Mathematisches Forschungsinstitut Oberwolfach

Report No. 39/2015

DOI: 10.4171/OWR/2015/39

Reductions of Shimura Varieties

Organised by
Laurent Fargues, Paris
Ulrich Görtz, Essen
Eva Viehmann, Garching
Torsten Wedhorn, Paderborn

23 August – 29 August 2015

Abstract. The aim of this workshop was to discuss recent developments in the theory of reductions of Shimura varieties and related topics. The talks presented new methods and results that intertwine a multitude of topics such as geometry and cohomology of moduli spaces of abelian varieties, p-divisible groups and Drinfeld shtukas, p-adic Hodge theory, and the Langlands program.

Mathematics Subject Classification (2010): 11xx, 14xx, 22xx.

Introduction by the Organisers

The workshop *Reduction of Shimura varieties* was attended by 50 participants with broad geographic representation, including a number of young participants. We had 19 talks of 60 minutes each.

Arithmetic properties of Shimura varieties which are encoded in their reduction to positive characteristic are an exciting topic which has roots in classical topics of number theory such as modular forms and modular curves and of algebraic geometry. On the other hand it is a currently very active research field that has contributed to some of the most spectacular developments in number theory and arithmetic geometry in the last twenty years. Shimura varieties are closely related to the Langlands program (classical as well as p-adic). A particular case is given by moduli spaces of abelian varieties, a classical object of study in algebraic geometry.

The topics of the talks covered the whole subject of reductions of Shimura varieties, and range from the development of new methods to study them, and results on their geometric and cohomological properties to applications both to the

Langlands program and to other parts of number theory and arithmetic geometry. Some talks reported on relations to group theoretic objects and constructions such as affine root systems and affine Grassmannians, or to analogous constructions in equal characteristic such as moduli spaces of shtuka.

Applications to number theoretic questions
Particular highlights were the talks on applications of the theory.

Wei Zhang explained joint work with Zhiwei Yun which gives expressions for special values of arbitrary derivatives of certain automorphic representations over global fields of positive characteristic in terms of intersection multiplicities on moduli spaces of shtukas. This is a completely new approach: Previously, usually only the leading term coefficient was considered.

Yifeng Liu's talk titled *Bad reduction of Shimura varieties, level raising and Selmer groups* provided an application of the theory of reductions of Shimura varieties to questions with a number-theoretic flavor, more precisely about the vanishing of certain Selmer groups, a geometric level raising theorem, and a reciprocity law for Gross-Schoen cycles.

Fabrizio Andreatta explained, in his talk "Heights of CM points on orthogonal Shimura varieties", his joint work with Eyal Goren, Benjamin Howard and Keerthi Madapusi Pera on the averaged Colmez conjecture about the heights of certain abelian varieties with complex multiplication. This is quite a powerful result: Tsimerman has shown that it gives rise to an unconditional proof of the André-Oort conjecture.

Shimura varieties and the Langlands program
Several talks explained aspects of the general paradigm that the cohomology of reductions of Shimura varieties should realize local Langlands correspondences — classical and p-adic. Related aspects such as p-adic Hodge theory and deformations of Galois representations were also considered.

Imai's and Ivanov's talk concerned the classical local Langlands correspondence: In his talk "Affinoids in the Lubin-Tate perfectoid space and simple epipelagic representations", Naoki Imai discussed joint work with Takahiro Tsushima on explicit constructions of part of the local Langlands and Jacquet-Langlands correspondences in the cohomology of the Lubin-Tate perfectoid space. In a similar vein, Alexander Ivanov proposed a construction of the local Langlands correspondence using covers of affine Deligne-Lusztig varieties, much in analogy with classical Deligne-Lusztig theory. The construction works in general; so far it is known that for GL_2 it does realize the local Langlands correspondence.

On the other hand, in his talk *Analytic functions on étale coverings of Drinfeld's upper half-plane*, Gabriel Dospinescu talked about his joint work with Arthur Cesar Le Bras about the geometric realization of Colmez' p-adic local Langlands correspondence. This proves a conjecture of Breuil showing that Colmez p-adic local Langlands correspondence for $GL_2(\mathbb{Q}_p)$ can be realized using the de Rham complex of higher coverings of Drinfeld upper half plane. They prove that the "classical" smooth part of the local Langlands is realized in the de Rham cohomology and the p-adic one is obtained by pulling back a filtration from the cohomology to the de Rham complex.

Xinwen Zhu explained joint work with Ruochuan Liu which shows that all the fibers of a $\mathbb{Q}_p$ local system over a smooth connected variety over a p-adic field are de Rham (as a representation of the absolute Galois group of the residue class field) as soon as this is true over a single point. This is an analogue in p-adic Hodge theory of Deligne's principle B (which refers to classical Hodge theory). It can be applied to many Shimura varieties since the de Rham property is easy to obtain at special points.

Brandon Levin talked on *Iwahori local models and deformation rings* and discussed interesting applications of the theory of local models of Shimura varieties to the study of deformation rings of Galois representations, in particular to proving instances of the Breuil-Mézard conjecture (specifically, in the case of GL_3 and Hodge-Tate weights $(2, 1, 0)$).

Geometry of the reduction of Shimura varieties

Several talks concerned the geometric structure of the reductions of Shimura varieties, in particular the natural stratifications on them.

The topic of Yichao Tian's talk on joint work with David Helm and Liang Xiao was the proof of the Tate conjecture for the special fibers of some unitary Shimura varieties. The cycles one has to construct to this end are found in the supersingular locus; the supersingular locus is a union of Deligne-Lusztig varieties, similarly as in the case studied by Vollaard and Wedhorn and other cases. This is also closely related to the stratifications discussed in Chen's talk; see below. To show that this produces enough cycles, one then has to study their geometry inside the full special fiber (in particular, their intersection matrix).

In her talk about joint work with Eva Viehmann, Miaofen Chen described a new stratification of affine Deligne-Lusztig varieties which generalizes many of the stratifications which were previously studied.

The question of the non-emptiness of Newton strata, was addressed by Chia-Fu Yu in his talk *Non-emptiness of the basic locus of Shimura varieties*; it also made an appearance in Mark Kisin's talk, who talked on joint work with Keerthi Madapusi Pera and Sug-Woo Shin about *Honda-Tate theory for Shimura varieties*, proving that in a Shimura variety $\mathrm{Sh}_K(G, X)$ with $G_{\mathbb{Q}_p}$ quasi-split, every isogeny class inside the special fiber of a suitable model contains a point which is the reduction of a CM point.

The theme of Benoît Stroh's talk *Bad reduction and boundary terms* on joint work with Kai-Wen Lan was that the phenomena of bad reduction and of singularities in the boundary (of a toroidal or the minimal compactification) should not interact with each other. This was illustrated by several results, and also played a role in other talks.

Gerd Faltings presented a case of a Shimura variety with bad, but semi-stable reduction — a rare, but very useful situation; his proof relies on an extension of the theory of "filtered modules" that allows to cover the semi-stable case.

Both George Boxer and Jean-Stefan Koskivirta talked on *Generalized Hasse invariants*. Boxer focused on the construction of generalized Hasse invariants in the Siegel case — while the classical Hasse invariant lives on the full moduli space and vanishes precisely on the non-ordinary locus, the generalized Hasse invariants live on the closure of some Ekedahl-Oort stratum and vanish exactly on its boundary. Boxer's results has applications to the construction of Galois representations attached to automorphic representations. Koskivirta reported on joint work with Goldring. The main focus of his talk was how to deal with Shimura varieties of Hodge type; a key tool is the framework of G-zips.

Michael Rapoport introduced an axiomatic framework for the reduction of general Shimura varieties concerning the existence and the interplay of several stratifications (Newton, Ekedahl-Oort, Kottwitz-Rapoport) which are well-known from the PEL case. In all cases where the axioms are satisfied, several nice consequences follow, such as the non-emptiness of "all" Newton strata that was also adressed above.

Xu Shen presented a generalization of results of Scholze, now allowing to view arbitrary Shimura varieties of abelian type "at infinite level" as perfectoid spaces and to equip them with a Hodge-Tate period map.

Finally, Eike Lau talked on *The image of the crystalline Dieudonné functor*, a very classical and central method, but whose image was up to now still not completely understood.

The unique environment provided by the Mathematisches Forschungsinstitut Oberwolfach stimulated intense discussions and initiated several new cooperations among the participants. All participants immensely enjoyed the workshop and are very grateful for the institute's hospitality.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, "US Junior Oberwolfach Fellows".

Workshop: Reductions of Shimura Varieties

Table of Contents

Abstracts

Boundary terms and the reduction of Shimura varieties

Benoit Stroh
(joint work with Kai-Wen Lan)

The subject of this talk is the interplay between the reduction and the boundary of Shimura varieties. The philosophy is that there is basically no interplay: the two phenomena are as much unrelated as they could be.

Denote by $X_{\mathcal{H},E} \to \mathrm{Spec}(E)$ a Shimura variety defined over its reflex field E associated to a reductive group G over $\mathbb{Q}$, with level structure $\mathcal{H} \subset G(\mathbb{A}^\infty)$ where $\mathbb{A}^\infty$ denotes the finite adeles of $\mathbb{Q}$. Let v be a place of E with completion E_v and ring of integers $\mathcal{O}_{E_v}$. Denote by p the prime number divided by v. We consider an integral model $X_\mathcal{H} \to \mathrm{Spec}(\mathcal{O}_{E_v})$ of $X_{\mathcal{H},E}$ which is flat, normal and is of one of the following types

- **(Sm) :** the smooth integral model considered by Kottwitz [2] in the PEL case, when $\mathcal{H}$ is hyperspecial at p so that $X_{\mathcal{H},E}$ has good reduction at v.
- **(Nm) :** normalization of a PEL Shimura variety in an auxiliary PEL Shimura variety associated to a different group with an hyperspecial level at p.
- **(Spl) :** normalization of a PEL Shimura variety in a splitting model of Pappas-Rapoport [9] when the level is deeper than a parahoric.
- **(Hdg) :** normalization of an Hodge type Shimura variety in an auxiliary Siegel variety as in [1] and [6].

Of course the case (Hdg) includes the cases (Sm) and (Nm). The case (Hdg) does not include the case (Spl). Let's mention that the parahoric level structures fall in the realm of case (Nm) when they are known to be flat and normal, which is the case if G splits over a tamely ramified extension of $\mathbb{Q}_p$ and p does not divide the order of $\pi_1(G^{\mathrm{der}}_{\mathbb{Q}_p})$ by [10]. Similarly, the splitting models of Pappas-Rapoport fall in case (Spl) when they are flat and normal.

The common interest of all cases in the previous list is that one can construct a family of toroidal compactifications $X^{\mathrm{tor}}_\mathcal{H}$ of $X_\mathcal{H}$ over $\mathrm{Spec}(\mathcal{O}_{E_v})$ and a minimal compactification $X^{\min}_\mathcal{H}$ still over $\mathrm{Spec}(\mathcal{O}_{E_v})$ by [3], [4], [5] and [6]. The typical strata of $X^{\min}_\mathcal{H}$ is still an integral model of a Shimura variety associated to a group with a lower rank, and this integral model still belongs to the previous list.

Our first theorem deals with the interaction of the bad reduction and the boundary. This lack of interaction will be expressed in a cohomological way. Choose ℓ any prime number different from p. Consider the complex of nearby cycles $\mathrm{R}\Psi_{X_\mathcal{H}}(\mathbb{Z}/\ell^s)$ with $s \geq 0$. Denote by $j^{\mathrm{tor}} : X_\mathcal{H} \hookrightarrow X^{\mathrm{tor}}_\mathcal{H}$ the open immersion inside a toroidal compactification.

Theorem 1. *We have a canonical isomorphism*

$$\mathrm{R}j^{\mathrm{tor}}_* \circ \mathrm{R}\Psi_{X_\mathcal{H}}(\mathbb{Z}/\ell^s) = \mathrm{R}\Psi_{X^{\mathrm{tor}}_\mathcal{H}} \circ \mathrm{R}j^{\mathrm{tor}}_*(\mathbb{Z}/\ell^s)\,.$$

This theorem is proved by a local study, using the explicit description of the toroidal compactification. We recall the reader that a commutation $\mathbb{R}j_* \circ R\Psi_X = \mathbb{R}\Psi_{\bar{X}} \circ \mathbb{R}j_*$ is easily seen to be false for a general open immersion $j : X \hookrightarrow \bar{X}$ of finite type schemes over $\mathrm{Spec}(\mathbb{Z}_p)$. On the contrary we see such a commutation for the constant sheaf as a cohomological precise meaning of the absence of interaction between the bad reduction of X, coded in $\mathrm{R}\Psi_X$, and the boundary, coded in $\mathrm{R}j_*$.

From this theorem, it is very easy to deduce that the cohomology of the geometric special fiber of $X_{\mathcal{H}}$ with value in $\mathrm{R}\Psi_{X_{\mathcal{H}}}(\mathbb{Z}/\ell^s)$ computes the cohomology of $X_{\mathcal{H},\bar{E}}$ with coefficient $\mathbb{Z}/\ell^s$, where $\bar{E}$ is an algebraic closure of E. This computation is moreover equivariant under all natural structures like the Hecke operators away from v.

We deduce many applications from that fact like for instance a Mantovan formula [7] expressing the étale cohomology of the generic fiber in terms of the cohomology of Igusa varieties and group theory in the unramified PEL case, or a Kottwitz-Scholze [12] counting point formula in the general ramified PEL case.

We then introduce the notion of well-positioned complex in $X_{\mathcal{H}}^{\mathrm{tor}}$. We will not give the precise definition here, but let's just mention that the constant sheaf supported on a stratum of any usual stratification, like the p-rank, the Newton, the Ekedhal-Oort or the Kottwitz-Rapoport one, is a well-positioned complex. The same is true for the intersection complex of the Zariski closure of a stratum. Moreover well-positioned complexes are preserved under basic sheaf theoretic operations.

Let $j^{\mathrm{min}} : X_{\mathcal{H}} \hookrightarrow X_{\mathcal{H}}^{\mathrm{min}}$ the open immersion inside the minimal compactification and let $i : Z \hookrightarrow X_{\mathcal{H}}^{\mathrm{min}}$ be a locally closed boundary strata, so that Z is itself an integral model of a Shimura variety. For any well-positioned complex $\mathcal{F}$, we give a Pink formula for $i^* \circ \mathrm{R}j_*^{\mathrm{min}}(\mathcal{F})$ which generalizes [11] when $\mathcal{F}$ is the constant sheaf. If moreover $\mathcal{F}$ is pure and perverse, we provide a Morel formula for $i^* \circ j_{!*}^{\mathrm{min}}(\mathcal{F})$ which generalizes [8] when $\mathcal{F}$ is the constant sheaf. We see both formulas as a cohomological precise explanation for the absence of interaction of the reduction, coded by the choice of the well-positioned complex $\mathcal{F}$, and the boundary, coded by $i^* \circ \mathrm{R}j_*^{\mathrm{min}}$ or $i^* \circ j_{!*}^{\mathrm{min}}$. Indeed in our Pink and Morel formulas, the answer is the product of a complex $\mathcal{G}$ on Z coming purely from the reduction of Z and the complex $i^* \circ \mathrm{R}j_*^{\mathrm{min}}(\mathbb{Z}/\ell^s)$ or $i^* \circ j_{!*}^{\mathrm{min}}(\mathbb{Q}_\ell)$ coming purely from the boundary.

References

[1] M. Kisin, *Integral models for Shimura varieties of abelian type*, J. Amer. Math. Soc. 23 (2010), no. 4, 967-1012.

[2] R. Kottwitz, *Points on some Shimura varieties over finite fields*, J. Amer. Math. Soc. 5 (1992), no. 2, 373-444.

[3] K.W Lan, *Arithmetic compactification of PEL-type Shimura varieties*, London Mathematical Society Monographs, vol. 36, Princeton University Press, Princeton, 2013.

[4] K.W Lan, *Compactifications of PEL-type Shimura varieties in ramified characteristics*, preprint, 2014.

[5] K.W Lan, *Compactifications of splitting models of PEL-type Shimura varieties*, preprint, 2015.

[6] K. Madapusi Pera, *Toroidal compactifications of integral models of Shimura varieties of Hodge type*, preprint, 2015.
[7] E. Mantova, *On the cohomology of certain PEL-type Shimura varieties*, Duke Math. J. 129 (2005), no. 3, 573-610.
[8] S. Morel, *Complexes pondérés sur les compactifications de Baily-Borel: le cas des variétés de Siegel*, J. Amer. Math. Soc. 21 (2008), no. 1, 23-61.
[9] G. Pappas and M. Rapoport, *Local models in the ramified case, II. Splitting models*, Duke Math. J. 127 (2005), no. 2, 193-250.
[10] G. Pappas and X. Zhu, *Local models of Shimura varieties and a conjecture of Kottwitz*, Invent. Math. 194 (2013), 147-254.
[11] R. Pink, *On l-adic sheaves on Shimura varieties and their higher direct images in the Baily-Borel compactification*, Math. Ann. 292 (1992), 197-240.
[12] P. Scholze *The Langlands-Kottwitz method and deformation spaces of p-divisible groups*, J. Amer. Math. Soc. 26 (2013), 227-259.

Stratifications in the reduction of Shimura varieties: an axiomatic approach

Michael Rapoport
(joint work with Xuhua He)

The talk was about joint work with X. He [6], and concerned the definition and study of characteristic subsets in the reduction modulo p of a general Shimura variety with parahoric level structure, more precisely the *Newton stratification*, the *Ekedahl-Oort stratification* and the *Kottwitz-Rapoport stratification*. I also discussed the *Ekedahl-Kottwitz-Oort-Rapoport stratification* which interpolates between the Kottwitz-Rapoport stratification in the case of an Iwahori level structure and the Ekedahl-Oort stratification of Viehmann in the hyperspecial case. The novelty of our approach comes from the proof of He [5] of the Kottwitz-Rapoport conjecture from [12]. In particular, our methods are purely group-theoretical and combinatorial, and do not use algebraic geometry.

1. Axioms on integral models

1.1. **The set-up.** Let $(\mathbf{G}, \{h\})$ be a Shimura datum and let $\mathbf{K} = K^pK$ be an open compact subgroup of $\mathbf{G}(\mathbb{A}_f)$, where $K^p \subset \mathbf{G}(\mathbb{A}_f^p)$ and where $K = K_p$ is a parahoric subgroup of $\mathbf{G}(\mathbb{Q}_p)$. Let $G = \mathbf{G} \otimes_{\mathbb{Q}} \mathbb{Q}_p$ and let $\{\mu\}$ be the conjugacy class of cocharacters of G corresponding to $\{h\}$.

Let $\mathrm{Sh}_{\mathbf{K}} = \mathrm{Sh}(\mathbf{G}, \{h\})_{\mathbf{K}}$ be the corresponding Shimura variety. It is a quasi-projective variety defined over the Shimura field $\mathbf{E}$. We will postulate the existence of an integral model $\mathbf{S}_{\mathbf{K}}$ over the ring of integers O_E of the completion E of $\mathbf{E}$ at a place $\mathbf{p}$ above the fixed prime number p, with certain properties, which we list below. Our aim is to study the special fiber $\mathrm{Sh}_K = \mathbf{S}_{\mathbf{K}} \times_{\mathrm{Spec} O_{\mathbf{E}}} \mathrm{Spec} \kappa_E$, resp. its set of geometric points, and some stratifications on it.

1.2. **Basic axioms on integral models.** We now list our first set of axioms.

(i) Our first axiom concerns the change in the parahoric subgroup. All parahoric subgroups are supposed to contain a fixed Iwahori subgroup I.

Axiom 1.1 (Compatibility with changes in the parahoric)**.** *For any inclusion of parahoric subgroups $K \subset K'$, and setting $\mathbf{K} = K^pK$ and $\mathbf{K}' = K^pK'$, there is a natural morphism*

$$\pi_{K,K'} \colon \mathbf{S}_{\mathbf{K}} \to \mathbf{S}_{\mathbf{K}'}, \tag{1.1}$$

which is proper and surjective, and is finite in the generic fibers.

(ii) We postulate the existence of a *local model* $\mathbf{M}_K^{\mathrm{loc}}$ attached to the triple $(G, \{\mu\}, K)$. Let $\mathcal{G} = \mathcal{G}_K$ be the group scheme over $\mathbb{Z}_p$ corresponding to K. Then $\mathbf{M}_K^{\mathrm{loc}}$ is a scheme which is projective and flat over $\mathrm{Spec} O_E$, equipped with an action of $\mathcal{G} \otimes_{\mathbb{Z}_p} O_E$, and with generic fiber equal to the partial flag variety associated to $(G, \{\mu\})$. Its formation should be functorial in the parahoric subgroup K, i.e., for $K \subset K'$, there should be a proper and surjective morphism,

$$p_{K,K'} \colon \mathbf{M}_K^{\mathrm{loc}} \to \mathbf{M}_{K'}^{\mathrm{loc}}. \tag{1.2}$$

Let M_K^{loc} be its special fiber. Then M_K^{loc} is a projective variety over κ_E, with an action of $\mathcal{G}_K \otimes_{\mathbb{Z}_p} \kappa_E$. We denote by $\tilde{W}$ the Iwahori-Weyl group of G and by W_K the finite subgroup of $\tilde{W}$ corresponding to K, cf. [4].

Axiom 1.2 (Existence of local models)**.** *There is a smooth morphism of algebraic stacks* [12, (7.1)]

$$\lambda_K \colon \mathbf{S}_{\mathbf{K}} \to [\mathbf{M}_K^{\mathrm{loc}}/\mathcal{G}_{O_E}],$$

compatible with changes in the parahoric subgroup K. The action of $\mathcal{G}_K \otimes_{\mathbb{Z}_p} \kappa_E$ on M_K^{loc} has finitely many orbits $\mathcal{O}_w$ which are indexed by $w \in \mathrm{Adm}(\{\mu\})_K$. Furthermore,

$$\mathcal{O}_w \subset \overline{\mathcal{O}}_{w'} \quad \textit{if and only if} \quad w \leq w'$$

in the partially ordered set $W_K\backslash\tilde{W}/W_K$.

Here $\mathcal{G} = \mathcal{G}_K$, and $\mathcal{G}_{O_E}$ denotes its base change to $\mathrm{Spec} O_E$. Furthermore, $\mathrm{Adm}(\{\mu\})_K$ denotes the *admissible subset* of $W_K\backslash\tilde{W}/W_K$, *cf.* [12].

Remark 1.3. Pappas and Zhu [11] have constructed such local models under a tameness assumption on G. However, in their set-up, the orbits in $\mathbf{M}_K^{\mathrm{loc}}$ are implicitly enumerated by a subset of the Iwahori Weyl group of a *loop group* version of G. Axiom 1.2 implicitly refers to Scholze's idea [1] that would construct local models of Shimura varieties whose special fibers are embedded as closed subschemes of a *Witt vector affine flag variety.*

We denote by $\lambda_K \colon Sh_K \to [M_K^{\mathrm{loc}}/\mathcal{G}_{\kappa_E}]$ the induced morphism of stacks on the special fiber. For any $w \in W_K\backslash\tilde{W}/W_K$, set

$$KR_{K,w} = \lambda_K^{-1}(\mathcal{O}_w) \subset Sh_K, \tag{1.3}$$

and call it the *Kottwitz-Rapoport stratum* (KR-stratum) of Sh_K attached to w, cf. [3, §8]. It is a locally closed subvariety of Sh_K. Note that, by definition, $KR_{K,w}$ is non-empty only if $w \in \mathrm{Adm}(\{\mu\})_K$.

(iii) Recall $B(G) = G(\breve{\mathbb{Q}}_p)/G(\breve{\mathbb{Q}}_p)_\sigma$, the set of σ-conjugacy classes of $G(\breve{\mathbb{Q}}_p)$, cf. [8]. Here $\breve{\mathbb{Q}}_p$ denotes the completion of the maximal unramified extension of $\mathbb{Q}_p$.

Axiom 1.4 (Existence of a Newton stratification). *There is a map*

$$\delta_K \colon Sh_K \to B(G),$$

compatible with changing the parahoric subgroup K (i.e., with $\pi_{K,K'}$), and such that for each $[b] \in B(G)$, the fiber of δ_K over $[b]$ is the set of $\bar{\kappa}_E$-rational points of a locally closed subvariety $S_{K,[b]}$ of Sh_K. Furthermore, if

$$S_{K,[b]} \cap \overline{S}_{K,[b']} \neq \emptyset,$$

then $[b] \leq [b']$ in the sense of the partial order on $B(G)$.

The subvariety $S_{K,[b]}$ of Sh_K is called the *Newton stratum* of Sh_K attached to $[b]$.

1.3. **Joint stratification and basic non-emptyness.** Let K be a parahoric subgroup, with corresponding subgroup $\breve{K}$ of $G(\breve{\mathbb{Q}}_p)$. Let $\breve{K}_\sigma \subset \breve{K} \times \breve{K}$ be the graph of the Frobenius map σ and $G(\breve{\mathbb{Q}}_p)/\breve{K}_\sigma$ be the set of $\breve{K}$-σ-conjugacy classes on $G(\breve{\mathbb{Q}}_p)$. The embedding $\breve{K}_\sigma \subset G(\breve{\mathbb{Q}}_p)_\sigma$ induces a projection map

$$d_K \colon G(\breve{\mathbb{Q}}_p)/\breve{K}_\sigma \to B(G). \tag{1.4}$$

On the other hand, the embedding $\breve{K}_\sigma \subset \breve{K} \times \breve{K}$ induces a map

$$\ell_K \colon G(\breve{\mathbb{Q}}_p)/\breve{K}_\sigma \to \breve{K}\backslash G(\breve{\mathbb{Q}}_p)/\breve{K}. \tag{1.5}$$

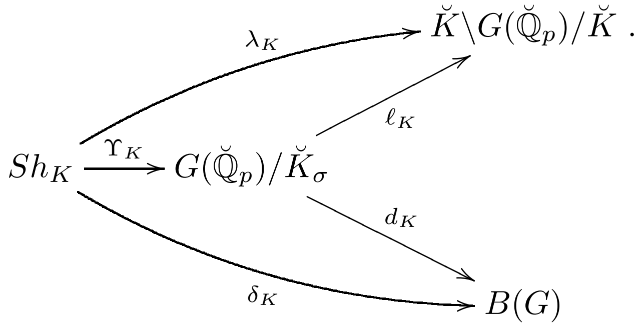
We now add the following axioms to our list.

(i) The first axiom relates the two maps λ and δ introduced in Axioms 1.2 and 1.4. Note that in its formulation, we identify $\breve{K}\backslash G(\breve{\mathbb{Q}}_p)/\breve{K}$ with $W_K\backslash\tilde{W}/W_K$, cf. [4, Prop. 8].

Axiom 1.5 (Joint stratification). *a) There exists a natural map*

$$\Upsilon_K \colon Sh_K \to G(\breve{\mathbb{Q}}_p)/\breve{K}_\sigma,$$

compatible with changes in the parahoric subgroup K, such that the following diagram commutes



Here the map λ_K is induced by the map (1.3).

b) Furthermore,

$$\Im\Upsilon_K = \ell_K^{-1}(\Im\lambda_K).$$

c) For $K' \subset K$, and any element $y' \in G(\breve{\mathbb{Q}}_p)/\breve{K}'_\sigma$ with image $y \in G(\breve{\mathbb{Q}}_p)/\breve{K}_\sigma$, the natural map

$$\pi_{K',K\,|\Upsilon_{K'}^{-1}(y')} : \Upsilon_{K'}^{-1}(y') \to \Upsilon_K^{-1}(y)$$

is surjective with finite fibers.

It should be pointed out that parts b) and c) of this axiom are principally used in connection with the study of EKOR-strata.

(ii) The second axiom is a weak non-emptiness statement. Let $\tau = \tau_{\{\mu\}}$ be the element of length zero in $\tilde{W}$ corresponding to $\{\mu\}$, cf. [4, Lemma 14].

Axiom 1.6 (Basic non-emptiness)**.** *The map*

$$KR_{I,\tau} \to \pi_0(Sh_I)$$

is surjective.

Here $\pi_0(Sh_K)$ denotes the set of geometric connected components of Sh_K. In other words, this axiom postulates that every geometric connected component of Sh_I intersects the KR-stratum $KR_{I,\tau}$.

2. Non-emptiness of KR- and Newton strata

From the axioms we deduce the following three statements.

Theorem 2.1. *Let K be a parahoric subgroup and let X_K be a geometric connected component of Sh_K. Then*

$$\lambda_K(X_K) = \mathrm{Adm}(\{\mu\})_K.$$

In other words, any geometric connected component of Sh_K intersects any KR-stratum (as their indices run over their natural range, i.e., $\mathrm{Adm}(\{\mu\})_K$).

Let $B(G,\{\mu\})$ denote the finite subset of $B(G)$ of [9, §6] which describes the Mazur inequality in group theoretic terms.

Theorem 2.2. *Let K be a parahoric subgroup and let X_K be a geometric connected component of Sh_K. Then*

$$\delta_K(X_K) = B(G,\{\mu\}).$$

In other words, any geometric connected component of Sh_K intersects any Newton stratum (as their indices run over their natural range, i.e., $B(G,\{\mu\})$).

The following closure relation between Newton strata is sometimes referred to as *Grothendieck's conjecture.*

Theorem 2.3. *Let K be a parahoric subgroup. Let $[b],[b'] \in B(G,\{\mu\})$. Then $\overline{S}_{K,[b']} \cap S_{K,[b]} \neq \emptyset$ if and only if $[b] \leq [b']$.*

3. EKOR-strata

3.1. **Definition of υ_K.** Let K be a parahoric subgroup, with corresponding subgroup $\breve{K}$ of $G(\breve{\mathbb{Q}}_p)$, and let $\breve{K}_1$ be the pro-unipotent radical of $\breve{K}$. Then

$$\breve{K}_\sigma \subset \breve{K}_\sigma(\breve{K}_1 \times \breve{K}_1) \subset \breve{K} \times \breve{K}.$$

Thus we have a factorization of λ_K,

$$Sh_K \to G(\breve{\mathbb{Q}}_p)/\breve{K}_\sigma \to G(\breve{\mathbb{Q}}_p)/\breve{K}_\sigma(\breve{K}_1 \times \breve{K}_1),$$

where the first map is Υ_K and the second map is the natural projection map. We denote the composition map by

$$\upsilon_K \colon Sh_K \to G(\breve{\mathbb{Q}}_p)/\breve{K}_\sigma(\breve{K}_1 \times \breve{K}_1). \tag{3.1}$$

We therefore obtain a commutative diagram

$$\begin{array}{ccc} Sh_K & & \\ & \overset{\lambda_K}{\searrow} & \\ \Big\downarrow \upsilon_K & & \breve{K}\backslash G(\breve{\mathbb{Q}}_p)/\breve{K}. \\ & \underset{\mathrm{nat}_K}{\nearrow} & \\ G(\breve{\mathbb{Q}}_p)/\breve{K}_\sigma(\breve{K}_1 \times \breve{K}_1) & & \end{array}$$

We identify in the sequel the coset space $G(\breve{\mathbb{Q}}_p)/\breve{K}_\sigma(\breve{K}_1 \times \breve{K}_1)$ with ${}^K\tilde{W}$, compatibly with the identification of $\breve{K}\backslash G(\breve{\mathbb{Q}}_p)/\breve{K}$ with $W_K\backslash\tilde{W}/W_K$, cf. [6]. Here ${}^K\tilde{W}$ denotes the set of elements of $\tilde{W}$ of minimal length in their left cosets modulo W_K.

Definition 3.1. The *Ekedahl-Kottwitz-Oort-Rapoport stratum* (EKOR-stratum) of Sh_K attached to $x \in {}^K\tilde{W}$ is the subset

$$EKOR_{K,x} = \upsilon_K^{-1}(x) \subset Sh_K.$$

Let $\mathrm{Adm}(\{\mu\})^K$ be the inverse image of $\mathrm{Adm}(\{\mu\})_K$ in $\tilde{W}$. Then $EKOR_{K,x}$ is non-empty only if $x \in \mathrm{Adm}(\{\mu\})^K$.

Remarks 3.2. (1) For a general parahoric subgroup, the EKOR-stratification is finer than the KR-stratification (the map λ_K factors through υ_K).

(2) If G is unramified and K is hyperspecial, the definition of the EKOR-stratification coincides with the Ekedahl-Oort stratification in the sense of Viehmann [13]. If $K = I$ is the Iwahori subgroup then the EKOR-strata coincide with the KR-strata. Therefore the EKOR-stratification for a general parahoric subgroup interpolates between the EO-stratification for the hyperspecial case and the KR-stratification for the Iwahori case.

Theorem 3.3. *Let K be a parahoric subgroup and $x \in \mathrm{Adm}(\{\mu\})^K \cap {}^K\tilde{W}$. Then $EKOR_{K,x}$ is locally closed and the closure of $EKOR_{K,x}$ is*

$$\overline{EKOR_{K,x}} = \sqcup_{x' \in {}^K\tilde{W}, x' \preceq_{K,\sigma} x} EKOR_{K,x'}.$$

Here $x' \preceq_{K,\sigma} x$ denotes a certain partial order on ${}^K\tilde{W}$, cf. [6].

The following theorem gives a relation between EKOR-strata and Newton strata.

Theorem 3.4. *For any parahoric K and any $[b] \in B(G, \{\mu\})$, there exists $x \in \mathrm{Adm}(\{\mu\})^K \cap {}^K\tilde{W}$ such that*

$$EKOR_{K,x} \subset S_{K,[b]}.$$

Remarks 3.5. 1) For a general parahoric K, there is no KR-stratum of level K that is entirely contained in a given Newton stratum.

2) For Shimura varieties of PEL type with hyperspecial level structure, the existence of an Ekedahl-Oort stratum in a given Newton stratum is proved by Viehmann/Wedhorn [14, Theorem 1.5(1)] and Nie [10, Corollary 1.6].

If $K' \subset K$, then the index set $\mathrm{Adm}(\{\mu\})^K \cap {}^K\tilde{W}$ for the EKOR-strata with level K is contained in the index set $\mathrm{Adm}(\{\mu\})^{K'} \cap {}^{K'}\tilde{W}$ for the EKOR-strata with level K' (the smaller the parahoric, the bigger the index set). In the sequel, we identify the index set for K with a subset of the index set for K'.

Proposition 3.6. *Let $K' \subset K$ be standard parahoric subgroups. Then for any $w \in \mathrm{Adm}(\{\mu\})^{K'} \cap {}^{K'}\tilde{W}$, there exists a finite subset $\Sigma_K(w)$ of $W_K w W_K \cap {}^K\tilde{W}$ such that*

$$\pi_{K',K}(EKOR_{K',w}) = \sqcup_{x\in\Sigma_K(w)} EKOR_{K,x}.$$

Furthermore, if $w \in \mathrm{Adm}(\{\mu\})^K \cap {}^K\tilde{W}$, then $\Sigma_K(w) = \{w\}$ and

$$\pi_{K',K}(EKOR_{K',w}) = EKOR_{K,w}.$$

This proposition is then used to prove the following non-emptiness statement.

Theorem 3.7. *Let X_K be a geometric connected component of Sh_K. For any parahoric K,*

$$\upsilon_K(X_K) = \mathrm{Adm}(\{\mu\})^K \cap {}^K\tilde{W}.$$

In other words, any geometric connected component of Sh_K intersects any EKOR-stratum (as their indices run through their natural range, i.e., $\mathrm{Adm}(\{\mu\})^K \cap {}^K\tilde{W}$.)

Finally, there is the following relation between EKOR-strata of level K and KR-strata of Iwahori level I.

Theorem 3.8. *Let K be a parahoric subgroup and $x \in \mathrm{Adm}(\{\mu\})^K \cap {}^K\tilde{W}$. Then*

$$\pi_{I,K\,|KR_{I,x}} : KR_{I,x} \to EKOR_{K,x}$$

is a finite morphism. In particular, $\dim EKOR_{K,x} = \dim KR_{I,x}$.

Remark 3.9. It may be conjectured that the morphism in Theorem 3.8 is finite étale. This would imply that all EKOR-strata are smooth, which we also conjecture. This is proved by Görtz/Hoeve [2] in the Siegel case.

References

[1] B. Bhatt, P. Scholze *Projectivity of the Witt vector affine Grassmannian*, arXiv:1507.06490
[2] U. Görtz, M. Hoeve, *Ekedahl-Oort strata and Kottwitz-Rapoport strata*, J. Algebra 351 (2012), 160–174.
[3] T. Haines, *Introduction to Shimura varieties with bad reduction of parahoric type*, Clay Mathematics Proceedings, Volume 4 (2005), 583–642.
[4] T. Haines, M. Rapoport, *On parahoric subgroups*, Adv. Math. 219 (2008), 188–198.
[5] X. He, *Kottwitz-Rapoport conjecture on unions of affine Deligne-Lusztig varieties*, arXiv:1408.5838.
[6] X. He, M. Rapoport, *Stratifications in the reduction of Shimura varieties*, in preparation.
[7] X. He, T. Wedhorn, *On parahoric reductions of Shimura varieties of PEL type*, preliminary notes, 2011.
[8] R. Kottwitz, *Isocrystals with additional structure*, Compositio Math. 56 (1985), 201–220.
[9] R. Kottwitz, *Isocrystals with additional structure. II*, Compositio Math. 109 (1997), 255–339.
[10] S. Nie, *Fundamental elements of an affine Weyl group*, Math. Ann. 362 (2015), 485–499.
[11] G. Pappas, X. Zhu, *Local models of Shimura varieties and a conjecture of Kottwitz*, Invent. Math. 194 (2013), no. 1, 147–254.
[12] M. Rapoport, *A guide to the reduction modulo p of Shimura varieties*, Astérisque (2005), no. 298, 271–318.
[13] E. Viehmann, *Truncations of level 1 of elements in the loop group of a reductive group*, Ann. of Math. (2) 179 (2014), 1009–1040.
[14] E. Viehmann, T. Wedhorn, *Ekedahl-Oort and Newton strata for Shimura varieties of PEL type*, Math. Ann. 356 (2013), 1493–1550.

Potentially crystalline deformation rings and Iwahori local models

Brandon Levin
(joint work with Daniel Le, Bao V. Le Hung and Stefano Morra)

In joint work with Daniel Le, Bao V. Le Hung and Stefano Morra, we compute potentially crystalline deformation rings for three dimensional representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. As an application, we deduce the weight part of Serre's conjecture for forms of $U(3)$ which are compact at infinity and split at places dividing p as conjectured by [14] for residual representations which are semisimple and generic at all primes above p. We also exhibit the geometric Breuil-Mézard conjecture. The method involves a detailed study of the moduli space of Kisin modules with descent datum. This builds on work of Breuil [2], Breuil-Mézard [4], Caruso-David-Mézard [7], Caraiani-Emerton-Gee-Savitt [5]. I also discussed joint work with Ana Caraiani which relates moduli of Kisin modules with descent data to Iwahori local models.

1. Potentially crystalline deformation rings

Fix a residual representation $\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{GL}_3\left(\overline{\mathbb{F}}_p\right)$. Let $\tau = \omega^a \oplus \omega^b \oplus \omega^c$ where ω is the mod p cyclotomic character of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. We assume τ is *generic*, i.e., $p - 4 \geq |a-b|, |b-c|, |a-c| \geq 3$. We want to describe the framed potentially crystalline deformation ring $R_{\overline{\rho}}^{(2,1,0),\tau,\Box}$ with Hodge-Tate weights $(2,1,0)$ and Galois type τ.

We consider the moduli space $X^{(2,1,0),\tau}$ of Kisin modules with p-adic Hodge type $(2,1,0)$ and descent data of type τ constructed in [6] building on work of [5] in the Barsotti-Tate case for GL_2. There is local model diagram relating $X^{(2,1,0),\tau}$ to the local model M^{loc} for $(\mathrm{GL}_3, \mu = (2,1,0), \text{Iwahori level})$. This induces a stratification of $X^{(2,1,0),\tau} = \bigcup_{w \in \mathrm{Adm}(2,1,0)} X_w^{(2,1,0),\tau}$ indexed by the $(2,1,0)$-admissible set. A mod p Kisin module in $X_w^{(2,1,0),\tau}(\overline{\mathbb{F}}_p)$ is said to have *shape* (or *genre*) w. This generalizes the notion of *genre* for rank 2 Breuil/Kisin modules which was crucial in [2, 7].

We can now outline our strategy for computing the deformation ring.

(1) Classify all Kisin modules of shape $w \in \mathrm{Adm}(2,1,0)$ over $\overline{\mathbb{F}}_p$.
(2) For $\overline{\mathfrak{M}} \in X_w^{(2,1,0),\tau}(\overline{\mathbb{F}}_p)$, construct the universal deformation space with height conditions. This amounts to constructing local coordinates for the local model.
(3) Impose monodromy condition on the universal family.

Previously, the finer properties of local Galois deformation rings were known for the most part only for Fontaine-Laffaille and potentially Barsotti-Tate deformation rings. Steps (1) and (2) generalize techniques of [2, 7, 10] used to compute potentially Barsotti-Tate deformation rings for GL_2. To extend these techniques to three dimensions, a key point was the correct notion of shape as discussed above. In addition, a more systematic approach to the p-adic convergence algorithm employed by [2, 7] was necessary for Step (2).

Step (3) requires a genuinely new method not present in the potentially Barsotti-Tate case where the link between moduli of finite flat groups schemes and Galois representations is stronger than in our situation (and hence, there is no monodromy condition). Let us briefly comment on the details of Step (3). Kisin [16] gave a characterization of when a torsion-free $\mathfrak{M}$ comes from a crystalline representation in terms poles of a monodromy operator $N_{\mathcal{M}}$. While one cannot compute $N_{\mathcal{M}}$ completely, one can approximate it using the genericity condition on τ. Vanishing of the poles corresponds to the vanishing of the sum of an explicit polynomial equation and an error term which is divisible by p^3. This suffices for determining the deformation ring and its special fiber.

As an illustration, we have the following theorem relating components of the deformation ring and predicted local weights. Here $\mathrm{JH}(\overline{\sigma}(\tau))$ denotes the Jordan-Hölder factors mod p of the principal series representation corresponding to τ under inertial local Langlands and $W^?(\overline{\rho})$ denotes the conjectural set of local weights predicted by [14].

Theorem 1 (LLLM). *Let $\overline{\rho}$ be a semisimple representation of* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ *and let τ be a* generic *tame inertial type. If $|W^?(\overline{\rho}) \cap \mathrm{JH}(\overline{\sigma}(\tau))| < 6$, then the irreducible components of* $\mathrm{Spec} R_{\overline{\rho}}^{(2,1,0),\tau,\square}$ mod p *are in bijection with $W^?(\overline{\rho}) \cap \mathrm{JH}(\overline{\sigma}(\tau))$.*

We expect the same result in the case of six common weights which is the maximal number possible, but there is one case remaining. Theorem 1 should be thought of as an instance of the geometric Breuil-Mézard conjecture of [8] (the intrinsic multiplicity of each local weight turns out to be one in this case). The bijection matches components labelled by Fontaine-Laffaille weights with the special fibers of Fontaine-Laffaille deformation rings.

2. Application: Weight part of Serre's conjecture

Serre's original modularity conjecture asserted that every odd irreducible continuous representation $\overline{r} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ arises from a modular form. Serre [18] gave a precise recipe for the minimal possible prime to p level and weight of such a modular form. The recipe for the minimal weight is given in terms of the restriction of $\overline{r}$ to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ (or even the inertia subgroup). The weight recipe and subsequent generalizations are often referred to as the weight part of Serre's conjecture (or even just Serre weight conjectures). There has much progress in recent years in formulating generalizations the weight part of Serre's conjecture (see [14, 12]) and in proving generalizations of the conjecture for 2-dimensional Galois representations over totally real fields. However, thus far, there are only a few theoretical results in the case of semisimple rank > 1: [11, 1] study modularity of 'obvious' weights and [9] proves Herzig's conjecture in our setting under the hypothesis that $\overline{r}$ is irreducible at all places above p. We reprove the results of [9] and extend them to the case where $\overline{r}$ is semisimple at all places above p.

Recall the setup for algebraic modular forms. Let F be an imaginary CM field with totally real subfield F^+ such that all primes of F^+ above p split in F. Let G is unitary group over F^+ which is isomorphic to $U(3)$ at each infinite place and split at each place above p. Let $\mathcal{G}$ be a reductive model over $\mathcal{O}_{F^+}[1/N]$ with $(N, p) = 1$.

Definition 1. A (global) *Serre weight* is an irreducible $\overline{\mathbb{F}}_p$-representation F_λ of $\mathcal{G}(\mathcal{O}_{F^+,p})$.

For each place $w \mid p$, let k_w denote the residue field. A (global) Serre weight is equivalent to a collection $(F_{\lambda_w})_{w|p}$ of irreducible representation of $\mathrm{GL}_3(k_w)$ which is conjugate self-dual, i.e., $(F^*_{\lambda_w})^c \cong F_{\lambda_{w^c}}$. For any Serre weight F_λ and any compact open $U \subset G(\mathbb{A}^{f,p}_{F^+})$, there is an associated space of mod p algebraic modular forms $S(U, F_\lambda)$. Let $\overline{r} : G_F \to \mathrm{GL}_3(\overline{\mathbb{F}}_p)$ be a continuous irreducible representation.

Definition 2. We say $\overline{r}$ is *modular of weight* F_λ if there exists some (nice) open compact U unramified above p such that

$$S(U, F_\lambda)_{\overline{m}} \neq 0$$

where $\overline{m}$ is the maximal ideal of Hecke algebra associated to $\overline{r}$.

We say $\overline{r}$ is *modular* if $\overline{r}$ is modular of some Serre weight $(F_{\lambda_w})_{w|p}$. Let $W(\overline{r})$ denote the set of modular Serre weights.

The weight part of Serre's conjectures predicts, in particular, that the set of modular Serre weights should be determined by the restrictions $\overline{r}|_{G_{F_w}}$ for all primes $w \mid p$. For $\overline{r}|_{G_{F_w}}$ semisimple and $F_w = \mathbb{Q}_p$, [14] gives a recipe for a collection $W^?(\overline{r}|_{G_{F_w}})$ of irreducible $\overline{\mathbb{F}}_p$-representation of $\mathrm{GL}_3(k_w)$.

Theorem 3 (LLLM). *Let $\overline{r} : G_F \to \mathrm{GL}_3(\overline{\mathbb{F}}_p)$ be an irreducible modular representation. Suppose p splits completely in F and for all places $w \mid p$ of F suppose $\overline{r}|_{G_{F_w}}$ is semisimple and generic. Suppose further that $\overline{r}$ satisfies the Taylor-Wiles conditions. Then*

$$W(\overline{r}) = (W^?(\overline{r}|_{G_{F_w}}))_{w|p}.$$

The weight elimination direction $W(\overline{r}) \subset (W^?(\overline{r}|_{G_{F_w}}))_{w|p}$ was already completed (or forthcoming) in [9, 15, 17]. The other inclusion (weight existence) is an application of our results on deformation rings following roughly the strategy outlined in [12, §3-4].

Briefly, the patching techniques of Gee-Kisin [13], Emerton-Gee [8] allow one to construct a patched module $M_\infty(\sigma(\tau))$ over the product of the local deformation rings (adjoin some patching variables). Furthermore, for each $\overline{\sigma} \in \mathrm{JH}(\overline{\sigma}(\tau))$, there is a subquotient $M_\infty(\overline{\sigma})$ of $M_\infty(\sigma(\tau)) \mod p$. We show that the generic fiber of the local deformation ring is connected and so $M_\infty(\sigma(\tau))$ has full support. Knowing this and the number of components in the special fiber, we deduce that $M_\infty(\overline{\sigma}) \neq 0$ for $\overline{\sigma} \in W^?(\overline{r}|_{G_{F_w}})$ by an inductive procedure involving a careful choice of tame types.

We expect our methods to carry over also to the case where p is unramified (but not necessarily split in F^+). We also aim to address the question of cyclicity of patched modules and the analogue of Breuil's lattice conjecture in future work.

References

[1] T. Barnet-Lamb, T. Gee, D. Geraghty, *Serre weights for $U(n)$*, J. Reine Angew. Math (to appear), 2014.

[2] C. Breuil, *Sur une problème de compatibilité local-global modulo p pour* GL_2, J. Reine Angew. Math. 692, (2012), 119-147.

[3] C. Breuil, A. Mézard, *Multiplicités modulaires et représentationes de* $\mathrm{GL}_2(\mathbb{Z}_p)$ *et de* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ *en* $\ell = p$, Duke Math J. 115, no. 2, (2002), 205-310.

[4] C. Breuil, A. Mézard, *Multiplicités modulaires raffinées*, Bull. Soc. Math. de France 142, (2014), 127-175.

[5] A. Caraiani, M. Emerton, T. Gee, D. Savitt, *Moduli spaces of Kisin modules with descent data*. In preparation. 2015.

[6] A. Caraiani, B. Levin, *Moduli spaces of Kisin modules with descent data and parahoric local models*. In preparation. 2015.

[7] X. Caruso, A. David, A. Mézard, *Un calcul d'anneaux de déformations potentiellement Barsotti-Tate*, preprint, arxiv:1402.2616.

[8] M. Emerton, T. Gee, *A geometric perspective on the Breuil-Mézard conjecture*, J. Inst. Math. Jussieu 13, no. 1, (2014), 183-223.

[9] M. Emerton, T. Gee, F. Herzig, *Weight cycling and Serre-type conjectures for unitary groups*, Duke Math. J. **162** (2013), no. 9, 1649-1722.
[10] M. Emerton, T. Gee, D. Savitt, *Lattices in the cohomology of Shimura curves*, Invent. Math. 200, no. 1, (2015), 1-96.
[11] T. Gee, D. Geraghty, *Companion forms for unitary and symplectic groups*, Duke Math. J. **161**, no. 2, (2012), 247-303.
[12] T. Gee, F. Herzig, D. Savitt, *General Serre weight conjectures*, preprint, arxiv:1509.02527.
[13] T. Gee, M. Kisin, *The Breuil-Mézard conjecture for potentially Barsotti-Tate representations*, Forum of Math, Pi 2, e1, (2014). 56 pp.
[14] F. Herzig, *The weight in a Serre-type conjecture for tame n-dimensional Galois representations*, Duke Math. J. **149** (2009), no. 1, 37-116.
[15] F. Herzig, D. Lee, S. Morra, *On mod p local-global compatibility for* GL_3 *in the ordinary case*, in preparation.
[16] M. Kisin, *Crystalline representations and F-crystals*, Algebraic geometry and number theory. In honour of Vladimir Drinfeld's 50th birthday, Prog. Math. 253, Birkhauser, (2006), 459-496.
[17] S. Morra, C. Park, *Serre weights for three dimensional ordinary Galois representations*, preprint, 2014.
[18] J.-P. Serre, *Sur les représentations modulaires de degré 2 de Gal*($\overline{\mathbb{Q}}/\mathbb{Q}$), Duke Math. J. **54**, no. 1, (1987), 179-230.

Analytic functions on some étale coverings of the *p*-adic upper half-plane

Gabriel Dospinescu
(joint work with Arthur-César Le Bras)

Let p be a prime number. If $l \neq p$ is another prime, the l-adic cohomology of the étale coverings of the p-adic upper half-plane realizes both the "classical" local Langlands correspondence for $G = \mathrm{GL}_2(\mathbf{Q}_p)$ and the Jacquet-Langlands correspondence for discrete series representations. This result (known as Carayol's conjecture) has already been vastly generalized to $\mathrm{GL}_n(F)$ (where F is a finite extension of $\mathbf{Q}_p$) thanks to work of Boyer, Carayol, Dat, Faltings, Fargues, Harris, Mieda, Taylor and others, see for instance [5, 9, 12, 13, 14]. The purpose of our work is to confirm a (version of a) conjecture of Breuil and Strauch, which shows that the coherent cohomology (more precisely the de Rham complex) of these étale coverings realizes the p-adic local Langlands correspondence for G (due to Berger, Breuil, Colmez, Emerton, Kisin, Paskunas [3, 6, 8, 11, 16, 18]) and the classical Jacquet-Langlands correspondence for de Rham non trianguline representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ (we will limit ourselves to representations with Hodge-Tate weights $0, 1$ in the sequel).

Let us start by fixing some notation. Let D be the quaternion division algebra over $\mathbf{Q}_p$, $\mathcal{O}_D$ its unique maximal order and let $\mathbf{X}$ be a special formal $\mathcal{O}_D$-module (in the sense of Drinfeld [10]) over $\overline{\mathbf{F}}_p$. Deforming $\mathbf{X}$ by $\mathcal{O}_D$-equivariant quasi-isogenies (of arbitrary height) yields a tower of Rapoport-Zink spaces $(\mathcal{M}_n)_{n\geq 0}$. More precisely $\mathcal{M}_n$ is the rigid analytic generic fiber of the corresponding p-adic formal scheme constructed by Rapoport and Zink [19], with level structure $1 + p^n\mathcal{O}_D$. For instance, $\mathcal{M}_0 \simeq \breve{\Omega} \times \mathbf{Z}$ by a fundamental theorem of Drinfeld [10], where

$\breve{\Omega} = \Omega \hat{\otimes} \widehat{\mathbf{Q}_p^{\rm nr}}$ and Ω is the rigid analytic space over $\mathbf{Q}_p$ such that $\Omega(\mathbf{C}_p) = \mathbf{P}^1(\mathbf{C}_p)\backslash \mathbf{P}^1(\mathbf{Q}_p)$. Moreover, $\mathcal{M}_n$ is endowed with commuting actions of G and D^*, the action of D^* being smooth ($1 + p^n\mathcal{O}_D$ acts trivially). Moreover, the quotient of $\mathcal{M}_n$ by the action of $p^{\mathbf{Z}}$ (seen as a subgroup of the center of G) descends (together with the actions of G and D^*) to a rigid analytic space Σ_n over $\mathbf{Q}_p$. We will describe the de Rham complex of Σ_n.

Let us fix an irreducible smooth representation ρ of D^*, trivial on $1 + p^n\mathcal{O}_D$, with trivial central character. We assume that ρ is nontrivial (the case $\rho = 1$ was dealt with by Breuil [4] and we warn the reader that it is fairly different from the case $\rho \neq 1$, in particular all results to follow are wrong when $\rho = 1$) and we let $\pi(\rho)$ be the (isomorphism class of) supercuspidal representation of G associated to ρ by the local Jacquet-Langlands correspondence. Let L be a finite extension of $\mathbf{Q}_p$ such that all these representations are defined over L. We consider L as field of coefficients from now on, in particular for a rigid space Z over $\mathbf{Q}_p$ we write $\mathcal{F}(Z) = H^0(Z, \mathcal{F}) \otimes_{\mathbf{Q}_p} L$ for a sheaf $\mathcal{F}$ on Z. One can attach (via a standard recipe of Fontaine) to[1] $\pi(\rho)$ a $(\varphi, \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p))$-module $M(\rho)$, of rank 2 over $L \otimes_{\mathbf{Q}_p} \mathbf{Q}_p{}^{\rm nr}$. Let $M_{\rm dR}(\rho) = (M(\rho) \otimes_{\mathbf{Q}_p{}^{\rm nr}} \overline{\mathbf{Q}}_p)^{\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)}$, a two-dimensional L-vector space.

Let $\mathcal{B}(\rho)$ be the set of (isomorphism classes of) absolutely irreducible admissible unitary completions of π on L-Banach spaces with action of G. The p-adic local Langlands correspondence for G establishes a canonical bijection

$$\mathcal{B}(\rho) \simeq \mathbf{P}(M_{\rm dR}(\rho)), \quad \Pi \mapsto \mathcal{L}(\Pi).$$

More precisely, each $\Pi \in \mathcal{B}(\rho)$ gives rise to a Galois representation $V(\Pi)$ via Colmez's Montreal functor [6]. By the compatibility of classical and p-adic Langlands correspondences for G (due to Colmez and Emerton [6, 11]) we know that

$$D_{\rm pst}(V(\Pi)) \simeq M(\rho)$$

canonically up to scalar (as $(\varphi, \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p))$-modules, *without* the Hodge filtration) and then

$$\mathcal{L}(\Pi) = \mathrm{Fil}^0(D_{\rm dR}(V(\Pi)))$$

is the Hodge filtration on $D_{\rm dR}(V(\Pi)) \simeq M_{\rm dR}(\rho)$.

If $\Pi \in \mathcal{B}(\rho)$, let $\Pi^{\rm an}$ (respectively $\Pi^{\rm sm}$) be the subspace of Π consisting of vectors v for which the orbit map $g \mapsto g.v$ is locally analytic (respectively locally constant). Then $\Pi^{\rm sm} \simeq \pi(\rho)$ and $\Pi^{\rm sm}$ is closed in $\Pi^{\rm an}$ (and dense in Π). Our first main result is then:

Theorem 1. *For each $\Pi \in \mathcal{B}(\rho)$ there is a unique (up to scalar) isomorphism of topological G-modules*

$$\mathrm{Hom}_{D^*}(\rho, \mathcal{O}(\Sigma_n)) \simeq (\Pi^{\rm an}/\Pi^{\rm sm})^*.$$

In other words, there is a canonical (up to scalar) exact sequence

$$0 \to \mathrm{Hom}_{D^*}(\rho, \mathcal{O}(\Sigma_n)) \to (\Pi^{\rm an})^* \to \pi(\rho)^* \to 0.$$

[1]More precisely, to the Weil representation attached to $\pi(\rho)$ by the classical local Langlands correspondence for G

If σ is a representation of $G \times D^*$, we write

$$\sigma^\rho = \mathrm{Hom}_{D^*}(\rho, \sigma).$$

The proof of the previous theorem is a mixture of local and global arguments. The global argument is used to produce a nonzero G-equivariant map

$$\mathcal{O}(\Sigma_n)^\rho \to (\Pi^{\mathrm{an}}/\Pi^{\mathrm{sm}})^*,$$

while the local argument (which is the most technical part of the proof) shows that any such map is an isomorphism. The key global ingredients are the Cerednik-Drinfeld uniformization theorem and a version of Emerton's local-global compatibility theorem for definite quaternion algebras split at p. Putting this together gives (after some work) a nonzero map $\mathcal{O}(\Sigma_n)^\rho \to (\Pi^{\mathrm{an}}/\Pi^{\mathrm{sm}})^*$ for *some* $\Pi \in \mathcal{B}(\rho)$. A deep theorem of Colmez ensures that the representation $\Pi^{\mathrm{an}}/\Pi^{\mathrm{sm}}$ is *independent* of the choice of $\Pi \in \mathcal{B}(\rho)$ (as it is also clear from the previous theorem!) thus we have such a map for *any* such Π. The local part of the argument is fairly technical, but the key idea is to use the theory of (ϕ, Γ)-modules and its link with p-adic Hodge theory (more precisely Berger's results [1, 2] on the p-adic differential equation attached to a de Rham representation) to promote $(\Pi^{\mathrm{an}}/\Pi^{\mathrm{sm}})^*$ to a G-equivariant $\mathcal{O}(\Omega)$-module (it will turn out to be a G-equivariant vector bundle over Ω, but this only at the very end of the argument!). The previous map becomes a map of G-equivariant sheaves on Ω and one can show its surjectivity using results of Kohlhaase [17] (concerning the transfer of some G-equivariant vector bundles on Ω to D^*-equivariant vector bundles on $\mathbf{P}^1$) and some functional analytic tricks. This was strongly influenced by very recent work of Colmez [7]. To show injectivity one can use a result of Colmez [7], stating that $\Pi^{\mathrm{an}}/\Pi^{\mathrm{sm}}$ is irreducible (we actually give a different argument, which we need to prove the theorems to follow).

The next theorem deals with the Hodge filtration and the de Rham complex of Σ_n. Note that since Σ_n is Stein and ρ is nontrivial, taking ρ-isotypic components in the de Rham complex of Σ_n yields an exact sequence of topological G-modules

$$0 \to \mathcal{O}(\Sigma_n)^\rho \to \Omega^1(\Sigma_n)^\rho \to H^1_{\mathrm{dR}}(\Sigma_n)^\rho \to 0.$$

Theorem 2. *There is a canonical (up to scalar) isomorphism*

$$H^1_{\mathrm{dR}}(\Sigma_n)^\rho \simeq M_{\mathrm{dR}}(\rho)^* \otimes \pi(\rho)^*$$

such that for any $\Pi \in \mathcal{B}(\rho)$ *the inverse image of* $\mathcal{L}(\Pi)^\perp \otimes \pi(\rho)^* \subset H^1_{\mathrm{dR}}(\Sigma_n)^\rho$ *in* $\Omega^1(\Sigma_n)^\rho$ *is* $(\Pi^{\mathrm{an}})^*$ *and the corresponding exact sequence*

$$0 \to \mathcal{O}(\Sigma_n)^\rho \to (\Pi^{\mathrm{an}})^* \to \mathcal{L}(\Pi)^\perp \otimes \pi(\rho)^* \simeq \pi(\rho)^* \to 0$$

is the exact sequence given by the previous theorem.

References

[1] L. Berger-Représentations p-adiques et équations différentielles, Invent. Math. 148 (2002), p. 219-284.

[2] L. Berger-Equations différentielles p-adiques et (φ, N)-modules filtrés, Astérisque 319 (2008), p. 13-38.

[3] L. Berger, C. Breuil-Sur quelques représentations potentiellement cristallines de $G_{\mathbf{Q}_p}$, Astérisque 330, p. 155-211.
[4] C. Breuil-Invariant L et série spéciale p-adique, Ann. Sci. Ecole Norm. Sup 37, p. 559-610, 2004.
[5] H. Carayol-Non-abelian Lubin-Tate theory, in L. Clozel and J.S. Milne, editors, Automorphic forms, Shimura varieties and L-functions, volume II, p. 1539. Academic Press, 1990.
[6] P. Colmez-Représentations de $\mathrm{GL}_2(\mathbf{Q}_p)$ et (φ,Γ)-modules, Astérisque 330 (2010), p. 281-509.
[7] P. Colmez-Correspondance de Langlands locale p-adique et changement de poids, preprint.
[8] P. Colmez, G. Dospinescu, V. Paškūnas-The p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$, Cambridge Journal of Mathematics, Volume 2 (2014), Number 1.
[9] J. F. Dat-Espaces symétriques de Drinfeld et correspondance de Langlands locale, Ann. Sci. école Norm. Sup. (4) 39 (2006), p. 1-74.
[10] V.G. Drinfeld-Coverings of p-adic symmetric domains. Func. Anal. and Appl., 10, p. 107115, 1976.
[11] M. Emerton-Local-global compatibility in the p-adic Langlands programme for $\mathrm{GL}_2/\mathbf{Q}$, preprint, disponible à http://www.math.northwestern.edu/ emerton/preprints.html.
[12] G. Faltings- A relation between two moduli spaces studied by V. G. Drinfeld, Algebraic number theory and algebraic geometry, Contemp. Math., vol. 300, Amer. Math. Soc., Providence, RI, 2002, p. 115-129.
[13] L. Fargues, A. Genestier, V. Lafforgue-L'isomorphisme entre les tours de Lubin-Tate et de Drinfeld, Progress in Mathematics, vol. 262, Birkhäuser Verlag, Basel, 2008.
[14] M. Harris-Supercuspidal representations in the cohomology of Drinfeld's upper half space; elaboration of Carayol's program. Invent. Math., 129, p. 75-119, 1997.
[15] M. Harris, R. Taylor-The geometry and cohomology of some simple Shimura varieties, Ann. of Math. studies 151, Princeton Univ. Press, 2001.
[16] M. Kisin-Deformations of $G_{\mathbf{Q}_p}$ and $\mathrm{GL}_2(\mathbf{Q}_p)$-representations, Astérisque 330 (2010), 511-528.
[17] J. Kohlhaase-Lubin-Tate and Drinfeld bundles, Tohoku Math. J. 63, No. 2, 2011, p. 217-254.
[18] V. Paškūnas-The image of Colmez's Montréal functor, Publ.Math. IHES 2013, Vol.118, Nr. 1, p.1-191.
[19] M. Rapoport, T. Zink- Period spaces for p-divisible groups, Annals Math. Studies 141, Princeton University Press, 1996.

Perfectoid Shimura varieties of abelian type

Xu Shen

Let p be a fixed prime. For a Shimura datum (G,X), we have the associated tower of Shimura varieties $(\mathrm{Sh}_K(G,X))_{K\subset G(\mathbb{A}_f)}$ over $\mathbb{C}_p$. Fix a sufficient small prime to p level $K^p\subset G(\mathbb{A}_f^p)$, and consider open compact subgroups K in the form $K=K_pK^p$ with $K_p\subset G(\mathbb{Q}_p)$. Let $\mathrm{Sh}_{K_pK^p}(G,X)^{\mathrm{ad}}$ be the associated adic spaces over $\mathbb{C}_p$. As usual, associated to the Shimura datum (G,X), we have the flag variety $\mathcal{FL}_G$ over $\mathbb{C}_p$, which will be viewed as an adic space. In [4], we proved the following theorem.

Theorem 1. *Assume that the Shimura datum (G,X) is of abelian type.*

(1) *There exists a perfectoid space S_{K^p} over $\mathbb{C}_p$ such that*

$$S_{K^p}\sim\varprojlim_{K_p}\mathrm{Sh}_{K_pK^p}(G,X)^{\mathrm{ad}}.$$

For the meaning of $\sim$*, see the Definition 2.4.1 of* [8]*.*

(2) *There is a* $G(\mathbb{Q}_p)$*-equivariant map of adic spaces*

$$\pi_{\mathrm{HT}} \colon S_{K^p} \longrightarrow \mathcal{FL}_G,$$

which is invariant for the prime to p Hecke action on S_{K^p}*, when* K^p *varies. Moreover, pullbacks of automorphic vector bundles over finite level Shimura varieties to* S_{K^p} *can be understood by using the map* π_{HT} *(for a precise statement, see* [4] *subsection 3.4).*

Recall that the basic theory of perfectoid spaces was developed in [5]. Recall also that Shimura varieties of abelian type are exactly those studied by Deligne in [2], where he proved that the canonical models of these Shimura varieties exist. When the weight is rational, Shimura varieties of abelian type (over characteristic 0) are known as moduli spaces of abelian motives. The class of abelian type Shimura varieties is strictly larger than the class of Hodge type Shimura varieties. By Deligne's classification, the class of abelian type Shimura varieties is also the main class of Shimura varieties. Natural examples of abelian type Shimura varieties (which are usually not of Hodge type) include those associated to quaternion algebras over a totally real field, and those associated to special orthogonal groups over $\mathbb{Q}$ with signature $(2, n)$ for some integer $n \geq 1$.

Before stating the ideas in the proof of the theorem, let us first give some remarks. If (G, X) is of Hodge type, then the theorem was proved by Scholze in [6] (and the part (2) for Hodge-Tate period map was completed by Caraini-Scholze in [1]). In fact, Scholze proved a stronger version for some compactification of Shimura varieties, which is the key geometric ingredient for his construction of automorphic Galois representations.

By definition, a Shimura datum (G, X) is called of abelian type if there exists a Shimura datum of Hodge type (G_1, X_1), together with a central isogeny between the derived subgroups $G_1^{\mathrm{der}} \to G^{\mathrm{der}}$, such that it induces an isomorphism of the associated adjoint Shimura datum $(G_1^{\mathrm{ad}}, X_1^{\mathrm{ad}}) \simeq (G^{\mathrm{ad}}, X^{\mathrm{ad}})$. Therefore, the geometry of Shimura varieties of abelian type and of Hodge type are closely related. The ideas in the proof of the theorem are as follows.

Step 1. For any Shimura datum (G, X), fix a connected component $X^+ \subset X$. We show that the statement (1) in the theorem is equivalent to the statement that, there exists a perfectoid space $S^0_{K^p}$ over $\mathbb{C}_p$, such that

$$S^0_{K^p} \sim \varprojlim_{K_p} \mathrm{Sh}^0_{K_pK^p}(G, X)^{\mathrm{ad}},$$

where $\mathrm{Sh}^0_{K_pK^p}(G, X)^{\mathrm{ad}}$ are the connected (adic) Shimura varieties which over $\mathbb{C}$ come from $X^+ \times \{e\}$ (e is the identity element in $G(\mathbb{A}_f)$).

Step 2. Let (G, X) be of abelian type and (G_1, X_1) be of Hodge type as above. Consider the scheme over $\mathbb{C}_p$ defined by

$$\mathrm{Sh}^0_{K^p}(G, X) = \varprojlim_{K_p} \mathrm{Sh}^0_{K_pK^p}(G, X).$$

Then we can show that there exists some $K_1^p \subset G_1(\mathbb{A}_f^p)$ such that

$$\mathrm{Sh}^0_{K^p}(G,X) = \mathrm{Sh}^0_{K_1^p}(G_1,X_1)/\Delta,$$

where Δ is some finite group, acting freely on the scheme

$$\mathrm{Sh}^0_{K_1^p}(G_1,X_1) = \varprojlim_{K_{1p}} \mathrm{Sh}^0_{K_{1p}K_1^p}(G_1,X_1).$$

We would like a perfectoid version of this construction. By Step 1 and Scholze's result for (G_1,X_1), there is a perfectoid Shimura variety $S^0_{K_1^p}(G_1,X_1)$, such that $S^0_{K_1^p}(G_1,X_1) \sim \varprojlim_{K_{1p}} \mathrm{Sh}^0_{K_{1p}K_1^p}(G_1,X_1)^{\mathrm{ad}}$. The key points are now

(1) Δ acts freely on $S^0_{K_1^p}(G_1,X_1)$, which implies that $S^0_{K_1^p}(G_1,X_1)/\Delta$ exists as a diamond (cf. [7]).
(2) In fact, $S^0_{K_1^p}(G_1,X_1)/\Delta$ exists as an adic space. Moreover, there is a finite étale Galois cover $S^0_{K_1^p}(G_1,X_1) \to S^0_{K^p} := S^0_{K_1^p}(G_1,X_1)/\Delta$ with Galois group Δ.

Then by a theorem of Kedlaya-Liu (cf. [3] Proposition 3.6.22), $S^0_{K^p}$ is perfectoid. By construction, we have $S^0_{K^p} \sim \varprojlim_{K_p} \mathrm{Sh}^0_{K_pK^p}(G,X)^{\mathrm{ad}}$. By Step 1 again, the statement (1) of the theorem holds.

Step 3. Let (G,X) and (G_1,X_1) be as in Step 2. Then we have $\mathcal{FL}_G = \mathcal{FL}_{G_1}$. By the results of Caraiani-Scholze, statement (2) of theorem holds for (G_1,X_1). Let $\pi'_{\mathrm{HT}}\colon S_{K_1^p}(G_1,X_1) \longrightarrow \mathcal{FL}_G$ be the Hodge-Tate period map for the Hodge type perfectoid Shimura variety $S_{K_1^p}(G_1,X_1)$. The key point is then $\pi'_{\mathrm{HT}}|_{S^0_{K_1^p}(G_1,X_1)}$ is Δ-invariant. So we get a map

$$\pi_{\mathrm{HT}}\colon S^0_{K^p} \to \mathcal{FL}_G.$$

Then applying the $G(\mathbb{Q}_p)$-action, and the theory of connected components of Shimura varieties, we get that the statement (2) of the theorem holds.

Here we give briefly some applications of the theorem. First, if (G,X) is a Shimura datum of abelian type such that the associated Shimura varieties are compact, then we can use the theorem to deduce the vanishing of degree i-th completed cohomology of these varieties, where $i > \dim \mathrm{Sh}_K$. Next, we can prove that the moduli spaces of polarized K3 surfaces with infinity level at p are perfectoid, by applying our theorem and the global Torelli theorem for K3 surfaces. We hope that this result will lead more interesting applications to the arithmetic of K3 surfaces.

References

[1] A. Caraiani, P. Scholze, *On vanishing of torsion in the cohomology of Shimura varieties*, in preparation.
[2] P. Deligne, *Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques*, in "Automorphic forms, representations and L-functions" (Corvallis 1977), Proc. Sympos. Pure Math. XXXIII, Amer. Math. Soc., 247-289, 1979.
[3] K. S. Kedlaya, R. Liu, *Relative p-adic Hodge theory: Foundations*, arXiv:1301.0792, to appear in Astérisque.

[4] X. Shen, *Perfectoid Shimura varieties of abelian type*, arXiv:1507.01824
[5] P. Scholze, *Perfectoid spaces*, Publ. Math. I.H.E.S.116 (2012), no.1, 245-313.
[6] P. Scholze, *On torsion in the cohomology of locally symmetric varieties*, Preprint, to appear in Ann. Math.
[7] P. Scholze, *Lectures on p-adic geometry*, notes taken by J. Weinstein, Fall 2014.
[8] P. Scholze, J. Weinstein, *Moduli of p-divisible groups*, Cambridge Journal of Mathematics 1 (2013), 145-237.

Bad reduction of Shimura varieties, level raising, and Selmer groups

Yifeng Liu

Let F be a totally real number field of odd degree d. Consider an elliptic curve E defined over F. From E, we may construct the following motive

$$M_E := \left(\otimes\text{-Ind}_F^{\mathbb{Q}}\, \mathrm{h}^1(E)\right)(\frac{d+1}{2})$$

over $\mathbb{Q}$, which has coefficient $\mathbb{Q}$ and rank 2^d, where $\otimes\text{-Ind}_F^{\mathbb{Q}}$ denotes the tensor induction of a motive over F to $\mathbb{Q}$. Moreover, M_E is canonically polarized, that is, equipped with a canonical pairing $M_E \times M_E \to \mathbb{Q}(1)$, of symplectic type induced from the Weil pairing on E. By definition, for every prime p, we have the p-adic realization $(M_E)_p$ of M_p, which is a Galois representation of $\mathbb{Q}$ on a $\mathbb{Q}_p$-vector space of dimension 2^d.

We may associate to the motive M_E two invariants. The first is the (complete) L-function, denoted as $L(s, M_E)$, which is originally defined for s with sufficiently large real part. The *modularity conjecture* asserts that $L(s, M_E)$ has a meromorphic continuation to the entire complex plane, and satisfies the following functional equation

$$L(s, M_E) = \epsilon(M_E)c(M_E)^{-s}L(-s, M_E),$$

where $\epsilon(M_E) \in \{\pm 1\}$ and $c(M_E)$ is a positive integer. The second is the (family of Bloch–Kato) Selmer groups: it has for each prime p, a $\mathbb{Q}_p$-subvector space $\mathrm{H}^1_f(\mathbb{Q}, (M_E)_p)$ of the Galois cohomology $\mathrm{H}^1(\mathbb{Q}, (M_E)_p)$.

For the above modularity conjecture, if we assume that E is modular, in other words, E can be attached to a cuspidal automorphic representation Π_E of $G(\mathbb{A})$ where $G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_{2,F}$, then one can compute that

$$L(s, M_E) = L(s + 1/2, \Pi_E, r_d)$$

where $r_d \colon {}^L G \to \mathrm{GL}_{2^d}(\mathbb{C})$ is the tensor product representation. In particular, the above modularity conjecture for M_E is known when $d = 1$ and by [1], $d = 3$.

Our main theorem is the following; see [3] for details.

Theorem 1. *Suppose that $d = 3$ and E is a modular elliptic curve over F satisfying some mild conditions. If $L(0, M_E) \neq 0$, then $\mathrm{H}^1_f(\mathbb{Q}, (M_E)_p) = 0$ for all but finitely many p.*

According to the Bloch–Kato Conjecture, which generalizes the B-SD Conjecture, we expect that the order of 0 of $L(s, M_E)$ at $s = 0$ (assuming the modularity conjecture) is equal to the dimension of $\mathrm{H}^1_f(\mathbb{Q}, (M_E)_p)$ over $\mathbb{Q}_p$ for all primes p. Thus, the above theorem confirms the Bloch–Kato Conjecture for the motive M_E when $d = 3$ and the analytic order is 0.

The proof follows essentially from two results: level raising of automorphic representation on certain Hilbert threefold, and an explicit reciprocity law for diagonal cycles on the Hilbert threefold. Suppose that we are in the situation of the previous theorem. The set of local epsilon factors of the motive M_E determines a definite quaternion algebra D over $\mathbb{Q}$; put $B = D \otimes_{\mathbb{Q}} F$. Then Π_E has a Jacquet–Langlands transfer Π to the group $G_B := \mathrm{Res}_{F/\mathbb{Q}} B^\times$, which is trivial at infinity. Fix a (neat) open compact subgroup U of $G_B(\widehat{\mathbb{Z}})$ induced by an Eichler order, such that Π^U has nonzero triple product period integral (a finite sum, in fact) against $D^\times$. This is possible as $L(0, M_E) \neq 0$ by the result of [2].

Fix a sufficiently large prime p (controlled by E and U) and an integer $\nu \geq 1$. For a prime ℓ inert in F at which U is hyperspecial maximal, we define a nearby quaternion algebra D' over $\mathbb{Q}$: it is indefinite, ramified at ℓ, and isomorphic to D away from $\{\infty, \ell\}$; put $B' = D' \otimes_{\mathbb{Q}} F$. The map $D'^\times \to \mathrm{Res}_{F/\mathbb{Q}} B'^\times$ induces a finite morphism of the corresponding Shimura varieties $Y \to X$ over $\mathbb{Q}$, with level structures determined by U and maximal at ℓ. Note that X is a (quaternionic) Hilbert threefold and Y is a Shimura curve. Let $\mathbb{T}$ be the spherical Hecke algebra for X. Then the Hecke eigenvalues of Π modulo p^ν give rise to an ideal $\mathfrak{m}$ of $\mathbb{T}$.

Now we can explain the meaning of level raising. For simplicity, assume that F is normal. For the complete version, see [3].

Theorem 2. *If ℓ satisfies the condition*

$$a_\ell(E) \equiv \ell^3 + 1 \mod p^\nu$$

and other mild congruence conditions, then the Galois representation on the Hecke quotient of the étale cohomology $\mathrm{H}^3(X_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(2))/\mathfrak{m}$ of $\mathbb{Q}$ is isomorphic to a direct sum of copies of $(\otimes\text{-}\mathrm{Ind}_F^{\mathbb{Q}}\, E(\overline{F})[p^\nu])(-1)$. Moreover, the singular quotient of the local Galois cohomology $\mathrm{H}^1_{\mathrm{sing}}(\mathbb{Q}_\ell, \mathrm{H}^3(X_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(2))/\mathfrak{m})$ at ℓ is canonically isomorphic to the dual $\mathbb{Z}/p^\nu$-module of $\mathrm{Map}(G_B\backslash G_B(\widehat{\mathbb{Z}})/U, \mathbb{Z}/p^\nu)[\mathfrak{m}]$.

In particular, $\mathrm{H}^3(X_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(2))/\mathfrak{m}$ is nontrivial as Π^U is. The reciprocity law is then a formula for the image of Y in $\mathrm{H}^1_{\mathrm{sing}}(\mathbb{Q}_\ell, \mathrm{H}^3(X_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(2))/\mathfrak{m})$ under the Abel–Jacobi map, which is given by the triple product integral of functions on $G_B\backslash G_B(\widehat{\mathbb{Z}})/U$ against $D^\times$, up to an elementary factor. Thus by our construction, for large ν, we have obtained elements in $\mathrm{H}^3(X_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(2))/\mathfrak{m}$ with nontrivial image in the singular quotient at ℓ.

The proof of the level raising and reciprocity law uses the integral model of X at ℓ studied by Zink [5], which has poly-nodal (but not semistable) reduction, together with the computation of the vanishing cycle on such model. It also uses the existence of Tate cycles (curves) on certain quaternionic Shimura surfaces, recently proved in [4].

References

[1] P. B. Garrett, *Decomposition of Eisenstein series: Rankin triple products*, Ann. of Math. (2) **125** (1987), 209–235.
[2] A. Ichino, *Trilinear forms and the central values of triple product L-functions*, Duke Math. J. **145** (2008), 281–307.
[3] Y. Liu, *Bounding cubic-triple product Selmer groups of elliptic curves*, preprint.
[4] Y. Tian, and X. Liang *Tate cycles on some quaternionic Shimura varieties mod p*, preprint.
[5] T. Zink, *Über die schlechte Reduktion einiger Shimuramannigfaltigkeiten*, Compositio Math. **45** (1982), 15–107.

Generalized Hasse invariants

George Boxer

Fix integers $g \geq 1$, $N \geq 3$, a prime p not dividing N, and let $X = X_{g,N}/\mathbf{F}_p$ be the moduli space of principally polarized abelian varieties of dimension g with a principal level N structure. We recall the Ekedahl-Oort stratification of X [4].

If k is an algebraically closed field of characteristic p, and $(A, \lambda) \in X(k)$ is a geometric point, then there are only finitely many possibilities for the p-torsion $A[p]$ of A as a finite group scheme over k up to isomorphism. This leads to a stratification

$$X = \coprod_{w \in W^I} X_w$$

of X into reduced locally closed subschemes X_w, which is characterized by the property that two geometric points $x, y \in X(k)$ lie in the same stratum X_w if and only if $A_x[p] \simeq A_y[p]$ as finite group schemes. The indexing set W^I is a certain set of Weyl group cosets whose precise definition will not play a role in this summary.

Let $\mathcal{A}/X$ be the universal abelian scheme, and let $\omega = \det e^*\Omega^1_{\mathcal{A}/X}$ be the determinant of the Hodge bundle. For each stratum X_w, Ekedahl and Oort [4] have constructed a canonical non-vanishing section $A_w \in H^0(X_w, \omega^{N_w})$ for a suitable positive integer N_w. We recall their construction.

For each stratum X_w they introduce the *canonical filtration*

$$0 = G_0 \subset G_1 \subset \cdots \subset G_c \subset \cdots \subset G_{2c} = \mathcal{A}[p]|_{X_w}.$$

Here $\mathcal{A}[p]|_{X_w}$ is the p-torsion of the universal abelian scheme over X_w, and the G_i/X_w are finite flat subgroup schemes of $\mathcal{A}[p]|_{X_w}$. Let $F : A[p] \to A[p]^{(p)}$ and $V : A[p]^{(p)} \to A[p]$ denote the relative Frobenius and Verschiebung. Then for each term G_i in the canonical filtration, $F^{-1}(G_i^{(p)})$ and $V(G_i^{(p)})$ exist as finite flat group schemes over X_w, and are again terms in the canonical filtration. In fact, the canonical filtration may be characterized as the coarsest filtration with this property.

Ekedahl and Oort show that there is a permutation $\sigma \in S_{2c}$ with the property that

$$V : (G_{\sigma(i)}/G_{\sigma(i)-1})^{(p)} \simeq G_i/G_{i-1} \quad \text{for } i = 1, \ldots, c, \text{ and}$$
$$F : G_i/G_{i-1} \simeq (G_{\sigma(i)}/G_{\sigma(i)-1})^{(p)} \quad \text{for } i = c+1, \ldots, 2c.$$

In particular for any $1 \leq i \leq 2c$ there is an isomorphism

$$G_i/G_{i-1} \simeq (G_{\sigma(i)}/G_{\sigma(i)-1})^{(p)},$$

either given by Frobenius or the inverse of Verschiebung. Let N denote the order of the permutation σ. Then we obtain isomorphism

$$G_i/G_{i-1} \simeq (G_{\sigma(i)}/G_{\sigma(i)-1})^{(p)} \simeq (G_{\sigma^2(i)}/G_{\sigma^2(i)-1})^{(p^2)} \simeq \cdots \simeq (G_i/G_{i-1})^{(p^N)}$$

by "going around the cycles of σ."

Now for a group scheme G/S we have the co-lie algebra $\omega_{G/S} = e^*\omega^1_{G/S}$, where $e : S \to G$ is the identity section. We always have that G_c, the middle term in the canonical filtration, is the kernel of Frobenius $\mathcal{A}[F]$. Hence we have $\omega|_{X_w} \simeq \det \omega_{G_c}$. From this and the filtration

$$0 = G_0 \subset G_1 \subset \cdots \subset G_c$$

we obtain

$$\omega|_{X_w} \simeq \bigotimes_{i=1}^{c} \omega_{G_i/G_{i-1}}$$

Differentiating the isomorphisms $G_i/G_{i-1} \simeq (G_i/G_{i-1})^{(p^N)}$ from above we obtain isomorphisms of sheaves

$$\omega_{G_i/G_{i-1}} \simeq \omega^{(p^N)}_{G_i/G_{i-1}}.$$

Finally taking determinants and multiplying them together for $i = 1, \ldots c$ we obtain an isomorphism

$$\omega|_{X_w} \simeq \omega|^{p^N}_{X_w}$$

or in other words a non vanishing section

$$A_w \in H^0(X_w, \omega^{p^N-1}).$$

I call the sections A_w generalized Hasse invariants because in the case that X_w is the ordinary locus, A_w is precisely the classical Hasse invariant.

The starting point of the present work is the question: does A_w extend to the Zariski closure $\overline{X}_w$ of X_w? Unfortunately at the present time I am unable to prove this, and it may not even be true. However I can prove [1] the following slightly weaker result, which is nonetheless sufficient for many applications.

Theorem 1. *For all sufficiently large integers M, A_w^M extends to a section of $\omega^{M(p^N-1)}$ on $\overline{X}_w$ whose non vanishing locus is precisely X_w.*

There has been quite a bit of of work on generalized Hasse invariants on Shimura varieties, including work of Goren, Ito, Goldring, Nicole, Geraghty, Wedhorn, and Koskivirta. However before this theorem was proved, all known results were either for geometrically simpler Shimura varieties (like the work of Goren on Hilbert modular varieties, or the work of Ito on split $U(n, 1)$ Shimura varieties) or for special strata (like the work of Goldring-Nicole and Koskivirta-Wedhorn on μ-ordinary Hasse invariants and an unpublished construction of Geraghty-Goldring for certain strata in the Siegel case.) Recently a similar theorem has been proved for all Hodge type Shimura varieties by Goldring-Koskivirta [2].

I will now briefly describe the proof of Theorem 1. The main difficulty in proving the theorem is that while the open strata X_w are smooth, their closures $\overline{X}_w$ can have rather nasty singularities. In particular they are often not normal. Moreover the canonical filtration, which is used to define the sections A_w on the open strata, need not extend on the closure (even in codimension one.)

Both of these difficulties can be dealt with at once in the following way. We may view the canonical filtration on an open stratum X_w as defining a parahoric level structure on the universal abelian scheme over X_w. In particular it defines a "canonical section"

$$s : X_w \to X_P$$

to the projection $\pi : X_P \to X$. Here X_P denotes a suitable moduli space of principally polarized abelian varieties with parahoric level structure at p.

It is known by a result of Görtz and Hoeve [3] that the image of the section s is a so called Kottwitz-Rapoport stratum of X_P. Denote its Zariski closure by $\tilde{X}_w$, so that the projection π restricts to a proper morphism

$$\pi : \tilde{X}_w \to \overline{X}_w$$

which restricts to an isomorphism on the dense open X_w.

Our strategy for proving the theorem is to first extend A_w to $\tilde{X}_w$, and then "descend" to $\overline{X}_w$. For the first step, we are now in considerably better shape because the canonical filtration extends to all of $\tilde{X}_w$ and moreover it follows from the theory of local models that $\tilde{X}_w$ is smooth locally a schubert variety (so in particular normal.) Thus extending A_w to $\tilde{X}_w$ reduces to a problem of computing the order of vanishing of a section of a line bundle on a Schubert variety. The second step is an elementary trick of commutative algebra: once we know that A_w extends to $\tilde{X}_w$ and vanishes on the complement of X_w, it follows that a sufficiently large power of A_w extends to $\overline{X}_w$.

References

[1] G. Boxer, *Torsion in the coherent cohomology of Shimura varieties and Galois representations*, Harvard University Thesis (2015)

[2] W. Goldring and J.-S. Koskivirta, Strata Hasse invariants, Hecke algebras and Galois representations, arXiv:1507.05032 (2015)

[3] U. Görtz and M. Hoeve, *Ekedahl-Oort strata and Kottwitz-Rapoport strata*, J. Algebra **351** (2012), 160–174.

[4] F. Oort, *A stratification of a moduli space of abelian varieties*, In *Moduli of abelian varieties* (2001), 345–416.

Generalized Hasse invariants

Jean-Stefan Koskivirta
(joint work with Wushi Goldring)

Let $S_{0,K} = Sh_K(G_0, X_0)$ be a Shimura variety of Hodge-type attached to a Shimura datum (G_0, X_0) and to an open compact subgroup $K \subset G_0(\mathbb{A}_f)$, and denote by E_0 its reflex field. Fix a prime number p and a place $\nu|p$ in E_0 where S_K has good reduction. Denote by $\mathcal{O}_\nu$ the ring of integers of the completion $E_{0,\nu}$ of E_0 at ν, and let κ be its residue field. Assume that the subgroup K decomposes as $K = K_p K^p$ with $K^p \subset G_0(\mathbb{A}_F^p)$ open compact and $K_p \subset G_0(\mathbb{Q}_p)$ hyperspecial. Kisin and Vasiu proved the existence of a smooth canonical $\mathcal{O}_v$-model $\mathscr{S}_K$ in [6] and [10] respectively. The special fiber $S_K := \mathscr{S}_K \otimes \kappa$ admits numerous stratifications (Newton, Ekedahl-Oort, p-rank,...). In this report we only consider the Ekedahl-Oort stratification. Let $\mathscr{A}$ be the abelian scheme over S_K given by pull-back from the Siegel Shimura variety, and let ω be the Hodge bundle on S_K. The main result of [3] is the following:

Theorem 1. *Assume either that (G_0, X_0) is of PEL-type, or that the character of ω is p-small. Let $C \subset S_K$ be an Ekedahl-Oort stratum and let $\overline{C}$ denote its Zariski closure, endowed with the reduced scheme structure. There exists an integer $N \geq 1$ and a section $H \in H^0(\overline{C}, \omega^N)$ such that C coincides with the non-vanishing locus of H.*

We will explain below the condition "p-small" of Theorem 1. It is always satisfied in the PEL case. For Hodge-type Shimura varieties, it seems that the character attached to ω is always a multiple of a minuscule character, so that even the case $p = 2$ is conjecturally covered by our result.

We now explain our approach to Theorem 1, which is entirely group-theoretical. Let $\mathcal{G}$ be a reductive model of G_0 over $\mathbb{Z}_p$ such that $\mathcal{G}(\mathbb{Z}_p) = K_p$, and define $G := \mathcal{G}_{\mathbb{F}_p}$. We may find a cocharacter $\mu : \mathbb{G}_{m,\mathcal{O}_\nu} \to \mathcal{G}_{\mathcal{O}_\nu}$ such that $\mu \otimes E_{0,\nu}$ lies in X_0. Denote again by $\mu : \mathbb{G}_{m,\kappa} \to G_\kappa$ its base change to κ. It defines a pair of opposite parabolic subgroups $P_\pm$ in G and a Levi $L = P_+ \cap P_-$. We may assume that there exists a Borel subgroup of G contained in P_+ and defined over $\mathbb{F}_p$. Let W be the Weyl group of G and let I be the type P_+. Let Ψ denote the set of roots of G and write Δ for the B-simple roots. Building on the work of Moonen-Wedhorn in the case $G = GL_n$, Pink-Wedhorn-Ziegler defined and studied the stack $G\text{-}\mathtt{Zip}^\mu$ of G-zips of type μ in [9] and [8]. They proved that the underlying topological space of $G\text{-}\mathtt{Zip}^\mu$ can be identified with ${}^I W$ (endowed with the topology induced by a certain order, finer than the Bruhat order). By results of Zhang in [13], the Hodge filtration and the conjugate filtration of $H^1_{dR}(\mathscr{A}/S_K)$ give rise naturally to a G-zip over S_K, and the induced morphism of stacks

$$\zeta : S_K \longrightarrow G\text{-}\mathtt{Zip}^\mu$$

is smooth (in the PEL case, a similar result was proved by Viehmann-Wedhorn in [11]). The Ekedahl-Oort strata are defined as the fibers of ζ. As a corollary, Zhang shows that the induced decomposition of S_K is a locally closed stratification and he determines the dimensions of strata (assuming they are nonempty). Furthermore he proves the density of the μ-ordinary locus of S_K.

Wedhorn-Yatsyshyn proved in [12] that the Ekedahl-Oort stratification is pure, i.e the inclusion of a stratum C in S_K is an affine morphism. Theorem 1 shows in almost all (and conjecturally all) Hodge-type cases that that this stratification is even principally pure, i.e C is the non-vanishing locus of a section of a line bundle over its Zariski closure. For example, the classical ordinary locus is the non-vanishing locus of a section in $H^0(S_K, \omega^{p-1})$. For general unitary Shimura varieties, Goldring-Nicole constructed a μ-ordinary Hasse invariant in [4], and this was generalized by the author and Wedhorn in [7] to arbitrary Hodge-type Shimura varieties. For split unitary Shimura varieties of signature $(n-1,1)$, Ito constructed such sections for all Ekedahl-Oort strata ([5]). Finally, Boxer constructed Hasse invariants for all strata in PEL cases A and C in his thesis ([1]), simultaneously to our work.

Before we can state the group-theoretical result that leads to Theorem 1, we first need the following definition:

Definition 2. We say that the character $\lambda \in X^*(L)$ is ample if $\langle \lambda, \alpha^\vee \rangle < 0$ for all $\alpha \in \Delta \setminus I$. We say that λ is p-small if $\langle \lambda, \alpha^\vee \rangle < p$ for all $\alpha \in \Psi$.

For a character $\lambda \in X^*(L)$, we denote by $\mathscr{L}(\lambda)$ the corresponding line bundle on $G\text{-}\mathtt{Zip}^\mu$. One of the main results of [3] is the following theorem:

Theorem 3 (Group-theoretical Hasse invariants). *Let $\mathscr{C} \subset G\text{-}\mathtt{Zip}^\mu$ be a zip stratum and let $\overline{\mathscr{C}}$ denote its closure, with the reduced structure. Let $\chi \in X^*(L)$ be an ample and p-small character of L. Then there exists an integer $N \geq 1$ and a section $H \in H^0(\overline{\mathscr{C}}, \mathscr{L}(\chi)^{\otimes N})$ whose non-vanishing locus is exactly the substack $\mathscr{C}$.*

Now Theorem 1 follows from Theorem 3 by pulling back along the map ζ, because the character of ω is always ample. Our construction of Hasse invariants uses a group-theoretical analogue $G\text{-}\mathtt{ZipFlag}^\mu$ of the flag space introduced in [2] by Ekedahl and Van der Geer. It is a quotient stack that classifies G-zips endowed with a complete flag refining the Hodge filtration. There is a natural morphism $\pi : G\text{-}\mathtt{ZipFlag}^\mu \to G\text{-}\mathtt{Zip}^\mu$ given by forgetting the flag, whose fibers are flag varieties. Furthermore, there exists a natural smooth morphism of stacks

$$\psi : G\text{-}\mathtt{ZipFlag}^\mu \longrightarrow [B \backslash G / B] .$$

We define the flag strata as the fibers of ψ. In general, the map π sends a flag stratum to a union of zip strata. However, for particular flag strata called minimal (resp. cominimal), the image consists of a single zip stratum. Moreover, the restriction of π to these strata is finite. Our technique to construct Hasse invariants is to pull back sections of line bundles on closures of Schubert strata via the map ψ to cominimal flag strata, and then descend them to zip strata using a descent lemma similar to Boxer's.

We list some advantages of our approach. First, the cocharacter μ need not be minuscule, and hence group-theoretical Hasse invariants exist even for groups that do not admit a Shimura variety. Second, we are able to construct sections of other Hecke-equivariant line bundles than simply ω. Third, the sections produced in this way are canonical, because the space of sections they belong to are all one-dimensional. This result has the following corollary:

Corollary 4. *Let $(G_1, \mu_1) \to (G_2, \mu_2)$ be a finite morphism of reductive groups over a finite field, mapping the cocharacter μ_1 to μ_2. Assume that there exists an ample and p-small character $\lambda \in X^*(L_2)$ whose restriction to L_1 is again p-small. Then the induced map $G_1\text{-}\mathtt{Zip}^{\mu_1} \to G_2\text{-}\mathtt{Zip}^{\mu_2}$ has topologically discrete fibers.*

This corollary shows that if two Ekedahl-Oort strata map to the same stratum by a finite morphism of Shimura varieties, there is no closure relation between them. We also deduce the following:

Corollary 5. *Assume S_K is projective. Then all Ekedahl-Oort strata are affine.*

In the Siegel case, we use results of Ekedahl-Van der Geer in [2] to show that the Ekedahl-Oort stratification extends naturally to the minimal stratification, and that all strata are affine. This result was generalized by Boxer in [1] for other PEL Shimura varieties.

References

[1] G. Boxer, *Torsion in the coherent cohomology of Shimura varieties and Galois representations.*
[2] T. Ekedahl and G. van der Geer, *Cycle classes of the E-O stratification on the moduli of abelian varieties*, Algebra, arithmetic and geometry (Penn State U.) (Y. Tschinkel and Y. Zarhin, eds.), vol. 269, Progress in Math., June 2009, pp. 567–636.
[3] W. Goldring and J.-S. Koskivirta, *Strata Hasse invariants, Hecke algebras and Galois representations.*
[4] W. Goldring and M.-H. Nicole, *The μ-ordinary Hasse invariant of unitary Shimura varieties*, To appear in Crelle's Journal, available at *https://sites.google.com/site/wushijig/.*
[5] T. Ito, *Hasse invariants for some unitary Shimura varieties*, Oberwolfach Report 28/2005 (C. Denniger, P. Schneider, and A. Scholl, eds.), 2005, pp. 1565–1568.
[6] M. Kisin, *Integral models for Shimura varieties of abelian type*, J. Amer. Math. Soc. **23** (2010), no. 4, 967–1012.
[7] J.-S. Koskivirta and T. Wedhorn, *Generalized Hasse invariants for Shimura varieties of Hodge type*, arXiv:1406.2178 (2014).
[8] R. Pink, T. Wedhorn, and P. Ziegler, *F-zips with additional structure*, Preprint, http://arxiv.org/abs/1208.3547v2.
[9] ———, *Algebraic zip data*, Doc. Math. **16** (2011), 253–300.
[10] A. Vasiu, *Integral canonical models of Shimura varieties of preabelian type*, Asian J. Math. **3** (1999), 401–518.
[11] E. Viehmann and T. Wedhorn, *Ekedahl-Oort and Newton strata for Shimura varieties of PEL type*, Math. Ann. **356** (2013), no. 4, 1493–1550.
[12] T. Wedhorn and Y. Yaroslav, *Purity of zip strata with applications to Ekedahl-Oort strata of Shimura varieties of Hodge type*, preprint 2014, arXiv:1404.0577.
[13] C. Zhang, *Ekedahl-Oort strata for good reductions of Shimura varieties of Hodge type*, Preprint, http://arxiv.org/abs/1312.4869v2.

A stratification of affine Deligne-Lusztig varieties

Miaofen Chen
(joint work with Eva Viehmann)

Let k be a finite field with $q = p^r$ elements and let $\overline{k}$ be an algebraic closure of k. We consider both the arithmetic case where $F = W(\mathbb{F}_q)[1/p]$ and the function field case where $F = k((t))$. In both cases let L denote the completion of the maximal unramified extension of F and let $\mathcal{O}_F$ and $\mathcal{O}_L$ be the valuation rings. We denote by ϵ the uniformizer t or p. We write $\sigma \colon x \mapsto x^q$ for the Frobenius of $\overline{k}$ over k and also for the induced Frobenius of L over F (mapping ϵ to ϵ).

Let G be a connected reductive group over $\mathcal{O}_F$ and let $K = G(\mathcal{O}_L)$. Since k is finite G is automatically quasi-split. Let $B \subset G$ be a Borel subgroup and $T \subset B$ a maximal torus in B, both defined over $\mathcal{O}_F$. We denote by $X_*(T)$ the set of cocharacters of T, defined over $\mathcal{O}_L$.

We fix a minuscule dominant cocharacter $\mu \in X_*(T)$ and an element $b \in G(L)$. The affine Deligne-Lusztig variety $X^G_\mu(b) = X_\mu(b)$ associated to the triple (G, b, μ) is defined as follows:

$$X_\mu(b)(\overline{k}) = \{g \in G(L)/K \mid g^{-1}b\sigma(g) \in K\epsilon^\mu K\}.$$

Here we use $\epsilon^\mu := \mu(\epsilon)$. In the function field case the affine Deligne-Lusztig variety $X^G_\mu(b)$ has a reduced scheme structure as it can be considered as reduced closed subscheme of the affine Grassmannian of G. In the arithmetic case, in many cases (i.e. when (G, μ) corresponds to a Shimura datum of Hodge type), $X_\mu(b)(\overline{k})$ is the set of $\overline{k}$-valued points of a Rapoport-Zink moduli space of p-divisible groups. For general (G, μ), Zhu endows the mixed characteristic affine Grassmannian of G with some reasonable algebro-geometric structure [11]. We have then the induced structure on $X_\mu(b)$. Therefore, in all these cases, we can always consider the geometric structure on the affine Deligne-Lusztig varieties.

Let

$$J_b(F) = \{g \in G(L) \mid g \circ b = b \circ \sigma(g)\}.$$

This is the set of F-points of an algebraic group over F, an inner form of some Levi subgroup of G (the centralizer of the Newton point ν_b of b, [6]). There is a natural action of $J_b(F)$ on $X_\mu(b)$.

The geometry of affine Deligne-Lusztig varieties has been studied by many people. For example we know about the sets of connected components ([1]), and for minuscule μ and $G = \mathrm{GL}_n$ or GSp_{2n} also their sets of irreducible components ([7],[8]). In several particular cases we even have a complete description of their geometry, for example Kaiser [5] for the moduli space of supersingular p-divisible groups of dimension 2, Vollaard-Wedhorn [10] for certain unitary groups of signature $(1, n-1)$, further generalized by Görtz and He in [3]. All of these results indicate a close relation between the geometry of the affine Deligne-Lusztig varieties and (the Bruhat-Tits building of) $J_b(F)$. However, so far a conceptual way to explain this is still lacking.

In this talk, we propose a new invariant on affine Deligne-Lusztig varieties which also induces a decomposition of the variety into locally closed subschemes. Our invariant has the property that it not only depends on the element $g^{-1}b\sigma(g) \in K\epsilon^{\mu}K$, i.e. on the p-divisible group or local G-shtuka at the point of the moduli space we are interested in, but also on the quasi-isogeny, resp. on the element g itself.

Based on the idea that the geometry of an affine Deligne-Lusztig variety should be studied in relation with the action of $J_b(F)$, we assign to an element $g \in G(L)/K$ the function

$$\begin{array}{rcl} f_g \colon J_b(F) & \to & X_*(T)_{\mathrm{dom}} \\ j & \mapsto & \mathrm{inv}(j,g). \end{array}$$

Here inv denotes the relative position, i.e. the uniquely defined element of $X_*(T)_{\mathrm{dom}}$ with $j^{-1}g \in K\epsilon^{f_g(j)}K$ given by the Cartan decomposition

$$G(L) = \coprod_{\xi \in X_*(T)_{\mathrm{dom}}} K\xi(\epsilon)K.$$

In general the closure of a stratum is not a union of strata.

It turns out that this stratification is the natural group-theoretic generalization of a number of other stratifications that were studied intensively over the past years, but only existed for special cases, and were up to now unrelated to each other. We discuss three classes of such stratifications.

1. The Bruhat-Tits stratification. In [10] Vollaard and Wedhorn consider the supersingular locus of Shimura varieties for unitary groups of signature $(1, n-1)$ at an inert prime. They show that a refinement of the Ekedahl-Oort stratification yields a stratification of this locus, such that the individual strata have a description in terms of fine Deligne-Lusztig varieties, and the closure relations are given in terms of the Bruhat-Tits building of the associated goup $J_b(F)$. In [3], Görtz and He generalize this result by computing a complete list of cases of affine Deligne-Lusztig varieties for which the same generalization of the Ekedahl-Oort invariant yields an analogous result. We can show that in the Vollaard-Wedhorn case, our invariant coincides with theirs. We conjecture that the same holds in all cases studied by Görtz and He and verify this conjecture in one additional case.

2. Semi-modules. The main tool for the study of affine Deligne-Lusztig varieties in the superbasic case is the stratification by semi-modules. It was first considered by de Jong and Oort in [2] for certain moduli spaces of p-divisible groups corresponding to the group GL_n and later extended to the superbasic case for unramified groups in [9] and [4]. We can show that it coincides with the special case for superbasic b of our stratification.

3. The a-number. Finally, we discuss the relation to the a-number of p-divisible groups. This invariant assigns with a p-divisible group X over k the natural number $\dim \mathrm{Hom}_k(\alpha_p, X)$. It is a particularly useful tool to study moduli spaces of

p-divisible groups with or without polarization, but so far does not have a good generalization for p-divisible groups together with endomorphisms. In general our invariant (for the group GL_n) seems to be not related to the a-number of p-divisible groups. However, in the crucial case of the generic a-number 1 (and basic b), we can show that this locus coincides with one $J_b(F)$-orbit of strata for our invariant. We conjecture that this result still holds if one drops the assumption that b is basic. In this way, our invariant defines also a group-theoretic generalization of the open stratum defined by $a = 1$.

Altogether these examples show that the functions f_g are an invariant that seems to be central in the study of the geometry of affine Deligne-Lusztig varieties. One might hope that it opens the way to a more systematic investigation of its relation to the Bruhat-Tits building of $J_b(F)$.

References

[1] M. Chen, M. Kisin, E. Viehmann, *Connected components of affine Deligne-Lusztig varieties in mixed characteristic*, arXiv:1307.3845, to appear in Compositio Mathematica.

[2] A. J. de Jong, F. Oort, *Purity of the stratification by Newton polygons*, J. Amer. Math. Soc., **13** (2000), 209–241.

[3] U. Görtz, X. He, *Basic loci in Shimura varieties of Coxeter type*, arXiv:1311.6263v1, to appear in Cambridge Journal of Math.

[4] P. Hamacher, *The dimension of affine Deligne-Lusztig varieties in the affine Grassmannian of unramified groups*, preprint, arXiv:1312.0486, to appear in IMRN.

[5] C. Kaiser, *Ein getwistetes fundamentales Lemma für die* GSp_4, Bonner Mathematische Schriften [Bonn Mathematical Publications], **303**, Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, 1997.

[6] R. Kottwitz, *Isocrystals with additional structure*, Compositio Math. **56** (1985), 201–220.

[7] E. Viehmann, *Moduli spaces of p-divisible groups*, J. Algebraic Geom., **17** (2008), 341–374.

[8] E. Viehmann, *The global structure of moduli spaces of polarized p-divisible groups*, Documenta Mathematica **13** (2008), 825–852.

[9] E. Viehmann, *The dimension of some affine Deligne-Lusztig varieties*, Ann. Sci. École Norm. Sup. **39** (2006), 513–526.

[10] I. Vollaard, T. Wedhorn, *The supersingular locus of the Shimura variety of* $\mathrm{GU}(1, n-1)$ *II*, Invent. Math., **184** (2011), 591–627.

[11] X. Zhu, *Affine Grassmannians and the geometric Satake in mixed characteristic*, arXiv:1407.8519.

On non-emptiness of the basic loci of Shimura varieties

Chia-Fu Yu

We show that the basic locus of a reduction mod p of a Shimura variety of abelian type is non-empty. Our motivation is to show non-emptiness of the minimal KR stratum, and hence non-emptiness of KR strata for many cases where the KR stratification has been established.

Description. Recall a Shimura datum is a pair (G, X) which consists of a connected reductive algebraic $\mathbb{Q}$-group G and a $G(\mathbb{R})$-conjugacy class X of $\mathbb{R}$-homomorphisms $h\colon \mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{\mathrm{m}} \to G_{\mathbb{R}}$ satisfying the usual axioms. For any open compact subgroup $U \subset G(\mathbb{A}_f)$, denote by

$$\mathrm{Sh}_U(G,X)_{\mathbb{C}} := G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/U$$

the associated complex Shimura variety and $\mathrm{Sh}_U(G,X)$ its canonical model defined over the reflex field $E = E(G,X) \subset \mathbb{C}$. The canonical model exists due to works of Shimura, Deligne, Milne and Borovoi.

Let p denote a prime number. We fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Let F be a field with $E \subset F \subset \overline{\mathbb{Q}}$, and v the corresponding place of the ring of integers O_F lying over p. Let $O_{(v)}$ denote the localization of O_F at v, which is assumed to be a DVR. We write $\mathrm{Sh}_{U,F}$ for $\mathrm{Sh}_U(G,X) \otimes_E F$ and $\mathcal{M}_U := \mathbf{M}_U \otimes \overline{k(v)}$ for the special fiber of an integral model $\mathbf{M}_U$ of $\mathrm{Sh}_{U,F}$ over $O_{(v)}$.

We make the following working definition, which allows us to discuss Newton strata. Let $B(G)$ denote the set of σ-conjugacy classes of $G(L)$, where L is the completion of the maximal unramified extension of $\mathbb{Q}_p$. Let $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$.

Definition 1. (1) An *enhanced integral model* of $\mathrm{Sh}_{U,F}$ is a pair $(\mathbf{M}_U, \mathbf{N})$, where

- $\mathbf{M}_U$ is an integral model of $\mathrm{Sh}_{U,F}$ over $O_{(v)}$, and
- $\mathbf{N}\colon \mathcal{M}_U \to B(G)$ is a map of sets with locally closed fibers.

(2) An *enhanced integral model* of a tower $\{\mathrm{Sh}_{U,F}\}_{U\in\mathcal{U}}$ of Shimura varieties is a datum $(\{\mathbf{M}_U\}_{U\in\mathcal{U}}, \mathbf{N})$, where

- $\{\mathbf{M}_U\}_{U\in\mathcal{U}}$ is an integral model of $\{\mathrm{Sh}_{U,F}\}_{U\in\mathcal{U}}$ over $O_{(v)}$ with continuous right $G(\mathbb{A}_f^p)$-action, and
- $\mathbf{N}\colon \mathcal{M}_U \to B(G)$ is a map of sets with locally closed fibers, for any $U \in \mathcal{U}$, which is invariant under the transition maps and prime-to-p Hecke operators.

(3) Suppose $f\colon (G_1, X_1) \to (G_2, X_2)$ is a morphism of Shimura data. Let $U_1 \subset G_1(\mathbb{A}_f)$ and $U_2 \subset G_2(\mathbb{A}_f)$ be open compact subgroups such that $f(U_1) \subset U_2$. We denote again by $f\colon \mathrm{Sh}_{U_1,F} \to \mathrm{Sh}_{U_2,F}$ the induced morphism of Shimura varieties. A morphism of enhanced integral models $(\mathbf{M}_{U_1}, \mathbf{N}_1)$ and $(\mathbf{M}_{U_2}, \mathbf{N}_2)$ of $\mathrm{Sh}_{U_1,F}$ and $\mathrm{Sh}_{U_2,F}$, respectively, is an $O_{(v)}$-morphism $\widetilde{f}\colon \mathbf{M}_{U_1} \to \mathbf{M}_{U_2}$ inducing f and commuting with maps $\mathbf{N}_i$.

We consider a class of integral models satisfying the functorial property for maps of special pairs, which is similar to Deligne's definition of canonical models. This is different from (and is much easier than) the theory of integral canonical models investigated by Milne, Vasiu, Moonen and Kisin.

Let (T, h_T) be a Shimura datum with T a torus over $\mathbb{Q}$. For any open compact subgroup $V \subset T(\mathbb{A}_f)$, the Shimura variety $\mathrm{Sh}_V(T,h) = \mathrm{Spec}(E_V)$ is a finite etale E-scheme, where E_V is a finite etale E-algebra. We define the integral model $\mathbf{S}_V$ of $\mathrm{Sh}_{V,F}$ by

$$\mathbf{S}_V := \mathrm{Spec}\left(O_{E_V} \otimes_{O_E} O_{(v)}\right)^{\mathrm{nor}},$$

where O_{E_V} denotes the maximal order in E_V and $R^{\rm nor}$ denotes the normalization of a reduced Noetherian ring R. Let $\mathbf{N}\colon \mathcal{S}_V := \mathbf{S}_V \otimes \overline{k(v)} \to B(T)$ be the constant map given by the class $[\mu_T] \in X_*(T_{\mathbb{Q}_p})_\Gamma$ through Kottwitz's isomorphism $X_*(T_{\mathbb{Q}_p})_\Gamma \simeq B(T)$, where $\mu_T := h_{T,\mathbb{C}}(1,z) \in X_*(T) = X_*(T_{\mathbb{Q}_p})$. It is easy to show that $(\{\mathbf{S}_v\}_{V\subset T(\mathbb{A}_f)}, \mathbf{N})$ is an enhanced integral model of $\{\mathrm{Sh}_{V,F}\}_{V\subset T(\mathbb{A}_f)}$.

Definition 2. Let (G,X) and $\mathrm{Sh}_{U,F}$ be as before. An enhanced integral model $(\mathbf{M}_U,\mathbf{N})$ of $\mathrm{Sh}_{U,F}$ is called *admissible* if for any special pair $f\colon (T,h_T) \hookrightarrow (G,X)$ and any open compact subgroup $V \subset T(\mathbb{A}_f)$ with $f(V) \subset U$, the morphism $f\colon \mathrm{Sh}_{V,F} \to \mathrm{Sh}_{U,F}$ extends to a morphism of enhanced integral models

$$\widetilde{f}\colon (\mathbf{S}_V, [\mu_T]) \to (\mathbf{M}_U, \mathbf{N}).$$

For any $h \in X$, let $\mu := h_{\mathbb{C}}(1,z)\colon \mathbb{G}_{\rm m} \to G$, viewed as a cocharacter over $\overline{\mathbb{Q}}_p$, and $\{\mu\}$ the conjugacy class of μ. Let $B(G,\{\mu\})$ be the Kottwitz set defined in Kottwitz [8]. Theorem 3 (2) uses a result of Kisin called the Langlands-Rapoport lemma.

Theorem 3. *Let* $(\mathbf{M}_U,\mathbf{N})$ *be an admissible enhanced integral model of any Shimura variety* $\mathrm{Sh}_U(G,X)\otimes_E F$.

(1) *One has* $\mathbf{N}(\mathcal{M}_U)\cap B(G)_b \neq \emptyset$, *where* $B(G)_b$ *is the set of basic* σ*-conjugacy classes.*
(2) *Assume that* $G_{\mathbb{Q}_p}$ *is quasi-split. We have* $\mathbf{N}(\mathcal{M}_U) = B(G,\{\mu\})$.

Consider the case where (G,X) is of Hodge type and $U = U_pU^p$, where $U_p \subset G(\mathbb{Q}_p)$ is a maximal open compact subgroup and $U^p \subset G(\mathbb{A}_f^p)$ is sufficiently small. We can construct an admissible integral model $\mathbf{M}_U$ of $\mathrm{Sh}_{U,F}$ through the normalization of the closure in the Siegel moduli scheme. This is the construction first made by Vasiu, and then studied by Kisin and others in the good reduction case. The same normalization construction also produces an admissible enhanced integral model for a tower of Shimura varieties, even those of abelian type. However, the transition maps in integral models are finite (the Kummer theorem). Therefore, this construction does not construct good integral models when the level group U_p is small.

Theorem 4. *Let* (G,X) *be a Shimura datum of PEL-type and* $U = U_pU^p \subset G(\mathbb{A}_f)$ *an open compact subgroup. Assume that* $p \nmid |\pi_1(G^{\rm der})|$, $G_{\mathbb{Q}_p}$ *splits over a tamely ramified extension of* $\mathbb{Q}_p$, U_p *is a parahoric subgroup and* U^p *is sufficiently small. Let* $\mathbf{M}_U$ *be the integral canonical model of* $\mathrm{Sh}_U(G,X)$ *constructed as in Pappas-Zhu* [11, Theorem 0.2]. *Then every KR stratum of* $\mathcal{M}_U$ *is non-empty.*

Proposition 5. *Let* (G,X) *be a Shimura datum of Hodge type and* $U = U_pU^p \subset G(\mathbb{A}_f)$ *an open compact subgroup. Assume that* $p \neq 2$, $G_{\mathbb{Q}_p}$ *is unramified,* U_p *is hyperspecial and* U^p *is sufficiently small. Let* $\mathbf{M}_U$ *be the integral canonical model of* $\mathrm{Sh}_U(G,X)$ *constructed by Kisin* [5]. *Then every EO stratum of* $\mathcal{M}_U$ *is non-empty.*

The EO strata of $\mathcal{M}_U$ are constructed in Wortmann [16] and C. Zhang [17] based on works of Wedhorn and Kisin. Non-emptiness of Newton strata for Shimura varieties as in Proposition 5 has been proved by Kisin [6] and D. U. Lee [10] using reduction of CM points.

References

[1] P. Deligne, Travaux de Shimura. *Sém. Bourbaki Exp.* 389 (1970/71), 123–165. Lecture Notes in Math., vol. 244, Springer-Verlag, 1971.

[2] P. Deligne, Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques. *Automorphic forms, representations and L-functions Part 2,* 247–289. 1979.

[3] G. Faltings, Integral crystalline cohomology over very ramified valuation rings. *J. Amer. Math. Soc.* **12** (1999), 117–144.

[4] Xuhua He, Kottwitz-Rapoport conjecture on unions of affine Deligne-Lusztig varieties. arXiv:1408.5838. 19 pp.

[5] Mark Kisin, Integral models for Shimura varieties of abelian type. *J. Amer. Math. Soc.* **23** (2010), no. 4, 967–1012.

[6] Mark Kisin, Mod p points on Shimura varieties of abelian type, Preprint, available at http://www.math.harvard.edu/∼kisin/preprints.html.

[7] R. E. Kottwitz, Isocrystals with additional structure. *Compositio Math.* **56** (1985), 201–220.

[8] R. E. Kottwitz, Isocrystals with additional structure. II. *Compositio Math.* **109** (1997), 255–339.

[9] R. P. Langlands and M. Rapoport, Shimuravarietäten und Gerben. *J. Reine Angew. Math.* **378** (1987), 113–220.

[10] D.-U. Lee, Non-emptiness of Newton strata of Shimura varieties of Hodge type. Preprint December, 2014, 40 pp.

[11] G. Pappas and Xinwen Zhu, Local models of Shimura varieties and a conjecture of Kottwitz, *Invent. Math.* **194** (2013), no. 1, 147–254.

[12] M. Rapoport, A guide to the reduction modulo p of Shimura varieties. *Automorphic forms. I. Astériques* **298** (2005), 271–318.

[13] M. Rapoport and M. Richartz, On the classification and specialization of F-isocrystals with additional structure. *Compositio Math.* **103** (1996), 153–181.

[14] A. Vasiu, Integral canonical models of Shimura varieties of preabelian type. *Asian J. Math.* **3** (1999), 401–518.

[15] E. Viehmann and T. Wedhorn, Ekedahl-Oort and Newton strata for Shimura varieties of PEL type. *Math. Ann.* **356** (2013), no. 4, 1493–1550.

[16] D. Wortmann, The μ-ordinary locus for Shimura varieties of Hodge type. arXiv:1310.6444, 34 pp.

[17] Chao Zhang, Ekedahl-Oort strata for good reductions of Shimura varieties of Hodge type, arXiv:1312.4869v2, 54 pp.

Affinoids in the Lubin-Tate perfectoid space and simple epipelagic representations

Naoki Imai

(joint work with Takahiro Tsushima)

Let K be a non-archimedean local field, and $\mathfrak{p}$ be the maximal ideal of $\mathcal{O}_K$. Let n be a positive integer.

For a non-negative interger i, let $\mathrm{LT}_n(\mathfrak{p}^i)$ be the connected Lubin-Tate space with level $\mathfrak{p}^i$ for the 1-dimensional formal $\mathcal{O}_K$-module of height n. The Lubin-Tate perfectoid space $\mathcal{M}$ is a projective limit of $\mathrm{LT}_n(\mathfrak{p}^i)$ with respect to i in some sense, which has a structure of a perfectoid space over $\widehat{K}^{\mathrm{ab}}$.

Let D be the central division algebra over K of invariant $1/n$. We put

$$G = GL_n(K) \times D^\times \times W_K.$$

Let $\mathrm{Art}\colon K^\times \to W_K^{\mathrm{ab}}$ be the Artin reciprocity map normalized such that a uniformizer is sent to a lift of the geometric Frobenius element. We put $\mathbf{C} = \widehat{\overline{K}}$. The base change $\mathcal{M}_{\mathbf{C}} = \mathcal{M} \otimes_{\widehat{K}^{\mathrm{ab}}} \mathbf{C}$ has an action of

$$G^1 = \{(g, d, \sigma) \in G \mid \det(g) = \mathrm{Nrd}_{D/K}(d)\mathrm{Art}_K^{-1}(\sigma)\}.$$

Boyarchenko-Weinstein constructed a family of affinoids in $\mathcal{M}_{\mathbf{C}}$, and related them with the local Langlands correspondence and the local Jacquet-Langlands correspondence for representations of "unramified type" in [1].

We want to treat some special case for representations of "ramified type". Simple epipelagic representations are the representations of "ramified type" with the smallest conductor.

Let k be the residue field of K, and p be the characteristic of k. We put $q = |k|$ and $f = \log_p q$. We write $n = p^e n'$, where n' is prime to p. We put $m = \gcd(e, f)$ if $e \geq 1$. Let ℓ be a prime number that is different from p.

Theorem. ([2], [3]) *We have a family of affinoids $\{\mathcal{X}_i\}_{i \in I}$ in $\mathcal{M}_{\mathbf{C}}$ and their formal models $\{\mathfrak{X}_i\}_{i \in I}$ such that*

- *the special fiber $\overline{\mathfrak{X}_i}$ of $\mathfrak{X}_i$ is isomorphic to the perfection of the smooth Artin-Schreier variety defined by*

$$z^q - z = \sum_{1 \leq i \leq j \leq n-1} y_i y_j \quad \text{in } \mathbb{A}^n_{k^{\mathrm{ac}}}$$

  if $p \nmid n$, and by

$$z^{p^m} - z = y^{p^e+1} - \frac{1}{n'} \sum_{1 \leq i \leq j \leq n-2} y_i y_j \quad \text{in } \mathbb{A}^n_{k^{\mathrm{ac}}}$$

  if $p \mid n$,
- *the stabilizer H_i of $\mathcal{X}_i$ in G^1 naturally acts on $\overline{\mathfrak{X}_i}$, and*
- $\text{c-Ind}_{H_i}^{G} H_{\mathrm{c}}^{n-1}(\overline{\mathfrak{X}_i}, \overline{\mathbb{Q}}_\ell)$ *realizes the local Langlands correspondence and the local Jacquet-Langlands correspondence for simple epipelagic representations.*

References

[1] M. Boyarchenko and J. Weinstein, *Maximal varieties and the local Langlands correspondence for GL(n)*, to appear in J. Amer. Math. Soc.

[2] N. Imai and T. Takahiro, *Affinoids in the Lubin-Tate perfectoid space and simple epipelagic representations I: tame case*, preprint.

[3] N. Imai and T. Takahiro, *Affinoids in the Lubin-Tate perfectoid space and simple epipelagic representations II: wild case*, in preparation.

Principle B for de Rham Representations

Xinwen Zhu
(joint work with Ruochuan Liu)

Let k be a p-adic field and we fix an algebraic closure $\bar{k}$ of k. Let $\rho : \mathrm{Gal}(\bar{k}/k) \to \mathrm{GL}(V)$ be a p-adic representation. Recall that ρ is called de Rham if

$$\dim_k (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{\mathrm{Gal}(\bar{k}/k)} = \dim_{\mathbb{Q}_p} V,$$

where B_{dR} is Fontaine's de Rham period ring. The importance of this notion lies in the fact that the p-adic Galois representations arising from geometry (i.e. those on the étale cohomology of smooth proper varieties over k) are always de Rham.

We are interested in a relative situation. More precisely, let X be a smooth connected rigid analytic variety over k and let $\mathbb{L}$ be a $\mathbb{Z}_p$-local system on X. Such a local system can be regarded as a geometric family of Galois representations parameterised by X. Then in [4], we proved the following theorem

Theorem. *Let k' be a finite extension of k, and $x \in X(k')$. If $\mathbb{L}_x$, the stalk of $\mathbb{L}$ at x, regarded as a p-adic representation of $\mathrm{Gal}(\bar{k}'/k')$, is de Rham, then $\mathbb{L}$ is a de Rham local system on X. In particular, $\mathbb{L}_y$ is de Rham at every classical point y of X.*

We first make two remarks.

Remark 1. There is the notion of arithmetic families of Galois representations. It is known that in that case, the de Rham loci is closed. The theorem indicates that the geometric family is very rigid. One probably can also compare it with Deligne's result [1] (called Principle B in *loc. cit.*): Let S be a smooth connected complex variety and $X \to S$ be a smooth proper morphism. Let t_s be a family of a Hodge classes (i.e. a global section of ...) such that it is absolutely Hodge for one point. Then t_s is absolutely Hodge for all $s \in S$.

Remark 2. We do not known whether the same statement would hold if we replace "de Rham" by "Hodge-Tate", but it fails if we replace "de Rham" by "crystalline".

There are two key ingredients in the proof. The first is the proétale site X_{proet} and period sheaves introduced in [5]. In particular, we have the period sheaf $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$. Let $\nu : X_{\mathrm{proet}} \to X_{\mathrm{et}}$ be the natural projection. Then we show that

Proposition 3. *$\mathcal{E}_i := R^i\nu_*(\mathbb{L} \otimes \mathcal{O}\mathbb{B}_{\mathrm{dR}})$ is a vector bundle on X.*

Proposition 4. *For every $x \in X(k')$, there is a canonical isomorphism $\mathcal{E}_i \otimes_{\mathcal{O}_X} k' \simeq H^i(\mathrm{Gal}(\bar{k}'/k'), \mathbb{L}_{\bar{x}} \otimes B_{\mathrm{dR}})$.*

We outline the proof of Proposition 1. Using the fact that $\mathcal{E}_i$ is an $\mathcal{O}_X$-module with an integrable connection, we reduce to prove that $\mathcal{E}_i$ is a coherent sheaf. The second key ingredient appears in the proof of this claim. Namely, using Falting's p-adic Simpson correspondence (cf. [2], but in fact we need a stronger version as in [3]), one can calculate $\mathcal{E}_i$ using some coherent object.

Note that although to deduce our main theorem, it is enough to use the statement of Proposition 2 for $\mathcal{E}_0$, the proof of Proposition 2 itself uses the local freeness of $\mathcal{E}_i$ for all i as given by Proposition 1.

We have the following application of our main theorem (as discussed with K.-W. Lan).

Let (G, X) be a Shimura datum. Let $K \subset G(\mathbb{A}_f)$ be a (neat) open compact subgroup. Let

$$\mathrm{Sh}_K(G, X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$$

be the corresponding Shimura variety. For a rational representation V of G, let $\mathbb{L}_V$ be the corresponding Betti local system on $Sh_K(G, X)$. The theory of canonical model gives a model of $\mathrm{Sh}_K(G, X)$ (still denoted by the same notation) defined over the reflex field $E \subset \mathbb{C}$, and for a choice of a prime p, a p-adic étale local system $\mathbb{L}_{V,p}$ over $\mathrm{Sh}_K(G, X)$. Note that if $G = T$ is a torus, $\mathrm{Sh}_K(T, X)$ is a finite set equipped with an action of $\mathrm{Gal}(\bar{E}/E)$ and one can show that in this case, $\mathbb{L}_{V,p}$ is de Rham at every point of $\mathrm{Sh}_K(T, X)$. Combining with our main theorem, we arrive at

Corollary 5. *Let F be a finite extension of E and $x \in \mathrm{Sh}_K(G, X)(F)$. Then the p-adic representation $\mathbb{L}_{V,p,x}$ of $\mathrm{Gal}(\bar{F}/F)$ is de Rham at p.*

This theorem is not new if (G, X) is of abelian type, as in the case $\mathrm{Sh}_K(G, X)$ parameterizes certain abelian motives, and $\mathbb{L}_{V,p}$ is the local system of their p-adic realizations. Deligne expected that a Shimura variety corresponding to a Shimura datum with rational weight should always parameterize certain motives. This is not known in general, but the above corollary gives an evidence.

References

[1] P. Deligne (notes by J.S. Milne): *Hodge cycles on abelian varieties*, in Hodge Cycles, Motives, and Shimura Varieties, LNM 900, Springer-Verlage, Berlin 1982.

[2] G. Faltings: *A p-adic Simpson correspondence*, Advances in Mathematics, Volume 198, Issue 2, 20 December 2005, Pages 847-862.

[3] K. Kedlaya, R. Liu: *Relative p-adic Hodge theory, II: Imperfect period rings*, preprint.

[4] R. Liu, X. Zhu: in preparation.

[5] P. Scholze: *p-adic Hodge theory for Rigid-analytic varieties*, Forum of Mathematics, Pi, 1, e1, 2013.

Affine Deligne-Lusztig varieties of higher level and Local Langlands correspondence for $\mathbf{GL_2}$

Alexander Ivanov

Following a fundamental example of Drinfeld for $\mathrm{SL}_2(\mathbb{F}_q)$, Deligne and Lusztig constructed in [6] certain varieties attached to a connected reductive group G over a finite field $\mathbb{F}_q$ and showed that any irreducible representation of $G(\mathbb{F}_q)$ occurs in the alternating sum of the ℓ-adic cohomology of these varieties. Those representations arise in families attached to characters of various tori in G. Thus the Deligne-Lusztig construction provides one with a geometric realization of the automorphic induction for G.

Since then one was trying to find similar constructions in the affine setting, aiming for a geometric realization of the automorphic induction and the local Langlands correspondence for connected reductive groups over local fields. However, usual geometric realizations of local Langlands make use of p-adic methods, formal schemes and adic spaces, also using the global theory. In our approach we consider a very natural generalization of the Deligne-Lusztig theory to the affine setup aiming a purely local approach to the local Langlands correspondence. First of all, the affine counterpart of Deligne-Lusztig varieties are the so called affine Deligne-Lusztig varieties defined by Rapoport in [12]. As the Deligne-Lusztig varieties are equipped with natural étale covers, it is also natural to expect such covers in the affine setup. In contrast to the finite case, there are covers of arbitrary deep level and not only of level zero. In the present talk, based on [8], we discuss a construction of such covers of arbitrary level attached to a connected reductive group over a local field F of positive characteristic. This extends [7], where the covers of level zero were constructed. Using these covers we realize the unramified part of the local Langlands correspondence for GL_2 over F using only schemes over $\mathbb{F}_q$ and purely local methods. Moreover, we give a detailed comparison of our construction with the theory of cuspidal types of Bushnell-Kutzko [3] (we use the language of Bushnell-Henniart [2]). On the 'algebraic' side we show an improvement of the Intertwining theorem [2] 15.1.

There are several closely related approaches: In [9],[10] Lusztig suggested a related construction generalizing the finite Deligne-Lusztig construction to affine setup in a slightly different manner. (A minor variation of) this construction was worked out for division algebras by Boyarchenko [1], Chan [5] and Boyarchenko-Weinstein [4]. A further closely related approach, was given by Stasinski in [13], who suggested a method to construct the so called extended Deligne-Lusztig varieties attached to groups over $\mathbb{F}_q[t]/t^r$. The advantages of our construction are that it (i) has a quite simple definition in terms of the Bruhat-Tits building of G, (ii) establishes a direct link with affine Deligne-Lusztig varieties, which are well-studied in various contexts. In particular, this allows to use the whole combinatoric machinery developed for their study.

To explain the results, we need some notation. Let k be a finite field with q elements, $\bar{k}$ its algebraic closure and let σ denote the Frobenius automorphism $x \mapsto x^q$ of $\bar{k}$. Let $F = k((t))$ resp. $L = \bar{k}((t))$ be the fields of Laurent series over k resp. $\bar{k}$ and $\mathcal{O}_F = k[\![t]\!], \mathcal{O}_L = \bar{k}[\![t]\!]$ their rings of integers. Let $\mathfrak{p}_L \subseteq \mathcal{O}_L$ denote the maximal ideal. We extend σ to an automorphism of L by setting $\sigma(\sum_n a_n t^n) = \sum_n \sigma(a_n) t^n$.

Let G be a connected reductive group over F. Let us recall the definition of affine Deligne-Lusztig varieties of Iwahori-level ([12] Definition 4.1). Let $\mathscr{B}_L$ be the Bruhat-Tits building of the adjoint group $G_{L,ad}$. The Bruhat-Tits building of G_{ad} over F can be identified with the σ-invariant subset of $\mathscr{B}_L$. Let S be a maximal L-split torus in G, which is defined over F. Let $I \subseteq G(L)$ be the Iwahori subgroup attached to a σ-stable alcove in the apartment corresponding to S. Let $\mathscr{F}$ be the affine flag manifold of G, seen as an ind-scheme over k. Its $\bar{k}$-points can be identified with $G(L)/I$. Let $\tilde{W}$ denote the extended affine Weyl group of G attached to S. The Bruhat decomposition of $G(L)$ induces the invariant position map

$$\mathrm{inv}\colon \mathscr{F}(\bar{k}) \times \mathscr{F}(\bar{k}) \to \tilde{W}.$$

For $w \in \tilde{W}$ and $b \in G(L)$ the affine Deligne-Lusztig variety attached to w and b is the locally closed subset

$$X_w(b) = \{gI \in \mathscr{F} \colon \mathrm{inv}(gI, b\sigma(g)I) = w\}$$

of $\mathscr{F}$, which is given its reduced induced sub-Ind-scheme structure. Let J_b be the σ-stabilizer of b, i.e., the algebraic group over F defined by

$$J_b(R) = \{g \in G(R \otimes_F L) \colon g^{-1} b\sigma(g) = b\}$$

for any F-algebra R. Then $J_b(F)$ acts on $X_w(b)$.

We sketch now the construction of natural covers of these varieties, which still admit an action by $J_b(F)$. Let $\Phi = \Phi(G, S)$ be the relative root system. We see 0 as the 'root' corresponding to the centralizer T of S in G (as G is quasi-split, this is a maximal torus). After choosing a σ-stable base point x in $\mathscr{B}_L$, with a concave function f on $\Phi \cup \{0\}$ one can associate a subgroup $G(L)_f \subseteq G(L)$. In [14], Yu defined a smooth model $\underline{G}_f$ of G_L over $\mathcal{O}_L$, such that $\underline{G}_f(\mathcal{O}_L) = G(L)_f$. Assume that $G(L)_f \subseteq I$ and that $G(L)_f$ is σ-stable. Then $\underline{G}_f$ descends to a smooth group scheme over $\mathcal{O}_F$. Further, $G(L)/G(L)_f$ is the set of $\bar{k}$-points of an Ind-scheme $\mathscr{F}^f$, which defines a natural cover of $\mathscr{F}$, as follows from [11] Theorem 1.4. Moreover, if $G(L)_f$ is normal in I, then $\mathscr{F}^f \to \mathscr{F}$ is a (right) principal homogeneous space under $I/G(L)_f$. There is a map

$$\mathrm{inv}^f \colon \mathscr{F}^f(\bar{k}) \times \mathscr{F}^f(\bar{k}) \to D_{G,f},$$

which covers the map inv. Here $D_{G,f}$ is a set of representatives of double cosets of $G(L)_f$ in $G(L)$. For $w_f \in D_{G,f}$, $b \in G(L)$, we define the *affine Deligne-Lusztig variety of level f* attached to w_f and b as the locally closed subset

$$X^f_{w_f}(b) = \{\bar{g} = gG(L)_f \in \mathscr{F}^f(\bar{k}) \colon \mathrm{inv}^f(\bar{g}, b\sigma(\bar{g})) = w_f\},$$

endowed with its induced reduced sub-Ind-scheme structure (in fact, this is a scheme locally of finite type over k). Assume $G(L)_f$ is normal in I. Then I acts on $D_{G,f}$ by σ-conjugation $w_f \mapsto i^{-1} w_f \sigma(i)$. Hence we can consider the stabilizer $I_{f,w_f} \subseteq I$ of w_f under this action. It acts on $X^f_{w_f}(b)$ on the right and this action commutes with the left action of $J_b(F)$. Moreover this I_{f,w_f}-action can be extended to an action of $Z(F)I_{f,w_f}$, where Z is the center of G. Thus we obtain the desired variety $X^f_{w_f}(b)$ with an action of $G(F) \times Z(F)I_{f,w_f}$.

Now we consider the case $G = \mathrm{GL}_2$. As the levels indexed by concave functions are cofinal, we can restrict attention to certain special functions f_m for integers $m \geq 0$ and write m instead of f_m everywhere. We determine the varieties $X^m_{w_m}(1)$ and the attached representations of $G(F)$ and compare our results with the algebraic construction of the same representations in [2] using the theory of cuspidal types. Let E/F be the unramified extension degree 2. If the image of w_m in the finite Weyl group is non-trivial, then $Z(F)I_{m,w_m}$ has a natural quotient isomorphic to E^*, and the $Z(F)I_{m,w_m}$-action in the ℓ-adic cohomology ($\ell \neq \mathrm{char}(F)$) of $X_{w_m}^m(1)$ factors through an E^*-action. In this way we obtain a $G(F)$-representation in the spaces $\mathrm{H}_c^i(X_{w_m}^m(1), \overline{\mathbb{Q}}_\ell)[\chi]$, where χ goes through smooth $\overline{\mathbb{Q}}_\ell^*$-valued characters of E^*. It turns out that if χ is minimal of level m (i.e., lies in sufficiently general position) then there is an integer i_0, such that $\mathrm{H}_c^i(X_{w_m}^m(1), \overline{\mathbb{Q}}_\ell)[\chi] = 0$ for all $i \neq i_0$ and

$$R_\chi = \mathrm{H}_c^{i_0}(X_{w_m}^m(1), \overline{\mathbb{Q}}_\ell)[\chi]$$

is an unramified irreducible cuspidal representation of $G(F)$, of level m (we also define R_χ for χ non-minimal). Let $\mathbb{P}_2^{\mathrm{nr}}(F)$ be the set of all isomorphism classes of admissible pairs over F attached to E/F (cf. [2] 18.2). Let $\mathscr{A}_2^{\mathrm{nr}}(F)$ be the set of all isomorphism classes of unramfied irreducible cuspidal representations of $G(F)$. We defined a map

$$R\colon \mathbb{P}_2^{\mathrm{nr}}(F) \to \mathscr{A}_2^{\mathrm{nr}}(F), \quad (E/F, \chi) \mapsto R_\chi. \tag{1}$$

Computing traces of enough elements acting in (finite-dimensional subspaces of) R_χ, one can show that this map is injective. Using the theory of cuspidal types and strata, Bushnell-Henniart attached to an admissible pair $(E/F, \chi)$ an irreducible cuspidal $G(F)$-representation π_χ ([2] §19). The tame parametrization theorem ([2] 20.2 Theorem) then shows that the map

$$\mathbb{P}_2^{\mathrm{nr}}(F) \xrightarrow{\sim} \mathscr{A}_2^{\mathrm{nr}}(F), \quad (E/F, \chi) \mapsto \pi_\chi$$

is a bijection (also for even q). Here is our main result (which also works for even q).

Theorem 1. *Let $(E/F, \chi)$ be an admissible pair. The representation R_χ is irreducible cuspidal, unramified, has level $\ell(\chi)$ and central character $\chi|_{F^*}$. Moreover, R_χ is isomorphic to π_χ. In particular, the map (1) is a bijection.*

The proof is purely local. Two ideas in the proof follow [1],[4]: it is Boyarchenko's trace formula and maximality of certain closed subvarieties of $X_{w_m}^m(1)$ (note that $X_{w_m}^m(1)$ itself is not maximal due to the presence of a 'level 0 part').

Finally, we remark that for $G = \mathrm{GL}_2$ and b superbasic, $J_b(F) = D^*$ for D a quaternion algebra over F and the varieties $X_{x_m}^m(b)$ seem to be very close (but unequal) to the varieties studied by Chan in [5].

References

[1] Boyarchenko M.: *Deligne-Lusztig constructions for unipotent and p-adic groups*, preprint, arXiv:1207.5876, 2012.

[2] Bushnell C.J., Henniart G.: *The Local Langlands Correspondence for* GL_2, Springer, 2006.

[3] Bushnell C.J., Kutzko P.C.: *The admissible dual of* $\mathrm{GL}(N)$ *via compact open subgroups*, Ann. of Math. Studies, vol. 129, Princeton University Press, 1993.

[4] Boyarchenko M, Weinstein J.: *Maximal Varieties and the local Langlands correspondence for* $\mathrm{GL}(n)$, to appear in JAMS, arXiv:1109.3522.

[5] Chan C.: *Deligne-Lusztig Constructions for Division Algebras and the Local Langlands Correspondence*, 2014, arXiv:1406.6122.

[6] Deligne P., Lusztig G.: *Representations of reductive groups over finite fields*, Ann. of Math. (2), **103**(1), 103-161, 1976.
[7] Ivanov A.: *Cohomology of affine Deligne-Lusztig varieties for* GL_2, J. of Alg. **383** (2013), 42-62.
[8] Ivanov A.: *Affine Deligne-Lusztig varieties of higher level and the local Langlands correspondence for* GL_2, 2015, arXiv:1504.00264.
[9] Lusztig G.: *Some remarks on the supercuspidal representations of p-adic semisimple groups*, in 'Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, Proc. Sympos. Pure Math., XXXIII, pp. 171-175., Amer. Math. Soc., Providence, R.I., 1979.
[10] Lusztig G.: *Representations of reductive groups over finite rings*, Represent. Theory **8** (2004), 1-14.
[11] Pappas G., Rapoport M., *Twisted loop groups and their affine flag varieties*, Adv. in Math. **219**, no. 1, (2008), 118-198.
[12] Rapoport M.: *A guide to the reduction modulo p of Shimura varieties*, Astérisque **298** (2005), 271-318.
[13] Stasinski A.: *Extended Deligne-Lusztig varieties for general and special linear groups*, Adv. in Math. **226**, no. 3, 2011, 2825-2853.
[14] Yu J.-K.: *Smooth models associated to concave functions in Bruhat-Tits theory*, preprint (2002).

Semistable $\mathcal{MF}$

Gerd Faltings

The category $\mathcal{MF}$ has been defined over base rings R which are smooth over an unramified valuation ring $V = V_0$ with perfect residue-field k. We assume given a Frobenius-lift ϕ on R. They consist of filtered finitely generated R-modules M (for technical reasons we usually assume that the Hodge numbers lie between 0 and $p-2$) which are p-torsion, and ϕ-linear maps Φ_i on $F^i(M)$, such that $\Phi_i|F^{i+1}(M) = p\Phi_{i+1}$, and such that the Φ_i induce an isomorphism

$$\Phi\colon \tilde{M} \otimes_{R,\phi} R \cong M.$$

Here $\tilde{M}$ denotes the universal object for which the Φ_i satisfying the condiition above define such a map Φ. Then Frobenius invariance implies that locally in Spec(R) M is isomorphic to $gr_F(M)$ and is isomorphic to the pushout of a V_0-module, and that maps (obvious definition) are strict. For example for maps one uses that ideals generated by minors of a given size contain their Frobenius transforms, thus are generated by powers of p. Also we assume given an integrable connection ∇ which satisfies Griffiths' transversality and for which the Φ_i are parallel. ∇ makes the theory independant of the choice of the Frobenius-lift ϕ.

Fontaine theory defines a fully faithful contravariant functor $\mathbb{D}$ from $\mathcal{MF}$ to Galois-representations by

$$\mathbb{D}(M) = Hom(M, A_{crys}(R) \otimes \mathbb{Q}_p/\mathbb{Z}_p).$$

Here $A_{crys}(R)$ is the ring defined by Fontaine starting with the maximal extension $\bar{R}$ of R which is unramified in characteristic zero. Fully faithfulness is shown over discrete valuation rings by analysing simple objects which correspond to tame

Galois-representations, and reducing the general case to this by defining an adjoint to $\mathbb{D}$.

If we assume that R has only semistable singularities one has to embed $Spec(R)$ into a smooth scheme and pass to the divided powerhull. This is because crystalline cohomology naturally defines crystals over $Spec(R)$ which are determined by their evaluation on the divided powerhull. This is a module with integrable connection. If we try to extend the previous theory we encounter two new difficulties:

Firstly the basering itself admits a Hodge filtrations, and secondly commutative algebra over it is more involved since it is no more regular and noetherian. However these difficulties can be overcome, firstly for objects annihilated by p, by lifting to a smooth ring. After that the previous proofs can be modified to work.

As an application we construct semistable models for certain Shimura varieties defined by spin-groups $GSpin(2n)$. On the associated orthogonal groups the level structure can be described as follows:

Without level structure smooth models have been constructed by A.Vasiu ([3], [4]). On them we have orthogonal vectorbundles E of rank $2n$, together with an isotropic line $F(E)$. For the level structure we assume given a maximal isotropic subspace (of dimension n) of E/pE.

References

[1] G.Faltings, *Crystalline cohomology and p-adic Galois representations*, in Algebraic analysis, geometry and number theory (J.Igusa ed.), John Hopkins University press, Baltimore (1989), p. 25–80.

[2] G.Faltings, *Integral crystalline cohomology over very ramified valuation rings*, Journ. of the AMS **12** (1999), p. 117–144.

[3] A.Vasiu, *Integral canonical models for Shimura varieties of Hodge type*, Thesis, Princeton University (1994).

[4] A.Vasiu, *Integral canonical models for Shimura varieties of preabelian type*, Asian J.Math. **3** (1999), p. 401–518.

Height of CM points on orthogonal Shimura varieties

Fabrizio Andreatta

(joint work with Eyal Goren, Ben Howard, Keerthi Madapusi Pera)

Let E be a CM field of degree $2d$ with totally real subfield F. Colmez [3] has proved that the Faltings height of an abelian variety A over $\mathbb{C}$ of dimension d with complex multiplication by the ring of integers of E and with CM type $\Phi \subset \mathrm{Hom}(E, \mathbb{C})$ depends only on (E, Φ) and not on A itself. We denote such quantity by $h^{\mathrm{Falt}}_{(E,\Phi)}$. Colmez has also provided a conjectural formula for the value of $h^{\mathrm{Falt}}_{(E,\Phi)}$ in terms of the logarithmic derivatives at $s = 0$ of certain Artin L-functions. The first main application of our results is an averaged version of his formula, namely

$$\frac{1}{2^d} \sum_{\Phi} h^{\mathrm{Falt}}_{(E,\Phi)} = -\frac{1}{2} \cdot \frac{L'(0,\chi)}{L(0,\chi)} - \frac{1}{4} \cdot \log \left| \frac{D_E}{D_F} \right| - \frac{d}{2} \log(2\pi).$$

Here χ is the quadratic Hecke character associated to the extension $F \subset E$, the sum on the left is taken over all CM types Φ and D_E and D_F are the discriminants of E and F respectively.

Remark 1. Shortly after our announcement X. Yuan and S.-W. Zhang announced a proof of the same result, but using different methods.

In our first announcement a controlled, error term in log(2) of the formula above appeared. The missing argument to get the precise formula is provided by work in progress by W. Kim and K. Madapusi Pera concerning the properties of integral models of Shimura varieties of orthogonal type for the prime 2.

Recently J. Tsimerman has proved that the averaged Colmez's conjecture implies the André-Oort conjecture for all Siegel modular varieties, without using GRH.

The result is obtained studying the arithmetic intersection between big CM cycles and certain arithmetic Heegner divisors on orthogonal type Shimura varieties, following the strategy outlined by T. Yang [5] and Howard-Yang [4] in the case $d = 2$.

1. GSpin Shimura varieties and Heegner divisors

Let (V, Q) be a quadratic space over $\mathbb{Q}$ of signature $(n, 2)$ with $n \geq 1$, and let $L \subset V$ be a maximal lattice. We set $D_L = [L^\vee : L]$ for the discriminant of L where $L^\vee$ is the dual of L with respect to the bilinear form defined by Q.

To this data one can associate a Shmura variety M defined over $\mathbb{Q}$. Consider the group $G = \mathrm{GSpin}(V)$, a particular compact open subgroup $K \subset G(\mathbb{A}_f)$, and the hermitian domain

$$\mathcal{D} = \{z \in V_{\mathbb{C}} : [z, z] = 0,\ [z, \overline{z}] < 0\}/\mathbb{C}^\times.$$

The complex points of M coincide with the n-dimensional complex manifold

$$M(\mathbb{C}) = G(\mathbb{Q})\backslash\mathcal{D} \times G(\mathbb{A}_f)/K.$$

Thanks to results of A. Vasiu, M. Kisin, K. Madapusi Pera and W. Kim and K. Madapusi Pera it admits a flat and normal integral model $\mathcal{M}$ over $\mathbb{Z}$, which is smooth over $\mathbb{Z}[1/2D_L]$ (there is a caveat for $p = 2$ that we ignore in this short note). Furthermore it is endowed with an arithmetic line bundle $\hat{\omega}$ (called the *tautological bundle*): over $\mathbb{C}$ the fiber over a point $[z] \in \mathcal{D}$ is the line $\mathbb{C}z$ with metric $-[z, \overline{z}]$.

Let $S_L = \mathbb{C}[L^\vee/L]$. For any half-integer k we define the space $S_k(\omega_L)$ of cusp forms, the space $M^!_k(\omega_L)$ of weakly holomorphic forms and the space $H_k(\omega_L)$ of harmonic weak Maass forms. These are vector valued modula forms transforming via the Weil representation ω_L. By a theorem of Bruinier-Funke [1] one has an exact sequence

$$0 \to M^!_{1-\frac{n}{2}}(\omega_L) \to H_{1-\frac{n}{2}}(\omega_L) \xrightarrow{\xi} S_{1+\frac{n}{2}}(\overline{\omega}_L) \to 0,$$

where ξ is a suitable differential operator.

Each form $f \in H_{1-\frac{n}{2}}(\omega_L)$ has a holomorphic part, which is a formal q-expansion

$$f^+ = \sum_{\substack{m \gg -\infty \\ \mu \in L^\vee/L}} c_f^+(m,\mu)\varphi_\mu \cdot q^m$$

valued in S_L. Here $\varphi_\mu \in S_L$ is the characteristic function of the coset $\mu \in L$. If $c_f^+(m,\mu) \in \mathbb{Z}$ for all $m \leq 0$ and $\mu \in L^\vee/L$, then one can associate to f a metrized line bundle $\widehat{\mathcal{Z}}(f)$.

2. Big CM cycles and the Bruinier-Kudla-Yang theorem

Given an element $\lambda \in F$ such that λ is negative for one real embedding ι_0 of F and positive for all the others, we can associate a quadratic space $V(Q)$ of signature $(2d-2,2)$ as follows. We put $V := E$ and $Q(x) = \mathrm{Tr}_{E/\mathbf{Q}}(\lambda x\bar{x})$. The torus $T_E := \mathrm{Res}_{E/\mathbb{Q}}(\mathbb{G}_m)$ acts via isometries on V: given $\alpha \in E^*$ we let it act on $V = E$ via multiplication by $x\bar{x}^{-1}$. One can lift such a map to a morphism of algebraic groups $T_E \to G$. The image $T \subset G$ is a torus, the choice of an embedding ι_0 of E over the given one of F and the choice of a suitable open compact subgroup of $T(\mathbb{A}_f)$ provides a 0-dimensional Shimura variety Y and a morphism of Shimura varieties $Y \to M$. The image consists of special points called *big CM points* in [2]. Upon taking integral models we get an arithmetic curve $\mathcal{Y}$ and a morphism $\mathcal{Y} \to \mathcal{M}$. Given a metrized line bundle $\widehat{\mathcal{L}}$ over $\mathcal{M}$ we define its arithmetic degree $[\mathcal{Y} : \widehat{\mathcal{L}}]$ along $\mathcal{Y}$ to be the arithmetic degree of its pull–back to $\mathcal{Y}$.

Our main result, from which the application to Colmez's conjecture follows, is the following theorem conjectured in [2]:

Theorem 1. *There exists an explicitly defined integer $D_{bad,L}$, depending the lattice L, such that for any $f \in H_{2-d}(\omega_L)$ with integral principal part, we have the equality*

$$\frac{[\widehat{\mathcal{Z}}(f) : \mathcal{Y}]}{\deg_{\mathbb{C}}(Y)} = -\frac{\mathcal{L}'(0,\xi(f))}{\Lambda(0,\chi)} + \frac{a(0,0)\cdot c_f^+(0,0)}{\Lambda(0,\chi)}$$

up to a $\mathbb{Q}$-linear combination of $\{\log(p) : p \mid D_{bad,L}\}$.

Here $\deg_{\mathbb{C}}(Y) = \sum_{y \in Y(\mathbb{C})} \frac{1}{|\mathrm{Aut}(y)|}$, the quantity $a(0,0)$ is defined in [2], using a certain Hilbert modular Eisenstein series of parallel weight 1, and $\Lambda(s,\chi)$ is the completed L-function of the quadratic character χ.

The archimedean contribution being computed already in [2], the key point is the computation of the finite intersection multiplicities of $[\widehat{\mathcal{Z}}(f) : \mathcal{Y}]$ and the comparison with the quantities appearing on the RHS.

3. Application to Colmez's conjecture

Choose an harmonic weak Maass form f so that $\xi(f) = 0$ via the Bruinier-Funke operator. Combining the Theorem with an explicit computation of $a(0,0)$ one gets

$$\frac{[\widehat{\mathcal{Z}}(f) : \mathcal{Y}]}{\deg_{\mathbb{C}}(Y)} \approx_L -c_f(0,0)\cdot\frac{2\Lambda'(0,\chi)}{\Lambda(0,\chi)}, \tag{1}$$

where $\approx_L$ means equality up to a $\mathbb{Q}$-linear combination of $\log(p)$ with $p \mid D_{bad,L}$.

After possibly replacing f by a positive integer multiple, the theory of Borcherds products provides a rational section $\Psi(f)$ of the line bundle $\omega^{\otimes c_f(0,0)}$, satisfying

$$-\log||\Psi(f)||^2 = \Phi(f) - c_f(0,0)\log(4\pi e^\gamma),$$

and $\mathrm{div}(\Psi(f)) = \mathcal{Z}(f)$. Combining this with (1) and dividing by $c_f(0,0)$ leaves

$$\frac{[\widehat{\omega}:\mathcal{Y}]}{\deg_{\mathbb{C}}(Y)} + d\cdot\log(4\pi e^\gamma) \approx_L -\frac{2\Lambda'(0,\chi)}{\Lambda(0,\chi)} + \frac{1}{c_f(0,0)}\frac{[\widehat{\mathcal{E}}_2(f):\mathcal{Y}]}{\deg_{\mathbb{C}}(Y)}.$$

One proves that the pullback to $\mathcal{Y}$ of the metrized line bundle $\widehat{\omega}$ computes Faltings heights of suitable CM abelian varieties, implying that

$$\frac{[\widehat{\omega}:\mathcal{Y}]}{\deg_{\mathbb{C}}(Y)} \approx_L \frac{1}{2^{d-2}}\sum_{\Phi} h^{\mathrm{Falt}}_{(E,\Phi)} + 2d\cdot\log(2\pi).$$

Putting it all together, we find that

$$\frac{1}{2^d}\sum_{\Phi} h^{\mathrm{Falt}}_{(E,\Phi)} = -\frac{1}{2}\cdot\frac{L'(0,\chi)}{L(0,\chi)} - \frac{1}{4}\cdot\log\left|\frac{D_E}{D_F}\right| - \frac{d}{2}\log(2\pi) + \sum_p b_E(p)\log(p)$$

for some rational numbers $b_E(p)$, with $b_E(p) = 0$ for all p not dividing $D_{bad,L}$.

The last step consists in proving that for every prime p there exists an L such that p does not divide $D_{bad,L}$. This implies that $b_p(E) = 0$ proving the averaged version of Colmez's conjecture.

References

[1] J. H. Bruinier, J. Funke, *On two geometric theta lifts.* Duke Math. J. **125** (2004), 45–90.

[2] J. H. Bruinier Bruinier, S. Kudla, T. Yang, *Special values of Green functions at big CM points.* IMRN **9** (2012), 1917–1967.

[3] P. Colmez, *Périodes des variétés abéliennes à multiplication complexe.* Ann. of Math. **138** (1993), 625–683.

[4] B. Howard, T. Yang, *Intersections of Hirzebruch-Zagier divisors and CM cycles.* LNM **2041** (2012).

[5] T. Yang, *Arithmetic intersection on a Hilbert modular surface and the Faltings height.* Asian J. Math. **17** (2013), 335–381.

Generic Tate cycles on some Shimura varieties over finite fields

Yichao Tian

(joint work with David Helm and Liang Xiao)

The supersingular locus of the characteristic p of a Shimura variety has been studied extensively. Most previous works such as [1, 2, 6, 8] mainly focus on the geometric properties of the supersingular locus. In this talk, which is based on [3], we will explain its relationship with Tate conjecture over finite fields. The general principle behind this work is that, in an even dimensional Shimura variety, the supersingular locus contributes to the generic Tate cycles of middle dimension.

We illustrate this principle by examining an explicit example of unitary Shimura variety.

Let F be a real quadratic field, and E be an imaginary quadratic extension of F. We fix a prime p, which is inert in F and splits in E. Denote by v and $\bar{v}$ the two places of E above p. Let $(D,*)$ be a E-central division algebra with positive involution of dimension n^2 such that the restriction of $*$ to E is the complex conjugation c and that $D^{\mathrm{opp}} \cong D \otimes_{E,c} E$. We assume that D splits at p, i.e. $D \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \mathrm{M}_{n\times n}(E_v) \times \mathrm{M}_{n\times n}(E_{\bar{v}})$.

Consider $V = D$ as a left D-module, equipped with a skew-Hermitian pairing

$$\psi : V \times V \to \mathbb{Q}$$

of the form $\psi(x,y) = \mathrm{Tr}_{D/\mathbb{Q}}(x\delta y^*)$, where δ is some element of $D^\times$ with $\delta^* = -\delta$ to be fixed later. We denote by $G = GU(V,\psi)$ the unitary similitude group of (V,ψ). We consider G as an algebraic group over $\mathbb{Q}$, and we choose δ such that $GU(\mathbb{R}) \cong G(U(1,n-1)\times U(n-1,1))$. Note that $G(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times \times \mathrm{GL}_n(E_v)$, since D is split at p.

Fix an order $\mathcal{O}_D$ of D maximal at p such that $\mathcal{O}_D \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is self-dual with respect to ψ. Let $K_p = \mathbb{Z}_p^\times \times \mathrm{GL}_n(\mathcal{O}_{E_v}) \subseteq G(\mathbb{Q}_p)$, and K^p be a sufficiently small open compact subgroup of $G(\mathbb{A}^{\infty,p})$. We consider the Shimura variety $\mathcal{S}h_{1,n-1}$ associated with G of level $K = K^pK_p$. This is a Shimura variety of PEL-type which parametrizes abelian varieties of dimension $2n^2$ equipped with an action by $\mathcal{O}_D$, a prime-to-p $\mathcal{O}_D$-linear polarization and a K^p-level structure, and satisfying certain Kottwitz's determinant condition of signature $(1,n-1)\times(n-1,1)$. Following Kottwitz [4], it is well known that $\mathcal{S}h_{1,n-1}$ admits a canonical integral model over $\mathbb{Z}_{p^2}$. We denote by $\mathrm{Sh}_{1,n-1}$ its special fiber over $\mathbb{F}_{p^2}$. This is a projective smooth algebraic variety over $\mathbb{F}_{p^2}$ of dimension $2(n-1)$.

We consider now the étale cohomology group $H^{2(n-1)}_{\mathrm{et}}(\mathrm{Sh}_{1,n-1,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(n-1))$, which is equipped with natural actions by the Galois group $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})$ and the Hecke algebra $\mathscr{H}_K := \overline{\mathbb{Q}}_\ell[K\backslash G(\mathbb{A})/K]$. Since $\mathrm{Sh}_{1,n-1}$ is projective and smooth, there is no continuous Hecke spectrum involved, and we have a canonical decomposition:

$$H^{2(n-1)}(\mathrm{Sh}_{1,n-1,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(n-1)) = \bigoplus_{\pi_f \in \mathrm{Irr}(G(\mathbb{A}^\infty))} \pi_f^K \otimes R_\ell(\pi_f), \tag{1}$$

where $\mathrm{Irr}(G(\mathbb{A}^\infty))$ denote the set of irreducible admissible representations of $G(\mathbb{A}^\infty)$, π_f^K denotes the K-invariant subspace for each $\pi_f \in \mathrm{Irr}(G(\mathbb{A}^\infty))$, and $R_\ell(\pi_f)$ is a certain representation of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})$ which are easy to describe in some special cases.

From now on, we assume that $\pi_f = \bigotimes'_{q<\infty} \pi_q \in \mathrm{Irr}(G(\mathbb{A}^\infty))$ is an object satisfying the following conditions:

(1) We have $\pi_f^K \neq 0$. In particular, π is unramified at p.

(2) There exists an irreducible admissible $(\mathrm{Lie}(G_{\mathbb{C}}), K_\infty)$-module π_∞ such that $\pi_f \otimes \pi_\infty$ is a cuspidal automorphic representation of G, and π_∞ is cohomological in degree $2(n-1)$ for the constant coefficient, i.e. we have

$$H^{2(n-1)}(\mathrm{Lie}(G_{\mathbb{C}}), K_\infty; \pi_\infty) \neq 0.$$

Here, K_∞ denotes a maximal compact subgroup of $G(\mathbb{R})$.

We now describe the Galois representation $R_\ell(\pi_f)$. Since $G(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times \times \mathrm{GL}_n(E_v)$, the p-component π_p of π_f has a natural decomposition $\pi_p = \pi_{p,0} \otimes \pi_v$, where $\pi_{p,0}$ and π_v are respectively irreducible admissible representations of $\mathbb{Q}_p^\times$ and $\mathrm{GL}_n(E_v)$. By (unramified) local Langlands correspondence, one can associate to π_v a local Galois representation

$$\rho_{\pi_v} : \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2}) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$$

normalized so that, if $\mathrm{Frob}_{p^2} \in \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})$ denotes the geometric Frobenius, $\rho_{\pi_v}(\mathrm{Frob}_{p^2})$ is semi-simple with characteristic polynomial

$$X^n + \sum_{i=1}^{n} (-1)^i p^{i(i-1)} a_v^{(i)} X^{n-i}, \tag{2}$$

where $a_v^{(i)}$ is eigenvalue of the i-th minuscule Hecke operator

$$T_v^{(i)} = \mathrm{GL}_n(\mathcal{O}_{E_v})\mathrm{diag}(\underbrace{p, \cdots, p}_{i}, \underbrace{1, \cdots, 1}_{n-i})\mathrm{GL}_n(\mathcal{O}_{E_v})$$

on $\pi_v^{\mathrm{GL}_n(\mathcal{O}_{E_v})}$. Then a result of Kottwitz [5] shows that we have, in the Grothendieck group of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})$-modules,

$$[R_\ell(\pi_f)] = m_G(\pi_f) \cdot [\rho_{\pi_v} \otimes \wedge^{n-1}\rho_{\pi_v} \otimes \chi_{p,0}^{-1} \otimes \overline{\mathbb{Q}}_\ell(\frac{(n-1)(n-2)}{2})],$$

where $\chi_{p,0} : \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2}) \to \overline{\mathbb{Q}}_\ell^{\times}$ is the character sending Frob_{p^2} to $\pi_{p,0}(p^2)$, and $m_G(\pi_f)$ is the automorphic multiplicity of π_f for the group G.

We put

$$H^{2(n-1)}(\mathrm{Sh}_{1,n-1,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(n-1))_{\pi_f} : = \pi_f^K \otimes R_\ell(\pi_f),$$

and

$$H^{2(n-1)}(\mathrm{Sh}_{1,n-1,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(n-1))_{\pi_f}^{G_{\mathbb{F}_{p^2}}-\mathrm{fin}} : = \bigcup_{\mathbb{F}_q/\mathbb{F}_{p^2}} \pi_f^K \otimes R_\ell(\pi_f)^{\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_q)},$$

where $\mathbb{F}_q$ runs through all finite extensions of $\mathbb{F}_{p^2}$. By Tate conjecture, one expects that the subspace $H^{2(n-1)}(\mathrm{Sh}_{1,n-1,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(n-1))_{\pi_f}^{G_{\mathbb{F}_{p^2}}-\mathrm{fin}}$ is generated by the cycle classes of algebraic cycles on $\mathrm{Sh}_{1,n-1,\overline{\mathbb{F}}_p}$ of codimension $n-1$. The main result of [3] shows this is indeed the case, and actually the desired algebraic cycles come from the supersingular locus of $\mathrm{Sh}_{1,n-1}$.

To make the statement precise, we still need the following notation. Note that there exists a unique unitary group G' over $\mathbb{Q}$ (for the CM extension E/F) such that $G'(\mathbb{R}) \cong G(U(0,n) \times U(n,0))$ and $G'(\mathbb{A}^\infty) \cong G(\mathbb{A}^\infty)$. Viewing K as an open compact subgroup of $G'(\mathbb{A}^\infty)$, there is a zero-dimensional Shimura variety $\mathrm{Sh}_{0,n}$ defined over $\mathbb{F}_{p^2}$ of level K attached to G'. It has a similar interpretation as moduli of abelian schemes equipped with an $\mathcal{O}_D$-action.

Thus π_f can be also viewed as an irreducible admissible representation of $G'(\mathbb{A}^\infty)$. Denote by $m_{G'}(\pi_f)$ the corresponding automorphic multiplicity of π_f for G'. The main result of [3] is the following

Theorem 1. *(i) The supersingular locus of* $\mathrm{Sh}_{1,n-1}$ *is the union of* n *closed subvarieties* Y_i *for* $1 \le i \le n$*, such that each* Y_i *is a fibration* $\mathrm{pr}_i : Y_i \to \mathrm{Sh}_{0,n}$ *with geometric fibers isomorphic to*

$$Z_k^{<n>} :\, = \{(H_1, H_2) \in \mathbf{Gr}(n,i) \times \mathbf{Gr}(n, i-1) | H_1^{(p)} \supseteq H_2, H_2^{(p)} \subseteq H_1\},$$

where $\mathbf{Gr}(n,i)$ *(resp.* $\mathbf{Gr}(n,i-1)$*) is the Grassmanian variety over* $\overline{\mathbb{F}}_p$ *which classifies* i*-dimensional (resp.* $(i-1)$*-dimensional) subspaces within an* n*-dimensional space, and* $H_1^{(p)}$ *and* $H_2^{(p)}$ *denotes the Frobenius twist of* H_1 *(resp.* H_2*).*

(ii) Let π_f *be an object of* $\mathrm{Irr}(G(\mathbb{A}_f))$ *as above. Assume that the eigenvalues of* $\rho_{\pi_v}(\mathrm{Frob}_{p^2})$ *(i.e. the roots of (2)) are distinct. Then the map*

$$\mathcal{JL}_{\pi_f} : H^0(\mathrm{Sh}_{0,n,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell) \xrightarrow{\mathrm{pr}_i^*} \bigoplus_{i=1}^n H^0(Y_{i,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)_{\pi_f} \xrightarrow{\sum_i \mathrm{Gys}_i} H^{2(n-1)}(\mathrm{Sh}_{1,n-1,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(n-1))_{\pi_f}$$

is injective, where Gys_i *is the Gysin map associated to the closed immersion* $Y_i \hookrightarrow \mathrm{Sh}_{1,n-1}$.

(iii) Suppose moreover that any two eigenvalues of $\rho_{\pi_v}(\mathrm{Frob}_{p^2})$ *do not differ by a root of unity, and* $m_G(\pi_f) = m_{G'}(\pi_f)$*. Then* $\mathcal{JL}_{\pi_f}$ *induces an isomorphism*

$$H^0(\mathrm{Sh}_{0,n,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)_{\pi_f} \xrightarrow{\sim} H^{2(n-1)}(\mathrm{Sh}_{1,n-1,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(n-1))_{\pi_f}^{G_{\mathbb{F}_{p^2}}-\mathrm{fin}}.$$

In particular, $H^{2(n-1)}(\mathrm{Sh}_{1,n-1,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(n-1))_{\pi_f}^{G_{\mathbb{F}_{p^2}}-\mathrm{fin}}$ *is generated by cycle classes of the irreducible components of the super-singular locus of* $\mathrm{Sh}_{1,n-1,\overline{\mathbb{F}}_p}$.

Remark 2. (1) The assumption that $m_G(\pi_f) = m_{G'}(\pi_f)$ is satisfied when π_f is the finite part of an automorphic representation of $G(\mathbb{A}_Q)$ which admits a cuspidal base change to $\mathbb{G}_m \times \mathrm{GL}_{n,E}$. Indeed, in this case, we have $m_G(\pi_f) = m_{G'}(\pi_f)$ by [9, Theorem E].

(2) The assumption in (ii) on the distinctness of the Frobenius eigenvalues is crucial for our method. If this assumption is not satisfied, it seems that the supersingular locus is not sufficient to give rise to all Tate cycles, and we do not know how to construct extra algebraic cycles to meet Tate conjecture.

(3) Similar results have been obtain in [7] for even dimensional Hilbert modular varieties when the prime p is inert in the totally real field.

Let us say a few words on the proof of the main theorem. First, the cycles Y_i's are explicitly constructed as the moduli spaces of isogenies between abelian schemes parametrized by $\mathrm{Sh}_{1,n-1}$ and those by $\mathrm{Sh}_{0,n}$. To prove Theorem 1(ii), we compute the intersection matrix of these cycles Y_i for $1 \leq i \leq n$. The most technical part is to compute the self-intersection of the each Y_i. It turns out that the image of this matrix in the π_f isotypical component of the cohomology group is non-degenerate if and only if the roots of (2) are distinct. Theorem 1(ii) then follows immediately from this fact. Statement (iii) is a direct consequence of (ii) together with some easy computation on the Galois representation $R_\ell(\pi_f)$.

References

[1] U. Görtz and X. He, Basic loci in Shimura varieties of Coxeter type, arXiv:1311.6263.

[2] B. Howard and G. Pappas, On the supersingular locus of the $GU(2,2)$ Shimura variety, *Algebra and Number Theory* **8** (2014), 1659–1699.

[3] D. Helm, Y. Tian, and L. Xiao, On Tate conjecture for the special fibres of some unitary Shimura varieties, Preprint in 2014, arXiv:1410.2343.

[4] R. Kottwitz, Points on some Shimura varieties over finite fields. *J. Amer. Math. Soc.* **5** (1992), no. 2, 373–444.

[5] R. Kottwitz, On the λ-adic representations associated to some simple Shimura varieties, *Inven. Math.* **108** (1992), 653–665.

[6] M. Rapoport, U. Terstiege, and S. Wilson, The supersingular locus of the Shimura variety for $GU(1, n-1)$ over a ramified prime, *Math. Z.* **276** (2014), no. **3-4**, 1165–1188.

[7] Y. Tian and L. Xiao, Tate cycles on quaternionic Shimura varieties over finite fields, arXiv:1410.2321

[8] I. Vollaard and T. Wedhorn, The supersingular locus of the Shimura variety of $GU(1, n-1)$ II, *Inventiones Math.* **184** (2011), 591–627.

[9] P. J. White, Tempered automorphic representation of the unitary group, preprint in 2012.

Shtukas and the Taylor expansion of *L*-functions over a function field

Wei Zhang

(joint work with Zhiwei Yun)

In this talk, we explained a joint work with Zhiwei Yun [3].

Let $k = \mathbb{F}_q$ be a finite field of characteristic $p > 0$. Let X be a geometrically connected smooth proper curve over k. Let $\nu : X' \to X$ is a finite étale cover of degree 2 such that X' is also geometrically connected. Let $F = k(X)$ and $F' = k(X')$ be their function fields.

Let $G = \mathrm{PGL}_2$ and $T = (\mathrm{Res}_{F'/F}\mathbb{G}_m)/\mathbb{G}_m$ the non-split torus associated to the double cover X' of X. Let Bun_2 be the stack of rank two vector bundles on X. The Picard stack Pic_X acts on Bun_2 by tensoring a line bundle. Then $\mathrm{Bun}_G = \mathrm{Bun}_2/\mathrm{Pic}_X$ is the moduli stack of G-torsors over X.

1. The Heegner–Drinfeld cycle

Let r be an *even* integer. Let $\mu \in \{\pm\}^r$ be an r-tuple of signs such that exactly half of them are equal to $+$. The Hecke stack Hk_2^μ is the stack whose S-points is the groupoid of the data $(\mathcal{E}_0, ..., \mathcal{E}_r, x_1, ..., x_r, f_1, ..., f_r)$ where $\mathcal{E}_i$'s are vector bundles of rank two over $X \times S$, x_i's are S-points of X, and each f_i is a minimal upper (i.e., increasing) modification if $\mu_i = +$, and minimal lower (i.e., decreasing) modification if $\mu_i = -$, and the modification takes place along the graph of x_i

$$\mathcal{E}_0 \overset{f_1}{\dashrightarrow} \mathcal{E}_1 \overset{f_2}{\dashrightarrow} \cdots \overset{f_r}{\dashrightarrow} \mathcal{E}_r \ .$$

The Picard stack Pic_X acts on Hk_2^μ by simultaneously tensoring a line bundle. Define $\mathrm{Hk}_G^\mu = \mathrm{Hk}_2^\mu / \mathrm{Pic}_X$. Assigning $\mathcal{E}_i$ to the data above descends to a morphism $p_i : \mathrm{Hk}_G^\mu \to \mathrm{Bun}_G$.

The moduli stack Sht_G^μ of Drinfeld G-Shtukas with r-modifications of type μ for the group G is defined by the following cartesian diagram

$$\begin{array}{ccc} \mathrm{Sht}_G^\mu & \longrightarrow & \mathrm{Hk}_G^\mu \\ \downarrow & & \downarrow{\scriptstyle (p_0,p_r)} \\ \mathrm{Bun}_G & \xrightarrow{(\mathrm{id},\mathrm{Fr})} & \mathrm{Bun}_G \times \mathrm{Bun}_G \end{array} \tag{1}$$

The stack Sht_G^μ is a Deligne-Mumford stack over X^r and the natural morphism

$$\pi_G^\mu : \mathrm{Sht}_G^\mu \longrightarrow X^r$$

is smooth of relative dimension r, and locally of finite type. We remark that Sht_G^μ as a stack over X^r is canonically independent of the choice of μ. The stack Sht_T^μ of T-Shtukas is defined analogously, with the $\mathcal{E}_i$ replaced by line bundles on X', and the points x_i on X'. Then we have a map

$$\pi_T^\mu : \mathrm{Sht}_T^\mu \longrightarrow X'^r$$

which is a torsor under the finite Picard stack $\mathrm{Pic}_{X'}(k)/\mathrm{Pic}_X(k)$. In particular, Sht_T^μ is a proper smooth Deligne-Mumford stack over $\mathrm{Spec} k$.

There is a natural finite morphism of stacks over X^r

$$\mathrm{Sht}_T^\mu \longrightarrow \mathrm{Sht}_G^\mu \ .$$

It induces a finite morphism

$$\theta^\mu : \mathrm{Sht}_T^\mu \longrightarrow \mathrm{Sht}_G'^\mu := \mathrm{Sht}_G^\mu \times_{X^r} X'^r \ .$$

This defines a class in the Chow group of proper cycles of dimension r with $\mathbb{Q}$-coefficient

$$\theta_*^\mu [\mathrm{Sht}_T^\mu] \in \mathrm{Ch}_{c,r}(\mathrm{Sht}_G'^\mu)_\mathbb{Q}.$$

In analogy to the classical Heegner cycles [1] in the number field case, we will call $\theta_*^\mu[\mathrm{Sht}_T^\mu]$ the *Heegner–Drinfeld cycle* in our setting.

2. The spectral decomposition of the cycle space

We denote the set of closed points (places) of X by $|X|$. For $x \in |X|$, let $\mathcal{O}_x$ be the completed local ring of X at x and let F_x be its fraction field. Let $\mathbb{A} = \prod'_{x \in |X|} F_x$ be the ring of adeles, and $\mathbb{O} = \prod_{x \in |X|} \mathcal{O}_x$ the ring of integers inside $\mathbb{A}$. Let $K = \prod_{x \in |X|} K_x$ where $K_x = G(\mathcal{O}_x)$. The (spherical) Hecke algebra $\mathscr{H}$ is the

$\mathbb{Q}$-algebra of bi-K-invaraint functions $C_c^\infty(G(\mathbb{A})/\!\!/K, \mathbb{Q})$ with the product given by convolution.

Let $\mathcal{A} = C_c^\infty(G(F)\backslash G(\mathbb{A})/K, \mathbb{Q})$ be the space of everywhere unramified $\mathbb{Q}$-valued automorphic functions for G. Then $\mathcal{A}$ is an $\mathscr{H}$-module. By an everywhere unramified cuspidal automorphic representation π of $G(\mathbb{A})$ we mean an $\mathscr{H}$-submodule $\mathcal{A}_\pi \subset \mathcal{A}$ that is irreducible over $\mathbb{Q}$. For every such π, $\mathrm{End}_{\mathscr{H}}(\mathcal{A}_\pi)$ is a number field E_π, which we call the *coefficient field* of π. Then by the commutativity of $\mathscr{H}$, $\mathcal{A}_\pi$ is a one-dimensional E_π-vector space.

The Hecke algebra $\mathscr{H}$ acts on the Chow group $\mathrm{Ch}_{c,r}(\mathrm{Sht}'^{\mu}_G)_{\mathbb{Q}}$ via Hecke correspondences. Let $\widetilde{W} \subset \mathrm{Ch}_{c,r}(\mathrm{Sht}'^{\mu}_G)_{\mathbb{Q}}$ be the sub $\mathscr{H}$-module generated by the Heegner–Drinfeld cycle $\theta^\mu_*[\mathrm{Sht}^\mu_T]$. There is a bilinear and symmetric intersection pairing

$$\langle \cdot, \cdot \rangle_{\mathrm{Sht}'^{\mu}_G} : \widetilde{W} \times \widetilde{W} \longrightarrow \mathbb{Q}. \tag{1}$$

Let $\widetilde{W}_0$ be the kernel of the pairing. The quotient $W := \widetilde{W}/\widetilde{W}_0$ is then equipped with a *non-degenerate* pairing induced from $\langle \cdot, \cdot \rangle_{\mathrm{Sht}'^{\mu}_G}$

$$(\cdot, \cdot) : W \times W \longrightarrow \mathbb{Q}\ .$$

The Hecke algebra $\mathscr{H}$ acts on W.

Let π be an everywhere unramified cuspidal automorphic representation of G with coefficient field E_π, and let $\lambda_\pi : \mathscr{H} \to E_\pi$ be the associated character, whose kernel $\mathfrak{m}_\pi$ is a maximal ideal of $\mathscr{H}$. Let

$$W_\pi = \mathrm{Ann}(\mathfrak{m}_\pi) \subset W \tag{2}$$

be the λ_π-eigenspace of W. This is an E_π-vector space. Let $\mathcal{I}_{\mathrm{Eis}} \subset \mathscr{H}$ be the Eisenstein ideal (cf. [3]). Informally speaking, this is the annihilator of the Eisenstein spectrum in the space of automorphic functions $\mathcal{A}$. Define

$$W_{\mathrm{Eis}} = \mathrm{Ann}(\mathcal{I}_{\mathrm{Eis}}).$$

Theorem 1. *We have an orthogonal decomposition of $\mathscr{H}$-modules*

$$W = W_{\mathrm{Eis}} \oplus \left(\bigoplus_\pi W_\pi \right), \tag{3}$$

where π runs over the finite set of everywhere unramified cuspidal automorphic representation of G, and W_π is an E_π-vector space of dimension at most one.

The $\mathbb{Q}$-bilinear pairing $(\cdot, \cdot)$ on W_π can be lifted to an E_π-bilinear symmetric pairing

$$(\cdot, \cdot)_\pi : W_\pi \times W_\pi \longrightarrow E_\pi \tag{4}$$

where for $w, w' \in W_\pi$, $(w, w')_\pi$ is the unique element in E_π such that $\mathrm{Tr}_{E_\pi/\mathbb{Q}}(e \cdot (w, w')_\pi) = (ew, w')$ for all $e \in E_\pi$.

3. Taylor expansion of L-functions

Let π be an everywhere unramified cuspidal automorphic representation of G with coefficient field E_π. The standard L-function $L(\pi, s)$ is a polynomial of degree $4(g-1)$ in $q^{-s-1/2}$ with coefficients in E_π, where g is the genus of X. Let $\pi_{F'}$ be the base change to F', and let $L(\pi_{F'}, s)$ be its standard L-function. This L-function is a product of two L-functions associated to cuspidal automorphic representations of G over F:

$$L(\pi_{F'}, s) = L(\pi, s)L(\pi \otimes \eta_{F'/F}, s),$$

where

$$\eta_{F'/F} : F^\times \backslash \mathbb{A}^\times / \mathbb{O}^\times \longrightarrow \{\pm 1\}$$

is the character corresponding to the étale double cover X' via class field theory. The function $L(\pi_{F'}, s)$ satisfies a functional equation

$$L(\pi_{F'}, s) = \epsilon(\pi_{F'}, s)L(\pi_{F'}, 1-s),$$

where $\epsilon(\pi_{F'}, s) = q^{-8(g-1)(s-1/2)}$. Let $L(\pi, \mathrm{Ad}, s)$ be the adjoint L-function of π. Denote

$$\mathscr{L}(\pi_{F'}, s) = \epsilon(\pi_{F'}, s)^{-1/2} \frac{L(\pi_{F'}, s)}{L(\pi, \mathrm{Ad}, 1)}, \tag{1}$$

where the the square root is understood as $\epsilon(\pi_{F'}, s)^{-1/2} := q^{4(g-1)(s-1/2)}$. In particular, we have a functional equation:

$$\mathscr{L}(\pi_{F'}, s) = \mathscr{L}(\pi_{F'}, 1-s).$$

Consider the Taylor expansion at the central point $s = 1/2$:

$$\mathscr{L}(\pi_{F'}, s) = \sum_{r \geq 0} \mathscr{L}^{(r)}(\pi_{F'}, 1/2) \frac{(s-1/2)^r}{r!},$$

i.e.,

$$\mathscr{L}^{(r)}(\pi_{F'}, 1/2) = \frac{d^r}{ds^r}\Big|_{s=0} \left(\epsilon(\pi_{F'}, s)^{-1/2} \frac{L(\pi_{F'}, s)}{L(\pi, \mathrm{Ad}, 1)} \right).$$

If r is odd, by the functional equation we have

$$\mathscr{L}^{(r)}(\pi_{F'}, 1/2) = 0.$$

Since $\mathscr{L}(\pi_{F'}, s) \in E_\pi[q^{-s-1/2}, q^{s-1/2}]$, we see that

$$\mathscr{L}^{(r)}(\pi_{F'}, 1/2) \in E_\pi \cdot (\log q)^r. \tag{2}$$

Then our main result in [3] relates the r-th Taylor coefficient to the self-intersection number of the π-component of the Heegner–Drinfeld cycle $\theta^\mu_*[\mathrm{Sht}^\mu_T]$ on the stack Sht'^μ_G.

Theorem 1. *Let π be an everywhere unramified cuspidal automorphic representation of G with coefficient field E_π. Let $[\mathrm{Sht}^\mu_T]_\pi \in W_\pi$ be the projection of the* *image of* $\theta^\mu_*[\mathrm{Sht}^\mu_T] \in \widetilde{W}$ *in* W *to the direct summand* W_π *under the decomposition* (3)*. Then we have an equality in* E_π

$$\frac{1}{2(\log q)^r}\,|\omega_X|\,\mathscr{L}^{(r)}(\pi_{F'},1/2) = \Big([\mathrm{Sht}^\mu_T]_\pi, \quad [\mathrm{Sht}^\mu_T]_\pi\Big)_\pi,$$

where ω_X *is the canonical divisor, and* $|\omega_X| = q^{-2g+2}$*.*

Remark 2. When $r = 0$, this formula is equivalent to the special case of Waldspurger formula [2] for unramified π, relating the automorphic period integral to the central value of the L-function of $\pi_{F'}$

$$\Big|\int_{T(F)\backslash T(\mathbb{A})} \varphi(t)dt\Big|^2 = \frac{1}{2}\,|\omega_X|\,\mathscr{L}(\pi_{F'},1/2),$$

where $\varphi \in \pi^K$ is normalized such that the Petersson inner product $(\varphi,\varphi) = 1$, and the measure on $G(\mathbb{A})$ is such that $\mathrm{vol}(K) = 1$, and the measure on $T(\mathbb{A})$ is such that the maximal compact open subgroup has volume one.

Remark 3. In [3] we only consider the everywhere unramified situation where the L-function has nonzero Taylor coefficients in even degrees only. But the same construction with slight modifications should work in the ramified case as well, where the L-function may have nonzero Taylor coefficients in odd degrees. The case $r = 1$ would then give an analog of the Gross–Zagier formula [1] in the function field case.

References

[1] B. Gross, D. Zagier, *Heegner points and derivatives of L-series.* Invent. Math. 84 (1986), no. 2, 225–320.

[2] J. Waldspurger, *Sur les valeurs de certaines fonctions L automorphes en leur centre de symétrie.* Compositio Math. 54 (1985), no. 2, 173–242.

[3] Zhiwei Yun, Wei Zhang, *Shtukas and the Taylor expansion of L-functions*, in preparation, 2015.

The image of the crystalline Dieudonné functor

Eike Lau

Let X be an $\mathbb{F}_p$-scheme. We study the crystalline Dieudonné functor

$$\mathbb{D}_X\colon (p\text{-div}/X) \to (\mathrm{DC}/X)$$

from the category of p-divisible groups over X to the category of (locally free) Dieudonné crystals. It is known by [3, 4, 1] that

a) when X is regular and F-finite, then $\mathbb{D}_X$ is an equivalence,
b) when X is an excellent l.c.i. scheme, then $\mathbb{D}_X$ is fully faithful,
c) in general $\mathbb{D}_X$ is not fully faithful. Example: $X = \operatorname{Spec} k[xy]/(x^2, xy, y^2)$.

In the l.c.i. case, in order to describe the essential image of $\mathbb{D}_X$ we have to take into account the Hodge filtration. We note that for a PD thickening of $\mathbb{F}_p$-schemes $U \to T$, the Frobenius $\sigma_T\colon T \to T$ factors over a map $\sigma_{U/T}\colon T \to U$. The following definition essentially appears in [5, §V.3].

Definition 1. Let $(\mathcal{M}, F, V)$ be a Dieudonné crystal over X. A submodule $\operatorname{Fil}\mathcal{M}_X \subseteq \mathcal{M}_X$ is called admissible if it is a locally direct summand and if for each open set $U \subseteq X$ and each PD thickening $U \to T$ of $\mathbb{F}_p$-schemes we have

$$\sigma^*_{U/T} \operatorname{Fil}\mathcal{M}_U = \operatorname{Ker}(F_T \colon \sigma^*_T\mathcal{M}_T \to \mathcal{M}_T)$$

inside $\sigma^*_{U/T}\mathcal{M}_U = \sigma^*_T\mathcal{M}_T$.

Let (DCF/X) be the category of Dieudonné crystals equipped with an admissible filtration. The Hodge filtration of a p-divisible group defines an extension of $\mathbb{D}_X$ to a functor

$$\tilde{\mathbb{D}}_X \colon (p\text{-div}/X) \to (\mathrm{DCF}/X).$$

Theorem 2. *If X is an l.c.i. scheme which can locally be embedded into a regular and F-finite scheme, then $\tilde{\mathbb{D}}_X$ is an equivalence.*

This is compatible with the previous results because the forgetful functor $(\mathrm{DCF}/X) \to (\mathrm{DC}/X)$ is an equivalence when X is regular and F-finite, and it is fully faithful when X is an l.c.i. scheme.

The proof of Theorem 2 uses the following construction. Let Z be an affine regular locally closed subscheme of X, and let Y be an infinitesimal neighbourhood of Z. We construct a category $\mathcal{C}_Y$ and a functor $\Phi_Y \colon (p\text{-div}/Y) \to \mathcal{C}_Y$ with the properties: (i) the functors Φ_Y for varying Y induce an equivalence of infinitesimal deformations; (ii) if Y is an l.c.i. scheme, then $\mathcal{C}_Y$ is equivalent to (DCF/Y) such that Φ_Y corresponds to $\tilde{\mathbb{D}}_Y$. The category $\mathcal{C}_Y$ involves a generalisation of the category of Dieudonné displays of [8] to a relative situation.

It is natural to ask for a modification of the category (DCF/X) for non-l.c.i. schemes which is equivalent to p-divisible groups in general. For those schemes Y that appear above, the category $\mathcal{C}_Y$ fulfils this requirement, but it is not clear how to extend the definition to a wider class of schemes. In the case of semiperfect schemes there is a more complete answer in a similar direction:

An $\mathbb{F}_p$-algebra R is called semiperfect if the Frobenius map $\sigma \colon R \to R$ is surjective. Then there is a universal p-adic PD thickening $A_{\mathrm{cris}}(R) \to R$, which carries a structure of a frame in the sense of [6] by [7, Lemma 4.1.8].

Proposition 3. *For each semiperfect ring R there is a functor*

$$\Phi_R \colon (p\text{-div}/Y) \to (\mathrm{windows}/A_{\mathrm{cris}}(R)).$$

An isogeny of semiperfect rings $R \to R'$ is a surjective homomorphism whose kernel is annihilated by a power of Frobenius. We call a semiperfect ring R balanced if the ideal $J = \operatorname{Ker}(\sigma \colon R \to R)$ satisfies $J^p = 0$.

Proposition 4. *For each semiperfect ring R which is isogeneous to a balanced semiperfect ring the functor Φ_R is an equivalence.*

This result can be applied to Kisin modules for perfectoid rings as follows. Let B be a perfectoid algebra of characterstic zero with ring of power-bounded elements B^0. Then $R = B^0/pB^0$ is a balanced semiperfect ring. Let $R^\flat = \varprojlim(R, \sigma)$ and

$A_{\text{inf}} = W(R^\flat)$. There is a natural surjective map $\theta\colon A_{\text{inf}} \to B^0$. In this setting, a Kisin module is a projective A_{inf}-module M of finite type equipped with a linear map $M^{(\sigma)} \to M$ whose cokernel is a projective B^0-module via θ.

Corollary 5. *If $p \geq 3$, p-divisible groups over B^0 are equivalent to Kisin modules.*

Sketch of proof: Kisin modules are equivalent to windows for $A_{\text{cris}} \to B^0$ by [2, Proposition 2.3.1], and these are equivalent to p-divisible groups over B^0 by Proposition 4 together with the Grothendieck-Messing deformation theorem.

References

[1] P. Berthelot, A. Ogus: F-Isocrystals and De Rham Cohomology. I. Invent. math. **72**, 159-199 (1983)

[2] B. Cais, E. Lau: Dieudonne crystals and Wach modules for p-divisible groups. arxiv:1412.3174.

[3] J. de Jong: Crystalline Dieudonné theory via formal and rigid geometry. Publ. Math. IHES **82**(1995), 5–96

[4] J. de Jong, W. Messing: Crystalline Dieudonné theory over excellent schemes. Bull. Soc. math. France **127** (1999) 333–348

[5] A. Grothendieck: Groupes de Barsotti-Tate et Cristaux de Dieudonné. Université de Montréal, 1974

[6] E. Lau: Frames and finite group schemes over complete regular local rings. Documenta Math. **15** (2010), 545–569

[7] P. Scholze, J. Weinstein: Moduli spaces of p-divisible groups. Cambridge Journal of Mathematics **1** (2013), 145-237

[8] Th. Zink: A Dieudonné theory for p-divisible groups. Class field theory—its centenary and prospect, 139–160, Adv. Stud. Pure Math. **30**, Math. Soc. Japan 2001

Reporter: Haifeng Wu

Participants

Prof. Dr. Jeff Achter
Department of Mathematics
Colorado State University
Weber Building
Fort Collins, CO 80523-1874
UNITED STATES

Prof. Dr. Fabrizio Andreatta
Dipartimento di Matematica
"Federigo Enriques"
Universita di Milano
Via Saldini 50
20133 Milano
ITALY

George Boxer
Department of Mathematics
The University of Chicago
5734 South University Avenue
Chicago, IL 60637-1514
UNITED STATES

Prof. Dr. Pascal Boyer
Département de Mathématiques
LAGA - CNRS: UMR 7539
Université Paris Nord (Paris XIII)
99, Avenue J.-B. Clement
93430 Villetaneuse Cedex
FRANCE

Dennis Brokemper
Institut für Mathematik
Universität Paderborn
Warburger Str. 100
33098 Paderborn
GERMANY

Dr. Miaofen Chen
Department of Mathematics
East China Normal University
No. 500, Dong Chuan Road
Shanghai 200 241
CHINA

Dr. Gabriel Dospinescu
Département de Mathématiques
École Normale Superieure de Lyon
46, Allee d'Italie
69364 Lyon Cedex 07
FRANCE

Prof. Dr. Gerd Faltings
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY

Prof. Dr. Laurent Fargues
Institut de Mathématiques de Jussieu
Case 247
Université de Paris VI
4, Place Jussieu
75252 Paris Cedex 05
FRANCE

Prof. Dr. Eyal Z. Goren
Department of Mathematics & Statistics
McGill University
805, Sherbrooke Street West
Montreal, P.Q. H3A 0B9
CANADA

Prof. Dr. Ulrich Görtz
Institut f. Experimentelle Mathematik
Universität Duisburg-Essen
45117 Essen
GERMANY

Prof. Dr. Thomas Haines
Department of Mathematics
University of Maryland
College Park, MD 20742-4015
UNITED STATES

Dr. Paul Hamacher
Department of Mathematics
Harvard University
One Oxford Street
Cambridge, MA 02138
UNITED STATES

Prof. Dr. Urs Hartl
Mathematisches Institut
Universität Münster
Einsteinstrasse 62
48149 Münster
GERMANY

Prof. Dr. Naoki Imai
Department of Mathematics
College of Arts and Sciences
The University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo 153-8914
JAPAN

Dr. Alexander Ivanov
Zentrum Mathematik
Technische Universität München
Boltzmannstraße 3
85748 Garching b. München
GERMANY

Daniel Kirch
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Mark Kisin
Department of Mathematics
Harvard University
Science Center
One Oxford Street
Cambridge MA 02138-2901
UNITED STATES

Dr. Jean-Stefan Koskivirta
Institut für Mathematik
Universität Paderborn
Warburger Strasse 100
33098 Paderborn
GERMANY

Dr. Arno Kret
School of Mathematics
Institute for Advanced Study
Princeton, NJ 08540
UNITED STATES

Prof. Dr. Kai-Wen Lan
School of Mathematics
University of Minnesota
127 Vincent Hall
Minneapolis MN 55455-0436
UNITED STATES

Prof. Dr. Eike Lau
Fakultät EIM - Elektrotechnik,
Informatik und Mathematik
Universität Paderborn
Warburger Str. 100
33098 Paderborn
GERMANY

Dr. Brandon W. Levin
Department of Mathematics
The University of Chicago
5734 South University Ave.
Chicago, IL 60637-1514
UNITED STATES

Dr. Yifeng Liu
Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA 02139-4307
UNITED STATES

Prof. Dr. Keerthi Madapusi Pera
Department of Mathematics
The University of Chicago
5734 South University Ave.
Chicago, IL 60637-1514
UNITED STATES

Prof. Dr. Yoichi Mieda
Graduate School of Mathematical Sciences
The University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo 153-8914
JAPAN

Andreas Mihatsch
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Abdellah Farid Mokrane
Département de Mathématiques
LAGA - CNRS: UMR 7539
Université Paris Nord (Paris XIII)
99, Avenue J.-B. Clement
93430 Villetaneuse Cedex
FRANCE

Stephan Neupert
Zentrum Mathematik
Technische Universität München
Boltzmannstrasse 3
85748 Garching b. München
GERMANY

Prof. Dr. Marc-Hubert Nicole
Département de Mathématiques
Faculté des Sciences de Luminy
Université d'Aix-Marseille II
70, route Leon Lachamp
13288 Marseille Cedex 9
FRANCE

Prof. Dr. Georgios Pappas
Department of Mathematics
Michigan State University
Wells Hall
East Lansing, MI 48824-1027
UNITED STATES

Prof. Dr. Michael Rapoport
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Dr. Shu Sasaki
Fakultät für Mathematik
Universität Duisburg-Essen
Thea-Leymann-Strasse 9
45127 Essen
GERMANY

Dr. Richard Shadrach
Department of Mathematics
Rice University
P.O. Box 1892
Houston, TX 77005-1892
UNITED STATES

Dr. Xu Shen
Fakultät für Mathematik
Universität Regensburg
93040 Regensburg
GERMANY

Prof. Dr. Brian Smithling
Department of Mathematics
Johns Hopkins University
Baltimore, MD 21218-2689
UNITED STATES

Prof. Dr. Benoit Stroh
Département de Mathématiques
LAGA - CNRS: UMR 7539
Université Paris Nord (Paris XIII)
99, Avenue J.-B. Clement
93430 Villetaneuse Cedex
FRANCE

Prof. Dr. Yichao Tian
Morningside Center of Mathematics
Chinese Academy of Sciences
55 Zhong Guan Cun East Road
Beijing 100 190
CHINA

Prof. Dr. Adrian Vasiu
Department of Mathematical Sciences
State University of New York at Binghamton
Binghamton, NY 13902-6000
UNITED STATES

Prof. Dr. Eva Viehmann
Zentrum Mathematik - M 11
Technische Universität München
Boltzmannstrasse 3
85748 Garching b. München
GERMANY

Prof. Dr. Torsten Wedhorn
Institut für Mathematik
Universität Paderborn
Warburger Strasse 100
33098 Paderborn
GERMANY

Prof. Dr. Jared Weinstein
Department of Mathematics & Statistics
Boston University
111 Cummington Mall
Boston, MA 02215
UNITED STATES

Haifeng Wu
Institut für Experimentelle Mathematik
Universität Duisburg-Essen
Ellernstraße 29
45326 Essen
GERMANY

Prof. Dr. Liang Xiao
Department of Mathematics
University of Connecticut
Storrs, CT 06269-3009
UNITED STATES

Prof. Dr. Chia-Fu Yu
Institute of Mathematics
Academia Sinica
No. 1 Roosevelt Rd. Sc. 4
Taipei 10617
TAIWAN

Dr. Chao Zhang
Yau Mathematical Sciences Center
Tsinghua University
Room 131, Jin Chun Yuan West Building,
Haidian District
Beijing 100 084
CHINA

Prof. Dr. Wei Zhang
Department of Mathematics
Columbia University
2990 Broadway
New York, NY 10027
UNITED STATES

Prof. Dr. Xinwen Zhu
Department of Mathematics
California Institute of Technology
1200 E. California Blvd.
Pasadena, CA 91125
UNITED STATES

Paul Ziegler
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY

Prof. Dr. Thomas Zink
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld
GERMANY