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Spectral Theory in Banach Spaces and Harmonic Analysis

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Abstract. The workshop brought together 49 researchers from 11 different countries, representing two different research areas, spectral theory in Banach spaces and harmonic analysis, in order to promote the exchange of methods and recent results of these two areas. The 28 talks focused on the H^∞-functional calculus for sectorial operators, related boundedness results on singular integrals, square function estimates and their application to the Kato square root problem, and regularity estimates for parabolic differential operators. They also raised many questions which lead to lively discussions, in particular during the afternoons, which were, for the most part, kept free of talks.

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Introduction by the Organisers

The conference was motivated by the recent solutions of two longstanding questions. The first one is Kato's square root problem, i.e. whether for an elliptic operator in divergence form on $L_2(\mathbb{R}^n)$ we have $\|L^{1/2}u\| \leq C\,\|\nabla u\| + C\,\|u\|$ for $u \in W_2^1(\mathbb{R}^n)$. After a long development over 40 years this was shown in a joint effort by P. Auscher, S. Hofmann, M. Lacey, A. McIntosh and Ph. Tchamitchian. For a new approach to this important result via Dirac operators and extensions of it, see the abstracts of A. McIntosh and S. Keith. The second problem, attributed to Brézis in the eighties, asks whether the Cauchy problem for every generator of an analytic semigroup A in an $L_q(\Omega)$-space with $1 < q < \infty$ has maximal L_p-regularity, i.e. does the solution y of the Cauchy problem $y' = Ay + f$, $y(0) = 0$, satisfy $\|y\|_{W_p^1(\mathbb{R}_+,L_q)} \leq C\,\|f\|_{L_p(\mathbb{R}_+,L_q)}$. Recently G. Lancien and N.J. Kalton gave counterexamples to this questions while L. Weis gave a characterization of maximal L_p-regularity in terms of R-boundedness. This criterium has since been shown to be useful in establishing maximal regularity for large classes of Cauchy problems for elliptic differential operators with rather general coefficients and boundary values, e.g. Schrödinger operators with singular potentials and Stokes operators (see the abstracts of M. Hieber, P. Kunstmann, J. Prüss and R. Schnaubelt). Applications of maximal regularity to non-linear differential equations were presented in the talks of H. Amann and G. Simonett.

The two problems of Kato and Brézis share a common background in the mathematical methods employed in their solution, e.g. the H^∞-functional calculus for sectorial operators in Banach spaces, boundedness of singular integral operators and square function estimates. Obviously, further progress will depend on an in-depth study of these methods and their interrelations. The workshop contributions featured some of the most recent progress in these directions.

N.J. Kalton used deep results from Banach space theory to crystallize the difficult perturbation theory of the H^∞-functional calculus. Some of the randomization techniques he used can be seen as an extension of the classical square function estimates from L_p-spaces to general Banach spaces. This method also underlies boundedness results for Fourier multiplier operators, Calderón-Zygmund operators and wavelet transforms for functions with values in a Banach space with the UMD-property (see the abstracts of W. Arendt, O. Blasco, T. Hytonen and C. Kaiser) and applications of those to spectral theory (A. Gillespie). Also in the scalar-valued case several extensions of basic methods in the theory of singular integral operators were presented, e.g. new spaces of BMO- and H^1-type associated to a given operator (X. Duong), Calderón-Zygmund operators associated to non-doubling measures (J. Garcia-Cuerva), bilinear pseudo-differential operators (A. Nahmod) and optimal domains for convolution operators (W. Ricker). As so often, new tools in harmonic analysis lead to new results for the H^∞-functional calculus of partial differential operators (M. Cowling, X. Duong).

Some of the talks reached beyond the circle of questions raised by the problems of Kato and Brézis. They were closely connected to the main topics by a shared methological background in spectral theory and harmonic analysis. Ch. Thiele discussed an approach to non-linear differential equations via the non-linear Fourier transform. B. Jefferies and J. van Neerven described the connection between spectral theory, square function estimates and stochastic integration on infinite dimensional Banach spaces. S. Grivaux and Y. Latushkin gave applications of spectral theory to instability and stability theorems for evolutionary systems. The talks of of J. Eschmeier, M. Haase and F. Sukochev were focused on the functional calculus and its applications in operator theory.

The varied background of the participants lead to a number of new collaborations started during the workshop. Last but not least, the unique setting of Oberwolfach and a week of beautiful sunshine contributed to a memorable and successful meeting.

Workshop: Spectral Theory in Banach Spaces and Harmonic Analysis

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Abstracts

Vector-valued multiplier theorems on Hölder and on Besov spaces

Wolfgang Arendt

We analyse operator-valued versions of the classical Marcinkiewicz theorem for periodic multipliers on Hölder and Besov spaces. In contrast to the situation in L^p, in these spaces the boundedness conditions of Marcinkiewicz- or of Michlin type suffice; R-boundedness ist not needed, but higher order conditions may have to be considered.

1. Periodic Hölder spaces

Let X be a Banach space, $0 < \alpha < 1$, and denote by

$$C^\alpha(\mathbb{T}, X) := \{u : \mathbb{T} \to X : \|u\|_\alpha < \infty\}$$

the *periodic Hölder space* where

$$\|u\|_\alpha := \sup_{t \neq s} \frac{\|u(e^{it}) - u(e^{is})\|}{|t-s|^\alpha} .$$

Let Y be a second Banach space and let $M_n \in \mathbb{L}(X, Y) \quad (n \in \mathbb{Z})$.

Definition. The sequence $(M_n)_{n\in\mathbb{Z}}$ is a C^α*-multiplier* if for each $u \in C^\alpha(\mathbb{T}, X)$ there exists $v \in C^\alpha(\mathbb{T}, Y)$ such that

$$M_n \hat{u}(n) = \hat{v}(n) \quad (n \in \mathbb{Z}) ,$$

where $\hat{u}(n) = \int_{\mathbb{T}} u(z) z^{-n} dz$ denotes the $n-th$ *Fourier coefficient* of u.

Then, by the uniqueness theorem and the closed graph theorem, there exists a bounded operator $T : C^\alpha(\mathbb{T}, X) \to C^\alpha(\mathbb{T}, Y)$ such that

$$(Tu)^\wedge(n) = M_n \hat{u}(n) \quad (n \in \mathbb{Z})$$

for all $u \in C^\alpha(\mathbb{T}, X)$.

We consider two conditions on the sequence $M := (M_n)_{n\in\mathbb{Z}}$ which we call the *Marcinkiewicz conditions of order* 1 *and* 2, respectively

$$(\mathcal{M}_1) \qquad \|M\|_1 := \sup_{n\in\mathbb{Z}} (\|M_n\| + |n| \; \|M_{n+1} - M_n\|) < \infty$$

and

$$(\mathcal{M}_2) \qquad \|M\|_2 := \|M\|_1 + \sup_{n\in\mathbb{Z}} (n^2 \|M_{n+1} - 2M_n + M_{n-1}\|) < \infty .$$

Theorem 1. [ABB]. *If condition $(\mathcal{M}_2)$ is satisfied, then $(M_n)_{n\in\mathbb{Z}}$ is a C^α-multiplier.*

This result holds for arbitrary Banach spaces, and it is proved in [ABB] by a direct convolution estimate without using dyadic decomposition. An alternative way is described in the following section. If we interprete $M_{n+1} - M_n$ as the first derivative, then $(\mathcal{M}_1)$ is analogous to Michlin's condition on $\mathbb{R}$ (instead of $\mathbb{T}$).

In fact, Marcinkiewicz considered the more general condition

$$(\mathcal{M}_{var}) \qquad \sup_{j\geq 0} \sum_{2^{j-1}\leq |k|\leq 2^j} \|M_{k+1} - M_k\| < \infty$$

which we call *Marcinkiewicz's variational condition.* He showed in the scalar case that this variational condition implies $(M_n)_{n\in\mathbb{Z}}$ to be an L^p-multiplier for $1 < p < \infty$. Clearly,

$$(\mathcal{M}_2) \Rightarrow (\mathcal{M}_1) \Rightarrow (\mathcal{M}_{var}) \ .$$

It is remarkable, that condition $(\mathcal{M}_{var})$ is not sufficient for $(M_n)_{n\in\mathbb{Z}}$ being a C^α-multiplier even in the scalar case [ABB]. The space C^α is a special Besov space and in fact, one can characterize precisely for which Besov spaces Marcinkiewicz's classical result holds. This is the topic of the following section.

2. Multipliers on periodic Besov spaces

For each $s \in \mathbb{R}$, $1 \leq p$, $q < \infty$ the *periodic Besov space* $B^2_{p,q}(\mathbb{T}, X)$ is defined via dyadic decomposition in the Fourier image with help of a partition of unity. It depends on the three indices s, p, q but not on the choice of the partition of unity up to some basic assumptions. We refer to [AB04a] for the details. $B^s_{p,q}(\mathbb{T}, X)$ is a space of distributions contained in $L^p(\mathbb{T}, X)$ if $s > 0$. Moreover, $B^m_{p,1}(\mathbb{T}, X) \subset W^p_m(\mathbb{T}, X) \subset B^m_{p,\infty}(\mathbb{T}, X)$ for each $m \in \mathbb{N}$. Hölder spaces coincide with special Besov spaces,

$$B^s_{\infty,\infty}(\mathbb{T}, X) = C^s(\mathbb{T}, X) \quad (0 < s < 1) \ .$$

Theorem 2. [AB04a]. *Let X, Y be a Banach spaces, $s \in \mathbb{R}, 1 \leq p, q \leq \infty$. The following assertions are equivalent:*

(i) *Each sequence $(M_n)_{n\in\mathbb{Z}} \subset \mathbb{L}(X, Y)$ satisfying $(\mathcal{M}_{var})$ is a $B^s_{p,q}$-multiplier.*
(ii) *X and Y are UMD-spaces and $1 < p, q < \infty$.*

The proof is based on a suitable choice of the partition of unity.

It was Amann [Am] who dicovered that Michlin's theorem holds for vector-valued Besov spaces over $\mathbb{R}$ for arbitrary Banach spaces if Michlin's condition of order 2 is imposed (see Section 3). A similar result had been announced by Weis [W01]. The above theorem is the periodic analogous result which is proved and applied to periodic Cauchy problems in [AB04a]. As in Girardi-Weis [GW] for the real line, also in the periodic case, condition $(\mathcal{M}_2)$ may be replaced by the weaker condition $(\mathcal{M}_1)$ if X and Y have non-trivial Fourier type (see [AB04a]).

3. Characterizations of Hilbert spaces

In the preceding section we saw that on Besov spaces it is of interest to consider higher order Marcinkiewicz conditions. For L^p-multipliers the situation is different. Pisier had discovered, as a consequence of Kwapien's theorem, that bounded lacunary sequences of operators are L^p-multipliers if and only if the underlying space is a Hilbert space. Because of the recent interest in operator-valued multiplier theorems and their applications to evolution equations the situation was analyzed in detail in [AB04b]. We formulate the result for multipliers on $\mathbb{R}$ considering merely the case where $X = Y$ (and refer to the periodic case and the case where $X \neq Y$ to [AB02]).

Theorem 3. [AB04b]. *Let X be a Banach space, $1 < p < \infty$. The following are equivalent:*

(1) *X is a Hilbert space;*

(ii) *for some $1 < p < \infty$, $m \in \mathbb{N}$ each function $M \in C^\infty(\mathbb{R}, \mathbb{L}(X))$ satisfying*

$$\sup_{x\in\mathbb{R}} \sup_{\ell=0,\cdots m} \|(1+|x|^\ell)M^{(\ell)}(x)\| < \infty$$

is an $L^p(\mathbb{R}, X)$-multiplier;

(iii) *for all $1 < p < \infty$ each $M \in C^1(\mathbb{R}\setminus\{0\}, \mathbb{L}(X))$ satisfying*

$$\sup_{x\neq 0}\{\|M(x)\| + \|xM'(x)\|\} < \infty$$

is an $L^p(\mathbb{R}, X)$-multiplier.

Thus, on Banach spaces different from Hilbert spaces, in Weis' multiplier theorem [W01] on $L^p(\mathbb{R}, X)$ R-boundedness is definitely needed even if higher order Michlin conditions are considered. This contrasts the situation in Besov spaces.

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H^∞-calculus for differential operators

Peer Christian Kunstmann

We present results on extrapolation and comparison for the H^∞-calculus, which address the following problem: Suppose that A is a sectorial operator in $L^2(\Omega)$ where, e.g., $\Omega \subset \mathbb{R}^n$ is open, and that A has a bounded H^∞-calculus in $L^2(\Omega)$. Under which conditions the realization of A in $L^p(\Omega)$ has a bounded H^∞-calculus for (suitable) $p \neq 2$?
In the first part we extrapolate using certain weighted norm bounds, which generalize kernel estimates of Gaussian type. In the second part we compare fractional domain spaces with those of another operator, which is already assumed to have a bounded H^∞-calculus.

Extrapolation: Let $\Omega \subset \mathbb{R}^n$ and suppose that $-A$ is the generator of a C_0-semigroup T_t in $L^2(\Omega)$. Let $m \geq 1$. It is not hard to see that the *Gaussian estimate*

$$|k(t,x,y)| \leq Ct^{\frac{n}{2m}} \exp\left(\left(\frac{|x-y|^{2m}}{t}\right)^{\frac{1}{2m-1}}\right), \tag{1}$$

where $k(t,x,y)$ denotes the integral kernel of T_t, is the special case $(p,q) = (1,\infty)$ of the following *generalized Gaussian estimate*

$$\|1_{B(x,t^{1/(2m)})} T_t 1_{B(y,t^{1/(2m)})}\|_{L^p \to L^q} \leq Ct^{\frac{n}{2m}(\frac{1}{p}-\frac{1}{q})} \exp\left(\left(\frac{|x-y|^{2m}}{t}\right)^{\frac{1}{2m-1}}\right). \tag{2}$$

The following theorem, obtained in joint work with Sönke Blunck ([1], [2]), generalizes results on maximal L^s-regularity in [6], [3], and on the H^∞-calculus in [5] and [4]. It is the first result in this context that allows for a restricted range in the L^p-scale.

Theorem 1. *Let A be a sectorial operator in $L^2(\Omega)$ of angle $\omega(A) < \pi/2$. If $(T_t) := (e^{-tA})$ satisfies (2) for some $1 \leq p < 2 < q \leq \infty$, then A is R-sectorial in $L^r(\Omega)$, $p < r < q$, of angle $\omega_R(A_r) = \omega(A_2)$*
If, in addition, A has an H^∞-calculus in $L^2(\Omega)$, then A has an H^∞-calculus in $L^r(\Omega)$, $p < r < q$, and the infimum $\omega_H(A_r)$ of the angles equals the sectoriality angle $\omega(A_2)$ of the operator in L^2.

The proof of R-sectoriality uses the following key estimate

$$N_{q,t^{1/(2m)}}(T_t f)(x) \leq CM_p f(x), \tag{3}$$

where $N_{q,\rho} g(x) = \rho^{-n/q} \|g\|_{L^q(B(x,\rho))}$ and $M_p f(x) = \sup_{\rho>0} N_{p,\rho} g(x)$ is the p-maximal function (cf. [1]). The proof of the H^∞-result relies on a weak type (p,p)-criterion which uses bounds of the form (2) and a suitable replacement of the well-known Hörmander condition to apply Calderon-Zygmund theory to non-integral operators (cf. [2]). The result in [2] is the first to use Calderon-Zygmund theory for operators which act boundedly only on a restriction of the L^p-scale, i.e., only on an interval $I \neq (1,\infty)$.

Theorem 1 may be applied to operators in divergence form but also to operators in non-divergence form. The main point is to establish (2) which is more easy than to prove (1) and possible also in a number of cases where (1) does not hold.

Example 1. Uniformly elliptic operators in divergence form on $\mathbb{R}^n$ of order $2m < n$, i.e., operators of the form

$$Au(x) = \sum_{|\alpha|=|\beta|=m} \partial^\alpha \left(a_{\alpha\beta(x)} \left(\partial^\alpha u\right)(x)\right),$$

where $a_{\alpha\beta} \in L^\infty(\mathbb{R}^n, \mathbb{C})$ (cf. [1]). For this class, the indices $p = 2n/(n+2m)$, $q = 2n/(n-2m)$ are known to be optimal.

Example 2. Schrödinger operators $-\Delta + V$ on $\mathbb{R}^n$ with singular potentials V, e.g., potentials from a pseudo-Kato class (cf. [1]). Here the indices p and q depend on certain form bounds of the potential. which are also known to fail (1).

Comparison (joint work with N.J. Kalton and L. Weis [7]): We present an extrapolation result for the H^∞-calculus which uses "comparison" of fractional domain spaces of a given operator A with those of a *simpler* operator B, which is known to have a bounded H^∞-calculus.

Theorem 2. *Let (X_0, X_1) be an interpolation couple of reflexive and B-convex spaces. Assume that B has an H^∞-calculus on X_0 and X_1, A is almost R-sectorial on X_0, X_1 and there are $\alpha < 0 < \beta$ such that*

$$(X_0)^{\cdot}_{\alpha,A} = (X_0)^{\cdot}_{\alpha,B}, \quad (X_1)^{\cdot}_{\beta,A} = (X_1)^{\cdot}_{\beta,B}. \tag{4}$$

Then A has an H^∞-calculus on all complex interpolation spaces $X_\theta = [X_0, X_1]_\theta$, $\theta \in (0,1)$.

Here, for a Banach space X and a sectorial operator C in X, the occurring spaces $(X)^{\cdot}_{\gamma,C}$ are defined as $(D(C^\gamma), \|C^\gamma \cdot\|)^\sim$, i.e., as completion of the fractional domain space for the homogeneous norm. The proof of this result relies on a new *Rademacher* interpolation method which is tailor-made for the H^∞-calculus: An operator has a bounded H^∞-calculus on a Banach space if and only if its fractional domain spaces interpolate by Rademacher interpolation (cf. [7]).

Example 3. Theorem 2 may be used to reobtain boundedness of the H^∞-calculus for elliptic operators A of order $2m$ in non-divergence form on $\mathbb{R}^n$ whose highest order coefficients are Hölder continuous: Take $B = 1+(-\Delta)^m$, $X_0 = L^2(\mathbb{R}^n)$, $X_1 = L^p(\mathbb{R}^n)$, and $\beta = 1$. B has a bounded H^∞-calculus in X_j and the domains of A and B are comparable in X_j. Moreover, due to the Hölder continuity assumption, it is also possible to show R-sectoriality of A in the Bessel potential space $H_2^{-\gamma}$ for some small $\gamma > 0$. This may be shown to yield the first comparison in (4) for $-\gamma/(2m) < \alpha < 0$ (cf. [7]).

For greater flexibility, we allow the operator A to act in a scale of spaces $Y_\cdot$ which are complemented in the spaces $X_\cdot$ of the scale the operator B acts in. The simplest case $X_j = Y_j$, $P_j = I$ gives back Theorem 2.

Theorem 3. *Let (X_0, X_1) be an interpolation couple of reflexive and B-convex spaces with complemented subspaces (Y_0, Y_1) and associated projections P_0, P_1 compatible with the interpolation couple. Assume that B has an H^∞-calculus on X_0 and X_1, A is almost R-sectorial on Y_0, Y_1 and there are $\alpha < 0 < \beta$ such that*

$$P_0((X_0)^{\cdot}_{\alpha,B}) = (Y_0)^{\cdot}_{\alpha,A}, \quad P_1((X_1)^{\cdot}_{\beta,B}) = (Y_1)^{\cdot}_{\beta,A}. \tag{5}$$

Then A has an H^∞-calculus on all complex interpolation spaces $Y_\theta = [Y_0, Y_1]_\theta$, $\theta \in (0, 1)$.

With $X_0 = L^2(\Omega)^n$, $X_1 = L^p(\Omega)^n$ and $Y_j = \mathbb{P}X_j$ where $\mathbb{P}$ denotes Helmholtz projection, Theorem 3 can be used to prove the following new result on the Stokes operator.

Theorem 4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary $\partial\Omega \in C^{1,1}$. Then the Stokes operator $A = -\mathbb{P}\Delta$ has a bounded H^∞-calculus on $L^p_\sigma(\Omega)$ for any $1 < p < \infty$, with angle $\omega_H(A_p) = 0$.*

Here $L^p_\sigma(\Omega) = \mathbb{P}L^p(\Omega)^n$. The proof is done by comparison with the Dirichlet Laplacian $B = -\Delta$ on $L^p(\Omega)^n$, which is known to have an H^∞-calculus for $1 < p < \infty$. Theorem 4 generalizes results due to Y. Giga ($\partial\Omega \in C^\infty$) and A. Noll and J. Saal ($\partial\Omega \in C^3$).

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Lipschitz spaces and Calderón-Zygmund operators associated to non–doubling measures

Jose Garcia-Cuerva

(joint work with A. Eduardo Gatto)

In the setting of a metric measure space (X, d, μ) with an $n-$dimensional Radon measure μ, we show that Lipschitz spaces are preserved by Calderón-Zygmund operators T associated to the measure μ. Also, for the Euclidean space $\mathbb{R}^d$ with an arbitrary Radon measure μ, we give several characterizations of Lipschitz spaces on the support of μ, $\mathcal{L}ip(\alpha,\mu)$, in terms of mean oscillations involving μ. This allows us to view the "regular "BMO space of X. Tolsa as a limit case for $\alpha \to 0$ of the spaces $\mathcal{L}ip(\alpha,\mu)$.

1. Calderón-Zygmund Operators

(X,d,μ) will be a **metric measure space** with the property

$$\mu(B(x,r)) \le Cr^n, \tag{1}$$

Lemma 1. *For every $\gamma > 0$ and every $r > 0$:*

$$\int_{B(x,r)} \frac{1}{d(x,y)^{n-\gamma}} d\mu(y) \le Cr^{\gamma} \text{ and } \int_{X \setminus B(x,r)} \frac{1}{d(x,y)^{n+\gamma}} d\mu(y) \le Cr^{-\gamma}$$

• **Lipschitz spaces** $f : X \to \mathbb{C}$ satisfies a Lipschitz condition of order $\beta \in]0,1[$ provided

$$|f(x) - f(y)| \le Cd(x,y)^{\beta} \text{ for every } x,y \in X. \tag{2}$$

The smallest constant in inequality (2) will be denoted by $\|f\|_{\mathcal{L}ip(\beta)}$. The linear space of all Lipschitz functions of order β, modulo constants, becomes, with the norm $\| \ \|_{\mathcal{L}ip(\beta)}$, a Banach space, which we shall call $\mathcal{L}ip(\beta)$.

• A **singular kernel** on (X,d,μ) will be a measurable function $K(x,y)$ on $X \times X \setminus \{x = y\}$ satisfying the following conditions:

(1) $|K(x,y)| \le \frac{A_1}{d(x,y)^n}$,

(2) $|K(x_1,y) - K(x_2,y)| \le A_2 \frac{d(x_1,x_2)^{\delta}}{d(x_1,y)^{n+\delta}}$ for $2d(x_1,x_2) \le d(x_1,y)$,

(3) $\left| \int_{\varepsilon < d(x,y) < R} K(x,y) d\mu(y) \right| \le B$ for all $0 < \varepsilon < R$.

(4) $\lim\limits_{\varepsilon \to 0} \int_{\varepsilon < d(x,y) < R} K(x,y) d\mu(y)$ exists for $\mu-$almost every point x.

• For K as above, we define the **truncated kernels**

$$K_{\varepsilon}(x,y) = K(x,y)\chi_{\{d(x,y) > \varepsilon\}}(x,y).$$

• For $f \in \mathcal{L}ip\,(\alpha),\ 0 < \alpha < \delta \le 1$, we define

$$\widetilde{T}_{\varepsilon} f(x) = \int_X \left(K_{\varepsilon}(x,y) - K_1(x_0,y)\right) f(y) d\mu(y)$$

• Finally, we can define the **singular integral operator** $\widetilde{T}$ by

$$\widetilde{T} f(x) = \lim_{\varepsilon \to 0} \widetilde{T}_{\varepsilon} f(x)$$

It follows from the properties of $K(x,y)$ and Lemma 1 that the limit exists $\mu-$almost everywhere.

We can prove the following result

Theorem 1.1. *Let K be a singular kernel as above and let $\widetilde{T}$ be the corresponding singular integral operator. Let $0 < \alpha < \delta \le 1$. Then $\widetilde{T}$ is a bounded operator on $\mathcal{L}ip\,(\alpha)$, if and only if $\widetilde{T}(1)$ is constant.*

2. Characterization of Lipschitz spaces

μ will be a fixed Radon measure on $\mathbb{R}^d$. All balls that we consider will be centered at points in the support of μ.

• Let β be a fixed constant. A ball B is called $\beta-$**doubling** if

$$\mu(2B) \leq \beta\mu(B).$$

Lemma 2. *Let $f \in L^1_{\text{loc}}(\mu)$. If $\beta > 2^d$, then, for almost every x with respect to μ, there exists a sequence of $\beta-$doubling balls $B_j = B(x, r_j)$ with $r_j \to 0$, such that*

$$\lim_{j\to\infty} \frac{1}{\mu(B_j)} \int_{B_j} f(y)d\mu(y) = f(x).$$

Theorem 2.1. *For a function $f \in L^1_{\text{loc}}(\mu)$, the conditions I, II, and III below, are equivalent*

(I) *There exist a constant C_1 and numbers f_B,one for each ball B, such that these two properties hold: For any ball B with radius r*

$$\frac{1}{\mu(2B)} \int_B |f(x) - f_B| d\mu(x) \leq C_1 r^\alpha, \tag{3}$$

and for any ball U such that $B \subset U$ and radius $(U) \leq 2r$.

$$|f_B - f_U| \leq C_1 r^\alpha, \tag{4}$$

(II) *There is a constant C_2 such that*

$$|f(x) - f(y)| \leq C_2 |x - y|^\alpha \tag{5}$$

for $\mu-$almost every x and y in the support of μ.

(III) *For any given p, $1 \leq p \leq \infty$, there is a constant $C(p)$, such that for every ball B of radius r, we have*

$$\left(\frac{1}{\mu(B)} \int_B |f(x) - m_B(f)|^p d\mu(x)\right)^{1/p} \leq C(p) r^\alpha, \tag{6}$$

where $m_B(f) = \frac{1}{\mu(B)} \int_B f(y)d\mu(y)$ and also for any ball U such that $B \subset U$ and radius $(U) \leq 2r$.

$$|m_B(f) - m_U(f)| \leq C(p) r^\alpha, \tag{7}$$

In addition, the quantities: $\inf C_1$, $\inf C_2$, *and* $\inf C(p)$ *with a fixed p are equivalent.*

• The linear space of all Lipschitz functions of order α, with respect to μ, modulo constants, becomes, with the norm $\inf C_2$ of Theorem 2.1, a Banach space, which we shall call $\mathcal{L}ip(\alpha, \mu)$.

• It is easy to see that $\mathcal{L}ip(\alpha, \mu)$ coincides with the space of Lipschitz functions of order α on the support of μ. By the extension theorem of Banach any Lipschitz function of order α with respect to μ has an extension to $\mathbb{R}^d$ that is a Lipschitz function of order α with an equivalent norm.

• For $0 < \alpha \leq 1$, a telescoping argument shows that (4) is equivalent to

$$|f_B - f_U| \leq C_1' \text{radius}(U)^\alpha \tag{8}$$

for any two balls $B \subset U$.

• If we further assume that μ is $n-$dimensional, we see that (8) is also equivalent to

$$|f_B - f_U| \leq C_1'' K_{B,U} \text{radius}(U)^\alpha, \tag{9}$$

for any two balls $B \subset U$, where $K_{B,U}$ is the constant introduced by X. Tolsa , given by

$$K_{B,U} = 1 + \sum_{j=1}^{N_{B,U}} \frac{\mu(2^j B)}{(2^j r)^n} \ ,$$

with $N_{B,U}$ equal to the first integer k such $2^k\text{radius}(B) \geq \text{radius}(U)$. Indeed (9) for comparable balls, that is, for $\text{radius}(U) \leq 2\text{radius}(B)$, reduces to (4) because, in this case, $K_{B,U}$ is controlled by an absolute constant.

Note that (3) and (9) make sense also for $\alpha = 0$ and the space defined by them is the space *RBMO*(μ) of X. Tolsa. Therefore, the spaces $\mathcal{L}ip(\alpha, \mu)$, $0 < \alpha \leq 1$ can be seen as members of a family containing also *RBMO*(μ).

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Optimal domains for convolution operators in $L^p(G)$

Werner J. Ricker

Let G be a compact abelian group with dual group Γ und μ be a finite regular Borel measure on G. For $p \in [1, \infty)$ the convolution operator $C_\mu^{(p)} : f \mapsto f * \mu$ is a bounded linear operator of $L^p(G)$ into itself. It always commutes with translations, that is, it is a p-multiplier operator. The $L^p(G)$-valued set function $m_\mu^{(p)} : E \mapsto C_\mu^{(p)}(\chi_E) = \chi_E * \mu$ is σ-additive, i.e. it is a vector measure on the Borel σ-algebra of G. Consequently, the domain space $L^p(G)$ of $C_\mu^{(p)}$ can be continuously imbedded into the space $L^1(m_\mu^{(p)})$ of all $m_\mu^{(p)}$-integrable functions, equipped with its mean convergence topology, and the integration operator $I_{\mu,p} : f \mapsto \int_G f dm_\mu^{(p)}$ is then an extension of $C_\mu^{(p)}$ from $L^p(G)$ to the larger domain space $L^1(m_\mu^{(p)})$. The important feature of this extension is that $L^1(m_\mu^{(p)})$ is the optimal lattice domain for $C_\mu^{(p)}$. That is, within the class of all Banach function spaces (based on G relative to normalized Haar measure λ) with order continuous norm, containing $L^p(G)$, and to which $C_\mu^{(p)}$ can be extended as a bounded linear operator (still maintaining its values in $L^p(G)$), the space $L^1(m_\mu^{(p)})$ is the maximal one. This is a consequence of some general results in [2].

The determination of the optimal domain and the extension of particular operators of analysis is an old problem. Consider a bounded linear operator $T : C(K) \to \mathbb{C}$, with K a compact Hausdorff space. By the Riesz representation theorem T has an integral representation with respect to some regular Borel measure ν on K. Then T has a continuous linear extension to all spaces $L^p(\nu)$ with $1 \le p \le \infty$, say, and $L^1(\nu)$ is the optimal one. In [4] the optimal domain for certain Sobolev imbeddings, within the class of (rearrangement invariant) Banach function spaces, is determined; see also [3] for further results in this direction. For the classical Volterra operator V in $L^p([0,1])$ given by $V(f) : t \mapsto \int_0^t f(s)\, ds$, the optimal domain space and the corresponding extended operator are identified in [7]. Other classes of Volterra operators are treated in [1].

This talk (joint work with Prof. S. Okada) reports on our work in [6] concerning the optimal domain and the extension of $C_\mu^{(p)}$ to its optimal domain, in the setting of $L^p(G)$ and for general finite regular Borel measures μ. For the special class of measures of the form $\mu = g d\lambda$ with $g \in L^1(G)$ we refer to [5].

Theorem 1. *Let $1 \le p < \infty$ and μ be a (non-zero) finite regular Borel measure on G. The following assertions hold.*

(i) *A Borel measurable function $f : G \to \mathbb{C}$ belongs to the optimal domain $L^1(m_\mu^{(p)})$ of $C_\mu^{(p)}$, if and only if,*

$$\int_G |f| \cdot |\varphi * \widetilde{\mu}|\, d\lambda < \infty, \qquad \varphi \in L^{p'}(G),$$

where $p^{-1} + (p')^{-1} = 1$ and $\widetilde{\mu} : E \mapsto \mu(-E)$ is the reflection of μ. Moreover,

$$\|f\|_{L^1(m_\mu^{(p)})} := \sup\Big\{ \int_G |f| \cdot |\varphi * \widetilde{\mu}|\, d\lambda : \|\varphi\|_{p'} \le 1 \Big\}.$$

(ii) *The natural inclusions*

$$L^p(G) \overset{J_\mu^{(p)}}{\hookrightarrow} L^1(m_\mu^{(p)}) \overset{\Lambda_\mu^{(p)}}{\hookrightarrow} L^1(G)$$

are continuous, with the estimates

$$\|\widehat{\mu}\|_\infty \le \|J_\mu^{(p)}\| \le |\mu|(G)$$

and

$$\|\chi_G\|_{L^1(m_\mu^{(p)})} \le \|\Lambda_\mu^{(p)}\| \le \|\widehat{\mu}\|_\infty^{-1}$$

holding, where $|\mu|$ is the variation measure of μ and $\widehat{\mu}$ is the Fourier-Stieltjes transform of μ. Moreover, $L^1(m_\mu^{(p)})$ is a translation invariant subspace of $L^1(G)$.

(iii) *The extension of* $C_\mu^{(p)}$ *to its optimal domain, namely the integration map* $I_{\mu,p} : L^1(m_\mu^{(p)}) \to L^p(G)$*, has operator norm* 1 *and is given by*

$$I_{\mu,p} : f \mapsto \int_G f \, dm_\mu^{(p)} = f * \mu, \qquad f \in L^1(m_\mu^{(p)}).$$

Consider the space of measures

$$M_0(G) := \{\mu \text{ a finite regular Borel measure on } G \text{ with } \widehat{\mu} \in c_0(\Gamma)\}.$$

For G infinite, it always contains $L^1(G)$ as a proper subspace and is itself a proper subspace of all measures on G. It is known, for $1 < p < \infty$, that $C_\mu^{(p)} : L^p(G) \to L^p(G)$ is compact if and only if $\mu \in M_0(G)$. When is the extended operator $I_{\mu,p} : L^1(m_\mu^{(p)}) \to L^p(G)$ compact? This can be described rather precisely via the following result (where $|m_\mu^{(p)}|$ denotes the variation measure of the vector measure $m_\mu^{(p)}$).

Theorem 2. *Let* $1 < p < \infty$ *and* μ *be a (non-zero) finite regular Borel measure on* G*. The following assertions are equivalent.*

(i) *The extension* $I_{\mu,p} : L^1(m_\mu^{(p)}) \to L^p(G)$*, of the operator* $C_\mu^{(p)}$*, is compact.*

(ii) *There exists* $g \in L^p(G)$ *such that* $\mu(E) = \int_E g \, d\lambda$*,* $E \subseteq G$ *Borel.*

(iii) *The* $L^p(G)$*-valued vector measure* $m_\mu^{(p)}$ *has finite variation.*

(iv) *The optimal domain* $L^1(m_\mu^{(p)}) = L^1(G)$ *is as large as possible.*

(v) $L^1(|m_\mu^{(p)}|) = L^1(G)$*.*

(vi) $L^1(|m_\mu^{(p)}|) = L^1(m_\mu^{(p)})$*.*

(vii) *There is a Bochner* λ*-integrable integrable function* $F : G \to L^p(G)$ *such that*

$$m_\mu^{(p)}(E) = \int_E F \, d\lambda, \qquad E \subseteq G \text{ Borel.}$$

(viii) *There is a Pettis* λ *integrable function* $H : G \to L^p(G)$ *such that*

$$m_\mu^{(p)}(E) = \int_E H \, d\lambda, \qquad E \subseteq G \text{ Borel.}$$

It can happen that the optimal domain $L^1(m_\mu^{(p)}) = L^p(G)$, e.g. if μ is the Dirac measure at some point of G (i.e. no genuine extension is possible).

Theorem 3. *Let* G *be infinite and* $1 < p < \infty$*.*

(i) *If* $\mu \in M_0(G) \setminus \{0\}$*, then* $L^p(G) \subseteq L^1(m_\mu^{(p)})$ *is a proper (dense) inclusion.*

(ii) *The inclusion* $L^1(m_\mu^{(p)}) \subseteq L^1(G)$ *is proper and dense whenever* μ *is not of the form* $g d\lambda$ *for some* $g \in L^p(G)$*.*

Question: For $\mu \geq 0$ it is known that

$$N_\mu^{(p)} := \{f \in L^1(G) : |f| * \mu \in L^p(G)\}$$

equals $L^1(m_\mu^{(p)})$, and for arbitrary μ that $N_{|\mu|}^{(p)} \subseteq L^1(m_\mu^{(p)})$. Is it always the case that $N_{|\mu|}^{(p)} = L^1(m_\mu^{(p)})$?

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L_p boundedness of pseudodifferential operators with operator valued symbols and application

Pierre Portal

(joint work with Željko Štrkalj)

Given a UMD Banach space X, we consider pseudodifferential operators of the form

$$T_a f(x) = \int_{\mathbb{R}^n} e^{ix.\xi} a(x,\xi) \widehat{f}(\xi) d\xi,$$

where a is a (symbol) map from $\mathbb{R}^{2n}$ to $B(X)$ (the space of bounded linear operators acting on X) and $\widehat{f}$ denotes the Fourier transform of some (Schwartz class) X-valued function. We look for classes of symbols a such that T_a extends to a bounded operator on $L_p(\mathbb{R}^n; X)$ (for $1 < p < \infty$). If X has finite dimension, this is a classical problem in harmonic analysis (see chapters VI and VII of [S93]). In the infinite dimensional setting the situation is, however, not well understood. The case of x independent, scalar-valued symbols has been treated by Bourgain in [B86] whereas the case of operator-valued Fourier multipliers had to wait until the results of Weis in [W01]. With a dependancy in x results have been obtained by Hieber and Monniaux (in [HM00]) under the additional assumption that X is a Hilbert space. The general case has then been treated by Štrkalj in his PhD thesis [St00]. The proof of his result was, however, rather involved and never published. The purpose of the present work, which is joint with Željko Štrkalj is to present a hopefully simpler proof of a similar result. The class of symbols under consideration is the following.

Definition 1. *Let* $0 \le \delta < 1$, $0 < r < 1$, $n \in \mathbb{N}$ *and* $m \in \mathbb{N}$. *Consider* $a : \mathbb{R}^{2n} \to B(X)$. *We say that*

$$a \in \mathcal{S}^0_{1,\delta}(r, m, X)$$

if the following hold.

(1) $\forall \alpha \in (\mathbb{Z}_+)^n \ |\alpha| \le m \quad \exists C_\alpha > 0 \quad \forall (x,y) \in \mathbb{R}^{2n}$

$$\mathcal{R}(\{(1+|\xi|)^{|\alpha|}\partial_\xi^\alpha a(x+ty,\xi)\,,\, \xi \in \mathbb{R}^n\,,\, t \in \mathbb{R}\}) \le C_\alpha,$$

(2) $\forall \alpha \in (\mathbb{Z}_+)^n \ 0 < |\alpha| \le m \quad \exists C_\alpha > 0 \quad \forall (x,y) \in \mathbb{R}^{2n} \quad \forall \xi \in \mathbb{R}^n$

$$\|\partial_\xi^\alpha a(x,\xi) - \partial_\xi^\alpha a(y,\xi)\| \le C_\alpha |x-y|^r (1+|\xi|)^{\delta r - |\alpha|}.$$

In the above definition $\mathcal{R}(\Phi) \le C$ means that $\Phi \in B(X)$ is R-bounded with constant not greater than C. Our main result is then the following.

Theorem 2. *Let* X *be a UMD space* , $1 < p < \infty$, $a \in \mathcal{S}^0_{1,\delta}(r, m, X)$ *and* $m > max(\frac{2n}{1-\delta}, 2n+4)$, *then* T_a *extends to a bounded operator on* $L_p(\mathbb{R}^n; X)$.

As an application, we consider the question of L_p-maximal regularity (i.e. the existence of a unique solution $u \in W^{1,p}([0,T]; X)$ such that $t \mapsto A(t)u(t)$ belongs to $L_p(0,T;X)$) of the following abstract parabolic problem, where $(A(t), D(A(t)))_{t\in[0,T]}$ is a family of generators of analytic semigroups.

$$(NACP)\begin{cases} u'(t) - A(t)u(t) &= f(t) \quad \forall t \in [0,T], \\ u(0) &= 0, \end{cases}$$

We make the following assumptions.

$$(AT) \quad \exists K > 0 \quad \exists 0 \le \alpha < \beta \le 1 \quad \exists \phi \le \psi < \frac{\pi}{2} \quad \forall (t,s) \in [0,T]^2 \quad \forall \lambda \notin \overline{\Sigma}_\psi,$$

$$\|A(t)R(\lambda, -A(t))(A(t)^{-1} - A(s)^{-1})\| \le K \frac{|t-s|^\beta}{1+|\lambda|^{1-|\alpha|}}.$$

$$(RA) \quad \exists \phi \in (0, \frac{\pi}{2}) \quad \exists M > 0 \quad \sigma(A(t)) \subset \Sigma_\phi \quad \forall t \in [0,T]$$

$$\text{and} \quad \mathcal{R}(\{(1+|\lambda|)R(\lambda, -A(t))\ ;\ \lambda \notin \overline{\Sigma}_\phi\ ,\ t \in [0,T]\}) \le M.$$

As a corollary of the above mentioned theorem, we then obtain the following.

Corollary 3. *Let* X *be a UMD Banach space,* $1 < p < \infty$ *and* $(A(t), D(A(t)))_{t\in[0,T]}$ *be a family of operators satisfying (RA) and (AT). Then (NACP) has* L_p*-maximal regularity.*

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Spectral mapping theorems for holomorphic functional calculi

Markus Haase

In this talk we present some recently obtained results on spectral mapping theorems of the type

$$f(\tilde{\sigma}(A)) = \tilde{\sigma}(f(A)), \tag{1}$$

where A is a closed operator on a Banach space X and f is a meromorphic function such that $f(A)$ is properly defined. The symbol $\tilde{\sigma}(A)$ denotes the *extended spectrum*

$$\tilde{\sigma}(A) := \begin{cases} \sigma(A) & \text{if } A \text{ is bounded} \\ \sigma(A) \cup \{\infty\} & \text{if } A \text{ is unbounded} \end{cases}$$

of the operator A. This is a then a non-empty, compact subset of the Riemann sphere $\mathbb{C}_\infty$.

Although most of our results and proofs are generic, we stick to a particular case for A and f. Namely, we require A to be a *sectorial operator* of angle ω $(0 \leq \omega < \pi)$ on X, by which we mean a closed operator A such that $\sigma(A) \subset \overline{\Sigma_\omega}$ and

$$\sup\{\|\lambda\, R(\lambda, A)\| \,|\, \lambda \in \mathbb{C} \setminus \overline{\Sigma_{\omega'}}\} < \infty$$

for each $\omega < \omega' < \pi$. (Here, $\Sigma_\omega := \{0 \neq z \in \mathbb{C} \,|\, |\arg z| < \omega\}$ is the open sector of angle 2ω symmetric with respect to the positive real axis.) The minimal ω with this property is called the *sectoriality angle* and is called $\omega(A)$. For convenience we also require A to be injective.

Since the late 1950's for such operators a definition of the so-called *fractional powers* A^α $(\alpha \in \mathbb{C})$ was known and Balakrishnan established the spectral mapping theorem $\tilde{\sigma}(A^\alpha) = \tilde{\sigma}(A)^\alpha$ for all $\alpha \in \mathbb{C} \setminus i\mathbb{R}$. Ten years later, namely in the late 1960's, also a definition of the *logarithm* $\log A$ was given, and Nollau could show the remarkable fact, that $B := \log A$ is a so-called *strip type* operator, i.e., has spectrum in a horizontal strip $\{z \,|\, |\mathrm{Im}\, z| \leq \omega\}$ with its resolvent bounded outside every larger strip. (We could show in [1] that the minimal strip height for $\log(A)$ equals the sectoriality angle $\omega(A)$.) However, it remained open up to now, whether also $\tilde{\sigma}(\log A) = \log(\tilde{\sigma}(A))$ holds.

The key to answer this question in the positive lies in setting up a whole functional calculus for A. Take $\omega(A) < \varphi < \pi$ and let

$$\mathcal{E} := H_0^\infty(\Sigma_\varphi) := \{f : \Sigma_\varphi \longrightarrow \mathbb{C}\,\text{hol.} \,|\, \exists\, s, C > 0 : |f(z)| \leq C \min\{|z|^s\, |z|^{-s}\}.$$

For $e \in \mathcal{E}$ define

$$e(A) := \frac{1}{2\pi i} \int_\Gamma e(z) R(z, A)\, dz$$

where $\Gamma = \partial\Sigma_{\omega'}$ for (any) $\omega(A) < \omega' < \varphi$. By the resolvent identity and Cauchy's theorem, this yields an algebra homomorphism

$$\Phi := (e \mapsto e(A)) : \mathcal{E} \longrightarrow \mathcal{L}(X)$$

with $\psi(A) = A(1+A)^{-2}$ for $\psi(z) = \frac{z}{(1+z)^2}$. This *elementary calculus* is now extended to a larger class of functions. Define

$$\operatorname{Reg}\mathcal{E} := \{e \in \mathcal{E} \mid e(A) \text{ is injective}\} \quad \text{and}$$
$$\mathcal{M}(\Sigma_\varphi) := \{f : \Sigma_\varphi \longrightarrow \mathbb{C}_\infty \mid f \text{ is meromorphic}\}$$

and call $f \in \mathcal{M}(\Sigma_\varphi)$ *regularizable*, if there exists $e \in \operatorname{Reg}\mathcal{E}$ such that also $ef \in \mathcal{E}$. (Recall that $\mathcal{M}(\Sigma_\varphi)$ is a field and $\mathcal{E}$ is a subalgebra of it.) For regularizable f the operator

$$f(A) := e(A)^{-1}(ef)(A) \tag{2}$$

is a closed operator and is independent of the chosen *regularizer* e. This gives a mapping

$$\mathcal{M}(\Sigma_\varphi)_A \longrightarrow \{\text{closed operators on } X\}$$

where we denote by $\mathcal{M}(\Sigma_\varphi)_A$ the set of all regularizable functions.

This construction is a generalization of a well-known procedure which goes back to Bade and McIntosh. There and in most of the more recent accounts, the regularizers are chosen to be rational functions, namely powers of the function ψ defined above. Let us make the following remarks.

(1) By abstract reasoning one obtains the usual rules:

$$f(A) + g(A) \subset (f+g)(A) \quad \text{and} \quad f(A)g(A) \subset (fg)(A),$$

with "=" if $g(A)$ is a bounded operator.

(2) Let $\lambda \in \mathbb{C}$ and $f \in \mathcal{M}(\Sigma_\varphi)_A$. Then

$$\frac{1}{\lambda - f(z)} \in \mathcal{M}(\Sigma_\varphi)_A \quad \Longleftrightarrow \quad \lambda - f(A) \text{ is injective,}$$

and in this case $(\lambda - f)^{-1}(A) = (\lambda - f(A))^{-1}$. This shows that one can detect the point spectrum of each operator $f(A)$ by the functional calculus.

(3) One immediately obtains a definition of $r(A)$ for each rational function $r = \frac{p}{q}$ in case that $\{q = 0\} \cap P\sigma(A) = \emptyset$.

(4) Our definition (2) is flexible in that one has certain freedom choosing a regularizer for a particular function. Also, since not only holomorphic but meromorphic functions are involved, one has access to each single spectral point of A. This is the key to spectral mapping results.

In the paper [3] we have cast the above procedure in an even more abstract setting. There one can find proofs for the above assertions and also related results.

Let us now come to the main spectral mapping results. The difficulty in obtaining a result of the form (1) lies in the fact that the function f is not necessarily defined on a neighborhood in $\mathbb{C}_\infty$ of $\overline{\Sigma_\omega}^{\mathbb{C}_\infty}$. (Otherwise one could reduce the question to the Dunford-Riesz calculus where the Spectral Mapping Theorem is

known to hold.) Hence the set $\mathcal{C} := \{0, \infty\} \cap \tilde{\sigma}(A)$ of *critical points* needs special consideration. However, if we disregard these points for a moment, we obtain an inclusion which holds without further restrictions.
Theorem 1.(H. 2004) Let $f \in \mathcal{M}(\Sigma_\varphi)_A$. Then $\overline{f(\tilde{\sigma}(A) \setminus \mathcal{C})}^{\mathbb{C}_\infty} \subset \tilde{\sigma}(f(A))$.

To obtain a "full" spectral mapping theorem, one has to require not only that f should be defined, i.e., have limits, at the critical points but rather that these limits are approached "fast enough".

Theorem 2. (H. 2004) Let $f \in \mathcal{M}(\Sigma_\varphi)_A$ have limits at the points of $\mathcal{C}$ which are approached "fast enough". Then $f(\tilde{\sigma}(A)) = \tilde{\sigma}(f(A))$.

The exact meaning of "fast enough" differs for both spectral inclusions. For "$\supset$" one only has to require a limit c such that $f - c$ is still integrable with respect to $|dz|/|z|$. For the inclusion "$\subset$" we have to require an "almost logarithmic" convergence rate. Polynomial convergence would suffice in any case.

Let us browse through some examples. Theorem 2 gives us the (well-known) equalities

$$\tilde{\sigma}(A^\alpha) = \tilde{\sigma}(A)^\alpha \quad (\alpha \in \mathbb{C} \setminus i\mathbb{R}) \quad \text{and}$$
$$\tilde{\sigma}(e^{-tA}) = e^{-t\tilde{\sigma}(A)} \quad (t > 0,\ \omega(A) < \pi/2).$$

Unfortunately, the case of the logarithm is *not* covered by Theorem 2. However, one can set up a functional calculus for strip type operators (as in [1]) and prove an inclusion theorem like Theorem 1 also for this calculus. Since one has $(e^z)(\log A) = A$ one obtains

$$\tilde{\sigma}(\log(A)) = \log(\tilde{\sigma}(A))$$

from these two inclusion theorems.

Finally, we remark that in case f shows wild behaviour in the critical points $\mathcal{C}$, even a "weak" spectral mapping theorem may fail. (This corresponds to the failing of a weak spectral mapping theorem for semigroups.) Namely by using Zabczyks example one can find a sectorial operator A of angle 1 on a Hilbert space H, having even a bounded H^∞-calculus, such that for $f(z) := z^i$ one has

$$\overline{f(\tilde{\sigma}(A) \setminus \mathcal{C})} \subset \{|z| \leq 1\}, \quad \text{but} \quad \sigma(A^i) \cap \{|z| = e\} \neq \emptyset.$$

Proofs of Theorem 1 and Theorem 2 as well as related results and references can be found in [2]. For more information on holomorphic functional calculi, see [4].

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Basic analytic questions on the nonlinear Fourier transform

Christoph Thiele
(joint work with Terence Tao)

In harmonic analysis, one studies continuity and mapping properties of operators such as the Fourier transform in various function spaces. We discuss analogues of some basic mapping properties of the Fourier transform for certain nonlinear variants of the Fourier transform that are sometimes referred to as scattering transforms. A series of articles by Beals and Coifman from the 1980s starting with [1] is a good precedent for the motivation and type of analysis we propose.

Scattering transforms come in many facets, and the algebraic and geometric part of the subject alone occupies a vast literature. But a recent surge of papers such as [4], [9], [7], and [5] on basic analytic properties shows that these properties are not fully understood even for simple models.

We restrict attention to one simple model of the scattering transform. Assume F is a sufficiently nice function on the real line and consider the following initial value problem with real parameter k:

$$g'(x) = g(x)F(x)e^{-2\pi ixk}$$
$$g(-\infty) = 1 \quad .$$

This initial value problem can be explicitly solved as

$$\log(g(x)) = \int_{-\infty}^{x} F(y)e^{-2\pi iyk}\, dy$$

where on the right hand side we have partial Fourier integrals. Thus $\log(g(\infty)) = \widehat{F}(k)$ is the Fourier transform of F. Now consider a matrix valued differential equation

$$G'(x) = G(x)\begin{pmatrix} 0 & F(x)e^{-2\pi ikx} \\ \overline{F(x)}e^{2\pi ikx} & 0 \end{pmatrix} \tag{1}$$

$$G(-\infty) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The solution to this differential equation can no longer be expressed as an explicit integral, because the coefficient matrices for different x do not commute. We define the nonlinear Fourier transform of F to be

$$\overbrace{F}(k) = G(\infty) = \begin{pmatrix} a(k) & b(k) \\ \overline{b(k)} & \overline{a(k)} \end{pmatrix}$$

where it is easy to see that $G(\infty)$ has the structure claimed here.

Using Picard iteration, one obtains a multilinear expansion of the functions a and b which obviously depend nonlinearly on F. One obtains that in linear approximation a is approximated by the constant 1 and b by the linear Fourier transform of F. This is further justification for calling $\overbrace{F}$ the nonlinear Fourier transform of F.

The first order system (1) is equivalent to a second order scalar differential equation. One can choose this equation to be the time independent Schrödinger equation with potential $F^2 - F'$ and energy $-k^2$. In this context the functions $1/a$ and b/a become the classical transmission and reflection coefficients of the Schrödinger operator, which manifests the connection to scattering theory.

The following analogues of classical estimates for the Fourier transform are known.

Nonlinear Riemann Lebesgue (a denotes the first entry in the matrix $\widehat{F}$):

$$\|\sqrt{\log|a|}\|_\infty \leq C\|F\|_1$$

Nonlinear Hausdorff Young for $1 < p < 2$

$$\|\sqrt{\log|a|}\|_{p'} \leq C_p\|F\|_p \tag{2}$$

Nonlinear Plancherel:

$$\|\sqrt{\log|a|}\|_2 = \|F\|_2 \tag{3}$$

The first inequality follows easily from Gronwall's inequality. The second inequality was only recently proved though not explicitly stated by M. Christ and A. Kiselev in [4]. The Plancherel identity follows from an argument using a contour integral which has been known for a long time. In linear approximations these conjectures become the corresponding classical (in)equalities, safe for the value of the constants.
The first two inequalities extend to hold for the maximal function $\sup_x \sqrt{\log|a(k,x)|}$ where $a(k,x)$ is the diagonal element of the solution $G(x)$ of (1). However, the inequality in the limiting case $p = 2$ remains unknown:
Conjecture 1:

$$\|\sup_x \sqrt{\log|a(.,x)|}\|_2 \leq C\|F\|_2$$

This conjecture is a nonlinear variant of Carleson's theorem [2]. With C. Muscalu and T., Tao, the author has proved a Walsh analogue of a slightly weaker form (weak type estimate) of this theorem [6].

Another interesting question is
Conjecture 2: The constant C_p in (2) can be chosen uniformly as p tends to 2.

In the linear case such a statement would follow from the endpoint estimates and an interpolation argument, but in the nonlinear situation no valid interpolation argument is known.

While an isometric identity such as (3) in the linear setting implies injectivity of the Fourier transform, the nonlinear Fourier transform turns out not to be injective on $L^2(\mathbb{R})$ (A. Volberg and P. Yuditskii, [9]). This raises the question of describing the fibers of the nonlinear Fourier transform. In joint work with T. Tao the author has worked out completely the case of rational functions a and b ([8]).

If one considers only functions F supported on the right half line, thus restricts the nonlinear Fourier transform to a map from $L^2([0,\infty)$, then it is injective and indeed bicontinuous in an explicit topology on the target space (J. Sylvester and D. Winebrenner, [7]). This target space is a nonlinear variant of a Hardy space.

Similarly there is a conjugate nonlinear Hardy space, which is the nonlinear Fourier transform of $L^2([-\infty, 0])$. A general function in $L^2(\mathbb{R})$ can be written as the sum of a function F_- supported on the left half line and a function F_+ on the right half line. Between the relevant Fourier transforms one has the simple identity

$$\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} = \begin{pmatrix} a_- & b_- \\ \overline{b_-} & \overline{a_-} \end{pmatrix} \begin{pmatrix} a_+ & b_+ \\ \overline{b_+} & \overline{a_+} \end{pmatrix}$$

This is a multiplicative analogue of the decomposition of $L^2(\mathbb{R})$ into Hardy space and conjugate Hardy space. However, as follows from the discussion above, for given left hand side the multiplicative splitting is not necessarily unique. The problem of decomposition of a matrix function into the product of two functions, each of which have some additional holomorphicity conditions, is called a Riemann Hilbert problem. Understanding non-uniqueness of the Riemann Hilbert problem is equivalent to understanding non-uniqueness of the inverse nonlinear Fourier transform.

A motivation for understanding the nonlinear Fourier transform comes from the study of integrable nonlinear partial differential equations. Just as the linear Fourier transform can be used to solve constant coefficient linear PDE, the nonlinear Fourier transform can be used to solve certain nonlinear PDE. As an example we discuss the Cauchy problem for the modified Korteweg de Vries equation, which is an equation for a function $F(t, x)$ in a time variable t and a one dimensional space variable x:

$$F_t = F_x xx + 6F^2 F_x$$

$$F(0, x) = F_0(x)$$

A formal solution to this problem is given as follows: If $(a(k), b(k))$ is the nonlinear Fourier transform of F_0, then

$$(a(k), e^{8ik^3 t} b(k))$$

is the nonlinear Fourier transform of $F(t, .)$. Thus modulo ability of taking and inverting the nonlinear Fourier transform, there is an explicit solution to the Cauchy problem for the modified Korteweg de Vries equation.

The fact that the nonlinear Fourier transform does not behave well in $L^2(\mathbb{R})$ has recently been extended by Colliander, Christ, and Tao to the theorem that mKdV is not well posed in $L^2(\mathbb{R})$ in the sense that the solution map from initial data to data at time t is not uniformly continuous in $L^2(\mathbb{R})$ [3].

A more detailed discussion of these topics will appear in the Park City Mathematics Series 2003, [8].

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A q-variation functional calculus for power-bounded operators on certain *UMD* spaces

Alastair Gillespie

(joint work with Earl Berkson)

We report on some recent work concerning the development of a functional calculus for an invertible operator U on a *UMD* Banach space X for which

$$c \equiv \sup_{n\in\mathbb{Z}} \|U^n\| < \infty.$$

This involves the Marcinkiewicz q-classes $\mathcal{M}_q(\mathbb{T})$, which are defined as follows. Let $\{s_k\}_{k\in\mathbb{Z}}$ be the dyadic points in the interval $(0, 2\pi)$ defined by

$$s_k = 2^{k-1}\pi \quad (k \le 0), \qquad s_k = 2\pi - 2^{-k}\pi \quad (k > 0),$$

and let $\{\Delta_k\}_{k\in\mathbb{Z}}$ be the dyadic arcs of the unit circle, given by

$$\Delta_k = \{e^{ix} : x \in [s_k, s_{k+1}]\}.$$

For $1 \le q < \infty$ and $\varphi : \mathbb{T} \to \mathbb{C}$, define the q-variation $\mathrm{var}_q(\varphi, \Delta_k)$ by

$$\mathrm{var}_q(\varphi, \Delta_k) = \sup \left\{ \sum_{n=1}^{N} |\varphi(e^{ix_{n-1}}) - \varphi(e^{ix_n})|^q \right\}^{1/q},$$

where the supremum extends over all partitions

$$s_k = x_0 < x_1 < \cdots < x_N = s_{k+1}$$

of $[s_k, s_{k+1}]$. The Marcinkiewicz q-class $\mathcal{M}_q(\mathbb{T})$ is defined as the class of all functions $\varphi : \mathbb{T} \to \mathbb{C}$ such that

$$\|\varphi\|_{\mathcal{M}_q(\mathbb{T})} \equiv \sup_{z\in\mathbb{T}} |\varphi(z)| + \sup_{k\in\mathbb{Z}} \mathrm{var}_q(\varphi, \Delta_k) < \infty.$$

It is readily seen that, with pointwise algebraic operations and norm $\|\cdot\|_{\mathcal{M}_q(\mathbb{T})}$, $\mathcal{M}_q(\mathbb{T})$ is a Banach algebra. Note that $\mathcal{M}_1(\mathbb{T})$ is the usual class of Marcinkiewicz multipliers on $\mathbb{T}$. Also, let $BV(\mathbb{T})$ denote the Banach algebra of all functions $\varphi : \mathbb{T} \to \mathbb{C}$ of bounded variation, with norm

$$\|\varphi\|_{BV(\mathbb{T})} = \sup_{z\in\mathbb{T}} |\varphi(z)| + \mathrm{var}_1(\tilde{\varphi}, [0, 2\pi]),$$

where $\tilde{\varphi}(\lambda) = \varphi(e^{i\lambda})$ for $0 \le \lambda \le 2\pi$. For $1 < q_1 < q_2 < \infty$, we have

$$BV(\mathbb{T}) \subset \mathcal{M}_1(\mathbb{T}) \subset \mathcal{M}_{q_1}(\mathbb{T}) \subset \mathcal{M}_{q_2}(\mathbb{T}),$$

with each inclusion norm-contractive.

It was shown in [5] that, given an invertible operator U on a *UMD* space X such that

$$c \equiv \sup_{n\in\mathbb{Z}} \|U^n\| < \infty,$$

there is a projection-valued function $E(\cdot) : [0, 2\pi] \to \mathcal{B}(X)$ (with certain natural properties) such that U has a spectral representation

$$U = \int_{[0,2\pi]}^{\oplus} e^{i\lambda}\, dE(\lambda).$$

Operators with a spectral representation of the above form (whether or not power-bounded) are called *trigonometrically well-bounded* and were introduced in [1]. They always have a continuous $BV(\mathbb{T})$ functional calculus given by

$$\varphi \to \varphi(U) \equiv \int_{[0,2\pi]}^{\oplus} \varphi(e^{i\lambda})\, dE(\lambda) \qquad (\varphi \in BV(\mathbb{T})).$$

Here, $\int_{[0,2\pi]}^{\oplus} \varphi(e^{i\lambda})\, dE(\lambda)$ denotes

$$\varphi(1)E(0) + \int_0^{2\pi} \varphi(e^{i\lambda})\, dE(\lambda),$$

the integral existing in the strong operator topology as a Riemann-Stieltjes integral. (For a detailed discussion of this type of integration, see [7] or the abbreviated account in [2].)

In later developments, using Bourgain's vector-valued version of the Marcinkiewicz multiplier theorem for *UMD* spaces, it was shown in [3] that, given an invertible operator U on a *UMD* space X for which

$$c \equiv \sup_{n\in\mathbb{Z}} \|U^n\| < \infty,$$

there is an $\mathcal{M}_1(\mathbb{T})$ functional calculus

$$\varphi \to \varphi(U) = \int_{[0,2\pi]}^{\oplus} \varphi(e^{i\lambda})\, dE(\lambda)$$

with norm at most c^2K_X, where K_X is a constant depending only on the space X. Furthermore, the Coifman-Rubio de la Francia-Semmes multiplier theorem [6] was used to show in [4] that when, in particular, X is a closed subspace of some space $L^p(\mu)$, $1 < p < \infty, 1 \leq q < \infty$ and $|p^{-1} - 2^{-1}| < q^{-1}$, U has an $\mathcal{M}_q(\mathbb{T})$-functional calculus, given by the same integral formula, with norm at most $c^2K_{p,q}$. In the present talk, we show that a similar result holds when U acts on a member of a class $\mathcal{I}$ of *UMD* spaces. This class is defined as those (necessarily *UMD*) spaces X for which there exists a compatible couple $\mathcal{X}_0$ and $\mathcal{X}_1$ for Calderón's complex method of interpolation, with $\mathcal{X}_0$ a Hilbert space and $\mathcal{X}_1$ a *UMD* space, such that X is isomorphic as a Banach space to a subspace an intermediate space $[\mathcal{X}_0, \mathcal{X}_1]_t$ for some t in the range $0 < t < 1$. The class $\mathcal{I}$ contains all L^p spaces and the von Neumann-Schatten ideals $\mathcal{C}_p$ for $1 < p < \infty$, as well as every *UMD* lattice of measurable functions on a σ-finite measure space. Whether or not *every UMD* space is in fact an intermediate space of the above form appears to be unknown. The main result presented in this talk is as follows.

Theorem. *Let X be a closed subspace of an intermediate space $[\mathcal{X}_0, \mathcal{X}_1]_t$ as above and let U be an invertible operator on X such that $c \equiv \sup_{n\in\mathbb{Z}} \|U^n\| < \infty$, so that U has a spectral representation $U = \int_{[0,2\pi]}^{\oplus} e^{i\lambda}\, dE(\lambda)$. Then, for each q in the range $1 \leq q < t^{-1}$, there is a constant $K_{X,q}$ such that U has an $\mathcal{M}_q(\mathbb{T})$ functional calculus given by*

$$\varphi \to \varphi(U) \equiv \int_{[0,2\pi]}^{\oplus} \varphi(e^{i\lambda})\, dE(\lambda)$$

and satisfying $\|\varphi(U)\| \leq c^2 K_{X,q} \|\varphi\|_{\mathcal{M}_q(\mathbb{T})}$ for $\varphi \in \mathcal{M}_q(\mathbb{T})$.

The proof of this result entails establishing a vector-valued $\mathcal{M}_q(\mathbb{T})$ multiplier theorem for $[\mathcal{X}_0, \mathcal{X}_1]_t$ and then using a transference argument. Notice that the result implies that, for $1 < p < \infty$ and $1 \leq q < \infty$, every invertible power-bounded operator on $\mathcal{C}_p$ has an $\mathcal{M}_q(\mathbb{T})$ functional calculus whenever $|p^{-1} - 2^{-1}| < (2q)^{-1}$.

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Function spaces associated with operators with kernel bounds

Xuan Thinh Duong

This lecture presents some recent results on function spaces associated with operators with kernel bounds. It is based on the joint works [ADM],[DY1],[DY2] of the author with Pascal Auscher, Alan McIntosh, and Lixin Yan.

The classical Hardy spaces and BMO spaces on $\mathbb{R}^n$ can be represented through the Laplacian $(-\Delta)$ on $\mathbb{R}^n$. They are, respectively, the natural substitutes for the spaces $L^1(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$ in the theory of Calderón-Zygmund singular operators. However, when we study singular operators whose kernels do not possess the required smoothness of kernels of Calderón-Zygmund operators such as Hölder continuity or the Hörmander (almost L^1) condition, then the classical Hardy spaces and BMO spaces are not necessarily the most suitable spaces for the study of these operators.

Let L be a linear operator acting on $L^2(\mathbb{R}^n)$. We assume the following conditions:

(i) L generates a bounded analytic semigroup e^{-tL} on $L^2(\mathbb{R}^n)$;

(ii) The semigroup e^{-tL} has a kernel $p_t(x,y)$ which satisfies the upper bound

$$|p_t(x,y)| \leq h_t(x,y) = t^{-n/m} g(\frac{|x-y|}{t^{1/m}})$$

in which m is a positive constant and g is a positive, bounded, decreasing function satisfying

$$\lim_{r\to\infty} r^{n+\epsilon} g(r) = 0$$

for some $\epsilon > 0$;

(iii) L has a bounded holomorphic functional calculus on $L^2(\mathbb{R}^n)$. For the definition and properties of operators with a bounded holomorphic functional calculus on L^2, we refer reader to [CDMY].

We now define the Hardy space and the BMO space associated to the operator L as follows.

Definition 1: We define the area integral function of a function u by

$$\mathbb{A}u(x) = \left\{ \int_{\Gamma(x)} |tLe^{-tL}u(y)|^2 \frac{dydt}{t^{n+1}} \right\}^{1/2}$$

where $\Gamma(x)$ is the cone of vertex x given by $\Gamma(x) = \{(y,t) \in \mathbb{R}^n \times [0,\infty) : d(x,y) \leq t\}$.

The Hardy space $H^1_L(\mathbb{R}^n)$ is defined by

$$H^1_L(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n) : ||f||_{H^1_L} = ||\mathbb{A}f||_{L_1(\mathbb{R}^n)} < \infty\}.$$

Definition 2: We denote

$$\mathbb{M} = \{f \in L^2_{loc}(\mathbb{R}^n) : \int_{\mathbb{R}^n)} \frac{|f(x)|}{1+|x|^{n+\epsilon}} dx < \infty\}.$$

The space $BMO_L(\mathbb{R}^n)$ is defined as the set of all functions $f \in \mathbb{M}$ such that

$$||f||_{BMO_L} = \sup \frac{1}{|B|} \int_B |f(x) - e^{-r_B^m L} f(x)| dx < \infty$$

where the supremum is taken on all the balls B.

We can show the following properties.

Theorem 3:

(a) When L is the Laplacian, the Hardy space $H^1_L(\mathbb{R}^n)$ and the $BMO_L(\mathbb{R}^n)$ space coincide with the classical Hardy space and BMO space, respectively.

(b) Assume that T is a linear operator which is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for some p, $1 < p \leq \infty$.

(i) If T is also bounded from $H^1_L(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$, then T is bounded from $L^q(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for all q, $1 < q \leq p$.

(ii) If T is bounded from $L^\infty(\mathbb{R}^n)$ to $BMO_L(\mathbb{R}^n)$, and $p < \infty$, then T is bounded from $L^q(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for all q, $p \leq q < \infty$.

(c) The space $BMO_L(\mathbb{R}^n)$ is the dual space of $H^1_{L^*}(\mathbb{R}^n)$ where L^* is the adjoint operator of L.

For the proofs of the Theorem 3 and other properties of the Hardy space $H^1_L(\mathbb{R}^n)$ and the BMO space $BMO_L(\mathbb{R}^n)$ such as the molecular decomposition and John-Nirenberg inequality, we refer the reader to [ADM], [DY1] and [DY2].

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$T1$ and Tb theorems in UMD spaces

Tuomas Hytönen

(joint work with Lutz Weis)

Let T be a linear operator from $\mathscr{S}(\mathbf{R}^n)$ to $\mathscr{S}'(\mathbf{R}^n)$ and $K(x, y)$ a locally integrable function on $\mathbf{R}^n \times \mathbf{R}^n$ minus the diagonal. We say that K is the *kernel* of T if $Tf(x) = \int K(x, y)f(y)dy$ for all $f \in \mathscr{D}(\mathbf{R}^n)$ and almost every x outside the support of f.

K is said to satisfy the *standard estimates* if, for some $0 < \gamma \leq 1$,

$$|x-y|^n K(x,y) \text{ is bounded on } x \neq y,$$
$$|x-y|^{n+\gamma}\frac{K(x,y)-K(x',y)}{|x-x'|^\gamma} \text{ is bounded on } 0 < \frac{|x-x'|}{|x-y|} < \frac{1}{2}, \text{ and}$$
$$|x-y|^{n+\gamma}\frac{K(x,y)-K(x,y')}{|y-y'|^\gamma} \text{ is bounded on } 0 < \frac{|y-y'|}{|x-y|} < \frac{1}{2}.$$

We say that T is a Calderón-Zygmund operator if it has a kernel K which satisfies the standard estimates.

Denote $\mathscr{A}_h^r f := r^{-n/2} f(r^{-1}(\cdot - h))$. T is said to have the *weak boundedness property* if $\langle \mathscr{A}_h^r \psi, T(\mathscr{A}_h^r \phi)\rangle$ is bounded when the variables range over $r > 0$, $h \in \mathbf{R}^n$, and all ϕ, ψ in a bounded subset of $\mathscr{D}(\mathbf{R}^n)$.

If T is a Calderón-Zygmund operator, the remarkable $T1$ theorem of G. David and J.-L. Journé [2] states that T is bounded in the norm of $L^2(\mathbf{R}^n)$ if and only if each of the following three conditions is satisfied: (i) T has the weak boundedness property, (ii) $T1 \in BMO(\mathbf{R}^n)$, and (iii) $T'1 \in BMO(\mathbf{R}^n)$.

Here the objects $T1$ and $T'1$ can be given a meaning as distributions modulo constants. The classical theory shows that a Calderón-Zygmund operator is bounded on $L^2(\mathbf{R}^n)$ if and only if it is bounded on $L^p(\mathbf{R}^n)$ for all $1 < p < \infty$.

Although the conditions of the $T1$ theorem are necessary and sufficient, it is sometimes not feasible in practice to check directly whether $T1 \in BMO(\mathbf{R}^n)$ and $T'1 \in BMO(\mathbf{R}^n)$ for some operators (such as the Cauchy integral on a Lipschitz graph). To overcome this problem, David, Journé and S. Semmes [3] proved a more general Tb theorem, in which the rôle played by the constant function 1 in $T1$ was taken by an arbitrary *para-accretive* function b on $\mathbf{R}^n$. The notion of para-accretivity generalizes that of accretivity, which simply requires that $\operatorname{Re} b(x) \geq \delta > 0$ for a.e. $x \in \mathbf{R}^n$.

One can equally well consider an operator T from $\mathscr{S}(\mathbf{R}^n)$ to the operator-valued distribution space $\mathscr{S}'(\mathbf{R}^n, \mathscr{L}(X))$, where X is a Banach space, and having an $\mathscr{L}(X)$-valued kernel $K(x,y)$ in the same sense as above. Then one may define the action of T on $X \otimes \mathscr{S}(\mathbf{R}^n)$ (a dense subspace of $L^p(\mathbf{R}^n, X)$ for $p < \infty$) by $\langle \psi, T(x \otimes \phi)\rangle := \langle \psi, T\phi\rangle x$, and inquire about its boundedness in the norm of $L^p(\mathbf{R}^n, X)$ for $1 < p < \infty$. There now exist even two approaches to a $T1$ theorem for such operators for a UMD space X.

The first one, based on an investigation of T on the Haar system and a decomposition of the operator into pieces which can be related to martingale transforms, was devised by T. Figiel [4] already in the 80's. He considers only scalar-valued kernels K; however, incorporating the recent R-boundedness techniques into his method, one can also use it to obtain an operator-valued version of the result. The other, Fourier-analytic approach, which was directly designed to handle also operator-valued kernels, was more recently invented by L. Weis and myself [6].

For scalar-valued kernels K, Figiel showed that the $T1$ theorem generalizes to UMD spaces word by word: If T is a Calderón-Zygmund operator, X is a UMD space and $1 < p < \infty$, then T is bounded in the norm of $L^p(\mathbf{R}^n, X)$ if and only if T satisfies the weak boundedness property and $T1, T'1 \in BMO(\mathbf{R}^n)$.

As for the operator-valued setting, in view of the principle that the way of generalizing classical theorems to Banach spaces is to replace boundedness conditions by R-boundedness, we define the *standard R-estimates* for an $\mathscr{L}(X)$-valued kernel K by replacing every occurrence of the word "bounded" in the definition of the standard estimates by "R-bounded". We also define the *weak R-boundedness property* of T by modifying the definition of the weak boundedness property in a similar way.

The following special $T1$ theorem is proved in [6]: Let X be a UMD space and T a Calderón-Zygmund operator whose $\mathscr{L}(X)$-valued kernel satisfies the standard R-estimates. Let further $T1 = 0 = T'1$. If T has the weak R-boundedness property, then it is bounded in the norm of $L^p(\mathbf{R}^n, X)$ for all $1 < p < \infty$. If X also enjoys Pisier's property α, then we even have a necessary and sufficient condition as follows: The collection $(\mathscr{A}_h^r)' T \mathscr{A}_h^r$, where $r > 0$ and $h \in \mathbf{R}^n$, is R-bounded on $L^p(\mathbf{R}^n, X)$ if and only if T satisfies the weak R-boundedness property.

Obtaining a full $T1$ theorem is then mainly a question of boundedness of the *paraproduct* operators $P(g, \cdot)$, where $g = T1$ and $g = T'1$; however, this is a delicate problem already in infinite-dimensional Hilbert spaces. A counterexample of F. Nazarov, S. Treil and A. Volberg [7] shows that the natural necessary condition [$g(\cdot)x \in BMO(\mathbf{R}^n, X)$ uniformly for $x \in B_X$, the unit ball of X] is not sufficient in general. We have shown in [6] that a sufficient condition is obtained by requiring that $g \in BMO(\mathbf{R}^n, U)$, where $U \hookrightarrow \mathscr{L}(X)$ is a UMD space such that B_U is an R-bounded subset of $\mathscr{L}(X)$ (and actually somewhat weaker but more technical assumptions would do); however, it is unclear how far this is from what is necessary.

I have also considered an operator-valued Tb theorem in [5], using a combination of ideas from Figiel's martingale approach to the vector-$T1$ theorem, and from R. Coifman and S. Semmes' new proof (found in [1]) of the scalar-Tb theorem. The idea is to replace the Haar system by a different martingale-like basis of $L^2(\mathbf{R}^n)$, which is carefully chosen in dependence on the particular para-accretive function b. Again, one obtains the $L^p(\mathbf{R}^n, X)$-boundedness of T (for X UMD and $1 < p < \infty$) under the assumptions of the standard R-estimates and the weak R-boundedness property, provided that $Tb = 0 = T'\tilde{b}$ for two para-accretive functions b and $\tilde{b}$. More generally one can allow the membership of Tb and $T'\tilde{b}$ in appropriate BMO-type spaces, just like above.

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Perturbing the $H^\infty-$calculus

Nigel Kalton
(joint work with Lutz Weis)

Let X be a complex Banach space and suppose A is a sectorial operator on X. Suppose A admits an $H^\infty-$calculus for some sector $\Sigma_\phi = \{z : \ |\arg\zeta| < \phi\}$. Next suppose that B is another sectorial operator such that the domain of B coincides with the domain of A and one has equivalence of A and B in the sense that

$$\|Ax\| \approx \|Bx\| \qquad x \in \text{ Dom(A)}. \tag{1}$$

It is known from an example of McIntosh and Yagi that these hypotheses do not imply that B will necessarily also have an $H^\infty-$calculus (for some sector) [5]. On the other hand Auscher, McIntosh and Nahmod [1] showed that if X is a Hilbert space and one additionally supposes that Dom (A^{-1}) = Dom (B^{-1}) and

$$\|A^{-1}x\| \approx \|B^{-1}x\| \qquad x \in \text{Dom }(A^{-1}) \tag{2}$$

then, indeed B must have an $H^\infty-$calculus.

This result can be generalized to arbitrary Banach spaces if one introduces the concept of an almost Rademacher sectorial operator. We say that B is almost Rademacher sectorial if $\{\lambda B(\lambda - B)^{-2}\}$ is Rademacher-bounded outside some sector Σ_ϕ. We then can let $\tilde{\omega}_R(A)$ be the infimum of all angles for which this condition holds. Similarly we let $\omega_H(B)$ be the infimum of all ϕ so that B has an $H^\infty(\Sigma_\phi)-$calculus.

Theorem 1. *Let A be a sectorial operator on a complex Banach space X which admits an $H^\infty-$calculus and suppose B is another sectorial operator satisfying (1) and (2). Then B admits an H^∞-calculus if and only if B is almost Rademacher sectorial and $\omega_H(B) = \tilde{\omega}_R(B)$.*

See [3] and [4] for two approaches to results of this type, and some generalizations. Let us say that B is a compact perturbation of A if $B = A + KA$ where $K : X \to X$ is a compact operator. Then the assumption of almost Rademacher sectoriality on B can be relaxed:

Theorem 2. *Let A be a sectorial operator on a complex Banach space X which admits an $H^\infty-$calculus and suppose B is a sectorial operator which is a compact perturbation of A and satisfies (1) and (2). Then B admits an H^∞-calculus (for some angle).*

It is however desirable to have theorems where one relaxes (2) and assumes only (1). Since counterexamples exist in this setting, one needs additional assumptions on B. These conditions can of course involve A. For example see the perturbation result in [2].

If we write $B = (1+T)A$ one can ask what assumptions on the bounded operator T will suffice, independent of the choice of A. In fact a rather complete answer can be given to this problem.

Let T be an operator on a Hilbert space $\mathcal{H}$. We recall the singular values of T are denoted by $s_n(T)$. We define $\mathcal{I}$ as the ideal of operators T so that

$$\sum_{n=1}^{\infty} \frac{s_n(T)}{n} < \infty. \tag{3}$$

This ideal is dual to the well-known Matsaev ideal. Note that every $T \in \mathcal{I}$ is compact if $s_n(T) = O((\log(n+1))^{-1-\epsilon}$ then $T \in \mathcal{I}$.

Theorem 3. *Let $\mathcal{H}$ be a Hilbert space and suppose $T \in \mathcal{I}$. Suppose A is a sectorial operator with an $H^\infty-$calculus and that $B = A + TA$ is a sectorial operator. Then B admits an $H^\infty-$calculus.*

The role of the ideal $\mathcal{I}$ is connected with the lower triangular projection (respect to some fixed orthonormal basis). It is a classical result of M.G. Krein that the lower triangular projection maps the trace-class into the Matsaev ideal (see [6] for a recent proof) and dually also maps $\mathcal{I}$ into $\mathcal{B}(H)$. Conversely if T has the property that Theorem 3 holds for every choice of A then the lower-triangle of T is bounded. This shows the result is essentially best possible. In particular compact perturbations do not preserve the $H^\infty-$calculus (unless we also assume (2)).

To extend this result to arbitrary Banach space involves some loss of precision. We define the approximation numbers of T by

$$a_n(T) = \inf\{\|T - F\| : \ \text{rank } (F) < n\}.$$

Theorem 4.*Let X be a complex Banach space and suppose A is a sectorial operator admitting an $H^\infty-$calculus. Let $T : X \to X$ be an operator satisfying the condition:*

$$\sum_{n=1}^{\infty} \frac{a_n(T)\log n}{n} < \infty.$$

Then if $B = A + TA$ is a sectorial operator, B also admits an $H^\infty-$calculus.

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Ergodic properties of measure-preserving bounded linear operators and unimodular point spectrum

Sophie Grivaux
(joint work with Frédéric Bayart)

We report here on some recent joint work with Frédéric Bayart (Université Bordeaux 1, France). The results presented here are taken out of the two preprints [2] and [3].

1. Introduction

The main topic of this report is hypercyclicity and frequent hypercyclicity of bounded operators on Banach spaces. In what follows, we will denote by X a separable infinite dimensional Banach space, and by T a bounded operator on X. The operator T is said to be *hypercyclic* if there exists a vector x in X whose orbit $\mathcal{O}rb(x,T) = \{T^n x; n \geq 0\}$ under the action of T is dense in X. Any hypercyclic operator is of course cyclic, and a simple Baire Category argument shows that T is hypercyclic if and only if it is topologically transitive, i.e. for every pair (U,V) of non empty open subsets of X, there exists an integer n such that $T^{-n}(U) \cap V$ is non empty. In particular the set of hypercyclic vectors for a given hypercyclic operator is a dense G_δ subset of X.

The notion of hypercyclicity is connected to the Invariant Subset Problem (T has no non trivial invariant closed set if and only if every non zero vector is hypercyclic for T). For instance, the fact that every separable infinite dimensional space supports a hypercyclic operator ([1]) can be used to show the following ([7]):

– any normed space of countable algebraic dimension supports an operator with no non trivial invariant closed set.

– any dense set $\{v_n; n \geq 0\}$ of linearly independent vectors of a Banach space X is the orbit of the first vector v_0 under the action of some bouded operator T on X: $\{T^n v_0; n \geq 0\} = \{v_n; n \geq 0\}$.

Hypercyclicity is also connected with the study of topological dynamics. For instance, the notion of chaos for operators has been introduced and investigated by Godefroy and Shapiro in [6]: T is chaotic if and only if it is hypercyclic and has a dense set of vectors with periodic orbit (there exists an $n \geq 1$ sucht that $T^n x = x$). We state here the Godefroy-Shapiro Criterion for hypercyclicity and chaos ([6]), which motivates part of the work we are going to present in the sequel:

Theorem 1.1. *Let $H_+(T)$ be the linear space spanned by the eigenspaces $ker(T - \lambda I), |\lambda| > 1$, and $H_-(T)$ the space spanned by the eigenspaces $ker(T - \lambda I), |\lambda| < 1$. If $H_+(T)$ and $H_-(T)$ are dense in X, then T is hypercyclic. If moreover the space $H_0(T)$ spanned by the eigenspaces $ker(T - \lambda I)$, where λ is an n^{th} root of unity, is dense in X, then T is chaotic.*

Thus the eigenvectors associated to eigenvalues of modulus 1 are important in investigating the chaotic behaviour of an operator. For more about hypercylicity and related topics, we refer the reader to the two surveys [9] and [10].

2. Frequently hypercylic operators

We now investigate another form of hypercyclicity, called frequent hypercyclicity: instead of studying the global behaviour of open sets under the action of T, we focus on one specific orbit and try to see how well it fills the space:

Definition 2.1. *The operator T is* frequently hypercyclic *if there exists a vector $x \in X$ such that for every non empty open set U, the set of integers n such that $T^n x$ belongs to U has positive lower density:*

$$\liminf_{N \to +\infty} \frac{1}{N} \# \{n \leq N;\ T^n x \in U\} > 0.$$

For instance, any multiple ωB, $|\omega| > 1$, of the standard backward shift on one of the spaces ℓ_p, $1 \leq p < +\infty$, is frequently hypercyclic. On the other hand, the backward shift on the Bergman space is an example of a hypercyclic operator which is not frequently hypercylic.

Frequent hypercyclicity differs deeply from the original notion of hypercyclicity, because the Baire Category Theorem is no longer available in this setting: it can be showed under mild assumptions that the set of frequently hypercyclic vectors for T is not a residual subset of X. Moreover, it seems that frequent hypercyclicity depends much more on the Banach structure of the space than hypercyclicity does: for instance, every infinite dimensional separable space supports a hypercyclic operator, but it may reasonably be conjectured that some spaces do not support any frequently hypercyclic operator.

There is a natural link between frequent hypercyclicity and ergodic properties of operators seen as measure-preserving transformations of the space X. We now focus on this aspect of the theory: suppose that $T : (X, \mathcal{B}, m) \longrightarrow (X, \mathcal{B}, m)$ is a measure-preserving transformation of the probability space $(X, \mathcal{B}, m)$: for every $A \in \mathcal{B}$, $m(T^{-1}(A)) = m(A)$. Then T is said to be *ergodic* if for every $A, B \in \mathcal{B}$ with $m(A) > 0$ and $m(B) > 0$, there exists an integer n such that

$m(T^{-n}(A) \cap B) > 0$. If T is ergodic, Birkhoff's ergodic theorem implies that for every $A \in \mathcal{B}$, $\frac{1}{N}\#\{n \leq N;\ T^n x \in A\}$ tends to $m(A)$ as N goes to infinity for m-almost every $x \in X$.

So suppose that for some bounded operator T, we manage to construct a non degenerate measure m (i.e. $m(U) > 0$ for every non empty open set U) with respect to which T defines a measure-preserving ergodic transformation. Then T will be frequently hypercylic. It turns out that such ergodic properties are best investigated in terms of eigenvectors associated to unimodular eigenvalues (which is not surprising in view of the Godefroy-Shapiro Criterion). We restrict ourselves to the case where T is a bounded operator on a complex infinite dimensional separable Hilbert space H.

Definition 2.2. *Let σ be a probability measure on $\mathbb{T}$. The operator T is said to have a* σ-spanning set of eigenvectors associated to unimodular eigenvalues *if for every measurable subset A of $\mathbb{T}$ such that $\sigma(A) = 1$, the kernels* $\ker(T - \lambda I)$ *for $\lambda \in A$ span a dense subspace of H. If σ is continuous ($\sigma(\{\lambda\}) = 0$ for every $\lambda \in \mathbb{T}$), then T is said to have a* perfectly spanning set of eigenvectors associated to unimodular eigenvalues.

For instance: any multiple ωB, $|\omega| > 1$, of the backward shift on ℓ_2 has a perfectly spanning set of unimodular eigenvectors (take σ to be the normalized length measure on the circle).

Theorem 2.3. *Suppose that T has a perfectly spanning set of unimodular eigenvectors. Then there exists a non degenerate gaussian measure on H with respect to which T defines an ergodic measure-preserving transformation. This implies that T is frequently hypercyclic.*

We consider here centered complex gaussian measures on H (which are in particular probability measures such that $\int_H ||x||^2 dm(x)$ is finite). Such measures are completely defined by their covariance operator S: $\langle Sx, y\rangle = \int_H \langle x, z\rangle \overline{\langle y, z\rangle} dm(z)$, which is positive, self-adjoint and of trace class. The support of m is the norm closure of the range of S. Finding a non degenerate gaussian invariant measure for T boils down to finding a positive, injective, self-adjoint and of trace class operator S such than $TST^* = S$, which is in turn equivalent to finding an isometry V and a Hilbert-Schmidt operator K with dense range such that $TK = KV^*$. Such questions have already been investigated by Flytzanis in [5]. The isometry V is built out of the multiplication operator by the independent variable λ on $L^2(\mathbb{T}, \sigma)$, and K is constructed using the eigenvectors of T. When the unimodular eigenvectors of T span a dense subspace of H, T admits a non degenerate invariant gaussian measure m ([5]). When T has a perfectly spanning set of unimodular eigenvectors, T is weak-mixing with respect to m, and even strong-mixing when σ can be chosen absolutely continuous with respect to the Lebesgue measure on the circle. The proof of these ergodicity results uses Wiener's theorem and the theory of Fock spaces.

3. The Hypercyclicity Criterion Problem

This work was motivated in part by the Hypercyclicity Criterion Problem (see [6] for instance), which was shown in [4] to be equivalent to the following open question of Herrero:

Question 3.1. *If T is hypercyclic on X, is the direct sum $T \oplus T$ of two copies of T hypercylic on $X \oplus X$?*

The answer is known to be yes when T satisfies some additional "regularity condition" ([8]): when T is upper-triangular for instance, or when T has a dense set of vectors with bounded orbit. There is a kind of formal similarity between the notions of topological transitivity and ergodicity, and in the case of a measure-preserving transformation T on a probability space $(X, \mathcal{B}, \mu)$, it is known that $T \times T$ is ergodic on $(X \times X, \mathcal{B} \otimes \mathcal{B}, \mu \otimes \mu)$ if and only if T is weak-mixing. But an operator which is not frequently hypercylic cannot be ergodic with respect to an invariant measure whose support is the whole space (if it were, Birkhoff's theorem would imply that it is frequently hypercyclic). It was conceivable to think that both problems were in some sense related, but the results presented above seem to point out that the situation is much more involved.

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Maximal L^p-regularity and H^∞-calculus for elliptic differential operators

Matthias Hieber
(joint work with R. Denk, G. Dore, J. Prüss and A. Venni)

In this talk we consider vector-valued elliptic boundary value problems subject to general boundary conditions and show that under suitable assumptions its realization A_B in $L^p(G;E)$ for $1 < p < \infty$ has maximal L^p-regularity or admits even a bounded H^∞-calculus.

More precisely, consider the problem

$$\begin{aligned} \lambda u + \mathcal{A}(x,D)u &= f \text{ in } G \\ \mathcal{B}_j(x,D)u &= g_j \text{ on } \partial G,\ j = 1,\dots,m. \end{aligned}$$

Here E is a UMD-Banach space, $G \subset \mathbb{R}^{n+1}$ is an open connected set with compact C^{2m}-boundary ∂G and $f : G \to E$ and $g_j : \partial G \to E$, $j = 1,\dots,m$ are given functions. The operator $\mathcal{A}$ is a differential operator of order $2m$ of the form $\mathcal{A}(x,D) = \sum_{|\alpha|\le 2m} a_\alpha(x)D^\alpha$ with variable $\mathcal{B}(E)$-valued coefficients $a_\alpha(x)$ and $\mathcal{B}_j$ is given by $\mathcal{B}_j(x,D) = \sum_{|\beta|\le m_j} b_{j\beta}(x)D^\beta$ with variable $\mathcal{B}(E)$-valued coefficients $b_{j\beta}(x)$ for $j = 1,\dots,m$. We assume that $m, n, m_1,\dots,m_m$ are natural numbers with $m_j < 2m$ $(j = 1,\dots,m)$. Assuming the Lopatinskii-Shapiro conditions, Agmon, Douglis, Nirenberg [ADN59] and [Sol66] proved the existence of a unique solution to the above problem satisfying

$$||u||_{W_p^{2m}} \le C[||f||_p + ||u||_p + \sum_j ||g_j||_{W_p^{2m-m_j-1/p}}]$$

Seeley [See68], [See69] proved in a series of papers that the operators A_B associated to this problem admits bounded imaginary powers on L^p provided the coefficients of $\mathcal{A}$ and $\mathcal{B}_j$ are smooth. For a generalization of this result, see [Duo90]. For related results see also [PS93], [DM96], [DR96], [DS97]. In this talk we show that A_B is a $\mathcal{R}$-sectorial operator and admits a bounded H^∞-calculus even in the vector-valued setting under mild assumptions on the coefficients. In fact, let $\mathcal{A}(\cdot)$ be a $\mathcal{B}(E)$-valued polynomial on $\mathbb{R}^n$ which is homogeneous of degree $2m \in \mathbb{N}$, i.e. $\mathcal{A}(\xi) = \sum_{|\alpha|=2m} a_\alpha \xi^\alpha,\quad \xi \in \mathbb{R}^n$. We call such a homogeneous $\mathcal{B}(E)$-valued polynomial $\mathcal{A}(\cdot)$ of degree $2m \in \mathbb{N}$ *parameter-elliptic* (see [Ama01],[DHP03]) if there is an angle $\phi \in [0,\pi)$ such that the spectrum $\sigma(\mathcal{A}(\xi))$ satisfies

$$\sigma(\mathcal{A}(\xi)) \subset \Sigma_\phi \quad \text{for all } \xi \in \mathbb{R}^n,\ |\xi| = 1. \tag{1}$$

We then call $\phi_{\mathcal{A}} := \inf\{\phi :\ (1) \text{ holds}\} = \sup_{|\xi|=1} |\arg \sigma(\mathcal{A}(\xi))|$ the *angle of ellipticity* of $\mathcal{A}$. For $D = -i(\partial_1,\dots,\partial_n)$ we call $\mathcal{A}(D) = \sum_{|\alpha|=2m} a_\alpha D^\alpha$ parameter elliptic, if its symbol $\mathcal{A}(\xi)$ is parameter-elliptic.

For fixed $p \in (1,\infty)$, we will assume the following conditions.

(RS) Smoothness Condition.

(i) $a_\alpha \in C(\overline{G}, \mathcal{B}(E))$ for each $|\alpha| = 2m$ and $\lim_{|x|\to\infty, x\in G} a_\alpha(x) = a_\alpha(\infty)$ exists in the case where G is unbounded;
(ii) $a_\alpha \in [L_\infty + L_{r_k}](G, \mathcal{B}(E))$ for each $|\alpha| = k < 2m$ with $r_k \geq p$ and $2m - k > n/r_k$;
(iii) $b_{j\beta} \in C^{2m-m_j}(\partial G, \mathcal{B}(E))$ for each j, β.

(E) Ellipticity Condition.
There exists $\phi_\mathcal{A} \in [0, \pi)$ such that the following assertions hold.
(i) The principal symbol $\mathcal{A}_\#(x, \xi) = \sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha$ is parameter-elliptic with angle of ellipticity $\leq \phi_\mathcal{A}$ for each $x \in \overline{G}$ and for $x = \infty$ in case G is unbounded.
(ii) (Lopatinskii-Shapiro Condition.) Set $\mathcal{B}_{j\#}(x, D) := \sum_{|\beta|=m_j} b_{j\beta}(x)D^\beta$, $B_\# := (B_{1\#}, \ldots, B_{m\#})$, and let $\nu(x)$ denote the outer normal of G in $x \in \partial G$. For each $x_0 \in \partial G$ and each ξ' in the tangent space of ∂G at x_0, the ODE-problem in $\mathbb{R}_+$

$$\begin{aligned} (\lambda + \mathcal{A}_\#(x_0, \xi' - \nu(x_0)D_y))v(y) &= 0 \quad y > 0, \\ \mathcal{B}_{j\#}(x_0, \xi' - \nu(x_0)D_y)v(0) &= h_j, \quad j = 1, \ldots, m \end{aligned}$$

has a unique solution $v \in C_0(\mathbb{R}_+; E)$ for each $(h_1, \ldots, h_m) \in E^m$ and each $\lambda \in \Sigma_{\pi-\phi_\mathcal{A}} \cup \{0\}$ with $|\xi'| + |\lambda| \neq 0$.

Suppose now that for $\phi_\mathcal{A} \in [0, \pi)$ the boundary value problem $(\mathcal{A}, \mathcal{B}_1, \ldots, \mathcal{B}_m)$ satisfies smoothness and ellipticity conditions (RS) and (E) above. Let A_B denote the realization of $\mathcal{A}(x, D)$ in $X = L_p(G; E)$ with domain

$$D(A_B) = \{u \in H_p^{2m}(G; E) : \mathcal{B}_j(x, D)u = 0, \quad j = 1, \ldots, m\}. \tag{2}$$

Then the following result is true (see [DHP03], Theorem 8.2 or also [DV02] for related results)

Theorem 1. *Assume* (RS) *and* (E). *Then for each $\phi > \phi_\mathcal{A}$ there exists $\omega_\phi \geq 0$ such that $\omega_\phi + A_B$ is $\mathcal{R}$-sectorial with $\phi^R_{\omega_\phi + A_B} \leq \phi$.*
In particular, if $\phi_\mathcal{A} < \frac{\pi}{2}$ then the parabolic initial-boundary value problem

$$\begin{aligned} \partial_t u + (A_B + \omega_\phi)u = f, \quad t > 0, \\ u(0) = 0, \end{aligned}$$

has the property of maximal regularity in $L_q(\mathbb{R}_+; L_p(G; E))$ for each $q \in (1, \infty)$.

The proof is based is on the characaterization of maximal L^p-regularity in terms of $\mathcal{R}$-bounds for $\lambda(\lambda + A)^{-1}$ due to Weis [Wei01]. For a different approach see the work of Kunstmann and Weis [KW04].

To state the second result of this talk we introduce another smoothness conditions on the coefficients of $\mathcal{A}$.

(H) Smoothness Conditions:

$a_\alpha \in BUC^\rho(\overline{G}, \mathcal{B}(E))$ for some $\rho \in (0,1)$ and each α with $|\alpha| = 2m$, $a_\alpha(\infty) = \lim_{|x|\to\infty} a_\alpha(x)$ exists if G is unbounded and $|a_\alpha(x) - a_\alpha(\infty)| \leq c|x|^{-\rho}$, $x \in G$ with $|x| \geq 1$.

Then our second result reads as follows (see [DDHPV04], Thm. 2.3).

Theorem 2. *Let E be a UMD-Banach space, $n, m \in \mathbb{N}$ and $1 < p < \infty$. Let G be a domain in $\mathbb{R}^{n+1}$ with compact C^{2m}-boundary ∂G. Suppose that for $\phi_{\mathcal{A}} \in [0, \pi)$ the boundary value problem $(\mathcal{A}, \mathcal{B}_1, \ldots, \mathcal{B}_m)$ satisfies smoothness and ellipticity conditions* (RS), (H) *and* (E) *above.*

Let A_B denote the realization of $\mathcal{A}(x, D)$ in $X = L_p(G; E)$ with domain

$$D(A_B) = \{u \in H_p^{2m}(G; E) : \mathcal{B}_j(x, D)u = 0, \quad j = 1, \ldots, m\}.$$

Then for each $\phi > \phi_{\mathcal{A}}$ there is $\mu_\phi \geq 0$ such that $\mu_\phi + A_B \in H^\infty(L_p(G; E))$ with $\phi^\infty_{\mu_\phi + A_B} \leq \phi$.

The proof is based on "randomizing norm techniques" developed by Kalton and Weis [KW01] and on kernel estimates for the resolvent obtained in [DHP03].

For different and very interesting approaches, see the work of Blunck and Kunstmann [BK03] and the recent work of Kalton, Kunstmann and Weis [KKW04]. For the case of VMO-coefficients on $\mathbb{R}^n$ see [DL02].

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Quadratic estimates and functional calculi of perturbed dirac operators

Alan McIntosh
(joint work with Andreas Axelsson and Stephen Keith)

By about 40 years ago the theory of differential operators in L_2 spaces was basically well understood. Such topics as accretive operators and sesquilinear forms, semigroups, fractional powers and interpolation theory had been developed, along with applications to nonlinear evolution equations. However a question posed by T. Kato [5] remained unsolved.

It can be expressed as follows. Let $\mathcal{V}, \mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces with $\mathcal{V}$ densely and continuously embedded in $\mathcal{H}_1$. If S is a closed linear operator from $\mathcal{H}_1$ to $\mathcal{H}_2$ with domain $\mathcal{D}(S) = \mathcal{V}$ and $\|Su\| + \|u\| \approx \|u\|_\mathcal{V}$, and $A \in \mathcal{L}(\mathcal{H}_2)$ satisfies $\mathrm{Re}(ASu, Su) \geq \kappa\|Su\|^2$ for some $\kappa > 0$, then the sesquilinear form $J : \mathcal{V} \times \mathcal{V} \to \mathbf{C}$ defined by $J[u, v] = (ASu, Sv)$ is ω-sectorial for some $\omega \in [0, \frac{\pi}{2})$. Its associated operator $L = S^*AS$ in $\mathcal{H}_1$ is ω-accretive, and thus has a square root $\sqrt{L}$ which is $\frac{\omega}{2}$-accretive and satisfies $\sqrt{L}\sqrt{L} = L$. The question is whether $\mathcal{D}(\sqrt{L}) = \mathcal{V}$ with equivalence of norms.

Indeed the answer to this question is negative as was shown in [7]. Nevertheless Kato's interest was in differential operators, so what became known as the Kato square root problem concerned the case when $\mathcal{H}_1 = L_2(\mathbf{R}^n)$, $\mathcal{H}_2 = L_2(\mathbf{R}^n; \mathbf{C}^n)$, $\mathcal{V} = W_2^1(\mathbf{R}^n)$, $S = \nabla$ and $A = (a_{jk}) \in L_\infty(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^n))$. In this case

$$J[u, v] = (A\nabla u, \nabla v) = \int \sum a_{jk}(x) \frac{\partial u}{\partial x_k} \frac{\partial \overline{v}}{\partial x_j}\, dx$$

and so $L = -\mathrm{div}A\nabla$. For such operators, it is true that $\mathcal{D}(\sqrt{L}) = \mathcal{V}$ with $\|\sqrt{L}u\| \approx \|\nabla u\|$. A consequence is that $J[u,v] = (\sqrt{L}u, \sqrt{L^*}v)$ for $u, v \in \mathcal{D}(\sqrt{L}) = \mathcal{D}(\sqrt{L^*}) = \mathcal{V}$.

In one dimension, this result was proved in the joint work of R. Coifman and Y. Meyer with the current author [4], along with the solution of a conjecture of A. Calderón on the boundedness of the Cauchy integral on Lipschitz curves. In full generality, it was proved by P. Auscher, S. Hofmann, M. Lacey and Ph. Tchamitchian, together with the current author [2]. In the meantime there were many developments and partial results. A good survey is given by C. Kenig in [6].

Kato also asked about analytic dependence of the square root on a parameter, as this question arose in his study of hyperbolic evolution equations. Suppose that $A^t \in L_\infty(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^n))$ are self adjoint and depend analytically on $t \in (-1, 1)$. Then $L^t = -\mathrm{div}A^t\nabla$ are positive self adjoint operators which satisfy $\|\sqrt{L^t}u\| \approx \|\nabla u\|$. This question was whether the operators $\sqrt{L^t} \in \mathcal{L}(W_2^1(\mathbf{R}^n); L_2(\mathbf{R}^n))$ depend analytically on t. The answer is positive, as follows as a corollary of the above estimate $\|\sqrt{L^z}u\| \approx \|\nabla u\|$ for the non-selfadjoint operators obtained on holomorphically extending A^t to A^z for z in a region of the complex plane.

In my recent paper with A. Axelsson and S. Keith [3], we built upon the proof of the Kato estimate to obtain the following result concerning first order elliptic systems acting on a Hilbert space $\mathcal{H} = L_2(\mathbf{R}^n; \mathbf{C}^N)$.

Let Γ be a homogeneous first order differential operator acting in the space $\mathcal{H}$ which satisfies $\Gamma^2 = 0$ as well as an ellipticity condition $\|\nabla u\| \le c(\|\Gamma u\| + \|\Gamma^* u\|)$ for all $u \in \Gamma(\mathcal{H}) \oplus \Gamma^*(\mathcal{H})$, and let $\Pi = \Gamma + \Gamma^*$. Consider perturbations of the type $\Pi_B = \Gamma + B^{-1}\Gamma^* B$ where $B \in L_\infty(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^N))$ satisfies $\mathrm{Re}(Bu, u) \ge \kappa\|u\|^2$.

Under these assumptions there is a (non–orthogonal) Hodge decomposition of $\mathcal{H}$ into closed subspaces: $\mathcal{H} = \overline{\mathcal{R}(\Gamma_B^*)} \oplus \mathcal{N}(\Pi_B) \oplus \overline{\mathcal{R}(\Gamma)}$, where $\mathcal{N}$ and $\mathcal{R}$ denote the nullspace and range of an operator. Moreover the operator Π_B has spectrum in the double sector $S_\omega = \{z \in \mathbf{C} : |\arg(\pm z)| \le \omega\}$ where $\omega = \sup|\arg(Bu, u)|$, and satisfies resolvent bounds $\|(\Pi_B - \lambda I)^{-1}\| \le C/\mathrm{dist}\,(\lambda, S_\omega)$ for $\lambda \notin S_\omega$.

These results follow from operator theory, but a proof of the quadratic estimate stated next requires the full strength of the harmonic analysis.

Our main result is that Π_B satisfies quadratic estimates

$$\int_0^\infty \|\Pi_B(I + t^2\Pi_B^2)^{-1}u\|^2\, t\, dt \;\le\; C\|u\|^2$$

for all $u \in \mathcal{H}$. This estimate implies that Π_B has a bounded functional calculus [1], i.e.

$$\|f(\Pi_B)u\|_2 \le C\|f\|_\infty \|u\|_2 \tag{1}$$

for all $u \in \overline{\mathcal{R}(\Pi_B)} = \overline{\mathcal{R}(\Gamma_B^*)} \oplus \overline{\mathcal{R}(\Gamma)}$ and all $f \in H^\infty(S_\mu^o)$, where S_μ^o is the open double sector $S_\mu^o = \{z \in \mathbf{C} : |\arg(\pm z)| < \mu\}$ for $\mu > \omega$.

It is a consequence of this result that $f(\Pi_B)$ depends holomorphically on B. This in turn implies perturbation estimates of the form

$$\|f(\Pi_{B+A})u - f(\Pi_B)u\|_2 \le C\|f\|_\infty \|A\|_\infty \|u\|_2 \tag{2}$$

provided $||A||_\infty$ is not too large.

This theorem implies many of the results in the Calderón program such as the boundedness of the Cauchy operator on Lipschitz curves and surfaces. Let us see how it generalises the Kato estimate.

On combining the Hodge decomposition with (1) in the case when $f(z) = z/\sqrt{z^2}$, we obtain the equivalence $\|\Gamma u\| + \|\Gamma^* B u\| \approx \|\Pi_B u\| \approx \|\sqrt{\Pi_B{}^2} u\|$.

The square root problem of Kato follows in the special case when we take $\mathbf{C}^N$ to be the complex exterior algebra $\wedge$ on $\mathbf{R}^n$ and Γ to be the exterior deriative d, i.e. set $N = 2^n$, $\mathbf{C}^N = \wedge = \wedge_{\mathbf{C}}\mathbf{R}^n = \oplus_{k=0}^n \wedge^k$, and $\Gamma = d = \nabla\wedge$. Suppose B splits as $B_k(x) : \wedge^k \to \wedge^k$, $0 \le k \le n$ for a.a. $x \in \mathbf{R}^n$, with $B_0 = I$ and $B_1(x) = A(x) : \mathbf{C}^n \to \mathbf{C}^n$. On making the identification $d : L^2(\mathbf{R}^n; \wedge^0) \to L^2(\mathbf{R}^n; \wedge^1)$ with $\nabla : L^2(\mathbf{R}^n; \mathbf{C}) \to L^2(\mathbf{R}^n; \mathbf{C}^n)$, and $d^* : L^2(\mathbf{R}^n; \wedge^1) \to L^2(\mathbf{R}^n; \wedge^0)$ with $-\mathrm{div} : L^2(\mathbf{R}^n; \mathbf{C}^n) \to L^2(\mathbf{R}^n; \mathbf{C})$, and restricting our attention to $u \in \wedge^0$, we obtain the Kato estimate $\|\nabla u\| \approx \|\sqrt{-\mathrm{div} A \nabla} u\|$ for all $u \in L^2(\mathbf{R}^n; \mathbf{C})$.

The new result however has implications for the whole Hodge–Dirac operator $d + d^*$. Rather than discussing these for operators acting in $\mathcal{H} = L_2(\mathbf{R}^n; \wedge)$, let us consider the implications for spectral projections of the Hodge–Dirac operator $d + d_g^*$ on a compact manifold M with a Riemannian metric g.

The operator $d+d_g^*$ is a selfadjoint operator in the Hilbert space $\mathcal{H} = L_2(M; \wedge T^*M)$, and so there is an orthogonal decomposition

$$\mathcal{H} = \mathcal{N}(d + d_g^*) \oplus \mathcal{H}_g^+ \oplus \mathcal{H}_g^-$$

where $\mathcal{H}_g^\pm$ are the positive and negative eigenspaces of $d + d_g^*$. The projections of $\mathcal{H}$ onto $\mathcal{H}_g^\pm$ are $\mathbf{E}_g^\pm = \xi^\pm(d + d_g^*)$ where the functions $\xi^\pm : S_\mu^o \cup \{0\} \longrightarrow \mathbf{C}$ defined by

$$\xi^\pm(z) = \begin{cases} 1 & \text{if } \pm \operatorname{Re} z > 0 \\ 0 & \text{if } \pm \operatorname{Re} z \le 0 \end{cases}$$

are holomorphic on S_μ^o. The subscript g denotes dependence on the metric g.

If the metric is perturbed to $g + h$, then the adjoint of d with respect to the perturbed metric has the form $d_{g+h}^* = B^{-1} d_g^* B$ for an associated positive selfadjoint multiplication operator B. The perturbation result (2) can be transferred to this context, thus giving

$$\|\mathbf{E}_{g+h}^\pm - \mathbf{E}_g^\pm\| \le C\|h\|_\infty := \operatorname{ess\,sup}_{x\in M} |h_x| \tag{3}$$

provided $\|h\|_\infty$ is not too large, where

$$|h_x| = \sup\{|h_x(v,v)| : v \in T_x M\,,\ g_x(v,v) = 1\}.$$

What (3) tells us is that these eigenspaces depend continuously on L_∞ changes in the metric. Indeed the eigenspaces depend analytically on L_∞ changes in the metric. This result is possibly surprising in that the local formula for d_{g+h}^* in terms of d_g^* depends on the first order derivatives of h.

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The Kato square-root problem on Lipschitz domains

Stephen Keith

(joint work with A. Axelsson and A. McIntosh)

In joint work with Andreas Axelsson and Alan McIntosh we solve the Kato square-root problem for elliptic systems on Lipschitz domains with mixed boundary conditions. This answers a question posed by J.-L. Lions in 1962. To do this we develop a general theory for quadratic estimates and the functional calculi of complex perturbations of Dirac-type operators on Lipschitz domains.

Let us now formulate the Kato square-root problem for elliptic operators on Lipschitz domains with mixed boundary conditions. Let $\Omega \subset \mathbf{R}^n$, $n \in \mathbf{N}$, be a bi-Lipschitz image of $\Omega' \subset \mathbf{R}^n$, where Ω' is a bounded smooth domain, or an unbounded smooth domain that coincides with upper half plane $\mathbf{R}^+ \times \mathbf{R}^{n-1}$ on the complement of a bounded set. In particular Ω may be a strongly Lipschitz domain. Let Σ_1 be an extension domain in the boundary Σ of Ω, with $\Sigma \setminus \overline{\Sigma_1}$ an extension domain, and let

$$V = \left\{ u \in H^1(\Omega; \mathbf{C}) : \gamma u \in H_0^{1/2}(\Sigma_1; \mathbf{C}) \right\} .$$

Here γ is the trace operator and $H^1(\Omega; \mathbf{C})$ and $H_0^{1/2}(\Sigma_1; \mathbf{C})$ are Sobolev spaces of complex-valued functions defined on Ω and Σ_1, respectively.

Given a matrix A with $(A)_{j,k} = a_{j,k}$ where $a_{j,k} \in L_\infty(\Omega; \mathbf{C})$ for each $j,k = 0,1,\ldots,n$, let $J_A : V \times V \longrightarrow \mathbf{C}$ be given by

$$J_A[u,v] = \int_\Omega \left(a_{00} u \overline{v} + a_{0,j} \frac{\partial u}{\partial x_j} \overline{v} + a_{j,0} u \frac{\partial \overline{v}}{\partial x_j} + a_{j,k} \frac{\partial u}{\partial x_k} \frac{\partial \overline{v}}{\partial x_j} \right) dx$$

for every $u, v \in V$. An implicit sum is taken over the recurring indices in the above formula.

Suppose that J_A satisfies the following accretivity condition: there exists $c > 0$ such that

$$\operatorname{Re} J_A[u,u] \geq c\left(\|\nabla u\|^2 + \|u\|^2\right) \tag{1}$$

for every $u \in V$. Here and below $(\cdot,\cdot)$ and $\|\cdot\|$ denote the inner product and norm on $L_2(\Omega;\mathbf{C})$. Then J_A is a densely defined, closed, sectorial sesqui-linear form. Consequently, there exists an operator L_A on $L_2(\Omega;\mathbf{C})$ with $\mathsf{D}(L_A) \subset V$ uniquely determined by the property that it is m-sectorial and satisfies $J_A[u,v] = (L_A u, v)$ for every $u \in \mathsf{D}(L_A)$ and $v \in V$. The square root $\sqrt{L_A}$ of L_A is then the unique m-accretive operator with $\left(\sqrt{L_A}\right)^2 = L_A$. For an explanation of this terminology and these results see [4, VI – Theorem 2.1, V – Theorem 3.35, VI – Remark 2.29]. The Kato square-root problem is to determine whether domain $\mathsf{D}\left(\sqrt{L_A}\right) = V$.

The Kato square-root root problem for second order elliptic operators on $\Omega = \mathbf{R}^n$ was solved by P. Auscher, S. Hofmann, M. Lacey, Ph. Tchamitchian, and the third author in [1], and for higher order elliptic operators and systems on $\mathbf{R}^n$ by Auscher, Hofmann, Lacey, Tchamitchian, and the third author in [2]. The Kato square-root problem for second order elliptic operators on strongly Lipschitz domains with Dirichlet or Neumann boundary conditions was solved by Auscher and Tchamitchian in [3] by reduction to [1]. This left open the Kato square-root problem with mixed boundary conditions.

The following theorem solves the Kato square-root problem for second order elliptic operators on Lipschitz domains with mixed boundary conditions. This result is new for both smooth and Lipschitz domains, and answers a question posed by J.-L. Lions in 1962 [5, Remark 6.1].

Theorem 1. *We have* $\mathsf{D}\left(\sqrt{L_A}\right) = V$ *with* $\|\sqrt{L_A}u\| \approx \|\nabla u\| + \|u\|$ *for every* $u \in V$*. The comparability constant implicit in the use of "$\approx$" depends only on* c*, the maximum* L_∞*-norm amongst* $a_{j,k}$*, and constants implicit in the definition of* Ω *.*

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Maximal regularity and regularity for parabolic equations

Gieri Simonett
(joint work with J. Escher and J. Prüss)

The theory of maximal regularity is a powerful tool for the treatment of nonlinear parabolic problems. In this note an outline is given on how maximal regularity, in conjunction with the implicit function theorem, can be used to establish regularity properties for a wide array of parabolic evolution equations.

In order to explain the main idea of our approach, let us consider the model problem of a family of graphs $\{\Gamma(t) = \mathrm{graph}(u(\cdot,t))\,;\, 0 \le t \le T\}$ over $\mathbb{R}^n$, evolving according to the mean curvature flow

$$\partial_t u - \left(\delta_{ij} - \frac{\partial_i u \partial_j u}{1+|\nabla u|^2}\right)\partial_i\partial_j u = 0, \qquad u(0) = u_0, \tag{1}$$

where $1 \le i, j \le n$, and where δ_{ij} denotes the Kronecker delta. Equation (1) is a quasilinear parabolic evolution equation of second order. To economize notation we set

$$F(u) := -\left(\delta_{ij} - \frac{\partial_i u \partial_j u}{1+|\nabla u|^2}\right)\partial_i\partial_j u$$

and restate equation (1) as

$$\partial_t u + F(u) = 0, \quad u(0) = u_0. \tag{2}$$

Let $E_j := buc^{2j+s}(\mathbb{R}^n)$, $j = 0, 1$, be the little Hölder spaces. The mapping F is real analytic, that is,

$$F \in C^\omega(E_1, E_0). \tag{3}$$

Given that F is differentiable, one can consider the linearized problem

$$\partial_t v + F'(u)v = f, \quad v(0) = v_0, \tag{4}$$

where $F'(u)$ is the Fréchet derivative of F at $u \in E_1$. Next we introduce the anisotropic spaces

$$\mathbb{E}_0(I) := C(I, E_0), \quad \mathbb{E}_1(I) := C^1(I, E_0) \cap C(I, E_1),$$

where $I = [0, T]$ is a fixed interval. Clearly, the trace operator $\gamma_0 : \mathbb{E}_1(I) \to E_1$, $v \mapsto v(0)$ is linear and continuous. It can be shown, and this is the essential part of the analysis, that the linear problem (4) enjoys the property of *maximal regularity*. By definition, this means that

$$(\partial_t + F'(u), \gamma_0) \in \mathrm{Isom}(\mathbb{E}_1(I), \mathbb{E}_0(I) \times E_1) \tag{5}$$

for any function $u \in E_1$. That is, the linear mapping $(\partial_t + F'(u), \gamma_0)$ is a topological isomorphism between the indicated spaces. It is here where maximal regularity begins to unfold. It implies that the linear problem (4) has a unique solution $v \in \mathbb{E}_1(I)$ for any given right hand side $(f, v_0) \in \mathbb{E}_0(I) \times E_1$. The solution v has optimal regularity, and therefore, no loss of regularity can occur for the linearized problem. Existence of a unique solution in $\mathbb{E}_1(I)$ to the nonlinear problem (2) can now be obtained by a reiteration argument and the contraction principle.

As an immediate outcome, one sees that there is also no 'loss of derivatives' for the nonlinear problem. (This is also true if F is fully nonlinear). It should be noted that iteration techniques based on the Nash-Moser implicit function theorem usually result in a loss of derivatives.

We give a brief account on how the property of maximal regularity in conjunction with a scaling argument (or a parameter trick) will show that the solution $u \in \mathbb{E}_1(I)$ of (2) is real analytic in space and time for any positive time.

Let u be the unique solution of (2) defined on a maximal interval of existence $[0, t^+(u_0))$. Let $T \in (0, t^+(u_0))$ be a fixed number and set $I := [0, T]$. For any given parameters $(\lambda, \mu) \in \mathbb{R} \times \mathbb{R}^n$ with $\lambda \in (-\varepsilon_0, \varepsilon_0)$ one can set

$$u_{\lambda,\mu}(t,x) := u(t + t\lambda, x + t\mu), \quad (t,x) \in I \times \mathbb{R}^n. \tag{6}$$

It is easy to see that $u_{\lambda,\mu} \in \mathbb{E}_1(I)$ for all (λ, μ), provided ε_0 is sufficiently small. Since the mapping F commutes with translations, that is,

$$\tau_a F(u) = F(\tau_a u), \quad u \in E_1, \quad a \in \mathbb{R}^n, \tag{7}$$

one finds that $v := u_{\lambda,\mu} \in \mathbb{E}_1(I)$ satisfies the parameter dependent equation

$$\partial_t v + (1+\lambda)F(v) - (\mu|\nabla v) = 0, \quad v(0) = u_0,$$

or equivalently, that $v := u_{\lambda,\mu}$ solves

$$\Phi(v, (\lambda, \mu)) = 0 \tag{8}$$

where $\Phi(v, (\lambda, \mu)) := (\partial_t v + (1+\lambda)F(v) - (\mu|\nabla v), \gamma_0 v - u_0)$. It follows from (3) that the mapping

$$\Phi : \mathbb{E}_1(I) \times ((-\varepsilon_0, \varepsilon_0) \times \mathbb{R}^n) \to \mathbb{E}_0(I) \times E_1$$

is real analytic. Moreover, $\Phi(\bar{u}, (0,0)) = (0,0)$, where $\bar{u} := u|_I$. It is a consequence of the maximal regularity property (5) that the Fréchet derivative $D_1\Phi(\bar{u}, (0,0))$ of Φ with respect to v satisfies

$$D_1\Phi(\bar{u}, (0,0)) = (\partial_t + F'(\bar{u}), \gamma_0) \in \operatorname{Isom}(\mathbb{E}_1(I), \mathbb{E}_0(I) \times E_1). \tag{9}$$

The implicit function theorem now allows to solve equation (8) for v in terms of (λ, μ) in an open neighborhood U of $(0,0) \in \mathbb{R} \times \mathbb{R}^n$. One concludes that

$$[(\lambda, \mu) \mapsto u_{\lambda,\mu}] \in C^\omega(U, \mathbb{E}_1(I)). \tag{10}$$

Consequently, the mapping

$$[(\lambda, \mu) \mapsto u_{\lambda,\mu}(t_0, x_0) = u(t_0 + t_0\lambda, x_0 + t_0\mu)] : U \to \mathbb{R} \tag{11}$$

is real analytic for any fixed $(t_0, x_0) \in I \times \mathbb{R}^n$ with $t_0 > 0$. Hence, the solution u of the mean curvature flow (1) is analytic in space and time for any positive time $t \in (0, t^+(u_0))$.

It is now clear that the only properties needed to carry through the arguments are (3), (7), and the crucial maximal regularity property (5). The nature of the mapping F is completely immaterial: it can be fully nonlinear, can act as a non-local mapping, and it can be of any order.

The idea of using parameters to prove regularity properties of solutions goes back to Angenent [1, 2]. The strategy of using translations to show analyticity in space was first employed in [5] for a free boundary problem for the flow of an incompressible fluid in a porous medium of infinite extent. In that context the mapping F happens to be fully nonlinear, nonlocal, and of first order. Translations were also used in [4] for the Stefan problem with surface tension in the case where the free interface is represented as the graph of a function over $\mathbb{R}^n$.

The advantage of applying maximal regularity relies on the fact that one can resort to the implicit function theorem. The difficulty, of course, lies in establishing maximal regularity for a given partial differential equation.

Our approach described so far relies on the fact that we can use translations on $\mathbb{R}^n$, and that the mapping F is equivariant with respect to translations. The approach can be generalized in two directions. First, it can be generalized to parabolic equations on a symmetric Riemannian manifold M, where one assumes that the nonlinear mapping F is equivariant with respect to the Lie group which acts as a transformation group on M. This has been done in [6]. Here we just give a sketch on how the translation-parameter trick can be localized. In order to do so, we pick $x_0 \in \mathbb{R}^n$ and choose a smooth cut-off function $\chi \in \mathcal{D}(\mathbb{R}^n)$ with

$$\text{supp}\,(\chi) \subset \mathbb{B}(x_0, \varepsilon_0), \tag{12}$$

where ε_0 can be chosen as small as we wish for. Instead of (6) we can now consider the parameter-dependent function

$$u_{\lambda,\mu}(t,x) := u(t + t\lambda, x + t\chi(x)\mu), \qquad (t,x) \in I \times \mathbb{R}^n. \tag{13}$$

The function $v := u_{\lambda,\mu}$ also satisfies a parameter-dependent equation

$$\partial_t v + F_{\lambda,\mu}(v) = 0, \qquad v(0) = u_0. \tag{14}$$

The new difficulty now lies in showing that the mapping $[(v, (\lambda, \mu)) \mapsto F_{\lambda,\mu}(v)]$ is analytic.

Complete proofs and examples are given in [7]. We also refer to [3] for further generalizations and refinements. The localized parameter-trick will be used to establish regularity results for free boundary problems, such as the Stefan problem with surface tension, and the Navier-Stokes equations with surface tension.

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On Taylor coefficients of vector-valued Bloch functions

Oscar Blasco

1. Introduction

The following notes are based upon the results on the paper with the same title (see [2]).

Let X, Y be complex Banach spaces. Let $Bloch(X)$ denote the space of X-valued analytic functions on the unit disc verifying that $||f||_{Bloch(X)} = ||f(0)|| + sup_{|z|<1}(1-|z|^2)||f'(z)|| < \infty$. We write $Bloch$ instead of $Bloch(\mathbb{C})$.

Definition 1.1. *A sequence $(T_n)_n$ in $\mathcal{L}(X,Y)$ is said to be a multiplier between $Bloch(X)$ and $\ell_1(Y)$, to be denoted $(T_n) \in \big(Bloch(X), \ell_1(Y)\big)$, if $(T_n(x_n))_n$ belongs to $\ell_1(Y)$ whenever $f(z) = \sum_{n=0}^{\infty} x_n z^n$ belongs to $Bloch(X)$.*

We endow the space with the norm in $\mathcal{L}(Bloch(X), \ell_1(Y)$.

Let us now recall the scalar-valued result on multipliers due to J.M. Anderson and A.L.Shields (see[1]) that we want to extend to the vector-valued setting:

$$(Bloch, \ell_1) = \ell(2,1) \tag{1}$$

where, for $1 \leq p, q \leq \infty$ we denote by $\ell(p,q)$ the spaces of sequences $(\alpha_n)_n$ in X such that

$$\Big(\sum_{k=0}^{\infty}\big(\sum_{n=2^k}^{2^{k+1}} |\alpha_n|^p\big)^{q/p}\Big)^{1/q} < \infty,$$

with the obvious modifications for $p = \infty$ or $q = \infty$.

For a complex Banach space X we write $\ell(p,q,X)$ for space of sequences in X for which $\|x_n\| \in \ell(p,q)$

The information about Taylor coefficients that can be achieved from the previous result is the following:

There exist a constants $C_1, C_2 > 0$ such that

$$C_1\|(\alpha_n)_n\|_{2,\infty} \leq \|\phi\|_{Bloch} \leq C_2\|(\alpha_n)_n\|_{1,\infty} \tag{2}$$

for any $\phi(z) = \sum_{n=0}^{\infty} \alpha_n z^n$.

The aim of this note is to understand whether (2) and (1) have natural extensions to vector-valued functions and how the vector-valued analogues of them depend on some geometrical properties on the Banach space X.

Out of the study on the problem and some partial result one can get the following general fact:

Theorem 1.2. *Let H be a Hilbert space and let Y be a Banach space. Then*

$$\big(Bloch(H), \ell_1(Y)\big) = \ell(2,1,\mathcal{L}(X,Y)).$$

2. The problems

Problem 1: For which Banach spaces X does it hold that

$$f(z) = \sum_{n=0}^{\infty} x_n z^n \in Bloch(X) \Rightarrow (x_n)_n \in \ell(2,\infty,X)? \tag{3}$$

To this aim let us give the following definitionnition.

Definition 2.1. *Let X be a complex Banach space. We definitionne*

$$\Lambda_{Bloch,\ell_1}(X) : \{(\lambda_n)_n \subset \mathbb{C} : T_n = \lambda_n.I \in (Bloch(X), \ell_1(X))\}.$$

It is easy to see that *Problem 1* can be rephrased as follows: For which Banach spaces X does it hold that $\Lambda_{Bloch,\ell_1}(X) = \ell(2,1)$?

Theorem 2.2. *Let H be a Hilbert space. Then there exists a constant $C > 0$ such that*

$$||(x_n)_n||_{2,\infty} \leq C||f||_{Bloch(H)}$$

for all $f(z) = \sum_{n=0}^{\infty} x_n z^n \in Bloch(H)$. Hence $\Lambda_{Bloch,\ell_1}(H) = \ell(2,1)$.

Let us give some sketch of the proof:

Step 1: Given $f \in Bloch(H)$ we associate a bounded operator $T_f : A_1 \to H$ by the formula $T_f(u_n) = x_n$, where $u_n(z) = (n+1)z^n$ where A_1 stands for the Bergman space of analytic functions ϕ such that $\int_{\mathbb{D}} |\phi(z)| dA(z) < \infty$.

Step 2: Use that $(A_1)^* = Bloch$ and show $\sum_n |\langle \lambda_n u_n, g\rangle| \leq C$ for any $||(\lambda_n)||_{2,1} \leq 1$ and $||g||_{Bloch} \leq 1$.

Step 3: Use that A_1 is isomorphic to ℓ_1 and, by invoking Grothendieck theorem, we obtain that T_f is absolutely summing and hence the result now follows from Step 2.

OPEN QUESTION: Does $\Lambda_{Bloch,\ell_1}(X) = \ell(2,1)$ imply X is isomorphic to a Hilbert space?.

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Boundedness of bilinear pseudodifferential operators

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(joint work with Árpád Bényi and Rodolfo Torres)

We study bilinear pseudodifferential operators beyond the so called Coifman-Meyer class, and aim at including those symbols that naturally from the non-smooth multipliers generalizing the bilinear Hilbert transform when they become x-dependent. Our goal is to understand how the bilinear pseudodifferential setup differs from the linear pseudodifferential one in terms of symbolic calculus and boundedness properties on products of Lebesgue and/or Sobolev spaces.

We considered two families of classes of bilinear symbols $BS^m_{\rho,\delta}$ and $\widetilde{BS^m_{\rho,\delta;\theta}}$ defined as:

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x,\xi,\eta)| \leq C_{\alpha\beta\gamma}(1+|\xi|+|\eta|)^{m+\delta|\alpha|-\rho(|\beta|+|\gamma|)}, \tag{1}$$

respectively

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x,\xi,\eta)| \leq C_{\alpha\beta\gamma}(1+|\eta-\tan\theta\ \xi|)^{m+\delta|\alpha|-\rho(|\beta|+|\gamma|)} \tag{2}$$

for all $(x,\xi,\eta)\in\mathbf{R}^{3n}$, all multi-indices α, β and γ, and some positive constants $C_{\alpha\beta\gamma}$. In one dimension, the latter condition expresses the decay of the derivatives of the symbol in terms of the distance from the frequency pair (ξ,η) to the line Γ_θ at angle θ with respect to the axis $\eta=0$ and up to a constant depending on θ is equivalent to the condition,

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x,\xi,\eta)| \leq C_{\alpha\beta\gamma}(1+dist((\xi,\eta);\Gamma_\theta))^{m+\delta|\alpha|-\rho(|\beta|+|\gamma|)} \tag{3}$$

When $\theta=-\pi/4$, condition (2) becomes $|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x,\xi,\eta)| \leq C_{\alpha\beta\gamma}(1+|\xi+\eta|)^{m+\delta|\alpha|-\rho(|\beta|+|\gamma|)}$ and we simply write $\widetilde{BS^m_{\rho,\delta}}$ to denote $\widetilde{BS^m_{\rho,\delta;-\pi/4}}$. In general, the classes $BS^m_{\rho,\delta}$ and $\widetilde{BS^m_{\rho,\delta}}$ are not comparable. We are interested in bilinear pseudodifferential operators, defined apriori from $\mathcal{S}(\mathbf{R}^n)\times\mathcal{S}(\mathbf{R}^n)$ into $\mathcal{S}'(\mathbf{R}^n)$, of the form

$$T_\sigma(f,g)(x)=\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\sigma(x,\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{ix\cdot(\xi+\eta)}d\xi d\eta. \tag{4}$$

The study of bilinear operators with symbols in the class $BS^0_{1,0}$ started in the works of Coifman and Meyer [9], [10], [11]. They used techniques related to Littlewood-Paley theory to prove that this class produces bilinear operators bounded on products of L^p spaces. The class $BS^0_{1,1}$ is the largest one that gives rise to bilinear pseudodifferential operators which can be realized as singular integral operators with 'bilinear kernels' of Calderón-Zygmund type. But the latter is a "forbidden" class, in that it yields unbounded operators on products of Lebesgue spaces and is not to close under transposition. Nevertheless, in [3] a substitute estimates is obtained in the spirit of the Leibniz rule for products of functions and proved that operators with forbidden symbols are bounded on products of Sobolev spaces with positive smoothness. Here we obtain new results concerning classes of order m, $BS^m_{1,0}$ and $BS^m_{1,1}$.

The investigation of classes $\widetilde{BS^0_{1,0;\theta}}$ was initiated by the pivotal work of Lacey and Thiele [20], [21] in the special case of the bilinear Hilbert transform. It was continued by Gilbert and Nahmod [14], [15] in the general bilinear multiplier setup. They proved that, in the one dimensional case, for x-independent symbols σ in $\widetilde{BS^0_{1,0;\theta}}$, $\theta\neq 0,\pi/2,-\pi/4$, T_σ is bounded on products of appropriate Lebesgue spaces. Muscalu, Thiele, and Tao [23] treated the multilinear setup. When $\theta = 0,\pi/2$, or $-\pi/4$, Γ_θ becomes a 'degenerate subspace' and these results do not apply anymore. However, the operators may still be bounded by other reasons. This is the case -for example- when the symbols in question also belong to the Coifman-Meyer class or when the bilinear operators can be represented as linear combinations of tensor products [7] of Calderón-Zygmund operators. In those cases, if an additional cancellation property along the antidiagonal is satisfied, it was in [8] that T_σ can be extended as a bounded operator from $L^p \times L^{p'}$ into the Hardy space H^1; see also [7], [12], [13] and [15]. Furthermore, note that if the symbol depends only on the sum of the frequency variables ξ and η, $\sigma(x,\xi,\eta) = \sigma_0(x,\xi+\eta)$, then σ_0 is a (classical) linear symbol in the class $S^0_{1,0}$ and $T_\sigma(f,g) = T_{\sigma_0}(fg)$, where T_{σ_0} is the linear pseudodifferential operator with symbol σ_0. It is well known that T_{σ_0} is bounded on Lebesgue spaces [25], therefore, using Hölder's inequality, one immediately gets boundedness of T_σ on product of Lebesgue spaces.

The general x-dependent case presents new fascinating challenges. The main question being, what are the precise additional conditions a symbol in the $\widetilde{BS}^m_{\rho,\delta;\theta}$ class must satisfy to ensure boundedness in the product of Lebesgue and - or - Sobolev spaces.

In what follows we continue our investigation into these issues and present some ideas to further our understanding of the rather drastic change in the passage from the class $BS^0_{1,0}$ to the class $\widetilde{BS}^0_{1,0}$. We do so by developing a symbolic calculus for the composition of classical linear pseudodifferential operators with bilinear pseudodifferential operators having symbols in the class $\widetilde{BS}^m_{1,0}$ and by making a connection to the bilinear Calderón-Vaillancourt class $BS^0_{0,0}$ [5] [2].

The Bilinear Classes $BS^m_{1,0}$ and $BS^m_{1,1}$. A bilinear operator $T : \mathcal{S} \times \mathcal{S} \to \mathcal{S}'$ has two formal transposes T^{*1} and T^{*2} defined via $\langle T(f,g),h\rangle = \langle T^{*1}(h,g),f\rangle = \langle T^{*2}(f,h),g\rangle$, for all f,g,h in $\mathcal{S}$. For example, for operators T_σ which are translation invariant, $\sigma(x,\xi,\eta) = \sigma(\xi,\eta)$, and the symbols of the transposes are given by $\sigma^{*1}(\xi,\eta) = \sigma(-\xi-\eta,\eta)$, $\sigma^{*2}(\xi,\eta) = \sigma(\xi,-\xi-\eta)$. For symbols that depend on x the situation is much more complicated. Nevertheless, in the case of symbols that belong to $BS^m_{1,0}$, one can compute the symbols of the transposes via an asymptotic formula. In [3] it is shown that $BS^m_{1,0}, m \geq 0$, produce a class of operators which is closed by transposition; but that the class $BS^m_{1,1}$ - which contains $BS^m_{1,0}$ - produces operators which are not necessarily bounded on product of Lebesgue spaces. Thus it follows from a theorem in [16] that $BS^m_{1,1}$ is *not* closed under transpositions. Here we prove a positive result about this 'forbidden' class by considering products of Sobolev spaces with positive smoothness.

Every operator T_σ with a symbol in the class $BS^m_{1,1}, m \geq 0$, has a bounded extension from $L^p_{m+s} \times L^q_{m+s}$ into L^r_s, provided that $1/p + 1/q = 1/r$, $1 < p,q,r < \infty$, and $s > 0$. Moreover,

$$\|T_\sigma(f,g)\|_{L^r_s} \leq C(p,q,r,s,n,m,\sigma)(\|f\|_{L^p_{m+s}}\|g\|_{L^q} + \|f\|_{L^p}\|g\|_{L^q_{m+s}}). \tag{5}$$

Actually a more general boundedness result from $L^p_{m+s} \times L^q_{m+t}$ into $L^r_{\min(s,t)}$ is true provided $1/p + 1/q = 1/r, 1 < p,q,r < \infty$, and $s,t > 0$. The proof is adapted after the proof given in [3] for the case $m = 0$. It makes use of a decomposition into elementary symbols due to Coifman and Meyer and a square function type estimate for the Sobolev space norm of a sum of functions with spectra supported in appropriate balls that can be traced back to Meyer's work [22]; see also Lemma 3 in [3]. The result above implies in particular, that the smaller class $BS^m_{1,0}$ also yields bounded operators from $L^p_{m+s} \times L^q_{m+s}$ into L^r_s for $s > 0$. Moreover, since $BS^m_{1,0}$ is closed under transpositions; duality and interpolation give also boundedness of operators in this class from $L^p_{m+s} \times L^q_m$ into L^r or from $L^p_m \times L^q_{m+s}$ into L^r, for $s > 0$. Since symbols in $BS^0_{1,0}$ yield bounded operators from $L^p \times L^q$ into L^r, interpolation with the above would give a weaker boundedness, say, from $L^p \times L^q_s$ into L^r for this class. But by using interpolation in the argument one loses the nice Leibniz rule estimates. We can fix these gaps by proving that for the smaller class $BS^m_{1,0}$ the result above also holds for $s = 0$ and moreover that the Leibniz rule property is preserved. Denote by $J^k = (I - \Delta)^{k/2}$ the Fourier multiplier operator with symbol $(1 + |\xi|^2)^{k/2} \in S^k_{1,0}$.

Given σ a symbol in $BS^m_{1,0}, m \geq 0$; there exists symbols σ_1 and σ_2 in $BS^0_{1,0}$ such that, for all $f, g \in \mathcal{S}$, we can decompose $T_\sigma(f, g) = T_{\sigma_1}(J^m f, g) + T_{\sigma_2}(f, J^m g)$.. Consequently, T_σ has a bounded extension from $L^p_m \times L^q_m$ into L^r, provided $1/p + 1/q = 1/r$, $1 < p, q, r < \infty$ and,

$$\|T_\sigma(f,g)\|_{L^r} \leq C(p,q,r,n,m,\sigma)\ (\|f\|_{L^p_m}\|g\|_{L^q} + \|f\|_{L^p}\|g\|_{L^q_m}). \tag{6}$$

One could replace the hand side of (6) with $C\ (\|f\|_{L^{p_1}_m}\|g\|_{L^{q_1}} + \|f\|_{L^{p_2}}\|g\|_{L^{q_2}_m})$, where $1/p_1 + 1/q_1 = 1/p_2 + 1/q_2 = 1/r, 1 < p_1, p_2, q_1, q_2, r < \infty$.

A natural question to ask is whether the same boundedness properties hold for the class $\widetilde{BS^m_{1,0}}$. If one assumes that *m is sufficiently large* then one can show the result above holds for this class as well. A heuristic explanation of this fact is that for the choice of m above one needs not worry about dealing with negative powers of the quantity $1 + |\xi + \eta|$ and a similar reduction to the class $BS^0_{1,0}$ holds again. One should compare this with the Kato-Ponce commutator estimates [18] that hold for all $m \geq 0$. Due to the special structure of the operators considered there (derivatives of product of two functions), one can interpolate the estimates that hold for both m large and $m = 0$ to cover the whole range of m's. Due to the generality of the problem we consider here, we miss the "endpoint" $m = 0$ as we explain below.

Symbolic calculus for the composition $S^k_{1,0} \circ \widetilde{BS^m_{1,0}}$. It is well known that the classes $S^m_{1,0}$ admit a symbolic calculus both for composition and transposition. This calculus shows, for example, that there is an asymptotic formula for the composition of two operators with symbols in such classes which has the product of the symbols as its main term. A similar asymptotic expansion exists for the transposes of such operators; see e.g. [25]. We have already discussed the existence of a symbolic calculus for transposes of the bilinear class $BS^m_{1,0}$. Concerning the composition of bilinear operators, several definitions are easy to imagine yet it is not clear which is the most natural or useful one. An interesting situation arises when one tries to compose a linear pseudodifferential operator with a bilinear pseudodifferential operator. Indeed, one important instance of such composition is provided by $J^k\ T_\sigma$. Motivated by the linear counterpart problem which has a positive answer, and since both classes $BS^m_{1,0}$ and $\widetilde{BS^m_{1,0}}$ could be viewed as bilinear extension of $S^m_{1,0}$, we asked the following natural questions:

1. *For $a \in S^k_{1,0}$ and $\sigma \in BS^m_{1,0}$, is it true that $L_a T_\sigma = T_\lambda$ with $\lambda \in BS^{m+k}_{1,0}$?*
2. *For $a \in S^k_{1,0}$ and $\sigma \in \widetilde{BS^m_{1,0}}$, is it true that $L_a T_\sigma = T_\lambda$ with $\lambda \in \widetilde{BS^{m+k}_{1,0}}$?*

The answer to the first question is negative. If we consider both a and σ to be x-independent, then it is easy to see that the composition symbol is $\lambda(\xi,\eta) = a(\xi+\eta)\sigma(\xi,\eta)$. Even in the simplest case where $a \in S^0_{1,0}$ and $\sigma \in BS^0_{1,0}$, the most we can say about λ is that it is in $\widetilde{BS^0_{1,0}}$. In fact, the failure of enough decay as to make λ belong to the smaller class $BS^0_{1,0}$ has nothing to do with the x-independence of a. Similarly, our argument fails if we replace the class $\widetilde{BS^m_{1,0}}$ with $BS^m_{1,0}$ or if we consider a general symbol $\sigma \in \widetilde{BS^m_{1,0;\theta}}$ with $\theta \neq -\pi/4$.
We show that the answer to the second question is affirmative. The ideas employed are inspired by some of the proofs given in the linear case by Hörmander [17], Kohn and Nirenberg [19], and Stein [25], but additional technical difficulties need to be overcome in the bilinear setting. We also obtain an asymptotic expansion for λ.

The interest in having a symbolic calculus for composition lies in its potential application to study boundedness on Sobolev spaces. The usual way to go about this is to reduce the study of operators of order m on Sobolev spaces to the study of operators of order 0 on Lebesgue spaces; see, e.g., [25] for a detailed treatment of the linear case. Unfortunately, being in the class $\widetilde{BS^0_{1,0}}$ *alone* is not enough to yield bounded operators on products of Lebesgue spaces [2]. There is, however, a connection of the symbolic calculus with the Calderón-Vaillancourt class $BS^0_{0,0}$ which gives boundedness of operators of order m [4] [2].

Further Sobolev extensions We conclude with a few interesting observations about the boundedness of generic bilinear multipliers on products of Sobolev spaces.

If the symbolσ is x-independent and T_σ is bounded from $L^p \times L^q$ into L^r, with $1/p+1/q = 1/r, r > 1$, then T_σ is also bounded from $L^p_s \times L^q_s$ into L^r_s for all $s \geq 0$.

Better local estimates can be achieved as long as we restrict the range of the index s. Let $a \wedge b$ denote the largest integer strictly smaller than $\min(a,\, b)$.

If the symbolσ is x-independent and T_σ is bounded from $L^p \times L^q$ into L^r, with $1/p + 1/q = 1/r, r > 1$, then T_σ is also bounded from $L^p_s \times L^q_s$ into $L^{r_s}_s$, where $1/r_s = 1/p + 1/q - s/n$ and $0 \leq s \leq n/p \wedge n/q$.

In particular, we recover a useful multiplication result that can be found, for example, in the book of Runst and Sickel [24]. An application of this result to the inverse conductivity problem is given in the work of Brown and Torres [6]. *Let $u \in L^p_s$ and let $v \in L^q_s$, with $1 < p,q < \infty, 1/p+1/q \leq 1$, and $0 \leq s < n\min(1/p,1/q)$. Then $uv \in L^{r^*}_s$, where $1/r^* = 1/p+1/q - s/n$.*

Remarks (i) The bilinear Hilbert transform and all its generalizations as bilinear (not necessarily smooth) multipliers in [14], [15] are bounded on products of Sobolev spaces like the ones considered in our propositions.
(ii) Similar arguments to the ones used in the proofs above show that the statements above remain true for x-dependent symbols in the class $BS^0_{1,0}$.

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Spectral properties of noncommuting operators

Brian Jefferies

The talk outlines the application of Clifford analysis techniques to functional calculi for finite systems of linear operators acting in a Banach space. A recent book [1] describes progress to date. Emphasis has been on obtaining the "richest" functional calculus for a system of operators in a given class, leading to a notion of *joint spectrum.* In particular, for n selfadjoint operators, $\boldsymbol{A} = (A_1, \dots, A_n)$ the joint spectrum $\gamma(\boldsymbol{A})$ obtained is the support of the Weyl functional calculus for $\boldsymbol{A}$. If the elements of $\boldsymbol{A}$ commute with each other, then $\gamma(\boldsymbol{A})$ is the support of the joint spectral measure. If $A_1, \dots, A_n$ are bounded linear operators with real spectra in a Banach space, then $\gamma(\boldsymbol{A})$ is just the Taylor spectrum [9].

The fundamental formula is the higher dimensional analogue

$$f(\boldsymbol{A}) = \int_{\partial\Omega} G_\omega(\boldsymbol{A})\boldsymbol{n}(\omega)f(\omega)\, d\mu(\omega). \tag{1}$$

of the Riesz-Dunford functional calculus for a single operator. Following a suggestion of A. McIntosh (C. 1988), the Cauchy kernel $\omega \longmapsto G_\omega(\boldsymbol{A})$ is defined via a plane wave decomposition formula in Clifford analysis. If $n = 2$ and $\sigma(\langle \boldsymbol{A}, s\rangle) \subset \mathbb{R}$ for $|s| = 1$, the relevant representation of the Cauchy kernel takes the form

$$G_\omega(\boldsymbol{A}) = -\frac{sgn(y_0)}{8\pi^2}\int_{S^1} (\langle yI - \boldsymbol{A}, s\rangle - y_0 sI)^{-2}\, ds$$

for $\omega = y_0 e_0 + y$, $y_0 \in \mathbb{R} \setminus \{0\}$, $y \in \mathbb{R}^2$ [1, 3].

The set $\gamma(\boldsymbol{A})$ of singularities of $\omega \longmapsto G_\omega(\boldsymbol{A})$ is a subset of $\mathbb{R}^{n+1}$ and serves as the *joint spectrum* of $\boldsymbol{A}$. The application of Clifford analysis methods is limited to operators whose spectra do not collectively stray too far from the real axis.

Applications include the study of the support of the fundamental solution of the symmetric hyperbolic system

$$\frac{\partial u}{\partial t} + \sum_{k=1}^{n} A_k \frac{\partial u}{\partial x_k} = 0$$

of partial differential equations and H^∞-functional calculi for commuting systems of operators acting in a Hilbert space, satisfying "square function extimates", for example, the differentiation operators on a Lipschitz surface. Formula (1) makes sense for functions that are left monogenic in the sense of Clifford analysis and have suitable decay at zero and infinity (following the argument of A. McIntosh [8] in $\mathbb{C}$). However, there is a correspondence between bounded monogenic functions $\tilde{f}$ and bounded holomorphic functions f in sectors via the formula

$$f(z) = \int_{\partial\Omega} G_\omega(z)\boldsymbol{n}(\omega)\tilde{f}(\omega)\,d\mu(\omega), \quad z \in \mathbb{C}^n. \tag{2}$$

For $\omega \in \mathbb{R}^{n+1}$, the Cauchy kernel $G_\omega(z)$ is a holomorphic extension in $z \in \mathbb{C}^n$ of the Cauchy kernel in Clifford analysis. In this manner, we obtain an H^∞-functional calculus $f \longmapsto f(A_1, \dots, A_n)$ for commuting systems of operators acting in a Hilbert space satisfying square function estimates in a similar way to the case $n = 1$ [1, 2].

Another point of contact is with functional calculi indexed by families of continuous probability measures $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)$ on $[0, 1]$, so called *Feynman operational calculi* [1, 4, 5, 6, 7]. The *equally weighted* functional calculus with $\mu_1 = \mu_2 = \cdots = \mu_n$ corresponds to the Weyl functional calculus. It is a natural idea to replace one of the time-ordering measures in Feynman's operational calculus for functions f of two variables by the one dimensional Wiener process $\langle W_t \rangle_{t \geq 0}$ to represent the solution X_t of the linear stochastic differential equation

$$dX_t + AX_t dt = B_t X_t dW_t$$

as $X_t = f_{dt,dW_t}(-A, B)X_0$ in terms of the stochastic functional calculus associated with the operator A, the operator valued function $\langle B_s \rangle_{s \geq 0}$, the time ordering measure dt and the time-ordering process $\langle W_s \rangle_{s \geq 0}$ over the interval $[0, t]$. Here f is the entire function $(z_1, z_2) \mapsto e^{z_1 + z_2}$ of two complex variables and $-A$ is the generator of an analytic semigroup acting on a Banach space. The possibility of obtaining such a representation for general processes B seems to be closely related to the maximal regularity of A and the existence of an H^∞-functional calculus.

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L^p–regularity of parabolic differential equations on $\mathbb{R}^d$ with unbounded coefficients

Roland Schnaubelt
(joint work with Giorgio Metafune, Diego Pallara, Jan Prüss, Abdelaziz Rhandi)

We study elliptic partial differential operators of the form

$$Au = \operatorname{div}(a\nabla u) + F \cdot \nabla u - Vu$$

on $\mathbb{R}^d$ with unbounded coefficients. Such operators occur as Schrödinger operators or as generators of transition semigroups arising in stochastic analyis. It is known that A endowed with a suitable domain $D(A)$ generates a strongly continuous semigroup $T(t)$, $t \geq 0$, on $L^p(\mathbb{R}^d)$, $1 \leq p < \infty$, if the dissipativity condition $pV + \operatorname{div} F \geq 0$ holds. Then $u(t,x) = T(t)f(x)$ solves the parabolic problem $\partial_t u(t,x) = Au(t,x)$, $t \geq 0$; $u(0,x) = f(x)$, $x \in L^p(\mathbb{R}^d)$, provided that $f \in D(A)$. But one needs additional conditions to obtain analyticity of $t \mapsto T(t)$ (for $t > 0$). Analyticity fails already for the Ornstein–Uhlenbeck operator $Au(x) = \operatorname{tr} aD^2u(x) + (bx, \nabla u(x))$, where $a = a^T > 0$ and $b \neq 0$ are real $d \times d$ matrices.

Our main interest, however, is directed to a precise description of the domain and to consequences of this description to qualitative properties of the semigroup. We concentrate on cases where $D(A)$ is given as the intersection of the domains of the summands of A. For brevity we do not state all assumptions precisely and give only a few references. In the cited papers one finds all details and more relevant literature.

First, we treat the case that $V = 0$. We assume that $a = a^T \in C_b^1(\mathbb{R}^d, \mathbb{R}^{d\times d})$ satisfies $a \geq \delta I > 0$ and an oscillation condition at infinity and that $(F'(x)a(x)\xi, \xi) \leq c\,|\xi|^2$ or $x, \xi \in \mathbb{R}^d$, see [MPR]. There are examples fulfilling our conditions such that $F(x)$ grows as $|x|\log|x|$. On the other hand, the operator $Au(x) = u''(x) - \operatorname{sign}(x)\,|x|^{1+\epsilon}u'(x)$ ($\epsilon > 0$) does not even generate a semigroup on $L^p(\mathbb{R})$. Employing a non–commutative Dore–Venni theorem, we prove that A with $D(A) = \{f \in W^{2,p}(\mathbb{R}^d) : F \cdot \nabla u \in L^p(\mathbb{R}^d)\}$ generates a C_0–semigroup on $L^p(\mathbb{R}^d)$, $1 < p < \infty$. By completely different methods this result was recently shown in [MPV] assuming in addition that F is globally Lipschitz. We then study global regularity of the density ρ of the invariant measure $d\mu$ of $T(t)$ assuming that it exists (i.e, it holds $\int \rho T(t) f\,dx = \int \rho f\,dx$ for all $f \in L^\infty(\mathbb{R}^d)$). It turns out that $\rho \in W^{2,q}(\mathbb{R}^d)$ for all $q < \infty$. This fact follows from the inclusion $D(A) \subset W^{2,p}(\mathbb{R}^d)$ for all $p \in (1,\infty)$ and standard semigroup theory. See [BKR] for a different approach to the global regularity of invariant measures.

In a second part, we investigate the case of a dominating potential. At first, consider the Ornstein–Uhlenbeck type operator $A_{OU}u = \Delta u - \nabla\Phi \cdot \nabla u$ where $\Phi \in C^2(\mathbb{R}^d)$, $e^{-\Phi} \in L^1(\mathbb{R}^d)$, and $|D^2\Phi| \leq \epsilon\,|\nabla\Phi|^2 + c_\epsilon$ for all $\epsilon > 0$. It turns out that it is convenient to study A_{OU} on the space $L^p(\mathbb{R}^d, d\mu)$, $1 < p < \infty$, with the (invariant) measure $d\mu = e^{-\Phi}\,dx$. If one transforms A_{OU} to $L^p(\mathbb{R}^d)$ (with the usual Lebesgue measure), one obtains an operator $Au = \Delta u + F \cdot \nabla u - Vu$, where the resulting F and V satisfy $V \geq -c_0$ and $|F| \leq \kappa(1 + c_0 + V)^{1/2}$. In this sense the potential 'dominates' the drift. Moreover, one deduces the dissipativity condition $\theta V + \operatorname{div} F \geq 0$ for some $\theta < p$ and the oscillation condition $|\nabla V| \leq \gamma V^{3/2} + c_\gamma$ for all $\gamma > 0$ (*). (Here we omit some technical details.) Under these assumptions, we show that A generates an analytic semigroup on $L^p(\mathbb{R}^d)$, $1 < p < \infty$, having maximal regularity of type L^q and that A has the domain $D(A) = \{u \in W^{2,p}(\mathbb{R}^d) : Vu \in L^p(\mathbb{R}^d)\}$, see [MPRS]. By means of the resulting elliptic L^p– estimates and duality, we further prove that A with domain $D(A) = D(\Delta) \cap D(V)$ generates an analytic semigroup on the spaces $C_0(\mathbb{R}^d)$ and $L^1(\mathbb{R}^d)$. Going back to the Ornstein–Uhlenbeck type operator on the weighted space, one sees that A_{OU} has the domain $D(A_{OU}) = W^{2,p}(\mathbb{R}^d, d\mu)$ and generates an analytic semigroup on $L^p(\mathbb{R}^d, d\mu)$, $1 < p < \infty$, having maximal regularity of type L^q. Under additional smallness conditions on the constant κ, similar theorems have been established in [CV] and [DPV] by different methods. The result on $L^p(\mathbb{R}^d)$ holds in fact for diffusion coefficients $a \in C_b^1(\mathbb{R}^d, \mathbb{R}^{d\times d})$ satisfying $a^T = a \geq \delta I > 0$. Moreover, the estimate (*) must only hold for a sufficiently small γ. There are examples showing that this smallness condition is quite sharp. We employ variational apriori estimates, localization/covering procedures and semigroup theory. A crucial ingredient is the weighted gradient estimate $\|V^{1/2}\,|\nabla u|\,\|_p \leq c(\|\Delta u\| + \|Vu\|_p)$ (**) which allows to control the drift term by the diffusion term and the potential.

We have also investigated the case of non–uniformly elliptic diffusion coefficients $a = a^T \in C^1(\mathbb{R}^d, (\mathbb{R}^{d\times d})$ such that $a > 0$ and $|a(x)|$ grows at most as $|x|^2 \log|x|$ at infinity, see [MPPS]. In this case A on the domain $D(A) = \{f \in W^{2,p}_{loc}(\mathbb{R}^d) : Vu, \operatorname{div}(a\nabla u) \in L^p(\mathbb{R}^d)\}$ generates an analytic semigroup with maximal L^q–regularity on $L^p(\mathbb{R}^d)$, $1 < p < \infty$. In the assumptions and the proofs, one has to replace at some points the euclidean norm $|y|$ by the quadratic form $(a(x)y, y)$. If $p > 2$, we require in addition an oscillation condition on a which involves V and the lower and upper bounds of $a(x)$. The latter assumption is needed to show a suitable extension of the gradient estimate (**). Finally, one allow for an isolated singularitiy x_0 of V and F if the assumptions hold on $\mathbb{R}^d \setminus \{x_0\}$.

The papers [MPPS], [MPRS], [MPS] can be found on
`http://cantor1.mathematik.uni-halle.de/reports/`

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Characteristic functions for row contractions

Jörg Eschmeier
(joint work with T. Bhattacharyya and J. Sarkar)

Let $T \in L(H)$ be a contraction on a complex Hilbert space. The defect operators $D = (1 - T^*T)^{1/2}$ and $D_* = (1 - TT^*)^{1/2}$ of T and T^* can be used to define the unitary matrix operator

$$U_T = \begin{pmatrix} T^* & D \\ D_* & -T \end{pmatrix} \in L(H \oplus \mathcal{D},\ H \oplus \mathcal{D}_*)\ ,$$

where $\mathcal{D} = \overline{\mathrm{Im} D}$ and $\mathcal{D}_* = \overline{\mathrm{Im} D_*}$ are the defect spaces of T and T^*. The transfer function of U_T, that is, the analytici operator-valued map $\Theta_T : \mathbb{D} \to L(\mathcal{D}, \mathcal{D}_*)$,

$$\Theta_T(z) = -T + D_*(1 - zT^*)^{-1} zD$$

is called the characteristic function of T. It was used by Sz.–Nagy and Foias [8]to build functional models for completely non-unitary (cnu) contractions and to develop an extensive theory of Hilbert space contractions.

The characteristic function is a complete unitary invariant for cnu contractions. More precisely, two analytic operator-valued functions $\Theta : \mathbb{D} \to L(\mathcal{D}, \mathcal{D}_*)$ and $\Theta' : \mathbb{D} \to L(\mathcal{D}', \mathcal{D}'_*)$ are said to coincide if there exist unitary operators $\tau : \mathcal{D} \to \mathcal{D}'$ and $\tau_* : \mathcal{D}_* \to \mathcal{D}'_*$ with $\Theta'(\lambda)\tau = \tau_*\Theta(\lambda)$ for all $\lambda \in \mathbb{D}$.

Theorem 1 (Sz.–Nagy and Foias [8]) Two cnu contractions $T \in L(H)$ and $R \in L(K)$ are unitarily equivalent if their characteristic functions coincide.

In the one-variable case this observation is only the starting point for a rich theory describing properties of contractions in terms of their characteristic functions. In more recent papers, e.g. by Drury [4], Müller-Vasilescu [6], Popescu [7], Arveson [1, 2], an analogous multivariable theory for contraction tuples on the unit ball $\mathbb{B}$ in $\mathbb{C}^n$ was initiated.

A commuting tuple $T = (T_1, \ldots, T_n) \in L(H)^n$ is called a row contraction if the row matrix operator $H^n \to H$, $(x_i) \mapsto \sum_{i=1}^n T_i x_i$, is a contraction. The adjoint of the row operator T is a column operator $T^* \in L(H, H^n)$. Exactly as in the one-variable case, one can define the characteristic unitary $U_T \in L(H \oplus \mathcal{D},\ H^n \oplus \mathcal{D}_*)$, where $\mathcal{D} \subset H^n$ and $\mathcal{D}_* \subset H$ are the defect spaces of T and T^*. We define the characteristic function of T as the transfer function $\Theta_T : \mathbb{B} \to L(\mathcal{D}, D_*)$,

$$\Theta_T(z) = -T + D_*(1 - ZT^*)^{-1}ZD$$

of U_T, where $Z : H^n \to H$, $(x_i) \mapsto \sum_{i=1}^n z_i x_i$, denotes the row multiplication with $z \in \mathbb{B}$.

Unlike the one-dimensional case, it is not enough to exclude unitary parts to make sure that the characteristic function becomes a complete unitary invariant. Indeed, if V is a cnu coisometry, then the characteristic functions of the cnu pairs $(V, 0)$ and $(0, V)$ coincide, although these pairs are far from being (componentwise) unitarily equivalent.

By definition a row contraction $T \in L(H)^n$ is completely non-coisometric (cnc) if there is no closed invariant subspace $M \neq \{0\}$ for T^* such that $T^*|M$ is a spherical isometry. A commuting tuple $V \in L(H)^n$ is a spherical isometry if $\sum_{i=1}^n V_i^* V_i = 1$. We say that T is of type $C_{\cdot 0}$ if the operators

$$\sum_{i_1,\ldots,i_k=1}^{n} T_{i_k} \ldots T_{i_1} T_{i_1}^* \ldots T_{i_k}^* = \sum_{|\alpha|=k} \gamma_\alpha T^\alpha T^{*\alpha} \quad \left(\gamma_\alpha = \frac{|\alpha|!}{\alpha!}\right)$$

converge strongly to zero.

Theorem 2 (Bhattacharyya–Eschmeier–Sarkar [3]) Two cnc row contractions $T \in L(H)^n$ and $R \in L(K)^n$ are unitarily equivalent if and only if Θ_T and Θ_R coincide.

The proof of this result uses functional models constructed with the help of the characteristic functions. Define $H(\mathcal{D})$ as the space of all analytic functions $f = \sum_\alpha a_\alpha z^\alpha \in \mathcal{O}(\mathbb{B}, \mathcal{D})$ with $\|f\|^2 = \sum_\alpha \|a_\alpha\|^2/\gamma_\alpha < \infty$. It is well known that $H(\mathcal{D})$ is a functional Hilbert space which, for $n = 1$, coincides with the $\mathcal{D}$-valued Hardy space on the unit disc. The characteristic function of a row contraction induces a contractive multiplier between $H(\mathcal{D})$ and $H(\mathcal{D}_*)$.

Theorem 3 (Ball–Trent–Vinnikov, Eschmeier–Putinar [5]) Let $S \subset \mathbb{B}$ and $f : S \to L(\mathcal{D}, \mathcal{D}_*)$ be arbitrary. Equivalent are:

(i) f extends to a multiplier $F \in M\big(H(\mathcal{D}), H(\mathcal{D}_*)\big)$ with norm bounded by 1;

(ii) $K_f(z, w) = \frac{1 - f(w)f(z)^*}{1 - \langle w, z\rangle}$ defines a positive definite function on $S \times S$;

(iii) there is a Hilbert space H and a unitary operator

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in L(H \oplus \mathcal{D},\, H^n \oplus \mathcal{D}_*)$$

such that $f(z) = D + C(1 - ZA)^{-1}ZB$ for $z \in S$.

The construction of the functional model is based on the following dilation theorem.

Theorem 4 Let $T \in L(H)^n$ be a row contraction. Then the map

$$j : H \to H(\mathcal{D}_*),\ j(h) = \sum_\alpha \gamma_\alpha (D_* T^{*\alpha} h) z^\alpha$$

intertwines $T^* \in L(H)^n$ and $M_z^* = (M_{z_1}^*, \ldots, M_{z_n}^*) \in L\big(H(\mathcal{D}_*)\big)^n$ such that

(i) $jj^* + M_{\Theta_T} M_{\Theta_T}^* = 1_{H(\mathcal{D}_*)}$,

(ii) j is isometric if and only if T is of type $C_{\cdot 0}$,

(iii) j is injective if and only if T is cnc.

The different parts of the proof can be found in Müller-Vasilescu [6], Arveson [2] and in Bhattacharyya–Eschmeier–Sarkar [3].

In the $C_{\cdot 0}$-case the map j induces a unitary equivalence between the given row contraction T and the compression of M_z to the model space $H_T = H(\mathcal{D}_*) \ominus \mathrm{Im} M_{\Theta_T}$.

If Θ_T and Θ_R coincide via unitaries τ and τ_*, then τ_* induces a unitary equivalence between the functional models of T and R. A complete proof can be found in [3]. In the completely non-coisometric case the above arguments have to be slightly modified. Details will be presented elsewhere.

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Stochastic integration in UMD spaces

Jan van Neerven
(joint work with Mark Veraar and Lutz Weis)

We report on a joint work with Mark Veraar and Lutz Weis [6].

Building upon previous work by Rosiński and Suchanecki [8] and Brzeźniak and the author [1], a systematic theory of stochastic integration for Banach space-valued functions with respect to Brownian motions has been constructed in [7] using a recent idea of Kalton and Weis to study vector-valued functions through certain operator-theoretic properties of the associated integral operators [4]. In the work presented here, the results of [7] are extended to a theory of stochastic integration for stochastic processes taking values in a UMD space.

Let (γ_n) be a sequence of independent standard Gaussian random variables on some probability space $(\Omega, \mathbb{P})$. A bounded operator $T : H \to E$ acting from a separable real Hilbert space H with orthonormal basis (h_n) into a real Banach space E is said to be *γ-radonifying* if the Gaussian sum $\sum_n \gamma_n Th_n$ converges in $L^2(\Omega; E)$. This definition is independent of the choice of (γ_n) and (h_n), and the vector space $\gamma(H, E)$ of all γ-radonifying operators from H to E is a Banach space with respect to the norm $\|\cdot\|_{\gamma(H,E)}$ defined by

$$\|T\|^2_{\gamma(H,E)} := \mathbb{E}\Big\| \sum_n \gamma_n Th_n \Big\|^2.$$

Let $W = (W(t))_{t\geqslant 0}$ be a Brownian motion on $(\Omega, \mathbb{P})$. The main result of [7] can be formulated as follows.

Theorem 1 ([7])**.** *For a function $\psi : [0, T] \to E$ such that $\langle \psi, x^*\rangle \in L^2(0, T)$ for all $x^* \in E^*$, the following assertions are equivalent:*

(1) *For every measurable set $A \subseteq [0, T]$ there exists an E-valued random variable η_A such that for all $x^* \in E^*$ we have*

$$\langle \eta_A, x^*\rangle = \int_A \langle \phi(t), x^*\rangle \, dW(t) \quad \textit{almost surely;}$$

(2) *There exists an operator $S_\psi \in \gamma(L^2(0, T), E)$ such that for all $f \in L^2(0, T)$ and $x^* \in E^*$ we have*

$$\langle S_\psi f, x^*\rangle = \int_0^T f(t)\langle \psi(t), x^*\rangle \, dt.$$

Writing $\eta_A = \int_A \psi(t)\, dW(t)$, for all $1 \leqslant p < \infty$ we have $\mathbb{E}\Big\| \int_0^T \psi(t)\, dW(t)\Big\|^p \eqsim_p \|S_\psi\|^p_{\gamma(L^2(0,T),E))}$, with equality for $p = 2$.

If the equivalent conditions of the theorem are satisfied, then ψ is said to be *stochastically integrable* with respect to W.

Denote by $\mathscr{F}^W = (\mathscr{F}^W_t)_{t\geqslant 0}$ the augmented filtration generated by W. A stochastic process $\phi : [0, T] \times \Omega \to E$ is said to be *$\mathscr{F}^W$-weakly progressive* if for all $x^* \in E^*$ the real-valued process $\langle \psi, x^*\rangle$ is progressively measurable with respect to $\mathscr{F}^W$. Such a process is said to be *elementary progressive* if it is of the form $\phi = \sum_{n=1}^N 1_{(t_n,t_{n+1}]} \otimes \xi_n$, where ξ_n is an $\mathscr{F}^W_{t_n}$-measurable simple random variable with values in E. Assuming that E is a UMD space, Garling [3] proved the following two-sided decoupling inequality for elementary progressive processes, valid for $1 < p < \infty$:

$$\mathbb{E}_\Omega \Big\| \int_0^T \phi(t)\, dW(t) \Big\|^p \eqsim_{p,E} \mathbb{E}_{\Omega\times\tilde\Omega} \Big\| \int_0^T \phi(t)\, d\tilde W(t) \Big\|^p.$$

Here $\tilde W$ is a Brownian motion on a probability space $(\tilde\Omega, \tilde{\mathbb{P}})$ and the integral on the right hand side is defined pathwise with respect to Ω. By Fubini's theorem, the Kahane-Khinchine inequalities and Theorem 1, the right hand side is proportional to

$$\mathbb{E}_{\Omega\times\tilde\Omega} \Big\| \int_0^T \phi(t)\, d\tilde W(t) \Big\|^p \eqsim_p \mathbb{E}_\Omega \Big(\mathbb{E}_{\tilde\Omega} \Big\| \int_0^T \phi(t)\, d\tilde W(t) \Big\|^2 \Big)^{p/2} = \mathbb{E}_\Omega \|S_\phi\|^p_{\gamma(L^2(0,T),E)},$$

where $S_\phi : \Omega \to \gamma(L^2(0,T), E)$ satisfies $\langle S_\phi(\omega) f, x^*\rangle = \int_0^T f(t)\langle \phi(t,\omega), x^*\rangle\, dt$ for all $f \in L^2(0,T)$ and $x^* \in E^*$ almost surely. As a consequence, the mapping $S_\phi \mapsto \int_0^T \phi(t)\, dW(t)$ extends to an isomorphism from the closure in $L^p(\Omega; \gamma(L^2(0,T), E))$ of the elementary progressive processes onto a certain closed subspace of $L^p(\Omega; E)$. Using a version of the Pettis measurability theorem for $\mathscr{F}^W$-measurable processes in combination with Itô's martingale representation theorem and approximation arguments, the range of this isomorphism can be identified as the subspace of all mean zero $\mathscr{F}^W_T$-measurable elements of $L^p(\Omega; E)$. The result is an extension of Itô's martingale representation theorem to UMD-valued processes, which is the main ingredient in the proof of the following theorem:

Theorem 2. *Let E be a UMD space and let $p \in (1,\infty)$. For a weakly progressive process $\phi : [0,T]\times\Omega \to E$ such that $\langle \phi, x^*\rangle \in L^p(\Omega; L^2(0,T))$ for all $x^* \in E^*$, the following assertions are equivalent:*

(1) *For every measurable set $A \subseteq [0,T]$ there exists a random variable $\eta_A \in L^p(\Omega; E)$ such that for all $x^* \in E^*$ we have*

$$\langle \eta_A, x^*\rangle = \int_A \langle \phi(t), x^*\rangle\, dW(t) \quad \textit{in } L^p(\Omega);$$

(2) *There exists a random variable $S_\phi \in L^p(\Omega; \gamma(L^2(0,T), E))$ such that for all $f \in L^2(0,T)$ and $x^* \in E^*$ we have*

$$\langle S_\phi(\omega) f, x^*\rangle = \int_0^T f(t)\langle \phi(t,\omega), x^*\rangle\, dt \quad \textit{for almost all } \omega \in \Omega.$$

Writing $\eta_A = \int_A \phi(t)\, dW(t)$ we have $\mathbb{E}\big\|\int_0^T \phi(t)\, dW(t)\big\|^p \eqsim_{p,E} \mathbb{E}\|S_\phi\|^p_{\gamma(L^2(0,T),E)}$.

If the equivalent conditions of the theorem are satisfied, then ϕ is said to be *L^p-stochastically integrable* with respect to W. Note that the scalar stochastic integral on the right hand side in (1) is well defined in $L^p(\Omega)$ by the Burkholder-Davis-Gundy inequalities. When combined with our generalized Itô representation theorem, the equivalence of norms in the last line of the theorem leads to Burkholder-Davis-Gundy inequalities for UMD-valued $\mathscr{F}^W$-martingales.

Theorem 2 can be applied to show that every continuous L^p-martingale $(M_t)_{t\geqslant 0}$ with respect to the filtration $\mathscr{F}^W$, with values in a UMD space E and satisfying $M_0 = 0$, is L^p-stochastically integrable with respect to W on every interval $[0,T]$ and satisfies

$$\mathbb{E}\Big\|\int_0^T M_t \, dW(t)\Big\|^p \lesssim_{p,E} T^{\frac{p}{2}} \, \mathbb{E}\|M_T\|^p.$$

In particular this applies to the continuous L^p-martingale $M_t := \int_0^t \phi(s)\, dW(s)$, where ϕ is an L^p-stochastically integrable process with values in E.

The idea to use decoupling inequalities to construct a theory of stochastic integration in UMD spaces is due to McConnell [5] who used convergence in probability rather than L^p-convergence. McConnell first generalized Garling's inequalities to obtain decoupling inequalities for tangent sequences with values in UMD spaces and used these to prove that a progressive process with values in a UMD space E is stochastically integrable if and only if its trajectories are stochastically integrable almost surely as E-valued functions. His arguments depend heavily on the equivalence of the UMD property and the geometric notion of ζ-convexity. By using stopping time arguments, our Theorem 2 can be localized to recover McConnell's result under somewhat weaker measurability assumptions. An advantage of this approach is that it uses the UMD property in a direct and elementary way through Garling's inequality. An Itô formula is obtained as well.

Our results can be extended to processes with values in $\mathscr{L}(H,E)$, where H is a separable real Hilbert space and E is a real UMD space; the integrator is then an H-cylindrical Brownian motion. In a subsequent paper we shall apply the results to the study of existence, uniqueness, and regularity of certain classes of nonlinear stochastic evolution equations in E, thereby extending parts of the theory of stochastic evolution equations in Hilbert spaces developed by Da Prato and Zabczyk [2] and many others, to the setting of UMD spaces.

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Spectral multipliers for the Kohn sublaplacian on the sphere in $\mathbb{C}^n$

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(joint work with O. Klima and A. Sikora)

The sphere S in $\mathbb{C}^n$ has a natural sublaplacian $\mathcal{L}$. We find the optimal s such that a Hörmander multiplier theorem with index s holds for $\mathcal{L}$.

Hörmander's theorem for Fourier multipliers [7] specialises to functions of the Laplacian. By a Hörmander theorem with index s, we mean a result of the following form, where H^s stands for the Sobolev space of functions with s derivatives in $L^2(\mathbb{R})$, and $\mathcal{L}$ for a nonnegative selfadjoint operator on $L^2(S)$.

Model Theorem. *Suppose that* $m\colon [0,\infty) \to \mathbb{C}$ *is bounded and Borel-measurable. Suppose also that* $0 \neq \varphi \in C_c^\infty(0,\infty)$ *and that*

$$\|m(t\cdot)\varphi(\cdot)\|_{H^s} \leq C \qquad \forall t > 0.$$

Then $m(\mathcal{L})$*, initially defined on* $L^2(S)$*, extends continuously to all the spaces* $L^p(S)$ *when* $1 \leq p \leq \infty$ *and is bounded if* $1 < p < \infty$*. Further,* $m(\mathcal{L})$ *is bounded from* $H^1(S)$ *to* $L^1(S)$ *and is of weak type* $(1,1)$*.*

The model theorem theorem implies that $\mathcal{L}$ has an $H^\infty(S_\epsilon)$ functional calculus for all positive ϵ, where $S_\epsilon = \{z \in \mathbb{C} : |\arg(z)| < \epsilon\}$, and

$$\|m(\mathcal{L})\| \leq C\,\epsilon^{-s}\|m(\mathcal{L})\|_{H^\infty} \qquad \forall m \in H^\infty(S_\epsilon).$$

Conversely, if such inequalities hold for all positive ϵ, then a model theorem holds, with a possibly different index [4, Theorem 4.10]. The model theorem implies that

$$\|\mathcal{L}^{iu}\|_{L^p \to L^p} \leq C(1+|u|)^{2s|1/p-1/2|} \qquad \forall u \in \mathbb{R}.$$

In $\mathbb{R}^n$, a Hörmander multiplier theorem for (minus) the Laplacian Δ with index s holds if and only if $s > n/2$, but A. Sikora and J. Wright [10] showed that

$$\|\Delta^{iu}\|_{L^p \to L^p} \leq C(1+|n|)^{n|1/p-1/2|} \qquad \forall u \in \mathbb{R}.$$

The Kohn sublaplacian $\mathcal{L}$ on the sphere in $\mathbb{C}^n$ is of interest in complex analysis, and as a model subelliptic operator. It may be defined by

$$\mathcal{L} = -\Delta + T^2\,,$$

where Δ denotes the Laplace–Beltrami operator on the sphere and T denotes the unit vector field on S pointing in the iz direction at the point z.

Theorem. *A Hörmander multiplier theorem with index* s *for the Kohn sublaplacian holds when* $s > n - 1/2$*, and this is best possible.*

Define the metric (distance function) d on S by

$$d(z,w) = |1 - z \cdot w|^{1/2},$$

where $z \cdot w$ is the usual Hermitian inner product. It is easy to check that the Lebesgue measure of the ball $B(w,t)$ with centre w and radius t satisfies

$$|B(w,t)| \leq C \min(t^{2n}, 1) \qquad \forall t > 0.$$

Equipped with this metric and Lebesgue measure, S is a space of homogeneous type in the sense of Coifman and Weiss [2]. Therefore the Hardy space $H^1(S)$ is a well defined "atomic H^1 space" [3]. Consequently, it is possible to prove a Hörmander theorem with index s when $s > n$ using the results of Coifman and Weiss.

The key step in proving the multiplier theorem is to associate to a multiplier m a kernel $k_m \colon S \times S \to \mathbb{C}$ and to establish that

$$\int_{d(x,y)>2d(y,y')} |k_m(x,y) - k_m(x,y')| \, dx \leq C \qquad \forall y, y' \in S. \tag{1}$$

Thus the task is to control a term of the form

$$\int_{d(x,y)>\epsilon} |k(x,y)| \, dx.$$

Fourier analysis is used to prove Hörmander's multiplier theorem. On $\mathbb{R}^n$,

$$\int |f(x)| \, dx \leq \left(\int \frac{1}{(1+|x|)^{2s}} \, dx \right)^{1/2} \left(\int (1+|x|)^{2s} |f(x)|^2 \, dx \right)^{1/2};$$

the first factor on the right hand side converges provided that $s > n/2$, and the second is essentially the H^s norm of $\hat{f}$, the Fourier transform of f; in the ultimate analysis, this is why $n/2$ is the critical index for the classical Hörmander theorem. In our case, we decompose the integral into integrals over annuli:

$$\int_{d(x,y)>\epsilon} |k(x,y)| \, dx = \sum_{n=0}^{\infty} \int_{2^{n+1}\epsilon \geq d(x,y) > 2^n \epsilon} |k(x,y)| \, dx,$$

and in each annulus we use the trivial estimate

$$\int_{B(y,2\delta)\setminus B(y,\delta)} |k(x,y)| \, dx \leq |B(y,2\delta) \setminus B(y,\delta)|^{1/2} \left(\int_S |k(x,y)|^2 \, dx \right)^{1/2} \tag{2}$$

An argument of R.B. Melrose [9] shows that the distribution $\cos(t\sqrt{\mathcal{L}})\delta_w$ (a solution to the wave equation involving $\mathcal{L}$) is supported in $B(w, \sqrt{2}t)$. We define the even function $M_n \colon \mathbb{R} \to \mathbb{C}$ by $M_n(\xi) = m_n(\xi^2)$. Then at least formally,

$$m_n(\lambda) = M_n(\sqrt{\lambda}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{M}_n(t) \cos(t\sqrt{\lambda}) \, dt,$$

where $\hat{M}_n$ is the Fourier transform of M_n, so

$$m_n(\mathcal{L}) = \frac{1}{2\pi} \int_0^{\infty} \left[\hat{M}_n(t) + \hat{M}_n(-t) \right] \cos(t\sqrt{\mathcal{L}}) \, dt$$

and

$$k_{m_n}(\cdot, y) = \frac{1}{2\pi} \int_0^\infty \left[\hat{M}_n(t) + \hat{M}_n(-t)\right] \cos(t\sqrt{\mathcal{L}})\delta_y \, dt;$$

by Melrose's finite propagation speed result,

$$\int_0^\epsilon \left[\hat{M}_n(t) + \hat{M}_n(-t)\right] \cos(t\sqrt{\mathcal{L}})\delta_y \, dt$$

is supported in $B(y, \sqrt{2}\epsilon)$. If m is smooth enough, then $\hat{M}_n$ vanishes fast enough to control the decay of $k_{m_n}(x, y)$ as x moves away from y. This argument, due to J. Cheeger, M. Gromov and M. Taylor [1], is a more abstract version of Hörmander's analysis, but it only yields a multiplier theorem when $s > n$, because $|B(y, t)|$ behaves like a multiple of t^{2n} for small t, and no smaller exponent will do.

The trick which is needed is the use of a weight w: we replace (2) by

$$\begin{aligned} &\int_{B(y,2\delta)\setminus B(y,\delta)} |k(x, y)| \, dx \\ &\le \left(\int_{B(y,2\delta)\setminus B(y,\delta)} w(x, y)^{-1} \, dx\right)^{1/2} \left(\int_S |k(x, y)|^2 w(x, y) \, dx\right)^{1/2}. \end{aligned}$$

If $w(x, y) = d(x, y)^\alpha$, then the first integral on the right hand side behaves as $\delta^{n-\alpha/2}$; the weight effectively lowers the homogeneous dimension. The cost of this is that one needs weighted L^2 estimates: ordinary L^2 estimates follow from the Plancherel theorems for Lie groups or for spherical harmonic expansions, but weighted Plancherel theorems are trickier.

Up to this point, everything is in my paper with Sikora [6], where we also prove weighted L^2 estimates for the sphere in $\mathbb{C}^2$ using harmonic analysis on SU(2). It is now possible to prove the general theorem for the sphere in $\mathbb{C}^n$ using the weighted L^2 estimates in the M.Sc. thesis of Klima [8] — the key to these is a careful study of complex spherical harmonics. A paper [5] with the details is in preparation.

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Evans functions and modified Fredholm determinants

Yuri Latushkin
(joint work with Fritz Gesztesy and Konstantin A. Makarov)

We announce a general formula relating the Evans function [JK, S] and the (modified) Fredholm determinant of a "sandwiched resolvent" [GM] for first-order nonautonomous differential equations. For simplicity, we formulate our results only for the case of L^1-perturbations of differential equations with L^∞-coefficients.

Let J denote one of the sets $\mathbb{R}_+ = [0,\infty)$, $\mathbb{R}_- = (-\infty,0]$, or $\mathbb{R}$. On J we consider the differential equation $y' = M(\cdot)y$, $M \in L^\infty(J;\mathbb{C}^{d\times d})$ and denote by $\Phi(\cdot)$ its fundamental matrix solution normalized by $\Phi(0) = I$. If Q is a given projection in $\mathbb{C}^d$ such that $m_{Q,J} := \sup_{x\in J} \|\Phi(x)Q\Phi(x)^{-1}\| < \infty$, then the upper and lower *general* (*or Bohl*) *exponents* are defined as follows (see, e.g., [DK]):

$$\varkappa_g(Q;J) = \inf\Big\{\alpha\in\mathbb{R}\,\Big|\,\sup_{x\geq x',x,x'\in J}\|e^{-\alpha(x-x')}\Phi(x)Q\Phi(x')^{-1}\|<\infty\Big\},$$
$$\varkappa_g'(Q;J) = \sup\Big\{\alpha\in\mathbb{R}\,\Big|\,\sup_{x\leq x',x,x'\in J}\|e^{-\alpha(x-x')}\Phi(x)Q\Phi(x')^{-1}\|<\infty\Big\}.$$

A system $\{Q_j\}_{j=1}^n$, $1\leq n\leq d$, of disjoint projections in $\mathbb{C}^d$ is called the *finest exponential splitting of order n over J* if the following holds: $Q_1+\cdots+Q_n = I$, $m_{Q_j,J}<\infty$, the segments $[\varkappa_g'(Q_j;J),\varkappa_g(Q_j;J)]$, $j=1,\dots,n$, are disjoint, and there is no exponential splitting of order $n+1$. In the following we assume that the projections Q_j are numbered such that $\varkappa_g(Q_j;J)<\varkappa_g'(Q_{j+1};J)$; we set $Q_0=0$, $Q_{n+1}=I$, and $\varepsilon_0 = (1/2)\min\{{\varkappa'}_g(Q_j;\mathbb{R})-\varkappa_g(Q_{j-1};\mathbb{R}) : j=1,\dots,n\}$.

We assume that the equation $y'=M(\cdot)y$ with $M\in L^\infty(\mathbb{R};\mathbb{C}^{d\times d})$, has an exponential dichotomy on $\mathbb{R}$ with the dichotomy projector Q. Moreover, we denote by $\{Q_j\}_{j=1}^n$ the finest exponential splitting over $\mathbb{R}$ for this equation and remark that the projections Q and Q_j are uniquely defined [DK, sec. IV.4]. In addition, there exists a $k\in\{1,\dots,n\}$ such that $Q=Q_1+\cdots+Q_k$ and $I-Q=Q_{k+1}+\cdots+Q_n$. Thus, $\varkappa_g'(Q_j;\mathbb{R})\leq\varkappa_g(Q_j;\mathbb{R})<0$ for $j=1,\dots,k$ and $0<\varkappa_g'(Q_j;\mathbb{R})\leq\varkappa_g(Q_j;\mathbb{R})$ for $j=k+1,\dots,n$. Clearly, $\{Q_j\}_{j=1}^n$ is also an exponential splitting for $y'=M(\cdot)y$ over $\mathbb{R}_+$ and over $\mathbb{R}_-$ with the corresponding segments $[{\varkappa'}_j^+,\varkappa_j^+]$ and $[{\varkappa'}_j^-,\varkappa_j^-]$ of general exponents; in the remainder of this announcement we denote ${\varkappa'}_j^\pm=\varkappa_g'(Q_j;\mathbb{R}_\pm)$, $\varkappa_j^\pm=\varkappa_g(Q_j;\mathbb{R}_\pm)$, $j=1,\dots n$.

Next, we consider the perturbed differential equation $y' = [M(\cdot) + R(\cdot)]y$ with $R \in L^1(\mathbb{R}; \mathbb{C}^{d\times d})$. Let $\{\widetilde{Q}_j^+\}_{j=1}^n$, respectively, $\{\widetilde{Q}_j^-\}_{j=1}^n$ be any exponential splitting for $y' = [M(\cdot) + R(\cdot)]y$ over $\mathbb{R}_+$, respectively, over $\mathbb{R}_-$. We will use "$\sim$" to denote all objects related to the splitting $\{\widetilde{Q}_j^\pm\}_{j=1}^n$ relative to the fundamental matrix solution $\widetilde{\Phi}(\cdot)$ of the equation $y' = [M(\cdot) + R(\cdot)]y$; for instance, $\widetilde{\varkappa}_j^+ = \inf\{\alpha \in \mathbb{R} : \sup_{x\geq x'\geq 0} \|e^{-\alpha(x-x')}\widetilde{\Phi}(x)\widetilde{Q}_j^+\widetilde{\Phi}(x')^{-1}\| < \infty\}$, etc. We recall (cf. [DK, Sec. IV.5.3]), that the general exponents $\varkappa_j^\pm$, $\varkappa'^\pm_j$ are stable under L^1-perturbations in the following sense: Let $\|R(\cdot)\| \in L^1(\mathbb{R}_\pm)$ and assume a given exponential splitting $\{Q_j^\pm\}_{j=1}^n$ over $\mathbb{R}_\pm$ for $y' = M(\cdot)y$. Then for each $\varepsilon \in (0, \varepsilon_0)$, there exists an exponential splitting $\{\widetilde{Q}_j^\pm\}_{j=1}^n$ over $\mathbb{R}_\pm$ for $y' = [M(\cdot)+R(\cdot)]y$ which is *ε-close to* $\{Q_j^\pm\}_{j=1}^n$, that is, the inequalities $\varkappa'^\pm_j - \varepsilon \leq \widetilde{\varkappa}'^\pm_j$ and $\widetilde{\varkappa}_j^\pm \leq \varkappa_j^\pm + \varepsilon$ hold for $j = 1, \dots, n$. For an explicit construction of $\{\widetilde{Q}_j^\pm\}_{j=1}^n$ we refer to [DK, Thm. IV.5.1].

Suppose that the perturbation R satisfies the following *hypothesis*: There exists a $\delta \in (0, \varepsilon_0)$ such that

$$\int_0^\infty e^{(\varkappa_j^+ - \varkappa'^+_j + \delta)x} \|R(x)\|dx < \infty, \quad j = 1, \dots, k,$$
$$\int_{-\infty}^0 e^{-(\varkappa_j^- - \varkappa'^-_j + \delta)x} \|R(x)\|dx < \infty, \quad j = k+1, \dots, n.$$

Theorem 1. *Let $\varepsilon \in (0, \delta)$ and assume that $\{\widetilde{Q}_j^\pm\}_{j=1}^n$ is any exponential splitting for $y' = [M(\cdot) + R(\cdot)]y$ over $\mathbb{R}_\pm$ which is ε-close to $\{Q_j\}_{j=1}^n$. Then for each $\gamma \in (\varepsilon, \delta)$, the perturbed equation $y' = [M(\cdot) + R(\cdot)]y$ has absolutely continuous matrix solutions $Y_j(\cdot)$, $j = 1, \dots, k$, on $\mathbb{R}_+$ and $Y_j(\cdot)$, $j = k+1, \dots, n$, on $\mathbb{R}_-$, respectively, such that for the forward filtration $\widetilde{E}_j^+(x) = \bigoplus_{i=1}^j \operatorname{Im}(\widetilde{Q}_j^+(x))$ and for the backward filtration $\widetilde{E}_j^-(x) = \bigoplus_{i=j}^n \operatorname{Im}(\widetilde{Q}_i^-(x))$ the following assertions hold:*

$$Y_j(x) \in \widetilde{E}_j^+(x) \backslash \widetilde{E}_{j-1}^+(x), \quad x \in \mathbb{R}_+,$$
$$\sup_{x\geq 0} e^{-(\varkappa'^+_j - \gamma)x} \|Y_j(x) - \Phi(x)Q_j\| < \infty, \quad j = 1, \dots, k,$$
$$Y_j(x) \in \widetilde{E}_j^-(x) \backslash \widetilde{E}_{j+1}^-(x), \quad x \in \mathbb{R}_-,$$
$$\sup_{x\leq 0} e^{-(\varkappa_j^- + \gamma)x} \|Y_j(x) - \Phi(x)Q_j\| < \infty, \quad j = k+1, \dots, n.$$

These solutions are unique up to terms of lower exponential order. Moreover, if $\{\widetilde{Q}_j^\pm\}_{j=1}^n$ is the splitting used in [DK, Thm. IV.5.1], *then $y' = [M(\cdot) + R(\cdot)]y$ has a unique set of absolutely continuous matrix solutions $Y_j(\cdot)$ such that $Y_j(\cdot) \in \operatorname{Im}(\widetilde{\Phi}(\cdot)\widetilde{Q}_j^+\widetilde{\Phi}(\cdot)^{-1})$, $j = 1, \dots, k$, on $\mathbb{R}_+$, respectively, $Y_j(\cdot) \in \operatorname{Im}(\widetilde{\Phi}(\cdot)\widetilde{Q}_j^-\widetilde{\Phi}(\cdot)^{-1})$, $j = k+1, \dots, n$, on $\mathbb{R}_-$, and the above assertions hold for each $\gamma \in (\varepsilon, \delta)$.*

In fact, the solutions $Y_j(\cdot)$ are solutions of some explicit integral equations.

Definition. *If $Y_j(\cdot)$, $j = 1, \dots, n$, are the matrix solutions in Theorem 1, then the* Evans determinant *$\mathcal{E}$ associated with the* (*ordered*) *pair of equations $y' = M(\cdot)y$*

and $y' = [M(\cdot) + R(\cdot)]y$ *is defined as follows:*

$$\mathcal{E} = \det[Y_1(0) + \cdots + Y_k(0) + Y_{k+1}(0) + \cdots + Y_n(0)].$$

One can show that $\mathcal{E}$ does not depend on the choice of solutions $Y_j(\cdot)$ in Theorem 1. Thus, the Evans determinant is uniquely determined by the pair of equations $y' = M(\cdot)y$ and $y' = [M(\cdot) + R(\cdot)]y$. If $M = M(\cdot; z)$ and $R = R(\cdot; z)$ depend (analytically) on a (spectral) parameter $z \in \Omega$ with Ω a domain in $\mathbb{C}$, then the above definition yields the *Evans function*, $E = E(z)$. The latter is a particularly useful tool in detecting isolated eigenvalues of differential operators obtained by linearizing PDEs along special solutions such as travelling waves; we refer to [JK, S] for basic notions and an extensive review of the literature on this subject. We emphasize that unlike our definition, the Evans function is usually defined only up to a nonvanishing (analytic) multiplier (cf. [JK, S]).

Using the polar decomposition $R(x) = V_R(x)|R(x)|$, we represent the perturbation R as a product $R = R_\ell R_r$, where $R_\ell(x) = V_R(x)|R(x)|^{\frac{1}{2}}$ and $R_r(x) = |R(x)|^{\frac{1}{2}}$, $x \in \mathbb{R}$. Moreover, we consider in $L^2(\mathbb{R}; \mathbb{C}^d)$ a first order differential operator, G, with domain the Sobolev space $W_1^2(\mathbb{R}; \mathbb{C}^d)$, defined by the formula $Gu = u' - M(\cdot)u$, $u \in W_1^2(\mathbb{R}; \mathbb{C}^d)$. Since by our assumptions the equation $y' = M(\cdot)y$ has an exponential dichotomy on $\mathbb{R}$, the operator G has a bounded inverse, G^{-1}, which is an integral operator with integral kernel K given by the formula

$$K(x, x') = \begin{cases} \Phi(x)Q\Phi(x')^{-1}, & x > x', \\ -\Phi(x)(I - Q)\Phi(x')^{-1}, & x < x'. \end{cases}$$

Since $\|R(\cdot)\| \in L^1(\mathbb{R})$, the "sandwiched resolvent" $R_r G^{-1} R_\ell$ is a Hilbert-Schmidt integral operator with integral kernel $R_r(x)K(x, x')R_\ell(x')$, $x, x' \in \mathbb{R}$.

Theorem 2. *The Evans determinant* $\mathcal{E}$ *for the pair of equations* $y' = M(\cdot)y$ *and* $y' = [M(\cdot) + R(\cdot)]y$ *and the* 2*-modified Fredholm determinant associated with the Hilbert-Schmidt operator* $R_r G^{-1} R_\ell$ *are related as follows:*

$$\det{}_2(I + R_r G^{-1} R_\ell) = \mathcal{E} \exp\left(\int_0^\infty \operatorname{tr}\big[\Phi(x)Q\Phi(x)^{-1}R(x)\big]\, dx - \int_{-\infty}^0 \operatorname{tr}\big[\Phi(x)(I - Q)\Phi(x)^{-1}R(x)\big]\, dx \right).$$

We refer to [KS] for a related result on Schrödinger equations.

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Double operator integrals and commutators

Fyodor A. Sukochev

1. Let $\mathcal{H}$ be a separable Hilbert space, $\mathcal{M} \subseteq \mathcal{L}(\mathcal{H})$ be a semi-finite von Neumann algebra equipped with faithful normal semi-finite trace τ. If $E(0,\infty)$ is a r.i. function space then $\mathcal{E} = E(\mathcal{M},\tau)$ denotes the corresponding "non-commutative symmetric operator space". We write $\mathcal{L}_p$ instead of $L_p(\mathcal{M},\tau)$. If $\mathcal{M} = \mathcal{L}(\mathcal{H})$ and $\tau = Tr$, then $\mathcal{E}$ is a (classical) symmetrically normed ideal of compact operators. In this case $\mathcal{L}_p$ coincides with $\mathfrak{S}_p$ — Schatten-von Neumann ideal.
An $\mathbb{R}$-*flow* on $(\mathcal{M},\tau)$ is an ultra-weakly continuous (equivalently, $\sigma(\mathcal{M},\mathcal{M}_*)$-continuous) representation $\gamma = \{\gamma_t\}_{t\in\mathbb{R}}$ on $\mathcal{M}$ by $*$-automorphisms of $\mathcal{M}$, which are τ-invariant. The $*$-automorphisms γ_t has a unique extension γ_t^E on $\mathcal{E}$. $\{\gamma_t^E\}_{t\in\mathbb{R}}$ is a *strongly continuous* group (equivalently, C_0-group) whenever $E(0,\infty)$ is a separable r.i. space. Let infinitesimal generator δ^E of γ_t^E be defined as follows

$$\mathcal{D}\mathrm{om}(\delta^E) = \{x \in \mathcal{E} :\ \|\cdot\|_{\mathcal{E}} - \lim_{t\to 0} \frac{\gamma_t^E(x) - x}{t} \text{ exists}\},$$

$$\delta^E(x) = \|\cdot\|_{\mathcal{E}} - \lim_{t\to 0} \frac{\gamma_t^E(x) - x}{t}, \quad x \in \mathcal{D}\mathrm{om}(\delta^E).$$

δ^E is a densely defined closed operator whenever $E(0,\infty)$ is a separable r.i. space, moreover

$$\delta^E(x^*) = (\delta^E(x))^*, \quad \delta^E(xy) = \delta^E(x)y + x\delta^E(y), \quad \forall x, y \in \mathcal{E} \cap \mathcal{M}.$$

Hence δ^E is a partially defined derivation on $\mathcal{M}$.
Problem. *For which scalar functions $f : \mathbb{R} \to \mathbb{R}$, is $f(x) \in \mathcal{D}\mathrm{om}(\delta^E)$, whenever $x = x^* \in \mathcal{D}\mathrm{om}(\delta^E)$?*
In particular, in order to guarantee that $f(x) \in \mathcal{E}$, whenever $x = x^* \in \mathcal{E}$ we shall assume everywhere below that function f satisfies the following

(NC) $$f \in C_b^1(\mathbb{R}), \quad f(0) = 0.$$

Here $C_b^1(\mathbb{R})$ is collection of all continuously differentiable functions with bounded derivative.
Proposition 1.[AdPS]*Let $\mathfrak{S} \subseteq \mathcal{L}(\mathcal{H})$ be a linear subspace equipped with a norm $\|\cdot\|_{\mathfrak{S}}$, $a : \mathcal{D}\mathrm{om}(a) \mapsto \mathcal{H}$ be a s.a. operator such that (i) $(\mathfrak{S}, \|\cdot\|_{\mathfrak{S}})$ is a Banach space; (ii) $\|x\| \leqslant \|x\|_{\mathfrak{S}}$ for every $x \in \mathfrak{S}$; (iii) $e^{ita}xe^{-ita} \in \mathfrak{S}$ for every $x \in \mathfrak{S}$; (iv) $\|e^{ita}xe^{-ita} - x\|_{\mathfrak{S}} \to 0$ as $t \to 0$ for every $x \in \mathfrak{S}$. Define the C_0-group $\gamma = \{\gamma_t\}_{t\in\mathbb{R}}$ of $*$-automorphisms in $\mathfrak{S}$ by $\gamma_t(x) = e^{ita}xe^{-ita}$ for every $x \in \mathfrak{S}$ and $t \in \mathbb{R}$. If $\delta : \mathcal{D}\mathrm{om}(\delta) \mapsto \mathfrak{S}$ is the infinitesimal generator of γ then $\mathcal{D}\mathrm{om}(\delta)$ admits the following description. Operator $x \in \mathfrak{S}$ belongs to $\mathcal{D}\mathrm{om}(\delta)$ if and only if the*

following two conditions hold (a) $x(\mathcal{D}\text{om}(a)) \subseteq \mathcal{D}\text{om}(a)$; (b) $[a, x] \in \mathfrak{S}$. Moreover, $\delta(x) = i[a, x]$ for every $x \in \mathcal{D}\text{om}(\delta)$.

Now we can consider the following example. Let $a : \mathcal{D}\text{om}(a) \mapsto \mathcal{H}$ be a s.a. operator, Define C^*-algebra $\mathfrak{S}_a$ as follows

$$\mathfrak{S}_a = \{x \in \mathcal{L}(\mathcal{H}) : \ \|e^{ita} x e^{-ita} - x\|_{\mathcal{L}(\mathcal{H})} \to 0, \text{ as } t \to 0\}.$$

Algebra $\mathfrak{S}_a$ satisfies the conditions (i)–(iv) of proposition 1 with respect to group $\gamma_t(x) = e^{ita} x e^{-ita}$, $x \in \mathfrak{S}_a$. Hence, $\gamma = \{\gamma_t\}_{t \in \mathbb{R}}$ is a C_0-group on $\mathfrak{S}_a$. The infinitesimal generator $\delta : \mathcal{D}\text{om}(\delta) \mapsto \mathfrak{S}_a$ of γ is a densely defined closed symmetric derivation on the C^*-algebra $\mathfrak{S}_a$. It follows from the proposition 1 that $\mathcal{D}\text{om}(\delta)$ admits the following description. $x \in \mathfrak{S}_a$ belongs to $\mathcal{D}\text{om}(\delta)$ if and only if (a) $x(\mathcal{D}\text{om}(a)) \subseteq \mathcal{D}\text{om}(a)$; (b) $[a, x] \in \mathfrak{S}_a$.

Let the C^1-function $f_\eta : (-1, 1) \mapsto \mathbb{R}$ $(0 < \eta < 1)$ be defined as follows

$$f_\eta(t) = |t| \left(\log \left| \log \frac{|t|}{e} \right| \right)^{-\eta}, \quad t \neq 0, \quad f_\eta(0) = 0.$$

Theorem 2.[AdPS] *There exist s.a. operators $a : \mathcal{D}\text{om}(a) \mapsto \mathcal{H}$, $x \in \mathcal{K}(\mathcal{H}) \subseteq \mathfrak{S}_a$ such that (i) $x(\mathcal{D}\text{om}(a)) \subseteq \mathcal{D}\text{om}(a)$; (ii) $[a, x] \in \mathfrak{S}_a$; (iii) $\|[a, f_\eta(x)]\| = \infty$.*

The latter indeed means that there exist a C^*-algebra $\mathfrak{S}_a$, closed densely defined symmetric derivation $\delta : \mathcal{D}\text{om}(\delta) \mapsto \mathfrak{S}_a$ on $\mathfrak{S}_a$, an operator $x = x^* \in \mathcal{D}\text{om}(\delta)$ such that $f_\eta(x) \notin \mathcal{D}\text{om}(\delta)$.

Let us consider a special case $\mathcal{M} = \mathcal{L}(\mathcal{H})$. Let $\mathfrak{S}_E$ be a symmetrically normed ideal, $a : \mathcal{D}\text{om}(a) \mapsto \mathcal{H}$ be a self-adjoint operator $\gamma_t(x) = e^{ita} x e^{-ita}$ $(x \in \mathfrak{S}_E)$ be a C_0-group of automorphisms, $\delta^E : \mathcal{D}\text{om}(\delta^E) \mapsto \mathfrak{S}_E$ be its infinitesimal generator. In this special case the problem can be reformulated as follows

Problem′.*For which scalar functions f, is $[f(x), a] \in \mathfrak{S}_E$, whenever $[x, a] \in \mathfrak{S}_E$?*

If $x = \sum_k \lambda_k p_k$ is a spectral decomposition of an operator $x = x^* \in \mathfrak{S}_E \subseteq \mathcal{K}(\mathcal{H})$ and $\psi_f(\lambda, \mu) = (f(\lambda) - f(\mu))/(\lambda - \mu)$, then

$$= \sum_{m,s} p_m [f(x), a] p_s = \sum_{m,s} p_m \left[\sum_k f(\lambda_k) p_k, a \right] p_s = \sum_{m,s} (f(\lambda_m) - f(\lambda_s)) p_m a p_s$$
$$= \sum_{m,s} \psi_f(\lambda_m, \lambda_s) \, (\lambda_m - \lambda_s) p_m a p_s = \sum_{m,s} \psi_f(\lambda_m, \lambda_s) \, p_m [x, a] p_s.$$

The last double sum is nothing else but "Double Operator Integral" that expresses the link between $[f(x), a]$ and $[x, a]$. The exact meaning of manipulations above is given below.

2. Let a,b be two s.a. operators affiliated with $\mathcal{M}$, e^a, e^b be the spectral measures of those operators. For all sets $A, B \in \mathcal{B}(\mathbb{R})$ (=σ-algebra of all Borel sets on $\mathbb{R}$) let us define the operators

$$P_E^a(A)(x) = e^a(A) x, \quad Q_E^b(B)(x) = x e^b(B), \quad x \in \mathcal{E}.$$

P_E^a, Q_E^b are σ-additive commuting measures, taking their values in $\mathcal{L}(\mathcal{E})$.

Now let the mapping $P^a_E \otimes Q^b_E : \mathcal{A}(\mathbb{R}^2) \mapsto \mathcal{L}(\mathcal{E})$ (from the algebra $\mathcal{A}(\mathbb{R}^2)$ of all Borel rectangles in $\mathbb{R}^2$) be defined as follows

$$P^a_E \otimes Q^b_E(A \times B) = P^a_E(A)Q^b_E(B).$$

In general, $P^a_E \otimes Q^b_E$ cannot be extended to a σ-additive measure on $\mathcal{B}(\mathbb{R}^2)$; it is a finitely additive measure on $\mathcal{A}(\mathbb{R}^2)$ of (in general) unbounded variation. However, in the special case $E = \mathcal{L}_2$, the finitely-additive measure $P^a_{L_2} \otimes Q^b_{L_2}$ can be extended to a σ-additive measure on $\mathcal{B}(\mathbb{R}^2)$, taking its values in orthogonal projections in $\mathcal{L}(\mathcal{L}_2)$. Consequently, for every function $\varphi \in B(\mathbb{R}^2)$ (=collection of all bounded Borel functions on $\mathbb{R}^2$) the following spectral integral exists

$$T^{a,b}_{\varphi,L_2} = \int_{\mathbb{R}^2} \varphi \, d(P^a_{L_2} \otimes Q^b_{L_2}), \; \|T^{a,b}_{\varphi,L_2}\|_{2,2} \leqslant C\|\varphi\|_\infty.$$

Definition 3. *A function $\varphi \in B(\mathbb{R}^2)$ will be called $P^a_E \otimes Q^b_E$ integrable if*

$$T^{a,b}_{\varphi,E}(\mathcal{E} \cap \mathcal{L}_2) \subseteq \mathcal{E} \cap \mathcal{L}_2,$$

and $T^{a,b}_{\varphi,E}$ is continuous on $\mathcal{E} \cap \mathcal{L}_2$ with respect to $\|\cdot\|_{\mathcal{E}}$. In this case $T^{a,b}_{\varphi,E}$ has a unique bounded extension on $\mathcal{E}$, denoted as

$$T^{a,b}_{\varphi,E} = \int_{\mathbb{R}^2} \varphi \, d(P^a_E \otimes Q^b_E).$$

Now we are in a position to introduce two classes of functions on $\mathbb{R}$. We write "$f \in (1)$" if either $f \in C^{1+\varepsilon}_b(\mathbb{R})$ (=the collection of all continuously differentiable functions with bounded derivative such that $|f'(t_1) - f'(t_2)| \leqslant L|t_1 - t_2|^\varepsilon$, $t_1, t_2 \in \mathbb{R}$, for some absolute constant L) or $\mathcal{F}f' \in M_b(\mathbb{R})$ (=collection of all bounded measures on $\mathbb{R}$); "$f \in (2)$" if f is a continuously differentiable function and derivative f' is of bounded total variation over $\mathbb{R}$. Let $E(0,\infty)$ be a r.i. separable function space, then our results are as follows

Theorem 4.[dPS1] *Let a, b be s.a. operators affiliated with $\mathcal{M}$, f be a function. If any of the following two conditions holds (i) $f \in$ (1); (ii) $f \in$ (2) and $E(0,\infty)$ has non-trivial Boyd indices; then function ψ_f is $P^a_E \otimes Q^b_E$ integrable.*

Theorem 5.[dPS2] *Let γ^E_t be a C_0-group generated by an $\mathbb{R}$-flow on $\mathcal{M}$, $f \in$ (NC). If any of the following conditions is satisfied (i) $f \in$ (1); (ii) $f \in$ (2) and $E(0,\infty)$ has non-trivial Boyd indices; then $f(x) \in \mathcal{D}\mathrm{om}(\delta^E)$ whenever $x = x^* \in \mathcal{D}\mathrm{om}(\delta^E)$ and*

$$\delta^E(f(x)) = T^{x,x}_{\psi_f,E}(\delta^E(x)).$$

Remark 6. For function f_η introduced above we have that $f_\eta \notin (1)$ but $f_\eta \in (2)$, because f_η has monotone derivative.

3. We know that in general the condition $f \in$ (NC) is NOT sufficient to guarantee that $f(x) \in \mathcal{D}\mathrm{om}(\delta)$ whenever $x = x^* \in \mathcal{D}\mathrm{om}(\delta)$. The natural question is for which r.i. spaces $E(0,\infty)$ the condition $f \in$ (NC) is indeed sufficient. The only space for which the positive result is obtained is $E = L_2$.

Corollary 7.[PS] *If a function* $f \in$ *(NC) then* $f(x) \in \mathcal{D}\mathrm{om}(\delta^2)$ *whenever* $x = x^* \in \mathcal{D}\mathrm{om}(\delta^2)$ *and*

$$\delta^2(f(x)) = T^{x,x}_{\psi_f, L_2}(\delta^2(x)).$$

Theorem 8.[PS] *Let* E *be a separable r.i. sequence space with trivial Boyd indices. There exist a* C^1*-function* $f_E : (-1,1) \mapsto \mathbb{R}$ *(that depends on r.i.* E*), a s.a. operator* $x \in \mathfrak{S}_E$ *and a s.a. operator* $a : \mathcal{D}\mathrm{om}(a) \mapsto \mathcal{H}$ *such that (i)* $x(\mathcal{D}\mathrm{om}(a)) \subseteq \mathcal{D}\mathrm{om}(a)$*; (ii)* $[a,x] \in \mathfrak{S}_E$*; (iii)* $[a, f_E(x)] \notin \mathfrak{S}_E$*.*
The latter means that for every symmetrically normed ideal $\mathfrak{S}_E$ with trivial Boyd indices there exist a C^1-function, a C_0-group γ_t with generator $\delta : \mathcal{D}\mathrm{om}(\delta) \mapsto \mathfrak{S}_E$ and an operator $x = x^* \in \mathcal{D}\mathrm{om}(\delta)$ such that $f_E(x) \notin \mathcal{D}\mathrm{om}(\delta)$.
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Wavelet transform for functions with values in Banach spaces

Cornelia Kaiser
(joint work with Lutz Weis)

Let $\psi \neq 0$ be in $L_2(\mathbb{R})$ such that its Fourier transform $\widehat{\psi}$ is in $L_2(\mathbb{R}^*)$. (With $\mathbb{R}^*$ we denote the multiplicative group $\mathbb{R} \setminus \{0\}$ with invariant measure $\frac{dt}{|t|}$.) Such a ψ we call an *admissible wavelet.* Without loss of generality we can assume that $\|\widehat{\psi}\|^2_{L_2(\mathbb{R}^*)} = \int_{\mathbb{R}\setminus\{0\}} |\widehat{\psi}(t)|^2 \frac{dt}{|t|} = 1$. By ψ_t we denote the dilated version of ψ, i.e., $\psi_t(u) = \frac{1}{|t|}\psi(\frac{u}{t})$, where $t \in \mathbb{R} \setminus \{0\}$ and $u \in \mathbb{R}$.

For a Banach space X and a function f in the Schwartz class $S(\mathbb{R}, X)$ we define the *continuous wavelet transform* $\mathcal{W}_\psi f$ of f with respect to ψ by

$$(\mathcal{W}_\psi f)(t,s) = (\psi_t * f)(s) = \int_{\mathbb{R}} \tfrac{1}{|t|}\psi(\tfrac{s-u}{t}) f(u) du, \qquad s \in \mathbb{R},\ t \in \mathbb{R}\setminus\{0\}.$$

If $X = \mathbb{C}$, it is well-known (see e.g. [1, 3]) that $\mathcal{W}_\psi$ extends to an isometry from $L_2(\mathbb{R})$ into $L_2(\mathbb{R}^* \times \mathbb{R})$, i.e.,

$$\|f\|^2_{L_2(\mathbb{R})} = \int_{\mathbb{R}} \int_{\mathbb{R}\setminus\{0\}} |(\mathcal{W}_\psi f)(t,s)|^2 \tfrac{dt}{|t|} ds.$$

Now one can ask the question if a similar result holds for $f \in L_2(\mathbb{R}, X)$, where X is an arbitrary Banach space. If X is a Hilbert space, we still obtain an isometry from $L_2(\mathbb{R}, X)$ into $L_2(\mathbb{R}^* \times \mathbb{R}, X)$. For Banach spaces X however we don't obtain equivalence of norms in general. But if we change the norm (and the space) on the right hand side in an appropriate way, we can still prove equivalence of norms, at least in the case that X is a UMD space.

More precisely, take an admissible wavelet $\psi \in L_2(\mathbb{R})$ such that its Fourier transform $\widehat{\psi}$ is absolutely continuous on $\mathbb{R} \setminus \{0\}$ and $\int_{\mathbb{R}\setminus\{0\}} |t\widehat{\psi}'(t)|^2 \frac{dt}{|t|}$ is finite. Then, if $p, q \in (1, \infty)$ and $X = L_q(\Omega, \mu)$ for some σ-finite measure space (Ω, μ), we obtain

$$\|f\|_{L_p(\mathbb{R},X)} \sim \left\| \left(\int_{\mathbb{R}\setminus\{0\}} |(\mathcal{W}_\psi f)(t, \cdot)|^2 \frac{dt}{|t|} \right)^{1/2} \right\|_{L_p(\mathbb{R},X)}.$$

For the proof we use an operator-valued Mihlin type multiplier theorem from [5]. For general UMD spaces X we can prove a similar result, using the generalized square functions introduced in [4]. For more details and the proofs we refer to [2].

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The operator sum method - revisited

Jan Prüss

1. Introduction

In recent years the method of operator sums has become an important tool for proving optimal regularity results for partial differential and integro-differential equations, as well as for abstract evolutionary problems, see for instance [5, 11]. This method has been invented in the fundamental paper of da Prato and Grisvard [2] and has been developed further in the case of two commuting operators, A and B, by Dore and Venni [4], Prüss and Sohr [13], and more recently by Kalton and Weis [6]. Since in these results the sum $A + B$ with natural domain $D(A + B) = D(A) \cap D(B)$ has similar properties as A and B, one obtains the important fact that the method can be iterated, and hence, complicated operators can be built up from simpler ones.

If the operators are noncommuting, matters are, naturally, much more involved. However, it is known that the Da Prato-Grisvard theorem remains valid if A and B satisfy certain commutator estimates. Such conditions were already introduced by Da Prato and Grisvard [2] and later on, Labbas and Terreni [7] proposed another, more flexible one. In Monniaux and Prüss [8], the Dore-Venni theorem was extended to the noncommuting case, employing the Labbas-Terreni condition.

An extension of the Kalton-Weis theorem to the noncommutative case for the Labbas-Terreni condition has been obtained by Strkalj [14] provided the underlying Banach space is B-convex. However, no such results are known for the Da Prato-Grisvard condition, and it is also not known whether or not the result of Monniaux and Prüss or Strkalj can be iterated. It is the purpose of this paper to present a noncommutative version of the Kalton-Weis theorem, employing the commutator condition of Labbas and Terreni, as well as that of Da Prato and Grisvard, without any assumption on the Banach space. Under stronger hypotheses we show that the sum $A+B$ admits an H^∞-calculus, so that the sum method can also be iterated in the noncommuting case.

A linear operator A on a Banach space X with domain $D(A)$, range $R(A)$, kernel $N(A)$ is called *sectorial* if

- $D(A)$ and $R(A)$ are dense in X,
- $(-\infty,0)\subset\rho(A)$ and $|t(t+A)^{-1}|\leqslant M$ for $t>0$.

Here $\rho(A)$ means the resolvent set of A. The class of all sectorial operators will be denoted by $\mathcal{S}(X)$. If A is sectorial, then it is closed, and it follows from the ergodic theorem that $N(A)=0$. We define the *spectral angle* ϕ_A of A by means of

$$\phi_A := \inf\{\phi>0 : \rho(-A)\supset\Sigma_{\pi-\phi},\ M_{\pi-\phi}<\infty\},$$

where $M_\phi := \sup\{|\lambda(\lambda+A)^{-1} :\ \lambda\in\Sigma_\phi\}$.

If A is sectorial, the functional calculus of Dunford given by

$$\Phi_A(f) := f(A) := \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda-A)^{-1}d\lambda$$

is a well-defined algebra homomorphism $\Phi_A : \mathcal{H}_0(\Sigma_\phi)\to\mathcal{B}(X)$, where $\mathcal{H}_0(\Sigma_\phi)$ denotes the set of all functions $f:\Sigma_\phi\to\mathbb{C}$ that are holomorphic and decay polynomially at 0 and at ∞. Here Γ denotes a contour $\Gamma=e^{i\theta}(\infty,0]\cup e^{-i\theta}[0,\infty)$ with $\theta\in(\phi_A,\phi)$. A is said to admit an $\mathcal{H}^\infty$*-calculus* if there are numbers $\phi>\phi_A$ and $M>0$ such that the estimate

$$|f(A)|\le M|f|_{\mathcal{H}^\infty(\Sigma_\phi)},\quad f\in\mathcal{H}_0(\Sigma_\phi), \tag{1}$$

is valid. In this case, the Dunford calculus extends uniquely to $\mathcal{H}^\infty(\Sigma_\phi)$, see for instance [3] for more details. We denote the class of sectorial operators which admit an $\mathcal{H}^\infty$-calculus by $\mathcal{H}^\infty(X)$. The infimum ϕ_A^∞ of all angles ϕ such that (1) holds for some constant $C>0$ is called the $\mathcal{H}^\infty$*-angle* of A.

Let $\mathcal{T}\subset\mathcal{B}(X)$ be an arbitrary set of bounded linear operators on X. Then $\mathcal{T}$ is called $\mathcal{R}$*-bounded* if there is a constant $M>0$ such that the inequality

$$\mathbb{E}(|\sum_{i=1}^N\varepsilon_iT_ix_i|)\le M\mathbb{E}(|\sum_{i=1}^N\varepsilon_ix_i|) \tag{2}$$

is valid for every $N\in\mathbb{N}$, $T_i\in\mathcal{T}$, $x_i\in X$, and all independent symmetric $\{\pm1\}$-valued random variables ε_i on a probability space $(\Omega,\mathcal{A},P)$ with expectation $\mathbb{E}$.

The smallest constant M in (2) is called the $\mathcal{R}$-*bound* of $\mathcal{T}$ and is denoted by $\mathcal{R}(\mathcal{T})$. A sectorial operator A is called $\mathcal{R}$-*sectorial* if the set

$$\{\lambda(\lambda+A)^{-1}:\ \lambda\in\Sigma_{\pi-\phi}\}\text{ is }\mathcal{R}\text{-bounded for some }\phi\in(0,\pi).$$

The infimum ϕ_A^R of such angles ϕ is called the $\mathcal{R}$-*angle* of A. We denote the class of $\mathcal{R}$-sectorial operators by $\mathcal{RS}(X)$. The relation $\phi_A^R\geq\phi_A$ is clear.

Finally, an operator $A\in\mathcal{H}^\infty(X)$ is said to admit an $\mathcal{R}$-*bounded* $\mathcal{H}^\infty$-calculus if the set

$$\{f(A):\ f\in\mathcal{H}^\infty(\Sigma_\phi),|f|_{\mathcal{H}^\infty(\Sigma_\phi)}\leq 1\}$$

is $\mathcal{R}$-bounded for some $\phi\in(0,\pi)$. Again, the infimum $\phi_A^{R\infty}$ of such ϕ is called the $\mathcal{RH}^\infty$-angle of A, and the class of such operators is denoted by $\mathcal{RH}^\infty(X)$.

We refer to the monograph of Denk, Hieber, and Prüss [3] for further information and background material, as well as to the preprint Prüss and Simonett [12] for the proofs of the results presented below.

2. The Main Result

In this section we formulate our main result for noncommuting operators. We first recall the commutator condition introduced by Da Prato and Grisvard [2]. Suppose that A and B are sectorial operators, defined on a Banach space X, and suppose that

$$(3)\qquad\left\{\begin{array}{l}0\in\rho(A).\text{ There are constants }c>0,\ \alpha,\beta>0,\ \beta<1,\ \alpha+\beta>1,\\ \psi_A>\phi_A,\ \psi_B>\phi_B,\ \psi_A+\psi_B<\pi,\\ \text{ such that for all }\lambda\in\Sigma_{\pi-\psi_A},\ \mu\in\Sigma_{\pi-\psi_B}\\ (\lambda+A)^{-1}D(B)\subset D(B)\text{ and}\\ |[B(\lambda+A)^{-1}-(\lambda+A)^{-1}B](\mu+B)^{-1}|\leq c/(1+|\lambda|)^\alpha|\mu|^\beta.\end{array}\right.$$

Then it was shown in [2] that the closure $L=\overline{A+B}$ is invertible, sectorial and $\phi_L\leq\max\{\psi_A,\psi_B\}$ holds, provided the constant c in (3) is sufficiently small.

A different, more flexible condition was later introduced by Labbas and Terreni [7]. It reads as follows.

$$(4)\qquad\left\{\begin{array}{l}0\in\rho(A).\text{ There are constants }c>0,\ \ 0\leq\alpha<\beta<1,\\ \psi_A>\phi_A,\ \psi_B>\phi_B,\ \psi_A+\psi_B<\pi,\\ \text{ such that for all }\lambda\in\Sigma_{\pi-\psi_A},\ \mu\in\Sigma_{\pi-\psi_B}\\ |A(\lambda+A)^{-1}[A^{-1}(\mu+B)^{-1}-(\mu+B)^{-1}A^{-1}]|\leq c/(1+|\lambda|)^{1-\alpha}|\mu|^{1+\beta}.\end{array}\right.$$

In Monniaux and Prüss [8], the Labbas-Terreni condition was employed to extend the Dore-Venni theorem to the noncommuting case. In particular, in this paper it is proved that $A+B$ with natural domain is closed and sectorial with spectral angle $\phi_{A+B}\leq\max\{\psi_A,\psi_B\}$ provided $X\in\mathcal{HT}$, $A,B\in\mathcal{BIP}(X)$, and (4) holds with a sufficiently small constant $c>0$. The Kalton-Weis theorem has been extended to the noncommuting case by Strkalj [14] provided the Labbas-Terreni conditions holds with sufficiently small $c>0$ and X is B-convex.

We are now in a position to state our main result.

Theorem 2.1. *Suppose* $A \in \mathcal{H}^\infty(X)$, $B \in \mathcal{RS}(X)$ *and suppose that* (3) *or* (4) *holds for some angles* $\psi_A > \phi_A^\infty$, $\psi_B > \phi_B^R$ *such that* $\psi_A + \psi_B < \pi$.
Then there is a constant $\nu \geq 0$ *such that* $\nu + A + B$ *is invertible and sectorial with* $\phi_{nu+A+B} \leq \max\{\psi_A, \psi_B\}$. *We may choose* $\nu = 0$ *if c is sufficiently small. Moreover, if in addition* $B \in \mathcal{RH}^\infty(X)$ *and* $\psi_B > \phi_B^{R\infty}$, *then* $\nu + A + B \in \mathcal{H}^\infty(X)$ *and* $\phi^\infty_{\nu+A+B} \leq \max\{\psi_A, \psi_B\}$.

3. Parabolic Equations on Wedges and Cones

In this section we consider an application of our main result to the diffusion equation on a domain of wedge or cone type, that is, on the domain $G = \mathbb{R}^m \times C_\Omega$, where $\Omega \subset S^{n-1}$ is open with smooth boundary $\partial\Omega \neq \varnothing$, and C_Ω denotes the cone

$$C_\Omega = \{x \in \mathbb{R}^n : \; x \neq 0, \; x/|x| \in \Omega\}.$$

We then consider the problem

$$\left\{ \begin{aligned} \partial_t u - \Delta u &= f && \text{in } G \times (0,T) \\ u &= 0 && \text{on } \partial G \times (0,T) \\ u|_{t=0} &= 0 && \text{on } G. \end{aligned} \right. \tag{5}$$

Here $m \in \mathbb{N}_0$ and $2 \leq n \in \mathbb{N}$. The function f is given in a weighted L_p-space, i.e.

$$f \in L_p(J \times \mathbb{R}^m; L_p(C_\Omega; |x|^\gamma dx)),$$

where $\gamma \in \mathbb{R}$ will be chosen appropriately, and $J = [0,T]$.

Introducing polar coordinates $x = r\zeta$ and using the Euler transform $r = e^\xi$, we obtain a problem in the standard unweighted space $X = L_p(J \times \mathbb{R}^m \times \Omega \times \mathbb{R})$ which reads

$$(\partial_t - \Delta_y)e^{2\xi}v + P(\partial_\xi)v - \Delta_\zeta v = g,$$

with some second order polynomial $P(z)$. Note that the first two operators do not commute. Applying our main result to this problem in X we obtain the following result, which originally is due to Nazarov [9].

Theorem 3.1. *Let* $1 < p < \infty$ *and suppose that* $\gamma \in \mathbb{R}$ *is subject to condition*

$$\lambda_1 > (2 - n/p - \gamma/p)(n - n/p - \gamma/p) \tag{6}$$

where $\lambda_1 > 0$ *denotes the first eigenvalue of the Laplace-Beltrami operator on* $\Omega \subset S^{n-1}$ *with Dirichlet boundary conditions.*
Then for each $f \in L_p(J \times \mathbb{R}^m; L_p(C_\Omega, |x|^\gamma dx)))$ *there is a unique solution u of (5) with regularity*

$$u, u/|x|^2, \partial_t u, \nabla^2 u \in L_p(J \times \mathbb{R}^m; L_p(C_\Omega, |x|^\gamma dx))).$$

The solution map $[f \mapsto u]$ *defines an isomorphism between the corresponding function spaces.*

For simplicity we have chosen the integrability exponent $p \in (1, \infty)$ to be the same for the variables t, x and y. By the arguments given above it also follows that we may choose different exponents for these variables, and we may arrange them in any order.

We note that the method described above can be applied to other problems on cone and wedge domains, like the Navier-Stokes equations, or free boundary value problems with moving contact lines and prescribed contact angles. These will be topics for our future work.

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Maximal regularity and quasilinear evolution equations

Herbert Amann

1. Abstract theory

Let E_0 and E_1 be Banach spaces such that $E_1 \overset{d}{\hookrightarrow} E_0$, set $J := J_{T_0} := [0, T_0)$ for some fixed positive T_0, and suppose that $1 < p < \infty$. Put

$$\mathbb{W}_p^1(J) := \mathbb{W}_p^1\big(J, (E_1, E_0)\big) := L_p(J, E_1) \cap W_p^1(\mathring{J}, E_0).$$

Then

$$B \in L_\infty\big(J, \mathcal{L}(E_1, E_0)\big)$$

possesses the property of **maximal L_p regularity** on J with respect to (E_1, E_0) if the map

$$\mathbb{W}_p^1(J) \to L_p(J, E_0) \times E, \quad u \mapsto \big(\dot{u} + Bu, u(0)\big)$$

is a bounded isomorphism, where E is the real interpolation space $(E_0, E_1)_{1/p',p}$ and the overdot denotes the distributional derivative on $\mathring{J}$. Since (e.g., [1, Theorem III.4.10.2])

$$\mathbb{W}_p^1(J) \hookrightarrow C(\overline{J}, E), \tag{1}$$

$u(0)$ is well defined. The set of all such maps B is denoted by

$$\mathcal{MR}_p(J) := \mathcal{MR}_p\big(J, (E_1, E_0)\big).$$

We also write $\mathcal{MR} := \mathcal{MR}(E_1, E_0)$ for the set of all $C \in \mathcal{L}(E_1, E_0)$ such that the constant map $t \mapsto C$ belongs to $\mathcal{MR}_p(J)$. Since the latter property is independent of p and the (bounded) interval (e.g., [3]), this notation is justified.

We are interested in quasilinear evolution equations of the form

$$\dot{u} + A(u)u = f(u) \text{ on } \mathring{J}, \qquad u(0) = u^0. \tag{2}$$

By a solution on J_T, where $0 < T \leq T_0$, we mean a $u \in \mathbb{W}_{p,\mathrm{loc}}^1(J_T)$ satisfying (2) in the sense of distributions on $\mathring{J}_T$ or, equivalently, a.e. on J_T.

Henceforth, we write C^{1-} for spaces of locally Lipschitz continuous maps, and $\mathcal{C}^{1-}$ if the Lipschitz continuity is uniform on bounded subsets of the domain (which is always the case if the latter is finite dimensional).

Due to (1) it is natural to assume that

$$(A, f) \in \mathcal{C}^{1-}\big(E, \mathcal{L}(E_1, E_0) \times E\big). \tag{3}$$

Indeed, this type of assumption has been used in practically all investigations of (2). In particular, Clément and Li [11] were the first to study (2) — in a concrete setting — by imposing the maximal regularity hypothesis that $A(e) \in \mathcal{MR}$ for each $e \in E$. Recently, Prüss [13] has extended this method to a nonautonomous abstract setting.

An assumption like (3) uses only part of the information contained in the statement: $u \in \mathbb{W}_p^1(J)$. Consequently, it imposes stronger restrictions on (A, f) than the hypothesis that this map be defined on $\mathbb{W}_p^1(J)$, which, after all, is the space in which solutions live.

Considering a map

$$(A, f) : \mathbb{W}_p^1(J) \to L_\infty\big(J, \mathcal{L}(E_1, E_0)\big) \times L_p(J, E_0)$$

we say that it possesses the Volterra property if, given $u \in \mathbb{W}_p^1(J)$ and $0 < T < T_0$, the restriction of $(A, f)(u)$ to J_T depends on $u \,|\, J_T$ only. Now we can formulate our main result, whose proof is found in [2].

Theorem *Suppose that*

- $A \in \mathcal{C}^{1-}\big(\mathbb{W}_p^1(J), \mathcal{MR}_p(J)\big)$;

- $f - f(0) \in \mathcal{C}^{1-}\big(\mathbb{W}_p^1(J), L_r(J, E_0)\big)$ *for some* $r \in (p, \infty]$, *and* $f(0) \in L_p(J, E_0)$;
- (A, f) *possesses the Volterra property;*
- $u^0 \in E$.

Then:

- *there exist a maximal* $T^* \in (0, T_0]$ *and a unique solution* u *of* (2) *on* $J^* := J_{T^*}$;
- *the map* $(A, f, u^0) \mapsto u$ *is locally Lipschitz continuous with respect to the natural Fréchet topologies of the spaces occurring above;*
- *if* $u \in \mathbb{W}_p^1(J^*)$, *then* $J^* = J$, *that is,* u *is global.*

The following proposition gives two important sufficient conditions for maximal regularity in the nonautonomous case.

Proposition (i) *If* $B \in C(\overline{J}, \mathcal{MR})$, *then* $B \in \mathcal{MR}_p(J)$.

(ii) *Let* $V \overset{d}{\hookrightarrow} H \overset{d}{\hookrightarrow} V'$ *be real Hilbert spaces and let* $B \in L_\infty\big(J, \mathcal{L}(V, V')\big)$ *be such that there exist constants* $\alpha > 0$ *and* $\beta \geq 0$ *with*

$$\big\langle v, B(t)v \big\rangle + \beta \|v\|_H^2 \geq \alpha \|v\|_V^2, \qquad a.a.\ t \in J, \quad v \in V,$$

where $\langle \cdot, \cdot \rangle : V \times V' \to \mathbb{R}$ *is the duality pairing. Then* $B \in \mathcal{MR}_2\big(J, (V, V')\big)$.

Proof (i) has been shown in [14] by constructing an evolution family. A simple direct proof is given in [3].

(ii) is a consequence of the well known Galerkin approach to evolution equations in a variational setting, essentially due to J.-L. Lions (see [2] for details). ■

2. Applications

To give an idea of the scope of the Theorem we consider two model problems. For this we suppose that

- Ω is a bounded Lipschitz domain;
- $a \in C^{1-}(\mathbb{R}, \mathbb{R})$ and $a(\xi) \geq \alpha > 0$ for $\xi \in \mathbb{R}$.

We also set $Q := \Omega \times J$ and $\Sigma := \partial\Omega \times J$.

Example 1 (nonlocal problems) Let $a_0, m \in L_\infty(\Omega)$ and $1 \leq \lambda < 1 + 4/n$. Denote by Ω' a measurable subset of Ω. Then the nonlocal parabolic problem

$$\begin{aligned} \partial_t u - \nabla \cdot \big(a\big(m{\textstyle\int_{\Omega'}} u(x, \cdot)\, dx\big)\nabla u\big) &= a_0\, |u|^{\lambda-1}\, u + f_0 && \text{on } Q, \\ u &= 0 && \text{on } \Sigma, \\ u(\cdot, 0) &= u^0 && \text{on } \Omega, \end{aligned}$$

has for each $f_0 \in L_2(Q)$ and $u^0 \in L_2(\Omega)$ a unique maximal weak solution u. If f_0 and u^0 are positive, then so is u. It is global if a is bounded and $a_0 \leq 0$. ■

By a maximal weak solution we mean, of course, a

$$u \in \mathbb{W}_{2,\mathrm{loc}}^1\big(J^*, \big(\mathring{H}^1(\Omega), H^{-1}(\Omega)\big)\big)$$

satisfying $u(0) = u^0$ and

$$\langle v, \dot{u} \rangle + \big\langle \nabla v, a\big(m \textstyle\int_{\Omega'} u(x,\cdot)\, dx\big) \nabla u \big\rangle = \big\langle v, a_0\, |u|^{\lambda-1}\, u + f_0 \big\rangle$$

a.e. on J^* and for every $v \in \mathcal{D}(\Omega)$.

We mention that an application of the results in [13], based on hypothesis (3), would require $\lambda = 1$.

Problems of this type have been intensively studied by M. Chipot and coworkers (cf. [4]–[10] and [15]–[18]). More precisely, in those papers the differential equations are either of the form

$$\partial_t u - a\big(\langle v, u \rangle\big) \Delta u = f_0,$$

where $v \in L_2(\Omega)$, or they are semilinear with nonlocal lower order terms. (The Laplace operator can be replaced by a general second order elliptic operator.) It is crucial that $a\big(\langle v, u(\cdot,t) \rangle\big)$ is a pure function of t, that is, independent of $x \in \Omega$. The proofs, except the ones in [18], rely on Schauder's fixed point theorem and are completely different from our approach.

Example 2 (equations with memory) Assume that Ω has a C^2 boundary, that $b, f \in C^{1-}(\mathbb{R}, \mathbb{R})$, that $k \in L_\rho(\mathbb{R}^+, \mathbb{R})$ for some $\rho > 1$, and μ is a bounded Radon measure on $[0, \infty)$ with $\operatorname{supp} \mu \subset [0, S)$ for some $0 < S \le \infty$. Also suppose that $2/p + n/q < 1$. Then

$$\begin{aligned} \partial_t u - \nabla \cdot \big(a(\mu * u) \nabla u\big) + k * \big(\nabla \cdot (b(u) \nabla u)\big) &= f(u) + f_0 && \text{on } Q, \\ u &= 0 && \text{on } \Sigma, \\ u &= \overline{u} && \text{on } \Omega \times (-S, 0], \end{aligned}$$

has for each $f_0 \in L_p\big(J, H_q^{-1}(\Omega)\big)$ and each

$$\overline{u} \in \mathbb{W}_p^1\big((-S,0), \big(\mathring{H}^1(\Omega), H^{-1}(\Omega)\big)\big)$$

a unique maximal weak solution

$$u \in \mathbb{W}_{p,\mathrm{loc}}^1\big((-S,T^*), \big(\mathring{H}_q^1(\Omega), H_q^{-1}(\Omega)\big)\big). \tag{4}$$

If there exists $r > 0$ such that $\operatorname{supp} \mu \subset [r, S)$, then the unique maximal weak solution in (4) of

$$\partial_t u - \nabla \cdot \big(a(\mu * u) \nabla u\big) = f_0 \quad \text{on } Q, \qquad u = 0 \quad \text{on } \Sigma$$

with $u \,|\, (-S, 0] = \overline{u}$ is global. ■

Choosing for μ the Dirac measure supported on $\{r\}$ for some $r > 0$, it follows that the retarded quasilinear parabolic problem

$$\begin{aligned} \partial_t u - \nabla \cdot \big(a\big(u(t-r)\big) \nabla u\big) &= f_0 && \text{on } Q, \\ u &= 0 && \text{on } \Sigma, \end{aligned}$$

has for each $f_0 \in L_p\big(J, H_q^{-1}(\Omega)\big)$ and each $\overline{u} \in \mathbb{W}_p^1\big((-S,0], \big(\mathring{H}_q^1(\Omega), H_q^{-1}(\Omega)\big)\big)$, where $S > r$, a unique global weak solution in (4) (with $T^* = T$).

It should be remarked that problems like the one of Example 2 cannot be treated at all by theorems invoking hypotheses of type (3).

There is a large literature on parabolic equations involving delays and memory terms. However, most of it concerns semilinear equations. Very little seems to be known about an L_p theory for quasilinear equations with memory terms in the top order part (see [12] and the references therein, and [19]). In fact, we do not know of any result for quasilinear equations in which (nondistributed) delay terms occur within the diffusion matrix.

For proofs of the above facts and many more examples we refer to [2].

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