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Mini-Workshop: Amalgams for Graphs and Geometries

Organised by
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Introduction by the Organisers

The Mini-Workshop: Amalgams for Graphs and Geometries, organised by A. A. Ivanov (London) and S. Shpectorov (Bowling Green) was held May 16-22. This meeting was well attended with 17 participants with broad geographic representation from 3 continents. There were 16 talks during the workshop including an invited talk by Anda Degeratu a participant in the competing String Theory workshop.

The method of group amalgams is a highly effective way of classifying mathematical objects possessing high degrees of symmetry. The idea of the method is separation of the study of the local structure of the acting group from the question of it's global isomorphism type.

The method of group amalgams has been successfully applied to problems in graph theory and diagram geometry. It also featured prominently in group theory. For example, the fact that the Monster sporadic simple group is the universal completion of the amalgam associated with the tilde geometry formed the foundation of the solution of the famous Y-group conjecture that the Y_{555} presentation defines the Bimonster (the direct product of two copies of the Monster sporadic simple group extended by a group of order 2). J.H. Conway coined for this theorem the name 'NICE' where **N** is for **N**orton, **I** for **I**vanov, **C** for **C**onway and **E** for anyone **E**lse. The proof of the NICE theorem based on the method of group amalgams is presented in the two volume monograph of the organisers published by Cambridge University Press.

Recently a dramatic progress was made within the study of flag-transitive diagram geometries. The importance of the notion of *constrained* completions of amalgams was realised. Within this framework many geometries of sporadic groups were characterised as the constrained completions of suitable amalgams. This approach also gives a general criterion about possible shapes of diagrams of flag-transitive geometries. This enables the area of diagram geometries to leave it's "botanical" stage of example collection and enter the stage of theory building.

Among other applications we would like to mention recent applications of the amalgam method to the cohomologies of finite groups. These ideas were described in the notes of M. Aschbacher on calculation of the Schur multiples of some finite simple groups.

During the workshop we had discussed in detail the proofs of a number of results obtained along the lines of the amalgam method, as well as of directions of future research. We believe that the abstract of the talks given at the workshop facilitate for the younger mathematicians access to these extremely important, yet very technically complex tools of mathematical research.

Mini-Workshop: Amalgams for Graphs and Geometries

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Abstracts

On the classification of distance-transitive graphs

John van Bon

Let Γ be a finite connected undirected graph without loops or multiple edges and let $G \leq \mathrm{Aut}(\Gamma)$. We say that G acts distance transitively on Γ if G acts transitively on the sets of ordered pairs of vertices $\Gamma_i = \{(x, y) \mid d(x, y) = i\}$, for each $i = 0, \ldots, \mathrm{diam}(\Gamma)$. The graph Γ will be called distance-transitive if it admits such a group action. Observe that if Γ is a distance-transitive graph, then $\mathrm{Aut}(\Gamma)$ acts distance transitively on it but there might be many subgroups that do so too. Distance-transitive graphs are the most symmetric among all graphs as they have, in a certain sense, the largest group of automorphisms possible. There are many examples among which are the Hamming graphs, Johnson graphs and Dual polar graphs.

We discuss the classification of these graphs and groups. In case the action of G on the vertex set of Γ is imprimitive then there is a natural way to obtain a new distance-transitive graph from Γ admitting a group acting primitively on its vertex set. Therefore in the classification project we assume that the action is primitive. In a later stage the graphs with an imprimitive group action can be determined.

A first step towards the classification of primitive distance-transitive graphs was made by C. Praeger, J. Saxl & K. Yokoyama [9] who proved that either Γ is known or G is an almost simple group or an affine group. Recently the classification of primitive distance-transitive graph admitting an affine group was completed, see [1, 2, 3, 4, 5, 6, 7, 8]. We will give an overview of the structure of the proof, which uses the classification of finite simple groups, and the main ideas involved. In case G is almost simple one can again invoke the classification of finite simple groups. We will survey the current status of the project in this case.

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Geometrical McKay Correspondece

Anda Degeratu

A Calabi-Yau manifold is a complex Kähler manifold with trivial canonical bundle. In the attempt to construct such manifolds it is useful to take into consideration singular Calabi-Yaus. One of the simplest singularities which can arise is an orbifold singularity. An orbifold is the quotient of a smooth Calabi-Yau manifold by a discrete group action which generically has fixed points. Locally such an orbifold is modeled on $\mathbb{C}^n/G$, where G is a finite subgroup of $SL(n, \mathbb{C})$.

From a geometrical perspective we can try to resolve the orbifold singularity. A resolution (X, π) of $\mathbb{C}^n/G$ is a nonsingular complex manifold X of dimension n with a proper biholomorphic map $\pi : X \to \mathbb{C}^n/G$ that induces a biholomorphism between dense open sets. We call X a *crepant resolution*[1] if the canonical bundles are isomorphic, $K_X \cong \pi^*(K_{\mathbb{C}^n/G})$. Since Calabi-Yau manifolds have trivial canonical bundle, to obtain a Calabi-Yau structure on X one must choose a crepant resolutions of singularities.

It turns out that the amount of information we know about a crepant resolution of singularities of $\mathbb{C}^n/G$ depends dramatically on the dimension n of the orbifold:

- $n = 2$: A crepant resolution always exists and is unique. Its topology is entirely described in terms of the finite group G (via the McKay Correspondence).
- $n = 3$: A crepant resolution always exists but it is not unique; they are related by flops. However all the crepant resolutions have the same Euler and Betti numbers: the *stringy* Betti and Hodge numbers of the orbifold [DHVW].
- $n \geq 4$: In this case very little is known; crepant resolutions exist in rather special cases. Many singularities are terminal, which implies that they admit no crepant resolution.

We would like to completely understand the topology of crepant resolutions in the case $n = 3$. In this paper we are concerned with the study of the ring structure in cohomology. This is related to the generalization of the McKay Correspondence. In what follows we give a description of the problem by moving back and forth between the case $n = 2$ and $n = 3$.

The case $n = 2$. The quotient singularities $\mathbb{C}^2/G$, for G a finite subgroup of $SL(2, \mathbb{C})$, were first classified by Klein in 1884 and are called *Kleinian singularities* (they are also known as *Du Val singularities* or *rational double points*). There are five families of finite subgroups of $SL(2, \mathbb{C})$: the cyclic subgroups $\mathcal{C}_k$, the binary dihedral groups $\mathcal{D}_k$ of order $4k$, the binary tetrahedral group $\mathcal{T}$ of order 24, the binary octahedral group $\mathcal{O}$ of order 48, and the binary icosahedral group $\mathcal{I}$ of order 120. A crepant resolution exists for each family and is unique. Moreover the finite group completely describes the topology of the resolution. This is encoded in the McKay Correspondence [McK1], which establishes a bijection between the set of

[1]Etymology: For a resolution of singularities we can define a notion of *discrepancy* [R1]. A crepant resolution is a resolution without discrepancy.

irreducible representations of G and the set of vertices of an extended Dynkin diagram of type ADE (the Dynkin diagrams corresponding to the simple Lie algebras of the following five types: A_{k-1}, D_{k+2}, E_6, E_7 and E_8).

Concretely, let $\{R_0, R_1, \ldots, R_r\}$ be the set of irreducible representations of G, where R_0 denotes the one-dimensional trivial representation. To G and its irreducible representations we associate an $(r+1)\times(r+1)$ adjacency matrix $A = [a_{ij}]$ with $i, j = 0, \ldots, r$. The entries a_{ij} are positive integers; they are defined by the tensor product decompositions

$$R_i \otimes Q = \sum_{i=0}^{r} a_{ij}\, R_j,$$

where Q denotes the natural two-dimensional representation of G induced from the embedding $G \subset SL(2,\mathbb{C})$. McKay's insight was to realize that the matrix A is related to the Cartan matrix C of a Dynkin diagram of type ADE, via

$$A = 2I - \widetilde{C}. \tag{0.1}$$

(Here $\widetilde{C}$ is the Cartan matrix of the extended Dynkin diagram; the matrix C is the $r \times r$-minor obtained by removing the first row and the first column from $\widetilde{C}$.)

Using McKay's correspondence it is easy now to describe the crepant resolution $\pi : X \to \mathbb{C}^2/G$. The exceptional divisor $\pi^{-1}(0)$ is the dual of the Dynkin diagram: the vertices of the Dynkin diagram correspond naturally to rational curves C_i with self-intersection -2. Two curves intersect transversally at one point if and only if the corresponding vertices are joined by an edge in the Dynkin diagram, otherwise they do not intersect. The curves above form a basis for $H_2(X,\mathbb{Z})$. The intersection form with respect to this basis is the negative of the Cartan matrix.

The first geometric interpretation of the McKay Correspondence was given by Gonzalez-Sprinberg and Verdier [GV]. To each of the irreducible representations R_i they associated a locally free coherent sheaf $\mathcal{R}_i$. The set of all these coherent sheaves form a basis for $K(X)$, the K-theory of X. Moreover, the first Chern classes $c_1(\mathcal{R}_i)$ form a basis in $H^2(X,\mathbb{Q})$ and the product of two such classes in $H^*(X,\mathbb{Q})$ is given by the formula

$$\left[\int_X c_1(\mathcal{R}_i)c_1(\mathcal{R}_j)\right]_{i,j=1,\ldots,r} = -\,C^{-1}, \tag{0.2}$$

where C^{-1} is the inverse of the Cartan matrix. The proof given by Gonzalez-Sprinberg and Verdier uses a case by case analysis and techniques from algebraic geometry. Kronheimer and Nakajima gave a proof of the formula using techniques from gauge theory [KroN].

To summarize, in the case of surface singularities, $\mathbb{C}^2/G$, the representation theory of the finite group G completely determines the topology the crepant resolution. The Dynkin diagram and the Cartan matrix (and hence the simple Lie algebra $\mathfrak{g}$ associated to it) encode everything we want to know about the topology of the crepant resolution.

The case $n = 3$. The finite subgroups of $SL(3, \mathbb{C})$ were classified by Blichfeldt in 1917 [Bl]: there are ten families of such finite subgroups. In the early 1990's a case by case analysis was used to construct a crepant resolution of $\mathbb{C}^3/G$ with the given stringy Euler and Betti numbers (see [Ro] and the references therein). As a consequence of these constructions, we know that all the crepant resolutions of $\mathbb{C}^3/G$ have the Euler and Betti numbers given by the stringy Euler and Betti numbers of the orbifold (since these numbers are unchanged under flops). In 1995 Nakamura made the conjecture that $\mathrm{Hilb}^G(\mathbb{C}^3)$ is a crepant resolution of $\mathbb{C}^3/G$. In general, for G a finite subgroup of $SL(n, \mathbb{C})$, the algebraic variety $\mathrm{Hilb}^G(\mathbb{C}^n)$ parametrizes the 0-dimensional G-invariant subschemes of $\mathbb{C}^n$ whose space of global sections is isomorphic to the regular representation of G. Nakamura made the conjecture based on his computations for the case $n = 2$ [INak]; then he proved it in dimension $n = 3$ for the case of abelian groups [Nak]. In 1999 Bridgeland, King and Reid gave a general proof of the conjecture in the case $n = 3$, relying heavily on derived category techniques [BKR]. In 2002 Craw and Ishii proved that (at least in the case G abelian) all the crepant resolutions arrive as moduli spaces [CI].

In the case of surface singularities, an important feature of the McKay Correspondence is that it gives the ring structure in cohomology in terms of the finite group. For the case $n \geq 3$, nothing is known about the multiplicative structures in cohomology or K-theory.

Let $G \subset SL(3, \mathbb{C})$ be a finite subgroup acting with an isolated singularity on $\mathbb{C}^3/G$. Let X be a crepant resolution of $\mathbb{C}^3/G$. On this resolution we associate a vector bundle $\mathcal{R}_i$ to each irreducible representation of G – this is the extension of the Gonzalez-Sprinberg-Verdier sheaves. These bundles form a basis of the K-theory of X, and via the Chern character isomorphism, we have that $\{\mathrm{ch}(\mathcal{R}_0), \mathrm{ch}(\mathcal{R}_1), \ldots, \mathrm{ch}(\mathcal{R}_\mathrm{r})\}$ basis of $H^*(X; \mathbb{Q})$.

The idea is to use the Atiyah-Patodi-Singer (APS) index theorem for studying multiplicative properties of the (Chern classes of the) bundles $\mathcal{R}_i$. In [De2] we show a that a generalization of Kronheimer and Nakajima's formula (0.2) holds in the compactly supported cohomology of X:

$$\left[\int_X (\mathrm{ch}(\mathcal{R}_\mathrm{i}) - \mathrm{rk}(\mathcal{R}_\mathrm{i}))\left(\mathrm{ch}(\mathrm{R}_\mathrm{j}^*) - \mathrm{rk}(\mathcal{R}_\mathrm{i})\right)\right]_{i,j=1,\ldots,r} = C^{-1}. \tag{0.3}$$

Here C is a matrix associated to the finite group G and its embedding into $SL(3, \mathbb{C})$, generalizing the Cartan matrix of the case $n = 2$.

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Classification of amalgams related to Phan theory

Ralf Gramlich

The project of establishing a unified theory for Phan's theorems [10], [11] originally concentrated on flag-transitive geometries. The following results have been established until now, see [3], [4], [5], [6].

Theorem. *Let $q \geq 4$, let $n \geq 3$, let Δ be the Dynkin diagram A_n, C_n, or D_n, and let G be a group that contains a weak Phan system of type Δ over $\mathbb{F}_{q^2}$. Then G is isomorphic to a factor group of*

- $SU_{n+1}(q^2)$, *if* $\Delta = A_n$;
- $Sp(2n, q)$, *if* $q \geq 8$ *and* $\Delta = C_n$
- $Spin^+(2n, q)$, *if* $\Delta = D_n$ *and* n *even; and*
- $Spin^-(2n, q)$, *if* $\Delta = D_n$ *and* n *odd.*

However, intransitive geometries arise naturally from the construction of flipflop geometries described in [2], for example the geometry of nondegenerate subspaces of an orthogonal space, see [1]:

Theorem. *Let $n \geq 3$, let $\mathbb{F}$ be an arbitrary field of characteristic not two distinct from $\mathbb{F}_3$ and $\mathbb{F}_5$, and let V be an $(n+1)$-dimensional vector space over $\mathbb{F}$ endowed with a nondegenerate orthogonal form. Then the geometry of nondegenerate subspaces of V is simply connected.*

One method to cope with the intransitivity is, of course, to restrict oneself to the study of the geometry consisting of the orbit of a single flag. The major drawback of this approach is the loss of elements of the geometry. Unfortunately, the more elements one removes from the geometry the more difficult it is to establish the simple connectedness of that geometry.

Therefore I would like to propose an alternative approach. Following Stroppel [12], one can drop the transitivity assumption for *one* class of types of the geometry, say the points, and still recover the geometry from the family of stabilizers. This allows for a number of amalgam-theoretic results for amalgams of intransitive geometries, see [7].

Finally, using Lie theory and the theory of Schur covers of topological groups [8], [9], one can use the methods developed for the proof of Theorem to prove the following result.

Theorem. *Let $n \geq 3$, let and let G be a group that contains a weak Phan system of type A_n over $\mathbb{C}$. Then G is isomorphic to a factor group of $SU_{n+1}(\mathbb{C})$.*

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Curtis-Phan-Tits theory

Corneliu Hoffman

This talk describes some recent results which establish a connection between the Curtis-Tits theorem and a series of results of Phan.

Here is the Curtis-Tits theorem as stated in [1]

Theorem (Curtis - Tits)**.** *Let K be the universal version of a finite Chevalley group of twisted rank at least 3 with root system Σ, fundamental system Π and root groups X_α, $\alpha \in \Sigma$. For each $J \subseteq \Pi$ let $K_J =< X_\alpha,\ \pm\alpha \in J >$. Let D be the set of all subsets of Π with at most 2 elements. Then K is the universal completion of the amalgam $\cup_{J\in D} K_J$.*

In the case of the diagram of type A_n, consider the following amalgam

$\cup_{i=1}^n L_i$ such that $L_i \cong SL_2(q) < L_i, L_{i+1} >\cong SL_3(q), [L_i, L_j] = (1)$ if $|i-j| \geq 2$

. then the universal cover of the amalgam is $SL_{n+1}(q)$

Theorem (Phan)**.** *Consider the following amalgam:*

$\cup_{i=1}^n U_i$, *such that* $U_i \cong SU_2(q^2) < U_i, U_{i+1} >\cong SU_3(q^2), [L-i, L_j] = (1)$ *if* $|i-j| \geq 2$

then the universal cover is $SU_{n+1}(q^2)$

The similarity between the two statements is striking. The following construction sheds some more light on the problem and provides other similar results.

Given a twin building $\mathcal{T} = (\mathcal{B}_+, \mathcal{B}_-, \delta_*)$,

$$\text{Opp}(\mathcal{B}) := \{(c_+, c_-) \in \mathcal{C}_+ \times \mathcal{C}_- | \delta_*(c_+, c_-) = 1_W\}$$

Chambers $x \in \mathcal{C}_+$ and $y \in \mathcal{C}_-$ with $\delta_*(x, y) = 1_W$ are called opposite, hence the notation. At least in the spherical case, $\text{Opp}(\mathcal{T})$ is a geometric chamber system. Its corresponding geometry will be denoted by Γ_{op} and will be called the opposites geometry.

Theorem (Muhler)**.** *If $\mathcal{T} = (\mathcal{B}_+, \mathcal{B}_-, \delta_*)$ is a spherical twin building associated to the spherical building $\mathcal{B}$ then the geometry Γ_{op} is simply connected.*

The stabilizers of the elements from a maximal flag are the Levi factors in the maximal parabolic subgroups of K. Therefore an inductive argument gives the Curtis-Tits theorem.

More generally let $\mathcal{B} = (B_+, B_-)$ be a twin buiding. A *flip* is an automorphism of $\mathcal{B}$ with the following properties:

(i) $\sigma^2 = Id$
(ii) $B_+^\sigma = B_-$
(iii) $d_+(x, y) = d_-(x^\sigma, y^\sigma)$, $d_*(x, y) = d_*(x^\sigma, y^\sigma)$
(iv) $\exists C_+ \in B_+$ such that $d_*(C_+, C_+^\sigma) = 1_W$.

Construct

$$\mathcal{C}_\sigma := \{(C_+, C_-) | C_+^\sigma = C_- \ ;\ d_*(C_+, C_-) = 1_W\} \neq \emptyset$$

We do not know if the chamber system $\mathcal{C}_\sigma$ is geometrizable in general, however this is the case in each of our examples. In the case of a classical spherical building we have constructed a series of examples of such flips for which the corresponding chamber system gives a simply connected geometry and corresponding amalgam presentation for a classical group. For a list of the results see Theorem from Ralf Gramlich's abstract and the references following it.

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Amalgams and representations

A.A. Ivanov

A locally projective amalgam is formed by the stabilizer $G(x)$ of a vertex x and the global stabilizer $G\{x,y\}$ of an edge (containing x) in a group G, acting faithfully and locally finitely on a connected graph Γ of valency 2^n-1 so that (i) the action is 2-arc-transitive; (ii) the subconstituent $G(x)^{\Gamma(x)}$ is the linear group $SL_n(2) \cong L_n(2)$ in its natural doubly transitive action and (iii) $[t, G\{x,y\}] \leq O_2(G(x) \cap G\{x,y\})$ for some $t \in G\{x,y\} \setminus G(x)$. D. Ž. Djoković and G.L. Miller [1] used the classical Tutte's theorem [2], to show that there are seven locally projective amalgams for $n=2$. In [3] used the most difficult and interesting case of Trofimov's theorem [4] to extend the classification to the case $n \geq 3$. It turned out that besides two infinite series of locally projective amalgams (embedded into the affine linear groups $AGL_n(2)$ and into the orthogonal linear groups $O^+_{2n}(2)$) there are exactly twelve exceptional ones as in Table 1. Some of the exceptional amalgams are embedded into sporadic simple groups M_{22}, M_{23}, Co_2, J_4 and BM. For each of the exceptional amalgam $n=3$, 4 or 5. In [5] for every locally projective amalgam $\mathcal{A}$ we calculate the minimal degree $m=m(\mathcal{A})$ of its complex representation (which is a faithful completion into $GL_m(\mathbf{C})$). For the exceptional amalgams the dimensions are given in Table 2. Analysing the minimal representations we answer three questions on exceptional locally projective amalgams left open in [3]: we haved shown that (1) $\mathcal{A}_4^{(1)}$ possesses $SL_{20}(13)$ as a faithful completion in which the third geometric subgroup is improper; (2) $\mathcal{A}_4^{(2)}$ possesses the alternating group Alt_{64} as a completion constrained at levels 2 and 3; (3) $\mathcal{A}_4^{(5)}$ possesses Alt_{256} as as a completion which is constrained at level 2 but not at level 3.

n	$\mathcal{A}$	$\frac{G^{[0]}/O_2(G^{[0]})}{O_2(G^{[0]})}$	$\frac{\widehat{G}^{[2]}}{K^{[2]}}$	$\frac{\widehat{G}^{[3]}}{K^{[3]}}$	$\frac{\widehat{G}^{[4]}}{K^{[4]}}$	some completions constrained at level 2
3	$\mathcal{A}_3^{(1)}$	$\frac{L_3(2)}{2^3}$	$\frac{S_3 \wr S_2}{2^4}$			-
	$\mathcal{A}_3^{(2)}$	$\frac{L_3(2)}{2^3}$	$\frac{S_5}{2^4}$			M_{22}
	$\mathcal{A}_3^{(3)}$	$\frac{L_3(2)}{2^3}$	$\frac{S_5}{2^4}$			-
	$\mathcal{A}_3^{(4)}$	$\frac{L_3(2)}{2^3 \times 2}$	$\frac{2^4{:}(S_3 \wr S_2)}{2^5}$			$(S_8 \wr 2)^+$
	$\mathcal{A}_3^{(5)}$	$\frac{L_3(2)}{2^3 \times 2}$	$\frac{S_5}{2^5}$			Aut M_{22}
4	$\mathcal{A}_4^{(1)}$	$\frac{L_4(2)}{1}$	$\frac{S_5}{2^4{:}3}$	$\frac{1}{1}$		M_{23}
	$\mathcal{A}_4^{(2)}$	$\frac{L_4(2)}{2^6}$	$\frac{S_5 \times 2}{2_+^{1+8}{:}S_3}$	$\frac{L_6(2)}{2^6}$		Alt_{64}
	$\mathcal{A}_4^{(3)}$	$\frac{L_4(2)}{2^{1+4+6}}$	$\frac{S_5}{2^{4+10}.S_3}$	$\frac{\text{Aut } M_{22}}{2^{10}}$		Co_2
	$\mathcal{A}_4^{(4)}$	$\frac{L_4(2)}{2^{4+4+6}}$	$\frac{S_5}{2^{3+12+2}.S_3}$	$\frac{3{\cdot}\text{Aut } M_{22}}{2_+^{1+12}}$		J_4
	$\mathcal{A}_4^{(5)}$	$\frac{L_4(2)}{2^{4+4+6}}$	$\frac{S_5 \times 2}{2^{3+12+2}.S_3}$	$\frac{L_6(2){:}2}{2_+^{1+12}}$		Alt_{256}
5	$\mathcal{A}_5^{(1)}$	$\frac{L_5(2)}{2^{10}}$	$\frac{S_5}{2^{3+12}.L_3(2)}$	$\frac{\text{Aut } M_{22}}{2_+^{1+12}{:}3}$	$\frac{1}{1}$	J_4
	$\mathcal{A}_5^{(2)}$	$\frac{L_5(2)}{2^{5+5+10+10}}$	$\frac{S_5}{2^3.[2^{32}].L_3(2)}$	$\frac{\text{Aut } M_{22}}{2^{2+10+20}.S_3}$	$\frac{Co_2}{2_+^{1+22}}$	BM

TABLE 1. **Exceptional Amalgams**

$\mathcal{A}$	$\mathcal{A}_3^{(1)}$	$\mathcal{A}_3^{(2)}$	$\mathcal{A}_3^{(3)}$	$\mathcal{A}_3^{(4)}$	$\mathcal{A}_3^{(5)}$	$\mathcal{A}_4^{(1)}$	$\mathcal{A}_4^{(2)}$	$\mathcal{A}_4^{(3)}$	$\mathcal{A}_4^{(4)}$	$\mathcal{A}_4^{(5)}$	$\mathcal{A}_5^{(1)}$	$\mathcal{A}_5^{(2)}$
$m(\mathcal{A})$	7	20	20	14	20	20	63	23	1333	255	1333	4371

TABLE 2. **Dimensions of Minimal Representations**

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On the classification of finite simple groups of both even and p -type

Inna Korchagina

In this talk we we discuss a classification of finite simple groups which are of simultaneously even- and p-type. In particular, we discuss a characterization of $A_{12}HN, Sp_8(2)$ and $F_4(2)$. This is a joint work with R. Lyons, and is a contribution to the revision project of Gorenstein, Lyons and Solomon of the Classification of Finite Simple Groups.

On Amalgams for Locally s-Arc Transitive Graphs

Cai Heng Li

Let Γ be a finite undirected simple graph that has no vertex of valency less than 3. For a vertex $v \in V$, denote by $\Gamma(v)$ the set of vertices adjacent to v. An s-arc of Γ is an $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices such that $v_{i-1} \in \Gamma(v_i)$ and $v_{i-1} \neq v_{i+1}$. For a group $G \leq \mathrm{Aut}\Gamma$, Γ is called *locally* (G, s)*-arc transitive* if, for any vertex v, the stabiliser G_v acts transitively on the set of s-arcs starting at v; further, Γ is called *locally* (G, s)*-transitive* if Γ is not locally $(G, s+1)$-arc transitive. A locally (G, s)-arc transitive graph Γ is called (G, s)*-arc transitive* if G is also transitive on the vertex set V. As usual, for a vertex v of Γ, denote by $G_v^{\Gamma(v)}$ the permutation group on $\Gamma(v)$ induced by G_v, and denote by $G_v^{[1]}$ the kernel of G_v acting on $\Gamma(v)$.

For an edge $\{v, w\}$ of Γ, the triple (G_v, G_w, G_{vw}) is called the *amalgam* of G and of Γ. The problem of determining the amalgam is fundamental for understanding the group G and the graph Γ.

The study of locally s-arc transitive graphs was initiated by a celebrated result of W. Tutte (1947), that is, s-arc transitive cubic graphs exist only for $s \leq 5$. Since then, studying locally s-arc transitive graphs has been one of the central topics in algebraic graph theory. The amalgam (G_v, G_w, G_{vw}) has been known for several special cases: the vertex transitive case with $s \geq 2$, see [9] and [7]; the case where 'Moufang condition' holds, see [10] for references; the cubic graph case, see [5]; the case where $G_v^{\Gamma(v)}$ and $G_w^{\Gamma(w)}$ are both rank one Lie type groups, see [1].

In particular, it was proved in [9] that there exists no non-trivial 8-arc transitive graphs. B. Stellmacher announced that for general locally s-arc transitive graphs, $s \leq 9$, and he and J. van Bon are jointly working on the project.

It is clear that for a locally $(G, 2)$-arc transitive graph Γ, $G_v^{\Gamma(v)}$ and $G_w^{\Gamma(w)}$ form a pair of 2-transitive permutation groups, and so each of them is known. The first step for classifying the amalgam (G_v, G_w, G_{vw}) is to determine the pairs $\{G_v^{\Gamma(v)}, G_w^{\Gamma(w)}\}$ of 2-transitive permutation groups. The following statements are proved in [6], which are independent of Stellmacher and van Bon's work.

(i) For $s = 2$, either $\mathsf{soc}(G_v^{\Gamma(v)}) \cong \mathsf{soc}(G_w^{\Gamma(w)})$ and Γ is regular, or some restricted conditions hold.
(ii) For $s \geq 3$, either both $(G_v^{[1]})^{\Gamma(w)\setminus\{v\}}$ and $(G_w^{[1]})^{\Gamma(v)\setminus\{w\}}$ are transitive, or some restricted conditions are satisfied.
(iii) For $s \geq 4$, either both $G_{vw}^{\Gamma(v)}$ and $G_{vw}^{\Gamma(w)}$ are soluble, or several very special cases occur.
(iv) For $s \geq 6$, both $G_{vw}^{\Gamma(v)}$ and $G_{vw}^{\Gamma(w)}$ are soluble, and both $|\Gamma(v)| - 1$ and $|\Gamma(w)| - 1$ are p-powers with p prime.

A global-action analysis of locally s-arc transitive graphs was developed in [2], which provides methods for constructing certain locally s-arc transitive graphs. Several families of locally 2- or 3-transitive graphs constructed in [3] and a family of locally 5-arc transitive graphs constructed in [4] justify the existence of several cases in the above statements. They particularly give new almalgams for locally s-arc transitive graphs.

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Semisymmetric graphs of twice odd order

C.W. Parker

Suppose that Γ is a connected graph and G is a subgroup of the automorphism group $\mathrm{Aut}(\Gamma)$ of Γ. Then Γ is *G-symmetric* if G acts transitively on the arcs (and so the vertices) of Γ and Γ is *G-semisymmetric* if G acts edge transitively but *not* vertex transitively on Γ. If Γ is $\mathrm{Aut}(\Gamma)$-symmetric, respectively, $\mathrm{Aut}(\Gamma)$-semisymmetric, then we say that Γ is *symmetric*, respectively, *semisymmetric*. If Γ is G-semisymmetric, then the orbits of G on the vertices of Γ are the two parts of a bipartition of Γ. If Γ is G-semisymmetric, then we say that G acts *semisymmetrically* on Γ.

Connected semisymmetric cubic graphs (connected graphs in which every vertex has degree 3) have been the focus of a number of recent articles, we mention specifically [3, 4, 5, 6, 8] where, for example, infinite families of such graphs are presented and where the semisymmetric graphs of order $2pq$ with p and q odd primes are determined (with the help of the classification of finite simple groups). We also remark that a catalogue of all the connected semisymmetric cubic graphs of order at most 768 has recently been obtained by Conder *et. al.* [1]. The objective of this talk is to partially describe all groups which act semisymmetrically on a connected cubic graph of order twice an odd number.

Suppose that Γ is a G-semisymmetric cubic graph. Let $\{u, v\}$ be an edge in Γ. Set $G_u = \mathrm{Stab}_G(u)$, $G_v = \mathrm{Stab}_G(v)$ and $G_{uv} = G_u \cap G_v$. Then as G acts edge transitively on Γ and u is not in the same G-orbit as v, we have $[G_u : G_{uv}] = [G_v : G_{uv}] = 3$. Suppose that $K \trianglelefteq G$ and $K \leq G_{uv}$. Then K fixes every edge of Γ and hence $K = 1$. As Γ is connected, the subgroup $\langle G_u, G_v\rangle$ acts transitively on the edges of Γ and so we infer that $G = \langle G_u, G_v\rangle$. We have shown that G satisfies

- G1 $G = \langle G_u, G_v\rangle$;
- G2 $[G_u, G_u \cap G_v] = [G_v, G_u \cap G_v] = 3$; and
- G3 no non-trivial subgroup of G_{uv} is normalized by both G_u and G_v (is normal in G).

This group theoretic configuration has been studied by Goldschmidt in [7] where he shows that the triple (G_u, G_v, G_{uv}) is isomorphic (as an amalgam) to one of fifteen possible such triples. So understanding semisymmetric cubic graphs is the same as understanding completions of the Goldschmidt amalgams. For the investigations in this talk, we are interested in finite groups G which are completions of a Goldschmidt amalgam (G_u, G_v, G_{uv}) and which have for which $[G : G_u] + [G : G_v] = 2[G : G_v]$ equal to twice an odd number. Since G_{uv} is a 2-group by Goldschmidt's Theorem, this means that G_{uv} is a Sylow 2-subgroup of G. If G is as above and $G_{uv} \in Syl_2(G)$, then we call G a *Sylow completion* of the amalgam (G_u, G_v, G_{uv}).

Suppose that G is a Sylow completion of a Goldschmidt amalgam (G_u, G_v, G_{uv}). Then a normal subgroup R of G is called a *regular* normal subgroup of G provided R acts semiregularly on the vertices of the coset graph $\Gamma = \Gamma(G, G_u, G_v, G_{uv})$.

Theorem. *Suppose that G is a Sylow completion of a Goldschmidt amalgam. Then there exists a regular normal subgroup R of G of odd order such that G/R is isomorphic to one of the groups indicated in column two of Table 3.*

We also give the graph theoretical version of this theorem .

Theorem. *Suppose that G acts semisymmetrically on a connected, cubic graph of twice odd order. Then there exists a normal subgroup R of G of odd order which acts semiregularly on the vertices of Γ such that G/R is isomorphic to one of the groups indicated in column two of Table 3.*

The proofs of these theorems use some carefully chosen parts of the classification of finite simple groups. Further details and relationship with results in [1], [3], [4], [5], [6] and [8] were explained in fuller details in the presentation.

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Cohen-Macauley geometries

Antonio Pasini

A flag complex of a diagram geometry has objects of the geometry as vertices and flags as simplices. A finite-dimensional complex is called spherical if all its maximal simplices have the same dimension n, and the reduced homology vanishes in all dimensions except (possibly) for n. A geometry is called spherical if its flag complex is spherical, and Cohen-Macaulay if all residues are spherical too.

This talk reports on a joint work by V. Burichenko and A. Pasini. The main result of this talk is the following: affine polar spaces are Cohen-Macaulay. As a byproduct, a new proof of 2-simple connectedness of affine polar spaces is obtained. Similar results for biaffine spaces are proved. Results of this kind can be used to compute low-dimensional cohomology of groups acting on geometries. The crucial step in the proof is a certain nontrivial geometric property of affine polar spaces.

Division	Type	G/R	Condition
1	G_1	3	
	G_1^1	Sym(3)	
2	G_1	3^2	
	G_1^1	$3^2.2$	
	G_1^2	3×6	
	G_1^3	$\mathrm{Sym}(3) \times \mathrm{Sym}(3)$	
3	G_2	$3^3.\mathrm{Alt}(4)$	
	G_2^3	$\mathrm{Sym}(3) \wr 3$	
	G_2^1	$3^3.\mathrm{Sym}(4)$	
	G_2^2	$3^3.\mathrm{Sym}(4)$	
	G_2^4	$\mathrm{Sym}(3) \wr \mathrm{Sym}(3)$	
4	G_1^3	$\mathrm{PSL}_2(p)$	p a prime, $p \equiv 11, 13 \pmod{24}$
5	G_2	$\mathrm{PSL}_2(p)$	p a prime, $p \equiv 11, 13 \pmod{24}$
	G_2^1	$\mathrm{PGL}_2(p)$	p a prime, $p \equiv 11, 13 \pmod{24}$
6	G_2^2	Alt(7)	
	G_2^4	Sym(7)	
7	G_2^1	$\mathrm{PSL}_2(p)$	p a prime, $p \equiv 23, 25 \pmod{48}$
8	G_2^1	$\mathrm{PSL}_2(p^2)$	p a prime, $p \equiv 5, 19 \pmod{24}$
	G_2^4	$\mathrm{P\Sigma L}_2(p^2)$	p a prime, $p \equiv 5, 19 \pmod{24}$
9	G_3	$\mathrm{PSL}_2(p)$	p a prime, $p \equiv 7, 9 \pmod{16}$
10	G_3	$\mathrm{PSL}_2(p^2)$	p a prime, $p \equiv 3, 5 \pmod{8}$
	G_3^1	$\mathrm{P\Sigma L}_2(p^2)$	p a prime, $p \equiv 3, 5 \pmod{8}$
11	G_4	$\mathrm{PSL}_3(p)$	p a prime, $p \equiv 5 \pmod{8}$
	G_4^1	$\mathrm{PSL}_3(p).2$	p a prime, $p \equiv 5 \pmod{8}$
12	G_4	$\mathrm{PSU}_3(p)$	p a prime, $p \equiv 3 \pmod{8}$
	G_4^1	$\mathrm{PSU}_3(p).2$	p a prime, $p \equiv 3 \pmod{8}$
13	G_5	M_{12}	
	G_5^1	$\mathrm{Aut}(M_{12})$	
14	G_5	$G_2(p)$	p a prime, $p \equiv 3, 5 \pmod{8}$
	G_5^1	$\mathrm{Aut}(G_2(3))$	

Table 3. Sylow Completions of Goldschmidt Amalgams

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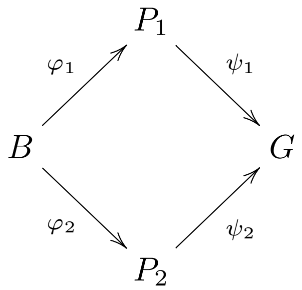
Completions of Goldsmidt amalgams

Peter Rowley

Let $\mathcal{A} = \mathcal{A}(P_1, P_2, P_3)$ be one of the fifteen Goldsmidt amalgams given in [3] and let G be a faithful completion of $\mathcal{A}$. We then have



Where $\varphi_1, \varphi_2, \psi_1, \psi_2$ are group monomorphismsm and $\varphi_1\psi_1 = \varphi_2\psi_2$. we identify P_i with $\mathrm{Im}\psi_i$, $i = 1, 2$.

Here we give a short survey of which groups are completions of Goldsmidt amalgams. In [8] Wester gave the matrix group descriptions of the universal completion for a number of the Goldsmidt amalgams.

In the case when G is finite Thiel showed in [7] that $SL_3(2^a)$ is a completion of the G_3 amalgam provided $a \neq 2$. This result was extended to

> **Theorem** (Parker and Rowley [4]). *The subgroups $SL_3(q)$ and $L_3(q)$ are completions of the Goldschmidt G_3-amalgam if and only if $q \notin \{4, 9\}$*
>
> *$SU_3(q)$ and $U_3(q)$ are completions of the Goldschmidt G_3 amalgam if and only if $q \notin \{3, 5\}$*

If G is a sporadic simple group we have

Theorem (Parker and Rowley [6]). *$M_{11}, M_{12}, J_1, M_{22}, M_23, HS, McL, C0_3, Co_2$ are not completions of the Goldschmidt G_3 amalgams whereas all the other sporadic groups, with the possible exception of M, are.*

For Goldsmidt G_4 amalagam we have

Theorem (Parker and Rowley [5]). *Suppose $G \leq GL_3(k)$ with k a finite field of characteristic p. if G is a completion of the Goldschmidt amalgam G_4 then p is odd and $G \cong SL_3(q)$ if $p \equiv 1 \pmod 4$ and $g \cong SU_3(p)$ if $p \equiv 3pmod4$.*

When G is symmetric we have following two results:

Theorem (Conder [2]). *For all but finitely many n the group S_n is a completion of G_3^1 amalgam.*

Theorem (Bundy, Rowley[1]). *If $n \notin \{1, 2, 3, 4, 6\}$ then S_n is a completion of the Goldschmidt G_1^3 amalgam.*

For a recent classification of Sylow-completion of the Goldschmidt amalgam see the abstract of C Parker's talk.

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Y-representation and 21-node system of the Monster and the moonshine module

Hiroki Shimakura

Our purpose is to study the Monster M (or interesting finite groups) and to explain its mysterious phenomena by using vertex operator algebras (VOAs). For details of the axiom of vertex operator algebras, see [FLM].

1. Automorphism group of V_L^+

One of the typical VOAs is the VOA V_L associated to an even lattice L. The automorphism group $\mathrm{Aut}(V_L)$ of V_L has the subgroup $O(\hat{L})$ induced by the automorphism group $O(L)$ of L: $O(\hat{L}) \cong \mathrm{Hom}(L, \mathbf{Z}_2).O(L)$. We take an element θ of $O(\hat{L})$ induced by the -1-symmetry. Then $V_L^+ = \{v \in V_L |\ \theta(v) = v\}$ is a subVOA of V_L. In [Sh1], a method of determining of $\mathrm{Aut}(V_L^+)$ is given. In particular, the following hold:

Proposition 1.1. [Sh1]

(i) $\mathrm{Aut}(V_L^+)$ *is finite if and only if* L *has no roots.*

(ii) *Suppose that* L *has no roots. Then* $\mathrm{Aut}(V_L^+) \subseteq C_{\mathrm{Aut}(V_L)}(\theta)\langle\theta\rangle$ *if and only if* L *is obtained by Construction B from a binary code.*

(iii) $\mathrm{Aut}(V_L^+)$ *has "nice" symmetries if* L *is isomorphic to* $\sqrt{2}E_8$ *and the Barnes-Wall lattice* BW_{16} *of rank* 16. *In particular,* $\mathrm{Aut}(V_{\sqrt{2}E_8}^+) \cong O_{10}^+(2)$ *and* $\mathrm{Aut}(V_{BW_{16}}^+) \cong 2^{16} \cdot \Omega_{10}^+(2)$.

V_L^+ is important from the viewpoint of finite group theory by (1). (For example, $\mathrm{Aut}(V_L)$ is infinite for any even lattice L.) A new relation between lattices and VOAs is given by (2). The VOAs $V_{\sqrt{2}E_8}^+$ and $V_{BW_{16}}^+$ are deeply related to the maximal 2-local subgroup of the Monster of shape $2^{10+16} \cdot \Omega_{10}^+(2)$.

2. Application to the moonshine module

The moonshine module $V^\natural$ was constructed in [FLM]: $V^\natural = V_\Lambda^+ \oplus V_\Lambda^{T,-}$, where Λ is the Leech lattice and $V_\Lambda^{T,-}$ is an irreducible V_Λ^+-module. Moreover, they showed that $V_2^\natural$ is a deformation of the algebra constructed by Griess and that $\mathrm{Aut}(V^\natural) \cong \mathrm{M}$. For a sublattice L of Λ, $V^\natural$ contains V_L^+ as a subVOA. By using this embedding, some maximal 2-local subgroups of the Monster are described in terms of V_L^+ in [Sh2].

Let us explain how to obtain symmetries of $V^\natural$ from VOAs in the case where $V = V_{\sqrt{2}E_8}^+ \otimes V_{BW_{16}}^+ \subset V^\natural$. Consider a certain decomposition of $V^\natural$ as irreducible V-modules into 2^{10} components: $V^\natural = \oplus_{d\in E} V^\natural(d)$, $|E| = 2^{10}$. By using the representation theory on V_L^+, for any $d_1, d_2 \in E$, there uniquely exists an element $d_3 \in E$ such that

$$V^\natural(d_1)_{(n)} V^\natural(d_2) \subset V^\natural(d_3),$$

where (n) is the binary operator on $V^\natural$. Hence we obtain a binary operation $(d_1, d_2) \mapsto d_3$ on E. In fact, this operation gives a group structure on E. For $f \in \mathrm{Hom}(E, \mathrm{C}^\times)$, we define a linear automorphism g_f of $V^\natural$: $g_f(v) = f(d)v$ if $v \in V^\natural(d)$. Then $g_f \in \mathrm{Aut}(V^\natural)$. Therefore we obtain an elementary abelian 2-subgroup $\mathrm{Hom}(E, \mathrm{C}^\times)$ of order 2^{10} of $\mathrm{Aut}(V^\natural)$. In [Sh2], it is shown that $N_{\mathrm{Aut}(V^\natural)}(E) \cong 2^{10+16} \cdot \Omega_{10}^+(2)$ without using the properties of the Monster. The author hopes that our arguments will give a "nice" explanation of some aspects of the Monster.

3. 21-node system and the moonshine module

One of the mysterious phenomena on the Monster M is the 21-node system (or the Y-representation): M is isomorphic to a certain Coxeter group with an extra relation. Our purpose is to explain the mysterious non-Coxeter relation in terms of vertex operator algebras.

Miyamoto [Mi1] found the 21 involutions of $V^\natural$ satisfying the Coxeter relation of the 21-node system. However, the non-Coxeter relation is not checked yet.

Combining the results of [Co] and [Mi2], we obtain the one-to-one correspondence between the idempotents of $V_2^\natural$ with norm $1/16$ and the $2A$-involutions of M. Moreover, Conway [Co] showed that the inner product of idempotents is determined by the conjugacy class of the product of the corresponding $2A$ involutions of M.

Question: Find 21-involutions (or 12-involutions forming the Y-diagram) with suitable inner products and explain the non-Coxeter relation in terms of vertex operator algebras.

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Simply connected geometries for $G_2(3).2$ and the Thompson sporadic group Th

S Shpectorov

The Thompson sporadic simple group Th is one of the few sporadic groups, whose uniqueness still has no computer-free proof. One of the reasons for this situation is the absence of a good, simply connected geometry, on which Th acts flag-transitively. Such a geometry, together with the result on the uniqueness of the related amalgam of maximal parabolics, would provide means for the identification of Th, which is a crucial step towards its uniqueness.

The Thompson group is a group with a large extraspecial subgroup. For such groups one can define, as in [MS], a family of elementary abelian subgroups called *singular subgroups*. In the case of Th, all singular subgroups can easily be classified, giving five conjugacy classes of orders 2^1 through 2^5. In terms of geometries, only singular subgroups of orders 2^1 and 2^5 are of interest, and they give two types of elements, which we call *lines* and *points*, respectively. Incidence between points and lines is defined by containment of the corresponding singular subgroups.

To be simply connected, a finite geometry must have rank at least three, so we need one further type of elements. The point-line incidence graph contains a family of subgraphs isomorphic to the incidence graph of the generalized hexagon of ${}^3D_4(2)$. These subgraphs become elements of the third type, called *hexagons*. The resulting rank three geometry $\Gamma(Th)$ satisfies, as a point-line geometry, a nice set of axioms, which we omit here.

Theorem. *The geometry $\Gamma(Th)$ is simply connected.*

We now outline the scheme of the proof.

The Thompson groups contains in the normalizer of a subgroup A of order three a section isomorphic to $G_2(3).2$. This section acts on the points of $\Gamma(Th)$ fixed by A and this leads to a similar flag-transitive geometry $\Gamma(G_2(3).2)$. We prove that this smaller geometry is simply connected, which via Tits' lemma, yields that the universal completion of the amalgam $\mathcal{A}(G_2(3).2)$ of the maximal parabolics in $G_2(3).2$ is isomorphic to $G_2(3).2$.

Let now $\mathcal{A}(Th)$ be the amalgam of maximal parabolics in Th, and let G be its universal completion. To prove Theorem , one only needs to show that $G \cong Th$. Within $\mathcal{A}(Th)$ one can choose a subgroup A of order three, so that the normalizers of A in the members of $\mathcal{A}(Th)$, taken modulo A, form an amalgam isomorphic to

$\mathcal{A}(G_2(3).2)$. By the above, we conclude that G must contain a new subgroup, which is an extension of A by $G_2(3).2$.

Now, the amalgam formed by two of the three members of $\mathcal{A}(Th)$ and the new subgroup, is exactly the amalgam looked at by Havas, Soicher, and Wilson [HSW]. They show that the latter amalgam necessarily generates Th, which leads to the conclusion that $G \cong Th$, proving Theorem . Notice, however, that this proof is not computer-free, as the result in [HSW] is obtained by Todd-Coxeter enumeration. We hope in the future work to establish simple connectedness of the geometry $\Gamma(Th)$ in a computer-free way.

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On groups of local characteristic p

G. Stroth

Looking at the finite simple groups one realizes that most maximal 2–locals have a very restricted structure. Let G be a finite simple group of characteristic 2–type. Let S be a Sylow 2–subgroup and M be a maximal 2–local containing S but such that $\Omega_1(Z(S))$ is not normal in M, then usually $\Omega_1(Z(O_2(M)))$ is a small module for $M/C_M(\Omega_1(Z(O_2(M))))$. Further $M/C_M(\Omega_1(Z(O_2(M))))$ has not many components. In the quasi thin group paper [AS] M. Aschbacher and S. Smith proved the following theorem

Theorem. *Let G be a quasi thin K-group of even type, T be a Sylow 2–subgroup of G, $T \le M_0 \le M$, M a maximal 2-local of G, and H a subgroup of G minimal subject to $T \le H \not\le M$ and $O_2(H) \neq 1$. Assume that V is a normal elementary abelian 2–subgroup of M_0 which satisfies $O_2(M_0/C_{M_0}(V)) = 1$, $O_2(M_0) = C_T(V)$, and $H \cap M$ normalizes V or $O^2(M_0)$. Then either $O_2(\langle M_0, H\rangle) \neq 1$ or V is a $2F$–module for $M_0/C_{M_0}(V)$ with cubic offender A.*

Here a $2F$–module V with cubic offender A means $C_A(V) = 1$, $|V : C_V(A)| \le |A|^2$ and $[V, A, A, A] = 1$. Recall that by Thompson replacement FF-modules are also $2F$–modules with cubic offenders. M. Aschbacher and S. Smith claimed that this theorem should also hold without the assumtion that G is quasi thin. In the first part of the talk we sketch a proof of that generalized theorem.

For this we need a few definitions

- We call a group H of characteristic p–type if $C_H(O_p(H)) \le O_p(H)$

- We call a group G of local characteristic p , if all nontrivial p-locals of G are of characteristic p-type
- Let H be of characteristic p -type, then let Y_H be the maximal normal elementary abelian subgroup of H with $O_p(H/C_H(Y_H)) = 1$.
- Let H be of characteristic p -type, S be a Sylow p–subgroup of H. Set $H_0 = N_H(S \cap C_H(Y_H))$. Then $Y_{H_0} = Y_H$ and $C_S(Y_{H_0}) = O_p(H_0)$.
- A group G is called a K_p–group if all simple composition factors of all nontrivial p–locals of G are cyclic, alternating, a group of Lie type or one of the 26 sporadics.

Now we can state the little theorem

Theorem. *Let G be a group of local characteristic p with $O_p(G) = 1$. Let S be a Sylow p–subgroup of G. Either there is exactly one maximal p-local containing S or there is some maximal p–local H with $S \leq H$ and Y_H is a $2F$–module with cubic offender A.*

Here is a sketch of the proof.

Choose some maximal p-local H with $Y_H \not\leq Z(S)$ and Y_H maximal. This is possible if there is more than one maximal p-local containing S. Suppose first that there is some p–local L of G such that $Y_H \not\leq O_p(L)$ and $S \cap L$ is a Sylow p–subgroup of L. The following argument is due to U. Meierfrankenfeld. Using this we were able to drop the asumption G to be a K_p–group from the version of the theorem presented at the conference. Choose $L_H = \langle Y_H, Y_H^g\rangle$ minimal such that $O_p(L_H) \neq 1$, $S \cap L_H$ is a Sylow p–subgroup of L_H and $Y_H \not\leq O_p(L_H)$. Now if $a \in Y_H \setminus O_p(L_H)$ and $b \in (Y_H^g \cap O_p(L_H)) \setminus Y_H$, then $[a, b] \notin Y_H \cap Y_H^g$. Hence $|Y_H O_p(L_H)/O_p(L_H)| \leq |[Y_H, b](Y_H \cap Y_H^g)| \leq |Y_H \cap O_p(L_H)/Y_H \cap Y_H^g| = |Y_H^g \cap O_p(L_H)/Y_H \cap Y_H^g|$. Hence $Y_H^g \cap O_p(L_H)/Y_H \cap Y_H^g$ is a $2F$–offender on Y_H. As $[Y_H^g \cap O_p(L), Y_H, Y_H^g \cap O_p(L_H)] \leq Y_H \cap Y_H^g$, it is a cubic offender.

So we may assume that there is no such L. This in particular shows that $Y_H \leq O_p(C_G(x))$ for all $1 \neq x \in Y_H$. Now choose P minimal with $S \leq P$, $O_p(P) \neq 1$ but $P \not\leq H$. As $Y_H \leq Y_{\langle H_0, P\rangle}$, we get with the maximality of Y_H that (H_0, P) is an amalgam. Suppose $Y_P \not\leq Z(S)$. Then there is some maximal p–local M such that $P \leq M$ and $Y_M \not\leq Z(S)$. Then (H_0, M_0) is an amalgam and so either Y_H or Y_M is an FF–module and we are done. So we may assume that $Y_P \leq Z(S)$. Let b be the parameter of that amalgam. If b is even, we have that Y_H is an FF-module. So we may assume that b is odd. As $Y_H \leq O_p(P)$, we get $b \geq 3$. Now a general argument, which also is independent of K_p–group assumptions (variation of the L–lemma [MSS], or a slightly stronger version of Stellmachers *qrc* - lemma [Ste], using the fact that $Y_H \leq O_p(C_G(x))$ for all $1 \neq x \in Y_H$, which replaces the possibility of being a dual of an FF-module by strong dual FF-module with quadratic offender, which then is an FF–module too.) gives that Y_H is a $2F$–module. Recall that the pushing - up situation of the *qrc*–lemma does not show up as by the choice of P we have that $Y_H \not\leq Z(O_p(P))$.

To get more information, maybe also the information about $M/C_M(Y_M)$ one has to implement more assumptions. This has been done by U. Meierfrankenfeld, B. Stellmacher and G. Stroth [MSS] in the so called structure theorem. For this we have to make a few definitions. As seen before $N_G(\Omega_1(Z(S)))$ plays a special role. Let G be as before (i.e. of local characteristic p and $O_p(G) = 1$).

- Denote by $\tilde{C}$ a maximal p-local containing $N_G(\Omega_1(Z(S)))$.
- Set $E = O^p(F_p^*(C_{\tilde{C}}(Y_{\tilde{C}})))$, here $F_p^*(X)$ is the preimage of $F^*(X/O_p(X))$.
- Non E -uniqueness. There is more than one maximal p-local of G containing E. Then the actual status is that we either have a nice amalgam, a nice pushing - up situation or $F^*(G) \cong M_{22}$, M_{23}, M_{24}, $L_n(q)$ or A_n, [MSS1], [MSS2] .

In what follows we will assume that we have E–uniqueness, so $\tilde{C}$ is the unique maximal p-local containing E. Set $Q = O_p(\tilde{C})$. Let M be any p-local, $M \not\leq \tilde{C}$. Then a consequence of E–uniqueness is $Q \not\leq O_p(M)$. Set $M_0 = \langle Q^M \rangle$ and let M be with M_0 maximal. Then the structure theorem describes $F^*(M_0/O_p(M_0))$ and the action of M_0 on Y_M. Here we will just give the version where $Y_M \not\leq O_p(\tilde{C})$.

Theorem. *Let G be a K_p–group of local characteristic p. Assume E–uniqueness, and let M_0 be as before. Set $M^0 = M_0S$ and $K = F^*(M^0/C_{M^0}(Y_M))$. Then one of the following holds*

i) *K is quasisimple and isomorphic to $SL(n,q)$, $Sp(2n,q)'$, $\Omega^{\pm}(n,q)$, or $E_6(q)$, q a power of p. Further $[K, Y_M]$ contains one of the modules below.*
ii) *$K \cong SL(n,q)' \times SL(m,q)'$, Y_M is the tensor product module of the two natural modules..*
iii) *$p = 2$, $K \cong 3A_6$, M_{22} or M_{24} and $[Y_M, K]$ is a 6–dimensional , 10–dimensional, 11-dimensional module, respectively.*
iv) *$p = 3$, $K \cong M_{11}$ or $2M_{12}$ and Y_M is the 5–dimensional or 6-dimensional module.*
v) *M^0 is a minimal parabolic*

group	prime	module	restriction
$SL(n,q)$	p	natural	
$SL(n,q)$	p	alternating square	
$SL(n,q)$	p	symmetric square	p odd
$SL(n,q^2)$	p	$V(\lambda_1)\otimes V(\lambda_1^\sigma)$	
$Sp(2n,q)$	p	natural	
$Sp(2n,q)$	p	spin	$n \leq 5$
$3A_6$	2	6-dim	
$\Omega^\pm(n,q)$	p	natural	
$\Omega^\pm(2n,q)$	2	half spin	$n \leq 6$
$E_6(q)$	p	$V(\lambda_1)$	
M_{11}	3	5-dimensional	
$2M_{12}$	3	6-dimensional	
M_{22}	2	10-dimensional	
M_{24}	2	11-dimensional	

Based on this theorem we have established the H-structure theorem [S], which reads as follows

Theorem. *Let G be a K_p–group of local characteristic p. Let R be the subgroup generated by all p-locals containing S. Assume that $O_p(R) = 1$ and E–uniqueness. Let M^0 be as before and assume $Y_M \not\leq Q$. Then one of the following holds*

(1) *There is a subgroup H of G with $M^0 \leq H$, $O_p(H) = 1$ such that for $F^*(H)$ the parabolics containing S are as in one of the following groups*
   i) *A group of Lie type in characteristic p and of rank at least three*
   ii) *$p = 2$ and we have He, Co_2, Co_1, $M(24)'$, J_4, Suz, F_2, F_1, or $U_4(3)$.*
   iii) *$p = 3$ and we have $M(24)'$, Co_1 or F_1*

(2) *$p = 2$ and M is an extension of an elementary abelian group of order 16 by $L_3(2)$, $\tilde{C}$ is an extension of an extraspecial group of order 32 by $\Sigma_3 \times \Sigma_3$. Further there are minimal parabolics P_1, P_2 with $P_1/O_2(P_1) \cong P_2/O_2(P_2) \cong \Sigma_3$ and $O_2(\langle P_1, P_2\rangle) = 1$.*

(3) *$p = 3$ and M and $\tilde{C}$ are as in Co_3. There are two minimal parabolics P_1, P_2 such that $P_1/O_3(P_1) \cong L_2(9)$, $P_2/O_3(P_2) \cong SL(2,9)$ and $O_3(\langle P_1, P_2\rangle) = 1$.*

(4) *M^0 is a minimal parabolic.*

Of course there is a problem. The H-structure theorem just gives information about H but not about the simple group G. In fact the group H might be a proper subgroup of G. If $p = 2$ and H has the structure of a group of Lie type, in fact then H is a group of Lie type, A. Hirn is working in classifying all examples where $H \neq G$ as a Ph. D. thesis. The case $p = 3$ and we have one of (1)(iii) above will be done by M. Salarian as a Ph. D. thesis. Here we will forget about K_p-groups and groups of local characteristic p. The theorems will read as follows.

Theorem. *Let G be a finite group containing two subgroups H_1 and H_2 where $H_1 \cong 3^{1+12}2Suz : 2$ and $H_2 \cong 3^8\Omega^-(8,3)$, where H_1 and H_2 share a Sylow 3–subgroup of G and $H_1 = N_G(Z(O_3(H_1)))$, then $G \cong F_1$.*

Theorem. *Let G be a finite group containing two subgroups H_1 and H_2, where $H_1 \cong 3^{1+4}Sp(4,3) : 2$ and $H_2 \cong 3^6 2M_{12}$, where H_1 and H_2 share a Sylow 3–subgroup of G and $H_1 = N_G(Z(O_3(H_1)))$, then $G \cong Co_1$.*

Theorem. *Let G be a finite group containing two subgroups H_1 and H_2, where $H_1 \cong 3^{1+10}U_5(2) : 2$ and $H_2 \cong 3^7O(7,3)$, where H_1 and H_2 share a Sylow 3–subgroup of G and $H_1 = N_G(Z(O_3(H_1)))$, then $G \cong M(24)'$.*

The first theorem has been proven.

The case (1) of the H–structure theorem with H a group of Lie type in odd characteristic is still open. It would be interesting to prove :

Let G be a group containing a subgroup H which is a group of Lie type in characteristic p and of rank at least three. If H is strongly p–embeded (i.e. $N_G(P) \leq H$ for any nontrivial p–subgroup P of H), then $G = H$.

Case (2) of the H-structure theorem has been dealt with by M. Aschbacher [Asch]. We get $F^*(G) = G_2(3)$.

The case (3) is still open. The claim is that in this case $G = Co_3$.

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Vertex stabilizers of graphs and tracks

V.I. Trofimov

The track method is a method of reconstruction of finite vertex stabilizers of groups of automorphisms of graphs from their restrictions to the neighborhood. In the first part of the talk, I discuss main ideas of this method which, in a sense, can be seen as the amalgam method realized along some special track of the graph.

In the second part of the talk, I discuss how the track method can be modified to be used in connection with the following conjecture.

Conjecture. Let Γ be an undirected connected locally finite graph, G be a vertex-transitive group of automorphisms of Γ, and $x \in V(\Gamma)$. Then at least one of the following holds:

1) there exists an imprimitivity system σ of G on $V(\Gamma)$ with finite (may be one-element) blocks such that $G^{\sigma}_{x^{\sigma}}$ is finite;

2) the graph Γ is hyperbolic (i.e. for some positive integer n, the graph Γ^n with the vertex set $V(\Gamma^n) = V(\Gamma)$ and the edge set $E(\Gamma^n) = \{\{y_1, y_2\} | 0 < d_\Gamma(y_1, y_2) \leq n\}$ contains the regular tree of valency 3).

In this context the following result is important.

Theorem. *Let Γ be an undirected connected locally finite non-hyperbolic graph, G be a vertex-transitive group of automorphisms of Γ, $x \in V(\Gamma)$, and $g \in G$. For each integer i, put $x_i = g^i(x)$. Let H be the subgroup of G generated by g and the pointwise stabilizer in G of the set $\{..., x_{-1}, x_0\}$, and let X be the H-orbit containing x. Then there exists an imprimitivity system σ of H^X on X with finite* (*may be one-element*) *blocks such that $(H^X)^\sigma$ is a cyclic group.*

Automorphisms of Moufang polygons

Richard Weiss

Let Γ be a Moufang polygon, let G denote its automorphism group and let $G^\dagger$ denote the subgroup of G generated by all the root groups of Γ. Then $G^\dagger$ is a normal subgroup of G and, except in three small exceptional cases, $G^\dagger$ is a simple group. In Chapter 37 of [2], the structure of the quotient group $G/G^\dagger$ is determined for almost all families of Moufang polygons. For example, if Γ is the Moufang triangle defined over a commutative field K, then $G/G^\dagger$ is isomorphic to $\mathrm{Aut}(K) \cdot K^*/(K^*)^3$. In recent work, we have determined the structure of $G/G^\dagger$ for the exceptional quadrangles of type E_6 and E_7. Tom De Medts has solved this problem for the exceptional quadrangles of type F_4 [1]. The two cases for which there is still no satisfactory result are the exceptional quadrangles of type E_8 and the hexagons of type E_8 (i.e. the hexagons defined over a 27-dimensional quadratic Jordan division algebra).

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On the Quillen dimension property

Satoshi Yoshiara

What is the most important prime for a given finite group? It should be 2 if the group is nonabelian simple, but the characteristic of a finite Chevalley group also plays an important role. The aim of my talk is to provide an approach to this problem from the view point of subgroup complexes, and to show the list of interesting primes to each sporadic simple group. Roughly speaking, a prime p is 'interesting' for a finite group G if the simplicial complex of chains of nontrivial p-subgroups of G can be shrinked much.

For a finite group G and a prime p, let $\mathcal{S}_p(G)$ (resp. $\mathcal{A}_p(G)$ and $\mathcal{B}_p(G)$) be a partially ordered set consisting of nontrivial p- (resp. elementary and radical, that is, R with $O_p(N_G(R)) = R$) subgroups of G. Each of these posets, $\mathcal{X}$, is associated with the simplicial complex $\Delta(\mathcal{X})$ of chains. The geometric realizations $|\Delta(\mathcal{X})|$ are homotopically equivalent to each other, whence they give the identical (reduced) homology groups $\tilde{H}_n(\mathcal{X}) := \tilde{H}_n(|\Delta(\mathcal{X})|, \mathbf{Q})$ for $n = 0, 1, \ldots$. It is evident that the dimension of the complex $\Delta(\mathcal{A})$ coincides with $m_p(G) - 1$, where $m_p(G)$ denotes the p-rank of G. Thus $\tilde{H}_n(\mathcal{X})$ vanishes if $n \geq m_p(G)$.

It is natural to observe the smallest dimension n for which $\tilde{H}_n(\mathcal{X})$ does not vanish. Note that we may not find such n. In that case, $|\Delta(\mathcal{X})|$ is contractible to a point, and the remarkable Quillen's conjecture says that we have $O_p(G) \neq 1$. The conjecture was affirmatively solved for $p \geq 7$ in [AS], where the following notion was introduced: We say that the *Quillen dimension property* (QD) holds for (G, p) if $n_p(G) := \min\{n \mid \tilde{H}_n(\mathcal{X}) \neq 0\}$ exists and $n_p(G) = m_p(G) - 1$.

A prime p is called *interesting* to a finite group G if (QD) fails. In [AS], it is shown that the possible interesting primes to most of the groups of Lie type (resp. the alternating groups) are 2, 3, 5 and the characteristics (resp. 2 and 3). To each sporadic group, the list of interesting primes is obtained [Yo3]. They are 2, 3 (to many), 5 (to McL, Ly, HN, $\mathbf{M}$), 11 (to J_4) and 13 (to $\mathbf{M}$). This is immediately verified in view of the list of radical p-subgroups (determined by Yoshiara and others e.g.[Yo1], [Yo2]), but the cases with $p = 5$ and $G = HN$, BM or M require certain amount of calculations.

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