Mathematisches Forschungsinstitut Oberwolfach

Report No. 13/2010

DOI: 10.4171/OWR/2010/13

C*-Algebren

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March 7th – March 13th, 2010

Abstract. The theory of C*-algebras plays a major rôle in many areas of modern mathematics, like Non-commutative Geometry, Dynamical Systems, Harmonic Analysis, and Topology, to name a few. The aim of the conference "C*-algebras" is to bring together experts from all those areas to provide a present day picture and to initiate new cooperations in this fast growing mathematical field.

Mathematics Subject Classification (2000): 46Lxx, 19Kxx, 37-xx, 22Dxx, 20G42, 81R50, 81R60.

Introduction by the Organisers

A (unital) C*-algebra is a non-commutative analogue of the space of continuous functions on a compact Hausdorff space. Since a compact Hausdorff space can be recovered as the maximal ideal space of the algebra of continuous functions, non-commutative C*-algebras represent virtual "non-commutative" spaces. The theory of C*-algebras goes back to work of Murray and von Neumann, who first studied a special variant now known as von Neumann algebras (which represent the non-commutative measure theory). The theory developed rapidly after some ground breaking work of Gelfand and Naimark in 1943. In the 70's and 80's of the last century, the point of view that non-commutative C*-algebras should be regarded as function spaces of "non-commutative" topological spaces became more and more a central theme of the theory. As a consequence, completely new areas in mathematics, like Non-commutative Geometry or Free Probability evolved and we now see that the theory of C*-algebras became a very active field with applications in and interactions with almost all areas of modern mathematics.

The aim of the workshop *C*-algebras*, organized by Claire Anantharaman-Delaroche, Siegfried Echterhoff, Mikael Rørdam, and Dan Voiculescu, is to bring together leading researchers from basically all areas related to the field. This gives a unique opportunity to maintain a broad view on the subject and to create new cooperations between researchers with different background. Among the 52 participants was a good number of young researchers, some of them already on the top of the field. There have been 27 lectures presented at the workshop with topics ranging from classification of C*-algebras, group actions on C*-algebras, orbit equivalence of dynamical systems, subfactor theory of von Neumann algebras, C*-algebras and logic, continuous fields of C*-algebras, group C*-algebras, C*-algebras in Quantum Field Theory, C*-algebras related to number theory, free probability, the relation between C*-algebras and Harmonic Analysis, and many others.

The abstracts presented in this report clearly show that there has been very exciting progress in many of the above mentioned areas. Much of this progress comes from interactions with other fields of mathematics. As an example for this we want to mention the beautiful results of Farah and Weaver on some very old conjectures regarding possible structures of non-seperable C*-algebras. They use methods from set theory and logic to give unexpected answers which show that existence or non-existence of certain phenomena very much depend on which set-theoretical axioms we assume to be true. Ergodic theory in interaction with operator algebras is another subject where major advances have been achieved, revealing new rigidity phenomena in the relations between groups, group actions and von Neumann algebras. Two very recent such spectacular advances are presented in the report: using Popa's powerful deformation/rigidity techniques, Ioana shows that everything is remembered in case of Bernoulli actions of Kazdhan property (T) groups; in the same spirit, another remarkable result of Ioana, Popa and Vaes provides the very first examples of groups entirely determined by their von Neumann algebras. As one of the many other exciting results we also mention the recent progress of Christensen and coworkers on pertubation theory of C*-algebras, in which they finish a long-term program by showing (among other things) that two nuclear C*-algebras which are "close" in a certain metric sense must be isomorphic.

It is a pleasure for the organizers to thank all participants of the workshop for their beautiful lectures and fruitful discussions. We also want to use this opportunity to thank the Mathematisches Forschungsinstitut Oberwolfach for providing a very stimulating environment and strong support for organizing this conference. Special thanks also go to the very competent and helpful staff of the institute.

Workshop: C*-Algebren

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Abstracts

W*-superrigidity for Bernoulli actions of property (T) groups

Adrian Ioana

The *group measure space construction* of Murray and von Neumann associates to every free ergodic measure preserving action $\Gamma \curvearrowright X$ of a countable group Γ on a probability space (X, μ) a II_1 factor $L^\infty(X) \rtimes \Gamma$ which contains $L^\infty(X)$ as a *Cartan subalgebra* ([3]).

The general question that arises is how much of the initial group/action data can be recovered from the group measure factor.

In this talk, I will present a result showing that, for a wide, natural family of group actions (Bernoulli actions of property (T) groups), their associated II_1 factor completely remembers the group and the action. We start by giving some motivation.

By a celebrated result of A. Connes, if Γ is infinite amenable, then the von Neumann algebra of any free ergodic action $\Gamma \curvearrowright X$ is isomorphic to the hyperfinite II_1 factor ([1]).

In contrast, the study of group measure space algebras arising from actions of non–amenable groups has led to a deep rigidity theory (see the introduction of [8]).

In particular, S. Popa's seminal work [5], [6] shows that, for actions belonging to a large class, if their group measure space algebras are isomorphic, then the actions are conjugate. More precisely, assume that Γ is an ICC group which has an infinite, normal subgroup with relative property (T) and let $\Gamma \curvearrowright X$ be a free ergodic action. Furthermore, suppose that $\Lambda \curvearrowright Y = Y_0^\Lambda$ is a Bernoulli action. Popa proves that if the actions $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ are W*–*equivalent*, i.e. if they produce isomorphic von Neumann algebras (also known as W*–algebras), then the groups are isomorphic and the actions are *conjugate* ([6]). The latter means that there exists an isomorphism of probability spaces $\theta : X \to Y$ such that $\theta \Gamma \theta^{-1} = \Lambda$.

The question underlying this result, explicitely formulated in the introduction of [6], is whether the same is true if one imposes all the conditions on only one of the actions. In other words, assume that Γ is an ICC group having an infinite, normal subgroup with relative property (T) and suppose that $\Gamma \curvearrowright X = X_0^\Gamma$ is a Bernoulli action. Is it true that any free ergodic action $\Lambda \curvearrowright Y$ which is W*–equivalent to $\Gamma \curvearrowright X$, must be conjugate to it?

The following theorem answers affirmatively this question.

Theorem 1. *Let Γ be a countable ICC group which admits an infinite, normal subgroup Γ_0 such that the inclusion $(\Gamma_0 \subset \Gamma)$ has relative property (T). Let (X_0, μ_0) be a non–trivial probability space and let $\Gamma \curvearrowright (X, \mu) = (X_0, \mu_0)^\Gamma$ be the Bernoulli action. Denote $M = L^\infty(X) \rtimes \Gamma$.*

Let $\Lambda \curvearrowright (Y, \nu)$ be a free ergodic measure preserving action of a countable group Λ on a probability space (Y, ν). Denote $N = L^\infty(Y) \rtimes \Lambda$.

If $N \cong M$, then $\Gamma \cong \Lambda$ and the actions $\Gamma \curvearrowright X$, $\Lambda \curvearrowright Y$ are conjugate.

Recall that an inclusion ($\Gamma_0 \subset \Gamma$) of countable groups has *relative property (T) of Kazhdan–Margulis* if any unitary representation of Γ which has almost invariant vectors must have a non-zero Γ_0–invariant vector. Examples of such inclusions are given by ($\mathbb{Z}^2 \subset \mathbb{Z}^2 \rtimes \Gamma$), for any non–amenable subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$, and by ($\Gamma_0 \subset \Gamma_0 \times \Gamma_1$), for a property (T) group Γ_0 (e.g. $\Gamma_0 = \mathrm{SL}_n(\mathbb{Z})$, $n \geq 3$) and an arbitrary countable group Γ_1.

If (X_0, μ_0) is a probability space, then the *Bernoulli action* $\Gamma \curvearrowright (X_0, \mu_0)^\Gamma$ is given by $\gamma \cdot (x_g)_g = (x_{\gamma^{-1}g})_g$, for all $(x_g)_g \in X_0^\Gamma$ and $\gamma \in \Gamma$.

Theorem A is proven in the framework of Popa's *deformation/rigidity theory* by playing against each other the "rigidity" of Γ (manifested here in the form of relative property (T)) and the "deformation properties" of Bernoulli actions $\Gamma \curvearrowright X$ (i.e. malleability and its weaker forms).

Before discussing its method of proof, let us put it into context.

In this talk, by a W*–rigidity result we mean a result deriving that two W*–equivalent actions $\Gamma \curvearrowright X$, $\Lambda \curvearrowright Y$ must be conjugate. If this happens when only one of these actions is in a fixed class, while the other can be *any* free ergodic action of *any* countable group, we have a *superrigidity* result.

Despite remarkable progress in the areas of W*–rigidity during the last decade, W*–superrigidity results remained elusive until very recently ([4], [8]).

The situation changed starting with the work of J. Peterson who was able to show the existence of virtually W*–superrigid actions ([4]).

Shortly after, S. Popa and S. Vaes discovered the first concrete families of W*–superrigid actions ([8]). They showed that for groups Γ in a certain class $\mathcal{G}$ of amalgamated free product groups, the II_1 factor of any free ergodic action $\Gamma \curvearrowright X$ has a unique group measure space Cartan subalgebra.

By using OE-superrigidity results from the literature, they respectively deduced that the following actions are W*–superrigid: a) Bernoulli actions, generalized Bernoulli actions, Gaussian actions of groups $\Gamma \in \mathcal{G}$ and b) any free mixing action of $\Gamma = \mathrm{PSL}_n(\mathbb{Z}) *_{T_n} \mathrm{PSL}_n(\mathbb{Z}) \in \mathcal{G}$ $(n \geq 3)$.

The proof of Theorem A uses a general strategy for analyzing group measure space decompositions of II_1. To sketch our approach, let $M = L^\infty(X) \rtimes \Gamma$ be as in Theorem A and denote $A = L^\infty(X)$.

Then, by using, among other things, ideas and techniques from [5], [6], [7], [2], [4] we prove the following classification result for embeddings of M into $M\overline{\otimes}M$:

Theorem 2. *Let $\theta : M \to M\overline{\otimes}M$ be a unital $*$–homomorphism and suppose that Γ is torsion free. Then one of the following holds:*

(1) $\theta(L(\Gamma_0)) \prec_{M\overline{\otimes}M} L(\Gamma) \otimes 1$ *or* $\theta(L(\Gamma_0)) \prec_{M\overline{\otimes}M} 1 \otimes L(\Gamma)$.
(2) $\theta(M) \prec_{M\overline{\otimes}M} L(\Gamma)\overline{\otimes}M$ *or* $\theta(M) \prec_{M\overline{\otimes}M} M\overline{\otimes}L(\Gamma)$.
(3) *We can find a character η of Γ, two group morphisms $\delta_1, \delta_2 : \Gamma \to \Gamma$ and a unitary $u \in M$ such that $\theta(A) \subset u(A\overline{\otimes}A)u^*$ and $\theta(u_\gamma) = \eta(\gamma)u(u_{\delta_1(\gamma)} \otimes u_{\delta_2(\gamma)})u^*$, for all $\gamma \in \Gamma$.*

Now, suppose that M also arises as the II_1 factor of a "mystery" action $\Lambda \curvearrowright Y$.

The decomposition $M = L^\infty(Y) \rtimes \Lambda$ induces a unital $*$–homomorphism $\theta : M \to M\overline{\otimes}M$ given by $\theta(av_\lambda) = av_\lambda \otimes v_\lambda$, for all $a \in L^\infty(Y)$ and every $\lambda \in \Lambda$ ([8]).

By applying Theorem B to θ we derive some information about the form of θ with respect to both group measure space decompositions of M. This relates the two decompositions and, as it turns out, the relationship is powerful enough to imply that they coincide, up to conjugation with a unitary element.

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A class of group factors LG that remember the group G

Stefaan Vaes

(joint work with Adrian Ioana and Sorin Popa)

To every countable group G is associated the group von Neumann algebra LG generated by the left translation unitary operators on $\ell^2(G)$. When G has infinite conjugacy classes (icc), the von Neumann algebra LG is a II_1 factor.

In this talk, I present a joint work in progress [6] with Adrian Ioana and Sorin Popa in which we construct the first examples of groups G with the following property: whenever Λ is a countable group and $\mathrm{L}G \cong \mathrm{L}\Lambda$, one must have $G \cong \Lambda$.

In general group factors LG depend in an extremely subtle way on G and up to now very little was known about when $\mathrm{L}G_1 \cong \mathrm{L}G_2$. The two main open problems in this respect – and both untouched by our methods – are the following.

- Are the free group factors $\mathrm{L}\mathbb{F}_n$ non-isomorphic?
- Connes' rigidity conjecture [2]: Is $\mathrm{L}G_1 \cong \mathrm{L}G_2$ equivalent with $G_1 \cong G_2$ in the world of property (T) groups?

By Connes' uniqueness theorem for injective II_1 factors [1] all amenable icc groups G give rise to isomorphic II_1 factors LG. On the opposite side, Connes and Jones [3] proved that $\mathrm{L}(\mathrm{SL}(n,\mathbb{Z})) \not\cong \mathrm{L}(\mathrm{SL}(2,\mathbb{Z}))$ when $n \geq 3$, while Cowling and Haagerup [4] showed that $\mathrm{L}\Gamma_n \not\cong \mathrm{L}\Gamma_m$ when the Γ_k are lattices in $\mathrm{Sp}(k,1)$ and $n \neq m$.

Popa's deformation/rigidity theory (see e.g. [7, 8]) has over the last 10 years lead to overwhelming progress in the understanding of *group measure space factors* $\mathrm{L}^\infty(X) \rtimes \Gamma$. In [6] we obtain the first results of similar strength for *group factors* $\mathrm{L}G$. We consider groups G given as *generalized wreath products* $G = H_0 \wr_I \Gamma = H_0^{(I)} \rtimes \Gamma$, associated with an abelian group H_0 and an action $\Gamma \curvearrowright I$ of a countable group Γ on a countable set I.

Theorem 1. *Let Γ be an icc group that admits an infinite normal subgroup with the relative property (T). Let $\Gamma_0 < \Gamma$ be an infinite amenable subgroup that is almost malnormal: for all $g \in \Gamma - \Gamma_0$, the intersection $g\Gamma_0 g^{-1} \cap \Gamma_0$ is finite. Put $I = \Gamma/\Gamma_0$ and let H_0 be a countable abelian group. Denote $G = H_0 \wr_I \Gamma$.*

If Λ is a countable group and $\theta : \mathrm{L}\Lambda \to p\mathrm{L}(G)p$ is a $*$-isomorphism, then $p = 1$ and $\Lambda \cong H_1 \wr_I \Gamma$ for some abelian group H_1 satisfying $|H_1| = |H_0|$. So, if we assume that $|H_0|$ is a square-free integer, then it follows that Λ is isomorphic with G.

In the even more specific case where $H_0 = \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z}$, we conclude moreover that the isomorphism $\theta : \mathrm{L}\Lambda \to \mathrm{L}G$ must be group-like: θ is the composition of an inner automorphism of $\mathrm{L}G$, the automorphism of $\mathrm{L}G$ given by a character $G \to S^1$ and the isomorphism $\mathrm{L}\Lambda \to \mathrm{L}G$ induced by a group isomorphism $\Lambda \to G$.

Concrete examples of groups $\Gamma_0 < \Gamma$ satisfying the assumptions in Theorem 1 can be given as follows. Take $\Gamma = \mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$. Let $A \in \mathrm{SL}(2,\mathbb{Z})$ be any hyperbolic matrix and put $\Gamma_0 := \{B \in \mathrm{SL}(2,\mathbb{Z}) \mid BAB^{-1} = A^{\pm 1}\}$. Viewing $\Gamma_0 < \mathrm{SL}(2,\mathbb{Z})$ as a subgroup of Γ, all conditions in Theorem 1 are satisfied. More examples, in which $\Gamma = \mathrm{SL}(n,\mathbb{Z})$, are given in [9, Example 7.4].

The conclusion of Theorem 1 does not hold for plain wreath products $G = H_0 \wr \Gamma = H_0^{(\Gamma)} \rtimes \Gamma$. Indeed, assume that Γ admits an infinite normal subgroup with the relative property (T). Choose a finite abelian group H_0 and put $G = H_0 \wr \Gamma$. Let Λ be a countable group such that $\mathrm{L}\Lambda \cong \mathrm{L}G$. In [6] we prove that Λ must be of the form $\Lambda \cong \Sigma \rtimes \Gamma$, where Σ is a countable abelian group and $\Gamma \curvearrowright \Sigma$ is an action by group automorphisms such that the probability measure preserving actions $\Gamma \curvearrowright \widehat{\Sigma}$ and $\Gamma \curvearrowright (\widehat{H_0})^\Gamma$ are conjugate.

Whenever $\mathbb{Z}$ is embedded as a subgroup of Γ, we construct as follows an action $\Gamma \curvearrowright \Sigma$ with the above properties and with Σ being torsion free. In particular, $\Lambda = \Sigma \rtimes \Gamma$ is not isomorphic with G. First put $\Sigma_0 = \mathbb{Z}[|H_0|^{-1}]$ and define the action $\mathbb{Z} \curvearrowright \Sigma_0$ where $n \in \mathbb{Z}$ acts through multiplication by $|H_0|^n$. Classic results in ergodic theory ensure that the probability measure preserving action $\mathbb{Z} \curvearrowright \widehat{\Sigma_0}$ is conjugate with a Bernoulli shift with base space $\widehat{H_0}$. Then define $\Sigma := \Sigma_0^{(\Gamma/\mathbb{Z})}$ and consider the *co-induced action* $\Gamma \curvearrowright \Sigma$. By construction, the probability measure preserving action $\Gamma \curvearrowright \widehat{\Sigma}$ is conjugate with the Bernoulli action $\Gamma \curvearrowright (\widehat{H_0})^\Gamma$. In specific examples, we can vary the embedding $\mathbb{Z} \hookrightarrow \Gamma$ and prove, for instance, that there are infinitely many non-isomorphic groups Λ_k satisfying $\mathrm{L}\Lambda_k \cong \mathrm{L}\big(\frac{\mathbb{Z}}{2\mathbb{Z}} \wr \mathrm{SL}(3,\mathbb{Z})\big)$.

The starting point to prove Theorem 1 is the following. Assume for simplicity that $p = 1$. The group von Neumann algebra $\mathrm{L}\Lambda$ carries a natural *comultiplication*

$\Delta : \mathrm{L}\Lambda \to \mathrm{L}\Lambda \overline{\otimes} \mathrm{L}\Lambda$ given by $\Delta(v_s) = v_s \otimes v_s$ for all $s \in \Lambda$. We view Δ as an embedding of $\mathrm{L}G$ into $\mathrm{L}G \overline{\otimes} \mathrm{L}G$. In turn, we view $\mathrm{L}G$ as a crossed product II_1 factor $\mathrm{L}G = A \rtimes \Gamma$, where $A = \mathrm{L}(H_0^{(I)}) = \mathrm{L}^\infty\big((\widehat{H_0})^I\big)$ and where $\Gamma \curvearrowright A$ is the *generalized Bernoulli action.*

Based on Popa's deformation/rigidity theory for Bernoulli actions [7, 8], Ioana [5] has given a very strong classification theorem of all possible embeddings of M into $M \overline{\otimes} M$ when M is the crossed product of a *plain Bernoulli action.* Through a generalization of these results to generalized Bernoulli actions, in [6] we are able to find a unitary $\Omega \in \mathrm{L}G \overline{\otimes} \mathrm{L}G$ such that

$$(1) \quad \Omega\Delta(A)\Omega^* \subset A \overline{\otimes} A \quad \text{and} \quad \Omega\Delta(u_g)\Omega^* = \omega(g)\, u_{\delta_1(g)} \otimes u_{\delta_2(g)} \quad \text{for all } g \in \Gamma$$

where $\omega : \Gamma \to S^1$ is a character and $\delta_i : \Gamma \to \Gamma$ are group homomorphisms. Exploiting that Δ is *symmetric* and *co-associative,* we may assume that $\delta_1 = \delta_2 = \mathrm{id}$ and next, that Ω is a *symmetric* 2*-cocycle:* for some scalars $\eta, \rho \in S^1$, we have

$$(2) \qquad (\Omega \otimes 1)(\Delta \otimes \mathrm{id})(\Omega) = \eta\, (1 \otimes \Omega)(\mathrm{id} \otimes \Delta)(\Omega) \quad \text{and} \quad \sigma(\Omega) = \rho\, \Omega$$

where $\sigma(a \otimes b) = b \otimes a$ denotes the flip automorphism.

We go on by proving the following *vanishing of* 2*-cohomology* theorem.

Theorem 2. *Whenever Λ is a countable group and $\Omega \in \mathrm{L}\Lambda \overline{\otimes} \mathrm{L}\Lambda$ is a unitary satisfying* (2) *for some scalars $\eta, \rho \in S^1$, we have $\eta = \rho = 1$ and there exists a unitary $w \in \mathrm{L}\Lambda$ such that $\Omega = (w \otimes w)\Delta(w^*)$.*

Combining (1) and Theorem 2, we find a subgroup $\Sigma < \Lambda$ and an isomorphic copy $\delta(\Gamma) < \Lambda$ of Γ inside Λ such that $w^* A w = \mathrm{L}\Sigma$ and $\{w^* u_g w \mid g \in \Gamma\}'' = \mathrm{L}(\delta(\Gamma))$. It will follow that $\Lambda \cong \Sigma \rtimes \Gamma$ in such a way that $\Gamma \curvearrowright \widehat{\Sigma}$ is conjugate with $\Gamma \curvearrowright (\widehat{H_0})^I$. From this point on, it will be easy to deduce Theorem 1.

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Subfactors, planar algebras and fusion categories

Dietmar Bisch

A subfactor $N \subset M$ with finite Jones index gives rise to an N-M bimodule $L^2(M)$ by completing M with respect to the norm induced by the canonical trace on the II_1 factor M ([15]). Tensoring these bimodules appropriately and then decomposing them into irreducible N-N, N-M, M-M, M-N sub-bimodules can be accomplished by computing the higher relative commutants of the subfactors (see e.g. [15], [7], [18]). These higher relative commutants are centralizer algebras of N resp. M in the II_1 factors in the Jones tower associated to $N \subset M$, and the collection of these finite dimensional C*-algebras is called the *standard invariant* of $N \subset M$. It is given by

$$\begin{array}{ccccccccc} \mathbb{C} = N' \cap N & \subset & N' \cap M & \subset & N' \cap M_1 & \subset & N' \cap M_2 & \subset & \cdots \\ & & \cup & & \cup & & \cup & & \\ & & \mathbb{C} = M' \cap M & \subset & M' \cap M_1 & \subset & M' \cap M_2 & \subset & \cdots \end{array}$$

and can be axiomatized as a *planar algebra* ([16], [22]). The *fusion graphs* or *principal graphs* Γ, Γ' of $N \subset M$ are obtained as the principal parts of the sequence of Bratelli diagrams of each row. They can be viewed as graphs that describe induction from irreducible N-N to irreducible N-M bimodules, respectively restriction from irreducible M-M to irreducible M-N bimodules. If the graphs are finite, then $[M : N] = \|\Gamma\|^2$ $(= \|\Gamma'\|^2)$ ([21]).

The standard invariant of a subfactor is a complete invariant, if M is the hyperfinite II_1 factor and $N \subset M$ is an *amenable* subfactor ([21]). This result of Popa implies that all subfactors of the hyperfinite II_1 factor R with Jones index ≤ 4 are classified by their standard invariant. The same holds true for subfactors of R whose principal graphs are finite (the so-called *finite depth* subfactors). On the other hand, it was shown in [11] that there are uncountably many non-isomorphic, irreducible, hyperfinite subfactors with index 6 which have the same standard invariant. They are all group-type subfactors as in [5], and can be distinguished by different cocycle actions of the associated group on R.

Each finite index, extremal subfactor yields two *fusion categories* (see [12] for the definition of *fusion category*) of N-N respectively M-M bimodules, if we allow infinitely many simple objects. The fusion graphs can be wild, and many such examples have been constructed in [5], [9]. The ideas and techniques of [5] can be used to construct fusion categories of bimodules of low rank. Note that the two fusion categories of bimodules arising from a subfactor are Morita equivalent via the N-M bimodule $L^2(M)$.

As mentioned above, subfactors of the hyperfinite II_1 factor with index ≤ 4 can be listed in terms of their standard invariants, and are usually given in terms of their principal graphs and associated canonical commuting squares (see e.g. [14], [21]). Haagerup showed in 1993 that the first "exotic" finite depth subfactor with Jones index above 4 occurs at index $\frac{5+\sqrt{13}}{2} = 4.3027756....$ The construction of this subfactor, and another exotic subfactor can be found in [1] (see also [20] for a planar algebra construction of this subfactor). The fusion algebras of the fusion categories associated to the Haagerup subfactor can be found for instance in [4]. It is worthwhile to note that one of these fusion algebras is non-abelian. Even more interestingly, Morrison and Snyder showed in [17] that the fusion category defined by the N-N bimodules of the Haagerup subfactor cannot be defined over any cyclotomic field, thus providing a counterexample to a conjecture in [12].

The classification of finite depth subfactors with small Jones indices ("small" means "close to 5 and below" at this point in time) has seen some spectacular advances in recent years. Haagerup presented a list of possible principal graphs of finite depth subfactors in 1993 ([13]). The list covered all possible graphs of subfactors up to index $3+\sqrt{3}$, and contained two infinite families of principal graph pairs. I ruled out one of these families in [8], and the second one was ruled out by Asaeda-Yasuda [2] with two exceptions. The first one were the Haagerup subfactors with index $\frac{5+\sqrt{13}}{2}$, and the second pair of graphs stayed open for more than 10 years. Recently, Bigelow, Morrison, Peters and Snyder constructed this remaining case using planar algebra techniques ([3], see also [16], [20]). They construct what they call the *extended Haagerup subfactor* using generators and relations for the associated planar algebra in the spirit of Peters' construction of the Haagerup subfactor. A key ingredient in the proof is Bigelow's jellyfish algorithm. This new subfactor has index 4.3772.... Together with the Asaeda-Haagerup subfactor of index $\frac{5+\sqrt{17}}{2} = 4.56155...$, and the GHJ/Okamoto subfactor of index $3+\sqrt{3} = 4.73205...$ ([14], [19]), the list of exotic finite depth subfactors in the index range from 4 to $3+\sqrt{3}$ is complete (to be precise, there are actually two GHJ/O subfactors). It is a testimony to the surprising rigidity phenomena occuring in the theory of subfactors that there are only 4 index values in the interval $(4, 3+\sqrt{3}]$ at which finite depth subfactors occur. Each of these subfactors comes with two interesting fusion categories, a topological quantum field theory and quantum invariants à la Turaev-Viro, which need to be explored.

What about classification of finite depth subfactors with index above $3+\sqrt{3}$? Classification work of subfactors with index at most 5 is currently being done by Jones, Morrison, Penneys, Peters, Snyder and others. Calegari and Snyder have found new number theoretic obstructions to the existence of principal graphs.

As soon as the index becomes bigger than 5 several interesting phenomena occur. A special index is the number $2\cdot(\frac{1+\sqrt{5}}{2})^2 = 3+\sqrt{5} = 5.23606....$ There exist irreducible, hyperfinite, finite *and* infinite depth subfactors at this index. For instance, infinite depth subfactors with index $3+\sqrt{5}$ are obtained as the *free product* (or *free composition*) of an A_3 and an A_4 subfactor ([10], see also [9], where the principal graphs of these subfactors can be found). In [6] two finite depth subfactors with index $3+\sqrt{5}$ were constructed, and an infinite sequence of pairs of possible principal graphs was computed that "converges" to the principal graph of the free product. Work is under way to try to realize these graphs as principal graphs of subfactors, or to show that they do not exist. It should be noted that they all give rise to fusion algebras.

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K-theory duality and hyperbolicity

Ian F. Putnam

(joint work with Jerome Kaminker, Michael Whittaker)

Two C^*-algebras, A and B are dual if the K-theory of one is naturally isomorphic to the K-homology of the other, and vice versa. The naturality is expressed by the existence of KK-classes,

$$\begin{array}{rcl}\delta & \in & KK(\mathbb{C}, A \otimes B), \\ \Delta & \in & KK(A \otimes B, \mathbb{C})\end{array}$$

The class δ induces maps, denoted δ_*, from $K_*(A)$ to $K^*(B)$ and from $K_*(B)$ to $K^*(A)$. Similarly, the element Δ induces maps in the other direction, denoted Δ_*.

It is a basic fact that if

$$(\delta \otimes 1_A) \otimes \sigma_{2,3}(1_A \otimes \Delta) = 1_A$$

in $KK(A, A)$, and a similar formula holds replacing A with B, then the maps δ_* and Δ_* are inverses. In this case, we say that A and B are dual.

We will present a specific example of such a dual pair and the elements δ and Δ. Afterward, we will describe how the example is a special case of a general result. In our example, the elements δ and Δ are actually in KK^1, so that the duality maps are of odd degree.

Begin with the matrix $M = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. It is an automorphism of $\mathbb{R}^2$ which preserves the integer lattice $\mathbb{Z}^2$ and so it induces a homeomorphism φ of the 2-torus $\mathbb{T}^2$. This is a hyperbolic toral automorphism: its eigenvalues are $\gamma^2 > 1$ and $0 < \gamma^{-2} < 1$, where γ is the golden mean. Let $\mathbb{R}^u$ and $\mathbb{R}^s$ be the associated eigenspaces, regarded as subsets of the torus. Here 's' stands for stable (meaning contracting) and 'u' stands for unstable (or expanding).

Choose finite φ-invariants subsets of the torus, P and Q. (The collection of periodic points of φ is dense.) Let $X^u = P + \mathbb{R}^u$ and $X^s = Q + \mathbb{R}^s$, which we regard as subsets of the torus, but with the topologies as a finite set of lines. On X^u, consider the equivalence relation $x \sim_s y$ if $x - y \in \mathbb{R}^s$ and on X^s, consider the equivalence relation $x \sim_u y$ if $x - y \in \mathbb{R}^u$. We let S denote the groupoid C^*-algebra, $C^*(X^u, \sim_s)$ and U denote the groupoid C^*-algebra, $C^*(X^s, \sim_u)$. The homeomorphism φ acts on each of these and the C^*-algebras for our duality are

$$A = S \times_\varphi \mathbb{Z}, B = U \times_\varphi \mathbb{Z}.$$

The duality element δ arises essentially because S and U are coming from transverse foliations of the torus. The element Δ is more subtle: it arises from a pair of repesentations of S and U on a Hilbert space $\mathcal{H}$ which commute up to compacts. This fact relies heavily on the hyperbolic nature of the dynamics. Then our element Δ is given as an extension

$$0 \to \mathcal{K}(\mathcal{H}) \to \mathcal{E} \to A \otimes B \to 0,$$

where $\mathcal{E}$ is the C^*-algebra generated by these two representations and the compact operators.

The fact that $\mathbb{T}^2$ is a compact group or that it is a manifold are not relevant; the only essential feature is the hyperbolic nature of the dynamics. That is, the result may be extended to the setting of a compact metric space and a homeomorphism which has canonical coordinates of contracting and expanding directions. The definition of such objects was given by David Ruelle, who called them Smale spaces.

These include hyperbolic toral automorphisms (as above), shifts of finite type (contracting and expanding spaces are both Cantor), various solenoids introduced by Bob Williams (Cantor times Euclidean), the space of Penrose tilings (Cantor times $\mathbb{R}^2$) and, more generally, the basic sets of Smale's Axiom A systems, where both directions can be expected to be fractal.

A variational principle for actions of sofic groups

David Kerr
(joint work with Hanfeng Li)

Recently Lewis Bowen introduced a collection of entropy invariants for measure-preserving actions of a countable sofic group G on a standard probability space (X, μ) [2]. Given a finite measurable partition $\mathcal{P}$ of X and a local modeling of G by permutations of a finite set $\{1, \dots, d\}$ as in the definition of soficity, one counts the number of partitions of $\{1, \dots, d\}$ which dynamically model $\mathcal{P}$ with respect to the action of a given finite subset of G. The logarithmic size of this quantity in proportion to d is then used to asymptotically generate a number by limiting along a given sequence Σ of sofic approximations for G and taking an infimum over local parameters. This number depends on Σ in general, but Bowen showed that for a fixed Σ one obtains a common value over all generating finite measurable partitions, yielding an invariant for the dynamical system. By a limiting procedure one can more generally define the entropy of any action admitting a countable measurable partition $\mathcal{P}$ whose entropy $H_\mu(\mathcal{P})$ is finite. Bowen used this sofic measure entropy in spectacular fashion to extend the Ornstein-Weiss entropy classification of Bernoulli shifts over countably infinite amenable groups to a large class of nonamenable groups, including all nontorsion countable sofic groups [2].

By taking an operator algebra viewpoint, we have developed a more general approach to sofic entropy which furnishes invariants for both measure-preserving actions on a standard probability space and continuous actions on a compact metrizable space [5]. We define the entropy of a self-adjoint sequence in the unit ball of the relevant unital commutative C^*-algebra (i.e., either $L^\infty(X, \mu)$ or $C(X)$) by computing the maximal cardinalities of ε-separated subsets of certain spaces of approximately equivariant unital positive linear maps into $\mathbb{C}^d$, where ε-separation is measured via weighted evaluation along the sequence. In both the measurable and topological contexts this quantity is shown to be the same for all dynamically generating self-adjoint sequences in the unit ball of the C^*-algebra in question, granted that one is working with a fixed sofic approximation sequence Σ for G. In the case of a measure-preserving action admitting a countable measurable partition with finite entropy, we recover Bowen's invariant. The topological entropy of the Bernoulli shift on $\{1, \dots, k\}^G$ is easily computed to be $\log k$, as one would expect. For amenable G Bowen showed that his invariant, within its domain of definition, coincides with classical measure entropy, and we prove that the same is true for both our measure and topological dynamical invariants.

The classical variational principle asserts that the topological entropy of a continuous action of an amenable group on a compact metrizable space is equal to the supremum of the measure entropies over all invariant Borel probability measures. We have established the variational principle in the sofic setting with respect to a fixed sofic approximation for the group. This is then applied as follows to the study of algebraic actions of a residually finite group G. For an element f in the integral group ring $\mathbb{Z}G$ which is invertible in the full group C^*-algebra of G, we show that the topological entropy of the canonical action of G on the dual $\widehat{\mathbb{Z}G/\mathbb{Z}Gf}$, with respect to any sofic approximation sequence arising from finite quotients of G, is equal to the Fuglede-Kadison determinant of f as an element in the group von Neumann algebra of G. This complements a result of Bowen [1], which asserts the same for measure entropy with respect to normalized Haar measure under the assumption that f is invertible in $\ell^1(G)$. In the case of amenable acting groups and classical entropy these relationships were developed in [7, 3, 4, 6].

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Large subalgebras and the Cuntz semigroup, with applications to transformation group C*-algebras

N. Christopher Phillips

We define a notion of a large subalgebra of a simple C*-algebra. For motivation, let h be a minimal homeomorphism of a compact metric space X, and recall Putnam's subalgebra of the crossed product $C^*(\mathbb{Z}, X, h)$. Let $u \in C^*(\mathbb{Z}, X, h)$ be the standard unitary corresponding to the generator of $\mathbb{Z}$. For $Y \subset X$ closed, Putnam defines $C^*(\mathbb{Z}, X, h)_Y \subset C^*(\mathbb{Z}, X, h)$ to be the subalgebra generated by $C(X)$ and all uf for $f \in C(X)$ such that f vanishes on Y. Taking Y to have nonempty interior gives a recursive subhomogeneous algebra in the sense of [4]. Taking Y to be a one point subset gives a subalgebra which contains much information about $C^*(\mathbb{Z}, X, h)$. For X the Cantor set, this construction was introduced in [7] and played a key role in [8]. For applications when X is not totally disconnected, see Section 4 of [5] and also [3]. The construction does not generalize well to actions of groups other than $\mathbb{Z}$, but subalgebras with a more complicated definition play a key role in [9] (applied to the C*-algebras of substitution tilings) and [6] (applied to the transformation group C*-algebra of a free minimal action of $\mathbb{Z}^d$ on the Cantor set).

Our definition abstracts the important properties of $C^*(\mathbb{Z}, X, h)_Y$ for a one point set Y, and its analogs in other situations. In the definition below, $QT(A)$ is the space of quasitraces on A, and $a \precsim b$ is Cuntz subequivalence for positive elements: there is a sequence $(z_n)_{n\in\mathbb{N}}$ such that $z_n b z_n^* \to a$ in norm. This definition remains tentative. For example, if A is exact, it is quite possible that condition (3) follows from the others.

Definition 1. Let A be an infinite dimensional stably finite simple separable unital C*-algebra. A subalgebra $B \subset A$ is said to be *large* in A if:

1. B contains the identity of A.
2. B is simple.
3. The restriction map $QT(A) \to QT(B)$ is surjective.
4. For every $m \in \mathbb{N}$, $a_1, a_2, \ldots, a_m \in A$, $\varepsilon > 0$, and $y \in B_+ \setminus \{0\}$, there are $c_1, c_2, \ldots, c_m \in A$ and $g \in B$ such that:
   (a) $0 \leq g \leq 1$.
   (b) For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$.
   (c) For $j = 1, 2, \ldots, m$ we have $(1-g)c_j$, $c_j(1-g) \in B$.
   (d) $g \precsim y$ relative to the subalgebra B.

Example 2. If X is a compact metric space, $h \colon X \to X$ is a minimal homeomorphism, and $Y \subset X$ is a finite set which intersects each orbit of h at most once, then $C^*(\mathbb{Z}, X, h)_Y$ can be shown to be large in $C^*(\mathbb{Z}, X, h)$. This is true even if X is infinite dimensional, and in particular applies when h is one of the examples in [2], for which $C^*(\mathbb{Z}, X, h)$ has perforation in its K_0-group or does not satisfy strict comparison of positive elements.

Example 3. If G is an almost AF Cantor groupoid in the sense of [6], whose reduced C*-algebra is simple, and if $G_0 \subset G$ is the AF subgroupoid required in the definition, then $C^*_{\mathrm{r}}(G_0)$ is a large subalgebra of $C^*_{\mathrm{r}}(G)$. In particular, the transformation group C*-algebra of a free minimal action of $\mathbb{Z}^d$ on the Cantor set has a large subalgebra which is AF.

Example 4. We have proved that if X is a compact smooth manifold with a free minimal action of $\mathbb{Z}^d$ via diffeomorphisms, then $C^*(\mathbb{Z}^d, X)$ has a large subalgebra which is a direct limit, with no dimension growth, of recursive subhomogeneous algebras.

The construction required for Example 4 was not discussed in the talk. It adapts ideas from [1], but requires technical point set topology.

The following definition is adapted from, and is equivalent, to Definition 6.1 of [10]. For a quasitrace τ, the corresponding dimension function d_τ is defined on positive elements by $d_\tau(a) = \lim_{n\to\infty} \tau(a^{1/n})$.

Definition 5. Let A be a stably finite unital C*-algebra.

(1) Let $r \in [0,\infty)$. We say that A has *r-comparison* if whenever $a, b \in M_\infty(A)$ satisfy $d_\tau(a) + r < d_\tau(b)$ for all $\tau \in QT(A)$, then $a \precsim b$.
(2) The *radius of comparison* of A, denoted $\mathrm{rc}(A)$, is

$$\inf\left(\left\{r \in [0,\infty) \colon A \text{ has } r\text{-comparison}\right\}\right).$$

(We take $\mathrm{rc}(A) = \infty$ if there is no r such that A has r-comparison.)

In particular, it turns out that if A is simple then A has strict comparison of positive elements if and only if $\mathrm{rc}(A) = 0$.

The main theorem of this talk is:

Theorem 6. *Let A be an infinite dimensional stably finite simple separable unital C*-algebra. Let $B \subset A$ be large in the sense of Definition 1. Then* $\mathrm{rc}(A) = \mathrm{rc}(B)$.

This theorem applies in particular to the examples of [2] (see Example 2), even though they have strictly positive radius of comparison.

Applying it to the situation of Example 4, we obtain:

Theorem 7. *Let X be a compact smooth manifold with a free minimal action of $\mathbb{Z}^d$ via diffeomorphisms. Then $C^*(\mathbb{Z}^d, X)$ has strict comparison of positive elements.*

Corollary 8. *In the situation of Theorem 7, the algebra $C^*(\mathbb{Z}^d, X)$ satisfies Blackadar's Second Fundamental Comparability Question.*

Corollary 8 is a statement about K-theory and projections. However, even if $C^*(\mathbb{Z}^d, X)$ is expected to have real rank zero, we do not know how to arrange that our large subalgebra has a large K_0-group. In particular, even in this case, our method of proof requires the Cuntz semigroup.

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On cocycle superrigidity for Gaussian actions

Jesse Peterson
(joint work with Thomas Sinclair)

A central motivating problem in the theory of measure-preserving actions of countable groups on probability spaces is to classify certain actions up to orbit equivalence, *i.e.*, isomorphism of the underlying probability spaces such that the orbits of one group are carried onto the orbits of the other.

One breakthrough which we highlight here is Popa's use of his deformation/rigidity techniques in von Neumann algebras to produce rigidity results for orbit equivalence (cf. [7, 8, 9, 10, 11, 12, 14]). One of the seminal results using these techniques is Popa's Cocycle Superrigidity Theorem [11, 12] (see also [2] and [17] for more on this) which obtains orbit equivalence superrigidity results by means of untwisting cocycles into a finite von Neumann algebra.

Let $\Gamma \curvearrowright^{\sigma} (X, \mu)$ be an ergodic, measure-preserving action on a standard probability space (X, μ), and let $\mathbb{A}$ be a Polish topological group. A cocycle is a measurable map $c : \Gamma \times X \to \mathbb{A}$ satisfying the cocycle identity $c(\gamma_1\gamma_2, x) = c(\gamma_1, \sigma_{\gamma_2}(x))c(\gamma_2, x)$, for all $\gamma_1, \gamma_2 \in \Gamma$, a.e. $x \in X$. To any homomorphism $\rho : \Gamma \to \mathbb{A}$ we can associate a cocycle $\tilde{\rho}$ by $\tilde{\rho}(\gamma, x) = \rho(\gamma)$. Using terminology developed by Popa (cf. [11]), a cocycle c is said to untwist if there exists a homomorphism $\rho : \Gamma \to \mathbb{A}$ such that c is cohomologous to $\tilde{\rho}$. We denote by $\mathcal{U}_{\text{fin}}$ the class of Polish groups consisting of closed subgroups of the unitary group of a finite von Neumann algebra.

Theorem (Popa's Cocycle Superrigidity Theorem, [11], [12])**.** *(for Bernoulli shift actions) Let Γ be a group which contains an infinite normal subgroup which either has property (T) or is the direct product of an infinite group and a nonamenable group, and let (X_0, μ_0) be a standard probability space. Then the Bernoulli shift action $\Gamma \curvearrowright \Pi_{g\in\Gamma}(X_0, \mu_0)$ is $\mathcal{U}_{\text{fin}}$-cocycle superrigid.*

The proof of this theorem uses a combination of deformation/rigidity and intertwining techniques that were initiated in [6]. Roughly, if we are given a cocycle into a unitary group of a II_1 factor, we may consider the "twisted" group algebra sitting inside of the group-measure space construction. The existence of rigidity can then be contrasted against natural malleable deformations from the Bernoulli shift in order to locate the "twisted" algebra inside of the group-measure space construction. Locating the "twisted" algebra allows us to "untwist" it, and, in so doing, untwist the cocycle in the process.

The existence of such s-malleable deformations (introduced by Popa in [9, 10]) actually occurs in a broader setting than the (generalized) Bernoulli shifts with diffuse core, but it was Furman [2] who first noticed that the even larger class of Gaussian actions are also s-malleable.

Given a real Hilbert space $\mathcal{H}$, there is a natural measure preserving action of $\mathcal{O}(\mathcal{H})$ on the probability space (X, μ) generated by $\dim(\mathcal{H})$ iid Gaussian random variables. If $\pi : \Gamma \to \mathcal{O}(\mathcal{H})$ is an orthogonal representation of Γ then the Gaussian action of Γ corresponding to π is the action of Γ on X induced by this representation.

The interplay between the representation theory and the ergodic theory of a group via the Gaussian action has been fruitfully exploited in the literature (cf. the seminal works of Connes & Weiss and of Schmidt, [1, 15, 16], *inter alios*).

In this talk, we will explore $\mathcal{U}_{\text{fin}}$-cocycle superrigidity within the class of Gaussian actions. The first theme we take up is the relation between the cohomology of group representations and the cohomology of their respective Gaussian actions. Under general assumptions, we show that cohomological information coming from the representation can be faithfully transferred to the cohomology group of the action with coefficients in the circle group $\mathbb{T}$.

Theorem 1. *Let Γ be a countable discrete group and $\pi : \Gamma \to \mathcal{O}(\mathcal{K})$ a weakly mixing orthogonal representation. A necessary condition for the corresponding Gaussian action to be $\{\mathbb{T}\}$-cocycle superrigid is for $H^1(\Gamma, \pi) = \{0\}$.*

The Bernoulli shift action of a group is precisely the Gaussian action corresponding to the left-regular representation, and the circle group $\mathbb{T}$ is contained in the class $\mathcal{U}_{\text{fin}}$. When combined with Corollary 2.4 in [5] which states that for a nonamenable group vanishing of the first ℓ^2-Betti number is equivalent to $H^1(\Gamma, \lambda) = \{0\}$ we obtain the following corollary.

Corollary 2. *Let Γ be a countable discrete group. If $\beta_1^{(2)}(\Gamma) \neq 0$ then the Bernoulli shift action is not $\mathcal{U}_{\text{fin}}$-cocycle superrigid.*

The second theme explored in this talk is the deformation/derivation duality developed by the speaker in [3]. The flexibility inherent at the infinitesimal level allows us to offer a unified treatment of Popa's theorem in the case of generalized Bernoulli actions and expand the class of groups whose Bernoulli actions are known to be $\mathcal{U}_{\text{fin}}$-cocycle superrigid. As a partial converse to the above results, we have that an *a priori* stronger property than having $\beta_1^{(2)}(\Gamma) = 0$, L^2-rigidity (see Definition 1.12 in [4]), is sufficient to guarantee $\mathcal{U}_{\text{fin}}$-cocycle superrigidity of the Bernoulli shift.

Theorem 3. *Let Γ be a countable discrete group. If $L\Gamma$ is L^2-rigid then the Bernoulli shift action of Γ is $\mathcal{U}_{\text{fin}}$-cocycle superrigid.*

Examples of groups for which this holds are groups which contain an infinite normal subgroup which has relative property (T) or is the direct product of an infinite group and a nonamenable group, recovering Popa's Cocycle Superrigidity Theorem for Bernoulli actions of these groups.

We also obtain new groups for which Popa's theorem holds. For example, we show that the theorem holds for any generalized wreath product $A_0 \wr_X \Gamma_0$, where A_0 is a non-trivial abelian group and Γ_0 does not have the Haagerup property. Also, if Γ is a lattice in a connected Lie group which does not have the Haagerup property.

We end with the following open problem (see [13]).

Problem 4 (Chifan, Ioana, Peterson, Popa)**.** *If Γ has vanishing first ℓ^2-Betti number, is the Bernoulli action of Γ $\mathcal{U}_{\text{fin}}$-cocycle superrigid? What about for $\Gamma = \mathbb{Z} \wr \mathbb{F}_2$?*

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Some necessary and some unnecessary applications of set theory to C*-algebras

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In recent years some longstanding open problems in the theory of C*-algebras have been solved by using set-theoretic methods. For example, Phillips and Weaver ([12]) used the Continuum Hypothesis, CH, to construct an outer automorphism of the Calkin algebra while I ([3]) used Todorcevic's Axiom (sometimes called Open Colouring Axiom, OCA) to prove that the Calkin algebra has no outer automorphisms. These results together show that the existence of an outer automorphism of the Calkin algebra can be neither proved nor refuted on the basis of the usual axioms of set theory, ZFC, since the theories ZFC, ZFC+CH and ZFC+TA are equiconsistent.

Our second example is concerned with the relative commutant $A' \cap A^U$ of a C*-algebra A in its ultrapower. Kirchberg asked whether all relative commutants of a separable C*-algebra A are isomorphic. Ge and Hadwin ([11]) proved that CH implies a positive answer and the author, Hart and Sherman ([6]) proved the converse: the positive answer to Kirchberg's question for any infinite-dimensional C*-algebra implies CH. Therefore the positive answer to Kirchberg's problem is not only independent from ZFC—it is equivalent to the Continuum Hypothesis! In [9] this result was sharpened, by showing that a separable C*-algebra A either has only isomorphic relative commutants or it has $2^{2^{\aleph_0}}$ nonisomorphic relative commutants. All of these results also apply to McDuff factors verbatim. The proofs of these results rely on classical results of Keisler, Dow and Shelah and show the benefits of applying the model theory of metric structures ([2], [7]) to operator algebras.

As the first two examples show, many problems independent from ZFC can be resolved using CH. Typically one uses a well-ordering of the reals in which all initial segments are countable to recursively construct an object with pathological properties. In both of our examples, this object is an isomorphism between two quotient structures. A result of W. Hugh Woodin, the so-called 'Σ^2_1-absoluteness theorem', gives a metamathematical explanation of the role of CH. For example, the existence of a K-theory reversing automorphism of the Calkin algebra, if consistent with ZFC, most likely follows from CH.[1]

A well-known problem of Naimark asks whether the only C*-algebras for which all irreducible representations are equivalent are the algebras of compact operators? Akemann and Weaver ([1]) proved that the existence of a counterexample is consistent with the ZFC. Their proof uses R. Jensen's $\diamondsuit_{\aleph_1}$ (frequently denoted by $\diamondsuit$), a strengthening of CH first used in order to prove the existence of a nonseparable linear ordering with no uncountable family of disjoint open intervals. It was conjectured by Suslin that no such ordering exists. Suslin's Hypothesis is also consistent with ZFC, and even with ZFC+CH. Another ingredient is the result of Kishimoto–Ozawa–Sakai, to the effect that the pure state space of a separable simple C*-algebra is homogeneous. As N.C. Phillips pointed out, since the construction starts from the CAR algebra and uses only crossed products with $\mathbb{Z}$ and inductive limits, the Akemann–Weaver counterexample to Naimark's problem is nuclear.

It is not known whether a counterexample can be constructed using ZFC+CH, or using ZFC alone. In other words, at present it is not known whether the use of set theory in [1] is necessary or not. It is conceivable that a counterexample can be

[1]More precisely, if there exists a proper class of measurable Woodin cardinals and there exists a K-theory reversing automorphism of the Calkin algebra in some forcing extension of the universe, then there exists a K-theory reversing automorphism of the Calkin algebra in every forcing extension of the universe that satisfies CH.

constructed without any extra set-theoretic axioms. By recent results of Shelah, instances of $\diamondsuit$ hold in all models of ZFC obtained without help of large cardinal axioms. Therefore, a proof that all simple nuclear C*-algebras have homogeneous pure state space would substantially weaken the assumptions needed to construct a counterexample to Naimark's problem.

Definition 1. Let A be a C*-algebra.

(1) A is UHF if A is a tensor product of full matrix algebras.
(2) A is AM (approximately matricial) if A is a direct limit of full matrix algebras.
(3) A is LM (locally matricial) if $\forall \epsilon > 0$ and for every finite $F \subseteq A$ there is a full matrix algebra $M \subseteq A$ such that $F \subseteq_\epsilon M$.

It is clear that UHF implies AM and that AM implies LM. Glimm proved that for separable unital C*-algebras these three notions coincide, and Dixmier asked whether the separability assumption is necessary for this conclusion.

A *character density* of A, $\chi(A)$, is the least cardinal of a dense subset of A. Hence A is separable if and only if $\chi(A) = \aleph_0$. Recall that $\aleph_1$ and $\aleph_2$ denote the first two uncountable cardinals. The following was proved in [8].

Theorem 2 (Farah–Katsura, 2008)**.**

(1) *AM $\not\Rightarrow$ UHF in any uncountable character density.*
(2) *LM $\Leftrightarrow$ AM in character density $\leq \aleph_1$.*
(3) *LM $\not\Rightarrow$ AM in character density $\geq \aleph_2$.*

The proofs of these results use some set theory, most notably combinatorics of the stationary subsets of $\aleph_1$. However, the theorem is proved within ZFC. As a matter of fact, we were not even using transfinite recursion—counterexamples were simply defined by a formula. Therefore, unlike the first two examples, the use of axiomatic set theory in this proof is not necessary.

Upon hearing about our results, M. Takesaki asked whether counterexamples to LM $\Rightarrow$ AM can be found among C*-algebras faithfully represented on a separable Hilbert space? Essentially using an idea of Bruce Blackadar, we were able to construct a nonseparable AM algebra faithfully represented on a separable Hilbert space. Such an algebra cannot be UHF.

In [5] I was able to extend these results and construct an AM (therefore simple nuclear) C*-algebra with faithful representations on both separable and nonseparable Hilbert space. As pointed out earlier, this shows that the Akemann-Weaver construction of a counterexample to Naimark's problem cannot be easily transferred to a situation in which $\diamondsuit$ holds on a cardinal larger than $\aleph_1$.

More examples of applications of set theory to C*-algebras can be found in [13], [10] or in N. Weaver's contribution to this volume (cf. [4]).

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Sofic groups, amalgamation and asymptotic freeness

Kenneth J. Dykema
(joint work with Benoît Collins)

We show that free products of sofic groups with amalgamation over amenable subgroups are sofic. We also show that families of independent uniformly distributed permutation matrices and certain families of non–random permutation matrices (essentially, those coming from quasi–actions of a sofic group) are asymptotically $*$–free as the matrix size grows without bound. These results are contained in the paper [3].

Sofic groups were introduced by M. Gromov [8] and named by B. Weiss [15]. In short, a group is sofic if it can be approximated (in a certain weak sense) by permutations. All amenable and residually amenable groups are sofic. Due in large part to work of Elek and Szabó [7], the class of sofic groups is known to be closed under taking direct products, subgroups, inverse limits, direct limits, free products, and extensions by amenable groups. See also [13] and [4] for recent interesting examples. It is unknown whether all groups are sofic, though Gromov's famous paradoxical dictum ("any statement about all countable groups is either trivial or false") would argue against it.

Several results illustrate the utility of knowing that a given group is sofic. Gromov [8] proved that Gottschalk's Surjunctivity Conjecture holds for the groups now called sofic. Elek and Szabó [5] proved that Kaplansky's Conjecture holds for sofic groups. In [6] they gave a description of sofic groups in terms of ultrapowers and proved that sofic groups are hyperlinear, which entails that their group von Neumann algebras embed in R^ω; thus, the topic of sofic groups makes contact with Connes' Embedding Problem, which is a fundamental open problem in the theory of von Neumann algebras. See the survey articles [10] and [11] for more on hyperlinear and sofic groups. A. Thom [12] proved some interesting results about the group rings of sofic groups. L. Bowen [1] classified the Bernoulli shifts of a sofic group, provided that the group is also Ornstein (e.g., if it contains an infinite amenable group as a subgroup).

Now we recall a few basic notions and give a definition of sofic groups. (See [5] for a proof that the definition in [15], which was for finitely generated groups, agrees with the one found below if the group is finitely generated.) The *normalized Hamming distance* $\mathrm{dist}(\sigma,\tau)$ between two permutations σ and τ, both elements of the symmetric group S_n, is defined to be the number of points not fixed by $\sigma^{-1}\tau$, divided by n. Note that if we consider S_n as acting on an n–dimensional complex vector space as permutation matrices, then this normalized Hamming distance is equal to $1-\mathrm{tr}_n(\sigma^{-1}\tau)$, where tr_n is the trace on $M_n(\mathbb{C})$ normalized so that the identity has trace 1.

A group Γ is *sofic* if for every finite subset F of Γ and every $\epsilon>0$, there exist an integer $n\geq 1$ and a map $\phi:\Gamma\to S_n$ such that

(i) for every $g\in F\backslash\{e\}$, $\mathrm{dist}(\phi(g),\mathrm{id})>1-\epsilon$, where e is the identity element of Γ,

(ii) for all $g_1,g_2\in F$, $\mathrm{dist}(\phi(g_1^{-1}g_2),\phi(g_1)^{-1}\phi(g_2))<\epsilon$.

We will call a map ϕ satisfying these properties an (F,ϵ)–*quasi–action* of Γ.

Our main theorem is:

Theorem 1. *Let $\Gamma=\Gamma_1 *_H \Gamma_2$ be a free product of groups with amalgamation over a subgroup H. Assume that Γ_1 and Γ_2 are sofic and that H is an amenable group. Then Γ is sofic.*

The main idea of the proof is to start with sufficiently good quasi–actions of Γ_1 and Γ_2 and from them to construct a quasi–action of Γ. Using Følner sets from H, we can describe these quasi–actions of the Γ_i when they are restricted to certain finite subsets of H. Now the idea is to conjugate one of the quasi–actions by a random permutation matrix, of some sort. The proof that this yields a quasi–action of Γ relies on calculation of certain matrix integrals over the permutation matrices. The proof is analogous in spirit to the proof of the main result in [2], which constructs matricial microstates in free products of von Neumann algebras with amalgamation over a hyperfinite subalgebra, and which implies that the class of hyperlinear groups is closed under taking free products with amalgamation over amenable subgroups.

The use of random matrices in this way is redolent of freeness. Asymptotic freeness of independent matrices (of various sorts) as the matrix size grows without bound is one of the mainstays of free probability theory, going back to seminal work [14] of Voiculescu, and has been a key element in applications of free probability theory to operator algebras and elsewhere. Asymptotic freeness of independent random permutation matrices was proved by A. Nica [9]. By combining Nica's result with our vanishing of moments result, we are able to extend Nica's asymptotic freeness result to the case of independent random permutation matrices *and* certain sequences of non–random permutation matrices; these are essentially sequences that arise from quasi–actions of sofic groups.

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Quantum symmetries in free probability

Stephen Curran

(joint work with Teodor Banica and Roland Speicher)

A sequence of random variables is called *exchangeable* if its joint distribution is invariant under permutations. In the 1930's, de Finetti gave his famous characterization of infinite exchangeable sequences as conditionally independent and identically distributed. It was later shown by Freedman [9] that conditionally i.i.d.

centered Gaussian sequences are characterized by the stronger condition of orthogonal invariance. While these results fail for finite sequences, approximation results have been obtained by Diaconis and Freedman [8].

De Finetti's theorem fails in the noncommutative context, as both classical and free independence give rise to exchangeability (see [10]). However, it was recently discovered by Köstler and Speicher ([11]) that de Finetti's theorem does have a natural free analogue if one requires the stronger condition of invariance under *quantum permutations*. More precisely, they call a sequence $(x_i)_{i\in\mathbb{N}}$ of noncommutative random variables *quantum exchangeable* if the joint distribution of $(x_1,\dots,x_n)$ is invariant under the natural action of Wang's *free permutation group* S_n^+ ([14]) which "quantum permutes" the variables. They then proved that for an infinite sequence of noncommutative random variables in a W*-probability space, quantum exchangeability is equivalent to being free and identically distributed with respect to a conditional expectation. In [5], we extended this result to more general sequences and gave an approximation result for finite quantum exchangeable sequences. The free analogue of Freedman's characterization of the Gaussian distribution was obtained in [6], where we showed that sequences of operator-valued free semicircular families with mean zero and common variance are characterized by invariance under Wang's *free orthogonal group* O_n^+ ([14]).

In this talk we present de Finetti theorems for the class of "easy" quantum groups, which is based on joint work with Teodor Banica and Roland Speicher [3]. The "easiness" condition for G a *compact orthogonal quantum group* ([16]) was introduced by Banica and Speicher in [1]. Very roughly, this condition states that the tensor category of G, consisting of intertwining operators between tensor powers of the fundamental representation of G, should be spanned by certain partitions coming from the tensor category of S_n. While this may at first appear to be a rather technical condition, we believe that it provides a good framework for studying certain common probabilistic features of S_n, O_n and their free versions, as is demonstrated in this talk (see also [4]).

There are exactly 6 classical orthogonal groups which are easy, 4 of which are listed in the table below with the corresponding collections of partitions:

Group	Partitions
Permutation group S_n	P: All partitions
Orthogonal group O_n	P_2: Pair partitions
Hyperoctahedral group H_n	P_h: Partitions with even block sizes
Bistochastic group B_n	P_b: Partitions with block size ≤ 2

There are also the 2 trivial modifications $S_n' = S_n \times \mathbb{Z}_2$ and $B_n' = B_n \times \mathbb{Z}_2$.

A quantum group G is called free if the corresponding partitions are noncrossing. There is a one to one correspondence between classical easy groups and free quantum groups, which on a combinatorial level corresponds to restricting to noncrossing partitions:

Quantum group	Partitions
S_n^+	NC: All non-crossing partitions
O_n^+	NC_2: Non-crossing pair partitions
H_n^+	NC_h: NC partitions with even block sizes
B_n^+	NC_b: NC partitions with block size ≤ 2

There are also free versions of S_n', B_n'. In addition there are the *half-liberated* easy quantum groups O_n^*, H_n^*, and the *hyperoctahedral series* $H_n^{(s)}$ and $H_n^{[s]}$, see [2].

We give the following extension of the classical results of de Finetti and Freedman to the context of easy quantum groups:

Theorem 1. *Let G be one of the easy quantum groups $S, H, O, B, S^+, H^+, O^+, B^+$.*

(1) *(Classical case): Suppose that $(X_i)_{i\in\mathbb{N}}$ is a sequence of random variables with moments of all orders such that the joint distribution of $(X_1, \ldots, X_n)$ is invariant under transformations from G_n for each $n \in \mathbb{N}$. Let $\mathcal{T}$ be the tail σ-algebra of the sequence, then:*
   (a) *If $G = S$, then $(X_i)_{i\in\mathbb{N}}$ are conditionally i.i.d. given $\mathcal{T}$.*
   (b) *If $G = H$, then $(X_i)_{i\in\mathbb{N}}$ are conditionally i.i.d. and have even distributions, given $\mathcal{T}$.*
   (c) *If $G = O$, then $(X_i)_{i\in\mathbb{N}}$ are conditionally i.i.d. Gaussian with mean zero, given $\mathcal{T}$.*
   (d) *If $G = B$, then $(X_i)_{i\in\mathbb{N}}$ are conditionally i.i.d. Gaussian, given $\mathcal{T}$.*

(2) *(Free case): Let $(x_i)_{i\in\mathbb{N}}$ be a sequence of self-adjoint random variables in a W*-probability space (M, φ), and suppose that the joint distribution of $(x_1, \ldots, x_n)$ is invariant under "quantum transformations" from G_n for each $n \in \mathbb{N}$. Then there is a φ-preserving conditional expectation E onto the tail algebra D such that:*
   (a) *If $G = S^+$, then $(x_i)_{i\in\mathbb{N}}$ are freely independent and identically distributed with respect to E.*
   (b) *If $G = H^+$, then $(x_i)_{i\in\mathbb{N}}$ are freely independent, and have even and identical distributions, with respect to E.*
   (c) *If $G = O^+$, then $(x_i)_{i\in\mathbb{N}}$ form a D-valued free semicircular family with mean zero and common variance.*
   (d) *If $G = B^+$, then $(x_i)_{i\in\mathbb{N}}$ form a D-valued free semicircular family with common mean and variance.*

The proof uses the *Weingarten formula* for evaluating integrals on easy quantum groups with respect to their Haar measures, and the combinatorial theory of classical and free cumulants. Roughly speaking, we show that if a sequence is G-invariant then the nonvanishing classical or free cumulants, taken with respect to the expectation onto the tail algebra, correspond precisely to the collection of partitions associated to the easy quantum group G. We also discuss approximation results for finite quantum invariant sequences.

It was shown by Ryll-Nardzewski [12] that de Finetti's theorem in fact holds under the weaker condition of *spreadability*, i.e., invariance under taking subsequences. In the latter part of this talk we introduce *quantum increasing sequence* *spaces* $A_i(k,n)$, which are certain universal C*-algebras whose spectrum is naturally identified with the space of increasing sequences $\mathbf{l} = (1 \leq l_1 < \cdots < l_k \leq n)$. These objects form *quantum families of maps* from $\{1,\ldots,k\}$ to $\{1,\ldots,n\}$, in the sense of Sołtan [13]. *Quantum spreadability* for a sequence of noncommutative random variables is then naturally defined by invariance under these families of quantum transformations. We give the following free analogue of the theorem of Ryll-Nardzewski:

Theorem 2 ([7]). *A sequence* $(x_i)_{i\in\mathbb{N}}$ *of self-adjoint random variables in a tracial W*-probability space* (M,τ) *is quantum spreadable if and only if the variables are free and identically distributed with amalgamation over the tail algebra.*

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Free probability and quantum information theory

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(joint work with Ion Nechita, Serban Belinschi)

1. Entanglement and additivity

In quantum information theory, a vector subspace C of a tensor product $A \otimes B$ of two Hilbert spaces is said to be *entangled* iff 0 is the only vector of C that can be written as $a \otimes b$ with $a \in A, b \in B$. The problems of quantifying entanglement, and of finding 'highly' entangled spaces are very important in quantum information theory.

For $x \in A \otimes B$, its *singular values* – also known as *Schmidt coefficients* in quantum information theory – are non-negative numbers $\lambda_1(x) \geq \ldots \geq \lambda_k(x) \geq 0$ such that

$$x = \sum_{i=1}^{k} \sqrt{\lambda_i} e_i(x) \otimes f_i(x)$$

Here, $k = \min(dim(A), dim(B))$; $e_i(x)$, $f_i(x)$ are orthonormal vectors. The sequence $\lambda(x) = (\lambda_1(x), \ldots)$ is uniquely defined.

Let $K_C = \{\lambda(x), x \in C, ||x|| = 1\}$. This compact set is a subset of the set

$$\Delta_k^+ = \{y \in \mathbb{R}_+^k : y_1 \geq y_2 \geq \cdots \geq y_k \geq 0, \sum_{i=1}^{k} y_i = 1\}.$$

Besides, we have $\Delta_k^+ \subset \Delta_k$, where $\Delta_k = \{x \in \mathbb{R}_+^k \mid \sum_{i=1}^k x_i = 1\}$ is the $(k-1)$-dimensional probability simplex. We shall also consider $\tilde{K}_C \subset \Delta_k$. It is the symmetrization of K_C under permuting the coordinates.

For a positive real number $p > 0$, we recall that the *Rényi entropy of order p* of a probability vector $x \in \Delta_k$ to be

$$H^p(x) := \frac{1}{1-p} \log \sum_{i=1}^{k} x_i^p.$$

Since $\lim_{p \to 1} H^p(x)$ exists, we define the *Shannon entropy* of x to be this limit:

$$H(x) = H^1(x) = -\sum_{i=1}^{k} x_i \log x_i.$$

Note that the following are equivalent:

- C is entangled
- $(1, 0, \ldots 0)$ does not belong to K_C
- $\min_{x \in K_C} H^p(x) > 0$

One important motivation for studying entanglement comes from the minimum output entropy problem. A *quantum channel* is a linear completely positive trace preserving map $\Phi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_k(\mathbb{C})$. For a quantum channel $\Phi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_k(\mathbb{C})$, we define its *minimum output Rényi entropy (of order p)* by

$$H^p_{\min}(\Phi) := \min_{\substack{\rho \in \mathcal{M}_n(\mathbb{C}) \\ \rho \geq 0, \operatorname{Tr} \rho = 1}} H^p(\Phi(\rho)).$$

where $H^p(\Phi(\rho)) := H^p(\text{eigenvalues}(\Phi(\rho))$.

The additivity problem – also known as *Minimum Output Entropy (MOE) additivity problem* – can be formulated as follows: For a given $p \geq 1$, do there exist two quantum channels Φ_1, Φ_2 such that

$$H^p_{\min}(\Phi_1 \otimes \Phi_2) < H^p_{\min}(\Phi_1) + H^p_{\min}(\Phi_2)?$$

The answer to this problem is affirmative: There exists such a pair of channels (Φ_1, Φ_2). It was proved by Hayden and Winter in the case $p > 1$, and by Hastings in the case $p = 1$ ([7, 9, 6]).

However there is no concrete counterexample so far: All available existence proofs rely on random techniques. Our aim is to explain the relation between this problem and free probability theory, and to show how results of free probability type in random matrix theory help to improve the bounds.

2. Random subspaces and free probability

Let k be an integer and $t \in (0, 1)$ be a real number. Let n be an integer less than kN. We assume that n is a function of N and that n, N vary and tend to infinity according to $n \sim tNk$.

Let V_N be a random subspace of dimension n of $\mathbb{C}^k \otimes \mathbb{C}^N$. In other words, V_N is a random element of the Grassman manifold of projections of rank n in $End(\mathbb{C}^{Nk})$, with respect to the invariant probability measure.

We are interested in the random set $K_{N,t,k} := \tilde{K}_{V_N}$ with the notations of the previous paragraph. Let us define the convex body $K_{t,k} \subset \mathbb{R}^k$ as follows:

$$K_{t,k} \subset \mathbb{R}^k := \{a \in \Delta_k, \forall b \in \Delta_k, \sum_{i=1}^k a_i b_i \leq \phi((b_i), t)\}$$

for all $(b_i) \in \Delta_k$, with $\phi((b_i), t) := ||pbp||_\infty$ where p and b are free selfadjoint elements of respective distribution $(1-t)\delta_0 + t\delta_1$ and $k^{-1} \sum \delta_{b_i}$ (see [12]).

Our main result is

Theorem 1. *Almost surely, $dist(K_{N,t,k}, K_{t,k}) \to 0$, where dist is the Hausdorff distance between sets.*

This result relies heavily on the following

Proposition 2. *Let Q_n be a random projection of rank n in $\mathcal{M}_{Nk}$ and let A be the diagonal matrix $diag(a_1, \ldots, a_k) \otimes I_N$. Then the operator norm of $Q_n A Q_n$ converges almost surely to $\phi((a_i), t)$ as $N \to \infty$.*

This proposition can be deduced with some work from results of almost sure convergence in norm of [5], that generalize results of asymptotic freeness of [11].

The idea to use results of almost sure convergence in operator norm of random matrices for the MOE additivity problem first appeared in [4], where the non-additivity of the MOE is reproved and extended in the case $p > 1$.

It is of natural interest to study the convex ball $K_{t,k}$ and in particular the minimum of Rényi functions on it. Here, we can prove:

Theorem 3. *For any $p \geq 1$, the minimum of the Rényi entropy on $K_{t,k}$ is reached at the point $x_{opt,t} =$*
$(\phi((1,0,\ldots,0),t),(k-1)^{-1}(1-\phi((1,0,\ldots,0),t),\ldots,(k-1)^{-1}(1-\phi((1,0,\ldots,0),t))$.
For any $p > 1$, this is the only minimizer. As a consequence, almost surely as $n \to \infty$:

$$H^p_{\min}(\Phi_n) \to H^p(x_{opt,t})$$

From this and our previous results in [3], we are able to prove that for any $k \geq 183$ there is a violation of the MOE additivity. The violation happens almost surely with the Bell state iff $k \geq 183$, and this violation can be made as close as possible to $\log 2$.

These results are proved in [1].

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Fundamental Facts about Nuclear C*-algebras

Nathanial P. Brown
(joint work with Wilhelm Winter)

In 1976 Alain Connes published the following remarkable result (cf. [6]): There is a unique injective II_1-factor with separable predual, namely the hyperfinite II_1-factor R of Murray and von Neumann. An important technical ingredient in the proof was the following "stability" result: If M is an injective II_1-factor, then $M \cong M\bar{\otimes}R$, i.e., M absorbs R tensorially; or, in more recent terminology, M is R-stable. With this stability result in hand, Connes was able to deduce classification.

In 1994 Eberhard Kirchberg announced a new and equally remarkable stability result: If A is simple, nuclear and purely infinite, then $A \cong A \otimes \mathcal{O}_\infty$, where $\mathcal{O}_\infty$ is the Cuntz algebra with infinitely many generators. (See [7] for a proof.) Knowing $\mathcal{O}_\infty$-stability, Kirchberg and Phillips were able to deduce classification results (independently) for simple, nuclear, purely infinite algebras (cf. [11]).

Very recently Wilhelm Winter has announced another fantastic stability result: If A is simple and has finite nuclear dimension (cf. [15]), then $A \cong A \otimes \mathcal{Z}$, where $\mathcal{Z}$ is the Jiang-Su algebra. From $\mathcal{Z}$-stability Winter has been able to deduce impressive classification results, and has laid out a clear strategy for attacking the general case. (See [14].)

On the other hand, there are now numerous examples of simple nuclear C*-algebras which are *not* $\mathcal{Z}$-stable and hence it becomes exceedingly important to know whether Winter's assumption of finite nuclear dimension is just one of convenience, or whether we've finally found the "right" noncommutative notion of dimension. Indeed, such considerations led Andrew Toms and Wilhelm Winter to the following conjecture:

Conjecture 1. *Let A be a simple, unital, nuclear C*-algebra. Then the following are equivalent:*

(1) *A has finite nuclear dimension;*
(2) *$A \cong A \otimes \mathcal{Z}$;*
(3) *A has strict comparison.*

Strict comparison is a C*-analogue of the following important property enjoyed by all II_1-factors: comparison of projections is determined by the trace. Thus we like to think of this conjecture in the following metamathematical way: A simple, nuclear C*-algebra is like an injective factor if and only if it is finite dimensional. Aside from its aesthetic appeal, confirmation of this conjecture would have deep and profound consequences for the structure theory of nuclear C*-algebras.

1. Technical results

I'd like to report on some technical work related to the Toms-Winter conjecture. It is known that (1) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (3), so my objective was to test the conjecture by constructing some potential counterexamples to the remaining implication.

In the end we don't get counterexamples; instead, they further suggest a positive answer to the Toms-Winter conjecture.

The starting point is the following fact: There exist nuclear C*-algebras which are both stably finite and purely infinite (in the sense of [8]). More precisely, it follows from Voiculescu's homotopy invariance theorem that cones and suspensions of Cuntz algebras, or any other purely infinite algebra, are quasidiagonal (cf. [13]). These paradoxical algebras exhibit exotic behavior and can be a nice source of counterexamples. The problem is, they aren't simple or unital (hence don't fall under the Toms-Winter conjecture). So we need a way of constructing simple algebras out of them, and the generalized inductive limits of Blackadar and Kirchberg (cf. [1]) allow us to do that. We now describe the construction, which is just a small modification of [4].

Let A be a separable unital QD C*-algebra and $\varphi_n\colon A \to M_{k(n)}(\mathbb{C})$ be u.c.p. maps such that $\|a\| = \lim_n \|\varphi_n(a)\|$ and $\|\varphi_n(ab)-\varphi_n(a)\varphi_n(b)\| \to 0$ for all $a, b \in A$. We assume $k(n) \to \infty$ since this can always be arranged, and is necessarily the case for all non-subhomogeneous QD algebras.

Next, choose natural numbers $s(n) > k(n)$ and define unital complete order embeddings (see Definition 11.2.1 and Remark 11.2.2 in [5]) $\Phi_n\colon A \to M_{s(n)}(A)$ as follows:

$$\Phi_n(a) = \left(\begin{array}{cccc|c} a & & & & \\ & a & & & \\ & & \ddots & & \\ & & & a & \\ \hline & & & & \varphi_n(a) \end{array}\right),$$

where all unspecified entries are zero and the corner $\varphi_n(a)$ is a scalar matrix. Define an inductive sequence

$$A \xrightarrow{\psi_1} M_{s(1)} \otimes A \xrightarrow{\psi_2} M_{s(1)} \otimes M_{s(2)} \otimes A \xrightarrow{\psi_3} M_{s(1)} \otimes M_{s(2)} \otimes M_{s(3)} \otimes A \xrightarrow{\psi_4} \cdots,$$

where

$$\psi_n\colon M_{s(1)} \otimes \cdots \otimes M_{s(n-1)} \otimes A \to M_{s(1)} \otimes \cdots \otimes M_{s(n-1)} \otimes M_{s(n)}(A)$$

is the unital complete order embedding

$$\psi_n = \mathrm{id}_{s(1)} \otimes \cdots \otimes \mathrm{id}_{s(n-1)} \otimes \Phi_n.$$

Checking that this defines a generalized inductive system in the sense of [1] is elementary – but a pain. The key points are the asymptotic multiplicativity of the maps $\{\varphi_n\}$ *and* the special form of our connecting maps.

Definition 2. Let $B = \mathrm{g}\varinjlim(M_{s(1)} \otimes \cdots \otimes M_{s(n-1)} \otimes A, \Psi_{m,n})$ be the generalized inductive limit C^*-algebra associated to the system above.

The first theorem shows that B is always simple, almost always has strict comparison (and probably always does), and many nice properties of A pass to B.

Theorem 3. *The generalized inductive limit B is unital, separable, simple, quasidiagonal and has stable rank one. Moreover,*

(1) *if* $\liminf_n(s(n) - k(n)) = 1$*, then* B *has real rank zero and stable rank one;*
(2) *if* $\liminf_n(s(n) - k(n)) \geq 2$*, then* B *is approximately divisible (cf.* [2]*);*
(3) *if* A *is nuclear (resp. exact; cf.* [5]*), then* B *is nuclear (resp. exact);*
(4) *if* A *satisfies the Universal Coefficient Theorem (cf.* [12]*), then so does* B*;*
(5) *if every tracial state on* A *is uniformly locally finite dimensional (cf.* [3, Definition 3.4.1]*), then the same is true for* B*;*

Returning to the Toms-Winter conjecture, we consider the case of quasidiagonal, purely infinite C*-algebras.

Theorem 4. *Let* A *be the unitization of an exact, QD, purely infinite C*-algebra. Then* B *is tracially AF in the sense of Huaxin Lin (cf.* [9]*).*

In particular, by Theorem 3, if A is nuclear and satisfies the Universal Coefficient Theorem, then Lin's classification theorem ([10]) applies and we deduce that B is an AH algebra with finite nuclear dimension.

The proofs of these results will be contained in a forthcoming paper.

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The C*-algebra of a vector bundle

Marius Dadarlat

Let $E \in \mathrm{Vect}(X)$ be a locally trivial complex vector bundle over a compact Hausdorff space X. If we endow E with a hermitian metric, then the space $\Gamma(E)$ of all continuous sections of E becomes a finitely generated Hilbert $C(X)$-bimodule. Let O_E denote the Cuntz-Pimsner algebra associated to $\Gamma(E)$ as defined in [4]. O_E is a locally trivial unital $C(X)$-algebra (continuous field) with fiber at x isomorphic to the Cuntz algebra $O_{n(x)}$, where $n(x)$ is the rank of the fiber E_x of E, see [5, Prop. 2].

The motivation for this research comes from an informal question of Cuntz: What are the invariants of E captured by the $C(X)$-algebra O_E? In other words, how are E and F related if there is a $C(X)$-linear $*$-isomorphism $O_E \cong O_F$. We have shown in [3] that if X has finite covering dimension, then all separable unital $C(X)$-algebras with fibers isomorphic to a fixed Cuntz algebra O_n are automatically locally trivial. Thus it is also natural to ask which of these algebras are isomorphic to Cuntz-Pimsner algebras associated to a vector bundle of constant rank n.

Theorem 1. *Let X be a compact metrizable space and let $E, F \in \mathrm{Vect}(X)$ of rank ≥ 2. Then O_E embeds as a unital $C(X)$-subalgebra of O_F if and only if there is $h \in K^0(X)$ such that $1 - [E] = (1 - [F])h$. Moreover, $O_E \cong O_F$ as $C(X)$-algebras if and only if there is h as above of virtual rank one.*

Thus the principal ideal $(1 - [E])K^0(X)$ determines O_E up to isomorphism and an inclusion of principal ideals $(1 - [E])K^0(X) \subset (1 - [F])K^0(X)$ corresponds to unital embeddings $O_E \subset O_F$. In particular if $E \in \mathrm{Vect}_{m+1}(X)$, then $O_E \cong C(X) \otimes O_{m+1}$ if and only if $[E] - 1$ is divisible by $m \geq 1$.

Let $\widetilde{K}^0(X) = \ker(K^0(X) \xrightarrow{rank} H^0(X, \mathbb{Z}))$ be the subgroup of $K^0(X)$ corresponding to elements of virtual rank zero, and set $[\widetilde{E}] := [E] - \mathrm{rank}(E) \in \widetilde{K}^0(X)$. Using the nilpotency of $\widetilde{K}^0(X)$ we derive the following:

Theorem 2. *Let X be a compact metrizable space of finite dimension n. Suppose that $\mathrm{Tor}(K^0(X), \mathbb{Z}/m) = 0$. If $E, F \in Vect_{m+1}(X)$, then $O_E \cong O_F$ as $C(X)$-algebras if and only if $([\widetilde{E}] - [\widetilde{F}])\left(\sum_{k=1}^{n} (-1)^{k-1} m^{n-k} [\widetilde{F}]^{k-1}\right)$ is divisible by m^n in $\widetilde{K}^0(X)$.*

In view of Theorem 1 it is natural to seek explicit and computable invariants (e.g. characteristic classes) of a vector bundle E that depend only on the principal ideal $(1 - [E])K^0(X)$ and hence which are invariants of O_E. We shall only consider vector bundles of rank $m + 1$ with $m \geq 1$.

For each $m \geq 1$, consider the sequence of polynomials $p_n \in \mathbb{Z}[x]$,

$$p_n(x) = \ell(n)\, m^n \, \log\left(1 + \frac{x}{m}\right)_{[n]} = \sum_{k=1}^{n} (-1)^{k-1} \frac{\ell(n)}{k} m^{n-k} x^k,$$

where $\ell(n)$ denotes the least common multiple of the numbers $\{1, 2, ..., n\}$ and the index $[n]$ indicates that the formal series of the natural logarithm is truncated after its nth term.

Theorem 3. *Let X be a finite CW complex of dimension d and let $E, F \in \mathrm{Vect}_{m+1}(X)$. If $O_E \cong O_F$ as $C(X)$-algebras, then $p_{\lfloor d/2 \rfloor}([\widetilde{E}]) - p_{\lfloor d/2 \rfloor}([\widetilde{F}])$ is divisible by $m^{\lfloor d/2 \rfloor}$ in $\widetilde{K}^0(X)$.*

For $x \in \mathbb{R}$, we set $\lfloor x \rfloor := \max\{k \in \mathbb{Z} : k \leq x\}$ and $\lceil x \rceil := \min\{k \in \mathbb{Z} : k \geq x\}$. Theorem 3 extends to finite dimensional compact metrizable spaces: if $n \geq 1$ is an integer such that $\widetilde{K}^0(X)^{n+1} = \{0\}$, then $p_n([\widetilde{E}]) - p_n([\widetilde{F}])$ is divisible by m^n in $\widetilde{K}^0(X)$ whenever $O_E \cong O_F$ as $C(X)$-algebras. The same conclusion holds for infinite dimensional spaces X but in that case n depends on E and F.

Concerning the completeness of the above invariant we have the following:

Theorem 4. *Let X be a finite CW complex of dimension d. Suppose that m and $\lfloor d/2 \rfloor!$ are relatively prime and that $\mathrm{Tor}(H^*(X, \mathbb{Z}), \mathbb{Z}/m) = 0$. If $E, F \in Vect_{m+1}(X)$, then $O_E \cong O_F$ as $C(X)$-algebras if and only if $p_{\lfloor d/2 \rfloor}([\widetilde{E}]) - p_{\lfloor d/2 \rfloor}([\widetilde{F}])$ is divisible by $m^{\lfloor d/2 \rfloor}$ in $\widetilde{K}^0(X)$.*

Next we exhibit characteristic classes of E which are invariants of O_E. For each $n \geq 1$ consider the polynomial $q_n \in \mathbb{Z}[x_1, ..., x_n]$:

$$\sum_{k_1 + 2k_2 + \cdots + nk_n = n} (-1)^{k_1 + \cdots + k_n - 1} m^{n - (k_1 + \cdots + k_n)} \frac{n!\,(k_1 + \cdots + k_n - 1)!}{1!^{k_1} \cdots n!^{k_n}\, k_1! \cdots k_n!}\, x_1^{k_1} \cdots x_n^{k_n}.$$

Thus $q_1(x_1) = x_1$, $q_2(x_1, x_2) = mx_2 - x_1^2$, $q_3(x_1, x_2, x_3) = m^2 x_3 - 3m x_1 x_2 + 2x_1^3$, etc. Let ch_n be the integral characteristic classes that appear in the Chern character, $ch = \sum_{n \geq 0} \frac{1}{n!} ch_n$.

Theorem 5. *Let X be a compact metrizable space and let $E, F \in \mathrm{Vect}_{m+1}(X)$. If $O_E \cong O_F$ as $C(X)$-algebras, then $q_n(ch_1(E), ..., ch_n(E)) - q_n(ch_1(F), ..., ch_n(F))$ is divisible by m^n in $H^{2n}(X, \mathbb{Z})$, for all $n \geq 1$.*

Reducing mod m^n it follows that the sequence

$$q_n(\dot{ch}_1(E), ..., \dot{ch}_n(E)) \in H^{2n}(X, \mathbb{Z}/m^n), \quad n \geq 1,$$

is an invariant of the $C(X)$-algebra O_E.

Let us denote by $\mathcal{O}_{m+1}(X)$ the set of isomorphism classes of unital separable $C(X)$-algebras with all fibers isomorphic to O_{m+1}. We study the range of the map $\mathrm{Vect}_{m+1} \to \mathcal{O}_{m+1}(X)$ using the computation of the homotopy groups of $\mathrm{Aut}(O_{m+1})$ of [3] and the Atiyah-Hirzebruch spectral sequence [1]. If T is a set, we denote by $|T|$ its cardinality. Let X be a finite connected CW complex of dimension d and let $m \geq \lceil (d-3)/2 \rceil$.

Theorem 6. *If $\mathrm{Tor}(H^*(X, \mathbb{Z}), \mathbb{Z}/m) = 0$, then each element of $\mathcal{O}_{m+1}(X)$ is isomorphic to O_E for some E in $Vect_{m+1}(X)$.*

Moreover $|\mathcal{O}_{m+1}(X)| = |\widetilde{K}^0(X) \otimes \mathbb{Z}/m| = |\widetilde{H}^{even}(X, \mathbb{Z}/m)|$.

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Classifying the C*-algebras of minimal homeomorphisms

Andrew S. Toms

Our talk focussed on joint work with Wilhelm Winter, summarized by the following result:

Theorem 1 (T-Winter, [2, 3]). *Let $\alpha : X \to X$ and $\beta : Y \to Y$ be minimal homeomorphisms of compact finite-dimensional metric spaces. It follows that $A := \mathrm{C}(X) \rtimes_\alpha \mathbb{Z}$ absorbs the Jiang-Su algebra tensorially. Moreover, if the projections in each of A and $B := \mathrm{C}(Y) \rtimes_\beta \mathbb{Z}$ separate traces, and if there is a graded order isomorphism*

$$\phi : (\mathrm{K}_0 A, \mathrm{K}_0 A^+, [1_A], \mathrm{K}_1 A) \to (\mathrm{K}_0 B, \mathrm{K}_0 B^+, [1_B], \mathrm{K}_1 B),$$

then there is a $*$-isomorphism $\Phi : A \to B$ which induces ϕ.

A theorem of Giol and Kerr shows that the finite-dimensionality of X is necessary in order to have $\mathcal{Z}$-stability for $\mathrm{C}(X) \rtimes_\alpha \mathbb{Z}$. (Something slightly weaker such as mean dimension zero for α may also suffice.) The bulk of the effort in the proof of Theorem 1 is concentrated on proving $\mathcal{Z}$-stability for $\mathrm{C}(X) \rtimes_\alpha \mathbb{Z}$. The classification portion of the theorem then follows from results of H. Lin and N. C. Phillips for real rank zero crossed products, and an earlier classification result of W. Winter valid for fairly general classes of $\mathcal{Z}$-stable C*-algebras.

The proof of $\mathcal{Z}$-stability for $\mathrm{C}(X) \rtimes_\alpha \mathbb{Z}$ uses the structure of subalgebras of the form

$$A_F = C^*(\mathrm{C}(X), u\mathrm{C}_0(X \backslash F)),$$

where $F \subseteq X$ is closed and u is the unitary implementing the action of α. These subalgebras were first studied by I. Putnam in the case that X is the Cantor set, and their theory was expanded considerably by N. C. Phillips in his work on recursive subhomogeneous algebras. Using results of H. Lin and N. C. Phillips, one can show that if $x, y \in X$ lie in distinct orbits, then the subalgebras $A_{\{x\}}$, $A_{\{y\}}$, and $A_{\{x,y\}}$ are all simple inductive limits of recursive subhomogeneous algebras with no dimension growth. In particular, they have finite decomposition rank, and are therefore $\mathcal{Z}$-stable by a result of W. Winter. The $\mathcal{Z}$-stability of these algebras can then be "pasted together" to obtain $\mathcal{Z}$-stability for the crossed product.

There is a second more recent approach to the $\mathcal{Z}$-stability of $A_{\{x\}}$, $A_{\{y\}}$, and $A_{\{x,y\}}$ which answers other interesting questions. It concerns inductive limits of subhomogeneous algebras (ASH algebras) with slow dimension growth, a class of C*-algebras which has long been a focus of Elliott's classification program for nuclear separable C*-algebras (and which includes the no dimension growth algebras mentioned above).

The first ingredient in this newer result is a theorem of W. Winter:

Theorem 2 (Winter, [4]). *Let D be a simple unital separable C*-algebra with locally finite decomposition rank, and suppose that the Cuntz semigroup of D is isomorphic to that of $D \otimes \mathcal{Z}$. It follows that $D \cong D \otimes \mathcal{Z}$.*

Locally finite decomposition rank is a bit too technical to define here. Roughly, it says that finite subsets of D are approximately contained in subalgebras with finite "topological dimension". The important thing for us is that ASH algebras have this property.

The second ingredient is the input for Theorem 2:

Theorem 3 (T, [1]). *Let D be a unital simple ASH algebra with slow dimension growth. It follows that the Cuntz semigroup of D is isomorphic to that of $D \otimes \mathcal{Z}$.*

As a corollary, we obtain the following equivalence:

Corollary 4. *Let D be a unital simple ASH algebra. It follows that $D \cong D \otimes \mathcal{Z}$ if and only if D has slow dimension growth.*

We moreover obtain a classification theorem for ASH algebras with slow dimension growth in the spirit of Theorem 1:

Corollary 5. *Let A, B be unital simple ASH algebras with slow dimension growth. Suppose that the projections separate traces in both A and B, and that there is a graded order isomorphism*

$$\phi : (\mathrm{K}_0A, \mathrm{K}_0A^+, [1_A], \mathrm{K}_1A) \to (\mathrm{K}_0B, \mathrm{K}_0B^+, [1_B], \mathrm{K}_1B).$$

It follows that there is a $*$-isomorphism $\Phi : A \to B$ which induces ϕ.

If one insists on classification by graded ordered K-theory alone, then the conditions of simplicity, slow dimension growth, and the separation of traces by projections cannot be relaxed in Corollary 5.

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Spectra of C^*-algebras and Extensions

Eberhard Kirchberg
(joint work with Oleg Boruch Ioffe)

We give a new result on non-commutative selection, and use it to derive a necessary and sufficient topological criterion for extensions to be semi-split in an (ideal system) equivariant way. Similar arguments show that a proof of the below stated conjecture on existence of extensions with prescribed ideal system implies also that (at least) all *coherent* locally compact T_0 spaces are primitive ideal spaces of amenable C^*-algebras.

In the following, spaces $P, X, Y, \ldots$ are second countable, algebras $A, B, \ldots$ are separable, — except corona spaces, multiplier algebras $\mathcal{M}(B)$, and ideals of corona algebras $\mathrm{Q}(B) := \mathcal{M}(B)/B$. We use the natural isomorphisms $\mathcal{I}(A) \cong \mathbb{O}(\mathrm{Prim}(A)) \cong \mathcal{F}(\mathrm{Prim}(A))^{op}$, and denote by $\mathbb{Q} := [0,1]^\infty$ the Hilbert cube (with its coordinate-wise order).

We call a map $\Psi\colon \mathbb{O}(X) \to \mathbb{O}(Y)$ *lower semi-continuous* if $(\bigcap_n \Psi(U_n))^\circ = \Psi((\bigcap_n U_n)^\circ)$, and *upper semi-continuous* if $\bigcup_n \Psi(U_n) = \Psi(\bigcup_n U_n)$, for each sequence $U_1, U_2, \ldots \in \mathbb{O}(X)$. If one works with *closed sets* $F_n \in \mathbb{F}(X)$, then one has to replace intersections by unions and interiors by closures.

The basic new observation is the following:

Proposition 1. *If B is stable and $\Psi\colon \mathcal{I}(B) \to \mathcal{I}(A)$ is a lower semi-continuous action of* $\mathrm{Prim}(B)$ *on A, then there exists a* lower s.c. *action $\mathcal{M}(\Psi)\colon \mathcal{I}(\mathcal{M}(B)) \to \mathcal{I}(A)$ of* $\mathrm{Prim}(\mathcal{M}(B))$ *on A, that has the following properties (i)–(iii):*

(i) *$\mathcal{M}(\Psi)$ is* monotone upper semi-continuous *(:= sup's of* upward directed *families of ideals will be respected).*
(ii) *$\mathcal{M}(\Psi)(J_1) = \mathcal{M}(\Psi)(J_2)$ if $J_1 \cap \delta_\infty(\mathcal{M}(B)) = J_2 \cap \delta_\infty(\mathcal{M}(B))$.*
(iii) *$\mathcal{M}(\Psi)(\mathcal{M}(B, I)) = \Psi(I)$ for all $I \in \mathcal{I}(B)$.*

The "extension" $\mathcal{M}(\Psi)$ of Ψ with (i)–(iii) is unique.

The map $\delta_\infty\colon \mathcal{M}(B) \to \mathcal{M}(B)$ in (ii) is the infinite repeat. For strongly p.i. (not necessarily separable) B and exact A, there is a nuclear *-morphism $h\colon A \to B$ with $\Psi(J) = h^{-1}(h(A) \cap J)$, if and only if Ψ is lower s.c. and *monotone* upper s.c., see [4]. This and Proposition 1 together yield the following theorem.

Theorem 2. *Suppose that B is stable, $A \otimes \mathcal{O}_2$ contains a regular* exact *C^*-algebra $C \subset A \otimes \mathcal{O}_2$, and that $\Psi\colon \mathcal{I}(B) \to \mathcal{I}(A)$ is a lower s.c. action of* $\mathrm{Prim}(B)$ *on A.*

*Then there is a *-morphism $h\colon A \to \mathcal{M}(B)$ such that $\delta_\infty \circ h$ is unitarily equivalent to h, $\Psi(J) = h^{-1}(h(A) \cap \mathcal{M}(B, J))$ and that*

$$[h]_J\colon A/\Psi(J) \to \mathcal{M}(B/J) \cong \mathcal{M}(B)/\mathcal{M}(B, J)$$

is weakly nuclear for all $J \in \mathcal{I}(B)$.

Here a subalgebra $C \subset D$ is *regular* if $C \cap (I + J) = (C \cap I) + (C \cap J)$ for all $I, J \in \mathcal{I}(D)$ and C separates the ideals of D.

Theorem 2 can be used to prove that the below described topological necessary and sufficient criterion (ii) in Theorem 4 is equivalent to the existence of an equivariant completely positive split map for extensions, Theorem 4(i).

Let $\epsilon\colon B \to E$ be a *-monomorphism onto a closed ideal of E and $\pi\colon E \to A$ an epimorphism such that $\epsilon(B)$ is the kernel of π. Further, we suppose that B is *stable* and that $\epsilon(B)$ is an essential ideal of E. We denote by $\gamma\colon A \to \mathrm{Q}(B) = \mathcal{M}(B)/B$ the Busby invariant of the extension

$$0 \to B \to E \to A \to 0\,.$$

We consider very general "actions" $\psi_B\colon S \to \mathcal{I}(B)$, $\psi_E\colon S \to \mathcal{I}(E)$, and $\psi_A\colon S \to \mathcal{I}(A)$, of a set S on B, E and A, and we require that the extension E is ψ-equivariant:

(i) $\epsilon(\psi_B(s)) = \epsilon(B) \cap \psi_E(s) = \epsilon(B)\psi_E(s)$, and
(ii) $\psi_A(s) = \pi(\psi_E(s))$ for all $s \in S$,

i.e., $0 \to \psi_B(s) \to \psi_E(s) \to \psi_A(s) \to 0$ is exact for each $s \in S$.

An action $\Phi\colon \mathcal{I}(A) \to \mathcal{I}(B)$ of $\mathrm{Prim}(A)$ on B is *upper semi-continuous* if Ψ preserves sup of families in $\mathcal{I}(A)$, i.e., $\Psi(I+J) = \Psi(I) + \Psi(J)$ and Ψ is monotone upper semi-continuous. Every upper semi-continuous action Φ has a lower semi-continuous adjoint map $\Psi\colon \mathcal{I}(B) \to \mathcal{I}(A)$ such that (Ψ, Φ) build a Galois connection (i.e., $\Psi(J) \supset I$ iff $J \supset \Phi(I)$). The rule is: *The upper adjoint is lower semi-continuous (preserves inf).* The following lemma is easy to see.

Lemma 3. *There is a unique maximal upper semi-continuous map $\Phi\colon \mathcal{I}(A) \to \mathcal{I}(B)$ with the property that $\Phi(\psi_A(s)) \subset \psi_B(s)$ for all $s \in S$.*

Now we can pass to the Galois *adjoint* of Φ and use Theorem 2 to get ([1]) that the criteriom (ii) in the following theorem implies (i). (The implication (i)⇒(ii) is trivial.)

Theorem 4. *Let B, E, A, ϵ, π, γ, $\psi_Y\colon S \to \mathcal{I}(Y)$ (for $Y \in \{B, E, A\}$) be as above, and let $\Phi\colon \mathcal{I}(A) \to \mathcal{I}(B)$ the map given in Lemma 3.*

Suppose, in addition, that A is exact and that B is weakly injective (has the WEP of Lance).

Then the following properties (i) and (ii) of the extension are equivalent:

(i) *The extension has an S-equivariant c.p. splitting map, i.e., there is a c.p. map $V\colon A \to E$ with $\pi \circ V = \mathrm{id}_A$ and $V(\psi_A(s)) \subset \psi_E(s)$ for all $s \in S$.*
(ii) *The Busby invariant $\gamma\colon A \to \mathrm{Q}(B)$ is nuclear, and, for all $J \in \mathcal{I}(A)$,*

$$\pi_B(\mathcal{M}(B, \Phi(J))) \supset \gamma(J)\,.$$

The criterion (ii) is in the class of amenable algebras of pure topological nature, and it shows that there can't be a sufficient criterion that only refers to the action on A (i.e., needs also assumptions on the action on B or on E), even if A and B are both amenable. The criterion (ii) is satisfied in the special case, where S is the lattice $\mathbb{O}(X)$ for some l.c. space X, and where E is exact, B is nuclear, the

[1]by the approximation argument in [1] for the existence of lifts in given operator convex cones

action of X on B is upper semi-continuous and where the action of X on A is continuous (:= upper and lower semi-continuous). In particular, (ii) is valid, if A and B are amenable, S is the lattice $\mathbb{O}(X)$ for some l.c. space X and the actions ψ_A *and* ψ_B of X on A and B are both continuous. Thus, by Theorem 4, *every* short exact sequence is (X-equivariant) *semisplit*, if we consider only the category of amenable algebras with continuous X-actions.

One gets from Theorem 4(ii) also the result from the Appendix of [2] for the category of *exact* $\mathrm{C}(P)$-algebras for Polish l.c. spaces P: Sufficient for the existence of a $\mathrm{C}(P)$-modular c.p. splitting of the extension is that B is amenable and that A is the algebra of C_0-sections of a continuous field over P. (Here are implicit requirements on all three algebras, e.g. that the E is an exact $\mathrm{C}(P)$-algebra.)

Similar considerations lead to a study of the following question on coherent locally compact spaces X.

Question 5. *Is every second-countable* coherent *(Def. 6) locally compact sober* T_0 *space* X *homeomorphic to the primitive ideal spaces* $\mathrm{Prim}(A)$ *of some* amenable A *?*

Definition 6. A subset C of T_0 space is X is *saturated* if $C = \mathrm{Sat}(C)$, where $\mathrm{Sat}(C)$ means the intersection of all $U \in \mathbb{O}(X)$ with $U \supset C$.

A sober T_0 space X is *coherent* if the intersection $C_1 \cap C_2$ of two *saturated* compact subsets $C_1, C_2 \subset X$ is again compact.

Notice that on the compact Hausdorff space $\mathbb{Q}$ there is also the (coarser) order topology, that is generated by the complements of the intervals $[0, \alpha]$ in $\mathbb{Q}$ and is a sober l.c. T_0 topology. We write Y_H and Y_{lsc} for $Y \subset \mathbb{Q}$ to distinguish the Y with Hausdorff topology from them with order topology (both induced from the corresponding topologies on $\mathbb{Q}$). The singleton $\{0\}$ is closed in both topologies.

Proposition 7.

(0) *A sober l.c. space* X *is coherent, if and only if, there is a subset* Y *of the Hilbert cube* $\mathbb{Q}$, *that is closed in the Hausdorff topology of* $\mathbb{Q}$, *such that* X *is homeomorphic to* $Y_{\mathrm{lsc}} \setminus \{0\}$.

(1) *Each closed subset* $F \subset \mathbb{Q}_H$ *is a coherent sober subspace* F_{lsc} *of* $\mathbb{Q}_{\mathrm{lsc}}$, *and is the intersection of an decreasing sequence* F_k *of closed subspaces of* $\mathbb{Q}_H$ *that are continuously order-isomorphic to spaces* $G_k \times \mathbb{Q}$ *with* $G_k \subset [0,1]^{n_k}$ *a finite union of* n_k*-dimensional cubes* $\alpha + t \cdot [0,1]^{n_k}$.

(2) *If* $F = \bigcap_k F_k$ *for a sequence* $F_1 \supset F_2 \supset \cdots$ *of closed subsets in* $\mathcal{F}(\mathbb{Q}_H)$, *and if each* $(F_k)_{\mathrm{lsc}}$ *is the primitive ideal space of an amenable C*-algebra, then* F_{lsc} *is the primitive ideal space of an amenable C*-algebra.*

Proposition 7 implies:

Corollary 8. *If there is a coherent sober l.c. space* X *that is not homeomorphic to the primitive ideal space of an amenable C*-algebra, then there is* $n \in \mathbb{N}$ *and a finite union* Y *of (Hausdorff) closed cubes in* $[0,1]^n$ *such that* Y *with induced order-topology is not the primitive ideal space of any amenable C*-algebra.*

Theorem 9 (O.B. Ioffe, E.K.)**.** *If $G \subset [0,1]^n$ is a finite union of cubes $\alpha + t[0,1]^n$, then the space G_{lsc} has a finite decomposition series $U_1 \subset U_2 \subset \cdots \subset U_k$, by open subsets $U_\ell \subset G_{\mathrm{lsc}}$ such that $U_{\ell+1} \setminus U_\ell$ is the primitive ideal space of an amenable C^*-algebra.*

Theorem 9 and Corollary 8 underline that it could be useful to study the following conjecture. This study needs refinements and generalizations of Proposition 1 in case of amenable B with $B \cong B \otimes \mathcal{O}_2 \otimes \mathbb{K}$ and commutative A.

Conjecture 10. *Suppose that X is a locally compact sober space, and there exists an open subset $U \subset X$ such that U and $X \setminus U$ are homeomorphic to primitive ideal spaces of amenable C^*-algebras.*

Then X is homeomorphic to the primitive ideal space of an amenable C^*-algebra.

A positive result would give that sober l.c. spaces are primitive ideal spaces of amenable C^*-algebras — if they have *decomposition series by open subsets $\{U_\alpha\}$ with coherent spaces $U_{\alpha+1} \setminus U_\alpha$.*

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Using duality theorems to compute K-theory for ring C*-algebras

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(joint work with Joachim Cuntz)

We present duality theorems for crossed products attached to affine transformations on adele rings. Moreover, we explain how these duality results can be used to determine the K-groups of certain ring C*-algebras.

1. Duality theorems

Let us fix a global field K, i.e. a finite separable field extension of $\mathbb{Q}$ or $\mathbb{F}_p(T)$ (p being a prime number). Let $\mathbb{A}_\infty$ and $\mathbb{A}_f$ denote the infinite and the finite adele ring over K, respectively. They are built from the local fields corresponding to infinite or finite places of K (see for instance [6] or [7] for an introduction). K sits canonically as a subring in $\mathbb{A}_\infty$ and $\mathbb{A}_f$. Thus we obtain actions of the $ax+b$-group P_K over K on the adele rings. These actions are given by affine transformations.

When we compare the associated crossed products, we obtain the following

Theorem 1 (Duality Theorem). *$C_0(\mathbb{A}_\infty) \rtimes P_K$ and $C_0(\mathbb{A}_f) \rtimes P_K$ are strongly Morita equivalent.*

The proof can be found in [3].

There are a couple of generalizations and modifications of this duality result. For instance, there is an equivariant version (see [3], Theorem 4.1). Moreover, we can obtain similar results for rings of matrices, i.e. for the actions of $P_{M_n(K)}$ on $M_n(\mathbb{A}_\infty)$ and $M_n(\mathbb{A}_f)$. And we also have local versions of the duality theorem.

Theorem 1 establishes an unexpected connection between infinite and finite adeles. For instance, if K is a number field, i.e. a finite extension of $\mathbb{Q}$, then the infinite adele ring is always homeomorphic to $\mathbb{R}^n$ (where n is the degree of K over $\mathbb{Q}$), whereas the finite adele ring is always totally disconnected. So $\mathbb{A}_\infty$ and $\mathbb{A}_f$ do not look very similar, at least from a topological point of view, and thus it is surprising that one can prove such a result as stated in Theorem 1.

On the one hand, these duality theorems make sense on their own, revealing interesting and unexpected phenomena. On the other hand, we were led to these duality results by our study of so-called ring C*-algebras, and indeed, we will see that the crossed products in the duality theorems are closely related to certain ring C*-algebras, and that these connections can be used to compute K-theory for these ring C*-algebras.

2. Ring C*-algebras

The first example of a ring C*-algebra is due to J. Cuntz (see [1]). Then, the theory of ring C*-algebras has been developed in [2] and [5]. Here is the construction:

Let R be a ring and let Z be the set of left zero-divisors of R. The set of left regular elements of R is denoted by $R^\times$, i.e. $R^\times = R \setminus Z$. Consider the Hilbert space $\ell^2(R)$ with its canonical orthonormal basis $\{\xi_r\colon r \in R\}$. Using the ring structure of R, we can define two families of bounded operators, $\{U^a\colon a \in R\}$ and $\{S_b\colon b \in R^\times\}$, by $U^a(\xi_r) = \xi_{a+r}$ and $S_b(\xi_r) = \xi_{br}$. The reduced ring C*-algebra is defined as the C*-subalgebra of $\mathcal{L}(\ell^2(R))$ generated by $\{U^a\colon a \in R\}$ and $\{S_b\colon b \in R^\times\}$. We denote it by $\mathfrak{A}_r[R]$.

We can think of $\mathfrak{A}_r[R]$ as the C*-algebra generated by the left regular representation of R. Moreover, we can also construct full ring C*-algebras which are given in terms of generators and relations. These constructions are studied in [5].

To build the bridge to the crossed products in the duality theorems, we consider the ring C*-algebra of the ring of integers R in a global field K. It turns out that for such a ring, the corresponding reduced and full ring C*-algebras coincide (see [2], Remark 1). This result leads to a characterization of the concrete C*-algebra $\mathfrak{A}_r[R]$ in terms of natural generators and relations. Using this characterization together with the theory of semigroup crossed products, we can finally establish the desired connection between the crossed products in the duality theorems and ring C*-algebras:

Theorem 2. $\mathfrak{A}_r[R]$ *is strongly Morita equivalent to* $C_0(\mathbb{A}_f) \rtimes P_K$.

A proof can be found in [2], Remark 3.

As an immediate consequence of Theorems 1 and 2, we deduce the following

Corollary 3. $\mathfrak{A}_r[R]$ *is strongly Morita equivalent to* $C_0(\mathbb{A}_\infty) \rtimes P_K$.

3. K-theory

This last observation is the key ingredient in computing K-theory for ring C*-algebras of rings of integers. Whereas it does not seem possible to determine the K-groups of $\mathfrak{A}_r[R]$ working over the finite adeles, i.e. merely using Theorem 2 without using the duality theorem, it turns out that once we apply the duality result and work over the infinite adele ring, the K-theory of $\mathfrak{A}_r[R]$ can be computed in a straightforward way.

In the number field case, we know that $\mathbb{A}_\infty$ is isomorphic to $\mathbb{R}^n$ as an additive topological group. Thus we can apply homotopy arguments to compute K-theory, and it is clear that this is impossible when we work over the totally disconnected finite adele ring.

In the function field case, there is not such a big difference between infinite and finite places. For instance, both $\mathbb{A}_\infty$ and $\mathbb{A}_f$ are totally disconnected, so there is no hope for homotopy arguments. However, there is a big difference between $\mathbb{A}_\infty$ and $\mathbb{A}_f$: $\mathbb{A}_f$ is the (restricted) product of infinitely many local fields, whereas $\mathbb{A}_\infty$ is given as the product of finitely many local fields. So from an algebraic point of view, the situation is much simpler when we work over the infinite adele ring. And it turns out that this is the reason why we can - working over $\mathbb{A}_\infty$ - compute K-theory without any extra arguments like homotopy.

Here are the final results:

Let K be a number field and R the ring of integers in K. We assume that the set of roots of unity in K is given by $\mu = \{\pm 1\}$. Choose a free abelian subgroup Γ of $K^\times$ with $K^\times = \mu \times \Gamma$.

Theorem 4.

$$K_*(\mathfrak{A}_r[R]) \cong \begin{cases} K_0(C^*(\mu)) \otimes_{\mathbb{Z}} \Lambda^*(\Gamma) \ \textit{if} \ \# \{v_{\mathbb{R}}\} = 0 \\ \Lambda^*(\Gamma) \ \textit{if} \ \# \{v_{\mathbb{R}}\} \ \textit{is odd} \\ \Lambda^*(\Gamma) \oplus ((\mathbb{Z}/2\mathbb{Z}) \otimes_{\mathbb{Z}} \Lambda^*(\Gamma)) \ \textit{if} \ \# \{v_{\mathbb{R}}\} \ \textit{is even and at least} \ 2. \end{cases}$$

Here we consider graded tensor products where $K_0(C^*(\mu))$ and $\mathbb{Z}/2\mathbb{Z}$ are trivially graded. We take the diagonal grading on the direct sum. $\# \{v_{\mathbb{R}}\}$ is the number of real places of K. The reader may consult [3] for the details.

Now let Γ be a free abelian subgroup of $\mathbb{F}_q(T)^\times$ with $\mathbb{F}_q(T)^\times = \mathbb{F}_q^\times \times \Gamma$.

Theorem 5. $K_*(\mathfrak{A}_r[\mathbb{F}_q[T]]) \cong \tilde{K}_0(C^*(\mathbb{F}_q^\times)) \otimes_{\mathbb{Z}} \Lambda^*(\Gamma)$.

Here $\tilde{K}_0(C^*(\mathbb{F}_q^\times))$ is the reduced K-theory of $C^*(\mathbb{F}_q^\times)$, i.e. the cokernel of the canonical map $K_0(\mathbb{C}) \to K_0(C^*(\mathbb{F}_q^\times))$. Moreover, the isomorphism in this theorem is meant as an isomorphism of $\mathbb{Z}/2\mathbb{Z}$-graded abelian groups where $\tilde{K}_0(C^*(\mathbb{F}_q^\times))$

is trivially graded, $\Lambda^*(\Gamma)$ is canonically graded and we consider graded tensor products. The proof can be found in [4].

As a last comment, we point out that - apart from technical details - there are many similarities between the number field case and the function field case. In both cases, the duality theorem is the key ingredient in our K-theoretic computations. And the final results look very similar as well. All these similarities demonstrate one of the leading principles in number theory which exactly predicts such a strong analogy between numbers and functions. So everything fits nicely into the general picture.

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Set theory and C*-algebras

Nik Weaver

We survey four results on the interface between set theory and C*-algebras. The common theme is that they are all counterexamples (to well-known old problems) that are built up in a transfinite series of stages.

1. A C*-algebra is *prime* if the intersection of any two nonzero ideals is nonzero; it is *primitive* if it has a faithful irreducible representation. Using Kadison's transitivity theorem, it is easy to show that any primitive C*-algebra must be prime. Conversely, Dixmier [7] proved that every separable prime C*-algebra is primitive. Thus, for separable C*-algebras the two properties are equivalent. Dixmier asked whether separability could be dropped from this assertion.

The answer is no [16]. A nonseparable C*-algebra that is prime but not primitive can be constructed by starting with an uncountable family of commuting projections and then sequentially adding in partial isometries which link selected projections. Two simple conditions can be stated which respectively ensure that the generated C*-algebra will be prime but not primitive. Once these conditions are identified, the details of the construction are just a matter of bookkeeping.

A superior example was provided later by Crabb [6]; it is constructed canonically, in a single step. The set-theoretic example is more complicated, but it came first because it was easier to find. We do not have to be clever to ensure that both conditions are satisfied, we just systematically take care of them one at a time.

2. Identify l^∞ with the diagonal masa (maximal abelian self-adjoint subalgebra) in $B(l^2)$. For any ultrafilter U on $\mathbb{N}$, the formula $\lim_U f(n)$ defines a pure state on l^∞, and every pure state is of this form. Anderson [3] showed that the formula $\lim_U \langle Ae_n, e_n\rangle$ defines a pure state on $B(l^2)$ that extends the original pure state on l^∞. Kadison and Singer [10] famously asked whether this extension is unique. But even if it were, this would not characterize the pure states on $B(l^2)$ since there could be pure states whose restriction to l^∞ is not pure.

In [10] Kadison and Singer had suggested that every pure state on $B(l^2)$ should restrict to a pure state on some masa. Anderson [4] formulated the stronger conjecture that every pure state on $B(l^2)$ restricts to a pure state on some atomic masa. Together with a positive solution to the Kadison-Singer problem, this would completely characterize the pure states on $B(l^2)$.

However, assuming the continuum hypothesis, even the weaker suggestion of Kadison and Singer is false [2]. The counterexample, a pure state whose restriction to any masa is not pure, is built up in $\aleph_1$ stages as a compatible family of pure states f_α on a nested family of separable C*-subalgebras A_α of $B(l^2)$. There are $\aleph_1$ masas M_α of $B(l^2)$ and each one is dealt with at a separate stage by including a projection from M_α in A_α and ensuring that f_α takes a value strictly between 0 and 1 on this projection. The continuum hypothesis is essential to the argument because the construction requires that each A_α be separable.

3. Brown, Douglas, and Fillmore [5] asked whether every automorphism of the Calkin algebra is inner. A negative solution, produced using the continuum hypothesis, was provided in [14]. The construction is similar in outline to the counterexample to Anderson's conjecture: we build up a compatible family of automorphisms ϕ_α of a nested family of separable C*-subalgebras A_α of the Calkin algebra. There are $\aleph_1$ unitaries u_α in the Calkin algebra and we must ensure that ϕ_α is not implemented by u_α on A_α. The technical details are somewhat involved.

(Farah [8] gave a simpler counterexample, also assuming the continuum hypothesis, and he proved the remarkable theorem that the open coloring axiom implies that all automorphisms of the Calkin algebra are inner.)

4. Naimark [12] showed that the algebra of compact operators on a Hilbert space has only one irreducible representation up to unitary equivalence, and he asked [13] whether this property characterizes these algebras. Rosenberg [15] showed that no other separable C*-algebras have this property.

As the representation theory of C*-algebras developed, Rosenberg's result was put in a broader context. C*-algebras were categorized into *type I* and *non type I*; type I C*-algebras have a completely transparent representation theory and there are trivially no type I counterexamples to Naimark's problem, whereas Glimm [9] showed that every separable non type I C*-algebra has uncountably many inequivalent irreducible representations. Naimark's question remained open in the nonseparable case, with the expectation being for a positive solution on the grounds that the representations of a nonseparable non type I C*-algebra should, if anything, be even worse than the representations of a separable non type I C*-algebra.

However, assuming a set-theoretic axiom called *diamond*, which is known to be consistent with but not provable from the standard axioms of set theory, there is a counterexample to Naimark's problem [1]. We build up the example in $\aleph_1$ stages, at each stage adding a unitary which makes some pair of pure states equivalent. This is accomplished using a powerful result of Kishimoto, Ozawa, and Sakai [11] which states that the pure state space of any simple separable C*-algebra is homogeneous. The fact that pure states proliferate exponentially as the construction proceeds is handled by diamond, which at each stage tells us how to choose the next pair of pure states to be made equivalent, and does this in such a way that when the construction is complete every pure state will have been dealt with.

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The Measurement Process in Local Quantum Theory and the Einstein Podolski Rosen Paradox

Sergio Doplicher

We describe in a qualitative way a possible picture of the Measurement Process in Quantum Mechanics [1], which takes into account: 1. the finite and non zero time duration T of the interaction between the observed system and the *microscopic part* of the measurement apparatus; 2. the finite space size R of that apparatus; 3. the fact that the *macroscopic part* of the measurement apparatus, having the role of *amplifying* the effect of that interaction to a macroscopic scale, is composed by a very large but finite number N of particles. The emphasis is on the local nature of realistic observables, which would not fit in Quantum Mechanics of a fixed and finite number of particles, but is an essential feature of Quantum Field Theory.

The conventional picture of the measurement, as an instantaneous action, which turns a pure state into a mixture and suppresses part of the state over the whole space (*reduction of wave packets*) arises only in the limit $N \to \infty, T \to 0, R \to \infty$.

After the discussion, motivated by ideas of Ludwig, by Daneri, Prosperi and Loinger in the early sixties, the limit where N tends to infinity has been most often recognised as the origin of decoherence.

We argue here that, as a consequence of the Principle of Locality (requiring that observables measured in mutually spacelike separated regions of spacetime should commute with one another), before those three limits are taken, no long range entanglement between the values of observables which are spacelike separated far away can be detected. As it is well known, however, entangled states for such observables will exist in local theories: simple examples are discussed explicitly, and show that the abundant experimental evidence of existence of entangled states is not in any contradiction with local commutativity of observables.

Our picture of the measurement process of a local observable implies, however, that in order to detect correlations, one of the observers has to wait until he enters the future causal shadow of the region employed by the apparatus of the other.

This fact is an immediate consequence of the local commutativity of the observables with the field operators (which in turn is no additional assumption, but a property of the canonical construction of the algebra of locally commuting or anticommuting field operators, based on the local observables alone; this construction is based on the crossed product of the C*-Algebra of local observables by the *rigid strict symmetric tensor C*-Category* describing the superselection sectors; for a survey, see [2]).

More precisely, adding to a state the microscopic part of the measuring apparatus will be described by the action of an isometry belonging, together with its time translates over the time duration T of the measurement, to a local field algebra; since this isometry commutes with all observables localised in a spacelike separated region, such observations will not distinguish between the state with or without the measuring apparatus, during the whole duration of the measurement.

Accordingly, there would be no Einstein Podolski Rosen Paradox.

We comment, however, on the fact that, while it is well known that, even in the standard view, the Einstein Podolski Rosen device cannot be used to transmit neither physical effect nor even information, if one adopt the point of view of Local Quantum Physics, local observables will never be one- (or finitely many-) particle observables, and entangled states will always appear as statistical mixtures when restricted to local algebras.

But, in the local picture of the measurement process, in addition, no correlation would be possible between the results of the measurements done by two observers in laboratories which are spacelike separated in far away regions.

Similar views had been proposed already in the early seventies by Hellwig and Kraus. A careful comparison with the growing experimental results of the recent decades might settle the question.

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Fusion for primary fields and twisted loop groups

Antony Wassermann

We explained how to use properties of operator product expansions (fusion) of bounded primary fields to prove the irreducibility of subfactors arising in conformal field theory. These OPEs are deduced indirectly using Haag duality and the particular case of C.C. Moore's result for finite cores of $SU(1,1)$, that representations without fixed vectors have matrix coefficients vanishing at infinity. In the application, these representations lie in the discrete series, so have the decay property because they are subrepresentations of the regular representation. In the last part of the talk we explain how the subfactors defined for loop group representations can be extended to twisted loop group representations. Unlike the untwisted case the twisted case does not approach a finite index classical inclusion. In fact for the twisted loop group for $SU(2N)$ corresponding to the period 2 automorphism with fixed point subgroup $Sp(N)$, the subfactor approaches the inclusion $R_1^{Sp(N)} \supset R_1^{SU(2N)}$ for a minimal action of $SU(2N)$ on the hyperfinite III_1 factor R_1.

Inner functions, real Hilbert subspaces and new Boundary QFT models

Roberto Longo
(joint work with Edward Witten)

We build up local time translation covariant Boundary Quantum Field Theory nets of von Neumann algebras A_V on the Minkowski half-plane M_+ starting with a local conformal net A of von Neumann algebras on $\mathbb{R}$ and an element V of a unitary semigroup $E(A)$ associated with A. The case $V = 1$ reduces to the net A_+ considered by Rehren and the speaker; if the vacuum character of A is summable A_V is locally isomorphic to A_+. We discuss the structure of the semigroup $E(A)$. By using a one-particle version of Borchers theorem and standard subspace analysis, we provide an abstract analog of the Beurling-Lax theorem that allows us to describe, in particular, all unitaries on the one-particle Hilbert space whose second quantization promotion belongs to $E(A^{(0)})$ with $A^{(0)}$ the $U(1)$-current net. Each such unitary is attached to a scattering function or, more generally, to a symmetric inner function. We then obtain families of models via any Buchholz-Mach-Todorov extension of $A^{(0)}$. A further family of models comes from the Ising model.

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The Baum-Connes conjecture for free orthogonal quantum groups

Christian Voigt

The aim of this talk is to discuss K-theoretic properties of the following C^*-algebras introduced by Wang [9].

Definition 1. Let $n \in \mathbb{N}$. The free orthogonal quantum group $A_o(n)$ is the universal C^*-algebra with self-adjoint generators $u_{ij}, 1 \leq i, j \leq n$ and relations

$$\sum_{k=1}^{n} u_{ik}u_{jk} = \delta_{ij}, \qquad \sum_{k=1}^{n} u_{ki}u_{kj} = \delta_{ij}.$$

If we write $u = (u_{ij})$, then the above relations are equivalent to saying that u is an orthogonal matrix. The abelianization of $A_o(n)$ is isomorphic to the algebra $C(O(n))$ of functions on the orthogonal group $O(n)$.
On $A_o(n)$ there exists a comultiplication $\Delta : A_o(n) \to A_o(n) \otimes A_o(n)$ given by

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}.$$

Together with this comultiplication, $A_o(n)$ is a compact quantum group in the sense of Woronowicz. However, in the sequel we shall rather consider it as the full

group C^*-algebra of a discrete quantum group instead. In this picture, $A_o(n)$ may be viewed as a quantum analogue of a free group.
The reduced C^*-algebra $A_o(n)_{\mathsf{red}}$ is the image of $A_o(n)$ in the GNS-representation of its Haar integral. It is known [2] that $A_o(n)$ is not amenable for $n > 2$, that is, the canonical map

$$\lambda : A_o(n) \to A_o(n)_{\mathsf{red}}$$

is not an isomorphism.
From the work of Meyer and Nest [6], [5] arises the formulation of an analogue of the Baum-Connes conjecture for the discrete quantum group corresponding to $A_o(n)$, and our main result is the following theorem [8].

Theorem 2. *Let $n > 2$. Then the discrete quantum group $A_o(n)$ satisfies the strong Baum-Connes conjecture.*

This may be formulated equivalently by saying that $A_o(n)$ has a γ-element and that $\gamma = 1$. As a consequence one obtains the following result.

Theorem 3. *Let $n > 2$. Then the free orthogonal quantum group $A_o(n)$ is K-amenable. In particular, the map*

$$K_*(A_o(n)) \to K_*(A_o(n)_{\mathsf{red}})$$

is an isomorphism.
The K-theory of $A_o(n)$ is

$$K_0(A_o(n)) = \mathbb{Z}, \qquad K_1(A_o(n)) = \mathbb{Z}.$$

These groups are generated by the class of 1 *in the even case and the class of the fundamental matrix u in the odd case.*

We remark that the notion of K-amenability, introduced by Cuntz for discrete groups in [4], carries over to the setting of quantum groups in a natural way.
In the proof of theorem 2 we use the theory of monoidal equivalence introduced by Bichon, de Rijdt and Vaes [3] to transfer the Baum-Connes problem for $A_o(n)$ into a problem concerning $SU_q(2)$ and the Podleś sphere. This step relies on fundamental work of Banica [1]. The crucial part of the argument is a detailed analysis of the equivariant KK-theory of the standard Podleś sphere. Our constructions in connection with the Podleś sphere are based on [7]. Finally, the K-theory computation for $A_o(n)$ involves some homological algebra for triangulated categories worked out in [5].
As a consequence of theorem 3 we obtain the following result, in the same way as in the classical case of free groups.

Theorem 4. *For $n > 2$ the reduced C^*-algebra $A_o(n)_{\mathsf{red}}$ does not contain non-trivial idempotents.*

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Factorization and dilation problems for completely positive maps on von Neumann algebras

Magdalena Musat
(joint work with Uffe Haagerup)

We study factorization and dilation properties for completely positive unital, trace-preserving maps (for short, c.p.u.t. maps) on von Neumann algebras. The starting point for our work has been the question of existence of non-factorizable Markov maps, as formulated by Anantharaman-Delaroche in [1]. We provide simple examples of non-factorizable c.p.u.t. maps on the $n \times n$ matrices for $n \geq 3$, as well as an example of a one-parameter semigroup $(T_t)_{t\geq 0}$ of c.p.u.t. maps on the 4×4 matrices such that T_t fails to be factorizable for all small values of $t > 0$.

Further, we study the noncommutative Rota dilation property introduced by Junge, Le Merdy and Xu in [3] in connection with their work on semigroups of operators acting on noncommutative L_p-spaces. We show that the most natural generalization of Rota's classical dilation theorem (cf. [5]) to the noncommutative setting does not hold, by providing an example of a self-adjoint c.p.u.t. map T on the $n\times n$ matrices for some large n, such that T^2 does not have the noncommutative Rota dilation property.

This work has revealed nice applications to finding estimates for the best constant in the noncommutative little Grothendieck inequality (see [2], [4]). Also, by using these techniques we have very recently solved an open question concerning an asymptotic version of the quantum Birkhoff conjecture.

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Stability of unitary representations

Andreas Thom
(joint work with Marc Burger, Narutaka Ozawa)

A unitary ε-representation of a discrete group G on a Hilbert space H is a map $\pi\colon G \to U(H)$ such that

$$\|\pi(gh) - \pi(g)\pi(h)\| \le \varepsilon$$

for all $g, h \in G$. David Kahzdan [1] showed that unitary ε-representations of amenable groups are (uniform) perturbations of unitary representations. In addition, a classical application of amenability is that uniformly close unitary representations are conjugate by a unitary which is close to the identity. Hence, the unitary representation obtained is unique up to unitary equivalence. We show in [2] that both phenomena fail for groups which contain free subgroups, i.e. there exist unitary ε-representations which are not close to unitary representations, and there exist uniformly close pairs of unitary representations which are not conjugate. It remains an interesting question to decide whether these properties characterize amenability. We also show that there is a close relationship with Dixmier's question about unitarisability of uniformly bounded representations.

Various constructions lead to finite-dimensional unitary ε-representations of the free group which are not close to unitary representations. We show in [2] that groups like $SL(3,\mathbb{Z})$ and $SL(2,\mathbb{Z}[\frac{1}{2}])$ do not allow for such representations. Hence, the dichotomy in the case of finite dimensions is more between groups of higher rank and rank one, as compared to the general case where the dichotomy is between amenable and non-amenable groups.

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K-theory for the Maximal Roe algebra of certain expanders

Hervé Oyono-Oyono
(joint work with Guoliang Yu)

In this joint work with G. Yu [4], we study the maximal version of the coarse Baum-Connes (BC) assembly map for family of expanders arising from a discrete group which satisfies the property τ. If Γ is such a group, the behaviours of the coarse BC conjecture for the associated family of expanders and of the BC conjecture for Γ differ substancially as we can see for $SL_2(\mathbb{Z})$:

- Since $SL_2(\mathbb{Z})$ has the Haagerup property, it satifies the BC conjecture for any coefficients [2];
- The family of Cayley graphs of $(SL_2(\mathbb{Z}/n\mathbb{Z}))_{n\in\mathbb{N}}$ (with respect to any finite set of generators arising from $SL_2(\mathbb{Z})$) is a family of expander graphs, therefore the coarse BC conjecture does not hold for $\coprod_{n\in\mathbb{Z}} SL_2(\mathbb{Z}/n\mathbb{Z})$ [3].

As we shall see, if we consider the maximal version of both conjectures, then their behaviours become quite similar. Recall that for a discrete proper metric space Σ, the aim of the coarse BC conjecture is to compute the K-theory of the Roe algebra of Σ [5]. This C^*-algebra is the completion of the $*$-algebra $C[\Sigma]$ of locally compact operators with finite propagation on $\ell^2(\Sigma)\otimes H$ (here H is any separable Hilbert space).

When Σ has bounded geometry, i.e the cardinal of balls of a given radius is uniformally bounded, then the next lemma, due to G. Gong, Q. Wang and G. Yu shows that $C[\Sigma]$ admits an envelopping algebra [1].

Lemma 1. *Let Σ be a discrete metric space with bounded geometry. For any T in $C[\Sigma]$, there exists a real number c, such that for any $*$-representation ϕ of $C[\Sigma]$ on a Hilbert space H_ϕ the inequality $\|\phi(T)\|_{\mathcal{B}(H_\phi)} \leq c\|T\|_{\mathcal{B}(\ell^2(\Sigma)\otimes H)}$ holds.*

Definition 2. The maximal Roe algebra of a discrete metric space Σ with bounded geometry, denoted by $C^*_{\max}(\Sigma)$, is the completion of $C[\Sigma]$ with respect to the $*$-norm

$$\|T\| = \sup_{(\phi,H_\phi)} \|\phi(T)\|_{\mathcal{B}(H_\phi)},$$

where (ϕ, H_ϕ) runs through representations ϕ of $C[\Sigma]$ on a Hilbert space H_ϕ.

The K-theory of the Roe algebra is the receptacle for generalized indices of (abstract) pseudodifferential elliptic operators on proper metric spaces quasi-isometric to Σ. This allows us to define an assembly map

$$\mu_{\Sigma,*} : \lim_r KK_*(C_0(P_r(\Sigma)), \mathbb{C}) \to K_*(C^*(\Sigma)),$$

where $P_r(\Sigma)$ is the Rips complex of order r for Σ. The coarse BC conjecture then asserts that $\mu_{\Sigma,*}$ is an isomorphism. For a finitely generated group Γ, if $|\Gamma|$ stands for the underlying metric space arising from any word metric, then according to the descent principle [5], the coarse BC conjecture for $|\Gamma|$ implies the Novikov conjecture for Γ. In particular, since G. Yu proved that the coarse BC conjecture holds for every proper metric space Σ with bounded geometry which coarsely embeds into a Hilbert space [7], we get that finitely generated discrete groups which are coarsely embeddable into a Hilbert space satisfy the Novikov conjecture. It is known that infinite families of expanders provide counterexamples for coarse embeddability into a Hilbert space.

The coarse BC assembly map admits a maximal version

$$\mu_{\Sigma,*,max} : \lim_r KK_*(C_0(P_r(\Sigma)),\mathbb{C}) \to K_*(C^*_{max}(\Sigma))$$

compatible with the previous one. Let us focus now on family of expanders associated with residually finite, finitely generated groups with property τ. Let $(\Gamma_i)_{i\in\mathbb{N}}$ be a decreasing family of normal subgroups of Γ with finite index such that $\cap_{i\in\mathbb{N}}\Gamma_i = \{e\}$. We equip Γ/Γ_i with the metric

$$d(a\Gamma_i, b\Gamma_i) = \min\{d(a\gamma_1, b\gamma_2), \gamma_1 \text{ and } \gamma_2 \text{ in } \Gamma_i\},$$

where d is any word metric on Γ and we fix on $X(\Gamma) = \coprod_{i\in\mathbb{N}} \Gamma/\Gamma_i$ an invariant metric which coincides on Γ/Γ_i with the above metric, such that $d(\Gamma/\Gamma_i, \Gamma/\Gamma_j) \geq i+j$ if $i \neq j$. N. Higson, V. Lafforgue and G. Skandalis proved that if Γ has property τ with respect to the family $(\Gamma_i)_{i\in\mathbb{N}}$, then $\mu_{X(\Gamma),*}$ fails to be surjective [3]. If we set $A_\Gamma = \ell^\infty(X(\Gamma),\mathcal{K}(H))/C_0(X(\Gamma),\mathcal{K}(H))$, then the bijectivity of $\mu_{\Sigma,*,\max}$ is related to the bijectivity of the maximal BC assembly map $\mu_{\Gamma,A_\Gamma,*,\max} : \lim_r KK^\Gamma_*(C_0(P_r(\Gamma)), A_\Gamma) \to K_*(A_\Gamma \rtimes_{\max} \Gamma)$ by the following result.

Theorem 3. *$\mu_{X(\Gamma),*,max}$ is an isomorphism if and only if $\mu_{\Gamma,A_\Gamma,*,max}$ is an isomorphism.*

Notice that we have similar statements for injectivity and surjectivity. In particular, we get that if Γ has the Haagerup property (for instance if $\Gamma = SL_2(\mathbb{Z})$), then $\mu_{X(\Gamma),*,\max}$ is an isomorphism. We can also get the following result concerning injectivity of the usual coarse BC assembly map.

Theorem 4. *If $\mu_{\Gamma,A_\Gamma,*} : \lim_r KK^\Gamma_*(C_0(P_r(\Gamma)), A_\Gamma) \to K_*(A_\Gamma \rtimes_{red} \Gamma)$ (the BC assembly map with coefficients in A_Γ) is one-to-one, then $\mu_{X(\Gamma),*}$ is one-to-one.*

Since G. Skandalis, J. L. Tu and G. Yu proved that for a finitely generated group which coarsely embeds into a Hilbert space, the BC assembly map is one-to-one for any coefficients [6], we get

Corollary 5. *If Γ is finitely generated and coarsely embeds into a Hilbert space, then $\mu_{X(\Gamma),*}$ is one-to-one.*

In particular, since $SL_n(\mathbb{Z})$ is exact and hence coarsely embeds into a Hilbert space, we get that the coarse BC assembly map for the disjoint union of expanders $\coprod_{i\in\mathbb{N}} SL_n(\mathbb{Z}/i\mathbb{Z})$ is one-to-one.

The proof of Theorem 3 suggests that the maximal coarse BC conjecture for $X(\Gamma)$ should be related to some asymptotic properties of the family of assembly maps $(\mu_{\Gamma_i,\max,*} : \lim_r KK^{\Gamma_i}_*(C_0(P_r(\Gamma)),\mathbb{C}) \longrightarrow K_*(C(\Gamma/\Gamma_i) \rtimes_{\max} \Gamma))_{i\in\mathbb{N}}$. Indeed, in the odd case, this family of assembly maps can be defined with values in almost ϵ-unitaries ($\|uu^* - 1\| < \epsilon$ and $\|u^*u - 1\| < \epsilon$) with finite propagation (i.e the support with respect to Γ is finite). More precisely, if we fix for every r a

Γ-invariant measure ν_r on $P_r(\Gamma)$ with $\operatorname{supp}\nu_r = P_r(\Gamma)$, then every element in $KK_*^{\Gamma_i}(C_0(P_r(\Gamma)),\mathbb{C})$ can be represented by a K-cycle $(L^2(\nu_r)\otimes H,\rho_r,F_i)$ where

- ρ_r is induced by the representation $C_0(P_r(\Gamma))\hookrightarrow L^2(\nu_r)$;
- F_i has finite propagation, is Γ_i-equivariant and satisfies the K-cycle conditions;

Let us denote by $\Psi^{\Gamma_i}(P_r(\Gamma))$ the set of operators F_i on $L^2(\nu_r)\otimes H$ satisfying the (odd) K-cycle conditions for $KK_1^{\Gamma_i}(C_0(P_r(\Gamma)),\mathbb{C})$, with finite propagation, which are Γ_i-equivariant and such that $\|F_i\|\leq 1$. Then we have

Proposition 6. *For any F_i in $\Psi^{\Gamma_i}((P_r(\Gamma))$ and any $\epsilon\in(0,1/72)$, there exists*

- *s (depending only on ϵ, d and the propagation of F_i);*
- *an ϵ-unitary $u_{F_i,\epsilon}$ in some $M_n(C(\Gamma/\Gamma_i)\rtimes_{max}\Gamma)$ of propagation less than s*

such that if x_{F_i} in $\lim_r KK_*^{\Gamma_i}(C_0(P_r(\Gamma)),\mathbb{C})$ comes from the K-cycle $(L^2(\nu_r)\otimes H,\rho_r,F_i)$ then $\mu_{\Gamma_i,max,*}(x_{F_i})=[u_{F_i,\epsilon}]$ in $K_1(C(\Gamma/\Gamma_i)\rtimes_{max}\Gamma)$.

For any integer i and any positive real r,r',s,s' and any ϵ in $(0,1/72)$, let us consider the following statements:

$\mathbf{QI_1}(\mathbf{i},\mathbf{r},\mathbf{r}',\mathbf{s},\epsilon)$: For any (odd) K-cycle F of $\Psi^{\Gamma_i}(P_r(\Gamma))$ with $u_{F,\epsilon}\sim_{\epsilon,s}1$ in some $M_N(C(\Gamma/\Gamma_i)\rtimes_{max}\Gamma)$, the class corresponding to F lies in the kernel of $KK_1^{\Gamma_i}(C_0(P_r(\Gamma)),\mathbb{C})\longrightarrow KK_1^{\Gamma_i}(C_0(P_{r'}(\Gamma)),\mathbb{C})$;

$\mathbf{QS_1}(\mathbf{i},\mathbf{r},\mathbf{s},\mathbf{s}',\epsilon)$: For any ϵ-unitary u in some $M_k(C(\Gamma/\Gamma_i)\rtimes_{max}\Gamma)$ with propagation less than s, there exists an (odd) K-cycle F of $\Psi^{\Gamma_i}(P_r(\Gamma))$ such that $u_{F,\epsilon}\sim_{2\epsilon,s'}u$ in some $M_N(C(\Gamma/\Gamma_i)\rtimes_{max}\Gamma)$,

where $\sim_{\epsilon,s}$ means homotopy within ϵ-unitaries of propagation less than s. Notice that we have similar statements $\mathbf{QI_0}$ and $\mathbf{QS_0}$ in the even case in terms of ϵ-projections (p self-adjoint and $\|p^2-p\|<\epsilon$) with finite propagation.

Theorem 7. *The following statements are equivalent:*

1. *For any positive real number r the following holds: There is an ϵ in $(0,1/72)$ such that for any positive real s, there exists an integer j and a positive real r' for which $QI_*(i,r,r',s,\epsilon)$ is true for all $i\geq j$.*
2. *$\mu_{X(\Gamma),max,*}$ is one-to-one.*

If we set $X^\infty(\Gamma)=\coprod_{i\in\mathbb{N}}\coprod_{j\geq i}\Gamma/\Gamma_j$, then the asymptotic statements QS_* provide obstructions for the surjectivity of the coarse BC assembly map

$$\mu_{X^\infty(\Gamma),\mathrm{max},*}:\lim_r KK_*(C_0(P_r(X^\infty(\Gamma))),\mathbb{C})\longrightarrow K_*(C^*_{max}(X^\infty(\Gamma))).$$

Similarly, we have QS_* statements for reduced cross-products. If we set $A_\Gamma^\infty=\ell^\infty(X^\infty(\Gamma),\mathcal{K}(H))/C_0(X^\infty(\Gamma),\mathcal{K}(H))$, and if Γ is exact in K-theory, we get obstructions for the surjectivity of the BC assembly map

$$\mu_{\Gamma,A_\Gamma^\infty,*}:\lim_r KK_*^\Gamma(C_0(P_r(X^\infty(\Gamma))),A_\Gamma^\infty)\longrightarrow K_*(A_\Gamma^\infty\rtimes_{\mathrm{red}}\Gamma)$$

from QS_* asymptotic statements.

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E-theory for C*-algebras over topological spaces

Ralf Meyer
(joint work with Marius Dadarlat)

This talk surveyed the main ideas and results of [5]. This article defines an E-theory for separable C*-algebras over second countable topological spaces and establishes its basic properties. The most interesting of these is an approximation theorem, which relates the E-theory over a general space to the E-theories over finite approximations to this space. This provides effective criteria for invertibility of E-theory elements, even over infinite-dimensional topological spaces. Furthermore, we prove a Universal Multicoefficient Theorem for C*-algebras over totally disconnected metrisable compact spaces.

Why do we need such a new E-theory?

Eberhard Kirchberg [6] proved a far-reaching classification theorem for non-simple, strongly purely infinite, stable, nuclear, separable C*-algebras. Roughly speaking, two such C*-algebras are isomorphic once they have homeomorphic primitive ideal spaces – call this space X – and are $\mathrm{KK}(X)$-equivalent in a suitable bivariant K-theory for C*-algebras over X. To apply this classification theorem, we need tools to compute this bivariant K-theory.

Recall that Kasparov theory only satisfies excision for C*-algebra extensions with a completely positive section. Similar technical restrictions appear for all variants of Kasparov theory, including Kirchberg's. This is a severe limitation. For instance, excision does not hold in general for extensions of the form $A(U) \rightarrowtail A \twoheadrightarrow A/A(U)$ for an open subset U, where $A(U)$ denotes the restriction of A to U, extended by 0 to a C*-algebra over the original space, even if A is nuclear.

In the non-equivariant case, such technical problems are resolved by passing to E-theory, which satisfies excision for all C*-algebra extensions (see [1]). The equivariant E-theory defined in [5] has the same advantages over its corresponding KK-theory. At the same time, the two theories are equal in sufficiently many cases to get information about KK from E-theory arguments. For instance, the natural map $\mathrm{E}_*(X;A,B) \to \mathrm{KK}_*(X;A,B)$ is invertible if X is locally compact and Hausdorff and A is a continuous field of nuclear C*-algebras over X. Hence we may use the good formal properties of E-theory to study KK.

Our definition of $\mathrm{E}_*(X;A,B)$ is based on asymptotic homomorphisms satisfying an approximate equivariance condition. An asymptotic homomorphism $\varphi_t\colon A \to B$, $t \in [0,\infty)$, is called *approximately X-equivariant* if for each open subset $U \subseteq X$, we have

$$\lim_{t\to\infty} \|\varphi_t(a)\|_{X\setminus U} = 0 \qquad \text{for all } a \in A(U),$$

where $\|\varphi_t(a)\|_{X\setminus U}$ denotes the norm of $\varphi_t(a)$ in the quotient $B(X \setminus U) = B/B(U)$ of B.

Let $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ be a countable basis for the topology of X. For each $n \in \mathbb{N}$, the open subsets $U_1, \dots, U_n$ generate a finite topology τ_n on X. Let X_n be the T_0-quotient of (X, τ_n), this is a finite T_0-space. The quotient map $X \twoheadrightarrow X_n$ allows us to view C*-algebras over X as C*-algebras over X_n for all $n \in \mathbb{N}$. Our first main result is a short exact sequence

$$\varprojlim_{n\in\mathbb{N}}{}^1 \mathrm{E}_{*+1}(X_n;A,B) \rightarrowtail \mathrm{E}_*(X;A,B) \twoheadrightarrow \varprojlim_{n\in\mathbb{N}} \mathrm{E}_*(X_n;A,B) \tag{1}$$

for all separable C*-algebras A and B over X. This is made plausible by the observation that an asymptotic homomorphism $A \to B$ is approximately X-equivariant if and only if it is approximately X_n-equivariant for all $n \in \mathbb{N}$. Hence the space of approximately X-equivariant asymptotic homomorphisms is the intersection of the spaces of approximately X_n-equivariant asymptotic homomorphisms for $n \in \mathbb{N}$.

As an important application of (1), we give an effective criterion for invertibility of E-theory elements: an element in $\mathrm{E}_*(X;A,B)$ is invertible if and only if its image in $\mathrm{E}_*\big(A(U), B(U)\big)$ is invertible for all $U \in \mathbb{O}(X)$. As a consequence, if all two-sided closed ideals of a separable nuclear C*-algebra A with Hausdorff primitive spectrum X are KK-contractible, then

$$A \otimes \mathcal{O}_\infty \otimes \mathbb{K} \cong \mathrm{C}_0(X) \otimes \mathcal{O}_2 \otimes \mathbb{K}.$$

This result solves the problem of characterising the trivial continuous fields with fibre $\mathcal{O}_2 \otimes \mathbb{K}$ within the class of strongly purely infinite, stable, continuous fields of C*-algebras. It is worth noting that in general the KK-contractibility of ideals does not follow from the KK-contractibility of the fibres. Indeed, there are examples of separable nuclear continuous fields A over the Hilbert cube with all fibres isomorphic to $\mathcal{O}_2$ and yet such that $\mathrm{K}_0(A) \neq 0$, see [2].

While (1), in principle, reduces the computation of $\mathrm{E}_*(X;A,B)$ for infinite spaces X to the corresponding problem for the finite approximations X_n, this does not yet lead to a Universal Coefficient Theorem. If $\mathrm{E}_*(X_n;A,B)$ is computable by Universal Coefficient Theorems for all $n \in \mathbb{N}$, the latter will usually involve short exact sequences. Thus we have to combine two short exact sequences, as in the computation of the K-theory for crossed products by $\mathbb{Z}^2$ using the Pimsner–Voiculescu exact sequence twice. This can only be carried through if we have some extra information. In terms of the general homological machinery developed in [7], we find that the homological dimension of E-theory over an infinite space X may be one larger than the homological dimensions of the finite approximations X_n. Thus it is usually 2, which does not suffice for classification theorems.

In fact, it is well-known that filtrated K-theory cannot be a complete invariant for C*-algebras over the one-point compactification of $\mathbb{N}$. The counterexample in [3] may be transported easily to any compact Hausdorff space.

If X is the Cantor space or, more generally, a totally disconnected metrisable compact space, then we may resolve the counterexamples mentioned above by taking into account coefficients. Our second main result is a Universal *Multi*coefficient Theorem for $\mathrm{E}_*(X; A, B)$ for two C*-algebras A and B over the Cantor space. It assumes that $A(U)$ belongs to the E-theoretic bootstrap class for all open subsets $U \subseteq X$ and yields a natural exact sequence

$$\mathrm{Ext}_{\mathrm{C}(X,\Lambda)}\bigl(\underline{\mathrm{K}}(A)[1], \underline{\mathrm{K}}(B)\bigr) \rightarrowtail \mathrm{E}(X; A, B) \twoheadrightarrow \mathrm{Hom}_{\mathrm{C}(X,\Lambda)}\bigl(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(B)\bigr),$$

where $\underline{\mathrm{K}}$ denotes the K-theory of A with coefficients, viewed as a countable module over the $\mathbb{Z}/2$-graded ring $\mathrm{C}(X, \Lambda)$ of locally constant functions from X to the $\mathbb{Z}/2$-graded ring Λ of Böckstein operations (see [4]). As a consequence, two C*-algebras A and B in the E-theoretic bootstrap class over X are $\mathrm{E}(X)$-equivalent if and only if $\underline{\mathrm{K}}(A)$ and $\underline{\mathrm{K}}(B)$ are isomorphic as $\mathrm{C}(X, \Lambda)$-modules.

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Perturbations of nuclear C*-algebras

Erik Christensen

(joint work with Allan Sinclair, Roger Smith, Stuart White, Wilhelm Winter)

In 1972 Kadison and Kastler [12] introduced a natural metric on the collection of all C*-subalgebras of the algebra of bounded operators on a separable Hilbert space. They conjectured that sufficiently close algebras are unitarily conjugate. We establish this conjecture when one of the algebras is separable and nuclear.

Definition 1. Let $\mathcal{A}$ and $\mathcal{B}$ be C*-subalgebras of a C*-algebra $\mathcal{C}$, then the distance $\|\mathcal{A} - \mathcal{B}\|$ is defined as the Hausdorff distance between their unit balls when considered as subsets of the unit ball of $\mathcal{C}$, and this space is equipped with the metric induced by the norm.

In connection to the original conjecture they also asked, if a unitary equivalence might be obtained via a unitary close to the identity. This last question was answered in the negative in 1982 by Johnson [10], for an example involving separable nuclear C*-algebras. In [2] from 1983 Choi and Christensen constructed examples of nuclear non isomorphic, but non separable C*-algebras being arbitrarily near to each other. The present result is then the final answer to the proposed conjecture as long as we work with the question inside the class of nuclear C*-algebras. We have no knowledge about the answer to the question for separable non nuclear C*-algebras. For von Neumann algebras the question was investigated in the period from 1972 to 1980, and the main result is that if two von Neumann algebras, say $\mathcal{M}$ and $\mathcal{N}$, are close and if one of them is injective, then $\mathcal{N} = u\mathcal{M}u^*$ for a unitary near to the identity. The vague statements such as *close, sufficiently close* and *near to* are imposed by the very nature of the question. Most of the proofs contain several approximations and applications of the functional calculus, so we get some terrible estimates on what the optimal constants might be. Since there is no hope that our methods will yield numbers close to the optimal bounds, we have just worked to get at least one concrete bound which is included in the main theorem:

Theorem 2. *Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras acting on a separable Hilbert space H. Suppose that $\mathcal{A}$ is separable and nuclear, and that $\|\mathcal{A} - \mathcal{B}\| < 10^{-11}$. Then there exists a unitary $u \in (\mathcal{A} \cup \mathcal{B})''$ such that $u\mathcal{A}u^* = \mathcal{B}$.*

Previously, results of this type were obtained in [4] and [15] for AF C*-algebras, in [16] for continuous trace C*-algebras, in [11] for subhomogeneous C*-algebras and in [13] for some nuclear C*-algebras.

For injective von Neumann algebras, the solution was based on the work by Connes [5], which tells that such an algebra, say $\mathcal{M}$, acting on a Hilbert space H has the property that for any operator x in $B(H)$ the ultraweak closure of the convex hull of the unitary translates uxu^* with u unitary in $\mathcal{M}$ has non empty intersection with the commutant $\mathcal{M}'$. This method does not work for C*-algebras for several reasons, but it turns out that Johnson's concept of an approximate virtual diagonal [8], [9] can be used as a replacement, but only under the cost of much more complicated proofs. We base this type of arguments on techniques taken from the article [7], where Haagerup shows that nuclearity implies amenability. Based on the results from [3] and [1] we can construct an increasing sequence of finite subsets $(\mathcal{X}_n)$ of the unit ball of $\mathcal{A}$ and a sequence of completely positive mappings $\gamma_n : \mathcal{A} \to \mathcal{B}$ with finite rank such that $\|(\gamma_n - \mathrm{id})_{|\mathcal{X}_n}\|$ is small. Being inspired by Eliott's approximate homomorphism technique [6], we then use the approximate virtual diagonals to construct a pair of sequences of completely positive mappings $\varphi_n : \mathcal{A} \to \mathcal{B}$ and $\psi_n : \mathcal{B} \to \mathcal{A}$ with finite rank, such that both of the sequences converge in *point norm* to homomorphisms which turn out to be inverses of each other. This establishes the isomorphism result for sufficiently close separable nuclear C*-algebras. The fact that they are also unitarily equivalent is based on a sequence of applications of a special version of the Kaplansky density theorem.

At the end of the paper we consider one-sided inclusions.

Definition 3. Let $\mathcal{A}, \mathcal{B}$ be C*-subalgebras of the C*-algebra $\mathcal{C}$ and let γ be a positive real number. We say that $\mathcal{A}$ is γ-nearly contained in $\mathcal{B}$ and write $\mathcal{A} \subseteq_\gamma \mathcal{B}$, if for any a in the unit ball of $\mathcal{A}$ there exists a b in $\mathcal{B}$ such that $\|a - b\| \leq \gamma$.

In this case it was previously shown in [11] that if $\mathcal{A}$ is subhomogeneous, then $\mathcal{A}$ is unitarily equivalent to a C*-subalgebra of $\mathcal{B}$. In this article we extend this result to the case where $\mathcal{A}$ has finite nuclear dimension.

Theorem 4. *Let $\mathcal{A} \subseteq_\gamma \mathcal{B}$ be a near inclusion of C*-algebras on a Hilbert space H and suppose that $\mathcal{A}$ is separable and has nuclear dimension at most n for some $n \geq 0$. Write $\eta = 2(n+1)(2\gamma + \gamma^2)(2 + 2\gamma + \gamma^2)$. Provided $\eta < 1/210000$, then $\mathcal{A}$ embeds into $\mathcal{B}$.*

The concept of nuclear dimension has been developed from the concept called covering dimension in the articles [14], [17], [18], [20]. In this context the concept of *order zero maps* [19] appears, and these maps behave well with respect to near inclusions.

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