, t_2 > \frac{1}{2}\dim M + 1$ such that $t_1 \geq t_2$. Then for all $G, H \in C^k_{t_2}(T\mathcal{D}^{t_1}_{\mu,D})$ we have $G \circ F_t, H \circ F_t \in C^k_{t_2}(T\mathcal{D}^{t_1}_{\mu,D})$ and $\{G \circ F_t, H \circ F_t\}^1 = \{G, F\}^1 \circ F_t$ on $T\mathcal{D}^{t_1}_{\mu,D}$.*

(4) $\tilde{F}_t$ is a Poisson map: *$\tilde{F}_t = \pi_R \circ F_t$ be the flow of α-Euler. Then*

$$\{f \circ \tilde{F}_t, g \circ \tilde{F}_t\}^1_+(u) = \left(\{f, g\}^1_+ \circ \tilde{F}_t\right)(u),\ \forall f, g \in C^k_r(\mathcal{V}^s_{\mu,D}),\ u \in \mathcal{V}^{s+2k}_{\mu,D},$$

where $k \geq 1$ and $r > \frac{1}{2}\dim M + 2$ such that $s + k \geq r$ (e.g. $k = 1$).

(5) Poisson bracket formulation of α-Euler: *Let $u(t) \subset \mathcal{V}^s_{\mu,D}$ be a curve satisfying $u \in C^0(I, \mathcal{V}^s_{\mu,D}) \cap C^1(I, \mathcal{V}^{s-1}_{\mu,D})$. Then $u(t)$ is a solution of α-Euler iff*

$$\frac{d}{dt} f(u(t)) = \{f, h\}^1_+(u(t)), \forall\, f \in C^1_s(\mathcal{V}^{s-1}_{\mu,D})$$

where $h(u) := \frac{1}{2}\langle u, u\rangle_1$ is the reduced Hamiltonian.

To obtain the same results for the Navier or mixed boundary conditions one notes first that the fundamental difference between these boundary conditions and the Dirichlet case is the following. For all vector fields u, v in $\mathcal{V}^s_D$, the vector field $\nabla_u v$ lies in $\mathcal{V}^{s-1}_D$. This is a fact that was used in many previous computations. Unfortunately, for vector fields u, v in $\mathcal{V}^s_{mix}$ this is not true since $\nabla_u v$ may not be in $\mathcal{V}^{s-1}_{mix}$. In this case one needs to use that $\nabla_u v - \nabla_v u = [u, v]$ is in $\mathcal{V}^{s-1}_{mix}$. With this observation in mind one revisits the entire program outlined above. First one shows that the geodesic spray is a smooth vector field. Then one modifies the proofs in several key lemmas. The upshot is that the theorems quoted above hold without change for the mixed (and hence also for the Navier) boundary conditions.

References

[1] R. Abraham and J. E. Marsden, *Foundations of Mechanics*. Second edition. Benjamin-Cummings Publ. Co, 1978. Updated 1985 version, reprinted by Perseus Publishing,

[2] V. I. Arnold, Sur la géométrie différentielle des groupes de Lie de dimenson infinie et ses applications à l'hydrodynamique des fluids parfaits, *Ann. Inst. Fourier, Grenoble*, **16**(1966), 319–361.

[3] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Graduate Texts in Math., **60**, Springer-Verlag, Second edition, 1989.

[4] V. I. Arnold and B. A. Khesin, *Topological Methods in Hydrodynamics*. Applied Mathematical Sciences, **125**. Springer-Verlag, New York, 1998.

[5] R. Camassa and D. D. Holm, An integrable shallow water equation with peaked solitons. *Phys. Rev. Lett.*, **71**(1993), 1661–1664.
[6] P. R. Chernoff and J. E. Marsden, *Properties of Infinite Dimensional Hamiltonian systems*, volume 425 of *Lecture Notes in Math.*, Springer-Verlag, New York, 1974.
[7] D. G. Ebin and J. E. Marsden, Groups of diffeomorphisms and the motion of an incompressible fluid, *Ann. of Math.* **92**(1970), 102–163.
[8] D. D. Holm, J. E. Marsden, and T. S. Ratiu, Euler-Poincaré models of ideal fluids with nonlinear dispersion, *Phys. Rev. Lett.*, **349**(1986), 4173-4177.
[9] J. E. Marsden and T. S. Ratiu, *Introduction to Mechanics and Symmetry*, Texts in Applied Mathematics, **17**, Springer-Verlag, 1994; Second Edition, 1999, second printing 2003.
[10] J. E.Marsden and T. S. Ratiu [1994], *Introduction to Mechanics and Symmetry*, Texts in Applied Mathematics, **17**, Springer-Verlag, 1994; Second Edition, 1999, second printing 2003.
[11] J. E.Marsden, T. S. Ratiu, and S. Shkoller, S., The geometry and analysis of the averaged Euler equations and a new diffeomorphism group, *Geom. Funct. Anal.*, **10**(3)(2000), 582–599.
[12] A. Odzijewicz and T. S. Ratiu, Banach Lie-Poisson spaces and reduction, *Comm. Math. Phys.*, **243**)(2003), 1–54.
[13] S. Shkoller, Geometry and curvature of diffeomorphism groups with H^1 metric and mean hydrodynamics, *J. Funct. Anal.*, **160**(1)(1998), 337–365.
[14] S. Shkoller, Analysis on groups of diffeomorphisms of manifolds with boundary and the averaged motion of a fluid, *J. Diff. Geom.*, **55**(1)(2000), 145–191.
[15] S. Shkoller, The Lagrangian averaged Euler (LAE-α) equations with free-slip or mixed boundary conditions. *Geometry, Mechanics, and Dynamics. Volume in Honor of the 60th Birthday of J.E. Marsden*, P. Newton, Ph. Holmes, A. Weinstein, editors, Springer-Verlag, New York, 2002, 169–180.
[16] S. Vasylkevych and J. E. Marsden, The Lie-Poisson structure of the Euler equations of an ideal fluid, preprint, 2004.

Discrete Lagrangian fluid dynamics & the Unified Model approach

Sebastian Reich

(joint work with Colin Cotter, Jason Frank, Nigel Wood, Andrew Staniforth)

The Lagrangian, particle following, approach to fluid dynamics has gained a renewed interest in the atmospheric fluid dynamics community over the last decade. The success of the Lagrangian formulation is largely due to its simplicity and geometric interpretation. However, relatively little work had been done on the direct numerical implementation[1] of the Lagrangian approach for atmospheric fluid dynamics till the publication of [2]. The, so called, Hamiltonian particle-mesh (HPM) method has since been extended to shallow-water flows on a sphere and hydrostatic multi-layer models [4]. The numerical robustness of the HMP method relies on several special features. (i) Thermodynamic quantities are evaluated over a fixed Eulerian mesh, while the (microscopic) transport is done via Lagrangian particle paths, which advect mass and (potential) temperature as passive quantities. (ii) To transfer microscopic (particle) quantities to thermodynamic (grid)

[1]It should be noted that the semi-Lagrangian method has, however, been widely implemented in numerical weather prediction codes.

quantities, it is essential that the date is spatially filtered over the fixed Eulerian grid. Indeed, our method can be interpreted as solving a regularized set of fluid equations [3, 5]. (iii) The HPM method is implemented such that the discretized (finite-dimensional) equations of motion are Hamiltonian (or variational) [2, 3, 6]. This property implies also that the HPM method satisfies a Kelvin circulation theorem and that it conserves balanced geostrophic motion [3, 1].

More recently it has been demonstrated that explicit time-stepping of the regularized equations is, on a linear equation level, equivalent to the effect of semi-implicit time-stepping of the unregularized shallow-water equations [5]. The semi-implicit method is, for example, used by the UK MetOffice to overcome the severe step-size restrictions due to unresolved waves in their non-hydrostatic Unified Model. The idea of the Unified Model is to only use unapproximated Euler equations for the dynamic core and to have the spatial and temporal approximations select the desired spatial and temporal resolution. The practical implementation of the Unified Model methodology poses challenging questions to the practitioners and theoreticians alike. Our own current work focuses on an interpretation of the semi-implicit method as a regularization of the unapproximated Euler equations and the implementation of such a regularization within an explicit time-stepping method. The goal is to implement regularized semi-Lagrangian and purely Lagrangian methods for the three-dimensional Euler equations and to test the methods within the Unified Model framework.

Other future work will include implementation of the HPM method over an unstructured finite-element grid, data assimilation for particle methods and subgrid modeling using particles.

References

[1] C.J. Cotter and S. Reich. Adiabatic invariance and applications: From molecular dynamics to numerical weather prediction. *BIT*, 44:439–455, 2004.

[2] J. Frank, G. Gottwald, and S. Reich. The Hamiltonian particle-mesh method. In M. Griebel and M.A. Schweitzer, editors, *Meshfree Methods for Partial Differential Equations*, volume 26 of *Lect. Notes Comput. Sci. Eng.*, pages 131–142, Berlin Heidelberg, 2002. Springer-Verlag.

[3] J. Frank and S. Reich. Conservation properties of Smoothed Particle Hydrodynamics applied to the shallow-water equations. *BIT*, 43:40–54, 2003.

[4] J. Frank and S. Reich. The Hamiltonian particle-mesh method for the spherical shallow water equations. *Atmos. Sci. Let.*, 5:89–95, 2004.

[5] J. Frank, S. Reich, A. Staniforth, A. White, and W. Nigel. Analysis of a regularized, time-staggered discretization method and its link to the semi-implicit method. *Athmos. Sci. Let.*, 6:97–104, 2005.

[6] B. Leimkuhler and S. Reich. *Simulating Hamiltonian Dynamics*. Cambridge University Press, Cambridge, 2005.

Low-order models for control of fluids

Clarence W. Rowley
(joint work with Vejapong Juttijudata)

We discuss methods for obtaining low-dimensional approximations of high-dimensional systems, useful in analyzing the dynamics of fluids, and designing feedback laws for their control. A common technique is to use proper orthogonal decomposition of a set of data, typically obtained from numerical solutions of the high-dimensional system, to determine a subspace that is then used for Galerkin projection of the governing equations. This procedure often works well, but can behave unpredictably: for instance, it can change the stability type of equilibria. We present several improvements to this standard technique.

We begin with a dynamical system on a (typically high-dimensional) vector space V, given by $\dot{x} = f(x)$, where $x \in V$. One wishes to approximate this system by dynamics on a lower-dimensional subspace $S \subset V$, and Galerkin projection specifies these dynamics by

$$\dot{r} = P_S f(r), \qquad r \in S,$$

where $P_S : V \to S$ is an orthogonal projection. Two main choices therefore govern the behavior of low-dimensional approximations obtained in this manner: the choice of the subspace S, and the choice of the inner product under which P_S is orthogonal. Both of these choices are important.

For instance, if one uses an inner product whose induced norm is a Liapunov function then this procedure preserves stability of an equilibrium at the origin, which as stated earlier is not guaranteed for arbitrary inner products. To see this, suppose the origin is a stable equilibrium point, with a quadratic Liapunov function

$$V(x) = x^T Q x, \quad \dot{V}(x) = 2x^T Q f(x) \leq 0.$$

Define an inner product by

$$\langle x, y \rangle = x^T Q y,$$

and let P be any orthogonal projection, i.e. $P^2 = P$, and $\langle x, Py \rangle = \langle Px, y \rangle$. Then V is a Liapunov function for the reduced-order system $\dot{r} = Pf(r)$:

$$\dot{V}(r) = 2r^T Q f(r) \leq 0.$$

Because energy typically decays or is preserved in a physical system, we call such an inner product an *energy-based* inner product, and such inner products have been useful for obtaining reduced-order models of compressible flows [5].

A common way of choosing a subspace for reduced order models is by using Proper Orthogonal Decomposition (POD) on a set of data gathered from simulations or experiments [1]. This method chooses a subspace for which the average error in the projection of the data onto the subspace is minimized (for a fixed dimension of the subspace), or equivalently, that the energy in the projected data is maximized. However, such a choice is not always optimal for capturing the

dynamics correctly, as low-energy phenomena (e.g., acoustic waves in a fluid) can produce a large dynamical response, by exciting large energy phenomena.

An alternative method, valid for stable linear systems, and popular in the control theory community, is balanced truncation, introduced in [3]. In this method, one changes into variables in which controllability and observability Gramians are equal and diagonal, and subsequently truncates the least controllable and observable modes in the model. This method enjoys upper bounds on the error (in the H_∞ norm) which are typically close to the minimum error achievable by any reduced order model of a given dimension, and has been extended to nonlinear systems in [2]. In this talk, we show that for stable linear systems, balanced truncation is identical to the standard POD/Galerkin projection, using the observability Gramian as an inner product, and impulse-state responses as the dataset. Since the quadratic form defined by the observability Gramian is always a Liapunov function, this inner product is indeed "energy-based" in the sense described above. We present a method of snapshots that enables efficient, approximate computation of balanced truncations even for very large systems such as fluids, and we show that in simulations of linearized channel flow that reduced-order models from balanced truncation have much smaller error in the H_∞ norm than models using the standard POD/Galerkin approach [4]. We also present controllers for low-dimensional models of oscillations in the compressible flow past a rectangular cavity, and are able to stabilize the oscillations in a 2D direct numerical simulation [6].

References

[1] P. Holmes, J. L. Lumley, and G. Berkooz. *Turbulence, Coherent Structures, Dynamical Systems and Symmetry*. Cambridge University Press, Cambridge, UK, 1996.

[2] S. Lall, J. E. Marsden, and S. Glavaški. Empirical model reduction of controlled nonlinear systems. In *Proceedings of the IFAC World Congress*, volume F, pages 473–478, July 1999.

[3] B. C. Moore. Principal component analysis in linear systems: Controllability, observability, and model reduction. *IEEE Trans. Automat. Contr.*, 26(1):17–32, Feb. 1981.

[4] C. W. Rowley. Model reduction for fluids using balanced proper orthogonal decomposition. *Int. J. Bifurcation Chaos*, 15(3):997–1013, Mar. 2005.

[5] C. W. Rowley, T. Colonius, and R. M. Murray. Model reduction for compressible flow using POD and Galerkin projection. *Phys. D*, 189(1–2):115–129, Feb. 2004.

[6] C. W. Rowley and V. Juttijudata. Model-based control and estimation of cavity flow oscillations. Proceedings of the 44th IEEE Conference on Decision and Control, Dec. 2005.

On temperature induced motion of shape memory materials

Jürgen Scheurle

The development of smart materials has captured considerable attention during the last decades. A particular class of solid-like, heat conducting materials for which temperature- and stress- induced solid-solid phase transitions lead to rather spectacular hysteretic pseudoelastic deformations, are the so-called shape memory alloys. The name "shape memory alloy" comes from the fact, that at low

temperature these materials (chemical compounds of two or more elements, e.g. AgCd, AuCd, CuAlNi, $BaTiO_3$, NiTi, to mention but a few examples) may sustain a residual deformation after the application of a stress. However, their original shape can be completely restored simply by heating them up to a temperature above a certain critical value (phase transition temperature). Due to their unique characteristics, shape memory alloys have a high potential for applications in various applied areas such as engineering. For instance, they have been tested as actuators and sensors in control systems; see [5]. In order to employ the full advantage of their capabilities, a good theoretical understanding and in-depth study of the materials' behaviour is needed. Several mathematical models have been proposed and studied in this regard. These models mostly consist of a system of coupled partial differential equations for the displacement and temperature fields. Existence and uniquiness of solutions for various types of initial and boundary data have been analysed by several authors. But the existing theory has still severe limitations from the point of view of mathematics. In particular, not much is known about qualitative (dynamical) aspects of the solutions. For instance, such properties could be used to control the materials' behaviour. See [1, 4, 3] for a few results on the (global) stability of equilibrium solutions.

As pointed out already, the shape memory efffect is due to a solid-solid phase transition on the microscopic level, i.e., the atomic crystal lattice of the material changes its configuration (symmetry). For high temperature, the most symmetric phase, called austenite, is energetically stable. Below the phase transition temperature, less symmetric phases, called martensites, are energetically stable, while the austenite phase becomes unstable. In fact, generally there exist several martensite phases that are conjugate by the symmetry group of the austenite phase. Very often different martensite phases even coexist in a material, which leads to the formation of patterened microstructures.

This kind of microscopic behaviour of shape memory alloys suggets, that there are solutions of the model equations which describe continual time-dependent defomations of the material on the macroscopic level, associated to transitions back and forth between the austenite phase and certain combinations of martensite variants, while the temperature oscillates around the phase transition temperature, say, induced by time-dependent distributed or boundary inputs of heat. In this contribution, we report on work in progress that confirms this conjecture mathematically on grounds of a model that is often named after Falk and Konopka [2]. It is to be noted that the kind of motion which we have in mind, has nothing to do with the standard phenomenon of thermal expansion of metallic materials. In fact, in view of the constraints and the (pinned) boundary conditions for the displacement field, which we shall impose, thermal expansion in the standard sense just increases the stress in a homogeneous material, but does not induce motion.

Let us consider a one-dimensional homogeneous shape memory solid B that is identified with the real interval [0,1], which it occupies in a reference configuration. Furthermore, this configuration will be identified with the austenite phase when no stress is applied. We suppose here, that the endpoints of the body B are pinned,

not necessarily stress-free, and held at the same time-periodic temperature. Finally, suppose that B is undergoing a purely longitudinal motion. Then, the conservation laws for momentum and energy governing the corresponding thermoviscoelastic processes give rise to the following system of partial differential equations.

$$\begin{aligned} \rho u_{tt} &= \sigma_x(u_x,\theta) + \beta u_{xxt} - \gamma u_{xxxx} \\ \underbrace{-\theta\varphi_0''(\theta)}_{\text{heat capacity}=1} \cdot \theta_t &= \kappa\theta_{xx} + \theta\sigma_\theta(u_x,\theta)u_{xt} + \beta u_{xt}^2 \end{aligned}$$

The variables, functions and parameters involved in these equations have the following physical meaning: spatial variable $x \in [0,1]$, time variable $t \in \mathbb{R}$, $u(x,t) =$ displacement, $\theta(x,t) =$ temperature, $\varrho =$ mass density, $\kappa =$ heat conductivity, $\beta =$ viscosity constant, $\gamma =$ capillarity coefficient. The function $\sigma = F_\varepsilon$ is the material stress, where $F = F(\varepsilon,\theta)$ represents the Helmholtz free energy density which is a function of the strain $\varepsilon = u_x$ and the temperature θ in case of a homogeneous body B. A typical form of F consistant with the Falk-Konopka theory is given by

$$F(\varepsilon,\theta) = \varphi_0(\theta) + \varphi_1(\theta)F_1(\varepsilon) + F_2(\varepsilon)$$

with

$$\varphi_0(\theta) = \alpha + \theta - \theta\ln\theta \qquad (\alpha = \text{const.})$$

and sufficiently smooth functions $\varphi_1, F_1, F_2 : \mathbb{R} \to \mathbb{R}$ which satisfy certain growth conditions at infinity. Furthermore, φ_1 is supposed to be monotonically increasing and to have a simple zero at the phase transition temperature $\theta = \theta_c$, while F_1 and F_2 are even, strictly convex functions which have a second order, respectively, fourth order zero at $\varepsilon = 0$.

Observe that for $\theta > \theta_c$, $F(\cdot,\theta)$ has a global minimum at the "austenite" $\varepsilon = 0$. Moreover, we assume that for $\theta < \theta_c$, we have exactly two martensite phases associated to two global minima of $F(\cdot,\theta)$ at certain values $\varepsilon = \pm M, M > 0$, while there is a local maximum of $F(\cdot,\theta)$ at $\varepsilon = 0$. So, in the latter range of temperature values, F is not a convex function of the strain ε. This is typical for shape memory alloys in contrast to the classical theory of elasticity. As a consequence, the above system of partial differential equations is rather singular for u_x near zero, when β or γ is equal to zero. Here, all parameters in that system are supposed to be positiv, and γ is considered to be a small regularizing parameter. Finally, we impose the initial and boundary conditions

$$\begin{gathered} u(x,0) = u_0(x),\ u_t(x,0) = v_0(x),\ \theta(x,0) = \theta_0(x) \\ u(0,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0, \qquad u(1,t) = \delta \qquad (\delta = \text{const.}) \\ \theta(0,t) = \theta(1,t) = \mu + a(t) \end{gathered}$$

where a is assumed to be a smooth, T-periodic function of time t with meanvalue zero and sufficiently small amplitude, and μ is viewed to be a real bifurcation parameter.

Then, standard parabolic PDE theory applies to construct a Poincaré (time-T) map for this model system in the Sobolev space $H^2 \cap \overset{\circ}{H}{}^1 \times L_2 \times L_2$ for the state vector $(u_0 - \delta x, v_0, \theta_0)$, cf. [6, 9]. The corresponding discrete dynamical system can be used to analyse qualitative aspects of the solutions of the continuous time model. In particular, fixed points of the Poincaré map correspond to T-periodic solutions of the latter (in an appropriate weak sense), and the respective stability properties are equivalent. Especially, using standard bifurcation and stability theory for maps in Banach spaces, one finds asymptotically stable T-periodic solutions of the kind conjectured above.

For $\delta = 0$, these solutions branch off from a branch of basic T-periodic solutions which undergo a subcritcal pitchfork bifurcation at a critical value μ_c of the bifurcation parameter μ. We have $\mu_c < \theta_c$, and $\mu_c \to \theta_c$ as $\gamma \to 0$. The basic solutions have a zero u-component and loose their stability by a simple real eigenvalue, as μ passes through μ_c from above. For small $\mid \delta \mid \neq 0$, an imperfect pitchfork bifurcation occurs nearby.

A similar result is to be expected in higher space dimensions. We have preliminary results in this regard, see [7, 8].

References

[1] J.M. Ball, P.J. Holmes, R.D. James, R.L. Pego, and P.J. Swart, *On the dynamics of fine structure*, J. Nonlinear Sci. **1** (1991), 17–70.

[2] F. Falk and P. Konopka, *Three-dimensional Landau theory describing the martensitic phase transition of shape memory alloys*, J. Phys.: Condensed Matter **2** (1990), 61–77.

[3] G. Friesecke and J.B. McLeod, *Dynamics as a mechanism preventing the formation of finer and finer microstructure*, Arch. Rat. Mech. Anal. **133** (1996), 199–247.

[4] W.D. Kalies, *Regularized models of phase transformations in one-dimensional nonlinear elasticity*, Ph.D. Thesis, Cornell University (1994).

[5] Martin Schleich, *Formgedächtnismaterialien als Fingerantrieb*, Fortschritts-Berichte VDI Reihe 17, **239** (2004), Düsseldorf: VDI-Verlag.

[6] R.S. Spies, *A State-Space Approach to a One-Dimensional Mathematical Model for the Dynamics of Phase Transitions in Pseudoelastic Materials*, J. Math. Anal. Appl. **190** (1995), 58–100.

[7] J. Zimmer, *Stored energy functions for phase transitions in crystals*, Arch. Rat. Mech. Anal. **172 (2)** (2004), 191–204.

[8] J. Zimmer, *Global Existence of a Nonlinear System in Thermoviscoelasticity with Nonconvex Energy*, J. Math. Anal. **292 (2)** (2004), 589–604.

[9] S.J. Watson, *Unique global solvability for initial-boundary value problems in one-dimensional nonlinear thermoviscoelasticity*, Arch. Rat. Mech. Anal. **153 (1)** (2000), 1–37.

Poisson brackets for rigid bodies in vortical fluids

Banavara N. Shashikanth

The Poisson brackets and the Hamiltonian structure of the dynamically interacting system of a single rigid body with the vorticity field of a Newtonian fluid are examined formally. Functional analytic details, as in, for example, Ebin and Marsden [2] for the case with no body, though very important, are not considered

here. The flow of the fluid is inviscid, of constant density and occupies the domain external to the body; further, it is assumed that the body is neutrally buoyant. Free-slip boundary conditions prevail at the body-fluid boundary ∂D and the fluid is at rest at infinity.

The system is investigated in two different settings. The first setting is in $\mathbb{R}^3$ with the vorticity field smooth everywhere and the second setting is in $\mathbb{R}^2$ with the vorticity field a singular distribution of N point vortices.

In the first setting, the Poisson brackets are examined in the framework of geometric mechanics, more specifically in the framework of cotangent bundle reduction theory [3, 4]. Starting with an appropriate choice of configuration space Q, the cotangent bundle T^*Q is constructed formally. For this problem Q is chosen as the following space of pairs of maps:

$$Q \ni q \equiv (\Phi_g, \eta_g),$$

where relative to some choice of the body and fluid reference configurations, B_{ref} and D_{ref}, respectively, $\Phi_g : B_{\text{ref}} \longrightarrow \mathbb{R}^3$, $g \in \text{SE}(3)$, is a rigid body embedding, and $\eta_g : D_{\text{ref}} \longrightarrow \mathbb{R}^3 \backslash \{\Phi_g(p)\}$, $p \in B_{\text{ref}}$, is a volume-preserving diffeomorphism that satisfies the boundary conditions on ∂D and at infinity. The tangent bundle TQ is the space of pairs $(q, \dot{q})$, where the tangent vectors $\dot{q} \in T_qQ$ are identified as:

$$\dot{q} \equiv (v_q, V_S),$$

where v_q is the divergence-free fluid velocity field (in the material or Lagrangian representation), not parallel to ∂D, and V_S is the body velocity (in the spatial representation). The cotangent bundle T^*Q is the space of pairs (q, α_q), where the covectors $\alpha_q \in T_q^*Q$ are identified as:

$$\alpha_q \equiv ((\mathbf{d}v_q^\flat, i^*v_q^\flat), P_S),$$

where, in the notation of exterior differential geometry [1], $v_q^\flat$ is the one-form associated with v_q via the metric on $\mathbb{R}^3$, $\mathbf{d}$ is the exterior derivative and i^* denotes pullback by the inclusion map $i : \partial D \longrightarrow \mathbb{R}^3$. In other words, a covector is identified with the pair $(\mathbf{d}v_q^\flat, i^*v_q^\flat)$: the vorticity two-form $\omega_q = \mathbf{d}v_q^\flat$ (on D, the fluid domain) and the tangential velocity one-form (on ∂D), and P_S, the body momentum (in the spatial representation).

The non-degenerate bilinear map $<,>: T_qQ \times T_q^*Q \longrightarrow \mathbb{R}$ that pairs tangent and cotangent elements is defined as follows. For elements $\dot{q} \equiv (v_q, V_S) \in T_qQ$, $\mu_q \equiv ((\mathbf{d}u_q^\flat, i^*u_q^\flat), R_S) \in T_q^*Q$, the map is written in terms of pairings of differential forms as:

$$\langle \mu_q, \dot{q} \rangle = \int_D \mathbf{d}u_q^\flat \wedge \star\beta_q - \int_{\partial D} i^*u_q^\flat \wedge i^* \star \beta_q + \langle R_S, V_S \rangle_{\text{SE}(3)},$$

where $\star$ is the Hodge star operator, β_q is the vector potential two-form which is related by the co-differential operator $\delta = \star\mathbf{d}\star$ to the velocity one-form as

$$v_q^\flat = \delta\beta_q$$

and $< , >_{\text{SE}(3)}$ is the bilinear pairing on SE(3).

Now consider the Hamiltonian $H : T^*Q \longrightarrow \mathbb{R}$, which is the kinetic energy of the body+fluid, obtained in the usual manner by the Legendre transform of the body+fluid kinetic energy Lagrangian:

$$H(q,\alpha_q) := \frac{1}{2}\left(\int_D \omega_q \wedge \star\,(\mathbf{d}\delta)^{-1}\,\omega_q - \int_{\partial D} i^* v_q^\flat \wedge i^* \star\,\delta^{-1} v_q^\flat + \left\langle\left\langle \zeta, A^{-1}\zeta\right\rangle\right\rangle_{\mathbb{R}^6}\right),$$

where the last term is the body kinetic energy written at the identity element of SE(3) ($\zeta \in \mathfrak{se}(3)^*, A : \mathfrak{se}(3) \longrightarrow \mathfrak{se}(3)^*$).

Following these preliminaries, work currently in progress is briefly described. The goal is to first write the Hamiltonian vector field on T^*Q relative to the choice of the canonical Poisson brackets on T^*Q:

$$\{F,G\}\,(q,\alpha_q) = \left\langle \frac{\delta F}{\delta q}, \frac{\delta G}{\delta \alpha_q}\right\rangle - \left\langle \frac{\delta G}{\delta q}, \frac{\delta F}{\delta \alpha_q}\right\rangle,$$

where $F, G : T^*Q \longrightarrow \mathbb{R}$. The next step is to Poisson reduce T^*Q by the symmetry groups in this problem. In particular, by the action of the groups, $\mathrm{Diff}_{\mathrm{vol}}(D_{\mathrm{ref}})$, the group of volume preserving diffeomorphisms of the fluid reference configuration, and SE(3). The Poisson brackets on the Poisson reduced spaces will then be derived as per the general theory outlined in [3, 4].

In the second setting, in $\mathbb{R}^2$, the system of a 2D rigid cylinder of arbitrary (smooth) shape dynamically interacting with N point vortices external to it is considered. The circulation about the cylinder is assumed to be zero. Here, the equations of motion of the system are first derived using conventional momentum balance arguments. Following this, the Poisson brackets of the system are obtained by inspection.

The equations of motion of this system relative to a body-fixed frame are [5, 6]:

$$\left(\frac{d}{dt} + \boldsymbol{\Omega}\times\right)\mathbf{L} = \Gamma\mathbf{k}\times\mathbf{V}, \qquad (\Gamma := \sum_{j=1}^{N}\Gamma_j) \tag{1}$$

$$\frac{d\mathbf{A}}{dt} + \mathbf{V}\times\mathbf{L} = 0, \tag{2}$$

$$\Gamma_j\left(\frac{d\mathbf{l}_j}{dt} + \boldsymbol{\Omega}\times\mathbf{l}_j + \mathbf{V}\right) = J\left(\frac{\partial W}{\partial \mathbf{l}_j}\right), j = 1,.,N \tag{3}$$

where $(\mathbf{L},\mathbf{A})$ are the system linear and angular momenta, respectively (i.e. fluid linear and angular impulse plus cylinder linear and angular momentum, respectively). These are related to the body velocities $(\mathbf{V},\boldsymbol{\Omega})$ and the position vectors and strengths of the vortices, $\mathbf{l}_j$ and Γ_j, by the relations:

$$\begin{pmatrix} \mathbf{L} \\ \mathbf{A} \end{pmatrix} = M \begin{pmatrix} \mathbf{V} \\ \boldsymbol{\Omega} \end{pmatrix} + \begin{pmatrix} \mathbf{p} \\ \pi \end{pmatrix},$$

where M is the 3×3 mass matrix containing the body mass and added mass terms, and $(\mathbf{p},\pi)$–functions of $\mathbf{l}_j$ and Γ_j–are terms that depend on the distribution of the point vortices about the body and the shape of the body. The function $W(\mathbf{l}_j,\mathbf{V},\boldsymbol{\Omega};\Gamma_j)$ appearing in (3) is the Kirchhoff-Routh function generalized to

moving rigid boundaries and J is the standard symplectic matrix. See the quoted references for details.

For the Hamiltonian function which is the body+fluid kinetic energy (minus infinite contributions due to the presence of the point vortices), the system (1), (2) and (3) is Hamiltonian on the space $P = \mathfrak{se}(2)^* \times (\mathbb{R}^{2N} \backslash \{\text{collision points}\}) \equiv P_b \times P_v$ equipped with the following Poisson brackets. For $F, G : P \longrightarrow \mathbb{R}$,

$$\begin{aligned}
\{F, G\} &= \{F\mid_{P_b}, G\mid_{P_b}\}_{\text{Lie}-\text{Poisson}} + \{F\mid_{P_v}, G\mid_{P_v}\}_{\text{point vortex}}, \qquad \text{when } \Gamma = 0,\\
\{F, G\} &= \{F\mid_{P_b}, G\mid_{P_b}\}_{\text{Lie}-\text{Poisson}} + \{F\mid_{P_v}, G\mid_{P_v}\}_{\text{point vortex}}\\
&\qquad + \{F\mid_{P_b}, G\mid_{P_b}\}_{2-\text{cocycle}}, \qquad \text{when } \Gamma \neq 0,
\end{aligned}$$

The component brackets are the Lie-Poisson bracket on $\mathfrak{se}(2)^*$, the canonical point vortex bracket and a 2-cocycle bracket (for the case $\Gamma \neq 0$) obtained from the 2-cocycle in the N point vortex problem, with no rigid boundaries, that arises due to the lack of equivariance of the momentum map under the coadjoint action (in *that* problem).

Another interesting feature of this system, not directly related to Poisson brackets, is the existence of a reciprocity (or reciprocal) relation that holds irrespective of the strengths of the vortices and the shape of the cylinder. Details may be found in [6].

Finally, some flow visualization experiments done at New Mexico State University, Mechanical Engineering department, by James Allen and students on a vortex ring impinging on a neutrally buoyant sphere are presented.

References

[1] R. Abraham, J. E. Marsden and T. Ratiu, *Manifolds, Tensor Analysis and Applications*, Second Edition, Springer, New York (1988).

[2] D. G. Ebin and J. E. Marsden, *Groups of diffeomorphisms and the motion of an incompressible fluid*, Ann. of Math. **92** (1970), 102–163.

[3] R. Montgomery, J. Marsden and T. Ratiu, *Gauged Lie-Poisson Structures*, Contemp. Math. **28** (1984), 101–113.

[4] M. Perlmutter and T. S. Ratiu, *Gauged Poisson Structures*, (preprint), (2005)

[5] B. N. Shashikanth, J. E. Marsden, J. W. Burdick and S. D. Kelly, *The Hamiltonian structure of a 2-D rigid circular cylinder interacting dynamically with N point vortices*, Phys. of Fluids, **14** (2002), 1214-1227.

[6] B. N. Shashikanth, *Poisson brackets for the dynamically interacting system of a 2-D rigid cylinder and N point vortices: The case of arbitrary smooth cylinder shapes*, Reg. Chaotic Dyn., **10** (2005), 1-14.

Numerical Simulations of the EPDiff Equation

Martin Staley
(joint work with Darryl Holm)

In this presentation, we numerically illustrated the evolution behavior of the EPDiff equation, "Euler-Poincaré equation on the diffeomorphisms," (see [1]) for a variety of initial conditions in two and three dimensions. In terms of divergence, gradient, and curl, EPDiff is:

$$m_t - u \times (\nabla \times m) + \nabla(u \cdot m) + m(\nabla \cdot u) = 0,$$

where the momentum vector m and the velocity vector u are related through the Helmholtz operator,

$$m = (1 - \alpha^2 \nabla^2)u.$$

Here, α is a scalar with dimensions of length.

Numerics were performed for both 2-D and 3-D on a Linux PC with 4 gigs of memory, using a carefully tuned C++ code for the time integration, and using Matlab for the creation of movie frames from the data.

We used compatible differencing for our evaluation of the divergence, gradient, and curl operators (see [2]). *Compatible* (or *mimetic*) difference operators are particular formulations of finite difference operators that are designed to preserve certain properties of continuous operators, such as $\nabla \cdot (\nabla \times f) \equiv 0$ and $\nabla \times (\nabla f) \equiv 0$. A key aspect of these operators is that their operands lie on different *spaces*, or grid centerings. In fact, different components of the same vectors lie at different places on the grid.

2-D numerical simulations were performed on grids with 1024^2 cells, a Runge-Kutta 4/5 explicit time-stepping method, a variable time step based on comparing the 4th-order predictor with the 5th-order corrector, and a variety of initial conditions with either peakon ($\exp(-|x/\alpha|)$) or Gaussian ($\exp(-(x/\alpha)^2)$) profiles. We solved for u, given m, in Fourier space.

3-D numerical simulations were performed on grids with 256^3 cells, a Runge-Kutta 3/4 explicit time-stepping method, a variable time step, and a variety of peakon-like and Gaussian-like initial conditions. As in two dimensions, we solved for u, given m, in Fourier space.

A variety of tests, computations, and observations helped verify that our numerics were of reasonable accuracy. For example, test runs of 1-D peakons, embedded in 2-D and 3-D, reproduced the known 1-D analytical solution. Each simulation was run backward from the ending time to the starting time, and the difference between initial conditions and reconstructed initial conditions was found to be small. Runs were done with different prescribed error tolerances and with 2nd-, 4th-, and 6th-order difference operators, with little effect. Characteristics known to hold in 1-D, and expected in higher dimensions (for example, the elastic collisions of peakons), were indeed observed. We also verified that the H_1 norm of the velocity field, known to be a conserved quantity, was in fact reasonably well conserved by the numerics.

As we illustrated animations of the time evolution of EPDiff for a variety of initial velocity profiles, we made several observations including the following: (1) all initial conditions produced moving, string-like (in 2-D) or brane-like (in 3-D) filaments; (2) filaments had peakon cross-sections, with characteristic width α; (3) filaments can reconnect with one another, a phenomenon that is not yet totally understood; (4) low-amplitude "memory wisps," with peakon profiles, trailed behind points of reconnection and may play a role in making the simulations reversible; (5) filaments colliding head-on undergo annihilation and subsequent reconstruction, as one might expect based on the behavior of peakon solutions of EPDiff in one dimension; (6) filaments tend to spread out, and decrease in amplitude, as time progresses; and (7) filaments approaching others from behind impart momentum to those in front.

References

[1] D. D. Holm and M. F. Staley, "Interaction dynamics of singular wave fronts." In preparation.

[2] J. M. Hyman and M. Shashkov, "The adjoint operators for the natural discretizations of the divergence, gradient, and curl on logically rectangular grids," *IMACS J. Appl. Num. Math.*, **25**, pp. 413-442 (1997).

Reporter: Jerrold E. Marsden

Participants

Dr. Emmanuel Audusse

Institut für Mathematik I (WE 1)
Freie Universität Berlin
Arnimallee 2-6
14195 Berlin

Christian Bauer

Zentrum Mathematik
TU München
Boltzmannstr. 3
85748 Garching bei München

Prof. Dr. Wolf-Jürgen Beyn

Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld

Prof. Dr. Alexey V. Borisov

Institute of Computer Science
Udmurt State University
1, Universitetskaya str,
Izhevsk 426034
Russia

Dr. Colin Cotter

Department of Mathematics
Imperial College London
Huxley Building
GB-London SW7 2AZ

Prof. Dr. Michael Dellnitz

Institut für Mathematik
Universität Paderborn
33095 Paderborn

Razvan Fetecau

Department of Mathematics
Stanford University
Stanford, CA 94305-2125
USA

Prof. Dr. Bernold Fiedler

Institut für Mathematik I (WE 1)
Freie Universität Berlin
Arnimallee 2-6
14195 Berlin

Prof. Dr. Susan Friedlander

Department of Mathematics
University of Illinois at Chicago
M/C 249, 322 SEO
851 S. Morgan Street
Chicago IL 60607-7045
USA

Francois Gay-Balmaz

Departement de Mathematiques
Ecole Polytechnique Federale
de Lausanne
CH-1015 Lausanne

Dr. Georg A. Gottwald

School of Mathematics & Statistics
The University of Sydney
Sydney NSW 2006
AUSTRALIA

Prof. Dr. George Haller

Department of Mechanical Engineering
MIT, Rm. 3-352
77 Massachusetts Ave.
Cambridge MA 02139
USA

Thomas John

Department of Mechanical and Environmental Engineering
University of California
Santa Barbara CA 93106-5070
USA

Dr. Oliver Junge

Institut für Mathematik
Universität Paderborn
33095 Paderborn

Eva Kanso

Control and Dynamical Systems
Caltech 107-81
1200 E.California Blvd.
Pasadena, CA 91125-8100
USA

Mustafa Sabri Kilic

Department of Mathematics
MIT
Rm. 2-487
77 Massachusetts Ave.
Cambridge MA 02139
USA

Alexander Kilin

Institute of Computer Science
Udmurt State University
1, Universitetskaya str,
Izhevsk 426034
Russia

Prof. Dr. Rupert Klein

Potsdam Institut für Klimafolgenforschung
Telegrafenberg A 31
Postfach 60 12 03
14412 Potsdam

Francois Lekien

Department of Mechanical and Aerospace Engineering
Princeton University
Engineering Quad, Olden Street
Princeton, NJ 08544-5263
USA

Prof. Dr. Pierre F. J. Lermusiaux

Division of Engineering and Applied Sciences
Harvard University
Pierce Hall
Cambridge, MA 02138
USA

Dr. Stefan Liebscher

Institut für Mathematik I
Freie Universität Berlin
Arnimallee 2-6
14195 Berlin

Ivan S. Mamaev

Institute of Computer Science
Udmurt State University
1, Universitetskaya str,
Izhevsk 426034
Russia

Prof. Dr. Jerrold E. Marsden

Control and Dynamical Systems
Caltech 107-81
1200 E.California Blvd.
Pasadena, CA 91125-8100
USA

Juan Melli-Huber

Department of Mechanical and
Aerospace Engineering
Princeton University
Engineering Quad, Olden Street
Princeton, NJ 08544-5263
USA

Prof. Dr. Kamran Mohseni

Aerospace Engineering Department
University of Colorado at Boulder
Campus Box 429
Boulder, CO 80309-0429
USA

Prof. Dr. Paul Newton

Dept. of Aerospace Engineering
University of Southern California
Los Angeles, CA 90089-1191
USA

Prof. Dr. Marcel Oliver

School of Engineering and Science
International University Bremen
Campus Ring 12
28759 Bremen

Kathrin Padberg

Institut für Mathematik
Universität Paderborn
Warburger Str. 100
33098 Paderborn

Prof. Dr. Tudor S. Ratiu

Departement de Mathematiques
Ecole Polytechnique Federale
de Lausanne
CH-1015 Lausanne

Prof. Dr. Sebastian Reich

Institut für Mathematik
Universität Potsdam
Postfach 601553
14415 Potsdam

Shane Ross

Department of Aerospace and
Mechanical Engineering
University of Southern California
Los Angeles CA 90089-1191
USA

Jens Rottmann-Matthes

Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld

Prof. Dr. Clarence Rowley

Department of Mechanical and
Aerospace Engineering
Princeton University
Engineering Quad, Olden Street
Princeton, NJ 08544-5263
USA

Prof. Dr. Jürgen Scheurle

Zentrum Mathematik
TU München
85747 Garching bei München

Shawn Shadden

Control and Dynamical Systems
Caltech 107-81
1200 E.California Blvd.
Pasadena, CA 91125-8100
USA

Prof. Dr. Banavara Shashikanth

Department of Mechanical
Engineering
New Mexico State University
P.O. Box 30001, Dept 3450
Las Cruces NM 88003-8001
USA

Dr. Roman Shvydkoy

Dept. of Mathematics, Statistics
and Computer Science, M/C 249
University of Illinois at Chicago
851 S. Morgan Street
Chicago, IL 60607-7045
USA

Dr. Stefan Siegmund

Fachbereich Mathematik
Universität Frankfurt
Robert-Mayer-Str. 6-10
60325 Frankfurt am Main

Martin Staley

Mathematical Modeling and Analysis
T-7, M.S. B284
Los Alamos National Laboratory
Los Alamos NM 87545
USA

Amit Surana

Department of Mechanical
Engineering
MIT, Rm. 3-352
77 Massachusetts Ave.
Cambridge MA 02139
USA

Vera Thümmler

Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld

Murat Turhan

Departement de Mathematiques
Ecole Polytechnique Federale
de Lausanne
CH-1015 Lausanne

Stefan Vater

Potsdam-Institut für Klimafolgen-
forschung (PIK)e.V.
Telegrafenberg A31
14412 Potsdam