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# Large Scale Stochastic Dynamics

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Abstract. Equilibrium statistical mechanics studies random fields distributed according to a Gibbs probability measure. Such random fields can be equipped with a stochastic dynamics given by a Markov process with the correspondingly high-dimensional state space. One particular case are stochastic partial differential equations suitably regularized. Another common version is to consider the evolution of random fields taking only values 0 or 1. The workshop was concerned with an understanding of qualitative properties of such high-dimensional Markov processes. Of particular interest are non-reversible dynamics for which the stationary measures are determined only through the dynamics and not given a priori (as would be the case for reversible dynamics). As a general observation, properties on a large scale do not depend on the precise details of the local updating rules. Such kind of universality was a guiding theme of our workshop.

*Mathematics Subject Classification (2000):* 82xx, 60xx, 35Lxx, 37xx.

## Introduction by the Organisers

"Large Scale Stochastic Dynamics" is at the crossroad of probability theory and statistical physics. One central theme of statistical physics is the emergent behavior resulting from the interaction of many identical components, the paradigm being a fluid or a gas. On the atomistic scale they consist of a huge number of identical molecules. Their motion is governed by Newton's equation of classical mechanics (ignoring quantum effects). The emergent description, valid only for particular initial states and on a sufficiently coarse space-time scale, are the compressible Navier-Stokes equations of fluid dynamics. Roland Dobrushin (1929-1995) and Frank Spitzer (1926-1992) had the vision that in the context of stochastic dynamics with many identical components the issue of emergent behavior is both mathematically challenging and important in modeling applications. The latter

judgment turned out to be more than true. Stochastic algorithms, such as kinetic Monte Carlo, importance sampling, Monte Carlo Markov chains, Glauber dynamics, and others, are daily practice. Their mathematical vision has evolved over the past twenty years into a rich, multifaceted research program. Our workshop is like a snap-shot of the current activities, in fact quite distinct from the snap-shot taken exactly three years ago. A partial list of topics reads

- energy transport in anharmonic lattices
- large deviations for diffusive systems
- motion of tracer particles
- nonequilibrium phase transitions
- superdiffusivity and the Airy processes
- hydrodynamic limits
- random polymers

We had 44 participants from 11 countries, mostly probabilists, but also experts from partial differential equations and statistical physics. They all enjoyed tremendously the unique and stimulating atmosphere at the Mathematische Forschungsinstitut Oberwolfach and hope to return some day.

Claudio Landim,
Stefano Olla,
Herbert Spohn.

## Workshop: Large Scale Stochastic Dynamics

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# Abstracts

## Current large deviations and vortices

Thierry Bodineau
(joint work with B. Derrida, J. Lebowitz)

Recently, it has been shown how to compute the large deviation function of the current in one dimensional diffusive systems [1]-[6]. The hydrodynamic large deviation theory [1, 11, 8], yields explicit expressions for the large deviation function as well as the cumulants of the current fluctuations (under some stability condition [4, 6]). The same hydrodynamical approach applies in principle also to currents in higher dimension. In this talk, we show however that this approach does not always catch the correct scaling of the large deviations or of the cumulants of the current in higher dimensions. This will be made explicit in the case of the 2 dimensional symmetric simple exclusion process (SSEP).

For a *one dimensional* diffusive system of length $L$ in contact at its left end with a reservoir at density $\rho_a$ and at its right end with a reservoir at density $\rho_b$, one can consider the total net number $Q(\tau)$ of particles leaving the left reservoir during a time interval $\tau$. This number $Q(\tau)$ fluctuates in time and one expects that in the long time limit

$$\text{Pro}\left(\frac{Q(\tau)}{\tau} \simeq J\right) \sim \exp\big[-\tau G_L(J;\rho_a,\rho_b)\big] \tag{1}$$

where $G_L(J;\rho_a,\rho_b)$ is the large deviation function of the flux through the system. In fact $G_L$ does not depend on where the flux, i.e. the integrated current, is measured along the one dimensional system, as long as particles cannot accumulate. For large $L$ and $J$ of order $\frac{1}{L}$, $G_L$ satisfies, the following scaling

$$G_L(J;\rho_a,\rho_b) \simeq \frac{1}{L} F(LJ;\rho_a,\rho_b) \tag{2}$$

The scaling (2) implies that for large $L$ all the cumulants of $Q(\tau)$ are of order $1/L$, i.e.

$$\lim_{\tau\to\infty} \frac{\langle Q(\tau)^n\rangle_c}{\tau} \simeq \frac{1}{L}\kappa_n(\rho_a,\rho_b)\ . \tag{3}$$

Explicit expressions of the $\kappa_n(\rho_a,\rho_b)$ have been obtained [4, 6] in terms of the diffusion constant $D(\rho)$ and the conductivity $\sigma(\rho)$ [13]. One can also show that the large deviation function $G_L$ of the current satistifies the fluctuation theorem [9, 12], i.e.

$$G_L(J;\rho_a,\rho_b) - G_L(-J;\rho_a,\rho_b) = J[\log z(\rho_b) - \log z(\rho_a)] \tag{4}$$

where $z(\rho)$ is the fugacity of a reservoir at density $\rho$.

In *higher dimension*, one can study, as in one dimension, the total current flowing through the system from one reservoir to the other, but one can also study

part of this current. In this talk, we consider the SSEP on a square lattice of size $L$, with periodic boundary conditions in the vertical direction and study the current flowing through a vertical slit of length $\ell < L$. The large deviation function $G_{L,\ell}(J;\rho_a,\rho_b)$, defined as in (1), depends of course on the size $\ell$ of the slit. One reason for considering the fluctuations of this partial current is that in experiments it is often only possible to measure the fluctuations of local quantities and not of global quantities.

In two dimensions, when $\ell = L$, i.e. when one considers the total current flowing through the system, the large deviation function derived from the hydrodynamic theory satisfies for large $L$ and $J$ of order 1 a scaling similar to the one dimensional case [3]

$$G_{L,L}(J;\rho_a,\rho_b) \simeq F(J;\rho_a,\rho_b) \tag{5}$$

(this would become $L^{d-2}F(L^{2-d}J;\rho_a,\rho_b)$ for a cube of size $L$ in dimension $d$ and $J$ of order $L^{d-2}$). We show that $G_{L,\ell}$ cannot satisfy the same scaling (5) as $G_{L,L}$ and that for large $L$, if one keeps the ratio $h = \ell/L$ fixed, then for all $0 < h < 1$ and $J$ of order 1

$$G_{L,Lh}(J;\rho_a,\rho_b) \to 0 \quad \text{as} \quad L \to \infty \ . \tag{6}$$

While, as in (3), one expects the $n-th$ cumulant of the total flux $Q(\tau)$ to have a limit as $L$ diverges

$$\lim_{\tau\to\infty} \frac{\langle Q(\tau)^n\rangle_c}{\tau} \to \kappa_n(\rho_a,\rho_b) \ . \tag{7}$$

(which would become $\frac{1}{2}\langle Q(\tau)^n\rangle_c \simeq \tau L^{d-2}\kappa_n(\rho_a,\rho_b)$ in dimension $d$). For the SSEP at mean density $\bar\rho$ on a periodic square domain, an explicit calculation shows a logarithmic divergence of the second cumulant for a slit of size $\ell = Lh$ and $0 < h < 1$

$$\lim_{\tau\to\infty} \frac{\langle Q^{(h)}(\tau)^2\rangle_c}{\tau} \sim \frac{2\bar\rho(1-\bar\rho)}{\pi}\log L \quad \text{as} \quad L \to \infty, \tag{8}$$

where $Q^{(h)}(\tau)$ is the flux of particles through the slit during time $\tau$.

The fluctuation theorem, which is satisfied as written in (4) for the two-dimensional SSEP when $J$ is the total current through the system (i.e. when $\ell = L$), has in fact no reason to remain valid for $\ell < L$. To illustrate this, we give a simple example of a two site model where one can see clearly that the fluctuation theorem is satisfied when one looks at the total current but is no longer valid when one considers only part of the current.

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## Fourier Law

Antti Kupiainen
(joint work with Jean Bricmont)

We discuss the problem of deriving Fourier's law of heat conduction starting from deterministic dynamics of a system with a large number of degrees of freedom coupled to noise at its boundary. We consider two kinds of systems

- Coupled weakly nonlinear systems
- Weakly coupled chaotic systems

The first class consists of a Hamiltonian system made of coupled anharmonic oscillators arranged on a three dimensional lattice $\mathbb{Z}_{2N} \times \mathbb{Z}^2$, and subjected to a stochastic forcing mimicking heat baths of temperatures $T_1$ and $T_2$ on the hyperplanes at 0 and $N$.

The stationary state of the resulting Markov process satisfies an infinite system of linear equations, so called Hopf equations. We introduce a truncation of the Hopf equations which leads to a nonlinear equation for the two-point stationary correlation functions. We prove in [1, 2] that these equations have a unique solution which, for $N$ large, is approximately a local equilibrium state satisfying Fourier law that relates the heat current to a local temperature gradient. The temperature exhibits a nonlinear profile.

In a kinetic scaling limit our system reduces to a Boltzmann equation of a classical phonon system. We consider in [3] the problem of approach to equilibrium for this equation. We prove that, if the initial condition is a small perturbation of an equilibrium state, and vanishes at infinity, the dynamics tends diffusively to equilibrium. The solution is the sum of a local equilibrium state, associated to

conserved quantities that diffuse to zero, and fast variables that are slaved to the slow ones. This slaving implies the Fourier law, which relates the induced currents to the gradients of the conserved quantities.

The second class of systems are motivated by some recent work of Eckmann and Yang [4] and of Dolgopyat, Keller and Liverani [5]. We consider deterministic dynamical systems where at each lattice site $x$ we have variables $(E_x, \theta_x)$. In the uncoupled dynamics $E_x$ is conserved and $\theta_x$ hyperbolic. We introduce a weak coupling that leaves $\sum_x E_x$ conserved. Interpreting $E_x$ as a PDF of a random walk we end up with the problem of studying diffusive behavior of a random walk in a space time dependent environment determined by the initial conditions of the $\theta$. For weak coupling the environment is weakly correlated in space time and deterministic diffusion follows from a proof of diffusion for such random walks (work in progress).

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## Microscopic Hamiltonian dynamics perturbed by a conservative noise


Cédric Bernardin

(joint work with Giada Basile and Stefano Olla)


When a gradient of temperature $\nabla T$ is applied to a material, we expect that, in the steady state, the heat current satisfies Fourier's law

$$< J >= \kappa \nabla T$$

where $\kappa$ is the conductivity of the material. If the system has length $N$ then finite conductivity means $< J >$ of order $1/N$.

A challenging question arising in statistical mechanics is to derive Fourier's law from microscopic Hamiltonian dynamics. The standard microscopic models are given by the Fermi-Pasta-Ulam (FPU) chains which are systems of coupled *anharmonic* oscillators. The Hamiltonian is given by

$$\mathcal{H} = \sum_x \frac{p_x^2}{2} + \sum_{x \sim y} V(q_y - q_x) + \sum_x U(q_x)$$

Here, the sum is indexed by the equilibrium positions of atoms (a sub-lattice of $\mathbb{Z}^d$), $p_x, q_x$ are momentum and position of atom with equilibrium position $x$, $V$ is the pair interaction potential and $U$ the one-site potential, or pinning.

Non-linearity is extremely important. In fact, in the linear case ($U', V'$ linear), the average energy current $< J >$ is independent of the length $N$ of the system, i.e. the conductivity $\kappa_N$ of the system diverges like $N$.

But the story is more complex and non-linearity is not sufficient to avoid diverging conductivity. Anomalous large conductivity is observed experimentally in carbon nanotubes and numerically in FPU systems without pinning in low dimension $d \leq 2$. It has been argued that the *conservation of momentum* (which is equivalent to the absence of a one-site potential) plays a major role in the divergence of the conductivity.

Since conductivity in non-linear systems is difficult to compute or estimate analytically, it is natural to model the nonlinearities by stochastic perturbations of the linear dynamics. In some sense these stochastic perturbations simulate (qualitatively) the long time (chaotic) effect of the deterministic non-linear model.

The models we propose are stochastic models which reproduce qualitatively the behavior observed for non-linear FPU chains. We consider a chain of harmonic oscillators with a quadratic one-site potential

$$\mathcal{H} = \frac{1}{2}\sum_x \left[p_x^2 + q_x \cdot (\nu I - \alpha\Delta) q_x\right]$$

Here, $\Delta$ is the discrete Laplacian and parameters $\alpha, \nu$ regulate the strength of the pair interaction and the strength of the pinning. Conservation of the momentum is equivalent to $\nu = 0$.

To this Hamiltonian dynamics, we add a stochastic noise. In terms of generator $L$ of the dynamics, it means that $L = A + \gamma S$ where $A$ is the Liouville operator associated to the Hamiltonian $\mathcal{H}$

$$A = \sum_x \left\{ \frac{\partial \mathcal{H}}{\partial p_x} \cdot \partial_{q_x} - \frac{\partial \mathcal{H}}{\partial q_x} \cdot \partial_{p_x} \right\}$$

and $S$ is the noise generator acting only on velocities. The strength of the noise is given by a positive constant $\gamma$.

Several noise can be considered. They conserve some physical quantities of interest.

- A noise conserving the total energy $\mathcal{H}$ and the total momentum $\sum_x p_x$.
- A noise conserving only the total energy $\mathcal{H}$.

The last noise simulates presence of the pinning since destroying conservation of momentum (as pinning does). Let us explain briefly how to construct such noise and for simplicity we restrict ourselves to the first noise for $d \geq 2$. Let $\mathbb{S}_{e,p}$ be the surface of constant kinetic energy $e$ and momentum $p$ for two n.n. atoms $x$ and $z$

$$\mathbb{S}_{e,p} = \left\{ (p_x, p_z) \in \mathbb{R}^{2d} : p_x^2 + p_z^2 = 2e;\ p_x + p_z = p \right\}.$$

The following vector fields are tangent to $\mathbb{S}_{e,p}$

$$X^{i,j}_{x,z} = (p^j_z - p^j_x)(\partial_{p^i_z} - \partial_{p^i_x}) - (p^i_z - p^i_x)(\partial_{p^j_z} - \partial_{p^j_x}).$$

We then define

$$S = \frac{1}{2(d-1)} \sum_x \sum_{i,j,k}^{d} \left( X^{i,j}_{x,x+e_k} \right)^2$$

with $e_1, \ldots, e_d$ the canonical basis of $\mathbb{Z}^d$.

By construction, we have $S(\mathcal{H}) = S(\sum_x p_x) = 0$ and consequently $\{ L(\mathcal{H}) = 0\}$ and $\{ L(\sum_x p_x) = 0$ iff $\nu = 0\}$.

Fourier's law and anomalous conductivity can be studied by two different approachs. The first one consists to put the system in contact with thermal baths (Langevin baths for example) with different temperatures and to analyse the behavior of the average current $< J >$ in the stationary state as a function of $N$ and of the difference of the boundary temperatures. The second one is based on linear response theory (Green-Kubo formula) which relates values of non-equilibrium transport coefficients to equilibrium fluctuations of currents. Consider the $d$-dimensional (isotropic) closed system of length $N$ with total energy $N^d e$ under periodic b.c. Green-Kubo formula gives the (infinite volume limit) conductivity in the direction $e_1$

$$\kappa^{1,1} = \lim_{t\to\infty} \frac{d}{2e^2 t} \lim_{N\to\infty} \sum_x \mathbb{E}_N \left( J_{x,x+e_1}(t) J_{0,e_1}(t) \right)$$

where $\mathbb{E}_N$ is the expectation starting with the microcanonical distribution and $J_{x,x+e_1}(t)$ is the total energy current between atom $x$ and atom $x + e_1$ up to time $t$. The total current energy is expressed as the sum of the integrated instantaneous current energy $j_{x,x+e_1}$ and a martingale term $M_{x,x+e_1}(t)$ due to the noise:

$$J_{x,x+e_1}(t) = \int_0^t j_{x,x+e_1}(s)\, ds + M_{x,x+e_1}(t)$$

The martingale term gives always a finite contribution ($= \gamma$) to the conductivity.

$$\kappa^{1,1} = \frac{d}{2e^2\gamma} \int_0^\infty C_J(t) dt + \gamma$$

where

$$C_J(t) = \lim_{N\to\infty} \sum_x \mathbb{E}_N(j_{0,e_1}(0) j_{x,x+e_1}(t))$$

Diverging conductivity and breakdown of linear response theory is the consequence of the slow decay of the time correlation of the instantaneous energy current $C_J(t)$

For the momentum-energy conservative noise, our model reproduces qualitatively what is expected for FPU chains and it explains the pertinence of the model:

**Theorem 1.** *([1]) $C_J(t) \sim_{t\to\infty} t^{-d/2}$ if the system is unpinned, while $C_J(t) \sim_{t\to\infty} t^{-(d/2+1)}$ if an on-site potential is present. Conductivity, defined by Green-Kubo formula, is then finite only in dimension $d \geq 3$ or for the pinned system.*

In the energy conservative case, conductivity, defined by Green-Kubo formula $\kappa_{GK}$, is finite in any dimension. But we have better:

**Theorem 2.** *([2]) Consider the OPEN system in contact with two reservoirs at different temperatures $T_\ell$ and $T_r$.*

$$\lim_{N\to\infty} N < J >_N = \kappa_{GK}(T_\ell - T_r)$$

The proofs of these theorems are based on more or less standard ingredients of hydrodynamic limits literature and on our ability to establish an *exact fluctuation-dissipation equation.* We conclude by few open questions:

- Can you analyse the energy-momentum conservative model in a non-equilibrium stationary state an deduce the order of divergence of the conductivity for low dimensional unpinned systems ? Our computations of the Green-Kubo formula predict a divergence of order $\sqrt{N}$ for $d = 1$, $\log N$ for $d = 2$.
- Imagine that atoms have random masses or interactions. What is the effect of the disorder on the conductivity? In particular, in the energy-momentum conservative model, does it decrease the divergence order of the conductivity as it is the case for a purely harmonic chain ?
- Can you say anything when the interactions are non-linear ? We have upper-bounds. Numerical simulations (G. Basile and R. Livi) show that increasing the strength of the noise $\gamma$ (meaning increasing ergodicity of the system) increases also the diverging order of the conductivity!!! Can you explain it ? For small $\gamma$ and anharmonic interactions, numerical simulations indicate our model has the same behavior as the corresponding FPU chains (same order of divergence for the conductivity) .
- Take harmonic oscillators and add an asymmetric noise of the form

  $$S = \sum_x (p_{x+1}\partial_{p_x} - p_x\partial_{p_{x+1}})^2 + p_x p_{x+1}(p_{x+1}\partial_{p_x} - p_x\partial_{p_{x+1}})$$

  This noise (considered alone) belongs to the KPZ class (I have rigorous results in this direction) and has hence large anomalous fluctuations. But it does not conserve total momentum. Add this noise to the Hamiltonian dynamics. Is the conductivity finite or not ?

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## Existence and uniqueness of an invariant measure for a heat conduction network

Philippe Carmona
(joint work with Alain Camanes)

Let us consider the *heat conduction network* model of Maes, Netočný and Verschuere [5]: $G = (V, \sim)$ is a finite connected graph with vertex set $V$. Two vertices $i \neq j$ are nearest neighbors if there is an edge between them : $i \sim j$. Every node $i \in V$ holds an atom of moment $p_i \in \mathbb{R}$ and position $q_i \in \mathbb{R}$. The total energy inside the system is given by the Hamiltonian

$$H = \sum_{i \in V} (\frac{p_i^2}{2} + V(q_i)) + \sum_{i \sim j} U(q_i - q_j)\,. \tag{1}$$

The *pinning potential* $V$ and the *interaction potential* $U$ are both assumed to be $C^\infty$ functions that are confining, that is

$$\lim_{|x| \to +\infty} V(x) = \lim_{|x| \to +\infty} U(x) = +\infty\,.$$

For sake of simplicity, we shall assume from now on that $U$ is a polynomial. We shall therefore not reach the level of generality of Eckmann and Hairer [2] in the case of a chain of oscillators.

There is a non empty subset $\partial V \subset V$, called the boundary set : the atom at the boundary site $i \in \partial V$ is connected to a *heat bath* at temperature $T_i$. This interaction is modeled by an Ornstein Uhlenbeck process (a Langevin process). Therefore, the dynamics of the system are described by the system of stochastic differential equations

$$(S) \qquad \begin{cases} dq_i(t) = p_i(t)\,dt \\ dp_i(t) = -\partial_{q_i} H\,dt + \mathbf{1}_{(i \in \partial V)} \left( -\frac{1}{2} p_i\,dt + \sqrt{T_i}\,dB_i(t) \right) \end{cases}$$

where $(B_i)_{i \in \partial V}$ are independent standard real Brownian motions.

The issue we want to address is the existence and uniqueness of an invariant probability measure. The obstructions to an easy solution of this problem are of two kinds:

- A *geometric obstruction.* If the graph $G$ and the boundary set $\partial V$ exhibit too much symmetry then, even for equal heat bath temperatures ($\forall i \in \partial V$, $T_i = T$), and for harmonic potentials $V(x) = U(x) = \frac{x^2}{2}$, there may exist more than one invariant probability measure.
- A *potential obstruction.* If the pinning potential $V$ is much stronger than the interacting potential $U$, then it is still an open problem to prove that there is sufficient energy transmission along the network.

We now introduce an assumption that turns out to be sufficient in the general potential case to ensure uniqueness , and necessary and sufficient in the harmonic case to ensure existence and uniqueness of an invariant probability measure.

Let $N = |V|$ be the number of vertices, $\Lambda$ be the linear operator defined on the canonical base $(e_i)_{i\in V}$ of $\mathbb{R}^N$ by

$$\Lambda e_i = \sum_{i\sim j} e_j \,,$$

the matrix of $\Lambda$ is often called the connectivity matrix of the graph $G$. Let $E_\Lambda$ be the smallest subspace of $\mathbb{R}^N$ containing the basis vectors $\{e_i, i \in \partial V\}$ and stable by $\Lambda$, that is

$$E_\Lambda = span(\Lambda e_i, k \geq 0, i \in \partial V) \,.$$

Define the couple $(G, \partial V)$ to be AET, *Asymmetrical Energy Transmitting*, if the vector space $E_\Lambda$ has full rank : $\mathrm{rank}(E_\lambda) = |V|$. We can now state the main result of this paper.

**Theorem 1.** *If $(G, \partial V)$ is AET, then there exists at most one invariant probability measure.*

The AET assumption is optimal in the harmonic case.

**Theorem 2.** *Let us consider a harmonic network with equal heat bath temperatures. If $(G, \partial V)$ is not AET, then there exists an infinite number of invariant probability measures of Gibbs type*

$$\mu_{\beta,\gamma}(dx) = \frac{1}{Z_{\beta,\gamma}} e^{-\beta H(x) - \gamma K(x)}\, dx \qquad (\beta = \frac{1}{T}, \gamma > 0) \,,$$

*where $K$ is a quadratic polynomial in variables $p_i, q_i$.*

**Remark 1.** *Let us stress the fact that the AET property ensures that there is enough energy transmitted from the reservoirs into the whole network, and that no symmetry will produce another invariant than the Hamiltonian.*

*2 We have not been able to overcome the technical assumption usually made in the literature when studying of a chain of oscillators (see Eckamnn, Hairer, Pillet, Rey-Bellet and Thomas* [7, 6, 9, 8, 3, 4, 2]*). We still need to assume that the* interaction dominates the pinning at infinity*, that is*

$$\liminf_{|x|\to+\infty} \frac{U(x)}{V(x)} > 0 \,.$$

In the first part of the talk we study the diamond, a counter example devised by Maes, Netočný and Verschuere [5]. This counterexample will give us a clear intuition about the nature of the geometric obstruction and was the origin of the AET definition.

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## Boundary-induced phase transition in the one-dimensional two-component symmetric exclusion process

Gunter M. Schütz
(joint work with Andreas Brzank)

We consider single-file diffusion in an open system with two species $A, B$ of particles. At the boundaries we assume different reservoir densities which drive the system into a non-equilibrium steady state. As a model we use an one-dimensional two-component simple symmetric exclusion process with two different hopping rates $D_A, D_B$ and open boundaries. Following [3] we consider a one-dimensional lattice with $L$ lattice sites (Fig. 1). Each site $i$ can be empty or occupied by a particle of type $A$ or $B$. Due to hard-core interaction any site carries at most one particle. Particles can hop to nearest neighbour sites (provided the target site is empty) with hopping rates $D_{A/B}$. In the case of open boundary conditions, particles are injected and removed according to the boundary rates $\alpha_{A/B}, \gamma_{A/B}, \beta_{A/B}$ and $\delta_{A/B}$ as illustrated in Fig. 1. Jump events occur after an exponentially random time which in dynamic Monte Carlo simulations (DMCS) is modelled by random sequential update [3].

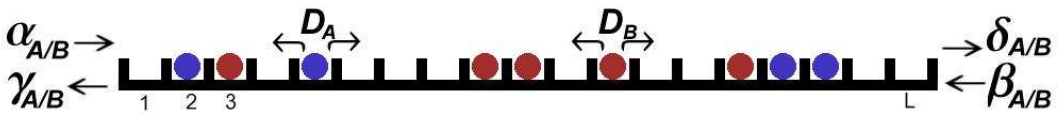


Figure 1. Two-component symmetric exclusion model with open boundaries. Each species of particles has its own hopping rates $D_A, D_B$. At the boundaries particles are extracted and injected with rates as indicated.

We consider

$$\alpha_{A/B} = D_{A/B}\rho^-_{A/B}, \quad \gamma_{A/B} = D_{A/B}(1-\rho^-_A-\rho^-_B) \tag{1}$$

$$\delta_{A/B} = D_{A/B}\rho^+_{A/B}, \quad \beta_{A/B} = D_{A/B}(1-\rho^+_A-\rho^+_B). \tag{2}$$

If the equilibrium condition $\rho^-_{A,B} = \rho^+_{A,B}$ of equal reservoir densities or chemical potentials $\mu^-_{A,B} = \mu^+_{A,B}$ holds then the process has an unique uncorrelated stationary equilibrium distribution with constant local particle densities equal to the reservoir densities. In a non-equilibrium setting the left reservoir densities $\rho^-_A$, $\rho^-_B$ are not equal to those on the right boundary ($\rho^+_A$, $\rho^+_B$). In this boundary-driven case the system evolves towards a complicated stationary state with non-vanishing particle currents.

For investigating the dynamics in the hydrodynamic limit we extend the rigorous approach of Quastel [2] and make an ansatz for the dynamics of a single particle (of species $A$ or $B$) localized at position $x$. This test particle acts as a tracer particle in the background of other particles, with a diffusive motion partially determined by its self-diffusion coefficient. Additionally, the test particle is subject to a background drift $b$ caused by the collective evolution of the entire system towards stationarity. This yields a system of coupled non-linear diffusion equations for the coarse-grained particle densities

$$\partial_t\rho_A(x,t) = \partial_x^2 D_s\rho_A(x,t) - \partial_x b(x,t)\rho_A(x,t) \tag{3}$$

$$\partial_t\rho_B(x,t) = \partial_x^2 D_s\rho_B(x,t) - \partial_x b(x,t)\rho_B(x,t). \tag{4}$$

The drift term $b$ can then be determined by using

$$\partial_t\left(\frac{\rho_A(x,t)}{D_A} + \frac{\rho_B(x,t)}{D_B}\right) = \partial_x^2(\rho_A(x,t)+\rho_B(x,t)). \tag{5}$$

Here we used diffusive rescaling of the time-coordinate. One finds

$$b = \frac{1}{\sigma}\partial_x(D_s\sigma - \rho). \tag{6}$$

In an infinite system the self-diffusion coefficient vanishes, as is indicated by the subdiffusive nature of single-file diffusion which was proved rigorously for tracer diffusion in the SEP in [1]. However, as argued in [3] we expect that in a finite system with open boundaries correction terms of leading order $1/L$ appear. This is confirmed by the exact result

$$D_s = \frac{1}{L}\frac{1-\rho}{\sigma} \tag{7}$$

proved in [4] for a finite period system with $L$ sites.

The relaxation of the initial density profile is analyzed by numerical integration and good agreement is found with Monte-Carlo simulation of the process [4]. In the steady state we find a discontinuous boundary-induced phase transition as

the total exterior density gradient between the system boundaries is varied. The asymptotic space-averaged mean density

$$\bar{\rho}_A = \lim_{L\to\infty} \frac{1}{L} \int_0^L dx \rho_A(x) \tag{8}$$

takes the value

$$\bar{\rho}_A = \frac{\rho_A^{\pm}}{\rho^{\pm}} \frac{\rho^+ + \rho^-}{2} \tag{9}$$

where the positive (negative) sign occurs for positive (negative) exterior gradient $\rho^+ - \rho^-$ of the total density $\rho^{\pm} = \rho_A^{\pm} + \rho_B^{\pm}$.

For the current we find for positive exterior grandient to leading order in $1/L$

$$j_A = \begin{cases} \frac{\rho_A^+}{\frac{\rho_A^+}{D_A} + \frac{\rho_B^+}{D_B}} \frac{\rho^- - \rho^+}{L} & \text{for } \rho^- < \rho^+ \\ \frac{\rho_A^-}{\frac{\rho_A^-}{D_A} + \frac{\rho_B^-}{D_B}} \frac{\rho^- - \rho^+}{L} & \text{for } \rho^- > \rho^+ \end{cases} \tag{10}$$

Hence, as one expects, the $A$-current is always opposite the total reservoir gradient $\Delta\rho \equiv \rho^+ - \rho^-$, but it changes in a non-analytic fashion at $\rho^- = \rho^+$ where it vanishes to leading order in $1/L$. Interestingly, however, the current of $B$-particles may have the same sign as the exterior reservoir gradient of $B$-particles. This counterintuitive phenomenon is a pumping effect [5], for which a quantitative physical understanding is still lacking.

The density profiles have an interesting structure. At one boundary a boundary layer develops inside which the current flows against the local density gradient. Generically the width of the boundary layer and the bulk density profiles do not depend on the two hopping rates. At the phase transition line, however, the individual density profiles depend strongly on the ratio $D_A/D_B$. Dynamic Monte Carlo simulation confirm our theoretical predictions.

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## Anomalous energy transport in the FPU-$\beta$ chain

Jani Lukkarinen
(joint work with Herbert Spohn)

In this talk we consider the energy current correlation function for the FPU-$\beta$ lattice. For small non-linearity one can rely on kinetic theory, and the issue reduces to a spectral analysis of the linearized collision operator. As we prove in [1], on the basis of kinetic theory, the energy current correlations decay in time as $t^{-3/5}$. This indicates that the thermal conductivity is anomalous, increasing as $N^{2/5}$ with the system size $N$.

Over the last decade there has been a lot of interest in understanding the energy transport through one-dimensional chains, amongst them the Fermi, Pasta, and Ulam (FPU) $\beta$-chain [2, 3]. Numerically, one common setup is to consider a chain of length $N$, and to couple the two particles on the boundary to thermal reservoirs at different temperatures, $T_-$ and $T_+$. For long times the chain relaxes to a steady state with a non-zero average energy current $j_{\mathrm{e}}(N) = (T_- - T_+)N^{-1}\kappa(N)$, and the interest lies in the dependence of $\kappa(N)$ on $N$ for large $N$. For a regular transport, i.e., for transport satisfying Fourier's law, one has $\kappa(N) \to$ const. for large $N$. Anomalous transport corresponds to $\kappa(N) \simeq N^{\alpha}$, with $0 < \alpha < 1$. In the $\beta$-chain molecular dynamics simulations point to an $\alpha$ of approximately 0.4 [4, 5]. However, even on the numerical level the accurate value of $\alpha$ is still being debated: for instance, in [6], a crossover at large $N$ to $\kappa(N) \simeq N^{\frac{1}{3}}$ is claimed.

Instead of studying a finite system with thermal boundary conditions, we adopt a different, but physically equivalent, procedure to study the thermal conduction in the FPU-$\beta$ chain. We consider the energy spread in an infinite chain with Hamiltonian dynamics, but with random initial data distributed according to the thermal equilibrium Gibbs measure with a given temperature $T > 0$.

The Hamiltonian of the FPU-$\beta$ chain is $H(q,p) = \sum_i \left[\frac{1}{2}p_i^2 + U_\beta(q_{i+1} - q_i)\right]$, with the canonical conjugate pair $q_i, p_i \in \mathbb{R}$, $i \in \mathbb{Z}$. The coupling is pure nearest neighbor, with a nonlinear potential $U_\beta(r) = \frac{1}{8}r^2 + \frac{1}{4}\beta r^4$, where $\beta > 0$. The equations of motion are thus explicitly $\frac{\mathrm{d}}{\mathrm{d}t}q_i = p_i$, $\frac{\mathrm{d}}{\mathrm{d}t}p_i = U_\beta'(q_{i+1} - q_i) - U_\beta'(q_i - q_{i-1})$. The initial conditions (for $t = 0$) of the Hamiltonian dynamics are distributed according to the Gibbs measure $Z^{-1}\mathrm{e}^{-H/T}\prod_{i\in\mathbb{Z}}[\mathrm{d}q_i\mathrm{d}p_i]$ (using a slightly formal notation). From now on, let $(q(t), p(t))$ denote the corresponding (random) solution to the above Hamiltonian dynamics.

We next introduce the local energy at the site $i \in \mathbb{Z}$ by the formula $e_i(q,p) = \frac{1}{2}\left[p_i^2 + U_\beta(q_{i+1} - q_i) + U_\beta(q_i - q_{i-1})\right]$. Employing the shorthand notation $e_i(t) = e_i(q(t), p(t))$, the normalized local average excess energy can then be defined as $\frac{1}{\chi}\left(\langle e_i(t)e_0(0)\rangle - \langle e_i\rangle\langle e_0\rangle\right)$. Here $\langle\cdot\rangle$ denotes the thermal average over the initial conditions, and $\chi$ is a convenient normalization constant. The energy spread at time $t$ is then defined as the spatial variance of the excess energy, $D(t) = \frac{1}{\chi}\sum_{i\in\mathbb{Z}} i^2\left(\langle e_i(t)e_0(0)\rangle - \langle e_i\rangle\langle e_0\rangle\right)$.

Fourier's law corresponds to a diffusive spreading, $D(t) = \mathcal{O}(t)$ for large $t$, while an exponent $\alpha > 0$ corresponds to a superdiffusive spreading with $D(t) =$

$\mathcal{O}(t^{1+\alpha})$. A more convenient form for $D(t)$ is found by introducing the energy current observable $j_{i,i+1}$, for each directed bond from $i$ to $i+1$. This satisfies $\frac{\mathrm{d}}{\mathrm{d}t}e_i + j_{i,i+1} - j_{i-1,i} = 0$, and for the FPU-$\beta$ model, such a current observable is given by $j_{i,i+1}(q,p) = -\frac{1}{2}(p_{i+1}+p_i)U'_\beta(q_{i+1}-q_i)$. In terms of the energy current-current correlation function

$$C_\beta(t) = \sum_{i\in\mathbb{Z}} \langle j_{0,1}(t) j_{i,i+1}(0)\rangle, \tag{1}$$

we then have $D(t) = D(0) + \frac{1}{\chi}\int_0^t \mathrm{d}s \int_0^t \mathrm{d}s' C_\beta(s-s')$. Clearly, if $\int_0^\infty \mathrm{d}t|C_\beta(t)| < \infty$, then $D(t) = \mathcal{O}(t)$. On the other hand, if $C_\beta(t) = \mathcal{O}(t^{\alpha-1})$ for large $t$ with $0<\alpha<1$, then $D(t) = \mathcal{O}(t^{1+\alpha})$, and the spreading is superdiffusive.

In the limit of small $\beta$, through methods from kinetic theory [7, 8, 9], $C_\beta(t)$ may be expressed in a more accessible form. The unperturbed ($\beta = 0$) dynamics are harmonic, with a dispersion relation $\omega(k) = (\frac{1}{2}(1-\cos k))^{1/2} = \left|\sin\frac{k}{2}\right|$. Let us choose as the basic periodic cell the interval $I = [0, 2\pi)$. Then for $x \in I$, $x \neq 0$, the derivative of $\omega$ reads $\omega'(x) = \frac{1}{2}\cos\frac{x}{2}$. For $x, y, z \in \mathbb{R}$, let also

$$\Omega(x,y,z) = \omega(x) + \omega(y) - \omega(z) - \omega(x+y-z). \tag{2}$$

Then the linearized collision operator of the FPU-$\beta$ lattice in the kinetic limit can be given in terms of

$$(Lf)(x) = \int_I \mathrm{d}y \int_I \mathrm{d}z\, \delta(\Omega(x,y,z))\,(f(x)+f(y)-f(z)-f(x+y-z)), \tag{3}$$

with $f$ periodically extended from $I$ to $\mathbb{R}$. We define further $\tilde{L} = \omega L \omega$, and consider this as a linear operator on $L^2(I)$. With the above definitions, $C_\beta(t)$ for small coupling $\beta$ is expected to satisfy the following limit.

**Kinetic conjecture:** *For any $t \in \mathbb{R}$, $T > 0$,*

$$\lim_{\beta\to 0^+} C_\beta(\beta^{-2}t) = \frac{T^2}{2\pi}\langle \omega', \exp\bigl[-\pi^{-1}(12T)^2|t|\tilde{L}\bigr]\omega'\rangle. \tag{4}$$

Here $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L^2(I)$. Thus for small $\beta$, to study the decay of $C_\beta(t)$, it is sufficient to study the decay of the scalar product

$$C(t) = \langle \omega', \mathrm{e}^{-|t|\tilde{L}}\omega'\rangle. \tag{5}$$

With these definitions, $\tilde{L}$ is a bounded positive operator with a decomposition $\tilde{L} = W - A$, where $A$ is compact and $W$ is a multiplication operator. In kinetic theory, it is a common practice to use the relaxation time approximation, which in our case amounts to approximating $\langle \omega', \mathrm{e}^{-|t|\tilde{L}}\omega'\rangle \approx \langle \omega', \mathrm{e}^{-|t|W}\omega'\rangle$. As we prove in [1], $W(x) = W(2\pi - x)$, and for $0 < x \ll 1$, $W(x)$ behaves asymptotically as $x^{5/3}$. Thus the relaxation time approximation predicts $\langle \omega', \mathrm{e}^{-|t|\tilde{L}}\omega'\rangle = \mathcal{O}(t^{-3/5})$ for large $t$, as also noted in [10].

The essential spectrum of $\tilde{L}$ coincides with the range of $W$, and thus starts from 0. Therefore, it is not obvious that the relaxation time approximation captures the true asymptotics. To understand the time decay leads to two distinct mathematical issues: (1) The so called collisional invariants, which in essence are zero modes of

$L$, could in principle prevent $C(t)$ from decaying to 0. (2) We use the resolvent expansion to estimate $\langle \omega', \mathrm{e}^{-|t|\tilde{L}}\omega'\rangle$. An inherent difficulty in such expansions is the estimation of the remainder term. Our method relies on the fact that the resolvent expansion is made up to an even order, as well as on the compactness of the operator $B = W^{-1/2}AW^{-1/2}$.

The main results proven in [1] are summarized below.

**Theorem 1.** *Suppose $\psi$ is periodic and locally integrable: $\psi|_I \in L^1(I)$. Then $\psi$ is a collisional invariant if and only if there are $c_1, c_2 \in \mathbb{C}$ such that $\psi(x) = c_1 + c_2\omega(x)$ for a.e. $x$.*

**Theorem 2.** *Let $R : (0,\infty) \to \mathbb{R}_+$ be defined by $R(\lambda) = \langle \omega', \frac{1}{\lambda+\tilde{L}}\omega'\rangle$. Then there is $0 < c_0 < \infty$ such that $\lim_{\lambda\to 0^+} \lambda^\alpha R(\lambda) = c_0$ for $\alpha = \frac{2}{5}$.*

**Corollary.** *With $\Gamma$ denoting the gamma function, we have also*

$$\lim_{t\to\infty} t^{1-\alpha} C(t) = \frac{c_0}{\Gamma(\alpha)}. \tag{6}$$

These results imply that, on the kinetic time scale, the energy spread is superdiffusive, with $D(t) \simeq c\, t^{7/5}$, $c > 0$, for large $t$. This corresponds to a heat conduction exponent $\alpha = \frac{2}{5}$ and is in agreement with the molecular dynamics simulations of [4, 5]. As the example of long time tails in classical fluids teaches us, kinetic theory might miss the true asymptotic decay of equilibrium correlation functions. Whether this is the case also for the FPU-$\beta$ chain, remains to be discovered.

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## A translation-invariant dynamics for chains of oscillators out of equilibrium.

Raphaël Lefevere

Consider chains of oscillators described by a Hamiltonian,

$$H(\underline{p},\underline{q}) = \sum_{i=1}^{N} \frac{1}{2}p_i^2 + V(q_i) + U(q_i - q_{i-1}).$$

Out of thermal equilibrium and in a stationary state, this type of system is characterized by a non-zero average current of energy. The microscopic current, i.e the rate of exchange of energy between site $i$ and $i+1$, is defined through the gain and loss of the local energies,

$$h_i = \frac{1}{2}p_i^2 + V(q_i) + \frac{1}{2}(U(q_i - q_{i-1}) + U(q_{i+1} - q_i))$$

$$\frac{d}{dt}h_i = j_i - j_{i-1}$$

$$j_i = \frac{1}{2}(p_i + p_{i+1})F(q_i - q_{i+1}).$$

where $F$ is the force derived from the interaction potential $U$, $F = -U'$.
The lack of translation-invariance of systems out of equilibrium is in general essential to their definition. Think of an extended system thermostatted at different temperatures at its boundaries. However, locally and in the bulk of the system, the effect of the temperature difference at the boundaries is roughly translation-invariant and its sole consequence is to break the time-reversal invariance of the dynamics in a very specific way, as I shall argue below. Following this idea, and in order to study the essential properties of the steady state out of equilibrium in a simpler way, I suggest to use the following translation-invariant dynamics [1],

$$dq_i = \frac{\partial H}{\partial p_i}dt$$

$$dp_i = -\gamma p_i dt - \frac{\partial H}{\partial q_i}dt + \frac{\tau}{2T}(F(q_{i-1} - q_i) + F(q_i - q_{i+1}))dt + \sqrt{2\gamma T}dw_i$$

where $w_i$ are standard independent Brownian motion, and $F(x) = -U'(x)$. I put periodic boundary conditions on the system. The basic idea behind the construction of this dynamics is that it is characterized by the same time-reversal breaking functional as the same Hamiltonian system heated at its boundaries but considered locally. Physically, it has two effects, the Langevin coupling fixes the average kinetic energy of the oscillators (the temperature) and the extra force whose intensity is regulated by the parameter $\tau$ creates some current of energy by breaking the equality of action and reaction. This term is responsible for the breaking of time-reversal symmetry of the system at $\tau = 0$. In order to see this, compare the

distribution of the above process $\xi = (\underline{p}(s), \underline{q}(s))_{-t\leq s\leq t}$ and of its time-reverse. This gives,

$$\frac{dP^t_\mu}{dP^t_\mu \Pi}(\xi) = \exp(R^t(\xi))$$

modulo temporal boundary terms,

$$R^t(\xi) \equiv A^t(\xi) - A^t(\Pi\xi) = \int_{-t}^{t} \sigma(s)ds + \frac{1}{T}(H(\xi_{-t}) - \beta H(\xi_t)),$$

and,

$$\sigma \equiv \frac{\tau}{T^2}\sum_{i=1}^{N} j_i = \frac{\tau}{T^2}\sum_{i=1}^{N}(p_i + p_{i+1})F(q_i - q_{i+1}).$$

Consider now the same Hamiltonian system but heated at its boundaries,

$$dq_i = \frac{\partial H}{\partial p_i}dt$$

$$dp_i = -\frac{\partial H}{\partial q_i}dt, \quad i \neq 1, N$$

$$dp_i = -\gamma p_i dt - \frac{\partial H}{\partial q_i}dt + \sqrt{2\gamma T_i}dw_i \quad i = 1, N$$

and $w_i$ are standard BM, $T_1 = T_L$,$T_N = T_R$. Rescaling the lattice on the interval $[0, 1]$ that we subdivide in boxes $B^N_\epsilon(x)$ of size $\epsilon$ and comparing again the distribution of this process and of its time-reverse, one gets, in the limit of large $N$ and small $\epsilon$, $\epsilon >> N^{-1}$, at lowest order in $N^{-1}$ and $\epsilon$,

$$\frac{dP^t_\mu}{dP^t_\mu \Pi}(\xi) = \exp(\sum_k R^t_{x_k}(\xi))$$

$x_k$ is the center of the $k$-th box and with,

$$R^t_x(\xi) = \frac{\nabla T(x)}{T^2(x)}\frac{1}{N}\sum_{i\in B^N_\epsilon(x)}\int_{-t}^{t} j_i(s)ds - \frac{1}{T(x)}\sum_{i\in B^N_\epsilon(x)}(h_i(\xi_t) - h_i(\xi_{-t}))$$

With $T(.)$ a smooth function on the interval $[0, 1]$ such that $T(0) = T_L$ and $T(1) = T_R$. So for each fixed $x$, $R_x = R$ when $T = T(x)$ and $\tau = \frac{1}{N}\nabla T(x)$, for the same functions $j_i$ and $H = \sum_i h_i$. This shows the equivalence claimed above. This type of dynamics has been used in the analysis of anharmonic chains out of equilibrium [2]. We finally note two facts concerning the case of the harmonic chain $U(x) = \omega^2\frac{x^2}{2}$, in that case, the stationary measure is,

$$\rho(\underline{q}, \underline{p}) = Z^{-1}\exp\left(-\beta H(\underline{q}, \underline{p}) + \frac{N\tau}{\gamma T^2}J(\underline{q}, \underline{p})\right)$$

with $J = \frac{1}{N}\sum_{i=1}^{N} j_i$. Except for the corrections due to the lack of translation-invariance, the case of the harmonic chain coupled to self-consistent heat baths is

extremely similar and one may expect that in that case the invariant measure is given by,

$$\exp\left(\sum_k \frac{\nabla T(x_k)}{T^2(x_k)}\frac{\kappa}{N}\sum_{i\in B_\epsilon^N(x)} j_i - \frac{1}{T(x_k)}\sum_{i\in B_\epsilon^N(x)} h_i(\underline{p},\underline{q})\right)$$

where $\kappa$ is the conductivity of the system and with the same notation as above and at lowest order in $N^{-1}$ and $\epsilon$. $T(x)$ is the temperature of the self-consistent temperature profile.

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## Tagged particle diffusion indeterministic dynamics – old and new results

Bálint Tóth

(joint work with Péter Bálint, Imre Péter Tóth)

The one dimensional Rayleigh gas consists of an infinite system of identical particles of mass $m = 1$ in which a tagged particle of mass $M$ is immersed. The dynamics of this infinitely extended system is governed by the laws of classical Newtonian dynamics: particles move uniformly between any two collisions and collide elastically, observing conservation of momentum and kinetic energy. The equilibrium Gibbs state of the system as seen from the tagged particle is the following: the positions of the gas particles form a Poisson point process of density 1 and the velocities of the particles are independent Gaussian of mean zero and variance equal to the reciprocal of their respective mass. (Density of the system, temperature and mass of the gas particles don't play a key role, the only truely relevant parameter of the problem is the mass ratio $M/m$ of the tagged particle and gas particles.) A major question is the understanding the trajectory $t \mapsto Q_t$ of the tagged particle under diffusive scaling: $t \mapsto A^{-1/2}Q_{At}$, when $A \to \infty$. In spite of the simplicity of the formulation, the problem is extremely difficult due to the long range memory of the process due to recollisions. The importance of the problem stems form the urge of understanding the dynamical nature of true physical Brownian motion, deriving rigorously from first principles of physics various mathematical models, such as Wiener process or integrated Ornstein-Uhlenbeck process.

First I surveyed the history of the problem and I presented a short summary of the existing rigorous mathematical results. This part of the talk was based on the papers [6], [10], [7], [4], [9], [11], [12].

In particular, in [9] and [11] $M$-independent lower and upper bounds were proved for the limiting variance of the tagged particle:

$$\underline{\sigma}^2 \le \liminf_{t\to\infty} \mathbf{Var}\big(t^{-1/2}Q_t\big) \le \limsup_{t\to\infty} \mathbf{Var}\big(t^{-1/2}Q_t\big) \le \overline{\sigma}^2,$$

where

$$\underline{\sigma}^2 := \sqrt{\pi/8} \approx 0.627\ldots, \qquad \overline{\sigma}^2 := \sqrt{2/\pi} \approx 0.798\ldots.$$

The question of mass-dependence of the limiting variance is still left totally open.

In [3] surprising *numerical* results were published. Reliable numerical simulations suggest that, contrary to the naïve folklore

$$\lim_{M\to 0}\lim_{t\to\infty} \mathbf{Var}\big(t^{-1/2}Q_t\big) \approx 0.740\cdots \neq \overline{\sigma}^2.$$

In [1] we elucidate theoretically this numerical fact. We show that in the $M \to 0$ limit the dynamics of the Rayleigh gas which was described above, converges in a properly defined sense to the dynamics of a system of particles where the central tagged particle of the Rayleigh gas is exchanged to two particles of mass $m = 1$ which interact between themselves via an inverse quadratic (Calogero-Moser type) potential $U(z)$, or, equivalently, inverse cubic repelling force $F(z)$:

$$U(z) = \frac{c^2}{2z^2}, \qquad F(z) = \frac{c^2}{z^3},$$

and collide elastically with the other gas particles. This is a completely deterministic statement. If we start the Rayleigh gas form random initial condition sampled according to the Gibbs equilibrium then in the $M \to 0$ limit we get a mixture of these dynamics with the strength parameter $c$ of the inverse square potential distributed according to $\varrho(c)dc$, where

$$\varrho(c) := \sqrt{\frac{2}{\pi}} \int_0^\infty \exp\{-z - \frac{c^2}{2z^2}\}dz.$$

This result explains rigorously the cause of the recurrent numerical instabilities in earlier numerical simulations of the Rayleigh gas for $M \ll 1$. (See e.g. [8], [2], [5].) Numerical simulations for this second model reproduce accurately the results reported in [3].

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## Nonequilibrium fluctuations for a tagged particle in one dimensional mean-zero zero-range processes

Sunder Sethuraman
(joint work with Milton Jara, Claudio Landim)

The 'zero-range' process follows the motion of a collection of dependent random walks on a lattice. The interaction is specified in terms of a function $g : \{0, 1, 2, \ldots\} \to \mathbb{R}_+$ and a transition probability $p(\cdot)$ where from a vertex with $k$ particles, a given particle displaces by $y$ with rate $(g(k)/k)p(y)$.

In this talk, we consider the process on the torus $\mathbb{T}_N = \mathbb{Z} \setminus N\mathbb{Z}$, and discuss the fluctuations of a tagged, or distinguished particle, say initially at the origin, when the system starts in a "local equilibrium" or "nonequilibrium" profile measure. We will assume $g(0) = 0$, $g(k) > 0$ for $k \geq 1$, as is standard, and also growth conditions

$$\exists a_0, a_1, b > 0 \text{ such that } |g(k+1) - g(k)| \leq a_0, \text{ and } g(k+b) - g(k) \geq a_1.$$

Also, we take $p(\cdot)$ to be finite range, and importantly mean-zero, $\sum yp(y) = 0$. Denote $\sigma^2 = \sum y^2 p(y)$ as its variance.

Let now $\rho_0 : \mathbb{T} \to \mathbb{R}_+$ be a positive, continuous function on the unit torus $\mathbb{T}$, and let $\nu_N = \prod_{k \in \mathbb{T}_N} \mu_{\rho_0(k/N)}$ where $\mu_\rho$ is a measure on $\{0, 1, 2, \ldots\}$ given by

$$\mu_\rho(k) = \frac{1}{Z_\rho} \frac{\alpha^k}{g(1) \cdots g(k)} \text{ for } k \geq 1, \text{ and } \mu_\rho(0) = \frac{1}{Z_\rho},$$

and $\alpha = \alpha(\rho)$ is chosen so that the mean particle number $\sum k\mu_\rho(k) = \rho$. The main result is that if $x_t^{(N)}$ is the position of the tagged particle at time $t$, then starting the system under $\nu_N(\cdot | \#\text{particles at origin} \geq 1)$, we have

$$\frac{1}{N} x_{N^2 t}^{(N)} \Rightarrow x_t,$$

with respect to the uniform topology on $D[0,1]$, where $x_t$ is the diffusion given by

$$dx_t \;=\; \sigma\sqrt{\frac{\alpha(\rho(t,x_t))}{\rho(t,x_t)}}d\mathbb{B}_t,$$

$\mathbb{B}_t$ is standard Brownian motion on $\mathbb{T}$, and $\rho(t,x)$ satisfies the hydrodynamic equation

$$\begin{cases} \frac{\partial\rho}{\partial t} \;=\; \sigma^2\frac{\partial^2}{\partial x^2}\alpha(\rho(t,x)) \\ \rho(0,x) \;=\; \rho_0(x). \end{cases}$$

This result is the first nonequilibrium fluctuation description for a tagged particle in a model with a general finite-range $p(\cdot)$. In [1], nonequilibrium fluctuations were proved in a nearest-neighbor simple exclusion model with an argument relying on the nearest-neighbor assumption. We also mention 'propagation of chaos' type results for finite-range simple exclusion in [3] which give the fluctuations for the 'averaged' tagged particle motion. More details, and related references can be found in [2].

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## Thermodynamic Limit of the Equilibrium Measures for Supercritical Zero Range Processes


Michail Loulakis

(joint work with Inés Armendáriz)


Consider a zero range process with $N$ particles on a lattice $\Lambda$ of $L$ sites. As an irreducible process on a finite state space it possesses a unique invariant measure which we denote by $\mu^{N,L}$. It is known [3] that for certain choices of the jump rate function $g(\cdot)$ the thermodynamic limit of $\mu^{N,L}$ (the limit as $N,L\to\infty$ such that $N/L\to\rho$) exhibits a phase transition in the following sense. There exists a critical density $\rho_c<\infty$ such that if $\rho\le\rho_c$ then the finite dimensional projections of $\mu^{N,L}$ converge to a product measure with expected number of particles per site $\rho$. If however $\rho>\rho_c$ then the expected number of particles per site under the limit product measure is $\rho_c$ and all the excess mass condensates in a single site[4, 3]. Precisely, if $\eta_x$ stands for the number of particles on the site $x\in\Lambda$ in the configuration $\eta\in\{0,1,\ldots\}^\Lambda$, then for any $\varepsilon>0$ we have

$$\lim_{\substack{N,L\to\infty\\ N/L\to\rho}}\mu^{N,L}\big[\frac{1}{L}\max_{x\in\Lambda}\eta_x>\rho-\rho_c-\varepsilon\big]=1.$$

To fix ideas we consider a standard model proposed by Evans for which the jump rate is given by

$$g(k) = \begin{cases} 1 + \frac{b}{k} & \text{if } k \geq 1 \\ 0 & \text{if } k = 0. \end{cases}$$

When $b > 2$ this model exhibits a phase transition at the critical density $\rho_c = \frac{1}{b-2}$. Let

$$M_L(\eta) = \max_{1 \leq x \leq L} \eta_x$$

and $m_L = \text{argmax}(\eta)$ be the position where the maximum occurs, be it the leftmost when it occurs on more than one sites. We define the operator $T : \Omega_L \longrightarrow \Omega_L$ so that:

$$(T\eta)_x = \begin{cases} \eta_x & \text{if } x \neq 1, m_L, \\ \eta_{m_L} & \text{if } x = 1, \\ \eta_1 & \text{if } x = m_L. \end{cases}$$

In short, the operator $T$ exchanges the first and the largest component in $\eta$. We also denote by $\nu_c$ the site marginal of the grand-canonical ensemble at critical density

$$\nu_c\big[\eta_x = k\big] = \frac{b-1}{b} \prod_{j=1}^{k} \frac{1}{g(j)}.$$

The main result of this work is the following theorem.

**THEOREM 1:** Let $\Sigma_2^L$ be the $\sigma$-field generated by $\eta_2, \ldots \eta_L$. If $\rho > \rho_c$, then

$$\lim_{\substack{N,L \to \infty \\ N/L \to \rho}} \sup_{A \in \Sigma_2^L} |\mu^{N,L} \circ T^{-1}(A) - \nu_c^L(A)| = 0. \tag{1}$$

This extends a result of Ferrari, Landim and Sisko[1] to the case where the number of sites increases to infinity together with the number of particles, and that of Großkinsky, Schütz and Spohn[3] in the sense that convergence to the critical grand canonical ensemble is extended beyond the finite dimensional marginals.

Several interesting facts about the invariant measures of the zero range process at supercritical densities are simple consequences of this Theorem. We would like to compute for instance the fluctuations of the maximum cluster around $(\rho - \rho_c)L$. A numerical investigation [2] suggests that for $b > 3$ the fluctuations of $M_L$ around $(\rho - \rho_c)L$ are Gaussian, while for $2 < b < 3$ the fluctuations scale as $L^{\frac{1}{b-1}}$. In view of Theorem 1 and the obvious equality:

$$M_L(\eta) = N - \sum_{x=2}^{L} (T\eta)_x, \qquad \mu^{N,L} - a.s,$$

the fluctuations of the maximum component reduce to the fluctuations of the sum of $L-1$ independent random variables with mean $\rho_c$ around $\rho_c(L-1)$, for which standard central limit theorems are available. The precise result is the following:

**Corollary 1:** Suppose $\rho > \rho_c$.
a) If $b > 3$, that is if $\nu_c$ has finite variance $\sigma^2 = \frac{(b-1)^2}{(b-2)^2(b-3)}$, then for all real $x$ we have

$$\lim_{\substack{N,L\to\infty \\ N/L\to\rho}} \mu^{N,L}\left(\frac{M_L(\eta) - (N-\rho_c L)}{\sigma L^{1/2}} \le x\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-u^2/2}du.$$

b) If $b = 3$, then for all real $x$ we have

$$\lim_{\substack{N,L\to\infty \\ N/L\to\rho}} \mu^{N,L}\left(\frac{M_L(\eta) - (N-\rho_c L)}{\sqrt{2L\log L}} \le x\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-u^2/2}du.$$

c) If $2 < b < 3$, then for all real $x$ we have

$$\lim_{\substack{N,L\to\infty \\ N/L\to\rho}} \mu^{N,L}\left(\frac{M_L(\eta) - (N-\rho_c L)}{[\Gamma(b)L]^{\frac{1}{b-1}}} \le x\right) = \int_{-\infty}^{x} L_{b-1}(u)du.$$

where $L_\alpha$ is the density of the completely asymmetric stable law with index $\alpha$ and characteristic function $\psi(t)$ given by:

$$\log\psi(t) = \int_{-\infty}^{0}\left(e^{itx} - 1 - itx\right)\frac{\alpha dx}{|x|^{\alpha+1}} = -C_\alpha|t|^\alpha(1 + i\ sgn(t)\tan\frac{\pi\alpha}{2})$$

It is worth noting that for $b = 3$ we still have Gaussian fluctuations after proper scaling.

The fluctuations of the bulk are intimately related to those of the maximum. Let $0 < \beta < \rho - \rho_c$ and define the bulk configuration $\eta^*$ by

$$\eta_x^* = \eta_x \chi_{\{\eta_x \le \beta L\}}.$$

An invariance principle for the bulk configuration follows from the Theorem 1.

**Corollary 2:** Let $b > 3$ so that $\nu_c$ has finite variance $\sigma^2$. If $\rho > \rho_c$ then under $\mu^{N,L}$ we have

$$\frac{1}{\sigma\sqrt{L}}\sum_{x=1}^{Lt}(\eta_x^* - \rho_c) \Longrightarrow W(t),$$

where $W(\cdot)$ is the standard Wiener process. Contrast this with the following result for the subcritical ($\rho < \rho_c$) fluctuations under $\mu^{N,L}$:

$$\frac{1}{\sigma\sqrt{L}}\sum_{x=1}^{Lt}(\eta_x - \rho_c) \Longrightarrow BB(t),$$

where $BB$ is a Brownian Bridge returning to 0 at time 1.

It is also clear that using Theorem 1 we can compute scaling limits for any order statistics. For instance, the second largest component is given by

$$M_L^{(2)}(\eta) = \max_{2\le x\le L}(T\eta)_x,$$

and the following limit theorem is immediate.

**Corollary 3:** Suppose $\rho > \rho_c$. Then, for any $x > 0$:

$$\lim_{\substack{N,L\to\infty \\ N/L\to\rho}} \mu^{N,L}\left(M_L^{(2)}(\eta) \le x\big[\Gamma(b)L\big]^{\frac{1}{b-1}}\right) \;=\; e^{-x^{1-b}}.$$

Finally, it is worth noting that the results extend to a variety of zero range models with finite critical density, in particular to models with streched exponential tails where the jump rate $g(\cdot)$ is given for $k > 0$ by

$$g(k) = A(1 + \frac{b}{k^\gamma}),$$

where $\frac{1}{2} < \gamma < 1$.

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## Condensation in a zero-range process with size-dependent jump rates


Stefan Grosskinsky

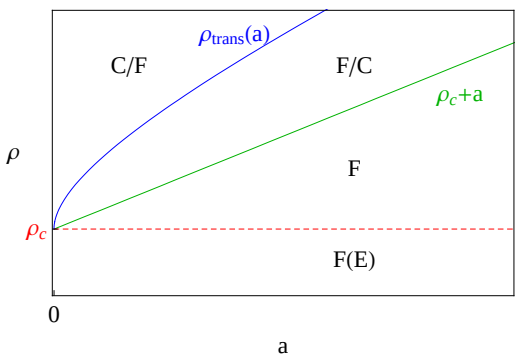
(joint work with Gunter M. Schütz)


We study a zero-range process (cf. [1]) on a periodic lattice $\Lambda_L$ of size $|\Lambda_L| = L$ with space-homogeneous, irreducible dynamics with jump rates

$$g_R(k) = \begin{cases} c_0 & , \; k \le R \\ c_1 & , \; k > R \end{cases} \qquad \text{for } k \ge 1\,, \quad g(0) = 0\,. \tag{1}$$

Here $c_0 > c_1 > 0$, and the parameter $R \ge 0$ depends on the system size, such that $R \to \infty$ and $R/L \to a \ge 0$ as $L \to \infty$. The total density of particles $\rho$ is conserved, and the stationary phase diagram in terms of the system parameters $a$ and $\rho$ consists of four phases.

In the fluid phases $F(E)$, $F$ and $F/C$ the stationary measure concentrates on homogeneous configurations with bulk density $\rho$. In phase $F(E)$ the canonical and grand-canonical ensembles are equivalent, and in phase $F/C$ there exists an additional metastable condensed phase, which has a lifetime exponential in the system size. Typical condensed configurations have a



homogeneous background distribution with density $\rho_c < \rho$, where the excess particles condense on a single lattice site. In phase $C/F$, i.e. for $\rho > \rho_{trans}$, the condensed phase becomes stable and the corresponding fluid phase metastable. In addition to metastability, the bulk density does not depend continuously on the total particle density $\rho$, so the transition is discontinuous.
We study this phenomenon rigorously on the level of the equivalence of ensembles analogous to [2]. For fixed system size $L$, it is well known [1, 3] that there exist *grand-canonical measures* of product form with single-site mass function

$$\nu^1_{\phi,R}(k) = \frac{1}{z_R(\phi)}\, w_R(k)\, \phi^k \quad \text{where} \quad w_R(k) = \prod_{i=0}^{k} g_R^{-1}(i)\ , \tag{2}$$

so in our case $w_R(k) = c_0^{-(k\wedge R)} c_1^{(R-k)\wedge 0}$. These measures are defined for fugacities $\phi < c_1$ with normalization $z_R(\phi) = \sum_{k\geq 0} w_R(k)\, \phi^k < \infty$. In the limit $L \to \infty$ they can be characterized by the entropy $s_{gcan}(\rho) := \sup_{\phi\geq 0} \big(\rho \log \phi - p(\phi)\big)$, i.e. the Legendre transform of the pressure

$$p(\phi) := \lim_{L\to\infty} \frac{1}{L} \log z_R^L(\phi) = \begin{cases} \log \frac{c_0}{c_0-\phi} & ,\ \phi < c_1 \\ \infty & ,\ \phi \geq c_1 \end{cases}\ . \tag{3}$$

Denoting by $s_{fluid}(\rho) = \rho(\log c_0 + \log \rho) - (1+\rho)\log(1+\rho)$ the entropy of the limiting measure with mass function $\nu^1_{\phi,\infty}(k) \sim (\phi/c_0)^k$, we get

$$s_{gcan}(\rho) = \begin{cases} s_{fluid}(\rho) & ,\ \rho \leq \rho_c \\ s_{fluid}(\rho_c) + (\rho - \rho_c)\log c_1 & ,\ \rho > \rho_c \end{cases}\ . \tag{4}$$

The *canonical measures* are given by $\pi_{L,N} := \nu^L_{\phi,R}\big(\,.\,\big|\, \sum_{x\in\Lambda_L} \eta_x = N\big)$, i.e. they are conditioned on a fixed number $N$ of particles. Their mass functions are independent of $\phi$ and given by

$$\pi_{L,N}(\boldsymbol{\eta}) = \frac{1}{Z_{L,N}}\, w_R^L(\boldsymbol{\eta})\, \delta\Big(\sum_{x\in\Lambda_L} \eta_x, N\Big)\ , \tag{5}$$

where the normalization is $Z_{L,N} = \sum_{\boldsymbol{\eta}} w_R^L(\boldsymbol{\eta})\, \delta\big(\sum_{x\in\Lambda_L} \eta_x, N\big)$. Our main result concerns the limiting behaviour of this quantity.

**Theorem 1.** *The limit*

$$s_{can}(\rho) := -\lim_{L\to\infty} \frac{1}{L} \log Z_{L,N}\ , \quad \textit{where } N/L \to \rho\ , \tag{6}$$

*exists and is called the canonical entropy. It is given by*

$$s_{can}(\rho) = \begin{cases} s_{fluid}(\rho) & ,\ \rho < \rho_{trans} \\ s_{fluid}(\rho_c) - \lim\limits_{L\to\infty} \frac{1}{L} \log w_R\big((\rho-\rho_c)L\big) & ,\ \rho > \rho_{trans} \end{cases}\ . \tag{7}$$

*The transition density $\rho_{trans}(a)$ is given by the unique solution of*

$$a = \Big(s_{fluid}(\rho) - (\rho-\rho_c)\log c_1 - s_{fluid}(\rho_c)\Big) \Big/ \log \frac{c_0}{c_1}\ , \tag{8}$$

*where $\rho_{trans}(a) \geq \rho_c + a$ with equality if and only if $a = 0$.*

Using the specific relative entropy $h(\pi_{L,N}, \nu^L_{\phi,R}) := \frac{1}{L} \left\langle \log \frac{\pi_{L,N}(\boldsymbol{\eta})}{\nu^L_{\phi,R}(\boldsymbol{\eta})} \right\rangle_{\pi_{L,N}}$, this result can be boosted by standard techniques [4] to show the equivalence of ensembles also on the level of weak convergence of measures. In particular

$$h(\pi_{L,N}, \nu^L_{\phi_R(\rho),R}) \to s_{can}(\rho) - s_{gcan}(\rho) \begin{cases} = 0 & , \ \rho \le \rho_c \\ > 0 & , \ \rho > \rho_c \end{cases} , \tag{9}$$

which implies that equivalence holds precisely in the phase $F(E)$. The phases $F$ and $F/C$ cannot be distinguished on this level. To study this point, we define

$$\Sigma^{bg}_L(\boldsymbol{\eta}) := \Sigma_L(\boldsymbol{\eta}) - \max_{x \in \Lambda_L} \eta_x \ , \tag{10}$$

which can be interpreted as the number of particles in the background, since typically at most one site contributes to the condensate.

**Theorem 2.** *Let $S_1, S_2, \ldots \in \mathbb{N}$ be any sequence with $S_L/L \to \rho_{bg} > 0$. Then*

$$I_\rho(\rho_{bg}) := - \lim_{L \to \infty} \frac{1}{L} \log \pi_{L,N}\big(\Sigma^{bg}_L = S_L\big) \in [0, \infty] \tag{11}$$

*exists for all $\rho > 0$ ($N/L \to \rho$), and defines the rate function for the events $\big\{\Sigma^{bg}_L = S_L\big\}$. For $\rho_{bg} > \rho$, $I_\rho(\rho_{bg}) = \infty$ and for $\rho_{bg} \le \rho$ it can be written as*

$$\begin{aligned} I_\rho(\rho_{bg}) &= s_{can}(\rho) - s_{fluid}(\rho_{bg}) + \\ &\quad + \begin{cases} (\rho - \rho_{bg}) \log c_0 & , \ \rho_{bg} \ge \rho - a \\ (\rho - \rho_{bg}) \log c_1 + a \log(c_0/c_1) & , \ \rho_{bg} \le \rho - a \end{cases} . \end{aligned} \tag{12}$$

For $\rho \le \rho_c$ this rate function has one minimum $I_\rho(\rho) = 0$, whereas for $\rho > \rho_c + a$ it has a second local minimum at $I_\rho(\rho_c)$, which becomes the global minimum for $\rho > \rho_{trans}$. Therefore, for $\rho > \rho_c + a$ we have two stable phases with average life-times of order $\exp(\xi L)$, which can be computed from (12) to be

$$\begin{aligned} \xi^{fluid}(\rho) &= s_{fluid}(\rho) - s_{fluid}(\rho - a) + a \log c_0 \ , \\ \xi^{cond}(\rho) &= s_{fluid}(\rho_c) - s_{fluid}(\rho - a) + (\rho_c + a - \rho) \log c_1 \end{aligned} \tag{13}$$

Our motivation to study this process comes from granular physics, where experiments on the clustering of particles can be described by a zero-range type model with size-dependent jump rates (see [5] and references therein). This exhibits a discontinuous condensation transition and metastability, both not present in previous results on zero-range processes without size-dependent jump rates [6, 7]. For such systems, these features have only been observed in a model with two particle species [8]. We restrict ourselves to the jump rates (1) as a simple prototype model, which allows a rigorous analysis. However, there is strong reason to believe that our results apply for a larger class of systems, which we have checked numerically for several examples. In contrast to condensation without size-dependent jump rates, our results cannot be generalized rigorously along the lines of [2], since the proof of Theorem 1 is based on an explicit computation of the canonical entropy. There are several other interesting differences to previous results regarding fluctuations and equilibration dynamics, which are explained in detail in [9].

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## Superdiffusivity through an example: the exclusion process with long jumps

Milton Jara

It is not uncommon to find physical systems that are not diffusive in nature, that is, systems for which the scaling exponent of different quantities is not 2 like in the case of standard diffusion. Many different approaches have been proposed to model this kind of behaviour. However, it is not so common to obtain rigorous scaling limits for stochastic systems that are local in nature, but have superdiffusive scaling. Examples of superdiffusive behavior can be found on tracer particles in one-dimensional shear flow, current fluctuations of the asymmetric simple exclusion, and heat conduction models in one dimension.

In order to understand what kind of limiting processes could appear, we fix a toy model: the exclusion process with long jumps. The exclusion process with long jumps is a system of random walks $\eta_t$ on the lattice $\mathbb{Z}^d$ with transition rate $p(z) = 1/|z|^{d+\alpha}$, conditioned to have at most one particle per site. It turns out that the right scaling exponent for this process is $\alpha$, in the sense that when space is scaled as $1/n$, time need to be scaled as $n^\alpha$ in order to obtain a non-trivial evolution. Define the fractional Laplacian $\Delta^{\alpha/2}$ by the integral formula

$$\Delta^{\alpha/2} f(x) = \int \frac{f(x+y) + f(x-y) - 2f(x)}{|y|^{d+\alpha}} dy.$$

Define the empirical density of particles $\pi_t^n$ by

$$\int G(x)\pi_t^n(dx) = \frac{1}{n^d} \sum_{x \in \mathbb{Z}^d} G(x/n)\eta_t(x)$$

for any continuous function $G : \mathbb{R}^d \to \mathbb{R}$ of compact support. Notice that $\pi_t^n = n^{-d} \sum_x \delta_{x/n}$, where $\delta_x$ is the Dirac-$\delta$ distribution in $x$. Our first result states

that the hydrodynamic limit of the process $\eta_t$ is given by the fractional heat equation. Denote by $\mu_n$ the initial distribution of $\eta_0$. Assume that $\pi_0^n$ converges in probability to the measure $u_0(x)dx$, where $u_0 : [0,1] \to \mathbb{R}$ is some initial profile. In that case we say that the sequence $\{\mu_n\}_n$ is associated to $u_0$. Then, for any $t > 0$, $\pi_{tn^\alpha}^n$ converges in probability to the measure $u(t,x)dx$, where $u(x,t)$ is the solution of the hydrodynamic equation

$$\begin{cases} \partial_t u(t,x) &= \Delta^{\alpha/2} u(t,x) \\ u(0,\cdot) &= u_0(\cdot). \end{cases}$$

Our next result states that the fluctuations of the density around the equilibrium are given by a fluctuation-dissipation result. Consider the process $\eta_t$ starting from a Bernoulli product measure of parameter $\rho \in [0,1]$. Then, the process

$$\mathcal{Y}_t^n = \frac{1}{n^{d/2}} \sum_{x \in \mathbb{Z}^d} (\eta_{tn^\alpha} - \rho)\delta_{x/n}(dx)$$

converges in distribution in the path space $\mathcal{D}([0,\infty), \mathcal{H}_{-2+d}(\mathbb{R}^d))$ to a generalized Ornstein-Uhlenbeck process formally defined by

$$d\mathcal{Y}_t = \mathcal{Y}_t \cdot \Delta^{\alpha/2} dt + \sqrt{\rho(1-\rho)}\nabla \cdot dW_t,$$

where $W_t$ is a white noise.

A third result about the scaling limit of this process relates the asymptotic of a distinguished particle with the Lévy process generated by the fractional Laplacian $\Delta^{\alpha/2}$. For an initial distribution of particles for which $\eta_0(0) = 1$ with probability one, let us denote by $X_t$ the position at time $t$ of the particle initially at the origin. In equilibrium the scaling limit of the process $X_t$ is the following. Consider the process $\eta_t$ starting from the measure $\nu_\rho(\cdot|\eta(0) = 1)$, where $\nu_\rho$ is the product Bernoulli mesure of parameter $\rho$. Then, the process

$$X_t^n = \frac{1}{n} X_{tn^\alpha} \xrightarrow{n\to\infty} (1-\rho)Z_t$$

in distribution, where $Z_t$ is the Lévy process generated by $\Delta^{\alpha/2}$. Once we know the hydrodynamic equation for the density of particles, this result can be generalized out of equilibrium.

**Theorem 1.** *Let $\{\mu_n\}_n$ be a sequence of initial measures associated to some profile $u_0$, with $\mu_n(\eta(0) = 1) = 1$ for every $n$. If $\alpha \geq 1$, assume further that the entropy density is bounded with respect to some equilibrium measure:*

$$\sup_n \frac{1}{n^d} \int \log\{d\mu_n / d\nu^\rho\} d\mu_n < +\infty$$

*Then, the process $n^{-1}X_{tn^\alpha}$ converges in distribution to the process $Z_t$ of independent increments of characteristic function*

$$\Psi(\beta) =: \log E[e^{i\beta Z_t}] = \int_0^t \int_{\mathbb{R}^d} \frac{1 - e^{i\beta x}}{|x|^{d+\alpha}} (1 - u(s,x)) dx ds,$$

*where $u(t,x)$ is the solution of the hydrodynamic equation.*

## Hyperbolic Scaling Limits: Traces and Maximum Principles

József Fritz

**Introduction:** Progress towards a satisfactory understanding of hyperbolic scaling limits in a regime of shocks is rather slow, we still have several open problems. Only one-dimensional lattice models are considered, the resulting macroscopic equations read as $\partial_t u + \partial_x f(u) = 0$, where $u$ is the vector of conserved quantities distinguished by the microscopic evolution law, $f$ denotes the associated flux. Attractive and related models [6] admit an effective coupling, in more general situations Lax entropy - flux pairs $(h(u), J(u))$ are the basic tools we can use; these are additional conserved quantities of the macroscopic system characterized by $\nabla J(u) = \nabla h(u) f'(u)$, where $f'$ is the Jacobian of $f$. Compensated compactness, that is the Div-Curl lemma for couples of entropy pairs, extends also to microscopic models [3], and in the presence of a rich entropy family, by means of a maximum principle it is possible to conclude that entropic measure solutions to the macroscopic equations are weak solutions. Let us remark that essentially any function of $u$ is an entropy for a single conservation law, but the construction of globally defined entropies might be a problematic issue in the case of systems; this fact might be responsible for difficulties we are facing with in the case of two-component models. A nontrivial entropy is not preserved by the microscopic evolution, therefore its evaluation leads to a non-gradient problem that should be regularized somehow. A bit more precisely, the microscopic process is generated by $\mathcal{L} := \mathcal{L}_0 + \sigma(\varepsilon)\,\mathcal{S}$, where $\mathcal{L}_0$ is the (asymmetric) generator of the basic model, $\mathcal{S}$ is a suitably chosen symmetric generator. The intensity, $\sigma(\varepsilon) > 0$ of regularization goes to $+\infty$ in such a way that $\varepsilon\sigma(\varepsilon) \to 0$ but $\varepsilon\sigma^2(\varepsilon) \to +\infty$ as the scaling parameter, $0 < \varepsilon \to 0$. The first condition means that the effect of regularization does vanish in the limit, but due to the second one, it is strong enough to do its job via LSI for $\mathcal{S}$. Various exclusion processes and Ginzburg - Landau models are to be discussed, see [3,4,5], the first steps of the argument are quite general.

**General remarks:** Let $\zeta_k(t)$, $t \geq 0$, $k \in \mathbb{Z}$ denote the vector conserved by the microscopic evolution, it is associated with a family $\lambda_u$ of stationary product measures such that $u = \lambda(\zeta_k)$. The empirical process we want to describe is $\bar{u}_\varepsilon(t,x) := \bar{\zeta}_{l,k}(t/\varepsilon)$ if $|x - k\varepsilon| < \varepsilon/2$, where $\bar{\zeta}_{l,k}$ is a block average of size $l = l(\varepsilon)$ such that $l/\sigma \to 0$ and $l^3\varepsilon/\sigma \to +\infty$. The first general step is the replacement of the microscopic current of $\bar{u}_\varepsilon$ by its canonical equilibrium expectation, the result can formally be written as $d\bar{u}_\varepsilon + \partial_x f(\bar{u}_\varepsilon)\,dt = d\partial_x R_\varepsilon$, where $R_\varepsilon$ is a vanishing term containing the martingale. There are two ways to interpret the empirical process in terms of measures. $m_\varepsilon(t,\phi) := \varepsilon \sum \phi(\varepsilon k)\bar{u}_\varepsilon(\varepsilon k)$ defines a trajectory in the space of signed measures. The distributions of this process form a tight family in the associated $C$ space, thus its *trace,* $m(t,\cdot)$ is defined also after the limit. However, $m$ does not yield too much information on the limiting behavior of $f(\bar{u}_\varepsilon)$, say.

The Young measures, $\Theta$ are defined by $d\Theta_\varepsilon := dt\,dx\,\theta^\varepsilon_{t,x}(du)$, where $\theta^\varepsilon_{t,x}$ is the Dirac mass at the actual value of $\bar{u}_\varepsilon(t,x)$. This family is tight in the space of measures, thus it controls the asymptotic behavior of various functions of the empirical process, and the Div-Curl lemma $\theta_{t,x}(h_1 J_2) - \theta_{t,x}(h_2 J_1) = \theta_{t,x}(h_1)\theta_{t,x}(J_2) -$

$\theta_{t,x}(h_2)\theta_{t,x}(J_1)$ is a general consequence of compensated compactness for microscopic systems.

Our next tool is the Lax inequality: $\partial_t\theta_{t,x}(h) + \partial_x\theta_{t,x}(J) \leq 0$ for convex entropy pairs, in the sense of distributions. The validity of this inequality depends very much on the structure of $\mathcal{S}$, the regularization must be elliptic even at the microscopic level, and this is not the case for general Ginzburg - Landau models. Unfortunately, even if we have proven the entropy inequality, its distributional form is not sufficient for a DiPerna argument [2] implying convergence to weak entropy solutions. At this step a maximum principle is needed: the Young measures, $\theta_{t,x}$ should be compactly supported, and the proof of [1] does work as soon as existence of traces of $h(\bar{u}_\varepsilon)$ at fix times have been established.

**Traces and the maximum principle:** The limiting trace $\Gamma(t, dx, du)$ of $\Theta$ does not exist in general. However, $m_{h,\varepsilon}(t,\phi) := \varepsilon \sum \phi(\varepsilon k) h(\bar{u}_\varepsilon(\varepsilon k))$ defines a trajectory in the space of measures, and its the distributions are tight in the D space if $h$ is an entropy, thus $h(\bar{u}_\varepsilon)$ admits a limiting trace. For a single conservation law this implies the maximum principle via the Lax inequality: if the initial values belong to $[a, b]$, then $\theta_{t,x}[a, b] = 1$ almost surely.

Careful large deviation estimates imply the Lax inequality for the one-component asymmetric Ginzburg - Landau model when its potential is a small perturbation of the quadratic one. There are not necessarily attractive models in this class, nevertheless the uniqueness of the hydrodynamic limit can be proven even in such situations, cf. [3].

The case of two-component systems is more complicated. LeRoux's system, $\partial_t v + \partial_x(v - v\rho) = 0$, $\partial_t \rho + \partial_x(\rho - u^2) = 0$ admits a rich entropy family, consequently a maximum principle can be established for the model of interacting exclusions in terms of its Riemann invariants. There is a hope to extend this approach also to the physically most interesting two-component Ginzburg - Landau model.

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## Relaxation scheme for a creation-annihilation model

Katalin Nagy
(joint work with Christophe Bahadoran and József Fritz)

We investigate the interaction of one-dimensional asymmetric exclusion processes of opposite speeds, where the exchange mechanism is combined with a creation-annihilation dynamics, and this asymmetric law is regularized with a nearest neighbor stirring of large intensity. This talk is an addendum to a previous paper by Fritz and Nagy [FN06], where an open problem has been left because of the lack of a suitable logarithmic Sobolev inequality.

**Model:** The configuration space of our system is the set of $\omega = (\omega_k \in \{0, 1, -1\} : k \in \mathbb{Z})$ sequences, $\eta_k := \omega_k^2$ denotes the occupation number, and $\omega_k$ is interpreted as the spin or charge of the particle at site $k \in \mathbb{Z}$. Our process is composed of the following local operations. Stirring $\omega \to \omega_s^k$ means that $\omega_k$ and $\omega_{k+1}$ are exchanged, the rest of the configuration is not altered. The action $\omega \to \omega_+^k$ creates a couple of particles: $(\omega_+^k)_k = +1$ and $(\omega_+^k)_{k+1} = -1$ if $\omega_k = \omega_{k+1} = 0$, other coordinates are not changed. Annihilation of a couple, $\omega \to \omega_\times^k$ is defined by $(\omega_\times^k)_k = (\omega_\times^k)_{k+1} = 0$ if $\omega_k = +1$ and $\omega_{k+1} = -1$, $(\omega_\times^k)_j = \omega_j$ otherwise. To define our dynamics we need the following formal generators.

$$\mathcal{L}_0\varphi(\omega) := \sum_{k\in\mathbb{Z}} c_k^s(\omega)\big(\varphi(\omega_s^k) - \varphi(\omega)\big)\,, \tag{1}$$

where $c_k^s(\omega) := (1/2)(\eta_k + \eta_{k+1} + \omega_k - \omega_{k+1})$. This generator lets $\oplus$ particles jump to the right, $\ominus$ particles jump to the left at rate 1, if allowed, while a collision $\oplus\ominus \to \ominus\oplus$ occurs at rate 2. Both particle number $\sum \eta_k$ and total charge $\sum \omega_k$ are preserved by this part of the dynamics. Moreover, all translation invariant product measures are stationary with respect to $\mathcal{L}_o$, cf. [TV03] for a first study of this model of interacting exclusions. The two-parameter family, $\lambda_{\rho,u}$, $0 < \rho < 1$, $|u| < 1$ of stationary states is characterized by $\lambda_{\rho,u}(\eta_k) = \rho$, $\lambda_{\rho,u}(\omega_k) = u$; we use the short hand notation $\lambda(\varphi) = \int \varphi\, d\lambda$ here.

The formal generator of the creation - annihilation process is acting as

$$\mathcal{G}_*\varphi(\omega) := \sum_{k\in\mathbb{Z}} c_k^+(\omega)(\varphi(\omega_+^k) - \varphi(\omega)) + \sum_{k\in\mathbb{Z}} c_k^\times(\omega)(\varphi(\omega_\times^k) - \varphi(\omega))\,, \tag{2}$$

where $c_k^+(\omega) := (1 - \eta_k)(1 - \eta_{k+1})$ and $c_k^\times := (1/4)(\eta_k + \omega_k)(\eta_{k+1} - \omega_{k+1})$. In this case only total charge $\sum \omega_k$ is preserved and we have a family, $\lambda_u^*$, $|u| < 1$ of translation invariant reversible measures such that $\lambda_u^*(\omega_k) = u$ and $\lambda_u^*(\eta_k) = \frac{1}{3}\big(4 - \sqrt{4 - 3u^2}\big)$.

Since we want to use the theory of compensated compactness, our process has to be regularized, say by an overall stirring

$$\mathcal{S}\varphi(\omega) := \sum_{k\in\mathbb{Z}} \big(\varphi(\omega_s^k) - \varphi(\omega)\big)\,. \tag{3}$$

This process is reversible with respect to any $\lambda_{\rho,u}$ and both $\sum \eta_k$ and $\sum \omega_k$ are preserved. The process of interacting exclusions generated by $\mathcal{L} := \mathcal{L}_o + \sigma\mathcal{S}$ has

been considered by Fritz and Tóth [FT04]. Here we are interested in the model given by $\mathcal{L}_2 = \mathcal{L}_o + \beta\mathcal{G}_* + \sigma\mathcal{S}$, where $\beta$ and $\sigma$ are positive parameters to be specified later. In [FN06] Fritz and Nagy discussed a model where the creation-annihilation process is replaced by a spin-flip dynamics and only particle number is conserved. The present model related to $\mathcal{L}_2$ seems to be interesting because it mimics electrophoresis, and its treatment requires some new ideas.

**Currents:** To understand the microscopic structure of our model, let us summarize some information, below $\mathbf{j}$ denotes the current of the conservative quantity indicated as a superscript. By some direct computations we get:

$\mathcal{L}_o\omega_k = \mathbf{j}^{\omega o}_{k-1} - \mathbf{j}^{\omega o}_k$ , $\lambda_{\rho,u}(\mathbf{j}^{\omega o}_k) = \rho - u^2$ , $\lambda^*_u(\mathbf{j}^{\omega o}_k) = \frac{1}{3}(4 - \sqrt{4-3u^2}) - u^2$;
$\mathcal{L}_o\eta_k = \mathbf{j}^{\eta o}_{k-1} - \mathbf{j}^{\eta o}_k$ , $\lambda_{\rho,u}(\mathbf{j}^{\eta o}_k) = u - u\rho$;
$\mathcal{G}_*\omega_k = \mathbf{j}^{\omega *}_{k-1} - \mathbf{j}^{\omega *}_k$ , $\lambda_{\rho,u}(\mathbf{j}^{\omega *}_k) = (1-\rho)^2 - \frac{\rho^2-u^2}{4}$ , $\lambda^*_u(\mathbf{j}^{\omega *}_k) = 0$
$\mathcal{G}_*\eta_k = c^+_k(\omega) - c^\times_k(\omega) + c^+_{k-1}(\omega) - c^\times_{k-1}(\omega)$ , $\lambda_{\rho,u}(\mathcal{G}_*\eta_k) = 2\big[(1-\rho)^2 - \frac{\rho^2-u^2}{4}\big]$.

**Macroscopic equations:** We use hyperbolic scaling of space and time, the scaled density fields are $\rho_\varepsilon(t,x) := \eta_k(t/\varepsilon)$ and $u_\varepsilon(t,x) := \omega_k(t/\varepsilon)$ if $|x - \varepsilon k| < \varepsilon/2$. A formal application of the principle of local equilibrium suggests that in the case of $\mathcal{L}$, $\rho_\varepsilon$ and $u_\varepsilon$ converge in a weak sense to weak solutions of the LeRoux system

$$\partial_t\rho + \partial_x(u - u\rho) = 0 \quad \text{and} \quad \partial_t u + \partial_x(\rho - u^2) = 0\,. \tag{4}$$

While the macroscopic equations for $\mathcal{L}_2$ read as

$$\partial_t u - \partial_x(u^2 + \tfrac{1}{3}\sqrt{4-3u^2}) = 0 \tag{5}$$

and

$$\partial_t\rho_\varepsilon + \partial_x(u_\varepsilon - u_\varepsilon\rho_\varepsilon) \approx \tfrac{2\beta}{\varepsilon}\left[(1-\rho)^2 - \tfrac{\rho^2-u^2}{4}\right] =: \tfrac{2\beta}{\varepsilon}C(\rho,u)\,, \tag{6}$$

both are to be interpreted in the weak (distributional) form. The second one is heuristic even then, the martingale part and the errors coming from the substitution of microscopic currents by their canonical equilibrium expectations have simply been omitted. It is presented only to exhibit the structure of the problem. Since $\rho_\varepsilon$ is not preserved by $\mathcal{L}_2$, its fluctuations do not vanish in the hydrodynamic limit. Nevertheless, the relaxation term on the right hand side gives rise to an a priori bound implying the required relation $\rho_\varepsilon \approx \frac{1}{3}(4 - \sqrt{4-3u_\varepsilon^2})$, this is what we need to get (5) from the second equation of (4). To do this crucial step, we extend relaxation theory of hyperbolic systems, see e.g. [Da00] to the microscopic models above. In this approach an appropriately chosen Lax entropy serves as the Liapunov function that implies relaxation.

**An entropy argument:** Suppose that we are given a Lax entropy pair $(h, J)$ for the LeRoux system (4), then from (5) and (6) we get

$$\partial_t h(\rho_\varepsilon, u_\varepsilon) + \partial_x J(\rho_\varepsilon, u_\varepsilon) \approx \tfrac{2\beta}{\varepsilon}\, h'_\rho(\rho_\varepsilon, u_\varepsilon)C(\rho_\varepsilon, u_\varepsilon)\,; \tag{7}$$

these equations are still heuristic. We have to find an entropy pair such that the right hand side above is negative. More precisely, it can be proven that for some

appropriately chosen entropy $h$ of the Leroux system there exists a constant $b > 0$ such that

$$h'_\rho(\rho_\varepsilon, u_\varepsilon)C(\rho_\varepsilon, u_\varepsilon) \le -b\big[\rho - \tfrac{1}{3}(4 - \sqrt{4 - 3u_\varepsilon^2})\big]^2, \tag{8}$$

and this estimate allows us to apply the substitution $\rho_\varepsilon \approx \frac{1}{3}(4 - \sqrt{4 - 3u_\varepsilon^2})$.

**Basic result:** Our main result reads as follows:

**Theorem 1.** *Suppose that* $\lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \varphi(x)\, u_\varepsilon(0,x)\, dx = \int_{-\infty}^{\infty} \varphi(x)\, u_0(x)\, dx$ *in probability for all* $\varphi \in C_c(\mathbb{R})$*, where* $u_0$ *is a given measurable function. Specify* $\sigma = \sigma(\varepsilon)$ *and* $\beta = \beta(\varepsilon)$ *such that* $\varepsilon\sigma(\varepsilon) \to 0$, $\varepsilon\sigma^2(\varepsilon) \to +\infty$, *while* $\sigma(\varepsilon)\beta(\varepsilon) \to +\infty$ *and* $\beta\varepsilon^{-1/3}\sigma^{1/3} \to 0$ *as* $\varepsilon \to 0$. *Then*

$$\lim_{\varepsilon\to 0} \int_0^\infty \int_{-\infty}^\infty \psi(t,x)\, u_\varepsilon(t,x)\, dx\, dt = \int_0^\infty \int_{-\infty}^\infty \psi(t,x) u(t,x)\, dx\, dt$$

*in probability for all* $\psi \in C_c(\mathbb{R}^2)$; *this* $u(t,x)$ *is the uniquely specified weak entropy solution to* $\partial_t u - \partial_x(u^2 + \frac{1}{3}\sqrt{4 - 3u^2}) = 0$ *with initial value* $u_0$.

**Main ideas of the proof:** In the proof we use the stochastic version of the theory of compensated compactness, see [F01, F04, FT04, FN06] for details, an LSI based estimation technique and the relaxation scheme presented above. The LSI based estimation technique used here was already present in the paper of Fritz and Tóth [FT04]. In the paper of Fritz and Nagy [FN06] a second logarithmic Sobolev inequality was needed to obtain the desired Burgers equation from the LeRoux equations. In the present case a second LSI was not available and a relaxation scheme was used instead to obtain (5) from the LeRoux equations.

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## Hyperbolic Conservation Laws with Discontinuous Fluxes and Hydrodynamic Limit for Particle Systems

Christian Klingenberg
(joint work with Gui-Qiang Chen, Nadine Even)

We are concerned with the following class of scalar hyperbolic conservation laws with discontinuous fluxes:

$$\partial_t \rho + \partial_x F(x, \rho(t,x)) = 0 \tag{1}$$

and with initial data:

$$\rho|_{t=0} = \rho_0(x), \tag{2}$$

where $F(\cdot, \rho)$ is continuous except on a set of measure zero.

The main feature of (1) is the discontinuity of the flux function in the space variable $x$. This feature causes new important difficulties in conservation laws. Kruzkov's approach in [11] for the $L^1$-contraction does not apply; entropy solutions even for the Riemann problem of (1) are not unique under the classical entropy conditions; several admissibility criteria have been proposed in [1, 2, 7, 9, 10] and the references cited therein. In particular, a uniqueness theorem was established in Baiti-Jenssen [3] when $F(x,\cdot)$ is monotone and Audusse-Perthame [1] for more general flux functions that especially include non-monotone functions $F(x,\cdot)$ in (1) under their notion. However, the existence of entropy solutions for the non-monotone case under the notion of Audusse-Perthame [1] has not been established. Unfortunately the entropy conditions proposed in the literature in general are not equivalent.

On the other hand, in statistical mechanics, some microscopic interacting particle systems with discontinuous speed parameter $\lambda(x)$, in the hydrodynamic limit, formally lead to scalar hyperbolic conservation laws with discontinuous flux of the form

$$\partial_t \rho + \partial_x \left(\lambda(x) h(\rho)\right) = 0 \tag{3}$$

and with initial data (2), where $\lambda(x)$ is continuous except on a set of measure zero and $h(\rho)$ is Lipschitz continuous. Equation (3) is equivalent to the following $2 \times 2$ hyperbolic system of conservation laws:

$$\begin{cases} \partial_t \rho + \partial_x(\lambda h(\rho)) = 0, \\ \partial_t \lambda = 0. \end{cases} \tag{4}$$

In particular, when $h(\rho)$ is not strictly monotone, system (4) is nonstrictly hyperbolic, one of the main difficulties in conservation laws (cf. [4, 6]). The natural question is which entropy solution the hydrodynamic limit selects, thereby leading to a suitable, physical relevant notion of entropy solutions of this class of conservation laws. This work is a first step and provides an answer to this question for a family of discontinuous flux functions via an interacting particle system, namely, the attractive zero range process (ZRP). This ZRP leads to a conservation law of

the form (3) with $\lambda(x) > 0$ and $h(\rho)$ being monotone in $\rho$, and its hydrodynamic limit naturally gives rise to an entropy condition of the type described in [1, 2].

Motivated by the hydrodynamic limit of the ZRP in [5] we adopt the notion of entropy solutions for a class of conservation laws with discontinuous flux functions, including the non-monotone case in the sense of Audusse-Perthame [1], and establish the existence of such an entropy solution via the method of compensated compactness in Section 3. This completes the well-posedness by combining a uniqueness result established in [1] for this class of conservation laws under the notion of entropy solutions.

In order to establish the hydrodynamic limit of large particle systems and the convergence of other approximate solutions to (1) rigorously in [5] we begin by establishing another compactness framework for (1)–(2). This mathematical framework is based on the notion and reduction of measure-valued entropy solutions, which is also applied for another proof of the existence of entropy solutions for the monotone case. We continue by establishing the hydrodynamic limit for a ZRP with discontinuous speed-parameter $\lambda(x)$ governed by the unique entropy solution of the Cauchy problem (2)–(3). Details can be found in [5].

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## Around the universality of the Airy$_1$ process

Patrik L. Ferrari
(joint work with Alexei Borodin, Tomohiro Sasamoto)

Half a decade ago, Prähofer and Spohn discovered the Airy$_2$ process in a surface growth model (the polynuclear growth (PNG) model) [12]. It is a universal process. It appears in directed last passage percolation percolation, various discrete growth models, domino tiling, random matrix theory (GUE Dyson's Brownian Motion) [9, 10]. The model we focus on in our recent research is the totally asymmetric simple exclusion process (TASEP), which can also be seen as a growth model (see Figure 1). Particles are on $\mathbb{Z}$ and they jump to their right with some given rate, provided the site is empty.

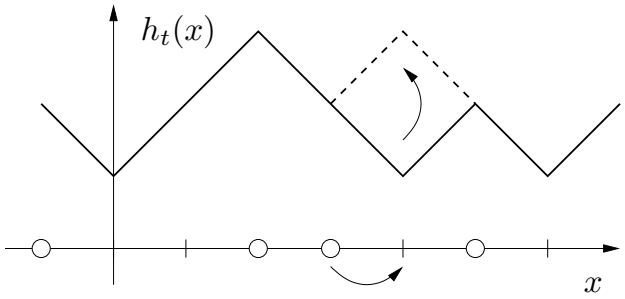


Figure 1. The growth model associated to the TASEP. The height function $h_t(x)$ increases by one unit if site $x$ is empty and decreases by one unit if site $x$ is occupied by a particle. Thus, when a particle jumps to the right, the the surface growth vertically as indicated by the dashed line.

*A question*: The TASEP belongs to the Kardar-Parisi-Zhang universality class [11]: the fluctuation of the position of a particle grows in time like $t^{1/3}$, while particles are correlated over distances of order $t^{2/3}$. What about the limit laws of fluctuations and limit process as time $t$ becomes large?

The answer depends on the type of initial conditions (but not too much), i.e., from the point of view of fluctuations, the universality class has still to be divided into a few subclasses. Below are results known starting from deterministic initial configurations. Any bounded ($t$-independent) fluctuations in the initial conditions discussed below leads to the same limit result by a standard coupling argument.
*Step initial conditions*: In the TASEP, the *Airy$_2$ process*, $\mathcal{A}_2$, occurs from step initial conditions, where particles initially occupy $\mathbb{Z}_-$ only [9]. The macroscopic density of particles has a (linearly) *decreasing region*, in which the fluctuations of particle positions ($\sim$ height function $h_t$) are governed by the Airy$_2$ process in the large time limit. In this case, from the growth point of view, the *limit shape is curved*.

*Periodic initial conditions*: The analogue of the $\text{Airy}_2$ process in the case of *flat limit shape*, has been and called *$Airy_1$ process*, $\mathcal{A}_1$ [14, 3]. This occurs in the TASEP starting with periodic initial conditions, e.g., one particle every $d \geq 2$ sites of $\mathbb{Z}$ [3, 2]. Recently we also were able to prove that this new process describes the large time fluctuations in the PNG model, as expected by universality [4]. A review on the $\text{Airy}_2$ and $\text{Airy}_1$ processes including their known properties is [7].

*Half-periodic initial conditions*: However, in a typical situation one can have coexistence of the two processes in separated regions, joined by a transition process, which takes place over distances of order $t^{2/3}$. In that case, there is a transition process, between the $\text{Airy}_2$ and the $\text{Airy}_1$ process: the *$Airy_{2\to 1}$ process*, $\mathcal{A}_{2\to 1}$. This was discovered and described in [5], where particles start from $2\mathbb{Z}_-$, see Figure 2.

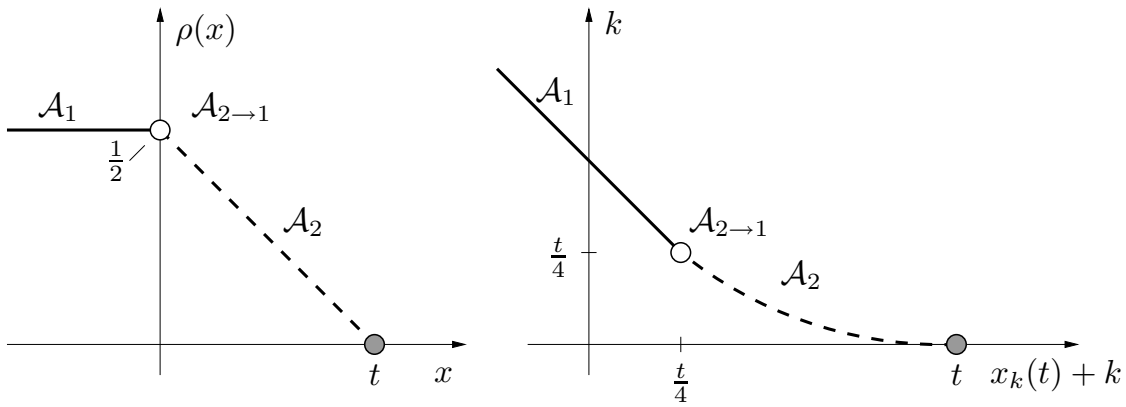


FIGURE 2. Left: the density $\rho$ for large time $t$ is linearly decreasing from $(0, 1/2)$ to $(t, 0)$. Right: The limit shape in an associated growth, obtained from the density ($k \in \mathbb{N}$ is the label of the particle which starts at $-2k$ and $x_k(t)$ is its position at time $t$).

*Remark on stationary initial conditions*: The stationary and translation invariant initial conditions are the Bernoulli-$\rho$ measures on $\mathbb{Z}$, with $\rho \in [0, 1]$. Interesting fluctuations occurs along the characteristics, given by $j = (1 - 2\rho)t$. There the fluctuations lives on $t^{1/3}$ scale. The asymptotic distribution of the current as seen from an observer sitting on the characteristics has been analyzed [13, 8]. For a slower or faster observer, what it is seen are essentially only the fluctuations of the initial distribution, i.e., Gaussian on a $t^{1/2}$ scale.

*Space-like paths extensions*: In the PNG model, it was shown in [6] that the $\text{Airy}_2$ process occurs not only for fixed time, but for any space-like path (a path $t = \pi(x)$ is space-like if $|\pi'(x)| \leq 1$). This is the case also for the TASEP, but instead of *space and time* we have *time and particle number*, see [1] for details. The boundary cases of space-like paths for TASEP are: (1) fixed time and (2) fixed particle number (tagged particle). For step initial conditions we prove convergence to the $\text{Airy}_1$

process [1, 4]. For step initial conditions the same kind of result holds but with the $\text{Airy}_2$ as limit process.

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## Current variance and the second class particle in particle systems


Márton Balázs

(joint work with Timo Seppäläinen)


[1] introduces a whole family of attractive stochastic interacting processes. Various interpretations of these include interacting particle systems, deposition growth models and last-passage percolation. Members of this family have product equilibrium distributions, and in the Eulerian hydrodynamic limit procedure their macroscopic densities $u(t, x)$ converge to a solution of the conservation law

$$\partial_t u(t, x) + \partial_x H(u(t, x)) = 0 \tag{1}$$

(see e.g. Rezakhanlou [11] for the Eulerian hydrodynamics). The function $H(u)$ is called the hydrodynamic flux function, and can be computed in an implicit form

from the model's jump rates and equilibrium distribution. Values of the solutions of (1), while they are smooth, are transported along the so-called *characteristic lines*, trajectories with a constant *characteristic speed* $C$. This speed turns out to be the derivative of the flux $H(u)$.

**Normal fluctuations.** Fluctuations of the evolution of these processes have been of great interest. When considered as particle systems, the precise question is the variance of the algebraic number $J_V(t)$ of particles which pass an observer moving with constant speed $V$ by time $t$. The following result was first shown by Ferrari and Fontes [6] for the simple exclusion process (SE):

$$\lim_{t\to\infty} \frac{\mathbf{Var}(J_V(t))}{t} = \text{const} \cdot |V - C|.$$

During my PhD. dissertation with the supervision and help of Prof. Bálint Tóth, I managed to extend this result to a whole family of processes in [1]. The extension shed light on the general structure of this limit, in accordance to the idea, shown explicitly in [6] for SE, that only fluctuations of the initial equilibrium state are transported and observed in this time-scale. Indeed, these fluctuations are rigidly transported along the characteristic curves. Both the fluctuation formulas and this idea show that the particle flux-fluctuations of this order disappear at the characteristic speed $V = C$.

**Dynamical (abnormal scaling) fluctuations.** The characteristic is where the more interesting dynamical fluctuations become visible. These have been conjectured on the $t^{1/3}$ scale for some models and on the $t^{1/4}$ scale for some others. The property which seems to make a difference is whether the hydrodynamic flux function $H(u)$ has a non-zero second derivative at the value of $u$ in question.

One instance of a model with linear hydrodynamics and $t^{1/4}$-scaling dynamical fluctuations is the *random average process* introduced by Ferrari and Fontes [7]. With Firas Rassoul-Agha and Timo Seppäläinen in [3] we proved this scaling using a reformulation of the process in terms of random walks in a space-time random environment (RWRE). The RWRE work originally motivated by the analysis of the random average process led to a series of quenched Central Limit Theorems for ballistic RWRE by Firas Rassoul-Agha and Timo Seppäläinen.

The asymmetric simple exclusion process is an example of models with non-zero second derivative of the hydrodynamic flux. By the pioneer work by Johannson [10], we know that the scaling $t^{1/3}$ is valid for the totally asymmetric simple exclusion process (TASEP) started with rarefaction wave initial data, and the scaling limit is connected to Tracy-Widom distributions. The result was extended to the equilibrium situation by Ferrari and Spohn [9]. Heavy combinatorics (called the RSK-correspondence) and random matrix theory are used in these papers. Cator and Groeneboom [5] on the other hand were able to show this scaling in the so-called *Hammersley's process* with purely probabilistic methods. Of course these methods do not give the precise limiting distribution, but they do not use model-specific combinatorial details either. With Eric Cator and Timo Seppäläinen we

adapted the arguments to the last-passage picture connected to the TASEP in [2]. We used an equilibrium last-passage picture involving the recently introduced *competition interface* by Ferrari and Pimentel [8].

We proceeded with Timo Seppäläinen by dropping the last-passage framework, which allowed us to extend the methods and the validity of the $t^{1/3}$-scaling to the general asymmetric simple exclusion process. This was a genuinely new result and an important open question for a long time.

A common tool in both the normal and the $t^{1/3}$-scaling fluctuations is an exact connection in [4] that relates current fluctuations to two-point functions, which are in turn further expressed in terms of the transition probabilities of the so-called second class particle. The talk described this correspondence, the normal fluctuation result as its consequence, and mentioned a usage to prove the $t^{1/3}$-scaling fluctuations in simple exclusion. A second talk by Timo Seppäläinen actually detailed this latter result.

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**Variance of the current in the asymmetric simple exclusion process.**

Timo Seppäläinen
(joint work with Márton Balázs)

In the 1-dimensional asymmetric simple exclusion process particles execute independent random walks on the lattice $\mathbb{Z}$ subject to the exclusion rule: at most one particle is permitted at each site, and any attempt to jump to an already occupied site is ignored. Only nearest-neighbor jumps are permitted. Each particle makes jump attempts to the right at rate $p$ and to the left at rate $q$. We assume $p > q$ to create the asymmetry, and also $p + q = 1$. The process at time $t$ is described by the occupation variables $\eta_i(t) \in \{0, 1\}$ whose value $\eta_i(t) = 1$ means that site $i$ is occupied by a particle at time $t$ and $\eta_i(t) = 0$ means that site $i$ is vacant at time $t$.

Independent, identically distributed Bernoulli occupation variables on the sites form an invariant distribution for this process. Fix a density value $\rho \in (0, 1)$ and consider henceforth the stationary process whose fixed-time marginal distribution on the occupation variables is always the density $\rho$ i.i.d. Bernoulli.

The quantity of interest is the net current of particles seen by an observer traveling at a fixed speed $V$ during the time interval $[0, t]$. Denote this quantity by $J^V(t)$. The main result is that when $V = V^\rho \equiv (p - q)(1 - 2\rho)$ (characteristic speed) then the variance of $J^{V^\rho}(t)$ is of order $t^{2/3}$. This is proved in [4] with some preliminary identities derived in [3]. The actual result is somewhat stronger for it gives moment bounds of the right order for the so-called second class particle for powers $p \in [1, 3)$.

The characteristic speed is defined by $V^\rho = f'(\rho)$ where $f(\rho) = (p - q)\rho(1 - \rho)$ is the particle flux at density $\rho$. It is expected that whenever $f''(\rho) \neq 0$ these fluctuations are of order $t^{1/3}$ and obey limit distributions related to Tracy-Widom distributions from random matrix theory. Precise limit distributions have been derived in the totally asymmetric case ($p = 1, q = 0$) by Johansson [2] and Ferrari and Spohn [1].

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## Diffusivity of Finite-Range Asymmetric Exclusion Processess on $\mathbb{Z}$

Benedek Valkó
(joint work with Jeremy Quastel)

A finite-range exclusion process on the integer lattice $\mathbb{Z}$ is a system of continuous time, rate one random walks with finite-range jump law $p(\cdot)$, i.e. $p(z) \geq 0$, and $p(z) = 0$ for $|z| > R$ for some $R < \infty$, $\sum_z p(z) = 1$, interacting via *exclusion*: Attempted jumps to occupied sites are suppressed. We consider asymmetric exclusion process with non-zero drift: $b = \sum_z zp(z) \neq 0$. The state space of the process is $\{0,1\}^{\mathbb{Z}}$. Particle configurations are denoted by $\eta$, with $\eta_x \in \{0,1\}$ indicating the absence, or presence, of a particle at $x \in \mathbb{Z}$.

Bernoulli product measures $\pi_\rho$, $\rho \in [0,1]$, with $\pi_\rho(\eta_x = 1) = \rho$ form a one-parameter family of invariant measures for the process. We consider the stationary process obtained by starting with $\pi_\rho$ for some $\rho \in (0,1)$. Although the fixed time marginals of this stationary process is easy to understand (since the $\eta$'s are just independent Bernoulli($\rho$) random variables), there are still lots of open questions about the full (space-time) distribution. Information about this process (and about the appropriate scaling limit) would be very valuable to understand such elusive objects as the Stochastic Burgers and the Kardar-Parisi-Zhang equations (see [QV] for a more detailed discussion).

We consider the two-point function,

$$S(x,t) = E[(\eta_x(t) - \rho)(\eta_0(0) - \rho)], \tag{1}$$

where the expectation is with respect to the stationary process obtained by starting from one of the invariant measures $\pi_\rho$. $S(x,t)$ satisfies the sum rules (see [PS])

$$\sum_x S(x,t) = \rho(1-\rho) = \chi, \qquad \frac{1}{\chi}\sum_x xS(x,t) = (1-2\rho)bt. \tag{2}$$

The diffusivity $D(t)$ is defined as

$$D(t) = (\chi t)^{-1} \sum_{x \in \mathbb{Z}} (x - (1-2\rho)bt)^2 S(x,t). \tag{3}$$

Using scaling arguments one conjectures [S],

$$S(x,t) \simeq t^{-2/3}\Phi(t^{-2/3}(x - (1-2\rho)bt)) \tag{4}$$

for some scaling function $\Phi$, as $t \to \infty$. A reduced conjecture is that

$$D(t) \simeq Ct^{1/3}, \tag{5}$$

as $t \to \infty$. Note that this means that the process has a superdiffusive behavior, as the usual diffusive scaling would lead to $D(t) \to D$. It is known that the mean-zero jump law would lead to this case, see [V].

If $f(t) \simeq t^\rho$ as $t \to \infty$ then as $\lambda \to 0$,

$$\int_0^\infty e^{-\lambda t} t f(t) dt \simeq \lambda^{-(2+\rho)}. \tag{6}$$

If $f$ satisfies (6) then we will say that $f(t) \simeq t^\rho$ *in the weak (Tauberian) sense.* Without some extra regularity for $f$ (for example, lack of oscillations as $t \to \infty$), such a statement will not imply a strong version of $f(t) \simeq t^\rho$ as $t \to \infty$. However, it does capture the key scaling exponent. The weak (Tauberian) version of the conjecture (5) is $\int_0^\infty e^{-\lambda t} t D(t) dt \simeq \lambda^{-7/3}$.

The first non-trivial bound on $D(t)$ was given in [LQSY] using the so called *resolvent approach*: the authors proved that $D(t) \geq Ct^{1/4}$ in a weak (Tauberian) sense. (They also proved the bound $D(t) \geq C(\log t)^{1/2}$ in $d = 2$, which was later improved to $D(t) \simeq C(\log t)^{2/3}$ in [Y].) This result shows that the stationary process is indeed superdiffusive, but does not provide the conjectured scaling exponent $1/3$.

The identification of this exponent was given in the breakthrough paper of Ferrari and Spohn [FS]. They treated the case of the totally asymmetric simple exclusion process (TASEP) where the jump law is $p(1) = 1$, $p(z) = 0$, $z \neq 1$. The focus of [FS] is not the diffusivity, their main result is a scaling limit for the fluctuation at time $t$ of a randomly growing discrete one dimensional interface $h_t(x)$ connected to the equilibrium process of TASEP. This random interface (the so-called *height function*) is basically the discrete integral of the function $\eta_x(t)$ in $x$. The scaling factor in their result is $t^{1/3}$ and the limiting distribution is connected to the The results of [FS] together with some additional tightness bounds would imply the existence of the limit $D^{TASEP}(t)t^{-1/3}$ and even the limiting constant can be computed (see [FS] and [QV] for details). Unfortunately, the needed estimates are still missing, but from [FS] one can at least obtain a lower bound of the right order:

$$D^{TASEP}(t) > Ct^{1/3} \tag{7}$$

with a positive constant $C$ (see [QV] for the proof).

In [QV] the resolvent approach is used to prove that the diffusivities of different AEP's are always of the same order. To be more precise: let $D_1(t), D_2(t)$ be the diffusivities of two finite range exclusion processes in $d = 1$ with non-zero drift. Then there exists $0 < \beta, C < \infty$ such that

$$C^{-1} \int_0^\infty e^{-\beta\lambda t} t D_1(t) dt \quad \leq \quad \int_0^\infty e^{-\lambda t} t D_2(t) dt \leq C \int_0^\infty e^{-\beta^{-1}\lambda t} t D_1(t) dt \tag{8}$$

Using this with (7) one gets that for any finite range exclusion process in $d = 1$ with non-zero drift, $D(t) \geq Ct^{1/3}$ in the weak (Tauberian) sense. [QV] also converts the Tauberian bound into pointwise bound in the nearest neighbor case to get $D(t) \geq Ct^{1/3}(\log t)^{-7/3}$.

Just a few months later Balázs and Seppäläinen [BS], proved that for any *nearest neighbor* asymmetric exclusion process in $d = 1$ there exists a finite constant $C$ such that for all $t \geq 1$,

$$C^{-1} t^{1/3} \leq D(t) \leq Ct^{1/3}. \tag{9}$$

Their proof uses refined and ingenious couplings to give bounds on the tail-probabilities of the distribution of the second class particle.

Using the correct upper and lower bounds for $D(t)$ from (9) in the nearest neighbor case, one can improve the Tauberian bounds of [QV] using the comparison lemma [QV2]. For any finite range exclusion process in $d = 1$ with non-zero drift, $D(t) = \mathcal{O}(t^{1/3})$ in the weak (Tauberian) sense: there exists a constant $0 < C < \infty$ such that

$$C^{-1}\lambda^{-7/3} \leq \int_0^\infty e^{-\lambda t} t D(t) dt \leq C\lambda^{-7/3}. \tag{10}$$

Getting strict estimates for a function using the asymptotic behavior of its Laplace transform usually requires some regularity and unfortunately very little is known qualitatively about $D(t)$. However, in our case (as noted in [QV]), one can get an upper bound for $D(t)$ using an inequality involving $H_{-1}$ norms, which gives the pointwise lower bound

$$D(t) \leq Ct^{1/3}. \tag{11}$$

for any finite range exclusion process in $d = 1$ with non-zero drift. (See [QV2] for details.)

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## Kinetic limit for a system of harmonic oscillators perturbed by a conservative noise

Giada Basile
(joint work with S. Olla, H. Spohn)

We consider a system of harmonic oscillators in $d$ dimensions, with $d \geq 1$, weakly perturbed by a stochastic dynamics which acts only on momenta preserving total momentum and total kinetic energy ([1]). Atoms are labelled by $\mathbf{y} \in \mathbb{Z}^d$, the $d$-dimensional lattice. The configuration space is $(\mathbb{R}^d \times \mathbb{R}^d)^{\mathbb{Z}^d}$ and a typical configuration is the state $(\mathbf{q_y}, \mathbf{p_y})_{\mathbf{y}\in\mathbb{Z}^d}$ where $\mathbf{p_y}$ is the momentum of $\mathbf{y}$ particle and $\mathbf{q_y}$ is the displacement of the $\mathbf{y}$ particle from its equilibrium position. The Hamiltonian of the system is given by

$$H(\mathbf{p}, \mathbf{q}) = \frac{1}{2} \sum_{\mathbf{y}\in\mathbb{Z}^d} \mathbf{p}_\mathbf{y}^2 + \mathbf{q_y} \cdot (\nu I - \alpha\Delta)\mathbf{q_y}$$

where $\Delta$ is the discrete Laplacian

$$\Delta f(\mathbf{z}) = \sum_{\mathbf{y}:|\mathbf{y}-\mathbf{z}|=1} (f(\mathbf{y}) - f(\mathbf{z}))$$

The global dynamics is determined by the generator

$$L = A + \varepsilon S$$

with $\varepsilon > 0$, where $A$ is the Hamiltonian vector field while $S$ is the generator of the stochastic perturbation. The operator $S$ acts only on the momenta $\{\mathbf{p_y}\}$ and generates a diffusion on the surface of constant kinetic energy and constant momentum. It is defined as

$$S \quad = \frac{1}{2(d-1)} \sum_{\mathbf{y}\in\mathbb{Z}^d} \sum_{i,j,k}^{d} \left(X^{i,j}_{\mathbf{y},\mathbf{y}+\mathbf{e}_k}\right)^2$$

where

$$X^{i,j}_{\mathbf{x},\mathbf{z}} = (p^j_\mathbf{z} - p^j_\mathbf{y})(\partial_{p^i_\mathbf{z}} - \partial_{p^i_\mathbf{y}}) - (p^i_\mathbf{z} - p^i_\mathbf{y})(\partial_{p^j_\mathbf{z}} - \partial_{p^j_\mathbf{y}}).$$

In dimension 1, in order to conserve total momentum and total kinetic energy, we have to consider a random exchange of momentum between three consecutive atoms, then we consider the vector fields

$$Y_z = (p_z - p_{z+1})\partial_{p_{z-1}} + (p_{z+1} - p_{z-1})\partial_{p_z} + (p_{z-1} - p_z)\partial_{p_{z+1}} + (p_{z-1} - p_z)\partial_{p_{z+1}}$$

Assuming that the initial state has finite energy, we introduce a new complex valued vector $\hat{\boldsymbol{\psi}}$ on $\mathbb{T}^d$ with components

$$\hat{\boldsymbol{\psi}}_+(\mathbf{k}) = \frac{1}{\sqrt{2}}\big(\omega(\mathbf{k})\hat{\mathbf{q}}(\mathbf{k}) + i\hat{\mathbf{p}}(\mathbf{k})\big), \quad \hat{\boldsymbol{\psi}}_-(\mathbf{k}) = \frac{1}{\sqrt{2}}\big(\omega(\mathbf{k})\hat{\mathbf{q}}(\mathbf{k}) - i\hat{\mathbf{p}}(\mathbf{k})\big)$$

$\forall \mathbf{k} \in \mathbb{T}^d$, where $\hat{\mathbf{q}}$, $\hat{\mathbf{p}}$ are the Fourier trasform of $\mathbf{p}, \mathbf{q}$ and $\omega(\mathbf{k})$ is the dispersion relation of the system. Given $\varepsilon > 0$, for any $J \in \mathcal{S}(\mathbb{R}^d \times \mathbb{T}^d)$ the Wigner distribution

$\bar{W}^{\varepsilon}(t)$ is defined trough its components

$$\big\langle J, \bar{W}_+^{i,\epsilon}(t)\big\rangle = \mathbb{E}\Big\{ \int_{\mathbb{R}^d} d\mathbf{u} \int_{\mathbb{T}^d} d\mathbf{k} \hat{\boldsymbol{\psi}}_+^{\,i}\big(\mathbf{k} - \frac{1}{2}\epsilon\mathbf{u}\big) \cdot \hat{J}(\mathbf{u},\mathbf{k})^* \hat{\boldsymbol{\psi}}_+^{\,i}\big(\mathbf{k} + \frac{1}{2}\epsilon\mathbf{u}\big)\Big\}$$

$$\big\langle J, \bar{W}_-^{i,\epsilon}(t)\big\rangle = \mathbb{E}\Big\{ \int_{\mathbb{R}^d} d\mathbf{u} \int_{\mathbb{T}^d} d\mathbf{k} \hat{\boldsymbol{\psi}}_-^{\,i}\big(\mathbf{k} - \frac{1}{2}\epsilon\mathbf{u}\big) \cdot \hat{J}(\mathbf{u},\mathbf{k})^* \hat{\boldsymbol{\psi}}_-^{\,i}\big(\mathbf{k} + \frac{1}{2}\epsilon\mathbf{u}\big)\Big\}$$

for every $i = 1, .., d$. We consider this distribution at the rescaled time $t/\epsilon$; in the limit $\epsilon \to 0$ (kinetic limit) $\{\bar{W}_+^{i,\epsilon}(t), i = 1, ..d\}$ converges in a weak sense to the solution of the following linear Boltzmann equation

$$\partial_t W^i(\mathbf{x},\mathbf{k},t) + \frac{1}{2\pi}\nabla\omega(\mathbf{k}) \cdot \nabla_{\mathbf{x}} W^i(\mathbf{x},\mathbf{k},t) =$$
$$\frac{1}{d-1}\sum_{j\neq i} \int_{\mathbb{T}^d} d\mathbf{k}' R(\mathbf{k},\mathbf{k}')\big(W^j(\mathbf{x},\mathbf{k}',t) - W^i(\mathbf{x},\mathbf{k},t)\big)$$

with $i = 1, .., d$, for $d \geq 2$. For $d = 1$ the equation reads

$$\partial_t W(x,k,t) + \frac{1}{2\pi}\partial_k\omega(k)\partial_x W(x,k,t) =$$
$$\int_{\mathbb{T}^1} dk' r(k,k')\big(W(x,k',t) - W(x,k,t)\big)$$

The right hand side of the Boltzmann equation describes the scattering of the normal modes due to the noise. In particular the mean "life" time of a $\mathbf{k}$-mode is $\phi(\mathbf{k})^{-1}$, where $\phi(\mathbf{k}) = \int_{\mathbb{T}^d} d\mathbf{k}' R(\mathbf{k},\mathbf{k}')$ for $d \geq 2$ and $\phi(k) = \int_{\mathbb{T}} dk' r(k,k')$ in $d = 1$. For small $\mathbf{k}$, $\phi$ behaves like $\sim |\mathbf{k}|^2$ which means that the small wavenumber modes have a small probability to be scattered, or, in other words, they survive for a long time. The behaviour of $\phi$ for small $\mathbf{k}$ depends on the properties of the noise, in particular from the conservation of momentum which assures $\phi(0) = 0$. If we consider a stochastic noise preserving only the kinetic energy but not the momentum, as the one defined in [5] for the one-dimensional chain, we get a collision rate $r(k,k') = 1$ and a mean life time $\tau = 1$ for every wavenumber $k$. The phonon Boltzmann equation has a probabilistic interpretation as a forward equation of a Markov process. For $d = 1$ it is the forward equation of a Markov process $(x(t), k(t))$, $t \geq 0$, where $x \in \mathbb{R}$ and $k \in \mathbb{T}$. For every $t \geq 0$, $k(t)$ is governed by the collision rate

$$\nu_k(dk') = dk' R(k,k'), \qquad k' \in \mathbb{T}$$

with a total collision rate $\phi(k) = \int_{\mathbb{T}} \nu(dk')$. Given $k(0) = k$, one has $k(t) = k$ for $0 \leq \tau \leq t$ with $\tau$ an exponentially distributed random variable of mean $\phi(k)^{-1}$. At time $\tau$, $k$ "jumps" to $dk'$ with probability $\nu_k(dk')/\phi(k)$. The joint process $(x(t), k(t))$, $t \geq 0$, is defined by

$$\frac{d}{dt}x(t) = \frac{1}{2\pi}\nabla\omega(k(t))$$

In more dimensions ($d \geq 2$) we have the Markov process $\{(\mathbf{x}(t), \mathbf{k}(t), i) \;\; i = 1, .., d\}$

Here the collision rate from $(i, \mathbf{k})$ to $(j, d\mathbf{k}')$ is given by

$$\nu_{\mathbf{k},i}(j, d\mathbf{k}') = \frac{1}{d-1}(1-\delta_{i,j})R(\mathbf{k}, \mathbf{k}')dk' \quad \forall i, j = 1, .., d$$

then transitions between states with the same index $i$ are forbidden. The total collision rate is given by

$$\phi(i, \mathbf{k}) = \sum_{j=1}^{d} \int_{\mathbb{T}^d} \nu_{\mathbf{k},i}(j, d\mathbf{k}') = \int_{\mathbb{T}^d} d\mathbf{k}' R(\mathbf{k}, \mathbf{k}')$$

$\forall i = 1, .., d$, and it doesn't depend on $i$, i.e. $\phi(i, \mathbf{k}) = \phi(\mathbf{k})$. Given a state $(i, \mathbf{k})$ at $t = 0$, it "jumps" at time $\tau$ to the state $(j, d\mathbf{k}')$ with a probability $\nu_{\mathbf{k},i}(d\mathbf{k}', j)/\phi(\mathbf{k})$, where $\tau$ is an exponentially distributed random variable of mean $\phi(\mathbf{k})^{-1}$. Again the joint process $(\mathbf{x}(t), \mathbf{k}(t), i)$ is defined as

$$\frac{d}{dt}\mathbf{x}(t) = \frac{1}{2\pi}\nabla\omega(\mathbf{k})$$

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## Existence of Entropy Solutions to Hyperbolic Conservation Laws with Discontinuous Fluxes


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(joint work with Gui-Qiang Chen, Christian Klingenberg)


We study scalar conservation laws with discontinuous fluxes of the following type:

$$\partial_t \rho + \partial_x \left(\lambda(x)h(\rho)\right) = 0 \tag{1}$$

with initial data:

$$\rho|_{t=0} = \rho_0(x). \tag{2}$$

Here $\lambda(x)$ is piecewise continuous and $h(\rho)$ is Lipshitz continuous.

The issue we address is the question of admissibility. Kruzkov's entropy condition [4] is not applicable because of the discontinuity of the flux. Many alternative entropy conditions have been suggested for which one can prove uniqueness of such

entropy solutions. Unfortunately these alternatives are not equivalent, in particular given an initial data, different entropy notions may lead to different unique entropy solutions.

One can ask the question which entropy solution is selected if one models this equation by taking the hydrodynamic limit of an interacting particle system. It turns out that the attractive zero range process (ZRP) leads to a conservation law of the form (1) with $\lambda(x) > 0$ and $h(\rho)$ being monotone in $\rho$, and its hydrodynamic limit naturally gives rise to an entropy condition in the sense of the following definition. Here we use the notation: $F(x,\rho) = \lambda(x)h(\rho)$.

**Definition 1** (Notion of entropy solutions in $L^\infty$)**.** *We say that an $L^\infty$ function $\rho : \mathbb{R}^2_+ := \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$ is an entropy solution of (1)–(2) provided that, for each $\alpha \in [M_0, \infty)$ (or $\alpha \in (-\infty, M_0]$) and the corresponding two steady-state solutions $m^\pm_\alpha(x)$ of (1), such that*

$$F(x, m^\pm_\alpha(x)) = \alpha, \tag{3}$$

*we have that*

$$\begin{aligned}\int \Big( &|\rho(t,x) - m^\pm_\alpha(x)|\, \partial_t J \\ &+ sgn(\rho(t,x) - m^\pm_\alpha(x))\big(F(x,\rho(t,x)) - \alpha\big)\, \partial_x J\Big)\, dtdx \\ &+ \int |\rho_0(x) - m^\pm_\alpha(x)| J(0,x)\, dx \geq 0\end{aligned} \tag{4}$$

*for any test function $J : \mathbb{R}^2_+ \mapsto \mathbb{R}_+$.*

This type of entropy solution was described in [2] for the monotone case and then for a more general class of flux functions, including the non-monotone case, in [1]. However, the existence of entropy solutions for the non-monotone case under the notion of Audusse-Perthame [1] has not been established

This motivates us to study well posedness of entropy solutions to (1)–(2) where entropy is meant in the sense of the definition given above. Uniqueness had been established by Audusse-Perthame [1] so we turn our attention towards existence and establish the existence of such an entropy solution by establishing the convergence of approximating entropy solutions via the method of compensated compactness.

**Theorem 1.** *Let $F(x,\rho)$ be strictly convex or concave in $\rho$ for a.e. $x \in \mathbb{R}$ and satisfy the properties introduced above. Let $\rho_0(x) \in L^\infty$. Define by $F_\varepsilon(x,\rho)$ the standard mollification of $F(x,\rho)$ in $x \in \mathbb{R}$. Then the sequence of entropy solutions (in the sense of Definition 2.1) $\rho^\epsilon$ of the Cauchy problem*

$$\begin{cases} \partial_t \rho + \partial_x F_\varepsilon(x,\rho) = 0, \\ \rho|_{t=0} = \rho_0(x) \geq 0. \end{cases} \tag{5}$$

*converges to the unique entropy solution of the Cauchy problem (1)–(2) in the sense of Definition 2.1, where again we use the notation $F(x,\rho) = \lambda(x)h(\rho)$.*

For more details, see [3].

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## Second and third class particles in TASEP

Patricia Gonçalves

(joint work with Pablo A. Ferrari and James Martin)

The Asymmetric Simple Exclusion process is one of the most studied interacting particle systems. In this process, particles evolve on $\mathbb{Z}$ according to interacting random walks with an exclusion rule which prevents to have more than a particle per site. The dynamics is as follows. Fixed a probability $p(\cdot,\cdot)$ on $\mathbb{Z}\times\mathbb{Z}$, each particle independently of the others, waits a mean 1 exponential time, after which, being at the site $x$ it jumps to a site $y$ at a rate $p(x,y) := p(y-x)$. If the site is occupied the jump is suppressed in order to respect the exclusion rule and after that it restarts. Without losing generality we assume $\sum_x p(x) = 1$. This continuous time Markov process $\eta_t$ has state space $\{0,1\}^{\mathbb{Z}}$ and for a site $x\in\mathbb{Z}$, $\eta_t(x)$ denotes the quantity of particles at that site at the macroscopic time $t$. Then, if $\eta_t(x) = 1$ the site $x$ is occupied otherwise it is empty.

When the transition probability rate $p(\cdot)$ has positive and finite mean $\gamma > 0$, the process is called Asymmetric Exclusion Process (AEP). The Asymmetric Simple Exclusion Process (ASEP) will be used when the jumps are nearest neighbor with $p(1) = p$ and $p(-1) = q$ with $p+q = 1$ and $p \neq 1/2$. The Totally Asymmetric Simple Exclusion process (TASEP) if in this last case $p = 1$.

Since the work of Rezakhanlou, it is know that starting this process from an initial measure associated to a profile (see [6]) it has an hydrodynamic limit given by the inviscid Burgers equation: $\partial_t u(r,t) + \gamma\nabla u(r,t)(1-u(r,t)) = 0$.

For $\rho\in[0,1]$ denote by $\nu_\rho$ the Bernoulli product measure of parameter $\rho$. It is know that $\nu_\rho$ is an invariant measure for this process and that all invariant and translation invariant measures are convex combinations of $\nu_\alpha$ if $p(\cdot,\cdot)$ is such that $p_t(x,y) + p_t(y,x) > 0$, $\forall x,y\in\mathbb{Z}^d$ and $\sum_x p(x,y) = 1$, $\forall y\in\mathbb{Z}^d$, see [4].

A *second class particle* is a particle that behaves with holes as a particle and with particles as a hole: if there is a second class particle at site $x$, then it jumps to $y$ with rate $p(y-x)$ if $y$ is empty and interchanges positions with a particle at $y$ at rate $p(x-y)$.

Let $\nu_{\lambda,\rho}$ be a product measure with density $\lambda$ to the left of the origin and $\rho$ to the right of it. Ferrari and Kipnis [1] start the TASEP with a configuration chosen accordingly to $\nu_{\lambda,\rho}$ with $0\le\rho<\lambda\le 1$ put a second class particle at site

0 and call $X_t$ its position at time $t$. Then they prove that $X_{tN}/N$ converges as $N \to +\infty$ to a Uniform random variable with support on $[-t, t]$. The extension of last result to the case of a AEP is straightforward, but for completeness we state this result in this general case and we make a sketch of its proof, following the same arguments as [1]. Almost sure convergence has been proven by Mountford and Guiol [5], Ferrari and Pimentel [3] and Ferrari, Martin and Pimentel [2].

We consider the process with different classes of particles. Holes can be considered as particles of class $\infty$. A class-$m$ particle at $x$ interchanges positions with class-$k$ particle at site $y$ at rate $p(y-x)$ if $m < k$ and at rate $p(x-y)$ if $m > k$. That is, a pair of class-$m$ and class-$k$ particles behaves as particle-hole if $m < k$ and as hole-particle if $m > k$; particles of the same class interact by exclusion. For example if a second class particle attempts to jump to a site occupied by a first class particle, the jump is suppressed but if instead it attempts to jump to a site occupied by a third class particle, then they exchange positions. As a consequence, the higher the degree of the class of a particle the less is its priority.

We start the process from a deterministic configuration, denoted by $\xi$, that has all negative sites occupied by (first class) particles, the origin and site 1 are occupied by a second class and a third class particle, respectively, and all sites to the right of site 1 are empty. We show that for the TASEP from the configuration $\xi$, the probability of the second class particle jumping to the right of the third class particle at time $t$ converges to $2/3$ as $t \to \infty$. The same argument shows that the limiting value equals $(p+1)/3p$ for the ASEP.

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## Stationary distributions of multi-type TASEPs


James Martin

(joint work with Pablo Ferrari)


The *totally asymmetric simple exclusion process* (TASEP) is defined as follows. Some sites $i \in \mathbb{Z}$ contain a particle; at the others we say there is a hole. The dynamics of the system are as follows: a bell rings as a Poisson process of rate 1 at each site independently; when the bell at site $i$ rings, if there is a particle at site $i$ and a hole at site $i-1$, they exchange. Put another way, each particle in the

system tries to jump to the left at rate 1; the jump succeeds whenever the site to its left is unoccupied.

Denote a configuration of the TASEP by $u = \big(u(j), j \in \mathbb{Z} \text{ or } \mathbb{Z}_N\big) \in \mathcal{U}_1$ or $\mathcal{U}_1^{(N)}$, where $\mathcal{U}_1 = \{1, \infty\}^{\mathbb{Z}}$ and $\mathcal{U}_1^{(N)} = \{1, \infty\}^{\mathbb{Z}_N}$. We set $u(i) = 1$ if there is a particle at $i$ and $u(i) = \infty$ if there is a hole at $i$. (This notation is not standard but will be convenient later).

There is a one-parameter family of translation-invariant extremal stationary measures $\nu_\lambda$, $\lambda \in [0, 1]$. Under $\nu_\lambda$, each site $i$ is occupied by a particle independently and with probability $\lambda$; the measure $\nu_\lambda$ is the *product measure* with density $\lambda$ of particles (or the measure of a *Bernoulli process with rate* $\lambda$). The only other extremal stationary measures are the so-called *blocking measures*: the measure concentrated on the configuration with particles at all negative sites and holes at all non-negative sites, and its translates.

We now consider a *TASEP with n classes of particle* or *n-TASEP*, for $n \geq 1$. Now a configuration $u = \big(u(j), j \in \mathbb{Z}\big)$ of the system is a member of $\mathcal{U}_n$, where

$$\mathcal{U}_n = \Big(\{1, 2, \ldots, n\} \cup \{\infty\}\Big)^{\mathbb{Z}}.$$

If $u(i) = r \leq n$ we say that there is a particle of class $r$ at site $i$, while if $u(i) = \infty$ we say that there is a hole at $i$.

The dynamics are as follows. Bells ring at rate 1 at each site independently. When the bell at site $i$ rings, the values at $i-1$ and $i$ swap if $u(i-1) > u(i)$, and remain unchanged otherwise. That is, the process jumps from $u$ to the configuration $u^{(i-1,i)}$ where

$$\begin{aligned} u^{(i-1,i)}(j) &= u(j) \text{ if } j \notin \{i-1, i\}, \\ u^{(i-1,i)}(i-1) &= \min\{u(i-1), u(i)\}, \\ u^{(i-1,i)}(i) &= \max\{u(i-1), u(i)\}. \end{aligned}$$

Put another way, each particle in the system tries to jump to the left at rate 1. The jump succeeds when the site to its left contains a hole or contains a particle with a higher class (in which case the two particles exchange positions).

One natural way in which $n$-type TASEPs arise is from couplings of several 1-type TASEPs. Let $\eta_t^{(1)}$, $\eta_t^{(2)}, \ldots, \eta_t^{(n)}$, $t \geq 0$ be $n$ processes realising a TASEP, started from initial conditions such that $\eta_0^{(1)}(j) \geq \eta_0^{(2)}(j) \geq \ldots \eta_0^{(n)}(j)$ for all $j$. (That is, whenever there is a particle at site $j$ in process $m$, there is also one in the processes $m+1$, $m+2, \ldots, n$). Suppose one couples the processes by using the same processes of bells at each site for all of them. Then the ordering $\eta_t^{(1)}(j) \geq \eta_t^{(2)}(j) \geq \ldots \eta_t^{(n)}(j)$ continues to hold for all $t$ (this is an instance of the *basic coupling* of Liggett [7]). Let $u_t(j) = \inf\{m : \eta_t^{(m)} = 1\}$, (with $u_t(j) = \infty$ if $\eta_t^{(m)}(j) = \infty$ for all $m$). Then $u_t$ realises an $n$-type TASEP.

The 2-TASEP has been studied from several perspectives. The existence of a translation-invariant stationary measure for the process on $\mathbb{Z}$, with densities $\lambda_1$ and $\lambda_2$ of first- and second-class particles (for $0 < \lambda_1 < \lambda_1 + \lambda_2 < 1$), was proved by

Liggett [7] to demonstrate ergodic properties of the TASEP. The uniqueness and extremality of this measure were shown by Ferrari, Kipnis and Saada [4] and Speer [8] (the only other extremal invariant distributions are those concentrated on a single state – "blocking measures" as for the 1-TASEP above). Derrida, Janowsky, Lebowitz and Speer [2] and Speer [8] construct the measure explicitly (both in finite and infinite volume) using a matrix method and show various regeneration and asymptotic properties; Ferrari, Fontes and Kohayakawa [3] give probabilistic interpretations and proofs of the measure and its properties.

Recently Omer Angel [1] gave an elegant construction of this stationary measure based on two independent product measures with densities $\lambda_1$ and $\lambda_1 + \lambda_2$ (and an analogous construction for the case of $\mathbb{Z}_N$); the proof involves providing bijections between certain families of binary trees and pairs of binary sequences.

We show that Angel's construction can be rewritten in terms of the operation of a queueing server (namely, an $M/M/1$ queue in discrete time). The two independent product measures correspond to the arrival process and the service process of the queue. The stationary measure for the 2-TASEP corresponds to the output process of the queue. Sites of the TASEP are interpreted as times in the queueing process. If the queue has a departure at time $i$, then one puts a first class particle at site $i$. If the queue has an "unused service" at time $i$ (that is, a service is available but there is no customer present) then one puts a second-class particle at site $i$. If there is no service available at time $i$, then there is a hole at site $i$. (Full descriptions are given in the next section).

We then generalise this result to the $n$-TASEP with $n \geq 3$. First a remark about the set of stationary measures. Let $\lambda_1, \ldots, \lambda_n \in (0,1)$ with $\sum_{r=1}^{n} \lambda_r < 1$. For the process on $\mathbb{Z}$, there is a unique translation-invariant stationary measure with density $\lambda_r$ of particles of type $r$, for each $r$. These stationary measures are extremal, and the only other stationary measures are blocking measures, concentrated on a single configuration. These facts can be proved almost identically to the proofs given in, for example, [4] and [8] for the case $n = 2$.

We construct a representation of these stationary measures based on a system of $n-1$ *queues in tandem*. Take the case of $\mathbb{Z}$, say. Consider a set of $n$ independent Bernoulli processes (product measures), such that the $m$th has density $\lambda_1 + \cdots + \lambda_m$, for each $m$. The service process of the $m$th queue is given by the $(m+1)$st product measure. The first product measure acts as the arrival process to the first queue. Thereafter, the arrival process to the $m$th queue, $m > 1$, is given by the output process from the $(m-1)$st queue. The $m$th queue is a "priority queue" with $m$ types of customer; at a service time, the customer who departs is the one whose class number is lowest out of those present. We show that the distribution of the output process of the $(n-1)$st queue provides the stationary distribution of the $n$-TASEP with densities $\lambda_1, \ldots, \lambda_n$.

The proof of stationarity has the following structure. First we define dynamics on a set of $n$ configurations of particles on the line (the "multi-line process"). On each line, the local transitions are those of a TASEP, and the lines are coupled in such a way that the bells on neighbouring lines always ring at the same time,

either at the same site or at neighbouring sites. We show that a collection of $n$ independent product measures is stationary under these dynamics. (In fact, it will turn out that in this equilibrium, the multi-line process provides a coupling of $n$ TASEP processes, whose marginal distributions at any fixed time are independent). The second step of the proof is to show that, under these dynamics, the final output process derived as in the previous paragraph is realising an $n$-TASEP.

In the case $n = 2$, the two lines of the multi-line process correspond to the arrival and departure processes of the queue.

An important motivation for much of the previous work on the 2-TASEP on $\mathbb{Z}$ was its application to the study of shock measures for the 1-TASEP; see for example [2], [8], [4], [3]. Several results developed in this context – in particular, concerning the "stationary measure as seen from a second-class particle" – can be proved very simply, and often strengthened, using the queueing representation. We also give extensions of some of these results to the case of the $n$-TASEP for $n > 2$.

Analogous results, with some interesting combinatorial consequences, can also be obtained for the $n$-TASEP defined on a cycle $\mathbb{Z}_N$ rather than the infinite line $\mathbb{Z}$.

Further details may be found in the papers and [5] and [6].

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## Scaling limits of reversible processes with log-concave invariant measures

Lorenzo Zambotti
(joint work with Luigi Ambrosio and Giuseppe Savaré)

We study Markov processes associated with stochastic differential equations, whose non-linearities are gradients of convex functionals. The main result is a stability property: if the associated invariant measures converge weakly, then the Markov processes converge in law. The proofs are based on the interpretation of a Fokker-Planck equation as the steepest descent flow of the relative Entropy functional in the space of probability measures, endowed with the Wasserstein distance. Applications include stochastic partial differential equations and convergence of equilibrium fluctuations for a class of random interfaces.

Let $U : \mathbb{R} \mapsto \mathbb{R}$ convex and $C^1$ such that $U(a) \geq A + B|a|$ with $B > 0$ and $\int_{\mathbb{R}} a e^{-U(a)} da = 0$. We fix $c > 0$ and we set:

$$V(x) := \begin{cases} \sum_{i=1}^k U(x_i - x_{i-1}) & \text{if } \ x_i \geq 0, \ \ \sum_i x_i = c \\ \\ +\infty & \text{otherwise} \end{cases}$$

and the probability measure

$$\gamma(dx) := \frac{1}{Z}\, e^{-V(x)}\, dx.$$

We want a natural reversible dynamics $(\phi_t)$ with invariant measure $\gamma$ and conservation of the average:

$$\sum_i \phi_t^i = \sum_i \phi_0^i \qquad \forall t \geq 0$$

(constant droplet volume), see [7] and [3]. We can choose:

$$\begin{cases} d\phi_t \ = - \partial\partial^* \{ \partial V'(\partial^* \phi)\, dt + dl \} + \sqrt{2}\, \partial dw \\ \\ \text{Neumann boundary conditions} \\ \phi^i \geq 0, \quad dl^i \geq 0, \quad \displaystyle\int_0^\infty \phi_t^i \ dl_t^i = 0 \end{cases}$$

There exists a unique stationary $\phi$ after fixing the droplet volume:

$\Phi_k(t,x) := \frac{1}{\sqrt{k}} \sum_i \phi^i_{k^4 t}\, 1_{[\frac{i-1}{k}, \frac{i}{k})}(x)$. $\Phi_k$ converges in law to the unique stationary solution of a Stochastic Cahn-Hilliard equation: (joint with A.Debussche)

$$\begin{cases} \dfrac{\partial u}{\partial t} = -\dfrac{1}{\sigma^2}\dfrac{\partial^2}{\partial x^2}\left(\dfrac{\partial^2 u}{\partial x^2} + \eta\right) + \sqrt{2}\,\dfrac{\partial}{\partial x}\dot{W} \\ \partial u(t,0) = \partial u(t,1) = 0, \\ \partial^3 u(t,0) = \partial^3 u(t,1) = 0, \\ u \geq 0,\ d\eta \geq 0,\ \int u\, d\eta = 0 \end{cases}$$

The above equations have the following abstract form

$$dX = -\nabla V(X)\, dt \,+\, \sqrt{2}\, dW_t$$

where $W$ is a Wiener process in a Hilbert space $H$ and $V$ is a convex function on $H$. An equivalent description of the law of $X$ is given by the associated Dirichlet Form $\mathcal{E}$:

$$\mathcal{E}(\varphi) \,:=\, \int \|\nabla\varphi\|^2_H\, d\gamma$$

where $\gamma$ is the unique invariant measure of $X$.

In finite dimension $V$ is convex iff $\gamma$ is *log-concave*: $\forall t \in [0,1]$

$$\log\gamma(tA + (1-t)B) \geq t\log(\gamma(A)) + (1-t)\log(\gamma(B)),$$

a stable property under weak convergence and linear projections. We set:

$$\mathcal{P}_2(H) := \left\{\mu \ \text{ probability on } H : \ \int_H \|x\|^2_H\, d\mu(x) < \infty\right\}.$$

$$W_{2,H}(\mu,\nu) := \inf\left\{\left[\int_{H\times H} \|y-x\|^2_H\, d\Sigma\right]^{\frac{1}{2}} :\ \Sigma \in \Gamma(\mu,\nu)\right\},$$

$\Gamma(\mu,\nu)$ is the set of *couplings* between $\mu$ and $\nu$ on $H \times H$.

For all probability measure $\mu$ on $H$ we set:

$$\mathcal{H}(\mu,\gamma) \,:=\, \int_H \rho\, \log\rho\, d\gamma$$

if $\mu = \rho\,\gamma$ with $\rho \in L^1(\gamma)$, and $+\infty$ otherwise.

**Theorem (Ambrosio/Gigli/Savaré)** *Let $\gamma$ be log-concave. Then for any $\mu_0 \in \mathcal{P}_2(H)$ there exists a unique continuous $\mu : (0,+\infty) \mapsto \mathcal{P}_2(H)$ such that:*

(1) *For all $\nu \in \mathcal{P}_2(H)$*

$$\frac{1}{2}\frac{d}{dt} W^2_{2,H}(\mu_t,\nu) + \mathcal{H}(\mu_t,\gamma) \,\leq\, \mathcal{H}(\nu,\gamma)$$

(2) *$\mu_t \to \mu_0$ in $\mathcal{P}_2(H)$ as $t \downarrow 0$.*

$(\mu_t : t \geq 0)$ is the *gradient flow* of the Relative Entropy functional $\mathcal{H}(\cdot, \gamma)$ in the Wasserstein space $\mathcal{P}_2(H)$.

**Theorem (Ambrosio/Savaré/Z.)** *Let $\gamma$ be log-concave. Then the form:*

$$\mathcal{E}(\Phi) := \frac{1}{2} \int \|\nabla \Phi\|_H^2 \, d\gamma$$

*is closable in $L^2(\gamma)$ and the associated semigroup $(P_t)_{t \geq 0}$ is*

$$P_t \varphi(x) = \int \varphi \, d\mu_t^x$$

*where $(\mu_t^x : t \geq 0)$ is the gradient flow of $\mathcal{H}(\cdot, \gamma)$ in $\mathcal{P}_2(H)$ with $\mu_0^x = \delta_x$.*

This result is based on previous work by (among others) Jordan-Kinderlehrer-Otto and Ambrosio-Gigli-Savaré.

We consider now a sequence $(\gamma_n, H_n)_n$ such that $H_n \subset H$ and

(1) $\gamma_n$ is log-concave and $\gamma_n \Longrightarrow \gamma$.
(2) $H_n$ "converges uniformly" to $H$, i.e. $\lim_{n \to \infty} \|\Pi_n h - h\|_H = 0$, $\forall\, h \in H$, where $\Pi_n : H \to H_n$ is the canonical projection, and $\exists\, \kappa > 0$:

$$\kappa^{-1} \|h\|_H \leq \|h\|_{H_n} \leq \kappa \|h\|_H, \qquad \forall\, h \in H_n, \quad \forall n.$$

**Theorem (Ambrosio/Savaré/Z.)**

- *Let $(\mu_t^n)_{t \geq 0}$ be the gradient flow of $\mathcal{H}(\cdot, \gamma_n)$ in $\mathcal{P}_2(H_n)$*
- *and $(\mu_t)_{t \geq 0}$ be the gradient flow of $\mathcal{H}(\cdot, \gamma)$ in $\mathcal{P}_2(H)$.*

*If $\mu_0^n \to \mu_0$ in $\mathcal{P}_2(H)$, then $\mu_t^n \to \mu_t$ in $\mathcal{P}_2(H)$.*

Under the above assumptions, let $X^n$ be the process associated with:

$$\mathcal{E}_n(\Phi) := \int \|\nabla \Phi\|_{H_n}^2 \, d\gamma_n$$

**Corollary***If the distribution of $X_0^n$ converges to the distribution of $X_0$ then $(X_t^n, t \geq 0)$ converges in law to $(X_t, t \geq 0)$.*

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## Ageing in mean-field spin glasses

Anton Bovier
(joint work with G. Ben Arous, J. Černý)

Aging has become one of the main paradigms to describe the long-time behavior of complex and/or disordered systems. Systems that have strongly motivated this research are *spin glasses*. The theoretical modeling of aging phenomena took a major leap with the introduction of so-called *trap models* by Bouchaud and Dean in the early 1990'ies [BD95] (see [BCKM98] for a review from the physics perspective, and [BČ06] for a review of rigorous results). These models reproduce the characteristic power law behavior seen experimentally while being sufficiently simple to allow for detailed analytical treatment. While trap models are heuristically motivated to capture the behavior of the dynamics of spin glass models, there is no clear theoretical, let alone mathematical derivation of these from an underlying spin-glass dynamics.

On a heuristic level, trap models reflect the idea that a stochastic process in a complex random environment will spend most of its time in the vicinity of *deep valley*, repsectively *metastable states*. The long-term behaviour of the process should then be governed by the sequence of transitions between such metastable state. The trap model heuristic then makes the following assumpions:

(i) The relative mean trapping times in different (relevant) metastable states are described by independent random variables.
(ii) The distributions of the mean trapping times are heavy tailed and in the domain of attration of an $\alpha$-stable distribution with $\alpha < 1$.
(iii) The distribution of the trapping times is exponential.

A justification of these assumptions was first attempted in [BBG03a, BBG03b] in the context of the *random energy model*. Here the first asumption poses no problem due to the assumed independence of the original random variables, while assumption (ii) is a consequence of extreme-value theory for iid variable. The proof of the last property required substatial work, involving a perturbative analysis of systems of renewal equations. An extension of these methods appeared to correlated environments appared prohibitively complicated.

In this talk we report on a new approach to this problem, undertaken in [BBČ07] that while not directly addressing the issue (iii), allows to prove ageing in a much simpler way and in particular allowed us to deal with corellated environments. The main idea is to separate the trajectories of the random process and the time spend along the trajectory. The aim is to show that the latter process, called the clock process, converges uder suitble scaling to a stable Lévy subordinator, for almost all trajectories. The point is that this analysis involves only the environment restricted to a single trajectory, which effectively can be seen a one-dimensional Gaussian process indexed by time. In suitable parameter ranges, this can then be analysed using more or less standard techniques from the theory of extremes of one-dimensional Gaussian processes.

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## Hydrodynamics for a non-conservative exclusion type process

Glauco Valle

The basic model of interacting particles system that gives a rough microscopic description of the evolution of the mass density profile of a system with two types of particles with hard-core interaction is the exclusion process. Let $p(\cdot)$ be a finite range symmetric transition probability function on $\mathbb{Z}$. The simple symmetric exclusion process (SSEP) on $\mathbb{Z}$ associated to $p(\cdot)$ is a Feller process with configuration space $\Omega = \{0,1\}^{\mathbb{Z}}$ whose evolution can be described in the following way: Initially each site of $\mathbb{Z}$ is occupied by one and only one particle of two possible types. If two particles of distinct type are at distance $z$ from each other, then, independently of any other pair of particles, after an exponencial time of parameter $p(z)$ they interchange positions. It is well-known that the SSEP has a hydrodynamic behavior under diffusive space-time scaling whose hydrodynamical equation is Laplace's equation.

We propose a basic microscopic model that generalizes the SSEP by allowing a local regular increase of mass in the system and we prove its hydrodynamic behavior. To provide a physically relevant description, we impose that the mass augmentation of first type particles occurs locally by an increase of one unit. When a first type particle enters the system it pushes the other particles to open space in a way that the spread of mass is symmetric around the position where the particle stays. To deal with this behavior, we enlarge the configuration space to $\tilde{\Omega} = \mathbb{N} \times (\Gamma_1 \cup \Gamma_2)$, where

$$\Gamma_1 = \left\{ \eta \in \{0,1\}^{\mathbb{Z} \cup (\mathbb{Z}+\frac{1}{2})} : \eta(x) = 0 \ \forall \ x \in \mathbb{Z} + \frac{1}{2} \right\}$$

and

$$\Gamma_2 = \left\{ \eta \in \{0,1\}^{\mathbb{Z} \cup (\mathbb{Z}+\frac{1}{2})} : \eta(x) = 0 \ \forall \ x \in \mathbb{Z} \right\}$$

for $\mathbb{Z}+\frac{1}{2}=\{x+\frac{1}{2}: x\in\mathbb{Z}\}$. The system alternates between a SSEP on $\mathbb{Z}$ and a SSEP on its dual lattice $\mathbb{Z}+\frac{1}{2}$, where the interchanges are a result of a superposed dynamic describing the increase of mass. This dynamic is described as follows:

(i) Let $h:[0,T]\times\mathbb{R}\to\mathbb{R}_+$ be a smooth bounded function which is directly Riemann integrable in the second variable for each fixed $t\in\mathbb{Z}$. Moreover suppose that

$$\sup_{t\in[0,T]} C(t)<\infty\,, \tag{1}$$

where

$$C(t):=\int_{-\infty}^{+\infty} h(t,u)du\,.$$

(ii) Fix $N>0$ as the scaling parameter and let $s$ denote a fixed macroscopic time. If the state of the system at macroscopic time $s$ is $(n,\eta)\in\tilde{\Omega}$ with $\eta\in\Gamma_1$ (resp. $\eta\in\Gamma_2$), then we have that independently at each site $x\in\mathbb{Z}$ (resp. $x\in\mathbb{Z}+\frac{1}{2}$) and at rate $h(s,x/N)$ the system changes to $(n+1,\eta')$ with $\eta'\in\Gamma_2$ (resp. $\Gamma_1$).

(iii) The configuration $\eta'$ is obtained from $\eta$ in the following way: all the system strictly at the right of site $x$ ($x$ included) is translated of $1/2$ units to the right; all the system strictly at its left is translated of $1/2$ units to the left; we set $\eta'(x)=0$ and $\eta'(x-\frac{1}{2})=1$.

We are only going to consider initial cofigurations on $\tilde{\Omega}$ of the kind $(1,\eta)$ for $\eta\in\Gamma_1$, and therefore the system is always in $\Gamma_1$ for $n$ odd and in $\Gamma_2$ for $n$ even. Thus as mentioned before during each interval of time with $n$ odd (resp. even) the system behaves as a SSEP on $\mathbb{Z}$ (resp. $\mathbb{Z}+\frac{1}{2}$) speeded up by $N^2$. From condition (1), we only have a finite increase of mass in finite time which is required for the system to be well described. In this text, the system we obtain will be called simply the non-conservative exclusion Process (NCEP).

The NCEP has hydrodynamical behavior whose Hydrodynamic equation is the convective diffusion equation

$$\begin{cases} \partial_t\rho(t,u)=\sigma\Delta\rho(t,u)-\frac{1}{2}\partial_u(\gamma(t,u)\rho(t,u))+h(t,u),\ (t,u)\in\mathbb{R}_+\times\mathbb{R},\\ \rho(0,u)=\rho_0(u),\ u\in\mathbb{R}, \end{cases}$$

where

$$\gamma(t,u)=\int_{-\infty}^{u} h(s,v)dv-\int_{u}^{+\infty} h(s,v)dv\,,\quad \sigma=\frac{1}{2}\sum_{z\in\mathbb{Z}} z^2p(z)\,.$$

In order to prove the hydrodynamic behavior of the NCEP we perform two transformations on the configuration space obtaining a system we can deal with more easily. The first transformation is obtained by reversing the roles between zeros and ones (or between first and second type particles). The second transformation makes the process into a SSEP on $\mathbb{Z}$ with superposed dynamics described by setting $N$ as the scaling parameter and $s$ as a fixed macroscopic time, and then for a site $x\in\mathbb{Z}$ at rate $b(s,x/N)$ and independently of any other site the system

from site $x$ is translated to the right, where $b : [0,T] \times \mathbb{R} \to \mathbb{R}_+$ is related to $h(\cdot,\cdot)$ by the equality

$$b(t,u) = h\Big(t, u - \int_0^t \frac{C(s)}{2} ds\Big). \tag{2}$$

The aim of the first transformation is to simplify the dynamics and of the second is to simplify the configuration space obtaining a gradient modified exclusion process. The system we get will be called the modified non-conservative exclusion process (MNCEP).

The intuitive reasoning behind (2) is that the MNCEP is obtained from the NCEP, after the fisrt transformation, by translating the system one half units to the right each time mass increases by one unit and thus $b$ at time $t$ should be the translation of $h$ in the space variable backwards by a half of the total mass that has entered the system before time $t$. Futhermore

$$\int_{-\infty}^{+\infty} b(t,u)du = C(t)\,,$$

and from (1) the MNCEP is also well-defined.

The MNCEP is a system without a proper parametrized family of equilibrium measures, indeed the unique invariant measure is the one concentrated on the empty configuration that with all sites unoccupied. Hence this model adds to the efforts that are been made to extend the results on hydrodynamics to new classes of non-conservative systems and systems without proper parametrized family of equilibrium measures, since for them no general method of proof exists. In other to prove the hydrodynamic limit we shall adapt one of the usual methods of proof to this case. The proof is based on the entropy method of Guo, Papanicolaou and Varadhan [1], see also [2]. Concerning the proof of the hydrodynamic behavior of non-conservative interacting particle systems, we have that similar arguments have been employed in [6], other papers as [3] and [4] also deal with adaptations of the entropy method and in [5] the relative entropy method is applied.

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### A topological glass

Jean-Pierre Eckmann

We propose and study a model with glassy behavior. The state space of the model is given by all triangulations of a sphere with $n$ nodes, half of which are red and half are blue. Red nodes want to have 5 neighbors while blue ones want 7. Energies of nodes with other numbers of neighbors are supposed to be positive. The dynamics is that of flipping the diagonal of two adjacent triangles, with a temperature dependent probability. We show that this system has an approach to a steady state which is exponentially slow, and show that the stationary state is unordered. We also study the local energy landscape and show that it has the hierarchical structure known from spin glasses. Finally, we show that the evolution can be described as that of a rarefied gas with spontaneous generation of particles and annihilating collisions.

### Hydrodynamic behavior and large deviations of boundary driven exclusion processes in dimension $d \geq 1$

Mustapha Mourragui

(joint work with J.S. Farfan Vargas and C. Landim)

A driven lattice gas with open boundaries is a system of particles jumping at random on a lattice, subject to the action of an external field and exchanging matter with a reservoir at his boundary. The combined action of the force field and the density gradient induced by the boundary conditions forces the system to reach a stationary non-equilibrium state. The hydrodynamic behavior starting from the stationary measures, has been derived in the dimension 1, for importante general classes of models ([4], [5], [6], [7], ...). Large deviation properties has been studied for boundary driven one-dimensional symmetric simple exclusion process (([1], [2], [3] and references therein).

We consider boundary driven symmetric exclusion processes with speed change with open boundaries in $d \geq 1$. The system is contained in a finite cylinder $\Omega_N = \{-N+1, \dots, N-1\} \times \pi_N^{d-1}$, with $\pi_N^{d-1}$ the $(d-1)$-dimensional microscopic torus of size $2N+1$ with the axes in the direction $x_1$, namely we impose periodic boundary conditions in all the directions but $x_1$. In the bulk particles jump according to the symmetric exclusion processes with speed change. On the boundaries ($x_1 = -N+1, N-1$) we allow for production and destruction of particles in the following way. Let $b(u)$ be smooth functions on $[-1,1] \times \pi^{d-1}$, where $\pi^{d-1}$ is $(d-1)$-dimensional macroscopic torus of lenght 2. A particle is added independently in each site when the site is empty, with rate $b(x/N)$, and removed independently in each site, when the site is occupied, with rate $(1 - b(x/N))$.

We first pove a law of large numbers starting from the stationary states. We obtain the following elliptic boundary value problem

$$\left\{ \begin{array}{rcl} \Delta\varphi(\rho) & = & 0\,, \\ \rho\big|_\Gamma & = & b \end{array} \right. \tag{1}$$

In our second result concerns, we show the large deviations principle for the empirical measures of the processes described above.

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## Macroscopic description of non equilibrium stationary states and fluctuations


Alberto De Sole

(joint work with Lorenzo Bertini, Davide Gabrielli, Giovanni Jona-Lasinio, Claudio Landim)


### 1. Description of thermodynamic systems out of equilibrium

We consider a macroscopic system (e.g. a gas) in a domain $V$ of the $d$-dimensional space, in contact with boundary reservoirs which will have, in general, different temperatures or chemical potentials. For simplicity, we will assume that the system is described by only one macroscopic variable $\rho$, which we interpret as the local density of particles in the system, and we say that the system is *macroscopically described* if we know the values of $\rho$ in each position $u \in V$.

In order to have such a macroscopic description we need that the system is in *local equilibrium*, namely $\rho$ should be approximately constant, $\rho(v) \simeq \rho(u)$, in all points $v$ at a distance from $u$ which can be very large at the microscopic scale, but which is very small at the macroscopic level. Moreover, we need to assume the existence of a *hydrodynamic limit*: namely when we look at *macroscopic time* intervals $\Delta\tau$, suitably accelerated w.r.t. the typical microscopic time evolution $\Delta t$ (for diffusive systems: $\Delta\tau = e^2\Delta t$), then the system evolves smoothly among situations of local equilibrium. So the macroscopic variable has a "smooth" time

evolution $\rho(\tau, u)$, which for conservative diffusive systems is solution of *hydrodynamic equation* of type

$$\partial_\tau \rho(\tau,u) + \nabla_u \cdot \Big( -\frac{1}{2} D(\rho) \nabla_u \rho(\tau,u) + \chi(\rho) E(u) \Big) = 0\,. \tag{1}$$

Assuming that the diffusion matrix $D$ is positive definite, the above hydrodynamic equation is globally attractive towards its unique *stationary solution* $\bar\rho(u)$.

## 2. Microscopic models: stochastic lattice gases

Stochastic lattice gases provide a large class of models for thermodynamic systems. They are defined on a *configuration space* $\mathcal{H} := X^{V_N}$, where $V_N \subset \mathbb{Z}^d$ is the finite lattice of microscopic points and $X \subset \mathbb{N}$ is the set of possible occupation numbers. A *configuration* $\eta = (\eta_x)_{x\in V_N}$ then describes the number of particles $\eta_x$ in each site $x$ of the lattice $V_N$. A *state* for this model is defined as a probability measure $\mu$ on the configuration space $X^{V_N}$. Finally, the time evolution is specified by saying that in each site $x \in V_N$ each particle, independently one from the other, wait an exponential time of rate $1/2 \sum_y c_{xy}(\eta)$ and then jump on a nearest neighborhood site $y$ according to the distribution described by the rates $\{c_{xy}(\eta)\}_{y\sim x}$.

In an *equilibrium* microscopic model the jump rates $c_{xy}(\eta)$ satisfy the *detailed balance* condition associated to a certain Hamiltonian function $H : X^{V_N} \to \mathbb{R}$ and to a certain value of the chemical potential $\lambda_0 \in \mathbb{R}$ at the boundaries. In order to describe models *out of equilibrium* we modify the jump rates by allowing the chemical potential of the reservoirs to be non constant: $\lambda_0 : \partial V \to \mathbb{R}$, and/or by introducing a (weak) external field which pushes the particles in a certain direction.

If there is sufficient ergodicity, for example if the jump rates are all positive $c_{xy}(\eta) > 0$, it is immediate to check that there is a unique *stationary state* $\bar\mu$, which is globally attractive: $\lim_{t\to\infty} \mu_t = \bar\mu$ for any initial state $\mu_0$. For *equilibrium* models the stationary state is the Gibbs measure w.r.t. the Hamiltonian $H$ and the chemical potential $\lambda_0$, $\mu_{eq}(\eta) \propto e^{-H(\eta)+\lambda_0 \sum_x \eta_x}$, while for *non equilibrium* models the stationary state is not known in general (only for some special models).

In order to talk about the hydrodynamic equation we need to introduce the *empirical measure*. It is a random variable $\pi^N$ which, to a given configuration $\eta$, associates a positive measure on the domain $V$, given by $\pi^N(\eta; du) = \frac{1}{N} \sum_{x\in V_N} \eta_x \delta_{x/N}(du)$. If we then assume that the initial state $\mu_0^N$ is in *local equilibrium* with macroscopic density profile $\rho_0(u)$, and we let the system evolve in time, we can show (for a large class of models) that for every *macroscopic time* $\tau = t/N^2$ the local equilibrium still holds, namely we have a law of large numbers

$$\pi^N(\eta, du) \qquad \xrightarrow[\mu^N_{N^2\tau}]{N\to\infty} \qquad \rho(\tau,u)du$$

where the macroscopic density profile $\rho(\tau, u)$ at time $\tau$ is solution of a hydrodynamic equation of type (1).

## 3. Thermodynamic functionals: free energy

In order to complete the theory of thermodynamic systems out of equilibrium, we need to introduce thermodynamic function(al)s such as free energy. Following the Boltzmann-Einstein theory for thermodynamic fluctuations, we can define the *free energy functional* by the equation

$$P(\rho(u) \simeq r(u),\, u \in V) \propto e^{-\epsilon^{-d}\mathcal{F}(r)}$$

namely the probability of observing the system in a macroscopic fluctuation described by the density profile $\rho(u) = r(u)$, $u \in V$, is exponentially small and the exponent is proportional to the free energy functional $\mathcal{F}(r)$. Notice that, while in equilibrium the free energy $\mathcal{F}$ is *local*, out of equilibrium, typically the free energy functional is a *non local* functional. In particular we can have long range density correlations.

It is quite clear how to formalize in the microscopic models that we are considering, i.e. for stochastic lattice gases, the definition of the free energy functional: by looking at *large deviations*. The *static Large Deviations Principle* states that (and it can be proved in various models of stochastic lattice gases)

$$\bar{\mu}^N(\pi^N(\eta, du) \simeq \rho(u)du, u \in V) \overset{N}{\sim} e^{-N\mathcal{F}(\rho)}$$

where $\simeq$ means closeness in the weak topology, and $\sim$ means logarithmic convergence, while the *dynamical Large Deviations Principle* states that, if we let the system evolve according to the stochastic dynamics, then

$$\mathbb{P}_{\bar{\mu}^N}(\pi^N(\eta_{N^2\tau}, du) \simeq \hat{\rho}(\tau, u)du, u \in V, \tau \in [0,T]) \overset{N}{\sim} e^{-N(\mathcal{F}(\hat{\rho}(0))+J_{[0,T]}(\hat{\rho})}$$

Here the rate function has two contributions: $\mathcal{F}(\hat{\rho}(0))$ is the "cost" of observing at time $\tau = 0$ the fluctuation $\hat{\rho}_0$, while $J_{[0,T]}(\hat{\rho})$ is the "cost" to follow, in the interval $[0,T]$, the fluctuation $\hat{\rho}(\tau)$.

There are few explicit models for which one can prove both the hydrodynamic behavior and the large deviations principles, and for which we can compute explicitly the free energy functional $\mathcal{F}(\rho)$. Among the most studied ones there the Zero Range process, for which the invariant measure is a product measure and hence the free energy is a local functional, and the Simple Exclusion process, for which we don't have an explicit expression for the invariant measure and the free energy functional is non local; It is expressed in terms of the variational problem

$$\mathcal{F}(\rho) = \sup_f \int_0^1 \Big(\rho(u)\log\frac{\rho(u)}{f(u)} + (1-\rho(u))\log\frac{1-\rho(u)}{1-f(u)} + \log\frac{f'(u)}{\rho_1-\rho_0}\Big)du$$

where the sup is over all function $f$ on $[0,1]$ such that $f(0) = \rho_0, f(1) = \rho_1, f' \geq 0$.

## 4. Hamilton-Jacobi equation and long range correlations

By comparing the direct process with the time reverse process we can express the free energy functional $\mathcal{F}(\rho)$ in terms of the variational problem

$$\mathcal{F}(\rho) = \inf_{\hat{\rho}|\hat{\rho}(-\infty)=\bar{\rho},\hat{\rho}(0)=\rho} J_{(-\infty,0]}(\hat{\rho})$$

from which, looking at the general expression of the dynamical rate functional $J_{[T_1,T_2]}$, we derive the associated Hamilton-Jacobi equation

$$\left\langle \nabla \frac{\delta \mathcal{F}}{\delta \rho}, \frac{1}{2}\chi(\rho)\nabla \frac{\delta \mathcal{F}}{\delta \rho} - \mathcal{D}(\rho) \right\rangle = 0$$

Note that the H-J equation does not have a unique solution, but there is a simple selection criterion for the free energy: $\mathcal{F}$ is the maximal among all the solutions of the H-J equation satisfying the appropriate boundary conditions.

The H-J equation has important applications in non-equilibrium thermodynamics. As an example we discuss how to use it to study long range correlations. By definition the *macroscopic density correlation function* $C(u,v)$, $u,v \in V$, is related to the free energy functional by $C^{-1}(u,v) = \frac{\delta \mathcal{F}}{\delta \rho(u)\delta \rho(v)}$. If we then expand the H-J equation to the second order in $\frac{\delta \mathcal{F}}{\delta \rho}$ and we make the change of variable $C(x,y) = C_{eq}(x)\delta(x-y) + B(x,y)$ we get the following equation on $B$

$$\mathcal{L}^\dagger B(x,y) = \alpha(x)\delta(x-y)$$

where $\mathcal{L}^\dagger$ is the adjoint w.r.t. the Lebesgue measure of an elliptic operator $\mathcal{L}$ and $\alpha(x) = \partial_{x_i}(\chi'_{ij}(\bar\rho(x))D^{-1}_{jk}(\bar\rho(x))\bar J_k(x))$. It follows that the sign of $B$ is determined by the sign of $\alpha$: if $\alpha(x) > 0$ (resp. $< 0$), then $B(x,y) < 0$ (resp. $> 0$).

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## Large Deviations Principles for Perturbed Conservation Laws

Mauro Mariani
(joint work with Lorenzo Bertini)

Many physical conservative models are described by the evolution of a scalar *density* $u$ which satisfies a continuity equation $\partial_t u + \text{divergence}(J) = 0$. In many cases, the *current* $J$ takes into account the basic phenomena of transport, diffusion and fluctuation. Each of these terms will in general depend on the variable $u$ itself. In these paper we are interested in the the limit in which diffusion and fluctuation vanish simultaneously. We thus introduce a Cauchy problem for the scalar $(1+1)$-dimensional stochastic PDE

$$\begin{cases} du^\varepsilon = -\Big[f(u^\varepsilon) + \frac{\varepsilon}{2}\big(D(u^\varepsilon)u^\varepsilon_x\big)_x\Big]dt + \varepsilon^\mu \left[\vartheta(u^\varepsilon)(\jmath^\varepsilon * dW)\right]_x \\ u^\varepsilon(0,x) = u_0(x). \end{cases} \tag{1}$$

where the unkown $u^\varepsilon = u^\varepsilon(t,x)$ is a scalar quantity, the time variable $t$ runs on a finite interval $[0,T]$, the space variable $x$ runs on the one-dimensional torus $\mathbb{T}$ and the subscrit $x$ stands for derivative w.r.t. the space variable. In (1) the functions $f$, $D$, $\vartheta$ are smooth with $D$ uniformly positive; $\mu$ is a real parameter $\mu > 1/2$; $W$ is cylindrical Browniam motion, and $\{\jmath^\varepsilon\}$ is a sequence of mollifiers on $\mathbb{T}$ acting by convolution. More precisely, the term $\left[\vartheta(u^\varepsilon)(\jmath^\varepsilon * dW)\right]_x$ in (1) represents the infinitesimal variation of a martingale $M^\varepsilon$ acting on $H^1(\mathbb{T})$ with quadratic variation

$$d\big[M^\varepsilon_\cdot(\varphi), M^\varepsilon_\cdot(\psi)\big] = dt\,\big(\jmath^\varepsilon * [\vartheta(u^\varepsilon)\phi_x], \jmath^\varepsilon * [\vartheta(u^\varepsilon)\psi_x]\big)$$

where $(\cdot,\cdot)$ stands for the inner product in $L_2(\mathbb{T})$. We refer to [1] for a more precise description of this equation and the main results stated below. Under general assumptions, it is possible to show existence and uniqueness of the problem (1) in a strong sense.

Motivated by interacting particle systems heuristics, we are interested in the analysis of (1) in the limit $\varepsilon \to 0$. Indeed in the hydrodynamical limit under Euler scaling, the empirical density of asymmetric particle systems can be roughly be approximated by a continuous density satisfying an evolution equation with vanishing diffusion and noise. From this point of view, the parameter $\varepsilon$ should be considered as the inverse of the number of particles in the system. A much studied model in interacting particles systems is the Totally Asymmetric Simple Exclusion Process (TASEP), see [5]. Hydrodynamical limits for this process are investigated in [6, 7], and a Large Deviations Principle (LDP) is investigated in [4, 8]. The TASEP formally corresponds to the case $f(u) = \vartheta^2(u) = u(1-u)$.

We are thus interested in convergence properties and Large Deviations Principles for (1) in the hydrodynamical-like limit $\varepsilon \to 0$. A first result in [1] states that the process $u^\varepsilon$ solution to (1) converges in a strong $L_p$-sense to the entropic

solution to

$$\begin{cases} \partial_t u + f(u)_x = 0 \\ u(0,x) = u_0(x) \end{cases} \tag{2}$$

As well known [3], the problem (2) does not admit classical solutions, while measure-valued or weak solutions are in general non-unique. An entropic condition is then generally required on the solution $u$ to (2) in order to ricover uniqueness.

Once convergence is established, we are intersted in LDP for the law $\mathbb{P}^\varepsilon$ of $u^\varepsilon$. From a technical point of view, the main issue here is that the drift term in (1) has a non-trivial limiting behaviour as $\varepsilon \to 0$. In particular, Friedlin-Wentcell like techniques do not work here. The methods introduced to investigate the LDP for this kind of equations involve a variational analysis of suitable (deterministic) functionals. The study of these functionals in terms of $\Gamma$-convergence is carried out in [2].

In [1] a LDP on the scale $\varepsilon^{2\mu}$ is established. The precise LDP statement is obtained in a Young-measures setting, and concerns the lift of $\mathbb{P}^\varepsilon$ to a suitable space of Young-measures. The corresponding LD rate functional basically quantifies, in terms of a suitable Hilbert norm, how much a Young-measure "does not satisfies" (2). In particular this rate functional vanishes on measure-valued solutions to (2). On the other hand, $u^\varepsilon$ converges in probability to the unique entropic solution to (2), and one needs therefore to study a second-order LDP.

The second-order LDP is investigated on the scale $\varepsilon^{2\mu-1}$. The candidate rate functional for this LDP is finite only on (a suitable class of) weak solutions to (1), and on such solutions it quantifies, in terms of $D$ and $\vartheta^2$, how much the entropic admissibility condition is violated by a weak solution to (2). In particular, the entropic solution to (1) is the only solution with vanishing second-order rate functional. Note however that, while the LDP upper bound is established in [2], lower bound results are partial.

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## Gibbs measures on Brownian currents

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(joint work with József Lőrinczi)

The subject of Gibbs measures (or better Gibbsian specifications) is a well developed subject in the context of discrete models like lattice spin systems (with continuous or discrete state space) or point processes. Applications to Quantum Mechanics and Statistical Physics sometimes require the understanding of general Gibbsian specifications over trajectories of diffusion processes. A general study of Gibbsian specifications over Brownian motion has been initiated by Spohn, Betz, Lőrinczi, Hiroshima in a series of papers [1, 7, 2, 5, 6, 7] where they consider essentially specifications associated to the energy $H_I$ in the form

$$H_I(X|Y) = \int_I V(X_t)dt + \int_I\int_I W(X_t - X_s, t-s)dtds + \int_I\int_{I^c} W(X_t - Y_s, t-s)dtds$$

where $I \subset \mathbb{R}$ is an interval, $X$ the trajectory inside $I$ and $Y$ a trajectory outside $I$ and $V, W$ are functions called respectively *pinning* potential and *interaction* potential. The kernels of the specification are given by

$$\rho_I(dX|Y) = \frac{e^{-H_I(X|Y)}}{Z_I(Y)}\mathcal{W}_I(dX|Y)$$

with $\mathcal{W}_I(dX|Y)$ the Wiener measure on $C(I, \mathbb{R}^d)$ with boundary condition $Y$ (i.e. if $I = [a, b]$ then under $\mathcal{W}_I(dX|Y)$ the path $X$ is conditioned to have $X_a = Y_a$ and $X_b = Y_b$). Then a Gibbs measure is a measure $\mu$ over continuous trajectories $\mathcal{X} = C(\mathbb{R}, \mathbb{R}^d)$ consistent with any $\rho_I$, $I \subset \mathbb{R}$: i.e. such that

$$\int\int f(X)\rho_I(dX|Y)\mu(dY) = \int f(X)\mu(dX).$$

In a recent paper [4] we studied another kind of Gibbs measures where the energy is now (formally) given by

$$H_I(X|Y) = \int_I V(X_t)dt + \int_I\int_I W_{\alpha\beta}(X_t - X_s, t-s)dX_t^\alpha dX_s^\beta + \int_I\int_{I^c} W_{\alpha\beta}(X_t - Y_s, t-s)dX_t^\alpha dX_s^\beta$$

where the stochastic integrals have to be understood in Itô sense. We showed that in order to have a natural setting for the concept of specification and Gibbs measure for this kind of energies one has to replace the configuration space $\mathcal{X}$ by $\mathcal{C} = \mathcal{X} \times C(\mathbb{R}^2, \mathcal{D}')$ where $\mathcal{D}'$ is a space of *currents*: continuous functionals over (a suitable class $\mathcal{D}$ of) time-dependent vectorfields on $\mathbb{R}^d$. An element of this new configuration space is then a couple $(x, C)$ where $x \in \mathcal{X}$ is the trajectory of the particle and $C \in C(\mathbb{R}^2, \mathcal{D}')$ can be though as giving the value of the work made by some test vectorfield $\varphi$ during the time interval $[s, t]$:

$$C_{ts}(\varphi) = \int_s^t \varphi_\alpha(u, x_u)dx_u^\alpha$$

When the trajectory is $C1$ this last quantity is completely determined by $x$, but this is not the case when the trajectories are samples drawn from Wiener measure. We prove that it is possible to lift the Wiener measure $\mathcal{W}$ on $\mathcal{X}$ to $\mathcal{C}$ and that we can construct a specification $\rho^\sharp$ on this extended configuration space for a large class of potentials $V, W$. Moreover by means of cluster expansion methods we can prove that under some decay and smallness condition for $W$ there exists a unique Gibbs measure $\mu^\sharp$ on $\mathcal{C}$ compatible with the specification $\rho^\sharp$.

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## Strict Convexity of the Surface Tension for Anharmonic Crystals


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(joint work with Codina Cotar, Stefan Müller)


We consider an an effective model with gradient interaction. The model describes a phase separation in $\mathbb{R}^{d+1}$, eg. between the liquid and vapor phase. For simplicity we consider a discrete basis $\Lambda_N \subset \mathbb{Z}^d$, and continuous height variables

$$x \in \Lambda_N \longrightarrow \phi(x) \in \mathbb{R}.$$

This model ignores overhangs like in Ising models, but gives a good approximation in the vicinity of the phase separation. The distribution of the interface is given in terms of its Gibbs distribution with nearest neighbor interactions of gradient type, that is, the interaction between two neighboring cites $x, y$ depends only on the discrete gradient,$\nabla\phi(x, y) = \phi(y) - \phi(x)$. More precisely, the Hamiltonian is of the form

$$H_N(\phi) = \sum_{x,y\in V_{N+1}, |x-y|=1} V(\phi(y) - \phi(x)) \tag{1}$$

where $V \in C^2(\mathbb{R})$ is an even function with quadratic growth at infinity:

$$V(\eta) \geq A|\eta|^2 - B, \qquad \eta \in \mathbb{R} \tag{2}$$

for some $A > 0, B \in \mathbb{R}$.

For given boundary condition $\psi \in \mathbb{R}^{\partial V_N}$ where $\partial V_N = V_{N+1} \setminus V_N$, the (finite) Gibbs distribution on $\mathbb{R}^{V_{N+1}}$ at inverse temperature $\beta > 0$ is given by

$$\mu_{V_N,\psi,\beta}(d\phi) \equiv \frac{1}{Z_{N,\psi,\beta}} \exp(-\beta H_N(\phi)) \prod_{x \in V_N} d\phi(x) \prod_{x \in \partial V_N} \delta_{\psi(x)}(d\phi(x)).$$

Here of course $Z_{N,\psi,\beta}$ is a normalizing constant such that

$$Z_{N,\psi,\beta} = \int_{\mathbb{R}^{V_{N+1}}} \exp(-\beta H_N(\phi)) \prod_{x \in V_N} d\phi(x) \prod_{x \in \partial V_N} \delta_{\psi(x)}(d\phi(x)).$$

One is particularly interested in tilted boundary conditions

$$\psi_u(x) =< x, u >= \sum_{i=1}^{d} x_i u_i$$

for some given 'tilt' $u \in \mathbb{R}^d$. This corresponds to an interface in $\mathbb{R}^{d+1}$ which stays normal to the vector $n_u = (u, -1) \in \mathbb{R}^{d+1}$.

An object of basic relevance in this context is the surface tension or free energy defined by the limit

$$\sigma(u) = \lim_{N \to \infty} -\frac{1}{|V_N|} \log \frac{Z_{N,\psi_u,\beta}}{Z_{N,0,\beta}}. \tag{3}$$

The existence of the above limit follows from a standard sub-additive argument. In case of *strictly* convex potential $V$ with

$$c_1 \leq V'' \leq c_2 \tag{4}$$

where $0 < c_1 \leq c_2 < \infty$, Funaki and Spohn showed in [1] that $\sigma$ is strictly convex.

The simplest strictly convex potential is the quadratic one with $V(\eta) = |\eta|^2$, which corresponds to a Gaussian model, also called gradient free field or harmonic crystal. Models with non quadratic potentials $V$ are sometimes called anharmonic crystals.

The strict convexity of the surface tension $\sigma$ plays a crucial role in the derivation of the hydrodynamical limit of the Landau-Ginsburg model in [1]. Under the condition (4), a large deviation principle for the rescaled profile with rate function given in terms of the integrated surface tension has been derived in [2]. Here also the strict convexity of $\sigma$ is very important. Both papers [1] and [2] use very explicitly the condition (4) in their proof. In particular they rely on the Brascamp Lieb inequality and on the random walk representation of Hellfer and Sjoestrand. which requires strict convex potential $V$.

The objective of our work is to prove strict convexity of $\sigma$ also for some non convex potential $V$. Our result is perturbative at high temperature (small $\beta$), and shows strict convexity of $\sigma(u)$ at every $u \in \mathbb{R}$ for potentials $V$ of the form

$$V(\eta) = V_0(\eta) + g(\eta)$$

where $V_0$ satisfies (4) and $g \in C^2(\mathbb{R})$ has a bounded second derivative such that $\sqrt{\beta} \cdot \|(g'')_-\|_{L^1(\mathbb{R})}$ is small enough.

Our proof is based on the scale decomposition of the free field as the sum of two independent free fields $\phi_1$ and $\phi_2$, where we choose the variance of $\phi_1$ small enough to match the non-convexity of $g$. The idea behind the proof is that one can gain convexity via integration!

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## Some results on the relaxation to equilibrium for a polymer with adsorption and repulsion

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(joint work with Fabio Martinelli, Fabio Lucio Toninelli)

Consider simple random walk paths on $\mathbb{Z}$ which start at 0 and end at 0 after $L$ steps, where $L$ is an even integer, i.e. elements of

$$\Omega_L = \{\eta \in \mathbb{Z}^{L+1} : \ \eta_0 = \eta_L = 0\,,\ |\eta_{x+1} - \eta_x| = 1\,,\ x = 0, \dots, L-1\}\,.$$

A well known polymer model (the *pinning model*) is obtained by assigning to each path $\eta \in \Omega_L$ a weight

$$\lambda^{N(\eta)}\,, \tag{1}$$

where $\lambda > 0$ is a parameter and $N(\eta)$ stands for the number of $x \in \{1, \dots, L-1\}$ such that $\eta_x = 0$, i.e. the number of *pinned* sites. If $\lambda > 1$ the weight (1) favors pinning of the path whereas if $\lambda < 1$ pinning is penalized. The case $\lambda = 1$ is referred to as the free case. Normalizing the weights (1) one has a probability measure $\mu = \mu_L^\lambda$ on the set $\Omega_L$ of all $\binom{L}{L/2}$ paths. This defines our first polymer model.

The second model is obtained by considering only paths that stay non–negative, i.e. elements of

$$\Omega_L^+ = \{\eta \in \Omega_L : \ \eta_x \geq 0\,,\ x = 1, \dots, L-1\}\,.$$

Normalizing the weights (1) one obtains a probability measure $\mu^+ = \mu_L^{+,\lambda}$ on the set $\Omega_L^+$ of all $\frac{2}{L+2}\binom{L}{L/2}$ non–negative paths. The positivity constraint will be often referred to as the presence of a *wall*.

The two models introduced above have been studied for several decades and very precise information is available on their asymptotic properties as $L$ becomes large. The reader is referred to the recent review [2] for more details. For the moment let us briefly recall that both models display a transition from a *delocalized* phase to a *localized* phase as $\lambda$ is increased. Namely, the following familiar scenario holds. For the system without the wall, if $\lambda \leq 1$ paths are delocalized (as in the free case $\lambda = 1$) with $|\eta_{L/2}|$ typically of order $\sqrt{L}$ and a vanishing density of pinned sites,

while as soon as $\lambda > 1$ paths are strongly localized with $|\eta_{L/2}|$ typically of order 1 with a positive density of pinned sites. The system with the wall has a similar behavior but the critical point is $\lambda = 2$ instead of $\lambda = 1$. Namely, due to the entropic repulsion induced by the wall, a small reward for pinning (as in the case $1 < \lambda \leq 2$) is not sufficient to localize the path.

These models and generalizations thereof, where the simple-random-walk paths are replaced by trajectories of more general Markov chains, are popular tools in the (bio)-physics literature to describe, e.g., pinning of polymers on defect lines in diffent dimensions, the Poland-Scheraga model of DNA denaturation, wetting models,...(we refer for instance to [2, Chap. 1] and references therein).

We are interested in the asymptotic behavior of a continuous time Markov chain naturally associated with these models. The process is described as follows. Independently, each site $x \in \{1, \ldots, L-1\}$ waits an exponential time with mean 1 after which the variable $\eta_x$ is updated according to the usual heat bath prescription.

The Markov chain is reversible with respect to $\mu$ for the system without the wall and it is reversible with respect to $\mu^+$ for the system with the wall. We shall study the speed at which the equilibria $\mu$ and $\mu^+$ are approached mostly by way of estimates on the *spectral gap* and the total variation *mixing time*. We refer to [1] for more background, and recall here that the inverse of the spectral gap (also known as relaxation time) measures convergence in the $L^2$–norm with respect to the equilibrium measure, while the mixing time measures convergence in total variation norm starting from the worst–case initial condition.

While essentially everything is known about the equilibrium properties of these polymer models, we feel that there is still much to understand as far as the approach to equilibrium is concerned. In particular, one would like to detect the dynamical signature of the phase transition recalled above. Our work is a first attempt in this direction. Before going to a description of our main results, we discuss some earlier contributions.

The problem is well understood in the free case $\lambda = 1$. In particular, for the system without the wall, the free case is equivalent to the so–called symmetric simple exclusion process which has been analyzed by several authors. We refer to the work of Wilson [4], where among other things the spectral gap of the chain is computed exactly as

$$\kappa_L = 1 - \cos\left(\frac{\pi}{L}\right) , \tag{2}$$

the principal eigenvalue of the discrete Laplace operator with Dirichlet boundary conditions, and the mixing time $T_{\text{mix}}$ is shown to be of order $L^2 \log L$ (with upper and lower bounds differing only by a factor 2 in the large $L$ limit).

As far as we know, Martin and Randall [3] is the only work where the dynamical problem for all $\lambda > 0$ was considered. They showed there is always a polynomial upper bound on the mixing time of the chain.

Our main results can be summarized as follows. We refer to [1] for the precise statements. We start with the system with the wall. A first result here is that for all $\lambda > 0$, the spectral gap is bounded below by the gap (2) of the free case, i.e.

$gap \geq \kappa_L \sim \pi^2/2L^2$. Also, we prove that for all $\lambda > 0$ the mixing time satisfies $T_{\mathrm{mix}} = O(L^2 \log L)$. Furthermore we can prove that these estimates are optimal (up to constant factors) in the delocalized phase, i.e. we can exhibit complementary bounds for $\lambda \leq 2$ on the gap and for $\lambda < 2$ on the mixing time. In the localized phase ($\lambda > 2$) we expect the relaxation to occur faster than in the free case. However, we prove a general lower bound on the mixing time giving $T_{\mathrm{mix}} = \Omega(L^2)$. Concerning the spectral gap we show an upper bound $gap = O(L^{-1})$. We conjecture these last two estimates to be of the correct order but a proof of the complementary bounds remains a challenging open problem (except for $\lambda = \infty$, where we can actually prove that $c_1 L^2 \leq T_{\mathrm{mix}} \leq c_2 L^2$).

The fact that the mixing time grows in every situation at least like $L^2$ does not exclude that, starting from a particular configuration, the dynamics can relax to equilibrium much faster. In the localized phase we can explicitly identify such a configuration and show that the dynamics started from it relaxes within a time of order $(\log L)^3$.

Concerning the system without the wall we can show that for all $\lambda > 1$ the relaxation is at least as fast as in the free case, i.e. $gap \geq \kappa_L$ and $T_{\mathrm{mix}} = O(L^2 \log L)$. However, for $\lambda > 1$ we believe the true behavior to be the same as described above for $\lambda > 2$ in the presence of the wall. On the other hand, the case $\lambda < 1$ is very different from the system with the wall. Here we prove that the spectral gap is no larger than $O(L^{-5/2})$, up to logarithmic corrections, establishing a clear dynamical transition from localized to delocalized phase. Describing the correct asymptotics of the gap (and of the mixing time) for $\lambda < 1$ remains an open problem, although a heuristic argument (see [1]) suggests that the $O(L^{-5/2})$ behavior may well be the correct one.

Finally, beyond focusing on global quantities like gap and mixing time, it is of interest to study how local observables like, e.g., the local height function $\eta_x$, relax to equilibrium. This question is particularly meaningful in the localized phase, where the infinite-volume equilibrium measure exists. As a consequence of the fact that the spectral gap vanishes for $L \to \infty$ as an inverse power of $L$, we show upper and lower bounds of stretched exponential type for the relaxation of local functions.

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## The quenched critical point of a diluted disordered polymer model

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(joint work with Francesco Caravenna, Béatrice de Tilière)

The issue addressed is the determination of the quenched critical point for the localization/delocalization phase transition of a polymer interacting with an attractive wall through a diluted disordered potential. The model we consider was first introduced by Bodineau and Giacomin in [1], as a *reduced model* for the so-called *copolymer near a selective interface model* [2], with the hope that it could have the same behavior as the full copolymer model, in the limit of weak coupling constants. Our main result shows that this is not the case.

Let $S = \{S_n\}_{n\geq 0}$ be the standard simple symmetric random walk on $\mathbb{Z}$, and denote by $P$ its law. For $N \in \mathbb{N}$ we denote by $P_N^+(\,\cdot\,) = P(\,\cdot\,|S_n \geq 0,\ \forall n \leq N)$ the law of the random walk conditioned to stay non-negative up to time $N$. The trajectories $\{(n, S_n)\}_{0\leq n\leq N}$ under $P_N^+$ model the configurations of a polymer chain of length $N$ above an *impenetrable wall*.

The interaction of the polymer with the wall is tuned by two parameters $\beta \geq 0$ and $p \in [0, 1]$. For fixed $\beta$ and $p$, we introduce a sequence $\omega = (\omega_n)_{n\geq 1}$ of i.i.d. random variables, taking values in $\{0, \beta\}$ and with law $\mathbb{P}$ given by:

$$\mathbb{P}(\omega_1 = \beta) = p, \quad \mathbb{P}(\omega_1 = 0) = 1 - p. \tag{1}$$

We define our model: for a fixed (typical) realization $\omega$ and $N \in \mathbb{N}$, we introduce the probability measure $P_{N,\omega}^{\beta,p}$ defined by

$$\frac{dP_{N,\omega}^{\beta,p}}{dP_N^+}(S) \stackrel{\text{def}}{=} \frac{1}{Z_{N,\omega}^{\beta,p}} \exp\left(\sum_{n=1}^{N} \omega_n \mathbf{1}_{\{S_n=0\}}\right), \tag{2}$$

where $Z_{N,\omega}^{\beta,p}$ is the usual partition function, i.e. the normalization necessary that $P_{N,\omega}^{\beta,p}$ is a probability measure.

We focus on the regime of large $\beta$ and small $p$: then $\omega$ represents a random sequence of charges sitting on the wall (i.e. on the $x$-axis), which are rare, but of strong intensity, and which attract the polymer. We are interested in the behavior of the polymer measure in the limit of large $N$: in particular, we want to understand if the attractive effect of the *environment* $\omega$ is strong enough to pin the polymer at the wall (*localization*), or if it is still more convenient for the polymer to wander away from it (*delocalization*), as it happens when there are no charges. It is clear that there is a competition between energy and entropy.

We define the free energy of the model by

$$f(\beta, p) \stackrel{\text{def}}{=} \lim_{N\to\infty} \frac{1}{N} \log Z_{N,\omega}^{\beta,p}. \tag{3}$$

The existence of this limit, both $\mathbb{P}(d\omega)$–a.s. and in $L^1(\mathbb{P})$, and the fact that $f(\beta, p)$ is non-random are proven in [5] via super-additivity arguments. By restricting to the set of random walk trajectories that stay strictly positive until epoch $N$ one

has

$$Z_{N,\omega}^{\beta,p} \geq \frac{\frac{1}{2}P\big(S_i \geq 0,\ 1 \leq i \leq N-1\big)}{P\big(S_i \geq 0,\ 1 \leq i \leq N\big)} \to \frac{1}{2},$$

i.e. $f(\beta,p) \geq 0$ for all $\beta, p$. Since this bound has been obtained by ignoring the contribution of the paths that touch the wall, it is customary to partition the phase space into:

- the Localized region $\mathcal{L} \stackrel{\text{def}}{=} \{(\beta,p) :\ f(\beta,p) > 0\}$
- the Delocalized region $\mathcal{D} \stackrel{\text{def}}{=} \{(\beta,p) :\ f(\beta,p) = 0\}$.

By a standard coupling on the environment, it is clear that the function $p \mapsto f(\beta,p)$ is non-decreasing. Therefore for every $\beta \geq 0$ there exists a critical value $p_c(\beta) \in [0,1]$ such that our model is localized for $p > p_c(\beta)$ and delocalized for $p < p_c(\beta)$ (in fact for $p \leq p_c(\beta)$, since the function $f(\beta,c)$ is continuous).

One may ask to what extent the definition of (de)localization given above in terms of the free energy corresponds to a real (de)localized behavior of the typical paths of $P_{N,\omega}^{\beta,p}$. Let us just mention that, by convexity arguments, one can prove that when $(\beta,p) \in \mathcal{L}$ the typical paths of $P_{N,\omega}^{\beta,p}$ for large $N$ touch the wall a positive fraction of time, while this does not happen when $(\beta,p)$ are in the interior of $D$. We will not focus on path properties in this paper: for deeper results, we refer to [5].

Some bounds on $p_c(\beta)$ can be obtained quite easily, as it is shown in [1].

**Lemma 1**

$$-\liminf_{\beta\to\infty} \frac{1}{\beta} \log p_c(\beta) \leq 1\,.$$

The proof can be given by a simple annealed computation. More important for us is the following result which is also proved in [1]. As it is important for our own results, we sketch the argument.

**Lemma 2**

$$-\limsup_{\beta\to\infty} \frac{1}{\beta} \log p_c(\beta) \geq \frac{2}{3}\,.$$

*Proof.* We have to bound the partition function from below. To this aim, we compute the contribution of the set of trajectories that touch the wall whenever there is a non-zero charge (on even sites, because of the periodicity of the random walk). Let $\Omega_N^\omega$ be the set of paths $\{S_n\}_{n\leq N}$ which visit every even site $n \in 2\mathbb{N}$ where $\omega_n > 0$. We also write $\xi_n$ for the location of the $n$-th positive $\omega$-charge sitting on an even site. We denote by $\iota_N := \max\{k \geq 0 :\ \xi_k \leq N\}$ the number of positive charges (sitting on even sites) up to $N$. Finally, we introduce the distribution of the first return time to zero of the simple random walk restricted to the non-negative half-line:

$$K^+(n) \stackrel{\text{def}}{=} P\big(S_i > 0,\ 1 \leq i \leq n-1,\ S_n = 0\big) \tag{4}$$

(observe that $K^+(n) = 0$ for $n$ odd) and we recall that with $C \stackrel{\text{def}}{=} 1/2\sqrt{\pi}$

$$K^+(2n) \stackrel{n\to\infty}{\sim} \frac{C}{n^{3/2}}, \quad \sum_{n\in\mathbb{N}} K^+(2n) = \frac{1}{2}\,. \tag{5}$$

Then we have

$$\begin{aligned} Z^{\beta,p}_{N,\omega} &\geq E^+_N\left(\exp\left[\sum\nolimits_{n=1}^N \omega_n \mathbf{1}_{\{S_n=0\}}\right]\,\mathbf{1}_{\{S\in\Omega^\omega_N\}}\right) \\ &= \frac{1}{P(S_1\geq 0,\dots,S_N\geq 0)}\cdot \mathrm{e}^{\beta\,\iota_N} \\ &\qquad\qquad \cdot\left(\prod_{\ell=1}^{\iota_N} K^+\left(\xi_\ell-\xi_{\ell-1}\right)\right)\cdot\left(\sum_{n=N-\iota_N+1}^{\infty} K^+(n)\right)\,. \end{aligned}$$

$\{(\xi_\ell - \xi_{\ell-1})/2\}_{\ell\geq 1}$ are independent geometrically distributed random variables with parameter $p$. Therefore $\iota_N/N \to p/2$ as $N\to\infty$, $\mathbb{P}(\mathrm{d}\omega)$–a.s.. Applying again the strong law of large numbers, from the last equation we get $\mathbb{P}(\mathrm{d}\omega)$–a.s.

$$\lim_{N\to\infty}\frac{1}{N}\,Z^{\beta,p}_{N,\omega} \;\geq\; \frac{p}{2}\left(\beta + \mathbb{E}\log K^+(\xi_1)\right).$$

Using (5) and Jensen's inequality we have with some constant $c_1 > 0$

$$\begin{aligned} \lim_{N\to\infty}\frac{1}{N}\,Z^{\beta,p}_{N,\omega} &\geq \frac{p}{2}\left(\beta + \log c_1 - \frac{3}{2}\mathbb{E}\log\xi_1\right) \\ &\geq \frac{p}{2}\left(\beta + \log c_1 - \frac{3}{2}\log\mathbb{E}\xi_1\right). \end{aligned}$$

Since $\mathbb{E}\xi_1 = 2p^{-1}$, we get with some constant $c_2$

$$\lim_{N\to\infty}\frac{1}{N}\,Z^{\beta,p}_{N,\omega} \;\geq\; \frac{p}{2}\left(\beta + \frac{3}{2}\log p + c_2\right),$$

so that

$$p \geq \mathrm{e}^{-\frac{2}{3}(\beta+c_2)} \Longrightarrow (\beta,p)\in\mathcal{L}\,,$$

and the lemma is proven. □

We can summarize the two lemmas by stating that for $\beta$ large we have

$$p_c(\beta) \asymp \mathrm{e}^{-c_{\mathrm{red}}\,\beta}$$

where $\frac{2}{3} \leq c_{\mathrm{red}} \leq 1$. Our main result is that in fact $c_{\mathrm{red}} = \frac{2}{3}$:

**Theorem**

*For every $c > \frac{2}{3}$ there exists $\beta_0 = \beta_0(c)$ such that*

$$f(\beta, \mathrm{e}^{-c\,\beta}) = 0,\ \forall\beta\geq\beta_0,$$

*i.e. $(\beta, \mathrm{e}^{-c\,\beta})\in\mathcal{D}$ for $\beta\geq\beta_0$. Therefore*

$$-\lim_{\beta\to\infty}\frac{1}{\beta}\log p_c(\beta) = \frac{2}{3}.$$

Let us discuss some consequences of this Theorem. We recall that our model $\mathbf{P}^{\beta,p}_{N,\omega}$ was first introduced in [1], as a simplified version ('reduced model') of the

so-called *copolymer near a selective interface model*, cf. [1] (see also [5] for a recent overview). It is known that the copolymer model undergoes a localization/delocalization phase transition. An interesting object is the *critical line* separating the two phases, in particular in the limit of weak coupling constants, where it becomes a straight line with positive slope $C_{cop}$.

A lot of effort has been put in finding the exact value of $C_{cop}$. This is motivated by the fact that $C_{cop}$ appears to be a *universal* quantity: it is independent of the law of the environment [6] and it determines the phase transition of a continuous copolymer model, arising as the scaling limit of the discrete one [2]. Up to now what is known is that $\frac{2}{3} \leq C_{cop} \leq 1$. Notice that $\frac{2}{3}$ and 1 are exactly the same bounds that were previously known for $c_{red}$, and this is not a case: in fact the definition of our model $\mathbf{P}_{N,\omega}^{\beta,p}$ has been inspired by the strategy behind the proof of $C_{cop} \geq \frac{2}{3}$, cf. [1].

The reason for introducing a reduced model was to have a more tractable model, which should possibly have the same behavior as the full copolymer model in the limit of weak coupling constants, i.e. for which possibly $c_{red} = C_{cop}$. However, the numerical results obtained in [4] provide strong indications for the fact that $C_{cop} > \frac{2}{3}$. If this is indeed the case, our result shows that the reduced model does not catch the full complexity of the copolymer model, i.e. the 'missing free energy' should come from a different strategy than the one which is at the basis of the lower bound $C_{cop} \geq \frac{2}{3}$.

By the Theorem, our model provides also a non-trivial example of a linear chain pinning model where, for large $\beta$, the quenched critical point $p_c(\beta)$ is *different from the annealed one* $p^a(\beta) = 1/(\mathrm{e}^\beta - 1)$ (see the proof of Lemma 2)

Our proof of the Theorem relies on *quenched* arguments, based on a rigorous *renormalization procedure* (somewhat in the spirit of [7]). The idea is to remove from the environment sequence $\{\omega_n\}_n$ the positive charges that are well-spaced (that therefore give no sensible contribution to the partition function) and to cluster together the positive charges that are very close. This procedure produces a new environment sequence $\{T(\omega)_n\}_n$, which has fewer charges but of stronger intensity. The key point is that replacing $\omega$ by $T(\omega)$ in the partition function yields an upper bound on the free energy. Then, by iterating the map $T$ several times, we obtain environment sequences for which the free energy can be estimated and shown to be arbitrarily small.

The same result has recently be obtained by Toninelli [9] with a simpler argument, avoiding the renormalization procedure we apply. We however believe that our direct procedure, eliminating "bad" regions in a recursive way for the quenched law, is of some value for other problems, e.g. for proving that the tangent of the critical line at 0 in the "true" is smaller than 1, but this is an open question up to now.

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