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Real Analysis, Harmonic Analysis, and Applications

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Abstract. The workshop focused on important developments within the last few years in real and harmonic analysis, including polynomial partitioning and decoupling as well as significant concurrent progress in the application of these for example to number theory and partial differential equations.

Mathematics Subject Classification (2010): 42B20.

Introduction by the Organisers

This workshop, which continued the triennial series at Oberwolfach on real and harmonic analysis that started in 1986, has brought together experts and young scientists working in harmonic analysis and its applications such as linear and nonlinear PDE, number theory, ergodic theory, and geometric measure theory, with the objective of furthering the important interactions between these fields.

Major areas and results represented at the workshop are:

(1) Decoupling, originating in the work of Wolff in the nineties and further developed by Garrigos and Seeger in part as a result of previous meetings in this series of Oberwolfach workshops, has seen a dramatic impact in recent years through the work of Bourgain, Demeter, Guth, and coauthors. They established sharp forms of decoupling inequalities that contain solutions to a long standing conjecture of Vinogradov and other problems in number theory such as Parsell–Vinogradov. Particularly surprising was the robustness of the proof of these number theoretic results requiring relatively little number theoretic information. In connection with the polynomial partitioning method, decoupling has found even more applications such as

a complete solution in dimension three to a longstanding open problem of almost everywhere convergence of solutions to the Schrödinger equation to the initial data.

(2) In the last year there has been a breakthrough in the Kakeya problem, a central problem in harmonic analysis and geometric measure theory that has already been featured in previous meetings of this series. This work was presented by one of its architects and extensively discussed. The Kakeya problem is at the heart of many problems in harmonic analysis such as the Fourier restriction problem and the behaviour of solutions to dispersive partial differential equations. Remarkably, the decoupling method discussed in the previous paragraph arose in large part from the study of the Kakeya and restriction problems.

(3) Multilinear inequalities have attracted an increasing attention in pure mathematics and computer science. One important class are the Brascamp–Lieb inequalities of which nonlinear variants have been found to be of interest in connection with various topics discussed at the meeting such as the Kakeya problem and non-commutative harmonic analysis. Singular Brascamp–Lieb inequalities of a particular entangled form have seen applications in enumerative combinatorics and ergodic theory.

(4) The theory of singular integrals and related questions in geometric measure theory. One central theme is the geometric characterization of analytic properties of measures such as L^2 boundedness of the Riesz transforms in terms of rectifiability and other regularity conditions. Also questions on directional singular integrals are intertwined with geometric measure theory. The weighted theory of singular integrals has been recently complemented with the technique of domination by sparse operators.

(5) Bellman function and other monotonicity techniques, discussed in the context of singular integral theory in previous meetings of this series, have led to a unified view on a wide class of sharp isoperimetric inequalities with Gaussian measures such as log-Sobolev inequality, Beckner's inequality, and Bobkov's inequality.

(6) Nonlinear Fourier analysis plays a role in solving integrable nonlinear PDEs. New conserved quantities for certain nonlinear flows have been constructed using a nonlinear Fourier transform, yielding long time information about the flow.

(7) Existence and properties of extremizers for inequalities such as Stein–Tomas Fourier restriction and a nonlinear Hausdorff–Young inequality were the subject of multiple presentations.

(8) Other topics of current interest were discussed including subelliptic operators and sharp spectral multiplier results on nilpotent Lie groups. New concepts of curvature suitable for analysis of geometric averaging operators associated to submanifolds of intermediate dimensions.

The meeting took place in a lively and active atmosphere and greatly benefited from the ideal environment at Oberwolfach. It was attended by 53 participants.

The program consisted of 28 lectures of 40 minutes. The organisers made an effort to include young mathematicians, and greatly appreciate the support through the Oberwolfach Leibniz Graduate Students Program, which allowed to invite several outstanding young scientists.

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Workshop: Real Analysis, Harmonic Analysis, and Applications

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Abstracts

On the Fourier transform norm for certain Lie groups

Ali Baklouti

Let G be a separable locally compact unimodular group of type I and $\hat{G}$ the unitary dual of G endowed with the Borel structure. We regard the Fourier transform $\mathcal{F}$ as a mapping of $L^1(G)$ to a space of μ-measurable field of bounded operators on $\hat{G}$ defined for $\pi \in \hat{G}$ by $L^1(G) \ni f \mapsto \mathcal{F}f : \mathcal{F}f(\pi) = \pi(f)$, where μ denotes the Plancherel measure of G. The mapping $f \mapsto \mathcal{F}f$ extends to a continuous operator $\mathcal{F}^p : L^p(G) \to L^q(\hat{G})$, where $1 < p \leq 2$ and q is its conjugate. We are concerned with the norm of this linear map $\mathcal{F}^p$. We give an estimate of this norm for some classes of solvable Lie groups and we discuss the sharpness problem. For arbitrary compact extensions of $\mathbb{R}^n$, an extremal function is given as an extension of a Gaussian function. Besides, as an example of non-compact extension, the universal covering group of the Euclidean motion group of the plane is also treated and an estimate of the norm is obtained.

Nonlinear perturbations of simple Brascamp–Lieb data

Jonathan Bennett

(joint work with N. Bez, S. Buschenhenke, T. Flock)

The general Brascamp–Lieb inequality simultaneously generalises a number of important functional inequalities in analysis, including the multilinear Hölder, Loomis–Whitney and Young convolution inequalities. It takes the form

$$\int_{\mathbb{R}^n} \prod_{j=1}^{m} (f_j \circ L_j)^{p_j} \leq C \prod_{j=1}^{m} \left(\int_{\mathbb{R}^{n_j}} f_j \right)^{p_j}, \tag{1}$$

where m, n, n_j are natural numbers, $L_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$ are linear surjections, and $p_j \in [0,1]$ for each $1 \leq j \leq m$. Following [BCCT1], we denote by $\mathrm{BL}(\mathbf{L}, \mathbf{p})$ the smallest constant C for which (1) holds for all nonnegative input functions $f_j \in L^1(\mathbb{R}^{n_j})$, $1 \leq j \leq m$; i.e.

$$\mathrm{BL}(\mathbf{L}, \mathbf{p}) = \sup_{\mathbf{f}} \mathrm{BL}(\mathbf{L}, \mathbf{p}; \mathbf{f}) := \frac{\int_{\mathbb{R}^n} \prod_{j=1}^{m} (f_j \circ L_j)^{p_j}}{\prod_{j=1}^{m} \left(\int_{\mathbb{R}^{n_j}} f_j \right)^{p_j}}.$$

Here $\mathbf{L}, \mathbf{p}, \mathbf{f}$ denote the m-tuples $(L_j), (p_j), (f_j)$ respectively.

Perhaps the most important results concerning $\mathrm{BL}(\mathbf{L}, \mathbf{p})$ are the following:

Theorem 1 (Lieb [L])**.** *The Brascamp–Lieb constant is exhausted by centred gaussian inputs; i.e.*

$$f_j(x) = e^{-\langle A_j x, x \rangle},$$

where A_j is a symmetric positive-definite $n_j \times n_j$ matrix.

Theorem 2 (B–Carbery–Christ–Tao [BCCT1], [BCCT2]). *$BL(\mathbf{L},\mathbf{p})<\infty$ if and only if*

$$\sum_{j=1}^m p_j n_j = n$$

and

$$\dim(V) \leq \sum_{j=1}^m p_j \dim(L_j V) \tag{2}$$

for all subspaces V of $\mathbb{R}^n$.

Recently, beginning with [BCW], certain nonlinear variants of (1) have arisen in harmonic analysis (see for example [BCW, BB, BHT, BH]). This involves replacing the linear maps L_j by smooth submersions B_j in a neighbourhood of a point (0, say) in $\mathbb{R}^n$. It seems reasonable to expect that if $\mathrm{BL}(\mathbf{dB}(0),\mathbf{p})<\infty$, then there exists a neighbourhood U of $0\in\mathbb{R}^n$ and a constant c such that

$$\int_U \prod_{j=1}^m (f_j\circ B_j)^{p_j} \leq c\mathrm{BL}(\mathbf{dB}(0),\mathbf{p})\prod_{j=1}^m \left(\int f_j\right)^{p_j}. \tag{3}$$

This conjectural inequality is known for data of "Loomis–Whitney type" – that is, for which

$$\bigoplus_{j=1}^m \ker(dB_j(0)) = \mathbb{R}^n;$$

see [BCW, BB, KS, C]. For data not of Loomis–Whitney type this has been verified in some special cases in unpublished work of Stovall. For arbitrary data a weaker version, requiring a small amount of regularity of the input functions f_j, is also available – see [BBFL].

The purpose of this note is to report further partial progress on (3). For this we require a definition.

Definition 1 (Simple Brascamp–Lieb data; see [BCCT1]). A Brascamp–Lieb datum is *simple* if (2) holds with strict inequality for all nonzero proper subspaces V.

It was shown in [V] that $\{\mathbf{L} : (\mathbf{L},\mathbf{p}) \text{ simple}\}$ is an open set, and so the simple Brascamp–Lieb data is a natural class to consider from the point of view of perturbations. Of particular relevance for us is that for simple data the functional $\mathrm{BL}(\mathbf{L},\mathbf{p};\cdot)$ has unique (up to elementary scalings) gaussian extremisers, and that these extremisers (appropriately normalised) depend smoothly on $\mathbf{L}$; see [BCCT1] and [V] respectively. Simple Brascamp–Lieb data frequently arises in interesting situations – for example the sharp Young convolution inequality for all nontrivial Lebesgue exponents.

Theorem 3. *Suppose $\mathbf{B}$ is an m-tuple of smooth submersions in a neighbourhood of $0\in\mathbb{R}^n$ for which $(\mathbf{dB}(0),\mathbf{p})$ is simple. Then, given any $\epsilon>0$ there exists a* *neighbourhood U of $0 \in \mathbb{R}^n$ such that*

$$\int_U \prod_{j=1}^m (f_j \circ B_j)^{p_j} \leq (1+\epsilon) BL(\mathbf{dB}(0), \mathbf{p}) \prod_{j=1}^m \left(\int f_j \right)^{p_j}. \tag{4}$$

We observe that the inequality (4) is a strengthening of (3) permitting c to be taken arbitrarily close to 1. In the case of Loomis–Whitney data at least, further refinements of this type follow from the factorisation approach of Carbery [C]. Our proof proceeds by the method of induction on scales, and is a variant of the approach in [BB].

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Applications of Sparse domination through localization

Frédéric Bernicot
(joint work with Cristina Benea)

In this talk, we have presented several results obtained in [BB], inspired by the very recent mathematical trend around the *sparse domination* : consider a L^2-bounded linear operator T on $\mathbb{R}^n$ (or a doubling Riemannian manifold), then we consider the sparse domination of the bilinear form, which means that for two functions $f, g \in L^2$ there exists a sparse collection of dyadic cubes $\mathcal{S} := (I)_I \subset \mathbb{D}$ such that

$$|\langle Tf, g\rangle| \lesssim \sum_{I\in\mathcal{S}} \left(\fint_I |f|\, dx \right) \cdot \left(\fint_I |g|\, dx \right) \cdot |I|.$$

We recall that for $\alpha \in \{0, \frac{1}{3}, \frac{2}{3}\}^n$, we denote the shifted dyadid grid

$$\mathbb{D}^\alpha := \left\{2^{-k}\left([0,1]^n + m + (-1)^k\alpha\right),\ k \in \mathbb{Z},\ m \in \mathbb{Z}^n\right\}$$

and by $\mathbb{D} := \cup_\alpha \mathbb{D}^\alpha$, the full collection of dyadic cubes. In such a setting, a collection of dyadic cubes $\mathcal{S} \subset \mathbb{D}$ is called *η-sparse* for some $\eta \in (0,1)$ if for all $\alpha \in \{0, \frac{1}{3}, \frac{2}{3}\}^n$, there exists a collection of measurable subsets $(E_Q)_{Q\in\mathcal{S}\cap\mathbb{D}^\alpha}$ such that for all $Q \in \mathcal{S} \cap \mathbb{D}_\alpha$

$$E_Q \subset Q \qquad \text{and} \qquad |E_Q| \geq \eta |Q|.$$

In [BB], we wanted to present the following general principle: if T is an operator satisfying some specific extra properties, then it is possible to track this property in the sparse domination and to obtain an improved domination. Indeed, we can expect a domination of the form

$$|\langle Tf, g\rangle| \lesssim \sum_{I\in\mathcal{S}} a_I(f)\cdot b_I(g)\cdot |I|,$$

with a sparse collection $\mathcal{S}$ and some coefficients $a_I(f)$ (resp. $b_I(g)$) which can be seen as a local version of the initial operator T (resp. T^*), and so could reflect some extra properties. We describe this principle in different situations:

- If T is a Calderón-Zygmund operator and satisfies a condition $T(1) = 0 \in$ BMO then $a_I(f)$ can be chosen as the oscillation of f on I (reflecting the $T(1) = 0$ condition). If $T^*(1) = 0 \in$ BMO then $b_I(g)$ can be chosen as the oscillation. If both $T(1) = 0 = T^*(1)$ then we can have simultaneously the oscillation on f and g. Such an observation can have some application for the composition of two Calderón-Zygmund operators. Similar observations can be done for a higher order regularity property than the oscillation (as the gradient of a Calderón-Zygmund operator).
- If T is a Haar multiplier, then this "frequency structure" can be tracked and we can have a sparse domination with

$$a_I(f) = \left(\fint_I |S_I(f)|^p\, dx\right)^{1/p}$$

(and similarly for $b_I(g)$) for an arbitrary exponent $p \in (0,\infty)$ and where S_I stands for the localized square function. Observe that the fact that we can play with the exponent p, is a consequence of the John-Nirenberg inequality. This result is obtained by a combination of this last inequality with a suitable stopping time argument, to select the good sparse collection. As an application, we have obtained that such a linear operator is bounded in every weighted Hardy space H^p_ω for arbitrary exponent $p \in (0,\infty)$ and arbitrary weight ω, with an implicit constant depending only on p. Here, the weighted Hardy space H^p_ω is the set of functions f whose the full square function Sf belongs to L^p_ω.
- An other application of a similar decomposition is the following improvment of the well-known fact: a H^1 (Hardy space) function admits an atomic decomposition. Indeed, it can be also proved that we can choose

an atomic decomposition whose the collection of the supports of the atoms is sparse.

- A last application that we have presented is the combination of the previous observations, applied to the operator $T = Id$. Indeed, for two generic functions $f, g \in L^2$ then it can be proved that

$$|\langle f, g\rangle| \lesssim \sum_{I \in \mathcal{S}} \mathrm{osc}_I(f) \cdot \mathrm{osc}_I(g) \cdot |I|$$

and so

$$|\langle f, g\rangle| \lesssim \int \mathcal{M}^\sharp f \cdot \mathcal{M}^\sharp g \, dx,$$

(where $\mathcal{M}^\sharp$ is the maximal sharp function) which can be viewed as a polarized version of the L^2-Fefferman-Stein inequality.

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Remarks on multijoints formed by lines and a k-plane.

Anthony Carbery
(joint work with Marina Iliopoulou)

We establish a result counting the multijoints formed by families of lines and a family of k-planes in $\mathbb{F}^n$ where $\mathbb{F}$ is an arbitrary field. Indeed, let $\mathcal{L}_1, \dots, \mathcal{L}_{n-k}$ be families of lines in $\mathbb{F}^n$ each with cardinality L and let $\mathcal{P}$ be a family of k-planes with cardinality P. We say that $x \in \mathbb{F}^n$ is a multijoint for $\mathcal{L}_1, \dots, \mathcal{L}_{n-k}, \mathcal{P}$ if there are $L_j \in \mathcal{L}_j$ and $K \in \mathcal{P}$ which meet at x and such that the the directions of $\{L_1, \dots, L_{n-k}, K\}$ span $\mathbb{F}^n$. We denote the set of multijoints by J. We prove that

$$|J| \leq C_n L P^{1/(n-k)}$$

where C_n depends only on n. This is work in progress.

In the talk we relate this result to others in the area of continuous and discrete multilinear k_j-plane multilinear Kakeya theorems, in particular to recent work of Ruixiang Zhang.

Entangled multilinear forms and applications

Polona Durcik
(joint work with Vjekoslav Kovač, Luka Rimanić, Kristina Ana Škreb, and Christoph Thiele)

As a model example we consider a quadrilinear singular integral form acting on two-dimensional functions $F_1, F_2, F_3, F_4 : \mathbb{R}^2 \to \mathbb{C}$, given by

$$\Lambda(F_1,\ldots,F_4) := \text{p.v.} \int_{\mathbb{R}^4} F_1(x,y)F_2(u,y)F_3(x,v)F_4(u,v)\kappa(x-u,y-v)dxdudydv,$$

where $\widehat{\kappa}$ is a smooth function on $\mathbb{R}^2 \setminus \{(0,0)\}$ satisfying the standard symbol estimates. That is, for any multi-index α there is a finite constant C_α such that

$$|\partial^\alpha \widehat{\kappa}(\xi,\eta)| \leq C_\alpha \|(\xi,\eta)\|^{-|\alpha|}$$

for all $(0,0) \neq (\xi,\eta) \in \mathbb{R}^2$. Informally we refer to this form as being entangled, because the two-dimensional functions share certain one-dimensional variables.

The case when F_4 is constant was proposed by Demeter and Thiele [DT] as the dual of a particular instance of the two-dimensional bilinear Hilbert transform. This was the only case which was left unresolved in [DT]. Its boundedness in a certain range of L^p spaces was obtained by Kovač [K12]. The general case of the above quadrilinear form, i.e. when the constant function is replaced by a general function F_4, was studied in [Dα], [Dβ]. The dyadic model of the above quadrilinear form had been previously addressed in [K12], [K11].

A related object is the so-called triangular Hilbert transform, which is a trilinear singular integral form given by

$$\Lambda_{\mathrm{T}}(F_1,F_2,F_3) := \text{p.v.} \int_{\mathbb{R}^3} F_1(x,y)F_2(y,z)F_3(z,x)\frac{1}{x+y+z}dxdydz.$$

By choosing the functions F_j properly, the triangular Hilbert transform specializes to the Carleson operator and the bilinear Hilbert transform. L^p boundedness of the triangular Hilbert transform is a major open problem.

The papers [Z] and [T] initiated the study of the truncated triangular, and more generally, the truncated simplex Hilbert transform. The truncated triangular Hilbert transform is defined by

$$\Lambda_{\mathrm{T},r,R}(F_1,F_2,F_3) := \int_{r<|x+y+z|<R} F_1(x,y)F_2(y,z)F_3(z,x)\frac{1}{x+y+z}dxdydz,$$

where $0 < r < R < \infty$. An application Hölder's inequality yields the bound

$$(1) \qquad |\Lambda_{\mathrm{T},r,R}(F_1,F_2,F_3)| \leq C\Big(\log\frac{R}{r}\Big)^\alpha \|F_1\|_{\mathrm{L}^{p_1}(\mathbb{R}^2)}\|F_2\|_{\mathrm{L}^{p_2}(\mathbb{R}^2)}\|F_3\|_{\mathrm{L}^{p_3}(\mathbb{R}^2)}$$

with $\alpha = 1$, for any choice of exponents $1 \leq p_1,p_2,p_3 \leq \infty$ which satisfy the Hölder scaling. An estimate with $\alpha = 0$ would imply an estimate for Λ_{T}.

The paper [DKT] establishes cancellation estimates (1) for the triangular Hilbert transform with a power $0 < \alpha < 1$ depending only on the exponents $1 < p_1, p_2, p_3 <$

∞, which satisfy the Hölder scaling. More generally, the paper [DKT] obtains cancellation estimates analogous to (1) for the truncated simplex Hilbert transform. Such estimates can be obtained by a structural induction, the base of the induction being of similar complexity as the form Λ and its higher-dimensional variants.

Bounds for multilinear singular integrals have applications to questions on distances in point configurations in positive density subsets of the Euclidean space. The upper Banach density of a set $A \subseteq \mathbb{R}^d$ is

$$\overline{\delta}_d(A) := \limsup_{N\to\infty} \sup_{x\in\mathbb{R}^d} \frac{\big|A \cap (x + [0,N]^d)\big|}{|x + [0,N]^d|}.$$

If $d \geq 2$, a set of positive upper Banach density in $\mathbb{R}^d$ contains all large distances. More precisely, if $d \geq 2$ and $\overline{\delta}_d(A) > 0$, there exists $\lambda_0(A)$ such that for any $\lambda \geq \lambda_0(A)$ the set A contains points $x, x+s$ with $\|s\|_{\ell^2} = \lambda$. This was shown independently by Bourgain [B], Falconer and Marstrand [FM], and Furstenberg, Katznelson and Weiss [FKW]. However, the same statement fails if $x, x+s$ is replaced by a three-term arithmetic progression $x, x+s, x+2s$. A counterexample was constructed in [B], and it uses the fact that the ℓ^2 norm satisfies the parallelogram identity.

One may ask what holds true if ℓ^2 is replaced by ℓ^p for $p \neq 2$. Cook, Magyar, and Pramanik [CMP] showed that if $1 < p < \infty$, $p \neq 2$, and the dimension d is large enough, then an arbitrary measurable subset of $\mathbb{R}^d$ contains three-term arithmetic progressions $x, x+s, x+2s$ such that the ℓ^p norm of the common difference s attains all sufficiently large real values. The authors of [CMP] reduced this problem to the bounds for certain singular integrals related to the bilinear Hilbert transform.

The paper [DKR] generalizes the result from [CMP] to corners in subsets of $\mathbb{R}^d \times \mathbb{R}^d$. The main result obtained in [DKR] is the following.

Theorem 1. *For any $1 < p < \infty$, $p \neq 2$, there exists $d_p \geq 2$ such that for every integer $d \geq d_p$ the following holds. For any measurable set $A \subseteq \mathbb{R}^d \times \mathbb{R}^d$ with $\overline{\delta}_d(A) > 0$ one can find $\lambda_0(A) > 0$ such that for any real number $\lambda \geq \lambda_0(A)$, there exist $x, y, s \in \mathbb{R}^d$ such that (x,y), $(x+s,y)$, $(x,y+s) \in A$ and $\|s\|_{\ell^p} = \lambda$.*

The proof of this theorem leads to studying bounds for certain higher-dimensional versions of the singular integral form

$$\text{(2)} \quad \text{p.v.} \int_{\mathbb{R}^4} F_1(x,y)F_2(u,y)F_3(x,v)F_4(u,v)\kappa(y-x-u, v-x-u)\,dxdudydv.$$

Generalizations of [CMP] to longer progressions and generalizations of [DKR] to higher-dimensional corners seem to be out of reach of the currently available techniques. The approach from [CMP], [DKR] leads to operators which are of similar complexity as the multilinear and simplex Hilbert transform. Bounds for the multilinear and simplex Hilbert transform remain an open problem.

A further application of estimates for multilinear forms is in the study of quantitative L^p norm convergence of ergodic averages

$$\frac{1}{n}\sum_{i=0}^{n-1} f(S^i x)g(T^i x) \tag{3}$$

as $n \to \infty$, where $S, T : X \to X$ are two commuting measure preserving transformations on a probability space $(X, \mathcal{F}, \mu)$, and $f, g \in \mathrm{L}^\infty(X)$. The paper [DKST] establishes quantitative bounds for convergence of the sequence of averages (3) in the norm by bounding its norm-variation.

The problem of showing quantitative bounds for norm convergence of (3) can be transferred to a problem on the Euclidean space. That is, one is lead to studying norm-variations of certain sequences of averages on $\mathbb{R}^2$. These in turn relate to multilinear forms such as (2).

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Wiener's lemma and orbits along primes

Tanja Eisner
(joint work with Bálint Farkas)

The classical Wiener lemma states that every complex Borel measure μ on the unit circle $\mathbb{T}$ satisfies

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} |\hat{\mu}(n)|^2 = \sum_{a \text{ atom}} |\mu(\{a\})|^2,$$

where $\hat{\mu}(n)$ denotes the nth Fourier coefficient of μ. As a corollary, a probability measure μ on $\mathbb{T}$ satisfies $\lim_{n\to\infty} \hat{\mu}(n) = 1$ if and only if μ is a Dirac measure.

Motivated by ergodic theorems along subsequences due to Furstenberg, Bourgain, Wierdl, Nair and others, we consider the following natural question.

Question. For which subsequences $(k_n) \subset \mathbb{N}$ do the assertion of Wiener's lemma and the corollary hold along (k_n)?

We first state a generalization of Wiener's lemma to *good* subsequences, i.e., subsequences $(k_n) \subset \mathbb{N}$ for which the limit

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \lambda^{k_n}$$

exists for every $\lambda \in \mathbb{T}$. Moreover, we call a subsequence $(k_n) \subset \mathbb{N}$ *extremal* if every probability measure μ on $\mathbb{T}$ with $\lim_{n\to\infty} \hat{\mu}(k_n) = 1$ is Dirac. We discuss classes of examples of extremal sequences such as sequences of positive upper density, primes, polynomials and polynomials of primes as well as connections to sequences coming from ergodic theory such as return times sequences and rigidity sequences.

We further discuss consequences for orbits of operators. For example, if a contraction T on a Hilbert space satisfies $\lim_{n\to\infty} T^{p_n} = I$ in the weak operator topology, where p_n denotes the nth prime, then $T = I$. Here, p_n can be replaced by p_n^2 or other polynomial of primes with certain algebraic property. More directions of research (such as extensions to Banach space operators) are indicated at the end of the talk.

Existence of optimizers for the Stein–Tomas inequality

Rupert L. Frank
(joint work with Elliott H. Lieb and Julien Sabin)

Many results in analysis concern the boundedness of linear operators. Once this boundedness is established, there are several natural follow-up questions: What is the norm of the operator and is the norm attained (that is, is the supremum defining the norm a maximum)? Moreover, are optimizing sequences (that is, normalized sequences approaching the supremum in the definition of the norm) precompact? In the presence of symmetries, one is interested in precompactness modulo symmetries. Note that precompactness of optimizing sequences implies that the norm is attained.

Here we are concerned with the precompactness of optimizing sequences for the Stein–Tomas inequality. It is illuminating to compare it to the corresponding problem for the Strichartz inequality.

The Strichartz inequality states that

$$\mathcal{S}_d := \sup_{\psi \not\equiv 0} \frac{\|e^{it\Delta/2}\psi\|^{2+4/d}_{L^{2+4/d}_{t,x}(\mathbb{R}\times\mathbb{R}^d)}}{\|\psi\|^{2+4/d}_{L^2(\mathbb{R}^d)}} .$$

For $d = 1, 2$ the sharp constant was found in [Fo07] and all optimizers were identified. Precompactness modulo symmetries was shown in [K] for $d = 1$ and in [Sh] for general d. It is conjectured that Gaussians are optimizers.

The Stein–Tomas inequality states that

$$\mathcal{R}_N := \sup \frac{\int_{\mathbb{R}^N} |\check{f}|^q \, dx}{\|f\|^q_{L^2(\mathbb{S}^{N-1})}} < \infty , \qquad q = \frac{2(N+1)}{N-1} , \qquad \check{f}(x) := \int_{\mathbb{S}^{N-1}} f(\omega) e^{i\omega\cdot x} \, d\omega .$$

For $N = 3$ the sharp constant was found in [Fo15]. Precompactness modulo symmetries was shown in [CS] for $N = 3$ and in [Sh] for $N = 2$. It is conjectured that constants are optimizers.

The restriction to dimensions $d = 1, 2$ in the Strichartz inequality and to $N = 2, 3$ in the Stein–Tomas inequality ensures that the exponents are even integers.

The following is our main result [FLS].

Theorem 1. *Let $N \geq 2$. If*

$$(\star) \qquad \mathcal{R}_N > \rho_N \, \mathcal{S}_{N-1} \qquad \text{with} \qquad \rho_N = \frac{2^{q/2}}{\sqrt{\pi}} \frac{\Gamma(\frac{q+1}{2})}{\Gamma(\frac{q+2}{2})} ,$$

then maximizing sequences for $\mathcal{R}_N$, normalized in $L^2(\mathbb{S}^{N-1})$, are precompact in $L^2(\mathbb{S}^{N-1})$ up to modulation (i.e., multiplication by $e^{ia\cdot\omega}$) and, in particular, there is a maximizer for $\mathcal{R}_N$.

That is, $(\star)$ guarantees precompactness and existence. Our proof also shows that $(\star)$ holds with $\geq$ and that, if $(\star)$ holds with $=$, then there is a maximizing sequence which is *not* precompact up to modulations. Thus, $(\star)$ is necessary and sufficient for precompactness.

Corollary. *Let $N \geq 2$ and assume that $\mathcal{S}_{N-1}$ is attained for e^{-x^2}. Then $(\star)$ holds and therefore, there is a maximizer for $\mathcal{R}_N$.*

The main difficulty in the proof of the theorem lies in proving that the weak limit of an optimizing sequence is not identically zero. To achieve this, one clearly needs to take care of modulations, but this is mostly technical. The real problem is to rule out concentration or, more precisely, concentration at an antipodal pair of points. One needs to show that such concentration leads at most to a value of $\rho_N \mathcal{S}_{N-1}$ of the variational quotient and by assumption $(\star)$ this is not possible for an optimizing sequence (for which the variational quotient tends to the strictly larger quantity $\mathcal{R}_N$). We note that if the sequence concentrates at a single point the maximal value is $\mathcal{S}_{N-1}$, and therefore the fact that $\rho_N > 1$ means that there is a non-local attractive interaction between antipodal concentration points. This is in stark contrast to other optimization problems with (approximate) dilation invariance and constitutes the main new feature of this problem.

The proof of the theorem is split into two parts. In the first part one shows that 'there is something somewhere' (that is, one excludes vanishing in Lions' terminology) and in the second part one shows that 'there is nothing anywhere else' (that is, one excludes dichotomy). The second step uses a method introduced by Lieb in his analysis of the sharp Hardy–Littlewood–Sobolev inequality and uses the Brézis–Lieb lemma and an elementary inequality. For the first part one uses the following refinement of the Stein–Tomas inequality.

Theorem 2. *There are $\sigma_N \in (0,1)$ and C_N such that for all $f \in L^2(\mathbb{S}^{N-1})$*

$$\|\check{f}\|_{L^q(\mathbb{R}^N)} \leq C_N \left(\sup_\alpha \sup_{Q\in\mathcal{D}} |Q|^{-1/2} \left\| (\chi_{L_{\theta_\alpha}(Q)} \chi_{C_\alpha} f)^\vee \right\|_{L^\infty(\mathbb{R}^N)} \right)^{1-\sigma_N} \|f\|_{L^2(\mathbb{S}^{N-1})}^{\sigma_N}.$$

Here $\mathcal{D}$ denotes the set of all dyadic cubes, and we have chosen finitely many caps C_α centered at $\theta_\alpha \in \mathbb{S}^{N-1}$ with projections L_{θ_α}.

This should be compared with the following refined Strichartz inequality, which says that there are $\sigma_d \in (0,1)$ and C_d such that for all $\psi \in L^2(\mathbb{R}^d)$

$$\|e^{it\Delta/2}\|_{L^{2+4/d}_{t,x}(\mathbb{R}\times\mathbb{R}^d)} \leq C_d \left(\sup_{Q\in\mathcal{D}} |Q|^{-1/2} \left\| e^{it\Delta/2} (\chi_Q \hat{\psi})^\vee \right\|_{L^\infty_{t,x}(\mathbb{R}\times\mathbb{R}^d)} \right)^{1-\sigma_d} \|\psi\|_{L^2(\mathbb{R}^d)}^{\sigma_d}.$$

Both inequalities are based on deep bilinear restriction estimates. These inequalities and their application are reminiscent of arguments in the construction of a minimal obstruction to global well-posedness of the mass-critical non-linear Schrödinger equation. There is an important difference, however, between the Strichartz and the Stein–Tomas case. Namely, in the former case concentration can be simply removed by scaling. In the latter case it needs to be discussed by a separate argument, and this is where ($\star$) comes into play.

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Sparse domination and weighted estimates for operators beyond Calderón-Zygmund theory

Dorothee Frey
(joint work with F. Bernicot, S. Petermichl, and B. Nieraeth)

In recent years, after a solution was found to the the well-known A_2 conjecture, the role of sparse operators has become increasingly important in the weighted theory of singular integral operators. Sparse domination yields optimal quantitative A_p estimates for $1 < p < \infty$ for Calderón-Zygmund operators. In this talk we report on recent progress on optimal weighted estimates for operators beyond the class of Calderón-Zygmund operators. The most prominent example of such an operator is the Riesz transform $\nabla L^{-1/2}$ associated with an elliptic operator $L = -\mathrm{div} A\nabla$, with $A \in L^\infty(\mathbb{R}^n; M^n(\mathbb{C}))$. Our setting also includes multipliers and paraproducts associated with L, and e.g. the Riesz transform associated with the Neumann Laplacian on a convex doubling domain in $\mathbb{R}^n$. Our assumptions are minimal and in particular apply to operators whose unweighted continuity is restricted to Lebesgue spaces with certain ranges of exponents (p_0, q_0) where $1 \leq p_0 < 2 < q_0 \leq \infty$. In [BFP], we show that these operators satisfy a sparse domination property. That is, for $p \in (p_0, q_0)$ there exists $C > 0$ such that for all $f \in L^p$ and $g \in L^{p'}$ both supported in $5Q_0$ for some $Q_0 \in \mathcal{D}$, there exists a sparse collection $\mathcal{S} \subset \mathcal{D}$ (depending on f, g) with

$$\left| \int_{Q_0} Tf \cdot g \, dx \right| \leq C \sum_{P \in \mathcal{S}} |P| \left(⨍_{5P} |f|^{p_0} \, dx \right)^{1/p_0} \left(⨍_{5P} |g|^{q_0'} \, dx \right)^{1/q_0'} . \tag{1}$$

Here, $\mathcal{D}$ is a family of a finite number of shifted dyadic grids, and a collection $\mathcal{S}$ is called sparse if it satisfies a certain Carleson condition. Given the sparse domination (1), we then establish weighted estimates for the operator T. We show that for $p \in (p_0, q_0)$, there exists $c_p > 0$ such that for all $\omega \in A_{p/p_0} \cap RH_{(q_0/p)'}$

$$\|T\|_{L^p_\omega \to L^p_\omega} \leq c_p([\omega]_{A_{p/p_0}} [w]_{RH_{(q_0/p)'}})^\alpha \tag{2}$$

with

$$\alpha := \max \left\{ \frac{1}{p - p_0}, \frac{q_0 - 1}{q_0 - p} \right\} .$$

Note that in the case $(p_0, q_0) = (1, \infty)$, one obtains the same estimate as for Calderón-Zygmund operators.

We moreover show that given the sparse domination (1), the weighted estimate in (2) is sharp. Since our assumptions on the operator T are minimal, we cannot show optimality of (2) in itself, but in [FN] we establish optimality of (2), given the asymptotic behaviour of the unweighted operator norm $\|T\|_{L^p \to L^p}$ for $p \to p_0$ and $p \to q_0$. The argument relies on a Rubio de Francia iteration argument. The asymptotic behaviour is for example known in the case of the Riesz transform on two copies of $\mathbb{R}^n$ glued smoothly along their unit circles, so that in this case sharpness of (2) is established.

In a second part, we report on progress on weighted weak type estimates. In [FN], we deduce quantitative weighted bounds involving A_1 directly from the sparse domination assumption (1). It has to be mentioned that we do not use any other properties of T. This is of particular importance since in the case $(p_0, q_0) = (1, \infty)$, the class of operators satisfying (1) is strictly larger than the class of Calderón-Zygmund operators. Our arguments use a Calderón-Zygmund decomposition adapted to sparse operators with the property that the "bad" part b cancels completely. We first obtain a strong type estimate involving A_1. That is, for $p \in (p_0, q_0)$, T satisfying (1) and $\omega \in A_1 \cap RH_{(q_0/p)'}$, we show that

$$\|T\|_{L^p(\omega)\to L^p(\omega)} \leq c c_p [\omega^{(q_0/p)'}]_{A_\infty}^{\frac{1}{p'}} [\omega^{(q_0/p)'}]_{A_1}^{\frac{1}{p(q_0/p)'}}, \tag{3}$$

with

$$c_p = \left[\left(\frac{p'}{q_0'}\right)'\right]^{\frac{1}{q_0'}} \left[\left(\frac{p_0'}{p'}\right)' \left(\frac{p}{p_0}\right)'\right]^{\frac{1}{p_0}}.$$

Similar to the case of Calderón-Zygmund operators, we deduce from the estimate (3) weak type (p_0, p_0) bounds for T.

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Averages over submanifolds of intermediate dimension and the Oberlin condition

Philip T. Gressman

The fundamental roles of affine arclength and affine hypersurface measures in harmonic analysis, particularly relating to the Fourier restriction problem and to the L^p-improving properties of convolution operators, have been widely recognized for many decades [CZ, DM, S] and lead naturally to the question of how these measures may be generalized to submanifolds of intermediate dimensions. Inspired by an observation of D. Oberlin [O] for affine hypersurface measure, we construct a measure $\mu_{\mathcal{A}}$ on immersed submanifolds of dimension d in $\mathbb{R}^n$ for any $d < n$ such that the following hold:

- If the immersed submanifold $\mathcal{M}$ is acted upon by a measure-preserving affine transformation of $\mathbb{R}^n$, then the affine measure associated to the new submanifold is simply the pushforward of the original measure via the affine transformation.
- If the immersion satisfies sufficient algebraic regularity conditions (which include the case when the immersion is real analytic in some coordinate system), then the measure $\mu_{\mathcal{A}}$ satisfies the so-called Oberlin condition:

there is a finite positive constant C such that for any compact, convex set $K \subset \mathbb{R}^n$,

$$\mu_{\mathcal{A}}(K) \leq C|K|^{\alpha}, \tag{1}$$

where $|\cdot|$ denotes the Lebesgue measure of K. Here the constant α is specifically fixed to be the largest possible exponent for which some nonzero measure on a d-dimensional submanifold of $\mathbb{R}^n$ exists satisfying (1). As in the case of Oberlin's original work, the value of α may be expressed as a ratio d/Q, where Q is the appropriate analogue of a homogeneous or scaling dimension. The condition (1) should be regarded as a kind of measure-theoretic curvature condition, because (1) cannot hold for any $\alpha > 0$ if the support of the measure is contained an affine hyperplane in $\mathbb{R}^n$ (simply because the μ-measure of a convex set K would be independent of the thickness of K in the direction transverse to the hyperplane while the Lebesgue measure could be taken arbitrarily small for sets K which are "thin" in this transverse direction).
- Any measure μ on the immersed submanifold $\mathcal{M}$ which satisfies (1) for the same exponent α must be pointwise dominated by some constant times the affine measure $\mu_{\mathcal{A}}$.

The construction of this measure involves two important new ideas. The first is that it is possible by a minor modification of a construction of Kempf and Ness [KN] to canonically associate a measure to any tensor of type $(0,k)$ on a manifold $\mathcal{M}$. This construction yields exactly the measure $\mu_{\mathcal{A}}$ when applied to an affine curvature tensor which is itself constructed rather directly from the immersion and its various derivatives. The second new idea in the proof is a generalization to higher dimensions of an earlier integral estimate [G] which shows that the average value of a polynomial p of degree m on some measurable set $E \subset \mathbb{R}$ is bounded below by a constant depending only on m times the supremum of p and its derivatives (appropriately weighted) on some interval I which is independent of p and contains some nontrivial proportion of the original set E (nontrivial meaning that the proportion is bounded below by some constant depending on m). In higher dimensions, the role of I is filled by a more general semialgebraic set, and corresponding derivative estimates are given for some family of smooth vector fields which are adapted both to the arbitrary measurable set $E \subset \mathbb{R}^n$ and to the geometry of degree m polynomials in $\mathbb{R}^n$ restricted to E.

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Two-dimensional Parsell-Vinogradov systems

Shaoming Guo
(joint work with Jean Bourgain, Ciprian Demeter; Ruixiang Zhang)

For $d, s \geq 1$ and $k \geq 2$, consider the integer solutions

$$x_{1,j}, x_{2,j}, ..., x_{d,j}, y_{1,j}, y_{2,j}, ..., y_{d,j},\ 1 \leq j \leq s \tag{1}$$

of the system of Diophantine equations (often referred to as the *Parsell–Vinogradov* system)

$$\sum_{j=1}^{s} x_{1,j}^{i_1} x_{2,j}^{i_2} ... x_{d,j}^{i_d} = \sum_{j=1}^{s} y_{1,j}^{i_1} y_{2,j}^{i_2} ... y_{d,j}^{i_d}. \tag{2}$$

Here $0 \leq i_1, i_2, ..., i_d \leq k$ are all integers such that $1 \leq i_1 + i_2 + ... + i_d \leq k$. For instance, when $d = 1$, the system (2) consists of k equation

$$\sum_{j=1}^{s} x_j^i = \sum_{j=1}^{s} y_j^i,\ \text{with } 1 \leq i \leq k. \tag{3}$$

When $d = k = 2$, the system (2) becomes

$$\begin{aligned}
&x_{1,1} + x_{1,2} + ... + x_{1,s} = y_{1,1} + y_{1,2} + ... + y_{1,s},\\
&x_{2,1} + x_{2,2} + ... + x_{2,s} = y_{2,1} + y_{2,2} + ... + y_{2,s},\\
&x_{1,1}^2 + x_{1,2}^2 + ... + x_{1,s}^2 = y_{1,1}^2 + y_{1,2}^2 + ... + y_{1,s}^2,\\
&x_{2,1}^2 + x_{2,2}^2 + ... + x_{2,s}^2 = y_{2,1}^2 + y_{2,2}^2 + ... + y_{2,s}^2,\\
&x_{1,1}x_{2,1} + x_{1,2}x_{2,2} + ... + x_{1,s}x_{2,s} = y_{1,1}y_{2,1} + y_{1,2}y_{2,2} + ... + y_{1,s}y_{2,s}.
\end{aligned} \tag{4}$$

For a large constant N, we let $J_{s,d,k}(N)$ denote the number of integer solutions (1) of the system of equations (2) with $0 \leq x_{1,j}, ..., x_{d,j}, y_{1,j}, ..., y_{d,j} \leq N$ for each $1 \leq j \leq s$.

Closely related to the number of solutions (1) of the system of equations (2) are several sharp decoupling inequalities. For $d \geq 1$ and $k \geq 2$, let $\mathcal{S}_{d,k}$ be the d dimensional surface in $\mathbb{R}^n$ with

$$n = \binom{d+k}{k} - 1, \tag{5}$$

defined by

$$\mathcal{S}_{d,k} = \{\Phi(t_1, t_2, ..., t_d) : (t_1, t_2, ..., t_d) \in [0,1]^d\}, \tag{6}$$

where the entries of $\Phi(t_1, t_2, ..., t_d)$ consist of all the monomials $t_1^{i_1} t_2^{i_2} ... t_d^{i_d}$ with $1 \leq i_1 + i_2 + ... + i_d \leq k$. For a subset $R \subset [0,1]^d$, define the extension operator associated to the set R by

$$E_R^{(d,k)} g(x) = \int_R g(t) \exp(x \cdot \Phi(t)) dt. \tag{7}$$

For each $p \geq 2$ and $0 < \delta \leq 1$, we denote by $V_p^{(d,k)}(\delta, p)$ the smallest constant such that

$$\|E_{[0,1]^d}^{(d,k)} g\|_{L^p(\mathbb{R}^n)} \leq V_p^{(d,k)}(\delta, p)\Big(\sum_{\substack{\Delta:\ \text{cube in } [0,1]^d \\ l(\Delta) = \delta}} \|E_\Delta^{(d,k)} g\|_{L^p(\mathbb{R}^n)}^p \Big)^p, \tag{8}$$

The main results we proved are the following. Take $d = 2$. For each $k \geq 2$, denote

$$p_k = \frac{1}{3} k(k+1)(k+2). \tag{9}$$

For $p = p_k$ and for each $\epsilon > 0$, we have

$$V_p^{(2,k)}(\delta, p) \leq C_\epsilon \Big(\frac{1}{\delta}\Big)^{2(\frac{1}{2} - \frac{1}{p}) + \epsilon}. \tag{10}$$

Here $C_\epsilon > 0$ is a constant which depends only on ϵ. Moreover, our result further implies

$$J_{s,2,k}(N) \leq C_\epsilon N^{2s+\epsilon} + C_\epsilon N^{4s - \frac{k(k+2)(k+1)}{3} + \epsilon}, \tag{11}$$

for every $\epsilon > 0$. The case $k = 3$ of the estimate (11) was proven in [BDGuo]. The other cases $k \geq 4$ were handled in [GZ]. For the sharpness of (11), we refer to [PPW]. For prior developments, we refer to [BDGuth], [W12], [W16] and [W17].

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Reflectionless measures for Calderón-Zygmund Operators

Benjamin Jaye
(joint work with Fedor Nazarov)

Fix $d \geq 2$. For $s \in (0, d)$, we call a function K an s-dimensional Calderón-Zygmund kernel if it is an odd function, smooth away from the origin, satisfying the decay estimates $|D^\alpha K(x)| \leq C(\alpha)|x|^{-s-|\alpha|}$ for every $x \in \mathbb{R}^d \backslash \{0\}$ and multi-index α.

The basic question motivating the mathematics of this talk is the following:

The Basic Question. Fix an s-dimensional Calderón-Zygmund kernel K. If μ is a finite (non-negative Borel) measure for which the inequality $|K * \mu| \leq 1$ holds pointwise outside of the support of μ, then what can we say about μ?

This question first arose in the study of the removable sets for Lipschitz continuous solutions of elliptic partial differential equations, and as such, the s-Riesz kernel $\frac{x}{|x|^{s+1}}$ is of particular interest as it is the gradient of the fundamental solution of the operator $(-\Delta)^\alpha$, $\alpha = \frac{d-s+1}{2}$.

The non-homogeneous $T(1)$-theorem [NTV] enables the point-wise condition in The Basic Question to be replaced by the $L^2(\mu)$ boundedness of the Calderón-Zygmund operator associated to K, meaning that there is a constant $C > 0$ such that

$$\sup_{\varepsilon>0} \int_{\mathbb{R}^d} \Big| \int_{\mathbb{R}^d \backslash B(x,\varepsilon)} K(x-y) f(y) d\mu(y) \Big|^2 d\mu(x) \leq C \|f\|^2_{L^2(\mu)}$$

for every smooth function f with compact support. The advantage of the latter condition is that it is more amenable to multiscale analysis, by decomposing the operator in a suitable wavelet system. Beginning with the pioneering work of Jones [Jo1, Jo2], David and Semmes [DS, DS2], a substantial literature has been developed concerning the kind of geometric conditions that one should expect on μ from the boundedness of an associated Calderón-Zygmund operator, and very concrete conjectures are in place [DS, JNT]. What is lacking in the theory are tools that, in high dimension, act as a bridge between the analytic condition and the geometry for fields of particular interest. The theory is particularly underdeveloped for the s-Riesz transform when $s \in (1, d-1)$, where neither the Menger curvature formula (exploited in [AT, Leg, MMV, MPV, Tol]) nor the strong maximum principle (exploited in [ENV, HMM, JNRT, NToV]) are available.

A *reflectionless measure* ν associated to a Calderón-Zygmund kernel K is a measure for which $K * \nu$ is constant on the support of ν in a suitable weak sense. The basic properties of reflectionless measures were systematically developed in [JN1]. The paper [JN2] contains several several results which show that a description of the reflectionless measures associated to a Calderón-Zygmund kernel K enables us to say a lot about the The Basic Question.

In the talk several open problems will be stated, the solutions to which would shed light on the description of reflectionless measures of the s-Riesz transform. One of the open problems, for which the reader can find much more information in [EN], is as follows:

Question. Fix $s \in (1, d-1)$, and set $K(x) = \frac{x}{|x|^{s+1}}$ to be the s-Riesz kernel. Is there a constant $C > 0$ such that whenever f is a *non-negative* smooth function with compact support,

$$\max_{\mathbb{R}^d} |K * f| \leq C \max_{\text{Support of } f} |K * f|.$$

The answer to this question is 'yes' in the range $s \in (0,1) \cup [d-1, d)$. The answer is 'no' in general if one drops the assumption that the function f is non-negative.

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A sharp nonlinear Hausdorff-Young inequality for small potentials

Vjekoslav Kovač
(joint work with Diogo Oliveira e Silva and Jelena Rupčić)

Let us begin by describing the setting that is sometimes informally called the *nonlinear Fourier transform.* Its more precise synonyms are the SU$(1,1)$-*scattering transform* and the *Dirac scattering transform.* More details can be found in the unpublished note by Tao [T], while a similar discrete-time model is studied in the lecture notes by Tao and Thiele [TT].

Take a measurable, bounded, and compactly supported function $f\colon \mathbb{R}\to\mathbb{C}$ and an arbitrary number $\xi\in\mathbb{R}$. The matrix-valued initial value problem

$$\frac{d}{dx}\begin{bmatrix} a(x,\xi) & b(x,\xi) \\ \overline{b(x,\xi)} & \overline{a(x,\xi)} \end{bmatrix} = \begin{bmatrix} a(x,\xi) & b(x,\xi) \\ \overline{b(x,\xi)} & \overline{a(x,\xi)} \end{bmatrix} \begin{bmatrix} 0 & f(x)e^{-2\pi i x\xi} \\ \overline{f(x)}e^{2\pi i x\xi} & 0 \end{bmatrix},$$

$$\begin{bmatrix} a(-\infty,\xi) & b(-\infty,\xi) \\ \overline{b(-\infty,\xi)} & \overline{a(-\infty,\xi)} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

has unique absolutely continuous solutions $a(\cdot,\xi)$ and $b(\cdot,\xi)$. This way we arrive at the functions $a,b\colon \mathbb{R}\to\mathbb{C}$ given by $a(\xi):=a(+\infty,\xi)$, $b(\xi):=b(+\infty,\xi)$, and we can study properties of the "forward transform" $f\mapsto a,b$. Observe that the matrices containing $a(x,\xi)$ and $b(x,\xi)$ remain in the matrix group SU$(1,1)$, since the matrices containing $f(x)e^{-2\pi i x\xi}$ belong to its Lie algebra $\mathfrak{su}(1,1)$.

It is useful to rewrite the problem as a system of two scalar differential equations, and then in turn as a system of two integral equations,

$$a(x,\xi) = 1 + \int_{-\infty}^{x} \overline{f(y)}e^{2\pi i y\xi} b(y,\xi)dy, \quad b(x,\xi) = \int_{-\infty}^{x} f(y)e^{-2\pi i y\xi} a(y,\xi)dy.$$

Applying Picard's iteration one arrives at multilinear expansions for a and b. By the work of Christ and Kiselev [CK01a], [CK01b] these expansions are known to converge and extend the definition of the transform to the functions $f\in \mathrm{L}^p(\mathbb{R})$ for $1\le p<2$. However, Muscalu, Tao, and Thiele [MTTβ] showed that these expansions cannot be used for a general $f\in\mathrm{L}^2(\mathbb{R})$.

It is useful to emphasize that we are not talking about the linear Fourier transform on the group SU$(1,1)$. Indeed, $f\mapsto a,b$ are "very" nonlinear transformations and, for instance, they do not allow us to use any general form of interpolation. On the other hand, they still share many symmetries with the linear Fourier transform (with respect to L^1-dilations, translations, modulations, etc.); see [T].

One source of motivation for this setting comes from the eigenproblem for the *Dirac operator*,

$$L := \begin{bmatrix} \frac{d}{dx} & -\bar{f} \\ f & -\frac{d}{dx} \end{bmatrix}, \quad L\begin{bmatrix} \varphi(\cdot,\xi) \\ \psi(\cdot,\xi) \end{bmatrix} = -\pi i\xi \begin{bmatrix} \varphi(\cdot,\xi) \\ \psi(\cdot,\xi) \end{bmatrix};$$

see [T]. This eigenvector equation for the imaginary eigenvalue $-\pi i\xi$ turns into the above system for a and b, after we substitute $a(x,\xi):=\varphi(x,\xi)e^{\pi i x\xi}$ and $b(x,\xi):=\psi(x,\xi)e^{-\pi i x\xi}$. Another source of motivation are the general AKNS-ZS systems [AKNS], [ZS]. In a very special case, one can consider two bodies in a plane with mutual interactions. If $u_1(t), u_2(t) \in \mathbb{C}$ determine their positions at time t, $\omega_1 \neq \omega_2$ are given angular velocities, and $\lambda \in \mathbb{R}$ is a certain spectral parameter, then the motion of the system is governed by the differential equation

$$\begin{bmatrix} u_1'(t) \\ u_2'(t) \end{bmatrix} = i\lambda \begin{bmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + \begin{bmatrix} 0 & \overline{f(t)} \\ f(t) & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}.$$

This time we substitute $a(t,\lambda) := u_1(t)e^{-i\omega_1\lambda t}$, $b(t,\lambda) := u_2(t)e^{-i\omega_2\lambda t}$ and once again we arrive at the same system for a and b as before. The central problem in this model is to determine if the system remains bounded for a.e. $\lambda \in \mathbb{R}$; it is related to the first question stated below.

One open question about our nonlinear Fourier transform is a nonlinear analogue of the Carleson theorem: if $f \in \mathrm{L}^2(\mathbb{R})$ and $\mathrm{supp}(f) \subseteq [0,+\infty)$, does the limit

$$\lim_{x\to+\infty} \begin{bmatrix} a(x,\xi) & b(x,\xi) \\ \overline{b(x,\xi)} & \overline{a(x,\xi)} \end{bmatrix}$$

exist for a.e. $\xi \in \mathbb{R}$? Even finiteness of $\sup_x |a(x,\xi)|$ for a.e. $\xi \in \mathbb{R}$ is open. Christ and Kiselev [CK01b] showed the analogous claim for $f \in \mathrm{L}^p(\mathbb{R})$ when $1 \leq p < 2$, while Muscalu, Tao, and Thiele [MTTα] established it in the Cantor group "toy-model", where the exponentials are replaced with characters of a different group.

Another open question is related to the nonlinear analogues of the Hausdorff-Young inequalities, also due to Christ and Kiselev [CK01a], [CK01b]:

$$\|(\log|a|^2)^{1/2}\|_{\mathrm{L}^{p'}(\mathbb{R})} \leq C_p \|f\|_{\mathrm{L}^p(\mathbb{R})}$$

for $1 \leq p \leq 2$ and its conjugated exponent p'. The case $p = 1$ is trivial (with $C_1 = 1$) by Grönwall's lemma, while $p = 2$ leads to the well-known scattering identity (with $C_2 = 1$). This identity appears for instance in the work of Faddeev and Buslaev [FB] and it is shown by the contour integration; see [MTTα]. It is not known if the optimal constants C_p are bounded uniformly in $1 \leq p \leq 2$; this problem arises in the absence of any available interpolation. This uniformity was confirmed by Kovač [K12], but only in the same Cantor group model mentioned before.

Let us formulate the main result by Kovač, Oliveira e Silva, and Rupčić from [KOR]. Fix an exponent $1 < p < 2$, a number $H > 0$ interpreted as the "height", and a number $W > 0$ interpreted as the "width". Let us also recall *Babenko-Beckner's constant* $\mathbf{B}_p := p^{1/2p} p'^{-1/2p'}$, which is known (by [Ba] and [Be]) to be the norm of the linear Fourier transform from $\mathrm{L}^p(\mathbb{R})$ to $\mathrm{L}^{p'}(\mathbb{R})$. We only consider functions f with controlled height and width, i.e. such that $|f| \leq H$ and that f is supported in an interval of length at most W.

Theorem 1. *There exist $\delta > 0$ and $\varepsilon > 0$ (depending on p, H, W) such that for each f satisfying $\|f\|_{\mathrm{L}^1(\mathbb{R})} \leq \delta$ one has*

$$\left\|(\log|a|^2)^{\frac{1}{2}}\right\|_{\mathrm{L}^{p'}(\mathbb{R})} \leq \left(\mathbf{B}_p - \varepsilon\|f\|^2_{\mathrm{L}^1(\mathbb{R})}\right)\|f\|_{\mathrm{L}^p(\mathbb{R})}.$$

The theorem claims no uniform boundedness of the constants C_p in any sense, because δ depends on p. Indeed, a uniform estimate for functions satisfying (say) $\|f\|_{\mathrm{L}^1(\mathbb{R})} \leq 1$ follows simply from Grönwall's lemma. Therefore, the emphasis of the theorem is on the fact that the nonlinear Hausdorff-Young ratio beats the linear one for sufficiently small values of $\|f\|_{\mathrm{L}^1(\mathbb{R})}$.

The strategy of the proof is to denote by $\mathfrak{G}$ the set of modulated Gaussians, $G(x) = Ce^{-Ax^2+Bx}$ for some $A > 0$, $B, C \in \mathbb{C}$, and to distinguish between the following two cases.

Case 1. If the relative L^p-distance of f from $\mathfrak{G}$ is greater than $\|f\|_{\mathrm{L}^1(\mathbb{R})}^{1/2}$, then we use Christ's sharpened linear Hausdorff-Young inequality [C],

$$\|\widehat{f}\|_{\mathrm{L}^{p'}(\mathbb{R})} \leq \Big(\mathbf{B}_p - c_p \frac{\mathrm{dist}_p^2(f, \mathfrak{G})}{\|f\|_{\mathrm{L}^p(\mathbb{R})}^2}\Big) \|f\|_{\mathrm{L}^p(\mathbb{R})}.$$

It compensates for the loss coming from an application of Grönwall's lemma.

Case 2. If the relative L^p-distance of f from $\mathfrak{G}$ is smaller than $\|f\|_{\mathrm{L}^1(\mathbb{R})}^{1/2}$, then we calculate a few terms of the multilinear expansion for $(\log |a|^2)^{1/2}$ and approximate f by a Gaussian. In the process of controlling the error terms we apply the standard Menshov-Paley-Zygmund estimate several times.

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A weak type estimate for rough singular integrals

Andrei Lerner

Let $Tf = \text{p.v.} f * K$ be a singular integral operator on $\mathbb{R}^n$. We assume that T is L^2 bounded and $|K(x)| \le \frac{C}{|x|^n}$ for $x \ne 0$. For $0 < t < 1/2$ define

$$\omega(t) = \sup_{x\ne 0, |y|\le t|x|} |x|^n |K(x-y) - K(x)|.$$

Given $0 < \eta < 1$, we say that a family of cubes $\mathcal{S}$ is η-sparse if for every cube $Q \in \mathcal{S}$, there exists a measurable set $E_Q \subset Q$ such that $|E_Q| \ge \eta |Q|$, and the sets $\{E_Q\}_{Q\in\mathcal{S}}$ are pairwise disjoint.

In [CR, HRT, La, L16, LN], it was proved that if T is smooth enough, then the following pointwise estimate holds:

$$|Tf(x)| \le C \sum_{Q\in\mathcal{S}} \Big(\frac{1}{|Q|}\int_Q |f|\Big)\chi_Q(x). \tag{1}$$

Here $\mathcal{S}$ is a sparse family with the sparseness constant depending only on n. To be more precise, first (1) was proved in [CR, LN] under the log-Dini condition saying that $\int_0^{1/2} \omega(t) \log\frac{1}{t}\frac{dt}{t} < \infty$. After that, the log-Dini condition was extended to the Dini condition $\int_0^{1/2}\omega(t)\frac{dt}{t} < \infty$ in [La], with subsequent elaborations in [HRT, L16].

Estimate (1), in particular, easily implies the A_2 theorem of Hytönen [H] establishing the linear weighted bound:

$$\|T\|_{L^2(w)} \le C[w]_{A_2}, \tag{2}$$

where $[w]_{A_2} = \sup_{Q\subset\mathbb{R}^n} \frac{w(Q)w^{-1}(Q)}{|Q|^2}$.

A short proof of (1) is given in [L16], where the following principle was obtained: if T is a sublinear operator of weak type $(1,1)$ and the grand maximal truncated operator

$$M_T f(x) = \sup_{Q\ni x} \|T(f\chi_{\mathbb{R}^n\setminus 3Q})\|_{L^\infty(Q)}$$

is of weak type $(1,1)$, then T satisfies the pointwise bound (1). In particular, if T is a singular integral operator with ω satisfying the Dini condition, then

$$M_T f(x) \le CMf(x) + T^\star f(x), \tag{3}$$

where M is the Hardy-Littlewood maximal operator and $T^\star$ is the maximal singular integral. This estimate implies the weak type $(1,1)$ of M_T, thus proving (1) for this class of operators.

Note that all the above mentioned results actually hold for more general non-convolution Calderón-Zygmund operators. Also observe that the Dini condition has played the crucial role for (3) and thus for (1) and (2). Therefore, it is natural to ask whether one can extend (2) to singular integrals not satisfying such a good smoothness condition.

We consider now a class of rough homogeneous singular integrals defined by $T_\Omega f = \text{p.v.} f * K$, with $K(x) = \frac{\Omega(x/|x|)}{|x|^n}$, where $\Omega \in L^\infty(S^{n-1})$ and $\int_{S^{n-1}} \Omega d\sigma = 0$.

In [HRT], Hytönen, Roncal and Tapiola proved that

$$\|T_\Omega\|_{L^2(w)} \le C\|\Omega\|_{L^\infty}[w]_{A_2}^2. \tag{4}$$

This is the currently best known dependence of $\|T_\Omega\|_{L^2(w)}$ on $[w]_{A_2}$, and the central open question is whether one can improve it to the linear one as in (2).

An attempt to adapt the previous approach, based on the use of the maximal operator M_{T_Ω}, fails due to the lack of smoothness. Hence, a possible analogue of (3) for M_{T_Ω} is also an open question. Moreover, it is still an open question whether the maximal singular integral $T_\Omega^\star$ is of weak type $(1,1)$.

In [CCPO], Conde-Alonso, Culiuc, Di Plinio and Ou obtained a sparse domination principle not relying on the endpoint estimates of the maximal truncations. In particular, they proved that for all $p > 1$,

$$|\langle T_\Omega f, g\rangle| \le Cp'\|\Omega\|_{L^\infty} \sup_{\mathcal{S}} \sum_{Q\in\mathcal{S}} \langle f\rangle_{p,Q}\langle g\rangle_{1,Q}|Q|, \tag{5}$$

where $\langle f\rangle_{r,Q} = \left(\frac{1}{|Q|}\int_Q |f|^r\right)^{1/r}$, and the supremum is taken over all sparse families with uniform sparseness constant.

It is not difficult to show, via a standard argument, that (5) implies (4). More generally, if T satisfies

$$|\langle Tf, g\rangle| \le C(p')^\alpha \sum_{Q\in\mathcal{S}} \langle f\rangle_{p,Q}\langle g\rangle_{1,Q}|Q| \quad (p > 1, \alpha > 0), \tag{6}$$

then $\|T\|_{L^2(w)} \le C[w]_{A_2}^{1+\alpha}$.

In this talk, we discuss an approach to (6) based on the following modification of the grand maximal truncated operator. Given an operator T, define the maximal operator $M_{\lambda,T}$ by

$$M_{\lambda,T} f(x) = \sup_{Q\ni x} (T(f\chi_{\mathbb{R}^n\setminus 3Q})\chi_Q)^*(\lambda|Q|) \quad (0 < \lambda < 1),$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing the point x, and f^* denotes the non-increasing rearrangement of f.

Notice that $M_{\lambda,T} f \uparrow M_T$ as $\lambda \to 0$. On the other hand, if T is an arbitrary operator of weak type $(1,1)$, then $\|M_{\lambda,T}\|_{L^1\to L^{1,\infty}} \le \frac{C}{\lambda}$. These observations suggest that the dependence of

$$\Phi_T(\lambda) = \|M_{\lambda,T}\|_{L^1\to L^{1,\infty}}$$

on λ is closely related to a sparse domination for T, where T is a given operator of weak type $(1,1)$. Also, the behaviour of $\Phi_T(\lambda)$ shows how good (or how bad) the maximal truncations of T are, which is of independent interest.

In a recent work [L17], a sparse domination principle based on $\Phi_T(\lambda)$ was obtained. In particular, it was shown there that if $\Phi_T(\lambda) \le C \log^\alpha \frac{\mathrm{e}}{\lambda}$ and T is essentially self-adjoint, then (6) holds.

Our main new result [L17] is the following:

Theorem 1. *If $\Omega \in L^\infty(S^{n-1})$, then*

$$\Phi_{T_\Omega}(\lambda) \le C_n \|\Omega\|_{L^\infty} \log \frac{\mathrm{e}}{\lambda} \quad (0 < \lambda < 1). \tag{7}$$

The proof of this theorem is long and technical. One of its main ingredients is a decomposition found by Seeger [S] in his proof of the weak type $(1,1)$ of T_Ω.

By what was mentioned above, Theorem 1 recovers (5), and so (4). Any improvement of the logarithmical power in (7) would lead to the corresponding improvement of (4).

It is also of interest to try to improve (7) by showing that the maximal operator (with $\Omega \in L^\infty$)

$$M_{\exp L, T_\Omega} f(x) = \sup_{0<\lambda<1} \frac{M_{\lambda, T_\Omega} f(x)}{\log \frac{\mathrm{e}}{\lambda}}$$

is of weak type $(1,1)$. If this were true, a sparse domination for T_Ω would be improved to

$$|\langle T_\Omega f, g\rangle| \le C \|\Omega\|_{L^\infty(S^{n-1})} \sum_{Q \in \mathcal{S}} \langle f\rangle_{L \log L, Q} \langle g\rangle_{1,Q} |Q|$$

(that is, comparing to (5), $p'\langle f\rangle_{p,Q}$ for $p > 1$ would be replaced by a smaller $\langle f\rangle_{L\log L, Q}$ average).

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Recent progress on the pointwise convergence problems of Schrödinger equations

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(joint work with Xiumin Du, Larry Guth)

The solution to the free Schrödinger equation

$$\begin{cases} iu_t - \Delta u = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R} \\ u(x,0) = f(x), & x \in \mathbb{R}^n \end{cases} \tag{1}$$

is given by

$$e^{it\Delta}f(x) = (2\pi)^{-n} \int e^{i\left(x\cdot\xi + t|\xi|^2\right)} \widehat{f}(\xi)\, d\xi.$$

An interesting and important problem in PDE is to determine the optimal s, for which $\lim_{t\to 0} e^{it\Delta}f(x) = f(x)$ almost everywhere whenever $f \in H^s(\mathbb{R}^n)$. This problem originates from L. Carleson, who proved convergence for $s \geq 1/4$ when $n = 1$. B. Dahlberg and C. Kenig showed that the convergence does not hold for $s < 1/4$ in any dimension. P. Sjölin and L. Vega proved independently the convergence for $s > 1/2$ in all dimensions. However, the pointwise convergence holds for $s < 1/2$. For instance, some positive partial results were obtained by J. Bourgain, Moyua-Vargas-Vega, Tao-Vargas. S. Lee used Tao-Wolff's bilinear restriction method to get $s > 3/8$ for $n = 2$. Recently J. Bourgain, via Bourgain-Guth's multilinear restriction method, proved that $s > 1/2 - 1/(4n)$ is a sufficient condition for the pointwise convergence when $n \geq 2$. In the two dimensinal case, Bourgain's result coincides with Lee's.

For many years, it had seemed plausible that convergence actually holds for $s > 1/4$ in every dimension. Only in 2012, Bourgain gave a counterexample showing that this is false in sufficiently high dimensions. Improved counterexamples were given by Lucá-Rogers and Demeter-Guo. Very recently, Bourgain gave counterexamples showing that convergence can fail if $s < \frac{n}{2(n+1)}$, by using results related to Gauss sums. Later, Lucá-Roger also provided a different proof of Bourgain's divergence result.

It is very natural to conjecture

Conjecture 1. The convergence holds for $f \in H^s(\mathbb{R}^n)$ provided that $s > \frac{n}{2(n+1)}$.

We are able to answer this conjecture in two dimensional case.

Theorem 1. *For every $f \in H^s(\mathbb{R}^2)$ with $s > 1/3$, $\lim_{t\to 0} e^{it\Delta}f(x) = f(x)$ almost everywhere.*

We use $B(c,r)$ to represent a ball centered at c with radius r on $\mathbb{R}^2$. It is routine and standard that Theorem 1 is a consequence of the boundedness of the associated maximal function, i.e. there exists a constant C_ϵ such that

$$\Big\| \sup_{0<t\leq R} |e^{it\Delta}f| \Big\|_{L^3(B(0,R))} \leq C_\epsilon R^\epsilon \|f\|_2, \tag{2}$$

holds for all $R > 0$ and all f with $\mathrm{supp}\widehat{f} \subset \{\xi \in \mathbb{R}^2 : |\xi| \sim 1\}$. The proof of the sharp maximal estimate (2) is based on the polynomial partitioning method and Bourgain-Demeter's decoupling theorem.

Conjecture 1 can be reduced to the following question:

Conjecture 2. Is the following sharp maximal inequality true, for any $f \in L^2$ with frequency support in $\{\xi \in \mathbb{R}^n : |\xi| \sim 1\}$,

$$\Big\| \sup_{0<t\le R} |e^{it\Delta} f| \Big\|_{L^{\frac{2(n+1)}{n}}(B^n(R))} \le R^{\epsilon} \|f\|_2? \tag{3}$$

Here $B^n(R)$ stands for a ball in $\mathbb{R}^n$, of radius R.

We expect that some refined Strichartz estimates can be used to resolve (3).

Sub-elliptic harmonic analysis

Alessio Martini

Harmonic analysis and the Laplace operator. Many fundamental results and open questions of classical harmonic analysis are related to the Laplace operator $\Delta = -\sum_{j=1}^{d} \partial_j^2$ on $\mathbb{R}^d$. Of particular interest are L^p-boundedness properties of operators of the form $F(\Delta)$ and their relations with size and smoothness properties of the corresponding multipliers $F : \mathbb{R} \to \mathbb{C}$. A paradigmatic example is the celebrated Bochner–Riesz conjecture on the L^p-boundedness of $(1-\Delta)_+^{\delta}$, which, despite recent breakthroughs, has so far been settled only in dimension $d \le 2$. On the other hand, for $p = 1$ a complete solution to the Bochner–Riesz problem has been long available and follows from a general result: if F is compactly supported and belongs to an L^2 Sobolev space of order $s > d/2$, then $F(\Delta)$ is bounded on L^1. The Mihlin–Hörmander theorem further extends this result to non-compactly supported F. All these results are sharp: the smoothness threshold $d/2$ related to (weak or strong) L^1 boundedness cannot be lowered. In addition, these results admit a robust version, where Δ is replaced by a more general elliptic operator on a manifold M: under some constraint on the geometry, one obtains essentially the same results as in the Euclidean case [SS, GHS], and the critical order of smoothness related to L^1 boundedness is $d/2$, where d is the dimension of M.

Ellipticity, however, is not always a natural assumption: in the context of sub-Riemannian geometry, the natural substitute $\mathfrak{L}$ of the Laplace operator, called sub-Laplacian, need not be elliptic, but satisfies sub-elliptic estimates. Weakening the ellipticity assumption has substantial consequences. On a sub-Riemannian manifold, the geodesic distance, despite being compatible with the manifold topology, is not locally equivalent to any Riemannian distance: the volume of balls of small radius goes as a power R^Q of the radius R, where Q, called "homogeneous dimension", is strictly larger than the topological dimension d of the manifold. Moreover, many basic questions of harmonic-analytic character on the functional calculus of a sub-Laplacian are still open, even in the "easier" L^1 theory.

Homogeneous sub-Laplacians on stratified groups. Consider the case of a homogeneous sub-Laplacian $\mathfrak{L}$ on a nonabelian stratified group G. Homogeneous sub-Laplacians are local models for more general sub-Laplacians on sub-Riemannian manifolds, in a similar way as the Euclidean Laplacian is the local model for second-order elliptic operators. A result due to Christ [C] and to Mauceri and Meda [MM90] tells us that weak type $(1,1)$ and L^p-boundedness of $F(\mathfrak{L})$ for all $p \in (1,\infty)$ hold if F satisfies a local scale-invariant smoothness condition of order $s > Q/2$. Since, for many purposes, Q is the appropriate dimensional parameter associated to the sub-Laplacian $\mathfrak{L}$ on G, one might expect that the threshold $Q/2$ is sharp. However the sharp threshold $\varsigma(\mathfrak{L})$ is known only in few particular cases, and, in those cases, it turns out to be strictly lower than $Q/2$. The mismatch between the sharp Mihlin–Hörmander threshold and the natural dimensional parameter associated to a sub-Laplacian $\mathfrak{L}$ was first discovered in the case of the Heisenberg groups H_n, for which Müller and Stein [MS] proved that $\varsigma(\mathfrak{L}) = d/2$; independently, Hebisch [H] proved that $\varsigma(\mathfrak{L}) \leq d/2$ on all Métivier groups (a class of 2-step groups including the H_n). In view of these results "of Euclidean type", it is natural to ask whether $d/2$ is the sharp threshold on all 2-step (or higher-step) groups. However an arbitrary 2-step group may be much more complicated than a Métivier group: for example, among the free 2-step groups $N_{n,2}$ ($n = 2, 3, \dots$), only $N_{2,2}$ is Métivier.

In the last few years, in collaboration with Müller, we have made significant progress on this problem. The class of 2-step groups where $\varsigma(\mathfrak{L}) \leq d/2$ is now known to be much wider than Métivier groups [MM13, MM14b, M15] and includes, e.g., all the groups with dimension $d \leq 7$ or arbitrarily large dimension but with at most 2-dimensional second layer. The newly developed techniques allow treating 2-step groups with a much greater structural complexity, and in particular they apply to the free group $N_{3,2}$. Moreover it is now proved that $d/2 \leq \varsigma(\mathfrak{L}) < Q/2$ for all homogeneous sub-Laplacians $\mathfrak{L}$ on all 2-step groups [MM16]; this confirms that all the known Euclidean-type results on 2-step groups are indeed sharp and moreover tells us that the Christ–Mauceri–Meda theorem is never sharp on 2-step groups.

Dropping dilation and translation symmetries. For more general sub-elliptic operators $\mathfrak{L}$ on manifolds, the problem of determining the sharp threshold $\varsigma(\mathfrak{L})$ presents additional challenges. In fact, even in the simplest cases where the local model is reasonably well understood, no robust sharp results are available.

Sharp results are known, however, in the presence of symmetries, as in the case of complex spheres [CS, CKS], where the action of the unitary groups yields precise information on invariant sub-Laplacians. Recently, we have obtained analogous sharp results for the Kohn Laplacian acting on forms on complex spheres [CCMS, M17] (joint work with Casarino, Cowling and Sikora) and for sub-Laplacians on quaternionic spheres [ACMM] (joint work with Ahrens, Cowling and Müller); the latter is the first result of this kind on a compact sub-Riemannian manifold whose horizontal distribution has codimension greater than one.

Another well-understood case is that of the Grushin operators $\mathcal{G} = -\Delta_x - |x|^2\Delta_y$ on $\mathbb{R}^{d_1}_x \times \mathbb{R}^{d_2}_y$. Unlike before, here we have no invariance with respect to a transitive group action. Nevertheless, in collaborations with Müller and Sikora, we could show that $\varsigma(\mathcal{G}) = (d_1 + d_2)/2$ [MS, MM14a]. Furthermore, in a recent work with Casarino and Ciatti [CCM], we proved a sharp multiplier theorem for a a Grushin-type operator on a sphere, which is neither homogeneous with respect to a system of dilations nor admitting a transitive group of symmetries.

Groups of exponential growth. In all the examples above the associated sub-Riemannian geometry is doubling. This need not be the case for an arbitrary sub-Riemannian manifold, as it is shown, e.g., by group-invariant sub-Laplacians on Lie groups of exponential growth. Recently, in collaboration with Ottazzi and Vallarino, we extended the Christ–Mauceri–Meda theorem to a particular class of exponential solvable groups and sub-Laplacians [MOVα]. The proof hinges on a Calderón–Zygmund decomposition adapted to the underlying non-doubling geometry, and on suitable gradient heat kernel estimates for large times. This appears to be the first "genuinely Mihlin–Hörmander-type" result for sub-elliptic non-elliptic operators $\mathfrak{L}$ with differentiable L^1 functional calculus on exponentially growing groups, and opens the perspective of investigating the sharp threshold $\varsigma(\mathfrak{L})$ in this context. In addition, this result can be applied to prove a multiplier theorem for sub-Laplacians with drift on Lie groups [MOVβ].

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Some new results concerning Hardy spaces on the hyperbolic disc

Stefano Meda
(joint work with Alessio Martini, Maria Vallarino, Sara Volpi)

The classical Hardy space $H^1(\mathbb{R}^n)$ may be defined in several equivalent ways, e.g., via the Riesz transform $\mathcal{R}$, the maximal heat operator $\mathcal{H}_*$ and the maximal Poisson operator $\mathcal{P}_*$. Recall that the Riesz transform $\mathcal{R}$ is the operator $\nabla(-\Delta)^{-1/2}$, where ∇ and Δ denote the standard gradient and Laplacian on $\mathbb{R}^n$, respectively, and that the heat and Poisson semigroups are defined by

$$\mathcal{H}_t f = \mathrm{e}^{t\Delta} f \qquad \text{and} \qquad \mathcal{P}_t f = \mathrm{e}^{-t(-\Delta)^{1/2}} f.$$

Set

$$\mathcal{H}_* f = \sup_{t>0} \big|\mathcal{H}_t f\big| \qquad \text{and} \qquad \mathcal{P}_* f = \sup_{t>0} \big|\mathcal{P}_t f\big|,$$

and define

$$\begin{aligned} H^1_{\mathcal{R}}(\mathbb{R}^n) &:= \{f \in L^1(\mathbb{R}^n) : \big|\mathcal{R} f\big| \in L^1(\mathbb{R}^n)\} \\ H^1_{\mathcal{H}}\mathbb{R}^n) &:= \{f \in L^1(\mathbb{R}^n) : \mathcal{H}_* f \in L^1(\mathbb{R}^n)\} \\ H^1_{\mathcal{P}}(\mathbb{R}^n) &:= \{f \in L^1(\mathbb{R}^n) : \mathcal{P}_* f \in L^1(\mathbb{R}^n)\}. \end{aligned}$$

A celebrated result [FeS] states that $H^1_{\mathcal{R}}(\mathbb{R}^n)$, $H^1_{\mathcal{H}}\mathbb{R}^n)$ and $H^1_{\mathcal{P}}(\mathbb{R}^n)$ are the same space, usually denoted by $H^1(\mathbb{R}^n)$ (see [S, Ch. 3 and 4]). Furthermore R.R. Coifman in one dimension [C] and R. Latter in higher dimensions [L] proved that $H^1(\mathbb{R}^n)$ admits an atomic characterisation. This is important for the applications, because it reduces the problem of establishing the boundedness of a linear operator T from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ to the easier problem of showing that T is uniformly bounded on atoms.

The three spaces defined above have natural analogues on every Riemannian manifold. Indeed, given a complete connected noncompact Riemannian manifold M, denote by ∇ the associated covariant derivative on M, and by $\mathcal{L}$ (minus) the Laplace–Beltrami operator. We write $\mathcal{R}$ for the *Riesz transform* $\nabla\mathcal{L}^{-1/2}$ on

M, $\mathcal{H}_t$ for the heat semigroup $\{e^{-t\mathcal{L}} : t \geq 0\}$ and $\mathcal{P}_t$ for the Poisson semigroup $\{e^{-t\mathcal{L}^{1/2}} : t \geq 0\}$. Then $H^1_{\mathcal{R}}(M)$, $H^1_{\mathcal{H}}(M)$ and $H^1_{\mathcal{P}}(M)$ may be defined much as above. A natural question is then whether $H^1_{\mathcal{R}}(M)$, $H^1_{\mathcal{H}}(M)$ and $H^1_{\mathcal{P}}(M)$ agree and admit an atomic decomposition.

There is an extensive literature concerning this problem in the case where the Riemannian measure of M is doubling. Not surprisingly, it turns out that, if in addition the Ricci curvature is bounded from below and the injectivity radius is positive, then $H^1_{\mathcal{H}}(M)$ and $H^1_{\mathcal{P}}(M)$ agree and admit an atomic decomposition [AMR, HLMMY, DKKP]. Also, in this generality, $H^1_{\mathcal{H}}(M)$ is contained in $H^1_{\mathcal{R}}(M)$, equivalently $\mathcal{R}$ is bounded from $H^1_{\mathcal{H}}(M)$ to $L^1(M)$. The question whether $H^1_{\mathcal{H}}(M)$ agrees with $H^1_{\mathcal{R}}(M)$ is much more delicate, and there are very few results in the literature. Amongst them it is worth mentioning the related work of Christ and Geller [CG], based on previous work of A. Uchiyama, who proved that on stratified groups G the Hardy space $H^1(G)$, as defined in the book of Folland and Stein [FoS], agrees with the Hardy space defined in terms of Riesz transforms associated to a sub-Laplacian on G, and the work of Dziubanski and his collaborators, who established a similar result for certain Schrödinger operators on $\mathbb{R}^n$ [DP, DZ].

On the contrary, to the best of our knowledge, there are no results in the literature concerning the analogue of the Fefferman–Stein characterisations in the case where M is, say, the hyperbolic disc. Quite recently, in a series of papers [MMV11, MMV12, MMV15] Mauceri, Meda and Vallarino introduced an atomic Hardy-type space $X^1(M)$ for a class of Riemannian manifolds that include the hyperbolic disc. The space $X^1(M)$ provides endpoint results when $p = 1$ for a class of interesting operators on M that includes the Riesz transform and the imaginary powers of $\mathcal{L}$. In view of the discussion above, it is natural to speculate whether $X^1(M)$ agrees with $H^1_{\mathcal{H}}(M)$, $H^1_{\mathcal{P}}(M)$ or $H^1_{\mathcal{R}}(M)$.

In my talk I shall describe the results we have proved so far in the special case of Riemannian symmetric spaces $\mathbb{X}$ of the noncompact type and real rank one. The prototype of such spaces is the hyperbolic disc $\mathbb{D}$. We have reasons to believe that similar results hold on higher rank symmetric spaces. I need to introduce the space

$$\widetilde{H}^1_{\mathcal{R}}(\mathbb{X}) := \{f \in \mathfrak{h}^1(\mathbb{X}) : \left|\mathcal{R}f\right| \in L^1(\mathbb{X})\},$$

which is contained in $H^1_{\mathcal{R}}(\mathbb{X})$. Here $\mathfrak{h}^1(\mathbb{X})$ denotes the Goldberg-type space introduced by M. Taylor (see [T]). Our main result is the following.

Theorem 1. *The following hold:*

- *(a)* $H^1_{\mathcal{H}}(\mathbb{X})$, $H^1_{\mathcal{P}}(\mathbb{X})$ *and* $H^1_{\mathcal{R}}(\mathbb{X})$ *are different spaces, and each of them is different from* $X^1(\mathbb{X})$*;*
- *(b)* $\widetilde{H}^1_{\mathcal{R}}(\mathbb{X})$ *does not admit an atomic decomposition, in the sense that the space of functions with compact support in* $\widetilde{H}^1_{\mathcal{R}}(\mathbb{X})$ *is not dense in* $\widetilde{H}^1_{\mathcal{R}}(\mathbb{X})$*.*

The comparison between $H^1_{\mathcal{H}}(\mathbb{X})$, $H^1_{\mathcal{P}}(\mathbb{X})$ and $H^1_{\mathcal{R}}(\mathbb{X})$ is accomplished by relating them to a one-parameter family $\{\mathfrak{X}^\gamma(\mathbb{X}) : \gamma > 0\}$ of isometric copies of $\mathfrak{h}^1(\mathbb{X})$. In the case where γ is a positive integer, the space $\mathfrak{X}^\gamma(\mathbb{X})$ was introduced in S. Volpi's thesis [V].

The proof that $\widetilde{H}^1_{\mathcal{R}}(\mathbb{X})$ does not admit an atomic decomposition hinges on the spherical analysis on rank one semisimple Lie groups, specifically on the Paley–Wiener theorem for the Helgason–Fourier transform.

Similar results hold in the discrete setting, e.g., on homogeneous trees [CM].

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Composition of singular integral operators with different homogeneities

Alexander Nagel
(joint work with Fulvio Ricci, Elias M. Stein, Stephen Wainger)

We study algebras of singular integral convolution operators on $\mathbb{R}^n$ and homogeneous nilpotent Lie groups that arise when considering the composition of Calderón-Zygmund operators with different homogeneities. Thus let $D^{\mathbf{a}}_\lambda(x_1,\ldots,x_n) = (\lambda^{a_1}x_1,\ldots,\lambda^{a_n}x_n)$. A *Calderón-Zygmund kernel adapted to the dilations* $D^{\mathbf{a}}_\lambda$ is a tempered distribution $\mathcal{K}$ on $\mathbb{R}^n$, given away from the origin by integration against a smooth function K, and satisfying the following.

(a) *Differential inequalities*: For every multi-index $\boldsymbol{\gamma}$ there is a constant $C_{\boldsymbol{\gamma}} > 0$ so that $|\partial^{\boldsymbol{\gamma}}_{\mathbf{x}} K(x_1,\ldots,x_n)| \le C_{\boldsymbol{\gamma}}(|x_1|^{\frac{1}{a_1}} + \cdots + |x_n|^{\frac{1}{a_n}})^{-\sum_{j=1}^n a_j(1+\gamma_j)}$.

(b) *Cancellation conditions:* Let $\psi_r(x_1,\ldots,x_n) = \psi(r^{a_1}x_1,\ldots,r^{a_n}x_n)$. There exists $C > 0$ so that for all $r > 0$ and all normalized bump functions ψ it follows that $|\langle \mathcal{K},\psi_r\rangle| \le C$.

We can now formulate the following problem. Let $\mathcal{K}_{\mathbf{a}},\mathcal{K}_{\mathbf{b}}$ be Calderón-Zygmund kernels on $\mathbb{R}^n$ associated to homogeneities $D^{\mathbf{a}}_\lambda$ and $D^{\mathbf{b}}_\lambda$. Let $T_{\mathbf{a}}[\varphi] = \varphi * \mathcal{K}_{\mathbf{a}}$ and $T_{\mathbf{b}}[\varphi] = \varphi * \mathcal{K}_{\mathbf{b}}$ where the convolution is on a homogeneous nilpotent Lie group G with automorphic dilations $D^{\mathbf{d}}_\lambda$. Suppose that both $T_{\mathbf{a}}$ and $T_{\mathbf{b}}$ extend to bounded operators on $L^2(\mathbb{R}^n)$. (This is always the case, for example, if $G = \mathbb{R}^n$ with standard vector addition, and on homogeneous nilpotent Lie groups if $\mathcal{K}_{\mathbf{a}}$ and $\mathcal{K}_{\mathbf{b}}$ have compact support and if $\frac{a_1}{d_1} \ge \frac{a_2}{d_2} \ge \cdots \ge \frac{a_n}{d_n}$ and $\frac{b_1}{d_1} \ge \frac{b_2}{d_2} \ge \cdots \ge \frac{b_n}{d_n}$.) Then $T_{\mathbf{a}} \circ T_{\mathbf{b}}$ is a bounded operator on $L^2(R^n)$ and if $\mathbf{a} = \mathbf{b}$ the composition is convolution with a Calderón-Zygmund kernel with the same homogeneity. However if $\mathbf{a} \ne \mathbf{b}$ we can ask what is the nature of the Schwartz kernel of $T_1 \circ T_2$ and what is the algebra of operators generated by such convolutions.

We introduce the following class of operators. Let $\mathbf{E} = \{e(j,k)\}$ be an $n \times n$ matrix of positive real numbers satisfying the *basic hypotheses* $e(j,j) = 1$ and $e(j,l) \le e(j,k)e(k,l)$. Set $N_j(x_1,\ldots,x_n) = |x_1|^{\frac{e(j,1)}{d_1}} + \cdots + |x_n|^{\frac{e(j,n)}{d_n}}$. Then $\mathcal{P}(\mathbf{E})$ is the class of tempered distributions $\mathcal{K}$, given by integration against a smooth function K away from the origin on $\mathbb{R}^n$, which satisfy

(a') *Differential inequalities:* If $\mathcal{K} \in \mathcal{P}(\mathbf{E})$ then away from the origin $\mathcal{K}$ is given by a smooth function K and for each multi-index $\boldsymbol{\alpha}$ there is a constant $C_{\boldsymbol{\alpha}}$ so that $\big|\partial^{\boldsymbol{\alpha}} K(x_1,\ldots,x_n)\big| \le C_{\boldsymbol{\alpha}} \prod_{j=1}^n N_j(x_1,\ldots,x_n)^{-d_j(1+\alpha_j)}$.

(b') *Cancellation Conditions:* Let $0 \le m \le n-1$ and let ψ be any normalized bump function of $n-m$ variables. If $\mathbf{r} = (r_{m+1},\ldots,r_n)$ with each $r_j > 0$ set $\psi_{\mathbf{r}}(x_{m+1},\ldots,x_n) = \psi(r_{m+1}x_{m+1},\ldots,r_nx_n)$. If $m = 0$ there is a constant C independent of ψ and $\mathbf{r}$ so that $\big|\langle \mathcal{K},\psi_{\mathbf{r}}\rangle\big| \le C$. If $m \ge 1$ define a tempered distribution $\mathcal{K}^{\#}_{\mathbf{r}}$ on $\mathbb{R}^m$ by setting $\langle \mathcal{K}^{\#}_{\mathbf{r}},\varphi\rangle = \langle \mathcal{K},\varphi\otimes\psi_{\mathbf{r}}\rangle$ for all $\varphi \in \mathcal{S}(\mathbb{R}^m)$. Then away from the origin $\mathcal{K}^{\#}_{\mathbf{r}}$ is given by a smooth function

$K_{\mathbf{r}}^{\#}$ and

$$\left|\partial_{x_1}^{\alpha_1}\cdots\partial_{x_m}^{\alpha_m}K_{\mathbf{r}}^{\#}(x_1,\ldots,x_m)\right|\leq C_{\boldsymbol{\alpha}}\prod_{j=1}^{m}N_j(x_1,\ldots,x_m,0,\ldots,0)^{-a_j(1+\alpha_j)}$$

with $C_{\boldsymbol{\alpha}}$ independent of ψ and $\mathbf{r}$. The same also holds for any permutation of the variables $x_1,\ldots,x_n$.

Theorem 1.

(a) *Rank* $(\mathbf{E})=1$ *if and only if there is a dilation structure on* $\mathbb{R}^n$ *for which* $\mathcal{P}(\mathbf{E})$ *is the space of Calderón-Zygmund kernels.*

(b) *If rank* $(\mathbf{E})>1$ *and* $\mathcal{K}\in\mathcal{P}(\mathbf{E})$ *then* K *is integrable at infinity.*

(c) *If rank*$(\mathbf{E})=m$ *and* $\mathcal{K}\in\mathcal{P}(\mathbf{E})$ *then* $\left|\{\mathbf{x}\in\mathbb{R}^n:|K(\mathbf{x})|>\lambda\}\right|\lesssim\lambda^{-1}\log(\lambda)^{m-1}$ *for* $\lambda\geq 1$. *Moreover there exists* $\mathcal{K}\in\mathcal{P}(\mathbf{E})$ *so that* $\left|\{\mathbf{x}\in\mathbb{R}^n:|K(\mathbf{x})|>\lambda\}\right|\gtrsim\lambda^{-1}\log(\lambda)^{m-1}$.

We can characterize elements of $\mathcal{P}(\mathbf{E})$ in terms of their Fourier transforms. Set $\mathcal{P}_0(\mathbf{E})=\{\mathcal{K}\in\mathcal{P}(\mathbf{E}):K\text{ is rapidly decreasing outside the unit ball}\}$. Such distributions can be characterized by the behavior of their Fourier transform $\widehat{\mathcal{K}}$. The norm dual to N_j is given by $\widehat{N}_j(\boldsymbol{\xi})=|\xi_1|^{\frac{1}{e(1,j)d_1}}+\cdots+|\xi_n|^{\frac{1}{e(n,j)d_n}}$. Define $\mathcal{M}_\infty(\mathbf{E})=\{m\in\mathcal{C}^\infty(\mathbb{R}^n):\left|\partial^{\boldsymbol{\alpha}}m(\boldsymbol{\xi})\right|\leq C_{\boldsymbol{\alpha}}\prod_{j=1}^n\left(1+\widehat{N}_j(\boldsymbol{\xi})\right)^{-\alpha_j d_j}\}$.

Theorem 2. $\mathcal{K}\in\mathcal{P}_0(\mathbf{E})$ *if and only if* $\widehat{\mathcal{K}}=m\in\mathcal{M}_\infty(\mathbf{E})$.

The class $\mathcal{P}(\mathbf{E})$ contains distributions which are flag kernels relative to two opposite flags. Suppose that $\mathcal{K}$ is a distribution satisfying appropriate cancellation conditions and the following *flag kernel* differential inequalities:

$$\left|\partial^{\gamma}K(\mathbf{x})\right|\lesssim\begin{cases}\prod_{j=1}^n\left(|x_1|^{\frac{a_j}{a_1}}+|x_2|^{\frac{a_j}{a_2}}+\cdots+|x_{j-1}|^{\frac{a_j}{a_{j-1}}}+|x_j|\right)^{-1-\gamma_j}\\ \prod_{j=1}^n\left(|x_j|+|x_{j+1}|^{\frac{b_j}{b_{j+1}}}+\cdots+|x_{n-1}|^{\frac{b_j}{b_{n-1}}}+|x_n|^{\frac{b_j}{b_n}}\right)^{-1-\gamma_j}\end{cases}$$

Note that the differential inequalities give no information when $x_1=x_n=0$ and $n\geq 3$.

Theorem 3. *Suppose that* $\frac{a_1}{b_1}\leq\frac{a_2}{b_2}\leq\cdots\leq\frac{a_n}{b_n}$ *with at least one strict inequality.*

(a) *The function* K *is integrable at infinity, and we can write* $\mathcal{K}=\mathcal{K}_0+\mathcal{K}_\infty$ *where* $\mathcal{K}_\infty\in L^1(\mathbb{R}^N)\cap\mathcal{C}^\infty(\mathbb{R}^N)$, *and* $\mathcal{K}_0$ *is a two-flag kernel supported in the unit ball.*

(b) *The kernel* $\mathcal{K}_0$ *belongs to the class* $\mathcal{P}_0(\mathbf{E})$ *associated to the matrix* $\mathbf{E}=\{e(j,k)$ *where* $e(j,k)=\frac{b_j}{b_k}$ *if* $j\leq k$ *and* $e(j,k)=\frac{a_j}{a_k}$ *if* $j>k$.

(c) *In particular the kernel* $\mathcal{K}$ *is given away from the origin by integration against a smooth function.*

When studying convolution on general homogeneous nilpotent Lie groups, we need to impose an additional condition of *double monotonicity*: $e(j,k)\leq e(j,k+1)$ and $e(j,k)\geq e(j+1,k)$.

Theorem 4. *If G is a homogeneous nilpotent Lie group, if $\mathbf{E}$ satisfies the basic hypotheses and is doubly monotone, and if $\mathcal{K} \in \mathcal{P}(\mathbf{E})$ then the operator $\varphi \to \varphi * \mathcal{K}$ extends to a bounded operator on $L^p(G)$ for $1 < p < \infty$.*

We now return to the original problem. Let $\mathcal{K}_{\mathbf{a}}, \mathcal{K}_{\mathbf{b}}$ be compactly supported Calderón-Zygmund kernels associated with the dilations $D_\lambda^{\mathbf{a}}$ and $D_\lambda^{\mathbf{b}}$. Let G be a homogeneous nilpotent Lie group with automorphic dilations $D_\lambda^{\mathbf{d}}$. Assume $\frac{a_1}{d_1} \geq \frac{a_2}{d_2} \geq \cdots \geq \frac{a_n}{d_n}$ and $\frac{b_1}{d_1} \geq \frac{b_2}{d_2} \geq \cdots \geq \frac{b_n}{d_n}$. Put $T_{\mathbf{a}}[\varphi] = \varphi * \mathcal{K}_{\mathbf{a}}$, $T_{\mathbf{b}}[\varphi] = \varphi * \mathcal{K}_{\mathbf{b}}$.

Theorem 5. *Let $\mathbf{E} = \{e(j,k)\}$ where $e(j,k) = \max\left\{\frac{a_j}{a_k}, \frac{b_j}{b_k}\right\}$. The operator $T_{\mathbf{a}} \circ T_{\mathbf{b}}$ is given by convolution with a tempered distribution $\mathcal{L}$ with $\mathcal{L} \in \mathcal{P}(\mathbf{E})$.*

Theorem 6. *Let $G \cong \mathbb{R}^n$ be a homogeneous nilpotent Lie group and let $\mathbf{E}$ be a doubly monotone matrix. If $\mathcal{K}, \mathcal{L} \in \mathcal{P}_0(\mathbf{E})$ then there exists $\mathcal{M} \in \mathcal{P}_0(\mathbf{E})$ such that $T_K \circ T_{\mathcal{L}} = T_{\mathcal{M}}$.*

Recent developments in sharp restriction theory

Diogo Oliveira e Silva
(joint work with Emanuel Carneiro, René Quilodrán, Mateus Sousa)

For the sake of concreteness, we start our discussion with the case of the unit sphere $\mathbb{S}^{d-1}$ equipped with surface measure σ, but the more general example of a smooth compact hypersurface should be kept in mind. Given $1 \leq p \leq 2$, for which exponents q does the *a priori* Fourier restriction inequality

$$\left(\int_{\mathbb{S}^{d-1}} |\widehat{f}(\omega)|^q d\sigma_\omega\right)^{\frac{1}{q}} \leq C\|f\|_{L^p(\mathbb{R}^d)} \tag{1}$$

hold? One easily checks that, for any value of q, inequality (1) holds if $p = 1$ and fails if $p = 2$. By duality, estimate (1) is equivalent to the adjoint restriction, or extension, inequality

$$\left(\int_{\mathbb{R}^d} |\widehat{g\sigma}(x)|^{p'} dx\right)^{\frac{1}{p'}} \leq C\|g\|_{L^{q'}(\mathbb{S}^{d-1})}, \tag{2}$$

where $p' = \frac{p}{p-1}$ denotes the exponent dual to p, and similarly for q. A complete answer for $q = 2$ is given by the classical Tomas–Stein inequality, which establishes the restriction inequality (1) for $q = 2$ in the sharp range $1 \leq p \leq \frac{2(d+1)}{d+3}$. The question of what happens for values of $q < 2$ is the starting point for the celebrated Fourier restriction conjecture.

Tomas–Stein restriction estimates are very much related to Strichartz estimates for linear partial differential equations of dispersion type. Let us illustrate this point in one particular instance, that of solutions $u(x,t)$ with $(x,t) \in \mathbb{R}^{d+1}$ to the Schrödinger equation $iu_t = \Delta u$, with prescribed initial data. Strichartz established

$$\|u\|_{L^{2+\frac{4}{d}}(\mathbb{R}^{d+1})} \leq C\|f\|_{L^2(\mathbb{R}^d)}, \tag{3}$$

provided that u is the solution of the Schrödinger equation satisfying $u(x,0) = f(x)$. It turns out that Strichartz estimates for the Schrödinger equation correspond to extension estimates on the paraboloid, a non-compact manifold which exhibits some scale invariance properties that allow the reduction to the compact setup of the Tomas–Stein inequality.

For the past several years, I have been very interested in extremizers and optimal constants for sharp variants of restriction and Strichartz-type inequalities. Apart from their intrinsic mathematical interest and beauty, sharp inequalities often allow for various refinements of existing inequalities. The following are natural questions, which can be posed in the particular case of Fourier restriction inequalities:

- What is the value of the optimal constant?
- Do extremizers exist?
  - If so, are they unique, possibly after applying the symmetries of the problem?
  - If not, what is the mechanism responsible for this lack of compactness?
- How do extremizing sequences behave?
- What are some qualitative properties of extremizers?
- What are necessary and sufficient conditions for a function to be an extremizer?

Questions of this kind have been asked in a variety of situations, and in the context of classical inequalities from Euclidean harmonic analysis they go back at least to the early work of Beckner on the sharp Hausdorff–Young inequality, and of Lieb on the sharp Hardy–Littlewood–Sobolev inequality. In comparison, sharp Fourier restriction inequalities have a relatively short history, with the first works on the subject going back to Kunze [K], Foschi [F] and Hundertmark–Zharnitsky [HZ]. These works concern extremizers and sharp constants for inequality (3) in the low dimensional cases $d \in \{1, 2\}$. These are the cases for which the Strichartz exponent $2 + \frac{4}{d}$ is an even integer, and one can rewrite the left-hand side of inequality (3) as an L^2 norm, and invoke Plancherel in order to reduce the problem to a multilinear convolution estimate.

Sharp Fourier restriction theory is becoming increasingly more popular, as shown by the large body of work that appeared in the last decade, and in particular in the last few years. We mention a recent survey [FO] on sharp Fourier restriction theory which may be consulted for information complementary to that on this abstract, and further references.

Perturbed paraboloids. Recent joint work with Quilodrán [OQ] focused on a family of sharp Strichartz estimates for higher order Schrödinger equations. More precisely, for an appropriate class of convex functions ϕ, we studied the Fourier extension operator on the surface

$$\{(\xi, \tau) \in \mathbb{R}^{2+1} : \tau = |\xi|^2 + \phi(\xi)\}. \tag{4}$$

One of our main tools was a new comparison principle for convolutions of certain singular measures supported on non-compact manifolds that holds in all dimensions. This is better illustrated in the following special case. Let μ_0 and μ_1 denote the projection measures on the surfaces given by (4) with $\phi(\xi) \equiv 0$ and $\phi(\xi) = |\xi|^4$, respectively. Then the pointwise inequality

$$(\mu_1 * \mu_1)\Big(\xi, \tau + \frac{|\xi|^2}{2} + \frac{|\xi|^4}{8}\Big) \leq (\mu_0 * \mu_0)\Big(\xi, \tau + \frac{|\xi|^2}{2}\Big)$$

holds for every $\tau > 0$ and $\xi \in \mathbb{R}^2$, and it is strict at almost every point of the support of the measure $\mu_1 * \mu_1$. This observation led to the exact determination of some optimal constants and to a proof that extremizers do not exist in this perturbed setting. Adapting ideas from the concentration-compactness principle of Lions, we further investigated the behaviour of general extremizing sequences. Generally speaking, the theory of concentration-compactness has proved a very efficient tool in exhibiting the precise mechanisms which are responsible for the loss of compactness in a variety of settings. In our concrete problem, extremizers fail to exist because extremizing sequences concentrate. Concentration can only occur at points where the convolution $\mu_1 * \mu_1$ attains its maximum value, or at spatial infinity. Last but not least, our methods further resolve a dichotomy from the recent literature [JSS] concerning the existence of extremizers for a family of fourth order Schrödinger equations.

Hyperboloids. In ongoing joint work with Carneiro and Sousa [COS], we are investigating optimal constants and the existence of extremizers for the adjoint Fourier restriction inequality on hyperboloids. The $L^2 \to L^p$ adjoint restriction inequality on the d-dimensional hyperboloid $\mathbb{H}^d \subset \mathbb{R}^{d+1}$ holds provided $6 \leq p < \infty$, if $d = 1$, and $\frac{2(d+2)}{d} \leq p \leq \frac{2(d+1)}{d-1}$, if $d \geq 2$. Quilodrán [Q] recently found the values of the optimal constants in the endpoint cases $(d,p) \in \{(2,4),(2,6),(3,4)\}$ and showed that the inequality does not have extremizers in these cases. We are able to answer two questions posed in [Q], namely: (i) we find the explicit value of the optimal constant in the endpoint case $(d,p) = (1,6)$ (the remaining endpoint for which p is an even integer) and show that there are no extremizers in this case; and (ii) we establish the existence of extremizers in all non-endpoint cases in dimensions $d \in \{1,2\}$. This completes the qualitative description of this problem in low dimensions.

An open problem. To finish, we would like to mention the following open problem: Do Gaussians extremize inequality (3) in all dimensions? It is known that Gaussians are critical points of the associated Euler–Lagrange equation in all dimensions. If Gaussians were known to be extremizers, it would then be possible to establish the unconditional existence of extremizers for the corresponding problem on the unit sphere $\mathbb{S}^{d-1}$. The methods outlined above are not enough to tackle this problem when $d \geq 4$, and we intend to gear the direction of our research towards a better understanding of this fundamental question.

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Borderline weighted estimates for the maximal function and for rough singular integral operators

Carlos Pérez

We will split the lecture in two parts:

Rough singular integral operators. One of the most relevant results in classical Harmonic Analysis is the classical good-lambda estimate between Calderón-Zygmund operators and the maximal functions due to R. Coifman and C. Fefferman. This result yields strong or weak L^p estimates with A_∞ weights from which the classical well known weighted estimates follow. We have shown in the recent work [LPRR] that a corresponding result holds for rough singular integrals T_Ω, $\Omega \in L^\infty$ and the Bochner-Riesz operator at critical level even though there is no such as good-λ inequality available. This result is key in the solution of some conjectures formulated by the author after the work of [P] and inspired by [DR], [C] and [Se]. One of the key points is to combine some extrapolation theorems for the class A_∞ obtained in [CMP04] and [CCMP] together with a sparse formula found by Conde-Culiuc-Di Plinio-Ou [CCPO].

Mixed weak type estimates. Muckenhoupt-Wheeden [MW] in the seventies and Sawyer [Sa] in the eightees, established some one-dimensional highly nontrivial extensions of the weak type $(1, 1)$ property of the maximal function involving weights. The one obtained by E. Sawyer provided another proof of the classical Muckenhoupt A_p theorem. These results were conjectured to hold for the Hilbert transform and for the maximal function in higher extensions. These conjectures were proved and extended in different directions in [CMP05] and [OP]. Further more difficult conjectures were formulated in [CMP05] that have been recently settled in [LOP]. These questions are of interest in the context of Multilinear Harmonic Analysis.

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Iterated Journé Commutators and multi-parameter BMO

Stefanie Petermichl

A classical result of Nehari [Ne] shows that a Hankel operator with anti-analytic symbol b mapping analytic functions into the space of anti-analytic functions by $f \mapsto P_{-}bf$ is bounded with respect to an L^2 norm if and only if the symbol belongs to BMO. This theorem has an equivalent formulation in terms of the boundedness of the commutator of the multiplication operator with symbol function b and the Hilbert transform $[H, b] = Hb - bH$. To see this correspondence one rewrites the commutator as a sum of Hankel operators with orthogonal ranges.

Let $H^2(\mathbb{T}^2)$ denote the Banach space of analytic functions in $L^2(\mathbb{T}^2)$. In [FS], Ferguson and Sadosky study the symbols of bounded 'big' and 'little' Hankel operators on the bidisk. Big Hankel operators are those which project on to a 'big' subspace of $L^2(\mathbb{T}^2)$ - the orthogonal complement of $H^2(\mathbb{T}^2)$; while little Hankel operators project onto the smaller subspace of complex conjugates of functions in $H^2(\mathbb{T}^2)$ - or anti-analytic functions. The corresponding commutators are

$$[H_1H_2, b],$$

and

$$[H_1, [H_2, b]]$$

where $b = b(x_1, x_2)$ and H_k are the Hilbert transforms acting in the k^{th} variable. Ferguson and Sadosky show that the first commutator is bounded if and only if the symbol b belongs to the so called little BMO class, consisting of those functions that are uniformly in BMO in each variable separately. They also show that if b belongs to the product BMO space, as identified by Chang and Fefferman [CF85], [CF80] then the second commutator is bounded. The fact that boundedness of the second commutator implies that b is in product BMO was shown in the groundbreaking paper of Ferguson and Lacey [FL]. The techniques to tackle this question in several parameters are very different and have brought valuable new insight and use to existing theories, for example in the interpretation of Journé's lemma [J86] in combination with Carleson's example [Ca]. Lacey and Terwilliger extended this result to an arbitrary number of iterates in [LT], requiring thus, among others, a refinement of Pipher's iterated multi-parameter version [Pi] of Journé's lemma.

When leaving the notion of Hankel operators behind, their interpretation as commutators allow for natural generalizations. Through the use of completely different real variable methods, Coifman, Rochberg and Weiss [CRW] extended Nehari's one-parameter theory to real analysis in the sense that the Hilbert transforms were replaced by Riesz transforms. The missing features of the Riesz transforms include analytic projection on one hand as well as strong factorisation theorems of $H^1(\mathbb{D})$ on the other.

The authors in [CRW] obtained sufficiency, i.e. that a BMO symbol b yields an $L^2(\mathbb{R}^d)$ bounded commutator for certain more general, convolution type singular integral operators. For necessity, they showed that the collection of Riesz transforms was representative enough:

$$\|b\|_{\mathrm{BMO}} \leq C \sup_{1\leq j\leq d} \|[R_j, b]\|_{2\to 2}.$$

Notably this lower bound was obtained somewhat indirectly through use of spherical harmonics in combination with the mean oscillation characterisation of BMO in one parameter.

These one-parameter results in [CRW] were extended to the multi-parameter setting in [LPPW]. Both the upper and lower estimate have proofs very different from those in one parameter. For the lower estimate, the methods in [FL] or [LT] find an extension to real variables through operators closer to the Hilbert transform than the Riesz transforms and an indirect passage on the Fourier transform side.

In a recent paper [DO] it is shown that iterated commutators formed with any arbitrary Calderón-Zygmund operators are bounded if the symbol belongs to product BMO.

The first part is concerned with mixed Hankel operators or commutators such as

$$[H_1, [H_2 H_3, b]].$$

We classify boundedness of these commutators by a mixed BMO class (little product BMO): those functions $b = b(x_1, x_2, x_3)$ so that $b(\cdot, x_2, \cdot)$ and $b(\cdot, \cdot, x_3)$ are uniformly in product BMO. Similar results can be obtained for any finite iteration of any finite tensor product of Hilbert transforms.

The second part is concerned with a real variable analog or commutators of the form

$$[R_{1,j_1}, [R_{2,j_2} R_{3,j_3}, b]],$$

where R_{k,j_k} are Riesz transforms of direction j_k acting in the k^{th} variable. We show necessity and sufficiency of the little product BMO condition when the R_{k,j_k} are allowed to run through all Riesz transforms by means of a two-sided estimate. Our argument works for all higher iterates and tensor products.

It is a general fact that two-sided commutator estimates have an equivalent formulation in terms of weak factorization. We find the pre-duals of our little product BMO spaces and prove a corresponding weak factorization result.

Much like discussed in the base cases of our results [CRW], [LPPW], boundedness of commutators involving Hilbert or Riesz transforms are a testing condition. If these commutators are bounded, the symbol necessarily belongs to a little product BMO. We then show that iterated commutators using a much more general class than that of tensor products of Riesz transforms are also bounded: commutators with Journé operators.

We make some remarks about the strategy of the proof.

In the Hilbert transform case, Toeplitz operators with operator symbol arise naturally.

While Riesz transforms in $\mathbb{R}^d$ are a good generalisation of the Hilbert transform, there is absence of analytic structure and tools relying on analytic projection or orthogonal spaces are not readily available. We overcome this difficulty through a first intermediate passage via tensor products of Calderón-Zygmund operators whose Fourier multiplier symbols are adapted to cones. This idea is inspired by [LPPW]. A class of operators of this type classifies little product BMO through two-sided commutator estimates, but it does not allow the passage to a classification through iterated commutators with tensor products of Riesz transforms. In a second step, we find it necessary to consider upper and lower commutator estimates using a well-chosen family of Journé operators that are not of tensor product type. These operators are constructed to resemble the multiple Hilbert transform. A two-sided estimate of iterated commutators involving operators of this family facilitates a passage to iterated commutators with tensor products of Riesz transforms. There is an increase in difficulty when the arising tensor products involve more than two Riesz transforms and when the dimension is greater than two.

The actual passage to the Riesz transforms requires for us to prove a stability estimate in commutator norms for certain multi-parameter singular integrals in terms of the mixed BMO class. In this context, we prove a qualitative upper estimate for iterated commutators using paraproduct free Journé operators. We make use of recent versions of $T(1)$ theorems in this setting. These recent advances are different from the corresponding theorem of Journé [J85]. The results we allude to have the additional feature to provide a convenient representation formula for bi-parameter in [Ma] and even multi-parameter in [Ou] Calderón-Zygmund operators by dyadic shifts. While a sufficient form of this result for characterisation results is contained in [OPS], the extension to all L^p as well as *all* Journé operators is more recent and appears as a special case in [HPW], where a two-weight question was addressed.

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Conserved energies and soliton stability in completely integrable PDEs

Daniel Tataru
(joint work with Herbert Koch)

This work is concerned with several one dimensional completely integrable pde models, which include the cubic NLS, mKdV and KdV. To keep the presentation simple we will just consider the the first two. The (de)focusing cubic Nonlinear Schrödinger equation (NLS) is

$$iu_t + u_{xx} \pm 2u|u|^2 = 0, \qquad u(0) = u_0,$$

and the complex (de)focusing modified Korteweg-de Vries equation (mKdV) has the form

$$u_t + u_{xxx} \pm 2(|u|^2u)_x = 0, \qquad u(0) = u_0,$$

with real or complex solutions in one space dimension on the real line.

These are part of an infinite family of commuting Hamiltonian flows, where each of the Hamiltonians can be viewed as conservation laws for each of the flows. The symplectic form is

$$\omega(u, v) = \Im \int u\bar{v}\, dx$$

and first several Hamiltonians are as follows:

$$\begin{aligned}
H_0 =& \int |u|^2 dx,\\
H_1 =& \frac{1}{i}\int u\partial_x\bar{u} dx,\\
H_2 =& \int |u_x|^2 + |u|^4 dx,\\
H_3 =& i\int \ u_x\partial_x\overline{u}_x + \frac{3}{2}|u|^2u\partial_x\bar{u} dx,\\
H_4 =& \int |u_{xx}|^2 + 2||u|_x^2|^2 + u^2(\bar{u}_x)^2 + (\bar{u}_x)^2u^2 + \frac{3}{2}|u|^6 dx.
\end{aligned}$$

Both of these equations have $\dot{H}^{-\frac{1}{2}}$ as a scale invariant critical Sobolev space. On the other hand the above energies only provide good H^k bounds when k is an integer. The goal of this work is to rectify this, and consider H^s norms for all $s > -\frac{1}{2}$. Rather than attempt to prove uniform bounds for H^s norms, as in our prior work, our first goal here is to construct new conservation laws which are equivalent to the H^s norms of the solutions. Precisely, our main result is as follows:

Theorem 1. *For each $s > -\frac{1}{2}$ and both for the focusing and defocusing case the energy functionals E_s are globally defined*

$$E_s : H^s \to \mathbb{R}$$

with the following properties:

(1) *E_s is conserved along the NLS and mKdV flow.*
(2) *If*[1] $\|u\|_{L^2+DU^2} \leq 1$ *then*

$$\left|E_s(u) - \|u\|_{H^s}^2\right| \lesssim \|u\|_{L^2+DU^2}^2 \|u\|_{H^s}^2.$$

(3) *The map*

$$H^\sigma \times (-\frac{1}{2}, \sigma] \ni (u,s) \to E_s(u)$$

is analytic in $u \in H^\sigma$ in the defocusing case. In the focusing case it is analytic provided $\frac{i}{2}$ is not an eigenvalue for L, and it is continuous in $u \in H^\sigma$ in general. It is also continuous in s, and analytic in s for $s < \sigma$.

Here L is the corresponding Lax operator

$$L = i \begin{pmatrix} \partial_x & -u \\ \pm\bar{u} & -\partial_x \end{pmatrix},$$

whose eigenvalues (which only exist in the focusing case) correspond to to soliton solutions, or soliton components of more general solutions.

Our conserved energies are defined in terms of the transmission coefficient T associated to the Lax operator, which is a meromorphic function in the upper half-plane. In order to be able to work with T for data which is only in H^s spaces, our construction uses T only away from the real axis.

A second goal of our work is to use these energies in order to study the orbital stability of multisoliton solutions for the focusing problem. Our main result here is as follows:

Theorem 2. *Multisoliton solution families are orbitally stable in H^s for both the cubic NLS flow and mKdV.*

This result is proved using Backlund type transforms which allow one to add or remove solitons from a given solution. The main challenge is to be able to deal with solitons which have nearly identical scale and velocity parameters.

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[1] Here the space $L^2 + DU^2$ is a convenient proxy for the obvious but ill-behaved choice $H^{-\frac{1}{2}}$.

Bochner-Riesz profile of an anharmonic oscillator

Adam Sikora
(joint work with Peng Chen and Waldemar Hebisch)

We investigate one dimensional Schrödinger type operator with the anharmonic potential

$$\mathcal{L} = -\frac{d^2}{dx^2} + |x|.$$

It is well-known that this type of operator is self-adjoint and admits a spectral resolution, which allows us to study the corresponding Bochner-Riesz means and more general spectral multipliers. To recall the notion of Bochner-Riesz means we set

$$\sigma_R^\alpha(\lambda) = \begin{cases} (1-\lambda/R)^\alpha & \text{for } 0 \le \lambda \le R \\ 0 & \text{for other } \lambda. \end{cases}$$

We then define the operator $\sigma_R^\alpha(\mathcal{L})$ and $\sigma_R^\alpha(\mathcal{H})$ using the spectral theorem. The main problem considered in Bochner-Riesz analysis is to find exponent $\alpha_{cr}(p)$ such that the operators $\sigma_R^\alpha(\mathcal{L})$ are bounded uniformly in R on L^p for all $\alpha > \alpha_{cr}(p)$. In most of the cases full description of Bochner-Riesz profile of general differential operators or even of the standard Laplace operator is an open problem, see e.g. [S].

Our study is motivated by results described in [AW, T], where combination of results obtained by Askey, Wainger and Thangavelu provide full description of convergence of Bochner-Riesz means for the harmonic oscillator $\mathcal{H} = -\frac{d^2}{dx^2} + x^2$ and it is one of very few examples when such full picture was obtained. Also in the case of the operator $\mathcal{L}$ which we consider here we obtain a complete description of the critical exponent $\alpha_{cr}(p)$ for all $1 \le p \le \infty$.

We show that the Bochner-Riesz profile of the operator $\mathcal{L}$ completely coincides with such profile of the harmonic oscillator $\mathcal{H} = -\frac{d^2}{dx^2} + x^2$. It is especially surprising because the Bochner-Riesz profile of the one-dimensional standard Laplace operator is known to be essentially different and the case of operators $\mathcal{H}$ and $\mathcal{L}$ resembles more the profile of multidimensional Laplace operators. To be more precise we recall the description of convergence of Bochner-Riesz means, which follows from Askey, Wainger and Thangavelu's results. It is stated in [T, Theorem 5.5] and can be summarised in the following way.

Proposition. *Consider the operator* $\mathcal{H} = -\frac{d^2}{dx^2} + x^2$. *Then* $\sigma_R^\alpha(\mathcal{H})$ *is uniformly bounded on* L^p *if the point* $(1/p, \alpha)$ *belongs to regions* A *or* B, *that is if* $\alpha > \max\{0, \frac{2}{3}|\frac{1}{2} - \frac{1}{p}| - \frac{1}{6}\}$, *see figure 1. Next if* $(1/p, \alpha)$ *belongs to regions* C, *that is if* $\alpha < \max\{0, \frac{2}{3}|\frac{1}{2} - \frac{1}{p}| - \frac{1}{6}\}$, *then* $\sup_{R>0} \|\sigma_R^\alpha(\mathcal{H})\|_{p\to p} = \infty$.

Our main result gives a complete picture of Bochner-Riesz convergence for the operator $\mathcal{L}$.

Theorem 1. *Suppose that* $\mathcal{L} = -\frac{d^2}{dx^2} + |x|$. *Then* $\sigma_R^\alpha(\mathcal{L})$ *is uniformly bounded on* L^p *if* $\alpha > \max\{0, \frac{2}{3}|\frac{1}{2} - \frac{1}{p}| - \frac{1}{6}\}$, *which means the point* $(1/p, \alpha)$ *belongs to*

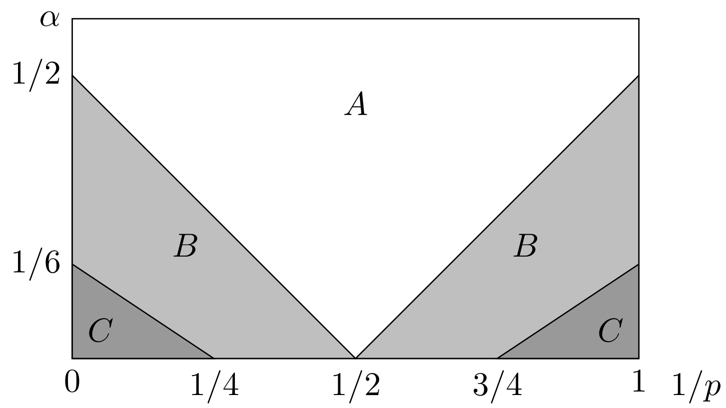

Figure 1. Convergence of Bochner-Riesz means for operators $-\frac{d^2}{dx^2} + |x|$ and harmonic oscillator $-\frac{d^2}{dx^2} + x^2$.

regions A *or* B*. Moreover if* $\alpha < \max\{0, \frac{2}{3}|\frac{1}{2} - \frac{1}{p}| - \frac{1}{6}\}$*, this is if* $(1/p, \alpha)$ *belongs to regions* C*, then* $\sup_{R>0} \|\sigma_R^\alpha(\mathcal{L})\|_{p\to p} = \infty$.

Figure 1 describes the convergence of Bochner-Riesz means for the operators $\mathcal{L}$ and $\mathcal{H}$. Note that the means are convergent in both regions A and B. The range A is common for all abstract operators in dimension 1. However the division between the parts B (convergent) and C (divergent) possibly depends on the operator. Indeed in case of the standard Laplace operator on $\mathbb{R}$ or on one dimensional torus Bochner-Riesz means converge in both regions B and C whereas for considered operators $\mathcal{L}$ and $\mathcal{H}$ the means are uniformly bounded only in B and they are not convergent in the part C.

The details discussion of the described results can be found in [CHS].

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Three Lower Bounds in Harmonic Analysis

Stefan Steinerberger

This talk discussed three problems in harmonic analysis that are concerned with bounding quantities from below. Interesting open problems arise.

0.1. **Poincaré inequalities.** The classical Poincaré inequality on the torus $\mathbb{T}^d$ states

$$\|\nabla f\|_{L^2(\mathbb{T}^d)} \geq \|f\|_{L^2(\mathbb{T}^d)}$$

for functions $f \in H^1(\mathbb{T}^d)$ with vanishing mean. A natural interpretation is that a function with small derivatives cannot substantially deviate from its mean on a set of large measure. This talk discussed a substantial improvement [S16].

Theorem 1. *There exists a set* $\emptyset \neq \mathcal{B} \subset \mathbb{T}^d$ *such that for every* $\alpha \in \mathcal{B}$ *there is a* $c_\alpha > 0$ *so that*

$$\|\nabla f\|_{L^2(\mathbb{T}^d)}^{d-1} \| \langle \nabla f, \alpha \rangle \|_{L^2(\mathbb{T}^d)} \geq c_\alpha \|f\|_{L^2(\mathbb{T}^d)}^{d}$$

for all $f \in H^1(\mathbb{T}^d)$ *with mean 0. If* $d \geq 2$*, then* $\mathcal{B}$ *is uncountable but Lebesgue-null.*

The exponents are optimal. The proof is simple and based on elementary properties of Fourier series – we believe it to be of great interest to understand under which conditions comparable inequalities exist on a general Riemannian manifold (M, g) equipped with a suitable vector field. The entire problem seems to relate to rather subtle dynamical properties of the vector field.

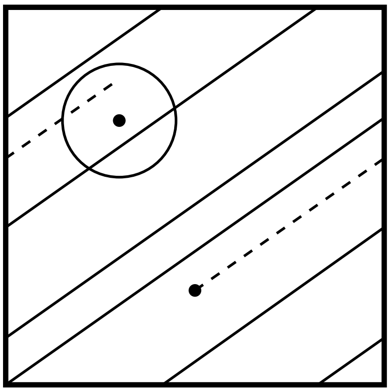

Figure 1. A well-mixing flow transports (dashed) every point relatively quickly to a neighborhood of every other point.

It would be of great interest to understand to which extent such inequalities can be true in a more general setup. It is not even clear to us whether comparable inequalities hold in $L^p(\mathbb{T}^d)$. Generally, for suitable vector fields Y on suitable Riemannian manifolds (M, g) it seems natural to ask whether there exists an inequality of the type

$$\|\nabla f\|_{L^p(M)}^{1-\delta} \| \langle \nabla f, Y \rangle f\|_{L^p(M)}^{\delta} \geq c\|f\|_{L^p(M)}$$

for some $\delta > 0$ and all $f \in W^{1,p}(M)$ with mean 0. The parameter δ can be expected to be related to the mixing properties of the flow – it is difficult to predict what the *generic* behavior on a fixed manifold might be (say, for a smooth perturbation of the flat metric on the torus).

0.2. **Integral operators.** The second part of the talk was concerned with results obtained jointly with Rima Alaifari (ETH), Lillian Pierce (Duke) [APS] and Roy Lederman (Princeton) [LS]. A sample result from [LS] is as follows.

Theorem 2. *There exists $c > 0$ such that for all $f \in L^2(\mathbb{R})$, normalized to $\|f\|_{L^2[-1,1]} = 1$ with compact support in $[-1,1]$*

$$\int_{-1}^{1} |\widehat{f}(\xi)|^2 d\xi \gtrsim \left(c\, \|f_x\|_{L^2[-1,1]}\right)^{-c\|f_x\|_{L^2[-1,1]}}.$$

A sample result for the Hilbert transform (from [APS]) is as follows.

Theorem 3. *For disjoint intervals $I, J \subset \mathbb{R}$, there exist $c_1, c_2 > 0$*

$$\|Hf\|_{L^2(J)} \geq c_1 \exp\left(-c_2 \frac{\|f_x\|_{L^2(I)}}{\|f\|_{L^2(I)}}\right) \|f\|_{L^2(I)}.$$

We have similar results for the Laplace transform. The proofs are based on Slepian's 'happy miracle' (the kernel of the integral operator arising as TT^* commutes with a simple differential operator). This cannot be reproduced at a greater level of generality. However, since Fourier/Laplace/Hilbert are rather diverse integral operators, it seems natural to assume that results of this type hold at a greater level of generality. We believe this question to be quite interesting.

0.3. **Topological Bounds on Fourier coefficients.** The final lower bound discussed was concerned with a recent inequality for elementary Fourier series that arose in the study of elliptic partial differential equations and has applications for level sets of the torsion function around the maximum [S17].

Theorem 4. *Assume $f \in C(\mathbb{T})$ has n sign changes. Then*

$$\sum_{k=0}^{n/2} |\langle f, \sin kx \rangle| + |\langle f, \cos kx \rangle| \gtrsim_n \frac{\|f\|_{L^1(\mathbb{T})}^{n+1}}{\|f\|_{L^\infty(\mathbb{T})}^{n}}.$$

The inequality is sharp up to the value of the implicit constant (that only depends on n). It can be understood as a generalization of the Sturm oscillation theorem.

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Riesz transforms, square functions, and rectifiability

Xavier Tolsa

The n-dimensional Riesz transform of a Radon measure μ in $\mathbb{R}^d$ is defined by

$$R_\mu f(x) = \int \frac{x-y}{|x-y|^{n+1}}\, f(y)\, d\mu(y).$$

The geometric characterization of measures μ such that the associated Riesz transform R_μ of codimension 1 (i.e. $n = d-1$) is bounded in $L^2(\mu)$ is a difficult problem with applications to other questions, such as the study of harmonic measure or the bilipschitz invariance of Lipschitz harmonic capacity.

When μ is Ahlfors-David regular, that is $\mu(B(x,r)) \approx r^n$ for all $x \in \operatorname{supp}\mu$ and all $0 < r < \operatorname{diam}(\operatorname{supp}\mu)$, by the solution of the David-Semmes problem in codimension 1 by Nazarov, Volberg and Tolsa [NToV], it turns out that R_μ is bounded in $L^2(\mu)$ if and only if μ is uniformly n-rectifiable. For more general measures, this characterization is much more delicate and, for the moment, there is not a complete solution.

A natural conjecture is the following: a Borel measure in $\mathbb{R}^{n+1}$ satisfying the growth condition

$$\mu(B(x,r)) \leq c_0\, r^n \quad \text{for all } x \in \operatorname{supp}\mu,\ r>0, \tag{1}$$

is bounded in $L^2(\mu)$ if and only if

$$\int_B \int_0^{r(B)} \beta_{\mu,2}(x,r)^2\, \Theta_\mu(x,r)\, \frac{dr}{r}\, d\mu(x) \leq C\, \mu(B) \quad \text{for any ball } B \subset \mathbb{R}^{n+1},\ r>0, \tag{2}$$

where $r(B)$ stands for the radius of B, and $\Theta_\mu(x,r)$ and $\beta_{\mu,2}(x,r)$ are the density and David-Semmes coefficients respectively defined by

$$\Theta_\mu(x,r) = \frac{\mu(B(x,r))}{r^n}$$

and

$$\beta_{\mu,2}(x,r) = \left(\inf_L \frac{1}{r^n} \int_{B(x,r)} \left(\frac{\operatorname{dist}(y,L)}{r} \right)^2 d\mu(y) \right)^{1/2},$$

where the infimum is taken over all n-planes $L \subset \mathbb{R}^{n+1}$.

Using a corona decomposition from [AT], one can show that the condition (2) is stable by bilipschitz mappings, and thus a positive solution to the conjecture above would imply that that the L^2 boundedness of codimension 1 Riesz transforms is also stable by bilipschitz mappings. More precisely, this means that if $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is bilipschitz and $\varphi\#\mu$ stands for the image measure of μ by φ, then the $L^2(\mu)$ boundedness of R_μ (assuming (1)) implies the $L^2(\varphi\#\mu)$ boundedness of $R_{\varphi\#\mu}$. In turn, this yields the invariance of the class of removable singularities of Lipschitz harmonic functions, by the results of [Vo].

Although the preceding conjecture is still open, there are several positive partial results:

- In the case $n = 1$ the conjecture was proved by Azzam and Tolsa in [AT], by using a corona type decomposition and the connection between the Cauchy kernel and Menger curvature.
- In the AD-regular case, $\Theta_\mu(x, r) \approx 1$ μ-a.e. and the conjecture is equivalent to saying that μ is uniformly n-rectifiable, by [DS]. Then the conjecture holds because of the solution of the David-Semmes problem in codimension 1 in [NToV].
- In [G], Girela-Sarrion proved the "easy" direction of the conjecture. That is, he showed that the condition (2) implies the $L^2(\mu)$ boundedness of R_μ. The main tool to prove this is the corona decomposition from [AT].
- In the recent work [JNT] by Jaye, Nazarov and Tolsa, it is shown that if one assumes, not only that the R_μ is bounded in $L^2(\mu)$, but also all the singular integral operators with a convolution type Calderón-Zygmund kernel of the form $K(x) = \psi(|x|)x$, so that $|\nabla^j K(x)| \leq 1/|x|^{n+j}$ for $j = 0, 1$, then the condition (2) holds.

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Hamming cube and martingales

Alexander Volberg

Let us consider smooth functions $M(x, y)$, $x \in I \subset \mathbb{R}$, $y \geq 0$, I is a convex subset of $\mathbb{R}$, such that

$$\mathcal{M} := \begin{pmatrix} M_{xx} + M_y/y & M_{xy} \\ M_{xy} & M_{yy} \end{pmatrix} \leq 0, \quad M_y \leq 0. \tag{1}$$

Each such function gives rise to an "isoperimetric" inequality with gaussian measure γ_n on $\mathbb{R}^n$:

$$\int M(f, |\nabla f|) d\gamma_n \leq M(\int f d\gamma_n, 0). \tag{2}$$

Examples are:

(1) log-Sobolev $M(x, y) = x \ln x - y^2/(2x)$,

(2) Bobkov's inequality with $M(x,y) = -\sqrt{[\Phi'(\Phi^{-1}(x))]^2 + y^2}$, $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-s^2/2}ds$,

(3) Beckner's inequality when $M(x,y) = x^{3/2} - \frac{3}{8}x^{-1/2}y^2$.

By saturating (1) to Monge-Ampère PDE $\det \mathcal{M} = 0$ one can seek for new or improved isoperimetries. Example is improved Beckner's inequality when

$$M(x,y) := \frac{1}{\sqrt{2}}(2x - \sqrt{x^2+y^2})\sqrt{x + \sqrt{x^2+y^2}}.$$

If one wants (2) on Hamming cube $Q_n = \{-1,1\}^n$, one needs to replace (1) by its correct discrete version. It turns out to be the so-called 2-*point inequality*

$$2M(x,y) \geq M(x+a, \sqrt{a^2+(y+b)^2}) + M(x-a, \sqrt{a^2+(y-b)^2}). \tag{3}$$

All functions mentioned above satisfy (3). How to ensure that M satisfy (3)?

Theorem 1. *If*

$$\min_{q\geq 0}\max_{p}(px - qy + U(p,q)) = \max_{p}\min_{q\geq 0}(px - qy + U(p,q)) =: M(x,y)$$

and U satisfies the main inequality

$$2U(p,q) \geq U(p+a, \sqrt{a^2+q^2}) + U(p-a, \sqrt{a^2+q^2}), \tag{4}$$

then M satisfies the 2-point inequality (3).

Theorem 2. *U satisfies* (4) *if and only if*

$$U(p,q) = \sup\{\int_0^1 F(f(x), \sqrt{q^2 + S^2 f(x)})dx : \int_0^1 f dx = p\} \tag{5}$$

for some F.

One can recognize in (5) the formula for Bellman function of martingales estimates.

Counting factorisations of X^n

Jim Wright

(joint work with Julia Brandes and Jonathan Hickman)

We give a precise count of the number solutions to the following system of polynomial congruences:

$$\begin{array}{rcl} x_1 + x_2 + \cdots + x_n & = & 0 \\ x_1^2 + x_2^2 + \cdots + x_n^2 & = & 0 \\ & \vdots & \\ x_1^n + x_2^n + \cdots + x_n^n & = & 0. \end{array} \quad \bmod N \tag{1}$$

By the Chinese remainder theorem, it suffies to consider the case when $N = p^s$ is a prime power. When $p > n$ we see that by the Newton-Girard formulae, the number of solutions to (1) is the sames as the number of solutions to the system

$$(2)\qquad \begin{array}{rcl} x_1 + x_2 + \cdots + x_n & = & 0 \\ x_1x_2 + \cdots + x_{n-1}x_n & = & 0 \\ & \vdots & \\ x_1 \cdots x_n & = & 0 \end{array} \qquad \mod\ p^s$$

of polynomial congruences given by the n elementary symmetric functions $e_1 = x_1 + \cdots + x_n, \ldots, e_n = x_1x_2\cdots x_n$ of $x_1, \ldots, x_n$. Since

$$(X - x_1)\cdots(X - x_n) = X^n - e_1X^{n-1} + \cdots (-1)^n e_n,$$

we see that the number of solutions to (2) counts the number of factorisations of the monomial $X^n = (X - x_1)\cdots(X - x_n)$ in the ring $\mathbb{Z}/p^s\mathbb{Z}$.

The problem of counting solutions to (1) is a special case of counting the solutions $(x_1, \ldots, x_n)$ to

$$(3)\qquad \begin{array}{rcl} x_1 + x_2 + \cdots + x_n & = & y_1 + y_2 + \cdots + y_n \\ x_1^2 + x_2^2 + \cdots + x_n^2 & = & y_1^2 + y_2^2 + \cdots + y_n^2 \\ & \vdots & \\ x_1^n + x_2^n + \cdots + x_n^n & = & y_1^n + y_2^n + \cdots + y_n^n \end{array} \qquad \mod\ N$$

for a given set of integers $y_1, \ldots, y_n$ which we currently do not know how to do. The number of solutions to (3) correspond to counting the number of factorisations of a general polynomial $P(X) = (X - y_1)\cdots(X - y_n)$ in the coefficient ring $\mathbb{Z}/N\mathbb{Z}$ which in turn arises from the study of the Fourier Resriction Problem along curves in the setting of local fields.

Our method giving a precise count of the number of solutions to (1) uses an elementary induction on scales argument, a basic technique from euclidean harmonic analysis. A simple lifting argument shows that (1) with $N = p^s$ is equivalent to the bounding the Haar measure of the sublevel set

$$\left\{\underline{x} = (x_1, \ldots, x_n) \in \mathbb{Z}_p^n : |P_1(\underline{x})|,\ \ldots,\ |P_n(\underline{x})| \le p^{-s}\right\}$$

in the p-adic ring $\mathbb{Z}_p$. Here $|\cdot|$ is the p-adic valuation and $P_j(\underline{x}) = x_1^j + \cdots + x_n^j$ are the symmetric power polynomials.

Our arguments are elementary and in particular do not use any special properties of the p-adic valuation $|\cdot|$ or Haar measure on $\mathbb{Z}_p$. Consequently the arguments also give precise bounds on the Lebesgue measure of the euclidean sublevel set

$$\left\{\underline{x} = (x_1, \ldots, x_n) \in [-1, 1]^n : |P_1(\underline{x})|,\ \ldots,\ |P_n(\underline{x})| \le \delta\right\}$$

where again $P_j(\underline{x}) = x_1^j + \cdots + x_n^j$.

A discretized incidence theorem in the plane

Joshua Zahl
(joint work with Nets Katz)

We prove an incidence theorem for points and lines in the plane that satisfy certain non-concentration conditions. This is an ingredient in the proof that every Kakeya set in $\mathbb{R}^3$ has Hausdorff dimension at least $5/2+\epsilon$ for some absolute constant $\epsilon > 0$.

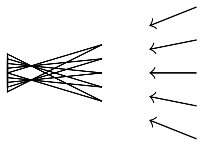
Square functions for bi-Lipschitz maps and directional operators

Pavel Zorin-Kranich
(joint work with Shaoming Guo, Francesco Di Plinio, and Christoph Thiele)

Let $v : \mathbb{R}^2 \to S^1$ be a 1/100-Lipschitz unit vector field. It is a long-standing open problem, attributed to Zygmund, whether the associated directional maximal operator

$$M_v f(x) := \sup_{0<r<1} \frac{1}{r} \int_{-r}^{r} f(x + v(x)t)dt$$

is bounded on any $L^p(\mathbb{R}^2)$, $1 < p < \infty$. The Lipschitz regularity and the restriction to small scales are necessary, as can be seen by considering a vector field pointing into a Perron tree (the arrangement of triangles used to construct a Kakeya set, see e.g. [S]):



A fruitful line of investigation haas been breaking the rotational symmetry of the problem. Indeed, decomposing S^1 into small arcs V and considering the set $X := \{x \in \mathbb{R}^2 : v(x) \in V\}$ on which v points in the direction V, one can extend $v|_X$ to a Lipschitz vector field on $\mathbb{R}^2$ pointing in the direction V by Kirszbraun's extension theorem. Hence without loss of generality one may assume that the vector field is almost horizontal.

A singular integral version of Zygmund's problem is attributed to Stein. The question is whether the directional Hilbert transform

$$H_u f(x, y) := \text{p. v.} \int_{-1}^{+1} f(x + r, y + u(x, y)r) \frac{dr}{r}$$

is bounded on any $L^p(\mathbb{R}^2)$ space, where $u : \mathbb{R}^2 \to [-1, 1]$ is 1/100-Lipschitz (here we have used the reduction to a small arc of directions).

Let ψ be a Schwartz function on $\mathbb{R}$ such that

$$1_{\pm[99/100,103/100]} \le \widehat{\psi} \le 1_{\pm[98/100,104/100]}.$$

Let Ψ be another Schwartz function on $\mathbb{R}$ such that $\widehat{\Psi}$ is supported on $\pm[1, 101/100]$. Let $P_t f := \psi_t * f$ be the Littlewood–Paley operators associated to ψ, where $\psi_t(x) = t^{-1}\psi(t^{-1}x)$. It has been proved by Lacey and Li [LL] that $H_u \circ P_t$ is bounded on L^p uniformly in $t \in (0, \infty)$, even if the function u is merely measurable.

They have also shown that L^2 estimates for $H_u \circ P_t$ imply an L^2 estimate for H_u provided that u is of the clss $C^{1+\epsilon}$. We have replaced [GPTZ] this regularity hypothesis by a Lipschitz hypothesis, which is sharp in view of the Perron tree example. This has been made possible by a one dimensional Littlewood-Paley diagonalization estimate for bi-Lipschitz maps:

Theorem 1. *Let $A : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function with $\|A\|_{\mathrm{Lip}} \le 1/100$ and consider the change of variable $T_A f(x) := f(x + A(x))$. Then*

$$\| \sum_{t \in 2^{\mathbb{Z}}} |(1 - P_t) T_A (\Psi_t * f)| \|_p \lesssim_{p,\psi,\Psi} \|A\|_{\mathrm{Lip}} \|f\|_p, \quad 1 < p < \infty.$$

If the sum over t is replaced by a square sum, then the estimte would follow from standard Littlewood–Paley theory and the Fefferman–Stein vector-valued maximal inequality. When the Lipschitz norm of A becomes too large, then in general T_A fails to be a bijection and the estimate of the theorem breaks down. Also, the Fourier support of Ψ is necessarily smaller than that of ψ.

In the case of u that is Lipschitz in the second variable, Theorem 1 allows us to reduce the L^p boundedness of H_u essentially to that of its diagonal part $\sum_{t \in 2^{\mathbb{Z}}} P_t H_u P_t$. Using the ideas in [BT] this leads to the following conditional results that generalize those obtained in [LL, BT].

Theorem 2. *Let $u : \mathbb{R}^2 \to \mathbb{R}$ be such that $u(x, \cdot)$ has Lipschitz constant $\le 1/100$ for almost every $x \in \mathbb{R}$. Assume further that*

$$\forall f \quad \sup_{0<t<t_0} \|H_u(\Psi_t *_2 f)\|_{p_0} \lesssim \|f\|_{p_0} \tag{1}$$

for some $1 < p_0 \le 2$ and $t_0 > 0$, where $*_2$ denotes convolution in the second variable. If $p_0 = 2$, then

$$\|H_u f\|_2 \lesssim \|f\|_2. \tag{2}$$

If $1 < p_0 < 2$, then

$$\|H_u f\|_p \lesssim \|f\|_p, \quad 1 + \frac{1}{3 - p_0} < p < \infty. \tag{3}$$

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