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Subfactors and Applications

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Abstract. The theory of subfactors connects diverse topics in mathematics and mathematical physics such as tensor categories, vertex operator algebras, quantum groups, quantum topology, free probability, quantum field theory, conformal field theory, statistical mechanics, condensed matter physics and, of course, operator algebras. We invited an international group of researchers from these areas and many fruitful interactions took place during the workshop.

Mathematics Subject Classification (2010): 17B37, 17B69, 18D10, 46L37, 46L60, 47L90, 81R15, 81T05, 81T40.

Introduction by the Organizers

Subfactor theory was initiated by Jones in the 1980's as a Galois type theory for operator algebras. His work led to the stunning discovery of Jones' surprising invariant for knots and links, the Jones polynomial. Since then, we have seen numerous unexpected and deep connections of subfactor theory to low-dimensional topology, quantum groups, quantum field theory and statistical mechanics. It was our aim to bring researchers and students in these different areas in mathematics and physics to Oberwolfach and create new interactions. For several of the junior participants it was the first time that they attended an Oberwolfach workshop. The event was highly successful in stimulating new interactions among all attendees. At least one participant said that of the many workshops he has attended at Oberwolfach, this was the most outstanding.

What follows is a more detailed description of the key topics central to the workshop.

(1) Subfactors and tensor categories
The language of tensor categories has proven to be highly effective in dealing with algebraic and combinatorial aspects of subfactor theory. In fact, the definitions of fusion and module categories were abstracted from the subfactor literature. Connections to conformal field theory, vertex operator algebras, quantum groups and quantum topology have also been well-studied in this context. Some talks at the workshop were about algebraic structures of tensor categories, and others were related to their operator algebraic realizations. Many new examples of tensor categories and formulations of structure originating in subfactor theory have been investigated. Arano, Giorgetti, Grossman, Henriques, Liu, Morrison, Neshveyev, Penneys, Plavnik, Ren, Snyder, and Tomatsu gave talks on these topics.

(2) Conformal field theory and vertex operator algebras
Algebraic quantum field theory is an operator algebraic approach to quantum field theory. Its version for a chiral half of 2-dimensional conformal field theory has been particularly successful. Another mathematical formulation of chiral conformal field theory is given by the theory of vertex operator algebras and the relations between the two approaches have caught much attention in the last ten years or so. Bischoff, Carpi, Huang, McRae, Osborne, Tanimoto, Tener and Wang discussed various aspects of conformal field theory.

(3) Other types of quantum field theories
More general quantum field theory and topological quantum field theory have been pursued by several investigators over the last few years. Runkel, Schweigert and Teschner presented their research results in this direction.

(4) Quantum groups
Quantum groups have been powerful tools to study new "symmetries" in a large body of mathematics. The Drinfel′d-Jimbo formulation is highly influential, and there is an operator algebraic formulation due to Woronowicz. Lechner, Valvekens, Wenzl and Makoto Yamashita talked about topics related to quantum groups.

(5) Other topics
We had a few talks on a related, but wider range of topics. Brothier gave a talk on Thompson's groups. They arise as symmetry groups in Jones' attempt to construct a continuum limit CFT from discrete structures. Hartglass discussed connections to free probability. There is an interesting Fock space construction that leads to natural free Araki-Woods factors. Ruth presented her work on discrete groups and von Neumann algebras, especially rigidity properties of ergodic, measure-preserving actions of certain classes of groups that includes certain lattices in Lie groups. Ogata talked on her studies on quantum spin chains. Reutter presented his results on biunitary connections and quantum information theory. Mayuko Yamashita gave a lecture on geometric quantization.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1641185, “US Junior Oberwolfach Fellows”. Moreover, the MFO and the workshop organizers would like to thank the Simons Foundation for supporting Yoshiko Ogata in the “Simons Visiting Professors” program at the MFO.

Workshop: Subfactors and Applications

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Abstracts

Haagerup CFT: A microscopic approach

Tobias J. Osborne
(joint work with Jacob Bridgeman, Alexander Hahn, Ash Milsted, and Ramona Wolf)

A fascinating open problem is to determine if, corresponding to every subfactor, there is a counterpart conformal field theory (CFT). There is already some promising evidence for this conjecture, with considerable focus falling on the Haagerup-type subfactors, for which there are currently no known counterpart CFTs. Since subfactors give rise to unitary fusion categories with algebra object, one can imagine attempting to construct counterpart CFTs via physical models built directly arising from these categories. In this talk, I report on progress generalising the construction given in the paper [1] where this technique was successfully applied to a simple example, namely Fibonacci anyons. I will employ this approach to build microscopic models of CFTs from fusion category data via such *anyon chains*. Such a model may be formulated for the H3 fusion category (corresponding to the Haagerup subfactor). Furthermore, I explain how it can be used to search for a critical model constructed from fusion categories that correspond to the Haagerup subfactor, and report on several numerical investigations we have done in this direction. I will report on the methods used to study such models and our (so far negative) progress in extracting a nontrivial CFT from the Haagerup chain.

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String nets and invariants of mapping class groups

Christoph Schweigert and Yang Yang

It is well-known that a spherical fusion category $\mathcal{A}$ allows to construct an extended oriented three-dimensional topological field theory. Such a theory assigns to a circle a modular tensor category, the Drinfeld center $\mathcal{Z}(\mathcal{A})$, to an oriented surface with boundary a vector space, depending functorially on the objects in $\mathcal{Z}(\mathcal{A})$ assigned to the boundary components, and to a three-manifold a linear map. In particular, mapping cylinders lead to representations of mapping class groups of surfaces.

In the Barret-Turaev-Viro-Westbury construction, the vector spaces assigned to surfaces are obtained as subspaces of auxilliary vector spaces. The string-net model provides an alternative construction of these vector spaces, as a quotient of a vector space freely generated by finite labelled graphs on the surface. The edges of the graphs are labelled by objects of the fusion category $\mathcal{A}$, the vertices by suitable morphisms of $\mathcal{A}$. The quotient is by a subspace of *null graphs* which are defined by local relations. For expositions of this construction addressing mathematicians, we refer to [4, 3].

The string-net approach provides two advantages that are crucial for the purpose of this talk: on the one hand side, specific elements in the vector spaces assigned to surfaces can be described by labelled graphs. On the other hand, the action of the mapping class group is completely geometric: it maps graphs to graphs (and preserves the null graphs, since they are defined by local relations).

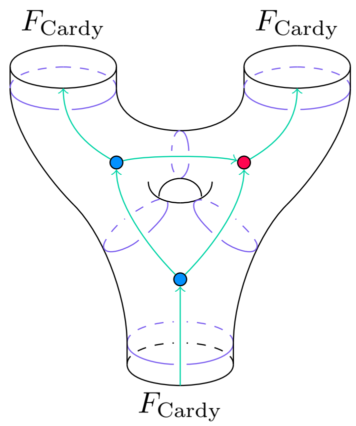
The string-net construction is defined for a spherical fusion category $\mathcal{A}$, i.e. a braiding on $\mathcal{A}$ is not required for the construction. We apply the string-net construction to the case where $\mathcal{A}$ is braided and even a modular tensor category. This category should be imagined as describing the chiral data of a two-dimensional rational conformal field theory. The fact that $\mathcal{A}$ is modular is equivalent to the statement that the Drinfeld center $\mathcal{Z}(\mathcal{A})$ is braided equivalent to $\mathcal{A}^{\mathrm{rev}} \boxtimes \mathcal{A}$. Bulk fields, where left and right movers are combined, are objects in this category. Their operator product leads to a Lagrangian algebra in $\mathcal{Z}(\mathcal{A})$, i.e. a commutative symmetric Frobenius algebra with a certain maximality property.

Denote by $(U_i)_{i\in I}$ representatives for the isomorphism classes of simple objects of $\mathcal{A}$. The object $L := \oplus_{i\in I} U_i^\vee \otimes U_i$ has a natural half-braiding that turns it into an object $F_{\mathrm{Cardy}} \in \mathcal{Z}(\mathcal{A})$. This object is known to have a natural structure of a Lagrangian algebra in $\mathcal{Z}(\mathcal{A})$.

Our aim is to give a consistent system of correlators, i.e. roughly speaking, to assign to any surface Σ with boundaries an element in the corresponding string-net space that is invariant under the action of the mapping class group. Our main result is that this can be achieved by the following string-net:

Theorem. [CS, YY, 2019 [5]] The following string-net describes an invariant of the mapping class group: choose a pair of pants decomposition for Σ; label the cutting lines and additional lines placed parallel to the boundary components by the canonical color $\sum_{i\in\mathcal{I}} \dim(U_i)U_i$. Choose a point in the interior of each pair of pants, on each boundary component and on each cutting line. Connect the point in the interior of a pair of pants to the three adjacent points on boundary components or cutting lines. Label these lines by F_{Cardy} and the points inside a pair of pants by the appropriate structure morphism of the Frobenius algebra F_{Cardy}, i.e. by a multiplication or a comultiplication.

We illustrate this string-net in the following example of a surface of genus one with three boundary components. Lines labelled by the canonical color are in purple, lines labelled by the Frobenius algebra F_{Cardy} in green:



The proof of this theorem uses a Lego-Teichmüller game (see [1, 2]). It is remarkable that the local relations in a string-net model allow to implement all moves of the Lego-Teichmüller game.

We finally also presented a conceptual explanation of our results: the Frobenius algebra F_{Cardy} turns out to be isomorphic in a non-trivial way to another Frobenius algebra. With this Frobenius algebra, the string-nets can be simplified in such a form that they are manifestly invariant under the action of the mapping class group. For details, we refer to [5].

Our results call for generalizations: to more general Lagrangian algebras in $\mathcal{Z}(\mathcal{A})$ and to the description of correlators of fields other than bulk fields, i.e. boundary or defect fields. Finally, it is tempting to raise the question in which way the string-net construction can be generalized to spherical finite tensor categories that are not necesesarily semisimple.

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Subfactors and unitary R-matrices

Gandalf Lechner

The Yang-Baxter equation is a cubic equation for a linear map $R \in V \otimes V \to V \otimes V$ on the tensor square of a vector space V, namely

$$\text{(YBE)} \qquad (R \otimes 1)(1 \otimes R)(R \otimes 1) = (1 \otimes R)(R \otimes 1)(1 \otimes R),$$

where 1 is the identity on V. This equation and its variants come from quantum physics, but also play a central role in various branches of mathematics, for instance in knot theory, quantum groups/Hopf algebras, and braid groups. Further recent interest in the solutions of the YBE stems from topological quantum computing.

Despite this widespread interest in the YBE, no satisfactory understanding of its solutions has been reached. In this talk, a new approach to the YBE was presented, based on operator algebras and subfactors [2]. We restrict to the case of most interest in applications, namely the case where V is a finite-dimensional Hilbert space and R is unitary. Such "R-matrices" exist in any dimension $d = \dim V$, simple examples being the identity 1 on $V \otimes V$, the tensor flip $F(v \otimes w) = w \otimes v$, diagonal R-matrices, and Gaussian R-matrices. The (unknown) set of *all* R-matrices of dimension d is denoted $\mathcal{R}(d)$.

The general strategy of our approach is to start from an arbitrary R-matrix $R \in \mathcal{R}(d)$ with base space V and derive operator-algebraic data (such as endomorphisms, subfactors, indices) from it that inform us about R. The main structural elements of our approach can be summarized in the following diagram:

$$(*) \qquad \begin{array}{ccccc} \varphi(\mathcal{N}) & \subset & \mathcal{N} & & \\ \cup & & \cup & & \\ \varphi(\mathcal{L}_R) & \subset & \mathcal{L}_R & \overset{\rho_R}{\longleftarrow} & B_\infty \\ \cap & & \cap & & \\ \lambda_R(\mathcal{N}) & \subset & \mathcal{N} & & \end{array}$$

Starting at the top of the diagram, $\mathcal{N}$ is the hyperfinite II_1 factor realised as an infinite tensor product $\mathcal{N} = \bigotimes_{n\geq 1} \mathrm{End}V$, weakly closed w.r.t. the normalised trace $\tau = \bigotimes_{n\geq 1} \frac{\mathrm{Tr}_V}{d}$, and equipped with the shift $\varphi : \mathcal{N} \to \mathcal{N}$, $\varphi(x) = 1 \otimes x$. We identify finite tensor powers $\mathrm{End}\, V^{\otimes n}$ with their natural embeddings into $\mathcal{N}$, so that $R \in \mathcal{N}$ and the YBE reads $\varphi(R)R\varphi(R) = R\varphi(R)R$.

The second line of the diagram is about the braid group structure: As is well known, any $R \in \mathcal{R}(d)$ defines a group homomorphism ρ_R from the infinite braid group B_∞ into the unitary group of $\mathcal{N}$ by mapping the standard generators b_n, $n \in \mathbb{N}$, of B_∞ to $\varphi^{n-1}(R) \in \mathcal{N}$. The von Neumann algebra generated by this representation is denoted $\mathcal{L}_R$.

The third line of the diagram introduces the *Yang-Baxter endomorphism* $\lambda_R \in \mathrm{End}\,\mathcal{N}$. It is defined in such a way that it restricts to the shift φ on $\mathcal{N}$. Explicitely,

$$(**) \qquad \lambda_R : \mathcal{N} \to \mathcal{N}, \qquad \lambda_R(x) := \underset{n\to\infty}{\text{w-lim}}\, R \cdots \varphi^n(R) x \varphi^n(R^*) \cdots R^*.$$

This definition is natural also from the point of view of the Cuntz algebra[1]. As particular examples, we note that the identity R-matrix gives the identity endomorphism, $\lambda_1 = \mathrm{id}_{\mathcal{N}}$, and the flip F gives the canonical endomorphism, $\lambda_F = \varphi$.

Let us list a few results from [2] (joint work with Roberto Conti):

(1) $\mathcal{L}_R$ is a factor (II_1 for non-trivial R). This provides us with three subfactors (I) $\lambda_R(\mathcal{N}) \subset \mathcal{N}$, (II) $\varphi(\mathcal{L}_R) \subset \mathcal{L}_R$, and (III) $\mathcal{L}_R \subset \mathcal{N}$ derived from R.

[1] Viewing $R \in \mathcal{R}(d)$ as a unitary in $\mathcal{O}_d$ yields a canonically associated endomorphism λ_R of $\mathcal{O}_d$. This endomorphism gives $(**)$ by extension to a type $\mathrm{III}_{1/d}$ factor $\mathcal{M} \supset \mathcal{N}$ and restriction.

(2) Subfactors (I),(II) have always finite index $\leq d^2$, but (III) may have infinite index. Its relative commutant coincides with the fixed point algebra $\mathcal{N}^{\lambda_R}$.
(3) The subfactors (I), (II) can be iterated by taking powers of λ_R and φ, respectively. One has $R \in \varphi^2(\mathcal{L}_R)' \cap \mathcal{L}_R \subset \lambda_R^2(\mathcal{N})' \cap \mathcal{N}$. Hence, for any non-trivial R-matrix, λ_R^2 is reducible and λ_R is not an automorphism [1].
(4) Both squares in $(*)$ are commuting squares. Denoting the τ-preserving conditional expectation $\mathcal{N} \to \lambda_R(\mathcal{N})$ by E_R, and the associated left inverse of λ_R by $\phi_R := \lambda_R^{-1} \circ E_R$, this implies $\phi_R(x) = \phi_F(x)$, $x \in \mathcal{L}_R$.

An interesting object to consider is $\phi_R(R)$. This is an element of $\varphi(\mathcal{L}_R)' \cap \mathcal{L}_R$, which thanks to (4) coincides with the (normalised) left partial trace $\phi_F(R)$ of R. We therefore have explicit elements of the relative commutant, and a connection from operator-algebraic structures to concrete properties of R. One finds [2]:

(5) Let $R \in \mathcal{R}$. Then the left and right partial traces of R coincide and are normal elements of $\operatorname{End} V$.
(6) Define the *character* τ_R of an R-matrix as the map $\tau_R : B_\infty \to \mathbb{C}$, $\tau_R := \tau \circ \rho_R$. If two R-matrices $R, S \in \mathcal{R}(d)$ have the same character, then $\phi_R(R)$ and $\phi_S(S)$ are unitarily equivalent.
(7) Any R-matrix with spectrum contained in a disc of radius less than $1 - 2^{-1/4}$ is trivial[2].

Item (6) suggests to consider R-matrices up to the natural equivalence relation $R \sim S$ given by coinciding characters and dimensions of R-matrices. Then $\phi_R(R)$ is an invariant for $\sim$, and in the involutive case ($R^2 = 1$), it is even a *complete* invariant: $R \sim S \iff \phi_R(R) \cong \phi_S(S)$ [4]. In the general non-involutive case, the partial trace is not a complete invariant.

As the last section in this overview, let us consider the problem of *classifying* all R-matrices up to the equivalence $\sim$ and announce some results from the upcoming article [5]. We consider here the case that the spectrum of R has cardinality 2, and normalise it to $\sigma(R) = \{-1, q\}$, $|q| = 1$, $q \neq -1$. In this situation, the representation ρ_R factors through the Hecke algebra $H_\infty(q)$, and we moreover have [5]:

(8) If $R \in \mathcal{R}(d)$ has no two opposite eigenvalues $\mu, -\mu$ in its spectrum, then $\varphi(\mathcal{L}_R) \subset \mathcal{L}_R$ is irreducible and τ_R is a (positive) Markov trace.

Hence for $q \neq 1$, any R-matrix gives a positive Markov trace on $H_\infty(q)$. We may therefore use Wenzl's classification of positive Markov traces on $H_\infty(q)$ [6]. Recall that his results state in particular that for a positive Markov trace to exist, one must have $q \in \{1, e^{2\pi i/\ell} : \ell \in \{4, 5, \ldots\}\}$, and at fixed ℓ, there exist finitely many possible Markov traces. In our Yang-Baxter setting, these possibilities are severely restricted [5]:

[2]This result has its origin in an estimate on the Jones index $[\mathcal{N} : \lambda_R(\mathcal{N})]$ in terms of $\phi_R(R)$.

(9) Let R be an R-matrix with spectrum $\{-1, q\}$, $q \neq 1$, and eigen projection P for the eigenvalue -1. Then $q \in \{\pm i, e^{i\pi/3}\}$. If $q = \pm i$, then $\tau(P) = \frac{1}{2}$, and if $q = e^{\pm i\pi/3}$, then $\tau(P) = \{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\}$. Two such R-matrices R, S are equivalent (in the sense of $\sim$) iff they have the same spectrum (q), dimension (d), and trace ($\tau(P)$).

The above result does *not* imply that all the possible combinations of eigenvalues q and traces $\tau(P)$ are indeed realised. We have found explicit R-matrices realising the combinations ($q = \pm i$, $\tau(P) = \frac{1}{2}$), ($q = e^{i\pi/3}$, $\tau(P) = \frac{1}{3}$), ($q = e^{i\pi/3}$, $\tau(P) = \frac{2}{3}$) and conjecture that the last possibility, ($q = e^{i\pi/3}$, $\tau(P) = \frac{1}{2}$), is not realised by any R-matrix. This is in line with observations made by Galindo, Hong, and Rowell [3], but so far no proof of this conjecture exists.

It is instructive to compare these findings with the situation at $q = 1$, which is completely different. For $q \neq 1$, we always have irreducible $\varphi(\mathcal{L}_R) \subset \mathcal{L}_R$, and the equivalence takes a simple form (it is given by the three parameters d, q, $\tau(P)$). For $q = 1$, on the other hand, $\varphi(\mathcal{L}_R) \subset \mathcal{L}_R$ is reducible except for the special cases $R \sim \pm 1, \pm F$, and the equivalence is more involved (it is given by the unitary equivalence class of $\phi_R(R)$). The case $q = 1$ corresponds to R being involutive and ρ_R factoring through the infinite *symmetric* group. In that case, a complete and explicit classification of R-matrices up to equivalence exists: R-matrices are parameterised by pairs of Young diagrams with d boxes in total, corresponding to the positive and negative eigenvalues of $\phi_R(R)$ [4]. We also mention that in this case, the index $[\mathcal{L}_R : \varphi(\mathcal{L}_R)]$ is a rational typically non-integer number.

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From quantum teleportation to biunitary connections (and back)

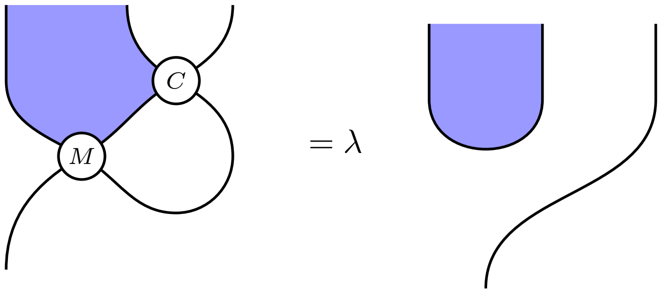
David Reutter

(joint work with Jamie Vicary)

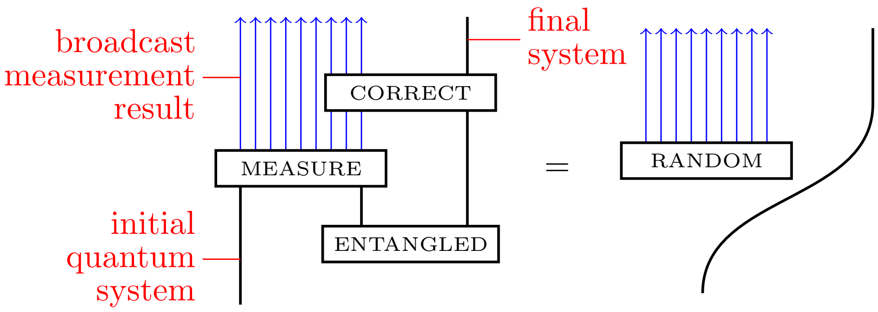
In my talk, I summarized joint work with Jamie Vicary on using techniques and structures developed in subfactor theory and planar algebra in quantum information theory. Most of my talk was based on our joint paper [8].

The basic premise of this work is that various protocols and constructions in quantum information theory can naturally be expressed in the setting of 2-categories (for more details, see [10, 11]). For example, a *quantum teleportation protocol* in a (dagger pivotal) 2-category is a pair of unitary 2-morphisms M and C, fulfilling

(1)

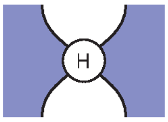


where λ is some scalar factor. Reading from bottom to top, such 2-categorical diagrams admit a direct physical interpretation in terms of basic quantum information theoretic processes:



To recover ordinary quantum information theory, we interpret these 2-categorical diagrams in the 2-category 2Hilb of 2-Hilbert spaces. The objects of this 2-category are finite-dimensional 2-Hilbert spaces (that is, $\mathbb{C}$-linear dagger category which are equivalent, as $\mathbb{C}$-linear dagger categories, to f.d.Hilb^n for some n), the 1-morphisms are $\mathbb{C}$-linear dagger functors and the 2-morphisms are natural transformations. In 2Hilb, the data of a pair of unitary 2-morphisms M and C fulfilling (1) precisely corresponds to the linear algebraic data describing quantum teleportation protocols in convential quantum information theory (as classified by Werner in [12]). On a fixed finite-dimensional Hilbert space H, this data is given by a *unitary error basis* [4]; a family of n^2 unitary matrices $\{U_i \in \text{End}(H)\}_{i=1}^{n^2}$ which form an orthogonal basis of $\text{End}(H)$ with the Hilbert-Schmidt inner product $\frac{1}{n}\text{Tr}(U_i^\dagger U_j) = \delta_{i,j}$.

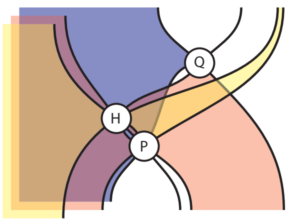
Biunitary connections. Direct graphical manipulation shows that a unitary 2-morphism C is part of a quantum teleportation protocol (1) if and only if it is unitary and if its 'quarter rotation' is unitary, up to some scalar factor. Such 'biunitary 2-morphisms' (or 'biunitary connections') were originally introduced by Ocneanu in the 90s [7] and have become a central tool in the classification of subfactors [2, 3]. Therefore, we can characterized teleportation protocols, or equivalently unitary error bases, as biunitaries in 2Hilb of the type of C in (1). Similarly, complex Hadamard matrices (unitary matrices all of whose entries have the same modulus) can be characterized as biunitaries in 2Hilb of the following type:



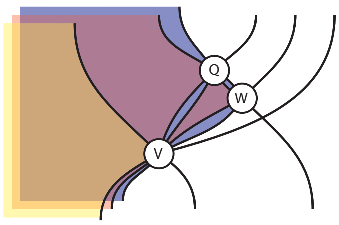
Expressed in the language of spin model planar algebras, this observation is due to Jones [2].

Hadamard matrices and unitary error bases provide the mathematical foundation for an extremely rich variety of quantum computational phenomena, amongst them the study of mutually unbiased bases, quantum key distribution, quantum teleportation, dense coding and quantum error correction [1, 4, 5, 12]. Nevertheless, their general structure is notoriously difficult to understand; in dimension n, Hadamard matrices have only been classified up to $n = 5$ (see e.g. [9]), and the general structure of unitary error bases is virtually unknown for $n > 2$.

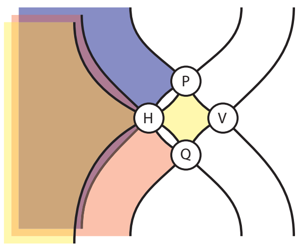
Given the description of quantum structures in terms of biunitaries as summarized above, one can immediately write down a large number of schemes for the construction of certain quantum structures from others, many of which are not previously known. This is based on the simple fact that the *diagonal* composite of two biunitaries is again biunitary. We give some examples[1] below; interpreting these diagrams in 2Hilb results in the explicit tensorial expression written below each diagram. Note that the biunitaries are connected diagonally in each case, as required.



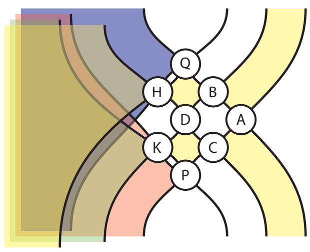
$U_{abc,de,fg} = H^{b,c}_{a,eg} P^{c,g}_{e,b,f}\, Q_{c,g,d}$



$U_{abc,def,gh} := \sum_r V^{b,c}_{a,rf,g} Q^{c}_{b,r,d} W_{rc,e,h}$



$U_{abc,de,fg} = \sum_r H^{b,c}_{a,r} P_{c,r,d} Q_{r,b,f} V_{r,e,g}$



$U_{abcd,ef,gh} = \frac{1}{n} \sum_{r,s} A_{f,h} B_{s,f} C_{r,h} D_{s,r} H^{d}_{a,s} K^{c}_{b,r} Q_{d,s,e} P_{r,c,g}$

[1] In these examples, we also use 'quantum Latin squares' [6], another type of biunitary connection whose shading pattern can for example be seen in the node P in the 4-fold composite.

Correctness of these constructions follows immediately from diagonality of the composition; no further details need to be checked. Our approach therefore offers advantages even for those constructions that are already known, since the traditional proofs of correctness are nontrivial.

In [8], we use the 4-fold composite from above to produce a unitary error basis on an 8-dimensional Hilbert space and show that it cannot be produced by any known constrution method. This is a proof of principle that the biunitary methods we propose can give rise to genuinely new quantum structures.

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Strong locality beyond linear energy bounds

Yoh Tanimoto

From a Haag-Kastler net, various subfactors arise. If there is a representation of a conformal net, one can construct the Jones-Wassermann subfactor. Alternatively, if there is an extension or a subtheory, each local algebra gives directly a subfactor. Therefore, while constructing new examples of Haag-Kastler net is an important problem by itself, it is also interesting from the point of view of subfactor.

Most examples are constructed from quantum fields, namely operator-valued distributions. A quantum field $\phi(x)$ (a Wightman field, or a field in a vertex operator algebra satisfying a polynomial energy bound) should satisfy locality: if f, g are test functions with spacelike separated supports, then $\phi(f), \phi(g)$ should commute (on a suitable domain). In order to construct a Haag-Kastler net, we need **strong locality**: $\phi(f), \phi(g)$ should strongly commute (their spectral projections commute). In various examples, strong locality is the last technical barrier to obtain a Haag-Kastler net.

The most commonly used tool to show strong locality is Driessler-Fröhlich theorem: if A and B commute on the domain of a H and satisfies $\|A\Psi\| \le C\|H\Psi\|$ with $C > 0$ and $[H, A]$ can be estimated as a sesquilinear form by H, and similar estimates hold for B, then A and B strongly commute. When H is the "Hamiltonian" satisfying $[H, \phi(f)] = i\phi(f')$, the bound $\|\phi(f)\Psi\| \le C\|H\Psi\|$ for all Ψ suffices. This is called a linear energy bound.

Linear energy bound does not apply to some interesting models, yet we can prove strong commutativity by a trick using the Driessler-Fröhlich theorem. Indeed, in this way we construct two new families of Haag-Kastler net: one is a chiral conformal net (the $\mathcal{W}_3$**-algebra**) and the other is a two-dimensional integrable QFT (the **Bullough-Dodd model**).

- The $\mathcal{W}_3$-algebra [3]. This is an extension of the Virasoro algebra as a vertex operator algebra. It is generated by the Virasoro field $L(z)$ and an additional field $W(z)$, satisfying the commutation relations

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}, \\
[L_m, W_n] &= (2m-n)W_{m+n}, \\
[W_m, W_n] &= \frac{c}{3\cdot 5!}(m^2-4)(m^2-1)m\delta_{m+n,0} \\
&\quad + b^2(m-n)\Lambda_{m+n} + \left[\frac{1}{20}(m-n)(2m^2-mn+2n^2-8))\right] L_{m+n},
\end{aligned}$$

where $\Lambda_n = \sum_{k>-2} L_{n-k}L_k + \sum_{k\le -2} L_k L_{n-k} - \frac{3}{10}(n+2)(n+3)L_n$ and $c \in \mathbb{C}, c \ne -\frac{22}{5}, b^2 = \frac{16}{22+5c}$. The $W(z)$ field has conformal dimension 3, and from this it follows that it cannot satisfy the linear energy bound, where $H = L_0$ is the conformal Hamiltonian. When the central charge c is larger than 2, then no coset realization is known.

We proved in [2] that for $c \ge 2$ the vacuum representation is unitary, hence the fields $L(z), W(z)$ are good candidates for quantum fields. The commutator $[W(z), W(w)]$ contains a non-linear expression in $L(z)$, and from this we can have a local energy bound: for $f \ge 0$, $\|W(f^{d-1})\Psi\| \le C\|(L(f) + c_f\mathbb{1})^{d-1}\|$. Note that, from the commutation relations, we have $[W(f^{d-1}), L(f)] = 0$. Then we can apply the Driessler-Fröhlich theorem with $H = (L(f) + L(g) + c\mathbb{1})^{d-1}$ for non-negative f, g. and obtain strong locality.

- The Bullough-Dodd model. This is a variation of the models of [4], where the S-matrix S contains poles at $\zeta = \frac{\pi i}{3}, \frac{2\pi i}{3}$:

$$S(\theta) = \frac{\tanh \frac{1}{2}\left(\theta + \frac{2\pi i}{3}\right)}{\tanh \frac{1}{2}\left(\theta - \frac{2\pi i}{3}\right)} \cdot \frac{\tanh \frac{1}{2}\left(\theta - \left(\frac{\pi}{3} - \epsilon\right) i\right)}{\tanh \frac{1}{2}\left(\theta + \left(\frac{\pi}{3} + \epsilon\right) i\right)} \frac{\tanh \frac{1}{2}\left(\theta - \left(\frac{\pi}{3} + \epsilon\right) i\right)}{\tanh \frac{1}{2}\left(\theta + \left(\frac{\pi}{3} - \epsilon\right) i\right)}$$

where $0 < \epsilon < \frac{\pi}{6}$. The S-symmetric Fock space is the Fock space based on $L^2(\mathbb{R})$ with the symmetry $\Psi(\theta_1, \theta_2) = S(\theta_2 - \theta_1)\Psi(\theta_2, \theta_1)$. With the (left-)creation and annihilation operator $z^\dagger, z$, one may form an operator $\phi(f) = z^\dagger(f) + z(f)$. This operator is localized in the left wedge if S is analytic [4], but for the S above we need to correct: $\tilde{\phi}(f) = \phi(f) + \chi(f)$, where $\chi(f) = \sum nP_n(\chi_1(f) \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1})P_n$, P_n is the projection onto the n-particle space and with $R = \operatorname*{Res}_{\zeta = \frac{2\pi i}{3}} S(\zeta)$

$$(\chi_1(f))\Psi(\theta) = 2\pi\sqrt{|R|} f^+(\theta + \tfrac{\pi i}{3})\Psi(\theta - \tfrac{\pi i}{3})$$

and f^+ is the one-particle function corresponding to a test function f.

We find a nice self-adjoint extension of $\chi(f)$. Similarly, $\tilde{\phi}'(g), \chi'(g)$ can be constructed for reflected wedges, Strong commutativity between $\tilde{\phi}(f), \tilde{\phi}'(g)$ can be shown by applying Driessler-Fröhlich theorem to $H = \tilde{\phi}(f) + \tilde{\phi}(g) + C(f,g)N$, where $C(f,g) > 0$ and N is the number operator. For the self-adjointness of H which follows from that of $\chi(f)$, we use a variation of the Kato-Rellich perturbation.

We also prove the existence of operators in (large enough) double cones, which completes the construction of Haag-Kastler net.

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Rigid C^*-tensor categories and discrete quantum groups with property (T)

Matthias Valvekens
(joint work with Stefaan Vaes)

Rigid C^*-tensor categories arise naturally in the study of subfactors as one of the various ways to axiomatise the standard invariant of a finite-index subfactor. Roughly speaking, a rigid C^*-tensor category is a monoidal C^*-category $\mathcal{C}$ with simple tensor unit ε where any object $\alpha \in \mathcal{C}$ admits a canonical conjugate $\overline{\alpha}$. Moreover, the structure morphisms implementing the left and right duality for α and $\bar{\alpha}$ should be compatible with the dagger structure on $\mathcal{C}$. We refer to [NT13, Chapter 2] for a comprehensive introduction to the subject. Concretely, any finite-index inclusion of II_1-factors $N \subset M$ gives rise to a rigid C^*-tensor category $\mathcal{C}(N \subset M)$ by taking all M-M-subbimodules of bimodules of the form $L^2(M) \otimes_N \cdots \otimes_N L^2(M)$, where $- \otimes_N -$ is Connes' relative tensor product. By a reconstruction theorem of Popa [Pop94], any finitely generated rigid C^*-tensor category can be obtained from a subfactor in this way (see also [NY15, Example 5.1]).

The simplest examples of rigid C^*-tensor categories come from discrete groups. Given any discrete group Γ, the category Hilb_f^Γ of finite-dimensional Γ-graded Hilbert spaces is a C^*-category with simple objects u_g corresponding to elements of Γ. One can then define a tensor product on Hilb_f^Γ by putting $u_g \otimes u_h = u_{gh}$ and extending naturally. It is not difficult to see that this turns Hilb_f^Γ into a rigid C^*-tensor category that precisely encodes the group operations in Γ.

Parallel to groups in the classical setting, rigid C^*-tensor categories can be thought of as encoding discrete "quantum" symmetries of some sort. In fact, they are closely related to discrete quantum groups: the category $\mathrm{Rep}_f(\mathbb{G})$ of finite-dimensional unitary representations of a compact quantum group $\mathbb{G}$ is a rigid C^*-tensor category. Since $\mathrm{Rep}_f(\mathbb{G})$ is a category of representations, it comes with a canonical monoidal forgetful functor into the category Hilb_f of finite-dimensional Hilbert spaces. Rephrased in this language, Woronowicz' celebrated Tannaka–Kreĭn duality theorem [Wor87] tells us that the data of a discrete quantum group —or its compact dual— are exactly given by a rigid C^*-tensor category $\mathcal{C}$ together with a fibre functor F into Hilb_f.

As generalisations of discrete groups, many avenues of research in the theory of rigid C^*-tensor categories are motivated by known results about discrete groups. In [VV18], we develop a spectral criterion to detect property (T) in the setting of rigid C^*-tensor categories, inspired by Żuk's criterion for discrete groups [Żuk01]. Using this criterion, we construct a family of examples of discrete quantum groups with property (T) given by relations coming from triangle presentations [CMSZ91] These discrete quantum groups are in some sense very different from classical discrete groups with property (T).

1. Property (T) for rigid C^*-tensor categories

A discrete group Γ has property (T) if and only if any unitary representation of Γ admitting almost-invariant vectors must admit a non-zero invariant vector. More formally, given any unitary representation of Γ on $\mathcal{H}$ such that there exists a sequence $(\xi_n)_n$ of unit vectors in $\mathcal{H}$ satisfying $g \cdot \xi_n - \xi_n \to 0$ for all $g \in \Gamma$, there must be a nonzero vector $\xi \in \mathcal{H}$ such that $g \cdot \xi = \xi$ for all $g \in \Gamma$.

The initial formulation of property (T) for standard invariants of subfactors goes back to [Pop99]. However, the representation-theoretic framework developed in [PV14, NY15, GJ15] allows for a definition in the language of monoidal categories that replicates the one cited above almost verbatim, e.g. by looking at representations of the *tube algebra* of $\mathcal{C}$.

In the case where $\mathcal{C}$ is the category of finite-dimensional unitary representations of a compact quantum group $\mathbb{G}$, there is a very interesting interaction between several closely related versions of property (T). Property (T) for discrete quantum groups was initially introduced by Fima in [Fim08], but it is not quite true for general $\mathbb{G}$ that property (T) for the representation category $\mathcal{C}$ corresponds exactly to the discrete dual $\widehat{\mathbb{G}}$ having property (T). The obstruction is that $\mathbb{G}$ must be of Kac type for $\widehat{\mathbb{G}}$ to have property (T). By results of [PV14] and [Ara14], the representation category $\mathcal{C}$ has property (T) if and only if the discrete dual $\widehat{\mathbb{G}}$ has Arano's *central* property (T) [Ara14]. For general $\mathbb{G}$, this is strictly weaker than the usual property (T), but all notions coincide when $\mathbb{G}$ is of Kac type [Ara14].

The results of [Ara14, Ara16] show that the representation categories of q-deformations G_q, $q \neq 1$ of higher rank Lie groups have property (T). This then yields examples of subfactors with property (T) standard invariants that are quite unrelated to discrete groups. However, the duals $\widehat{G}_q$ are *not* property (T) discrete quantum groups, since they are not of Kac type. In fact, until recently, all known examples of discrete quantum groups with property (T) were commensurable with discrete groups [FMP15]. Since property (T) passes between finite-index inclusions, these discrete quantum groups in some sense “inherit” property (T) from the discrete group, thus motivating our search for more examples.

2. A spectral criterion: new examples

Given a discrete group Γ with a finite, symmetric generating set S not containing the identity, we define a finite graph $L(S)$. The vertices are the elements of S, and there is an edge between $x, y \in S$ if $x^{-1}y \in S$. The *graph Laplacian* of $L(S)$ is the positive operator Δ that takes a function $f : S \to \mathbb{C}$ and maps it to

$$(\Delta f)(s) = f(s) - \frac{1}{\deg(s)} \sum_{t \sim s} f(t).$$

Clearly, the constant functions always lie in the kernel of f. If $L(S)$ is moreover connected, these are the only ones. Under these conditions, Żuk showed in [Żuk01] that Γ has property (T) if the smallest non-zero eigenvalue of Δ is strictly greater than $1/2$. Moreover, this bound is optimal. The crucial advantage of Żuk's criterion is that it can be computed from finitary data once one knows all relations of the form $xyz = e$ with $x, y, z \in S$, since the operator Δ is quite simply given by a finite matrix. Our spectral criterion is a monoidal category version of Żuk's result.

Theorem ([VV18]). *Let $\mathcal{C}$ be a rigid C^*-tensor category and S a symmetric generating set of irreducibles not containing the tensor unit. Then there is a canonical way to define a Laplacian operator Δ on $\bigoplus_{i\in \mathrm{Irr}(\mathcal{C}),\alpha\in S} \mathrm{Hom}(\alpha, i \otimes \alpha)$. If zero is a simple eigenvalue of Δ and the smallest non-zero eigenvalue is strictly greater than $1/2$, then $\mathcal{C}$ has property (T).*

Considering that finding concrete examples to apply Żuk's criterion for discrete groups to is not entirely trivial, it is somewhat remarkable that the categorical formulation gives an elementary proof of property (T) for $\mathrm{Rep}_f(\mathrm{SU}_q(3))$, $q \neq 1$ with respect to the generating set given by the fundamental representation and its conjugate. This recovers a part of the result of [Ara14].

To obtain examples of Kac-type compact quantum groups of which the discrete duals have property (T), we use the combinatorial data of *triangle presentations*. These objects arise naturally in the study of groups acting simply transitively on Euclidean Tits buildings of type $\tilde{A}_2$ [CMSZ91]. Given any triangle presentation T, we construct a Kac-type compact quantum group $\mathbb{G}_T$ through a generators-and-relations approach inspired by Woronowicz' construction of $\mathrm{SU}_q(3)$ [Wor87].

Theorem ([VV18]). *Let T be a triangle presentation of order ≥ 2. Suppose that the Euclidean building associated with T is a classical Bruhat–Tits building. Then $\mathrm{Rep}_f(\mathbb{G}_T)$ has property (T), and admits irreducible objects of arbitrarily high dimension.*

Since $\mathbb{G}_T$ is of Kac type, the first part of the conclusion implies that the dual of $\mathbb{G}_T$ is a discrete quantum group with property (T). On the other hand, the second part ensures that $\widehat{\mathbb{G}}_T$ is not commensurable with a discrete group, so the $\widehat{\mathbb{G}}_T$ are in some sense "truly quantum" examples of discrete quantum groups with property (T). Our proof relies heavily on the results of [CMSZ91] and on the geometry of classical Bruhat–Tits buildings. Conjecturally, one should be able to give a completely combinatorial proof that does not rely on any geometric assumptions, but this seems out of reach for now.

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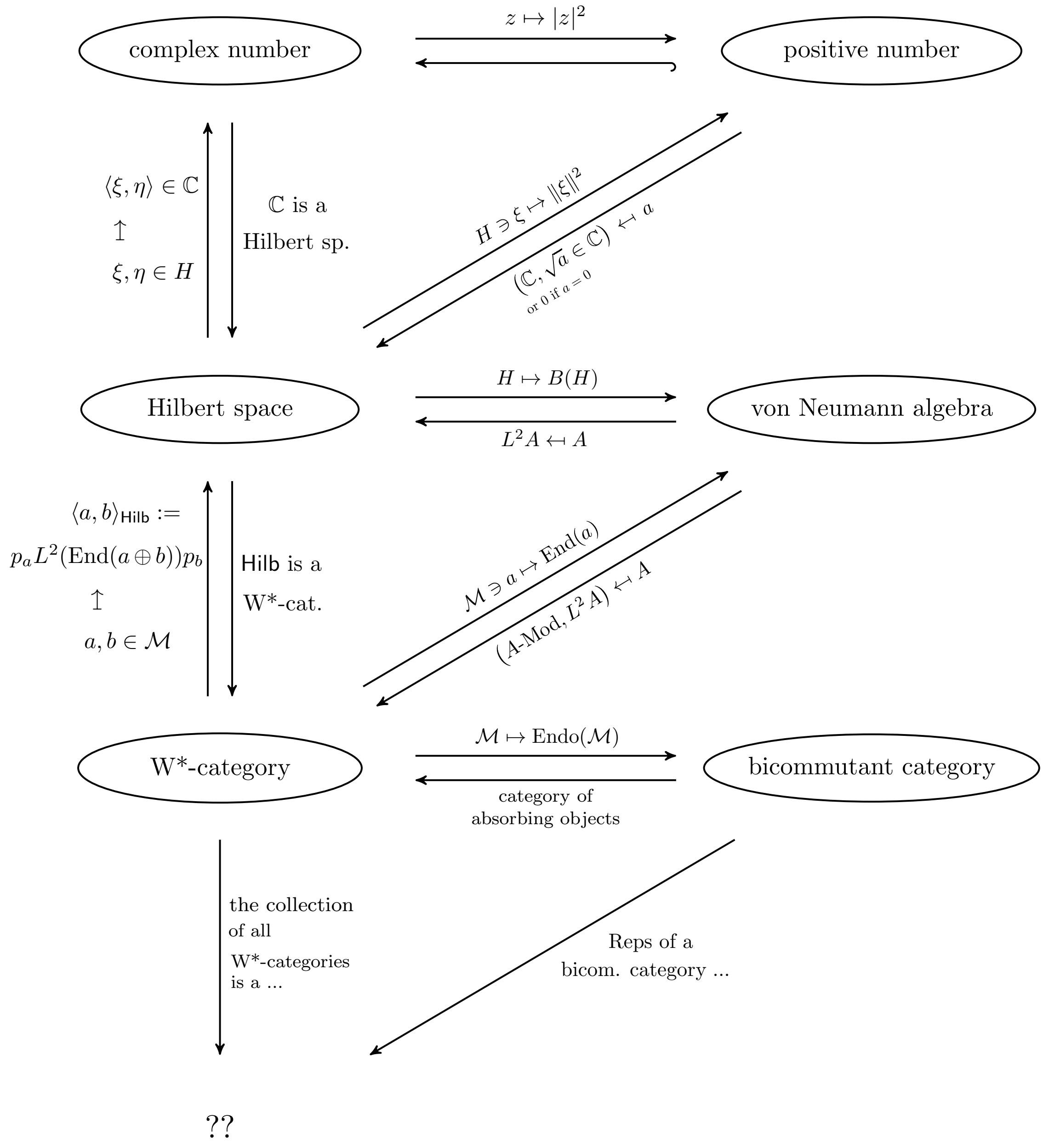
Bicommutant categories

Andre Henriques

Bicommutant categories are higher categorical analogs of von Neumann algberas. Examples of bicommutant categories can be constructed from unitary fusion categories, and from conformal nets. We review these constructions, and present a new result, joint with Dave Penneys: given two Morita equivalent unitary fusion categories, their associated bicommutant categories are equivalent as tensor categories (not just Morita equivalent). We conjecture that, similarly, there exist many non-isomorphic conformal nets whose associated bicommutant categories are equivalent as tensor categories.

Below is a chart of higher (and lower) categorical analogs of Hilbert spaces, and von Neumann algebras:



Quantum symmetric spaces from reflection equation and module categories

Makoto Yamashita

(joint work with Kenny De Commer, Sergey Neshveyev, Lars Tuset)

Reflection equation was introduced by Cherednik to understand quantum inverse scattering problems on the half line and soon became a powerful guiding principle to quantize Poisson homogenous spaces into actions of quantum groups in the hands of Sklyanin, Gurevich, Donin, and others.

Analogous to the case of the celebrated Kohno–Drinfeld theorem on the braided tensor categories arising from the Knizhnik–Zamalodchikov equations and representation theory of q-deformation quantum groups, there are two constructions of reflection operators in the framework of module categories. On the one hand, Enriquez and Etingof gave module categories from monodromy of cyclotomic Knizhnik–Zamolodchikov equations through quantization problem of dynamical r-matrices, following earlier work of Leibman, and Golubeva and Leksin. On the other, Letzter, and Kolb and Balagovic gave coideal subalgebras of the q-deformed universal enveloping algebra and universal reflection operator in a certain completion of tensor product.

One central question is whether these constructions give equivalent module categories, thereby inducing equivalent representations of the type B braid groups. We solve this affirmatively in the formal 'multiplier' algebra setting, and moreover show that the reflection operator becomes a complete invariant of categorical structure.

A multiplier algebra is simply a direct product of matrix algebras; more formally, the algebra for braided monoidal category associated with a semisimple complex Lie group G is given by

$$\mathcal{U}(G) = \prod_{\pi:\ \operatorname{Irr} G} \operatorname{End}(V_\pi)$$

where direct product is taken over the irreducible finite dimensional representations of G.

The algebra $\mathcal{U}(G)[\![h]\!]$ of formal power series with this coefficient becomes a quasi-Hopf algebra with the standard coproduct $\mathcal{U}(G)[\![h]\!] \to \mathcal{U}(G^2)[\![h]\!]$ and the associator $\Phi \in \mathcal{U}(G^3)[\![h]\!]$ coming from the Knizhnik–Zamolodchikov equation, and also becomes a Hopf algebra with the deformed coproduct $\Delta_h \colon \mathcal{U}(G)[\![h]\!] \to \mathcal{U}(G^2)[\![h]\!]$ coming from the Drinfeld–Jimbo quantization. Then the Kohno–Drinfeld theorem can be summarized as Φ being a coboundary of some element $F \in \mathcal{U}(G^2)[\![h]\!]$ with respect to Δ_h.

If θ is an involutive automorphism of G, by considering the representations of G^θ which appear in some representation of G we obtain an analogous multiplier algebra $\mathcal{U}(G^\theta)$. We then have two quasi-coactions of the quasi-bialgebra $(\mathcal{U}(G)[\![h]\!], \Delta, \Phi)$ on $\mathcal{U}(G^\theta)[\![h]\!]$, one coming from the cyclotomic KZ equation (hence with the trivial coaction map and a nontrivial associator $\Psi \in \mathcal{U}(G^\theta \times G^2)[\![h]\!]$), and another from the Letzter coideal (hence with a nontrivial coaction map α_h and the trivial associator). The problems becomes: first conjugate the map α_h to the standard coaction map (the restriciton of Δ), then to write the associator Ψ as a 'mixed coboundary' of some $G \in \mathcal{U}(G^\theta \times G)[\![h]\!]$ and F up to this conjugation.

The proof relies on elementary but curious combination of Lie algebra cohomology, Hochschild cohomology, and formality machinery from noncommutative geometry inspired by works of Calaque and Brochier. The above problems have easy solutions when the compact symmetric space U/U^θ, with U being a compact form of G compatible with θ, does not have Hermitian structure, or equivalently, if G^θ is semisimple since these cohomological obstruction vanish.

In the Hermitian case the situation is more interesting: these symmetric spaces appear as coadjoint orbits, and at the quasi-classical limit, we expect a one-parameter family of equivariant Poisson structures coming from the Kirillov bracket. At the level of algebras, since G^θ has a one-dimensional center, it is diffcult to obtain cohomological rigidity if one just looks at the universal enveloping algebra of its Lie algebra. Nonetheless, the multiplier setting allows us to reduce the first problem to Letzter's description of spherical vectors for the coideal subalgebras. On the side of cyclotomic KZ equation, again we need to modify the standard one by an action of formal characters to achieve nontrivial quasi-classical limit, apparetly only possible in the multiplier setting. We then identify the lower order coefficients of the monodromy for cyclotomic KZ equation and show a universality property to solve the second problem.

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Fusion, positivity, and finite-index subfactors in chiral conformal field theory

James Tener

Conformal nets, vertex operator algebras, and Segal CFTs are three possible axiomatizations of chiral conformal field theory. The goal of each approach is to encode all of the relevant physical models (as examples of the axioms) and expected phenomena (as theorems), while discovering exciting new mathematics along the way. The difficulty of proving a given physical fact (e.g. "WZW models are rational") in one of these three languages can differ greatly between the different frameworks. It is therefore highly desirable to develop mathematical infrastructure for rigorously translating between the languages of VOAs, conformal nets, and Segal CFTs.

The traditional approach to constructing algebras of observables from fields is to take the von Neumann algebra generated by fields smeared by functions supported in a region of space time. This approach was given a careful mathematical treatment in recent work of Carpi-Kawahigashi-Longo-Weiner [2] which gives the first general-purpose bridge between the theories of VOAs and conformal nets. One of the challenges which arises in this approach is that the smeared fields are unbounded operators, which come along with a variety of technical obstacles.

My talk focused on an alternate approach to relating conformal nets and VOAs using only bounded operators which was developed in a recent series of articles [12, 13, 14]. In this approach, local observables are constructed from fields not by smearing, but instead by considering point insertion operators localized in 'partially thin' annuli. This is based upon breakthrough ideas of Henriques which relate conformal nets, Segal CFT, and partially thin Riemann surfaces [7]. The first article in the series [12] describes a correspondence $V \leftrightarrow \mathcal{A}_V$ between unitary VOAs V and conformal nets $\mathcal{A}_V$ in the vacuum sector. The correspondence in question was conjectured at a talk at the 2015 Oberwolfach workshop Subfactors and Conformal Field Theory [1]. The second article [13] extends this correspondence to non-vacuum representations, and the third [14] develops tools for comparing the 'fusion' product theories between VOAs and conformal nets. These tools are then applied to resolve several open problems, including:

Theorem ([14])**.** *Let V be a VOA which is a WZW model corresponding to a simple finite-dimensional complex Lie algebra at positive integer level, or a $\mathcal{W}$-algebra of type ADE in the discrete series. Then all subfactors arising from irreducible representations of the conformal net $\mathcal{A}_V$ corresponding to V have finite index. Hence $\mathcal{A}_V$ is rational.*

The method of proof generalizes the approach pioneered by Wassermann [16], and also relies on the work of Huang on rigidity [10, 9] and that of Henriques classifying representations of WZW conformal nets [8]. Proofs of certain special cases of this theorem had previously appeared, such as in the case of WZW models of type ACG (all levels), type D (odd levels), and $\mathcal{W}$-algebras of type A_1 (the Virasoro minimal models) [16, 15, 11, 3], as well as other isolated examples.

A second application of our framework is to positivity phenomena in VOAs. It was shown by Gui that the category of unitary modules for a unitary VOA is naturally a unitary modular category whenever a certain family of matrices associated to the VOA are positive (semi)definite [5, 6, 4]. The positivity of the matrices associated to WZW models associated to exceptional Lie algebras was demonstrated in [14], and combined with Gui's work this sufficed to close the problem of unitarity of representation categories for WZW models.

Theorem ([14, 5, 6, 4])**.** *Let V be a VOA which is a WZW model corresponding to a simple finite-dimensional complex Lie algebra at positive integer level, or a $\mathcal{W}$-algebra of type AE in the discrete series. Then the category of unitary V-modules is naturally a unitary modular tensor category.*

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Quantum Operations on Conformal Nets

Marcel Bischoff
(joint work with Simone Del Vecchio and Luca Giorgetti)

Conformal Nets. Let $\mathcal{I}$ be the set of proper open intervals $I \subseteq S^1$ on the circle. A *conformal net* $\mathcal{A}$ associates with each $I \in \mathcal{I}$ a von Neumann algebra $\mathcal{A}(I)$ on a fixed Hilbert space $\mathcal{H}$, such that $\mathcal{A}(I) \subseteq \mathcal{A}(J)$ for $I \subseteq J$ and $\mathcal{A}(I) \subseteq \mathcal{A}(J)'$ for $I \cap J = \emptyset$. Here $M' = \{m' \in \mathrm{B}(\mathcal{H}) : mm' = m'm \text{ for all } m \in M\}$ is the *commutant* of M. There is a (projective) unitary positive-energy representation U of the group of orientation preserving diffeomorphisms $\mathrm{Diff}_+(S^1)$ of the unit circle S^1, such that $U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI)$ for $g \in \mathrm{Diff}_+(S^1)$. The vector $\Omega \in \mathcal{H}$ is called the *vacuum* and is asked to be the (up to a phase) unique vector satisfying $U(R(\theta))\Omega = \Omega$ for any rotation $R(\theta)\colon z \mapsto \mathrm{e}^{\mathrm{i}\theta}$. Then each $\mathcal{A}(I)$ is a type III_1 factor (or $\mathbb{C}$). The Reeh–Schlieder property gives that Ω is cyclic and separating for $\mathcal{A}(I)$.

There is a notion of a subnet $\mathcal{B} \subseteq A$ and extension $\mathcal{C} \supseteq \mathcal{A}$ of a given local conformal net $\mathcal{A}$. The problem of finding local extensions $\mathcal{C} \supseteq A$ is well-understood at least in the finite index case. Here local extensions correspond to local Q-systems in $\mathrm{Rep}(\mathcal{A})$ [LR95] and it is enough to know $\mathrm{Rep}(\mathcal{A})$ to classify such extensions. In other words, it is a tensor categorical problem. Similarly, discrete extensions can be described by local generalized Q-systems [DVG18].

Question 1. *Given a conformal net $\mathcal{A}$ how can we characterize $\mathcal{B} \subseteq A$?*

We note that for type III subfactors we have duality between finite index subfactors and extensions. But in the situation of conformal nets the the dual Q-system does not live in $\mathrm{Rep}(\mathcal{A})$. Therefore knowing $\mathrm{Rep}(\mathcal{A})$ alone can only give us necessary but no sufficient conditions for subnets $\mathcal{B} \supseteq \mathcal{A}$ to exist.

If $G \leq \mathrm{Aut}(\mathcal{A})$ is a closed subgroup then the *orbifold net* $\mathcal{A}^G$ is a subnet of $\mathcal{A}$. If G is finite then subgroups $H \leq G$ are in bijective correspondence with intermediate nets $\mathcal{B} \subseteq \mathcal{C} \subseteq \mathcal{A}$. We call a such a subnet Galois subnet. More precisely a subnet $\mathcal{B} \subseteq \mathcal{A}$ is called *Galois* if $\mathcal{B} = \mathcal{A}^{\mathrm{Aut}(\mathcal{A}|\mathcal{B})}$. Here by $\mathrm{Aut}(\mathcal{A}|\mathcal{B})$ we mean the subgroup of automorphisms which fix the net $\mathcal{B}$ pointwise.

Example 2. *If we consider the inclusion* $\mathcal{A} := \mathcal{A}_{\mathrm{Spin}(N)_1} \subseteq \mathcal{A}_{\mathrm{SU}(2)_{10}} =: \mathcal{B}$ *then* $\mathrm{Aut}(\mathcal{A}|\mathcal{B})$ *is trivial but* $[\mathcal{A} : \mathcal{B}] = 3 + \sqrt{3}$. *Thus* $\mathcal{B} \subseteq \mathcal{A}$ *is* not *Galois.*

Then a natural question is if we can generalize to a more general notion than that of automorphisms. Evans and Gannon asked:

Question 3 ([EG11])**.** *Can we orbifold a VOA[or conformal net] by something more general than a group?*

Quantum Operation on Conformal Nets. Quantum operations are described by unital completely positive maps.

Let $M \subseteq \mathrm{B}(\mathcal{H})$ be a von Neumann algebra and $\Omega \in \mathcal{H}$ a cyclic and separating vector which induces the faithful state $\omega(\,\cdot\,) = (\Omega, \cdot\,\Omega)$. We denote by $\mathrm{UCP}(M, \Omega)$ the convex set of normal unital completely positive maps $\phi\colon M \to M$ which preserve the state, i.e. $\omega \circ \phi = \omega$.

Let $\mathcal{A}$ be a conformal net. We denote by $\mathrm{QuOp}(\mathcal{A})$ the set of *(extremal) quantum operations* on $\mathcal{A}$ defined as follows. An element $\phi \in \mathrm{QuOp}(\mathcal{A})$ is a family $\phi = \{\phi_I \in \mathrm{UCP}(\mathcal{A}(I), \Omega)\}_{I \in \mathcal{I}}$ with the following properties. The family is *compatible*, i.e. $\phi_{I_2}|_{\mathcal{A}(I_1)} = \phi_{I_1}$ for all $I_i \in \mathcal{I}$ with $I_1 \subseteq I_2$; each ϕ_I is an *extreme point* in the convex space $\mathrm{UCP}(\mathcal{A}(I), \Omega)$; the map ϕ_I "*preserves the (local) conformal symmetry*", i.e. $\phi_I(U(\gamma)) = U(\gamma)$ for any $\gamma \in \mathrm{Diff}_+(S^1)$ supported in I; and there is a *Markov adjoint* $\bar{\phi}$ which is a family with all the above properties such that $\omega(\bar{\phi}_I(x)y) = \omega(x\phi_I(y))$ for all $x, y \in \mathcal{A}(I)$.

One can show that invertible (under composition) quantum operations are automorphisms of the net, in other words $\mathrm{QuOp}(\mathcal{A})^\times = \mathrm{Aut}(\mathcal{A})$. In this sense: quantum operations on $\mathcal{A}$ *generalize* automorphisms of $\mathcal{A}$.

Proposition 4. *Let* $S \subseteq \mathrm{QuOp}(\mathcal{A})$ *be a set, then the fixed point net* $\mathcal{A}^S$ *defined by* $\mathcal{A}^S(I) = \{a \in \mathcal{A}(I) \,|\, \phi_I(a) = a \text{ for all } \phi \in S\}$ *is an irreducible subnet of* $\mathcal{A}$.

We obtain the following Galois like theory for finite index subnets.

Theorem 5 ([Bis17])**.** *Let* $\mathcal{A}$ *be a conformal net. There is a bijective correspondence*

$$\{\mathcal{B} \subseteq \mathcal{A} \,|\, \text{ subnet with } [\mathcal{A} : \mathcal{B}] < \infty\} \longleftrightarrow \{K \subseteq \mathrm{QuOp}(\mathcal{A}) \,|\, K \text{ finite hypergroup}\}$$

The correspondence is given by $\mathcal{B} \mapsto \mathrm{QuOp}(\mathcal{A}|\mathcal{B})$ *and* $K \mapsto \mathcal{A}^K \subseteq \mathcal{A}$.

Here by K being a hypergroup[1] we mean that $\mathrm{id} \in K$, the set K is closed under $\bar{\cdot}$ and that the convex hull $\mathrm{Conv}(K)$ is closed under composition and we have the property that for any $\phi_1, \phi_2 \in K$ we have $\mathrm{id} \prec \phi_1 \circ \phi_2$ if and only if $\phi_1 = \bar{\phi}_2$. The number $w \in [1, \infty)$ determined by $\bar{\phi} \circ \phi = w^{-1} \cdot \mathrm{id} + \cdots$ is called the weight of ϕ.

[1]in the sense of Sunder–Wildberger [Sun92, SW03]

Proposition 6 ([BR19]). *Let $\mathcal{A}$ be a rational conformal net and $K \subseteq \mathrm{QuOp}(\mathcal{A})$. Then each w_ϕ for $\phi \in K$ is the index of an irreducible finite depth subfactor and*

$$[\mathcal{A} : \mathcal{A}^K] = \sum_{\phi \in K} w_\phi \in \{1, 2, 3, \tfrac{5+\sqrt{5}}{2}, 4, 3+\sqrt{3}, 5\} \cup (5\tfrac{1}{4}, \infty)\,.$$

Example 7. *In Example 2 we get $K = \mathrm{QuOp}(\mathcal{A}|\mathcal{B}) = \{\mathrm{id}, \phi\}$ with the only non-trivial relation in $\mathrm{Conv}(K)$ being*

$$\phi \circ \phi = \tfrac{1}{2+\sqrt{3}}\,\mathrm{id} + \tfrac{1+\sqrt{3}}{2+\sqrt{3}}\phi\,.$$

Let F be a fusion ring. A *(unitary) categorification* $\mathcal{F}$ of F is a (unitary) fusion category whose fusion ring is isomorphic to F. Every fusion ring gives an abstract hypergroup $K_F = \{\frac{1}{\mathrm{FPdim}\,X} X \in \mathbb{C}[F] \mid X \in F\}$ and we can talk about a *categorification of a hypergroup.*

Theorem 8 ([Bis17]). *Let $\mathcal{A}$ be* holomorphic net*, i.e. a rational conformal net with* $\mathrm{Rep}(\mathcal{A})$ *trivial. If K is a finite hypergroup in* $\mathrm{QuOp}(\mathcal{A})$ *then there is a unitary fusion category $\mathcal{F}$ which is a* unitary categorification *of K such that* $\mathrm{Rep}(\mathcal{A})$ *is braided equivalent to the Drinfel'd center $Z(\mathcal{F})$.*

Problem 9. *For $\mathcal{A}$ a conformal net associated with a lattice* [DX06] *or a strongly local vertex operator algebra* [CKLW18] *find $\phi \in \mathrm{QuOp}(\mathcal{A})$ with $\phi \circ \phi = w^{-1}\,\mathrm{id} + (1 - w^{-1})\phi$ for small positive real numbers $w > 1$.*

For example for the even $A_2 \times E_6$ lattice the hypothetical (see [EG11]) conformal net associated with the Haagerup subfactor would give such an element ϕ with $w = \frac{11+3\sqrt{13}}{2}$.

Infinite index subnets. Finally we are interested in infinite index subnets. We equip $\mathrm{QuOp}(\mathcal{A})$ with the bounded weak topology. We have evidence to believe the following conjecture:

Conjecture 10. *Let $\mathcal{A}$ be a strongly additive conformal net. Then the correspondence of Theorem 5 extends to a bijective correspondence*

$$\{\mathcal{B} \subseteq \mathcal{A} \mid \textit{ discrete subnet}\} \longleftrightarrow \{K \subseteq \mathrm{QuOp}(\mathcal{A}) \mid K \textit{ compact hypergroup}\}$$

Here we should use a certain notion of a discrete subnet (called subnet of compact type in [Car04]) and a still need to find the precise notion of compact hypergroup. A crucial step is to associate a compact hypergroup with any discrete braided local inclusion $(N \subseteq M, \Omega)$ presented in the talk by Luca Giorgetti.

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Weak quasi-Hopf algebras, VOAs and conformal nets

Sebastiano Carpi

(joint work with Sergio Ciamprone, Claudia Pinzari)

Weak quasi-Hopf algebras, introduced by Mack and Schomerus in [13, 14], are a generalization of Drinfeld's quasi-Hopf algebras. Every fusion category is tensor equivalent to the representation category of a weak quasi-Hopf algebra [11]. After these early works there seems has not been no relevant progress in the theory until the recent work by Ciamprone and Pinzari [2] where some specific examples from quantum groups at roots of unity in the type A case where studied in detail.

In a subsequent work by Ciamprone and Pinzari and me [3] we develop various aspects of the theory and consider many examples and applications. Here, I will briefly report of some the results contained in the latter work with emphasis on the unitarity and conformal field theory aspects emerging in connection with vertex operator algebras (VOAs) and conformal nets. These results indicate that weak quasi-Hopf algebras give a useful and natural tool to study certain relevant properties of fusion categories and conformal field theory.

A a weak quasi-Hopf algebra is a quintuple $(A, \Delta, \varepsilon, S, \Phi)$ satisfying various assumptions. Here A is a unital associative algebra (over $\mathbb{C}$), the coproduct $\Delta : A \to A \otimes A$ is a homomorphism, the counit $\varepsilon : A \to \mathbb{C}$ is a nonzero homomorphism, the antipode $S : A \to A$ is an antiautomorphism, Φ is the associator.

In contrast with the quasi-Hopf algebra case the coproduct is not assumed to be unital so that $\Delta(1_A)$ is an idempotent in $A \otimes A$ commuting with $\Delta(A)$ which is in general different from $1_A \otimes 1_A$. This fact allows a much more flexibility.

The coproduct gives a tensor structure on the representation category $\mathrm{Rep}(A)$. The tensor product $\pi_1 \underline{\otimes} \pi_2$ on objects of $\mathrm{Rep}(A)$ is then given by the restriction of $\pi_1 \otimes \pi_2 \circ \Delta$ to the invariant subspace $\pi_1 \otimes \pi_2 \circ \Delta(1_A) V_{\pi_1} \otimes V_{\pi_2}$. If A is finite-dimensional and semisimple then $\mathrm{Rep}(A)$ is a fusion category.

Now, for a given (finite-dimensional and semisimple) A, the additive function $D : \mathrm{Gr}(\mathrm{Rep}(A)) \to \mathbb{Z}$ defined by $D([\pi]) := \dim(V_\pi)$ is a weak integral dimension function i.e. it satisfies $D([\pi_1 \underline{\otimes} \pi_2]) \leq D([\pi_1])D([\pi_2])$, $D([\iota]) = 1$ and $D([\overline{\pi}]) = D([\pi]) \geq 0$. All fusion categories have integral weak dimension functions.

The following result is due to Häring-Oldenburg [11].

Theorem ([11]). *Let $\mathcal{C}$ be a fusion category and $D : \mathrm{Gr}(\mathcal{C}) \to \mathbb{Z}$ be an integral weak dimension. Then there exists a finite dimensional semisimple weak quasi-Hopf algebra $(A, \Delta, \varepsilon, S, \Phi)$ and a tensor equivalence $\mathcal{F} : \mathcal{C} \to \mathrm{Rep}(A)$ such that $D([X]) = \dim(V_{\mathcal{F}(X)})$ for all $X \in \mathrm{Obj}(\mathcal{C})$.*

Extra structures on $\mathcal{C}$ give extra structures on A (see [11] and [3]): braidings give R-matrices; C*-tensor structures on $\mathcal{C}$ give Ω-involutive structures on A (in particular the algebras A become a C*-algebras). The weak quasi-Hopf algebra associated to a fusion category $\mathcal{C}$ is highly non-unique. It depends on the choice of the integral weak dimension function D and, once D is fixed, is only defined up to a "twist".

Now let $\mathcal{C}^+$ be a linear C*-category, $\mathcal{C}$ be a fusion category and $\mathcal{F} : \mathcal{C}^+ \to \mathcal{C}$ be a linear equivalence. In the proof of the following theorem weak quasi-Hopf algebras plays a crucial role.

Theorem 1. ([3]). *If $\mathcal{C}$ is tensor equivalent to a unitary fusion category $\mathcal{D}^+$ then $\mathcal{C}^+$ can be upgraded to a unitary fusion category so that $\mathcal{F} : \mathcal{C}^+ \to \mathcal{C}$ becomes a tensor equivalence. This unitary tensor structure on C^+ is unique up to unitary equivalence and makes $\mathcal{C}^+$ unitary tensor equivalent to $\mathcal{D}^+$.*

In fact the result is still valid if $\mathcal{C}$ is only assumed to be rigid and semisimple provided that it has an integral weak dimension function. As a corollary we find a positive answer to a question by Cesar Galindo in [6]

Corollary 2. ([3]). *Two tensor equivalent unitary fusion categories must be unitary tensor equivalent.*

A different proof of the latter result was found independently by Reutter [15].

We now apply the previous theorem to the unitarizability of the representation categories of unitary affine VOAs. Let $\mathfrak{g}$ be a complex simple Lie algebra, let k be a positive integer and let $V_{\mathfrak{g}_k}$ be the corresponding simple level k affine VOA. It is known that $V_{\mathfrak{g}_k}$ is a unitary strongly rational VOA and that every $V_{\mathfrak{g}_k}$-module is unitarizable. We denote by $\mathrm{Rep}^u(V_{\mathfrak{g}_k})$ the linear C*-category of unitary $V_{\mathfrak{g}_k}$-modules. Because of the unitarizability of the $V_{\mathfrak{g}_k}$-modules the forgetful functor $\mathcal{F} : \mathrm{Rep}^u(V_{\mathfrak{g}_k}) \to \mathrm{Rep}(V_{\mathfrak{g}_k})$ is a linear equivalence. By a result of Finkelberg [4, 5] we know that $\mathrm{Rep}(V_{\mathfrak{g}_k})$ is tensor equivalent to the "semisimplified" tensor category $\widetilde{\mathrm{Rep}}(G_q)$ associated to the representations of the quantum group G_q, with G the simply connected compact Lie group corresponding to $\mathfrak{g}$ and and the rooth of unity q is given by $q = e^{\frac{i\pi}{d(k+h^\vee)}}$. Here $h^\vee$ is the dual Coxeter number, $d = 1$ if $\mathfrak{g}$ is ADE, $d = 2$ if $\mathfrak{g}$ is BCF and $d = 3$ if $\mathfrak{g}$ is G_2.

It was shown by Wenzl and Xu [17, 18] that $\widetilde{\mathrm{Rep}}(G_q)$ is tensor equivalent to a unitary fusion category. As a consequence we have the following result.

Theorem 3. ([3]). $\mathrm{Rep}^u(V_{\mathfrak{g}_k})$ *has a structure of unitary fusion category which is unique up to unitary equivalence.*

Unitary tensor structures on $\mathrm{Rep}^u(V_{\mathfrak{g}_k})$ have been constructed directly in a series of papers [7, 7, 9] by Bin Gui for the Lie types A, B, C, D, and G_2 and more recently by James Tener in [16] for the remaining cases E_6, E_7, E_8 and F_4 by completely different methods. By our uniqueness result these structures agree with those we have found.

Our method works also for many other VOAs such as e.g. lattice VOAs and certain holomorphic orbifolds.

As another application of the theory of weak quasi-Hopf algebra we give a classification of pseudo-unitary type A ribbon fusion categories. The starting point is the work of Kazhdan and Wenzl [12] on the classification of type A tensor categories. As a consequence of our results we have in particular the following theorem.

Theorem 4. ([3])*Let $\mathcal{C}$ be a modular fusion category with modular matrices S, T coinciding with the Kac-Peterson matrices for the $\mathfrak{sl}(n)$ affine Lie algebra at positive integer level k. Then $\mathcal{C}$ is ribbon equivalent to* $\mathrm{Rep}(V_{\mathfrak{sl}(n)_k})$.

Now let $\mathcal{A}_{V_{\mathfrak{sl}(n)_k}}$ be the conformal net on S^1 associated to the strongly local unitary VOA $V_{\mathfrak{sl}(n)_k}$ [1]. As a first consequence of Theorem 4 we have

Corollary 5. *We have a unitary ribbon equivalence* $\mathcal{F} : \mathrm{Rep}^u(V_{\mathfrak{sl}(n)_k}) \to \mathrm{Rep}(\mathcal{A}_{V_{\mathfrak{sl}(n)_k}})$.

The same result has been independently obtained by Bin Gui [10] by different methods (direct analytic proof instead of classification). As a second consequence of Theorem 4 we obtain e new proof of Finkelberg's equivalence in the type A case.

Corollary 5. *If $q = e^{\frac{i\pi}{(k+n)}}$ there is a unitary ribbon equivalence* $\mathcal{F} : \mathrm{Rep}(V_{\mathfrak{sl}(n)_k}) \to \widetilde{\mathrm{Rep}}(\mathrm{SU}(n)_q)$.

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Non-rational conformal field theories

Joerg Teschner

Theoretical physics suggests that beyond the much-studied class of rational CFT there should exist large classes of non-rational CFT. Such CFT should be mathematically very rich, due to links with harmonic analyisis, quantum Teichmueller theory and two-dimensional quantum gravity. The main goal of my talk was be to present hints that a rigorous mathematical study of important non-rational conformal field theories may not be as elusive as it might seem. Liouville theory and some recent results on the relation between free fermion and Virasoro conformal blocks at c=1 offer a reasonably complete (partly conjectural) picture of the relevant representation categories.

A conjecture was formulated in my talk for the decomposition of the Connes fusion of two unitary highest weight representations of the Virasoro algebra with central charge c¿25 and highest weights h satisfying 24 h ¿ c-1. It is conjectured to decompose as the direct integral of representation from this series weighted with a density that can be expressed in terms of the so-called DOZZ formula for the three-point function of Liouville CFT, together with the formula proposed by brothers Zamolodchikov for the expectation value of exponential Liouville fields on the Poincare disc. In a suitable generalisation of the approach to Connes fusion developed by Jones, Wassermann and Gui one would expect this decomposition to be aconsequence of a partly conjectural braid relation of intertwining operators derived in 2001.

Classification of pure split invariant state with on-site symmetry

Yoshiko Ogata

A quantum spin chain is given by $\mathcal{A} := \bigotimes_{\mathbb{Z}} M_d$, where M_d is the algebra of $d \times d$-matrices. On quantum spin chain, we consider the on-site symmetry β given by a finite group G and a unitary representation U on $\mathbb{C}^d$ with $U(g) \notin \mathbb{C}1$, for $g \neq e$. The automorphism β_g, for $g \in G$ is defined by

$$\beta_g(A) := Ad\left(\bigotimes_{\mathbb{Z}} U(g)\right)(A), \quad A \in \mathcal{A}, \quad g \in G. \tag{1}$$

We consider the classification of the set $SPG(\mathcal{A})$, the set of all pure split β-invariant states. The criterion for the classification is as follows: $\omega_0, \omega_1 \in SPG(\mathcal{A})$ are equivalent if there are automorphisms α_L, α_R on $\mathcal{A}_L$, $\mathcal{A}_R$ such that

(1) ω_1 is quasi-equivalent to $\omega_0 \circ (\alpha_L \otimes \alpha_R)$, and
(2) $\alpha_L \circ \beta_g|_{\mathcal{A}_L} = \beta_g|_{\mathcal{A}_L} \circ \alpha_L$, and $\alpha_R \circ \beta_g|_{\mathcal{A}_R} = \beta_g|_{\mathcal{A}_R} \circ \alpha_R$, for each $g \in G$.

It turns out that the complete invariant is the second cohomology class h_ω associated to a projective representation given for each $\omega \in SPG(\mathcal{A})$. Our main theorem is that ω_0 and ω_1 are equivalent if and only if $h_{\omega_1} = h_{\omega_0}$.[O] This is a problem motivated by the classification problem of SPT phases.

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Compact hypergroups from discrete subfactors

Luca Giorgetti
(joint work with Marcel Bischoff, Simone Del Vecchio)

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Extensions and subtheories in CFT. My talk is a follow-up of Marcel Bischoff's talk contained in this volume, and it aims to extend his results to the non-rational CFT setting [2]. A *local conformal net* is an operator algebraic axiomatization of chiral Conformal Field Theory, it consists of a collection of von Neumann algebras $\mathcal{A}(I)$ on a common Hilbert space $\mathcal{H}$, associated with open proper intervals of the unit circle $I \subset \mathbb{S}^1$ and subject to various physically motivated assumptions, the most important of which is locality: $\mathcal{A}(I)$ and $\mathcal{A}(J)$ elementwise commute if $I \cap J = \emptyset$. We refer to Marcel's talk for the definition, and to the literature [4], [12], [13], [10].

Given a local conformal net $\mathcal{A}$, both *extensions* $\mathcal{A} \subset \mathcal{C}$ and *subtheories* $\mathcal{B} \subset \mathcal{A}$ are described by nets of subfactors. The study of extensions has been initiated in [19], the one of subtheories more recently in [1]. They have been spelled out in the case of subfactors with finite Jones index/theories with finitely many inequivalent DHR superselection sectors (rationality assumption [16]).

Due to the locality constraint, the two problems, seemingly very similar, are of a quite different nature [1]. To see this, let $\mathcal{N} \subset \mathcal{M}$ be an irreducible infinite subfactor on $\mathcal{H}$ with finite index, let $\xi \in \mathcal{H}$ be a jointly cyclic and separating vector with associated Tomita's conjugations $J_{\mathcal{N}}$ and $J_{\mathcal{M}}$. Let $\gamma = \mathrm{Ad}_{J_{\mathcal{N}} J_{\mathcal{M}}} \in \mathrm{End}(\mathcal{M})$ be Longo's canonical endomorphism [18], with dual canonical $\theta = \gamma_{|\mathcal{N}} \in \mathrm{End}(\mathcal{N})$. Denote by $\iota : \mathcal{N} \to \mathcal{M}$ the inclusion morphism, with dual morphism $\bar{\iota} : \mathcal{M} \to \mathcal{N}$ defined by $\gamma = \iota\bar{\iota}$. Thus also $\theta = \bar{\iota}\iota$. By the finiteness of the index, there are solutions $w \in \mathcal{N}$ and $v \in \mathcal{M}$ of the conjugate equations for ι and $\bar{\iota}$. Namely, $wn = \theta(n)w$ for all $n \in \mathcal{N}$, and $vm = \gamma(m)v$ for all $m \in \mathcal{M}$, such that $v^*\iota(w) = w^*\bar{\iota}(v) = 1$. The triples $(\theta, w, x = \bar{\iota}(v))$ and $(\gamma, v, y = \iota(w))$ are two C*-Frobenius algebras [3] respectively in $\mathrm{End}(\mathcal{N})$ and $\mathrm{End}(\mathcal{M})$ (Q-systems), and it is well known that they determine the inclusion $\mathcal{N} \subset \mathcal{M}$ either, respectively, from $\mathcal{N}$ or $\mathcal{M}$ only. Moreover, all the Q-systems arise in this way. So the description of over/subfactors is completely symmetric in the case of a single inclusion.

Back to the CFT situation, if $\mathcal{N} = \mathcal{B}(I) \subset \mathcal{M} = \mathcal{A}(I)$, the main result of [19] states that θ extends to a localized and transportable (DHR) endomorphism of the net $\mathcal{B}$, in symbols $\theta \in \mathrm{Rep}(\mathcal{B})$, and that the Q-system (θ, w, x) in $\mathrm{Rep}(\mathcal{B})$ determines the whole net extension $\mathcal{B} \subset \mathcal{A}$.

Now, symmetrically, assume that also $\gamma \in \mathrm{Rep}(\mathcal{A})$ with Q-system (γ, v, y) in $\mathrm{Rep}(\mathcal{A})$. Then the Jones basic construction $\mathcal{M} \subset \mathcal{M}_1$, which has γ as dual canonical endomorphism, is part of a net extension $\mathcal{A} \subset \mathcal{D}$ by [19] with $\mathcal{M}_1 = \mathcal{D}(I)$. On the one hand, the net extension is necessarily relatively local with respect to $\mathcal{A}$ by [19], i.e., $\mathcal{A}(I)$ and $\mathcal{D}(J)$ commute if $I \cap J = \emptyset$. On the other hand, the Jones projection $e_I = [\mathcal{B}(I)\Omega]$, with $\Omega \in \mathcal{H}$ the vacuum vector of $\mathcal{A}$, is independent on the interval I by the Reeh-Schlieder property. Thus $e_I = e_J$. In particular, $e_I m e_I = m e_I$ for every $m \in \mathcal{M} = \mathcal{A}(I)$, and $\mathcal{B}(I) = \mathcal{A}(I)$. From this we conclude that $\gamma \in \mathrm{Rep}(\mathcal{A})$ is only possible when $\mathcal{B} = \mathcal{A}$. The above discussion says that the problem of determining relatively local extensions of a given local conformal net $\mathcal{A}$ is a matter of finding Q-systems in $\mathrm{Rep}(\mathcal{A})$. Instead, subnets of $\mathcal{A}$ cannot be read in $\mathrm{Rep}(\mathcal{A})$, just think of holomorphic nets where $\mathrm{Rep}(\mathcal{A})$ is trivial, or Virasoro minimal models [5]. We propose to look at the structure of *generalized (global) gauge transformations* of $\mathcal{A}$ (also called *quantum operations*) in order to describe subtheories.

Non-rational CFTs and infinite index subfactors. If $\mathcal{A}$ is non-rational, it may have conformal net extensions $\mathcal{A} \subset \mathcal{C}$ with infinite index, i.e., such that every subfactor $\mathcal{A}(I) \subset \mathcal{C}(I)$ has infinite index. If the subfactors are "tamely" infinite index, namely if they are *discrete* in terminology of [14], see the next section for explanations, and assuming a slight strengthening of the locality assumption on $\mathcal{A}$ (strong additivity), then relatively local extensions are described by *generalized Q-systems of intertwiners* in $\mathrm{Rep}(\mathcal{A})$ [7] (a generalization of a C*-Frobenius algebra object with infinitely many (co)multiplications). Weakening the discreteness assumption, the machinery of generalized Q-systems in $\mathrm{End}(\mathcal{N})$ still works at the level of a single subfactor [9]. Another recent treatment of discrete subfactors, close in spirit to generalized Q-systems of intertwiners, is due to [15]. Again assuming discreteness almost everywhere, we present below our generalization of the study of conformal subnets of $\mathcal{A}$, from finite [1] to infinite index [2].

A duality theorem for discrete subfactors. We abstract the properties of subfactors coming from conformal inclusions, and formulate a duality theorem for subfactors.

Let $\mathcal{N} \subset \mathcal{M}$ be an irreducible subfactor of type *III* with separable predual.

Definition 1. *$\mathcal{N} \subset \mathcal{M}$ is called semidiscrete if it admits a (necessarily unique) normal faithful conditional expectation $E : \mathcal{M} \to \mathcal{N} \subset \mathcal{M}$.*

This property is guaranteed for conformal inclusion by the Bisognano-Wichmann property, namely the modular group of $(\mathcal{A}(I), \Omega)$ coincides with the one-parameter unitary group implementing the dilations of I. Given that the dilations are implemented on $\mathcal{A}$ and $\mathcal{B} \subset \mathcal{A}$ by the same unitaries (the inclusion is conformal), the existence of E (preserving the vacuum state) follows by Takesaki's theorem.

Definition 2. *$\mathcal{N} \subset \mathcal{M}$ is called in addition discrete if the normal semifinite faithful operator-valued weight $\hat{E} : \mathcal{M}_1 \to \mathcal{M} \subset \mathcal{M}_1$ dual to E* [17] *is semifinite on $\mathcal{N}' \cap \mathcal{M}_1$.*

Every finite index subfactor is in particular discrete, being $\hat{E}$ an everywhere defined conditional expectation.

Proposition 3 ([14], [7])**.** *Let $\mathcal{N} \subset \mathcal{M}$ be an irreducible semidiscrete subfactor of type III. The following conditions are equivalent:*

(i) *$\mathcal{N} \subset \mathcal{M}$ is discrete.*
(ii) *The dual canonical endomorphism $\theta \in \mathrm{End}(\mathcal{N})$ is a countable direct sum of irreducible subendomorphism with finite dimension* [20]*, i.e., $\theta = \bigoplus_i \rho_i$ and $d_{\rho_i} < \infty$ for every i.*
(iii) *$\mathcal{M}$ admits a Pimsner-Popa basis $\{\psi_i\} \subset \mathcal{M}$ over $\mathcal{N}$ with respect to E made of charged fields, namely $\psi_i n = \rho_i(n)\psi_i$, $n \in \mathcal{N}$, for every i.*

The following condition is equivalent to the locality of the net $\mathcal{A}$ [19], [7] when $\mathcal{N} \subset \mathcal{M}$ comes from a local net extension $\mathcal{B} \subset \mathcal{A}$. It is a generalization of the finite index notion of *commutativity* for C*-Frobenius algebras.

Definition 4. *An irreducible discrete subfactor $\mathcal{N} \subset \mathcal{M}$ of type III is called local if it is braided, namely if θ lives in a unitary braided tensor subcategory of* $\mathrm{End}(\mathcal{N})$ *with braiding denoted by $\{\varepsilon_{\rho,\sigma}\}$, and if it admits a commutative Pimsner-Popa basis of charged fields, namely $\psi_i\psi_j = \varepsilon_{\rho_j,\rho_i}\psi_j\psi_i$ for every i, j.*

Once a Pimsner-Popa basis of charged fields is commutative, then all such bases are commutative. Clearly, the commutativity condition can be equivalently formulated with the opposite braiding $\varepsilon^{\mathrm{op}}_{\rho,\sigma} = \varepsilon^*_{\sigma,\rho}$. In CFT applications, the braiding is the DHR braiding of $\mathrm{Rep}(\mathcal{B})$. See [11] for explanations.

Theorem 5. *For every irreducible discrete local subfactor $\mathcal{N} \subset \mathcal{M}$ of type III there is a canonically associated set $K = K(\mathcal{N} \subset \mathcal{M})$ of quantum operations acting on $\mathcal{M}$ via unital completely positive (ucp) maps and such that $\mathcal{N} = \mathcal{M}^K$. Here $\mathcal{M}^K$ is the fixed points subalgebra. Moreover, K has the structure of a compact metrizable hypergroup.*

Roughly speaking, a compact hypergroup is a compact Hausdorff topological space K such that the space of bounded Radon measures $M(K)$ admits a Banach algebra structure with respect to a "convolution product". Denoted by δ_k the Dirac measure concentrated in $k \in K$ (an extreme point in the set of probability measures $P(K)$), then the convolution $\delta_k * \delta_h$ is a probability measure. In the group case, $\delta_k * \delta_h$ is again extreme in $P(K)$ and it is equal to δ_{kh}. There is an antilinear involution on measures $\delta_k \mapsto \delta_{\bar{k}}$ which exchanges the order of products and which generalizes the group inversion. There is a convolution neutral element δ_{id}. Further continuity properties are required in some notion of hypergroup present in the literature, and omitted in others. All the topological properties collapse in the case of finite hypergroups.

No compact quantum gauge groups. Instead of digging further into the definition of hypergroup, we mention the main consequence of our construction for conformal inclusions: Hopf algebras or compact quantum groups cannot act as (global) gauge symmetries on a local conformal net.

Theorem 6. *Let $\mathcal{N} \subset \mathcal{M}$ be an irreducible semidiscrete local subfactor of type III with depth 2, then K is a compact group.*

The depth 2 condition, namely the factoriality of $\mathcal{N}' \cap \mathcal{M}_2$ where $\mathcal{N} \subset \mathcal{M} \subset \mathcal{M}_1 \subset \mathcal{M}_2$ is (the beginning of) a Jones tower, is equivalent to $\mathcal{N}$ being the fixed points under the action on $\mathcal{M}$ of a compact quantum group [14], [21]. Moreover, note that discreteness is *not* assumed in the previous theorem, as it follows from semidiscreteness and depth 2 by a result of [8]. In particular

Corollary 7. *Let $\mathcal{B} \subset \mathcal{A}$ be a conformal inclusion of local conformal nets. If $\mathcal{B}(I) \subset \mathcal{A}(I)$ has depth 2, then $\mathcal{B}$ is a compact group orbifold subnet of $\mathcal{A}$.*

In the case of finite index subnets, this fact is known both in the von Neumann algebraic and in the vertex operator algebraic axiomatization of chiral CFT.

Examples in CFT. A family of compact (non-finite) examples of hypergroups in chiral CFT come from local conformal extensions of $\mathcal{B} = \mathrm{Vir}_{c=1}$. They are all irreducible and the discrete ones are intermediate in $\mathcal{B} \subset \mathcal{A} = \mathcal{A}_{\mathrm{SU}(2)_1}$ [6], [22]. In this case, $\mathcal{B} = \mathcal{A}^G$ with $G = \mathrm{SO}(3)$ and every $\mathcal{B} \subset \mathcal{C} \subset \mathcal{A}$ is of the form $\mathcal{C} = \mathcal{A}^H$ with H a closed non-normal subgroup of G. Then we have $\mathcal{B} = \mathcal{C}^{G//H}$, where $K = G//H$ is a double coset compact hypergroup.

Further examples of "disconnected" compact hypergroups which are not double cosets can be produced by tensoring with finite index local conformal net extensions $\mathcal{B}_1 \subset \mathcal{A}_1$, namely $\mathcal{B} \otimes \mathcal{B}_1 \subset \mathcal{A} \otimes \mathcal{A}_1$. We are not aware by now of any more exotic "connected" example arising in CFT.

Compact hypergroups of ucp maps. We briefly sketch the construction of the hypergroup K and we mention the main technical result (Theorem 9) of [2]. Let $\Omega \in \mathcal{H}$ be a cyclic and separating unit vector for $\mathcal{M}$ such that $(\Omega, \cdot\,\Omega) = (\Omega, E(\cdot)\Omega)$, e.g., the vacuum vector of a local conformal net.

The hypergroup appearing in Theorem 5 is defined by

$$K := \mathrm{Extr}(\mathrm{Markov}_{\mathcal{N}}(\mathcal{M}, \Omega))$$

where $\mathrm{Markov}_{\mathcal{N}}(\mathcal{M}, \Omega)$ is the set of normal faithful ucp maps $\phi : \mathcal{M} \to \mathcal{M}$ which are $\mathcal{N}$-bimodular, Ω-stochastic and which admit an Ω-Markov adjoint. We refer to Marcel's talk for the precise definitions. Here we only mention the following

Lemma 8. *If $\mathcal{N} \subset \mathcal{M}$ is as in Theorem 5, then every ucp $\mathcal{N}$-bimodular map $\phi : \mathcal{M} \to \mathcal{M}$ is automatically in* $\mathrm{Markov}_{\mathcal{N}}(\mathcal{M}, \Omega)$.

The hypergroup "convolution" of ucp maps ϕ_1, ϕ_2 is the composition $\phi_1 \circ \phi_2$, the neutral element is the identity map id, and the involution is the Markov adjunction $\phi \mapsto \bar{\phi}$. The topology on K is induced by Arveson's bounded weak topology, or equivalently, in this case, the pointwise ultra-strong/weak operator topology. The non-trivial steps of the construction, namely the compactness of the subset of extreme points and the measure theoretical interpretation of ucp maps are settled by the following

Theorem 9. *Let $\mathcal{N} \subset \mathcal{M}$ be as in Theorem 5. There is a homeomorphism between the set* $\mathrm{Markov}_{\mathcal{N}}(\mathcal{M}, \Omega)$ *and the set of states of a commutative unital separable C^*-algebra $C^*_{\mathrm{red}}(\mathcal{N} \subset \mathcal{M})$ associated with the subfactor.*

In particular, K is identified with the spectrum of $C^*_{\mathrm{red}}(\mathcal{N} \subset \mathcal{M})$.

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Conformal Field Theories as Scaling Limit of Anyonic Chains

Zhenghan Wang

This talk is based on the joint work [1]. In the study of two dimensional topological phases of matter, the bulk physics is modelled by anyon models or unitary modular tensor categories (UMTCs). In fractional quantum Hall physics, the boundary physics is modelled by conformal field theories. It is conjectured that every anyon model can be reconstructed as the representation category of a vertex operator algebra (VOA). In this paper, we investigate this conjecture from the point-view of anyonic chains (ACs).

We provide a mathematical definition of a low energy scaling limit of a sequence of general non-relativistic quantum theories in any dimension, and apply our formalism to anyonic chains. We formulate a conejcture on conditions when a chiral unitary rational (1+1)-conformal field theory would arise as such a limit and verify the conjecture for the Ising minimal model $M(4,3)$ using Ising anyonic chains. Part of the conjecture is a precise relation between Temperley-Lieb generators $\{e_i\}$ and some finite stage operators of the Virasoro generators $\{L_m + L_{-m}\}$ and $\{i(L_m - L_{-m})\}$ for unitary minimal models $M(k+2, k+1)$. Assuming the conjecture, most of our main results for the Ising minimal model $M(4,3)$ hold for unitary minimal models $M(k+2, k+1), k \geq 3$ as well. Our approach is inspired by an eventual application to an efficient simulation of conformal field theories by quantum computers, and supported by extensive numerical simulation and physical proofs in the physics literature.

Anyons are modelled by simple objects in unitary MTCs. ACs are the anyonic analogues of quantum Heisenberg spin chains investigated purely as an academic curiosity first. Abstractly, ACs' conceptual origin can be traced back at least to Jones' Baxterization of braid group representations and his idea of generalized spin chains regarding "spins" as something each with a large algebra of observables at sites and being tensored together with generalized tensor products such as Connes fusion. In the scaling limit, ACs are exactly solvable but not known to be rigorously solvable mathematically. We reverse the logic in this paper to regard ACs as localization of CFTs, thus provide a space locality for VOAs. Our philosophy, as inspired by algorithmic discrete mathematics, is that instead of using ACs to approximate VOAs, VOAs serve as good approximations of sufficiently large finite ACs in their low energy spectrum.

Our limit of a sequence of quantum theories $\{(\mathcal{W}_n, H_n)\}$ will be dictated by both space and energy localities. The Hilbert space $\mathcal{W}$ of a quantum theory has two important bases: the basis B_S encoding the spacial locality, and the basis B_E of energy eigenstates of H. We will refer to the two bases as space basis and energy basis, respectively. Operators can be local with respect to one of the two bases, but there is a tension of locality with respect to both bases. To define a limit of the sequence of quantum theories $\{(\mathcal{W}_n, H_n)\}$, we need to embed the Hilbert space $\mathcal{W}_n$ into $\mathcal{W}_{n+1}$. Which locality of space and energy is preserved by the embedding leads to different notions of limit. We will construct the scaling limit of a sequence of quantum theories $\{(\mathcal{W}_n, H_n)\}$ from their low energy behaviors when the lattice sizes go to zero, therefore we preserve energy locality. Preservation of space locality will lead to the thermodynamic limit.

Besides the Hilbert space and Hamiltonian, another essential feature of any quantum theory is the algebra of observables. Since our quantum theories are non-relativistic, time needs to be addressed differently. The algebra structure of observables encodes consecutive measurements as multiplication, hence somewhat reflects time in the limit. As noted above, our formulation of scaling limit will have **everything** that can be computed using some limit of physical objects. Compared to other well-established formulations of chiral CFTs such as VOAs following Wightman's axioms, and LCNs, our scaling limit results in a much bigger set of observables. In fact, we will show, in the case of Ising anyonic chain, our resulting observables contain as a subset corresponding to smeared fields (or Wightman's) observables $\phi(f)$, a subset corresponding to bounded observables of LCN and a subset corresponding to observables in the VOA $M(4,3)$. We conjecture the same holds for all unitary minimal models $M(k+2,k+1)$ for $k \geq 3$.

An important desideratum of our scaling limit is finitely complete and accessible in the sense that any sequence that should have a limit indeed has one in the scaling limit, and anything in the scaling limit is a limit of some sequence. So the theory in the limit should be completely describable by the sequence of finite theories and there should be no *extra* object that is not some limit of finite objects. Our scaling limits VOA $\mathcal{V} = \oplus_{n=0}^{\infty}\mathcal{V}_n$ should be regarded as computable using the AC approximations. Philosophically, such VOAs $\mathcal{V} = \oplus_{n=0}^{\infty}\mathcal{V}_n$ from ACs categorify computable integral sequences such that each vector space V_i serves as a categorification of the integer $\dim V_i$.

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Modular distortion for II_1 multifactor bimodules

David Penneys

(joint work with Marcel Bischoff, Ian Charlesworth, Samuel Evington, Luca Giorgetti, and André Henriques)

This project started at the 2018 AMS Mathematics Research communities program on Quantum Symmetries: Subfactors and Fusion Categories.

Bimodules over factors and unitary fusion categories. Let A, B be II_1 factors and ${}_AH_B$ an $A-B$ bimodule. We call H *dualizable* if there are maps $\mathrm{ev}_H \in \mathrm{Hom}_{B-B}(\overline{H} \boxtimes_A H \to L^2B)$ and $\mathrm{coev}_H \in \mathrm{Hom}_{A-A}(L^2A \to H \boxtimes_B \overline{H})$ satisfying the zig-zag equations. By [Bis97] (see also [EK98, BDH14]), dualizability is equivalent to H being *bifinite*: $\dim({}_AH)\cdot\dim(H_B) < \infty$, in which case H breaks up as a finite direct sum of simple bimodules. As an example, given a finite index II_1 subfactor, the *state independent* Haagerup L^2 space L^2B [Haa75] is an $A-B$ bimodule. Below, we assume all bimodules are dualizable.

We call ${}_AH_B$ *finite depth* if the unitary multitensor category (semisimple rigid tensor C* category)

$$\mathcal{C} = \mathcal{C}(H) := \begin{pmatrix} {}_A\mathcal{C}_A & {}_A\mathcal{C}_B \\ {}_B\mathcal{C}_A & {}_B\mathcal{C}_B \end{pmatrix} \subset \mathsf{Bim}(A \oplus B)$$

generated by H under $\boxtimes, \oplus, \subseteq, \overline{\cdot}$ is *multifusion* in the sense of [EGNO15].

Definition 1. The *modular distortion* of ${}_AH_B$ is

$$\delta = \delta(H) := \left(\frac{\dim({}_AH)}{\dim(H_B)}\right)^{1/2} \in \mathbb{R}_{>0}.$$

We say ${}_AH_B$ has *constant distortion* if for all sub-bimodules ${}_AK_B \subseteq {}_AH_B$, $\delta(K) = \delta(H)$. We call ${}_AH_B$ *extremal* if ${}_AH_B$ has constant distortion $\delta = 1$.

One can view the modular distortion as an analog of the modular function on a locally compact group, i.e., the ratio of left to right Haar measure.

Remark 11. The set of modular distortions of invertible $A-A$ bimodules is the *fundamental group* of A.

Given a unitary tensor category $\mathcal{C}$ and a group G, a *G-grading* on $\mathcal{C}$ is a decomposition $\mathcal{C} = \bigoplus_{g\in G} \mathcal{C}_g$ such that $\otimes : \mathcal{C}_g \times \mathcal{C}_h \to \mathcal{C}_{gh}$. There is a finest grading called the universal grading group $\mathcal{U}_\mathcal{C}$ [EGNO15]. For a II_1 factor, we denote the universal grading group of the dualizable bimodules $\mathsf{Bim}_\mathsf{d}(A)$ by $\mathcal{U}_A$.

Question 2. *What is $\mathcal{U}_R$ where R is the hyperfinite II_1 factor?*

Observe that δ gives a multiplicative map from the simple dualizable $A-A$ bimodules to $\mathbb{R}_{>0}$, which gives a group homomorphism $\delta : \mathcal{U}_A \to \mathbb{R}_{>0}$. Using this, we have an extremely quick proof of the following folklore result.

Proposition 3 (Folklore, [EK98])**.** *If ${}_AH_A$ is finite depth, then ${}_AH_A$ is extremal.*

Proof. Since $\mathcal{C}(H)$ is fusion, $\mathcal{U}_A$ is finite. Hence $\delta(\mathcal{U}_A) \subset \mathbb{R}_{>0}$ is a compact group, so it must be $\{1\}$. □

By [Pop90], a finite depth hyperfinite II_1 subfactor $A \subset B$ is completely determined by its standard invariant $\mathcal{C}({}_AL^2B_B)$. As a corollary, every unitary fusion category $\mathcal{C}$ admits an essentially unique embedding $\mathcal{C} \hookrightarrow \mathsf{Bim}(R)$, and every embedding is realized by a II_1 subfactor. [FR13, Izu17].

Bimodules over multifactors and unitary multifusion categories. Inspired by our investigation of *bicommutant categories* [HP17], we would like to extend this result to $n \times n$ unitary multifusion categories $\mathcal{C}$. Here, $n \times n$ means $\mathcal{C}$ is indecomposable and $\dim(\mathrm{End}(1_\mathcal{C})) = n$, so we can orthogonally decompose $1_\mathcal{C} = \bigoplus_{i=1}^n 1_i$ into n simples, and $\mathcal{C} = (\mathcal{C}_{ij})_{i,j=1}^n$ where $\mathcal{C}_{i,j} = 1_i \otimes \mathcal{C} \otimes 1_j$.

We observe that an $n \times n$ multifusion category is faithfully graded by the *groupoid* $\mathcal{G}_n$ with n objects and a unique isomorphism between any two objects. Only thinking about the arrows of the groupoid, an operator algebraist may prefer to think of $\mathcal{G}_n$ as a system of matrix units for $M_n(\mathbb{C})$.

One can already see there will be a slight difference for embeddings of 2×2 unitary multifusion categories.

Proposition 4. *Any 2×2 unitary multifusion category admits an essentially unique embedding $\mathcal{C} \hookrightarrow \mathsf{Bim}(R^{\oplus 2})$ up to the modular distortion on $\mathcal{C}_{12}$.*

All distortions can arise from embeddings. However, not all embeddings arise from subfactors $A \subseteq B$ where $\mathcal{C} \hookrightarrow \mathsf{Bim}(A \oplus B)$, as we *always* have $\delta({}_AL^2B_B) = [B : A]^{1/2}$, and the indices of possible subfactors realizing a 2×2 unitary multifusion category will be a discrete subset of $\mathbb{R}_{>0}$ in some interval above 1.

Example 5. Given any projection $p \in P(R)$ with $\mathrm{tr}(p) \in (0, 1]$, we have an embedding

$$\mathsf{Mat}_2(\mathsf{Hilb}_{\mathsf{fd}}) \hookrightarrow \mathsf{Bim}(R \oplus pRp) \qquad \begin{pmatrix} L^2R & L^2Rp \\ pL^2R & pL^2Rp \end{pmatrix}$$

Observe that $\delta(L^2Rp) = \mathrm{tr}(p)^{-1}$ which can take any value in $[1, \infty)$.

In order to embed multifusion categories, we must use II_1 *multifactors*, which are finite direct sums of II_1 factors. Below, A and B will denote multifactors where $A = \bigoplus_{i=1}^a A_i$ and $B = \bigoplus_{j=1}^b B_j$, where $Z(A) = \mathrm{span}_\mathbb{C}\{p_i\}_{i=1}^a$ with $A_i = p_iA$ and $Z(B) = \mathrm{span}_\mathbb{C}\{q_j\}_{j=1}^b$ with $B_j = q_jB$.

A II_1 multifactor bimodule ${}_AH_B$ is dualizable if and only if $H_{ij} := p_iHq_j$ is bifinite for all i, j. Again, we will only consider dualizable bimodules. We will also restrict our attention to *connected* bimodules, i.e., those which satisfy

$Z(A) \cap Z(B) \cap B(H) = \mathbb{C}1_H$. The definition of finite depth is the same as above for multifactor bimodules.

Definition 6. The *modular distortion* of ${}_AH_B$ is a partially defined matrix $\delta = \delta(H) \in M_{a\times b}(\mathbb{R}_{>0})$ where $\delta_{ij} = \delta(H_{ij})$ when $H_{ij} \neq 0$. We say ${}_AH_B$ is *extremal* if every $A_i - A_i$ bimodule generated by H in $\mathcal{C}(H)$ is extremal.

Using the fact that a unitary multitensor category has a *universal grading groupoid* $\mathcal{U}_\mathcal{C}$ [Pen18], a similar proof as in Proposition 3 above shows that finite depth implies extremal for multifactor bimodules.

Theorem 7. *The following are equivalent for a multifactor bimodule* ${}_AH_B$*.*

- *H is extremal.*
- *H_{ij} has constant distortion for each i, j, and (δ_{ij}) extends to a well-defined groupoid homomorphism $\mathcal{G}_{a+b} \to \mathbb{R}_{>0}$, i.e.,*

$$\delta_{ij}\delta_{i'j'} = \delta_{ij'}\delta_{i'j} \qquad \forall\, 1 \leq i \leq a \text{ and } \forall\, 1 \leq j \leq b.$$

The analog of Popa's uniqueness theorem for finite depth connected II_1 multifactor inclusions only holds under the additional assumption that the two inclusions have identical distortions.

Example 8 ([Pop95b])**.** Consider the inclusion $P = \mathbb{C} \oplus \mathbb{C} \subset M_2(\mathbb{C}) \oplus \mathbb{C} = Q$ whose bipartite adjacency matrix is

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

where the rows are indexed by i and the columns by j. The inclusion $A = P \otimes R \subset Q \otimes R = B$ does not admit any downward Jones basic construction [Jon83]. Taking the next two steps in the Jones tower $A_0 \subset A_1 \subset A_2 \subset A_3$, we get a Morita equivalent inclusion $A_2 \subset A_3$ with the same standard invariant which manifestly admits two downward basic constructions. One quickly observes these inclusions have different distortions:

$$\delta({}_{A_0}L^2A_{1A_1}) = \begin{pmatrix} 1 & 3/2 \\ 2 & 3 \end{pmatrix} \qquad \delta({}_{A_2}L^2A_{3A_3}) = \begin{pmatrix} 5/2 & 3/2 \\ 5/3 & 1 \end{pmatrix}.$$

One calculates that

$$\delta({}_{A_{2n}}L^2A_{2n+1A_{2n+1}}) \xrightarrow{n\to\infty} \begin{pmatrix} \phi^2 & \phi \\ \phi & 1 \end{pmatrix}$$

where ϕ is the golden ratio.

We calculate general formulas for the behavior of the distortion under Morita equivalence and taking basic constructions using some results from [GdlHJ89]. An inclusion $A \subset B$ admits an infinite Jone tunnel if and only if the distortion is *standard*. This condition is calculated from matrix (D_{ij}) of statistical dimensions of $(L^2B)_{ij}$. We show this is equivalent to Popa's *homogeneity* criterion [Pop95b] when we endow B with the unique Markov trace, and with Giorgetti-Longo's notion of *super-extremality* [GL19]. Using techniques from [Ocn88] and [Pop90], we prove the following.

Theorem 9. *An $n \times n$ unitary multifusion category admits an essentially unique embedding $\mathcal{C} \hookrightarrow \mathsf{Bim}(R^{\oplus n})$ up to the modular distortion.*

Again, not all embeddings are realized from multifactor inclusions, and we have explicit formulas to determine which distortions arise from inclusions.

Remark 12. At this workshop, we learned of the result [Tom18] which could also be used to prove the uniqueness part of the above results.

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On super-modular categories

Julia Plavnik

Elementary particles such as electrons and photons are either fermions or bosons. But elementary excitations of topological phases of matter behave like exotic particles called anyons. Unitary modular categories play an important role since they model the emergent anyon systems of bosonic topological phase of matter. Because of this and their connections to other fields in mathematics, modular tensor categories have been largely studied. While a substantial part of the theory of anyons can be developed using unitary modular categories by bosonization, to fully capture topological properties of anyons in fermionic topological phases of matter require super-modular categories.

A **super-modular category** is a unitary pre-modular category with Müger center braided equivalent to the unitary pre-modular category sVec of super vector spaces. An algebraic motivation for studying super-modular categories is that any unitary braided fusion category is the equivariantization of either a modular or super-modular category, see [13, Theorem 2]. Topological motivations include the study of spin 3-manifold invariants ([13, 1, 2]) and $(3+1)$-TQFTs ([14]).

One way to construct examples of super-modular categories is from modular tensor categories via the Deligne product, more precisely, $\mathcal{B} = \mathcal{C} \boxtimes \mathrm{sVec}$ is a super-modular category if $\mathcal{C}$ is modular. This class of examples is called split super-modular categories and all pointed super-modular categories are of this form. But not all examples of super-modular categories are split. An interesting family of non-split super-modular categories is $\mathrm{PSU}(2)_{4m+2}$. The quantum group modular category $\mathrm{SU}(2)_{4m+2}$, is constructed from $U_q(su_2)$ at the root of unity $q = e^{\frac{\pi i}{4m+4}}$. This modular tensor category has rank $4m+3$ and has a fusion subcategory $\mathrm{PSU}(2)_{4m+2}$ that is generated by simple objects with even labels. So, $\mathrm{PSU}(2)_{4m+2}$ has rank $2m+2$ and it has only two integral objects, the first and last one. Both integral objects are invertible ones, and can be read from the formulas for the S and T-matrices of $\mathrm{PSU}(2)_{4m+2}$ (which are explicitely described in [4]) that the non-trivial invertible generates the Müger center and it has twist -1, i.e., $\mathrm{PSU}(2)_{4m+2}$ is a super-modular category. Moreover, this category is non-split super-modular since $r+1$ does not divide $4m+3$, which shows that the ranks would not work.

Another way to encounter super-modular categories is from **spin-modular categories**, which are modular categories with a distinguish fermion. A distinguished fermion is just an invertible object of order two with twist equal to -1. This fermion gives rise to a $\mathbb{Z}_2$-grading on the modular tensor category, where the trivial component corresponds to the centralizer of the sucategory sVec generated by the fermion. Then, the trivial component has half of the dimension of the original category and it is, by construction, a super-modular category. The family $\mathrm{PSU}(2)_{4m+2}$ is an example of a super-modular category arising from a spin-modular category, the category $\mathrm{SU}(2)_{4m+2}$. It is conjectured that every super-modular categoy arises in this way. This conjecture was first posted by Müger in the most general case of pre-modular fusion categories [12]. Unpublished counter-examples due to Drinfeld exist in the symmetric case and Galindo and Venegas gave some explicit counter-examples in that context in [8]. It is still an open question if every super-modular category admits a minimal modular extension, that is, if there exist a modulartensor category of double of the dimension of the super-modular category that contains it as a fusion subcategory. This is known as the minimal modular extension conjecture and it was proved for sVec by Kitaev [10]. Moreover, it is known that if a minimal modular extension exists then there are exactly 16 non-equivalent ones [11]. In [4], an explicit construction of the minimal modular extensions of $\mathrm{PSU}(2)_{4m+2}$ is given via the zesting construction. The open question of existence of minimal modular extensions was also posed in the language of Witt equivalence classes in [7].

The theory of modular categories is solid and has achieved a good level of maturity. Given the proven success of this theory, the goal is to develop a parallel theory for the super-modular case. There has been important progess on this direction. For example, even if the S-matrix of a super-modular category is non-invertible, it can always be factorized as $S = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes \hat{S}$, where $\hat{S}$ is an invertible and symmetric matrix. Moreover, the T-matrix can be expresed as a product $T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \hat{T}$, with $\hat{T}$ a diagonal matrix. Notice that $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are the S- and T-matrices from sVec, respectively. Since the fermion f is transparent, $f \otimes X \ncong X$, for any simple X, which implies that super-modular categories have even rank. Moreover, $f \otimes X \ncong X^*$, for any X simple. Then, there is a non-canonical partition of the classes of isomorphisms of simple objects $\Pi = \Pi_0 \Pi_1$ in the super-modular category $\mathcal{B}$, such that $\Pi_1 = f\Pi_0$, $\mathbf{1} \in \Pi_0$, and if $X \in \Pi_0$ then $X^* \in \Pi_0$. Associated to this partition, we have the so-called naive fusion rules $\hat{N_{i,j}^k} = N_{i,j}^k + N_{i,j}^{fk}$. One nice feature of the naive fusion rules is that they can be recovered from $\hat{S}$ via a Verlinde type formula. There is also a balancing equation and a formula for the second Frobenious-Schur indicator of self-dual simple objects similar to the ones for modular tensor categories but in terms of $\hat{N}$, $\hat{S}$, and $\hat{T}$. For more details about the properties of super-modular categries see [4], [6] .

Another crucial result is the generalization of the celebrated congruence subgroup Theorem for modular categories by Ng and Schauenburg to the context of super-modular categories. We consider the modular group $\mathrm{SL}(2, \mathbb{Z}) =< \mathrm{s}, \mathrm{t} >$, where $\mathrm{s} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\mathrm{t} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The assignment $\mathrm{s} \to \hat{S}$, $\mathrm{t}^2 \to \hat{T}^2$ gives a projective representation $\hat{\rho}$ of $\Gamma_\theta = \langle \mathrm{s}, \mathrm{t}^2 \rangle \subset \mathrm{SL}(2, \mathbb{Z})$. Moreover, it was proved in [3] that this projective representation has finite image, that is $\hat{\rho}(\Gamma_\theta)$ is a finite group, when the super-modular category admits a minimal modular extension (which is conjectured to be always true).

A fundamental advance for the theory of super-modular categories was the proof of rank-finiteness, i.e., given a rank (number of isomorphism classes of simple objects) there are only finitely many super-modular cateogries (up to equivalence) of that rank. This was proved in the more general setting of G-crossed braided fusion categories in[9]. This results makes the classification by rank plausible. The classification of super-modular categories up to rank 6 was achieved in [6] and all of them are either split or of the form $\mathrm{PSU}(2)_{4m+2}$, for $m = 0, 1, 2$. Also, this classification gives rise to a classification of spin-modular categories up to rank 11. To be able to advance in the classification program, more tools are needed. An essential technique is the study of the Galois symmetries for super-modular categories, which has been deeply developed in [5]. For example, the Galois group of a super-modular category $\mathcal{B}$ is an abelian subgroup of $\mathfrak{S}_r$, where r is half of the rank of $\mathcal{B}$. This has been crutial to push the classification (up to fusion rules) of super-modular up to rank 8, when the quantum dimensions and naive fusion rules are bounded by 14. The study of the possible $\hat{S}$-matrices for each abelian subgroup of $\mathbb{S}_4$ and the possible splitting of the naive fusion rules give 3 prime realizations of super-modular categories with the given bounds [5].

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Free products of finite-dimensional von Neumann algebras and free Araki-Woods factors

Michael Hartglass
(joint work with Brent Nelson)

The study of von Neumann algebra free products can be traced to the development of the field of free probability by Voiculescu in the late 1980's. In 1993, Dykema was able to classify the structure of free products of finite-dimensional von Neumann algebras $(A_1, \phi_1) * (A_2|, \phi_2)$ equipped with tracial states ϕ_1 and ϕ_2 in terms of the interpolated free group factors $(L(\mathbb{F}_t))_{t>1}$ [1]. In 1997, Dykema studied free products of finite-dimensional von-Neumann algebras equipped with non-tracial states, and determined a necessary and sufficient condition for factorality as well as the spectrum of the modular operator in terms of spectra of the modular operators for each ϕ_i [2]. Dykema asked the whether $(A_1, \phi_1) * (A_2, \phi_2) \cong (B_1, \psi_1) * (B_2, \psi_2)$ whenever the groups generated by $\sigma(\Delta_{\phi_1}) \cup \sigma(\Delta_{\phi_2})$ and $\sigma(\Delta_{\psi_1}) \cup \sigma(\Delta_{\psi_2})$ coincide. Here Δ_φ is the modular operator of a faithful normal state, φ, and $\sigma(x)$ is the spectrum of an operator x.

Utilizing a Fock space construction, Shlyakhtenko constructed the free Araki–Woods factors, a natural type III analogue for the free group factors [7]. In the almost periodic case, these are denoted as (T_G, φ_G) where G is a countable multiplicative subgroup of the positive real numbers. The state φ_G is almost periodic with point spectrum G, and $(T_G, \varphi_G) \cong (T_H, \varphi_H)$ if and only if $G = H$. The construction of these factors, along with Dykema's work on non-tracial free products, prompted Shlyakhtenko to ask whether the free product factors of Dykema are in fact isomorphic to free Araki–Woods factors.

Houdayer gave a partial answer to this question in 2007 when he investigated certain free products of the form $(\mathbb{C} \oplus \mathbb{C}) * M_2(\mathbb{C})$ and $M_2(\mathbb{C}) * M_2(\mathbb{C})$ and identified them with free Araki–Woods factors [5]. Houdayer's result was a significant step in understanding non-tracial free products, but his work could not determine every finite dimensional free product. Even free products of the form $M_m(\mathbb{C}) * M_m(\mathbb{C})$ were not able to be identified.

In joint work with Nelson, [4], we have been able to completely classify the structure of free products of finite dimensional von Neumann algebras in terms of free Araki Woods factors (possibly direct sum a finite-dimensional piece), therefore giving a positive answer to questions of Dykema and Shlyakhtenko. We also extend the classification to free products of certain hyperfinite von Neumann algebras with almost periodic states.

Our new tool is analyzing a von Neumann algebra assigned to a weighted graph [3]. We are able to identify more complicated free products in terms of this weighted graph, and use techniques developed in our other work on these weighted graphs to identify these free products. The study of these weighted graph von Neumann algebras can be traced to Jones and Penneys when they used these to construct a discrete subfactor $N \subset M$ where $N \cong L(\mathbb{F}_\infty)$ and M is a type III_λ factor for $\lambda \in (0,1]$ [6]. A corollary of our analysis is identifying M with an appropriate free Araki–Woods factor.

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New Spherical Planar Algebras and Fusion Rules

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(joint work with Christopher Ryba)

We constructed a continuous family of spherical planar algebras C with a generic parameter q in [1]. When $q = e^{\frac{2\pi i}{2N+2}}$, we constructed unitary fusion categories $C^{N,k,\ell}$ with three discrete parameters $N, k, \ell \in \mathbb{N}$, $N > 0$, $k + \ell > 0$. We also obtain infinitely many non-unitary semi-simple pivotal monoidal category, and non-semisimple ones for different $q \in \mathbb{C}$. For each $C^{N,k,\ell}$, we obtain a 3D Turaev-Viro TQFT. Two sequences $C^{N,0,1}$ and $C^{N,1,1}$, $N \in \mathbb{N}^+$ are particularly interesting. We showed that the unitary fusion category $C^{N,0,1}$ (or $C^{N,1,1}$) is a module category of the representation category of quantum $SU(N)_{N+2}$ (or $SU(N+2)_N$). Such modules categories are called exceptional quantum subgroups by Adrian Ocneanu. We obtain a new type of Schur-Weyl duality for families of exceptional or (type E) quantum subgroups. (This is different from the Schur-Weyl duality from the co-ideal construction of conformal pairs. In that approach, one obtains type D quantum subgroups of quantum $SU(N)$).

Feng Xu constructed an exceptional quantum subgroup of $SU(N)_{N+2}$ using the α-induction of the conformation inclusion $SU(N)_{N+2} \subset SU(N(N+1)/2)_1$ in 1998 [2]. It is not hard to believe that it is isomorphic to $C^{N,0,1}$, because exceptional quantum subgroups are rare in general, but it is not easy to prove two (higher-rank) unitary fusion categories are isomorphic. In recently joint work with Xu [3], we are able to prove that they are isomorphic using the classification result in [1].

We remark that the fusion categories $C^{N,1,0}$, $N \in \mathbb{N}^+$, can be parameterized as a spherical category $C^{q,1,0}$ over $\mathbb{C}(q)$. We conjecture that the corresponding three-manifold invariant from Turaev-Tiro TQFTs can also be parameterized. The exceptional quantum subgroups $C^{N,0,1}$ can not be parameterized as a spherical category $C^{q,0,1}$ over $\mathbb{C}(q)$, because the pivotal structure is not well-defined in the limit $N \to \infty$.

Computing the fusion rule of the exceptional quantum subgroups $C^{N,0,1}$, $N \in \mathbb{N}^+$, has been an challenging open question, since Xu constructed this sequence from conformal inclusions in 1998. The fusion rules were known only for small rank cases ($N \leq 5$). Ocneanu illustrates the fusion rule of the fundamental representations acting on the modules for quantum $SU(N)$, $N = 2, 3, 4$, as a colored graph in [5].

In recent joint work with Christopher Ryba [4], we are able to compute the fusion rule of $C^{q,1,0}$ in a closed from in terms of the Littleword-Richardson coefficients. The simple objects of $C^{q,1,0}$ are labelled by all Young diagrams. We obtain a new fusion rule on Young diagrams. We also obtained the characters of these simple objects and their generating function in a closed form in terms of symmetric polynomials with infinitely many variables. Furthermore, for the fusion category $C^{N,k,\ell}$, we obtain the fusion rule for the action of the fundamental representations in a closed form. In particular, we answered the question of Xu on the fusion rule of the exceptional quantum subgroup $C^{N,0,1}$ posed in 1998. This is the first known fusion rule for a sequence of exceptional quantum subgroups.

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Rokhlin actions of fusion categories

Yuki Arano

The classification problem of a single operator algebra has been worked out by many people both in C*-algebra and von Neumann algebra setting. As a next step, it is natural to classify relative positions of two algebras, namely the subalgebras. In the von Neumann algebra setting, the classification of subfactor, which is a main topic of this conference, has been fascinated many operator algebraists.

One can summarize a strategy of the classification as follows:

First from a finite-index inclusion of C*-algebras $A \subset B$ (or a subfactor), we consider the rigid C*-tensor category Bimod(A) of all Hilbert A-bimodules of finite index, so that B sits in this category. Furthermore B admits an algebra object structure (i.e. the multiplication internal to the tensor category)

$$B \otimes_A B \to B.$$

Now this B and its algebra structure recovers $A \subset B$. The algebra object coming in this way is axiomatized in terms of Q-systems of Longo, so this gives a one-to-one correspondence between the Q-systems in $\mathrm{Bimod}(A)$ and finite-index extension B of A. This does not reduce the problem easier, since the category $\mathrm{Bimod}(A)$ is usually too huge to hundle. Instead consider the C*-tensor category $\mathcal{C}$ generated by A (which is automatically rigid), which is often small enough to handle, especially when the index is small enough. Using this, we can decompose the problem into the following three parts.

(1) Determine what kind of rigid C*-tensor categories arises in this way.
(2) Classify all Q-system in a given rigid C*-tensor categories.
(3) Classify all realization of a given rigid C*-tensor category as bimodules over A. In other words, classify all (fully faithful) tensor functor $\mathcal{C} \to \mathrm{Bimod}(A)$.

As a particular example of rigid C*-tensor category, consider the rigid C*-tensor category $\mathrm{Hilb}_G^{\mathrm{f}}$ of all G-graded Hilbert spaces where G is a discrete group. In this case, the problem (1) corresponds to the classification of the group itself, where as the Q-system (modulo Morita equivalence) is in one-to-one correspondence with the pair of the finite subgroup $H \subset G$ and the second cohomology class $\omega \in H^2(H, T)$. We would like to think these data as an invariant of the inclusion instead of classifying all of them. We are particularly interested in the problem (3), since the functor $\mathrm{Hilb}_G^{\mathrm{f}} \to \mathrm{Bimod}(A)$ is nothing but the (stabilized cocycle) action of G on A. In this way, we will see the functor $\mathcal{C} \to \mathrm{Bimod}(A)$ as an action of a tensor category on an operator algebra.

In the von Neumann algebra case, one can classify all group actions on a hyperfinite factor. Furthermore, by Popa's striking result [4], such action of tensor category is also known to be unique (up to some equivalence relation) for hyperfinite II_1-factor. This is related to Tomatsu's talk in this workshop.

In the C*-algebra case, it is difficult to classify the finite group actions on a C*-algebra without any additional assumption. In [2], Izumi introduced the property so-called the Rokhlin property and used it to classify finite group actions on a C*-algebra. We mimic this in the actions of tensor categories and give a classification result for actions of rigid C*-tensor categories on a C*-algebra, which also gives a first classification result for inclusions of C*-algebras. The formal statement is as follows. We refer [1] for the detailed definition appearing in the theorem.

Theorem 1. Let $\mathcal{C}$ be a C*-fusion category. Let $\alpha, \beta \colon \mathcal{C} \to \mathrm{Bimod}(A)$ be two actions of $\mathcal{C}$ with the Rokhlin property. Assume $\alpha(X)$ and $\beta(X)$ are approximately unitarily equivalent for each $X \in \mathcal{C}$. Then α and β are naturally tensor equivalent modulo $\mathrm{Aut}(A)$.

This particular shows the following.

Theorem 2. Let $A \subset B$ and $A \subset B'$ be two inclusions of simple C*-algebras where the associated tensor category $\mathcal{C}$ is fusion and the action $\mathcal{C} \to \mathrm{Bimod}(A)$ has the Rokhlin property. Assume B and B' are approximately unitarily equivalent as A-bimodules. Then there exists an isomorphism $\theta\colon B \to B'$ such that $\theta(A) = A$.

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Centrally free actions of amenable C*-tensor categories on von Neumann algebras

Reiji Tomatsu

A *cocycle action* of a C*-tensor category $\mathscr{C}$ on a properly infinite von Neumann algebra M is a unitary tensor functor $(\alpha, c)\colon \mathscr{C} \to \mathrm{End}(M)_0$. Namely, we have endomorphisms α_X with conjugate endomorphisms, $T^\alpha \in M$ for $T \in \mathscr{C}(X,Y)$ and also unitaries $u_{X,Y} \in M$ for $X, Y \in \mathscr{C}$ satisfying

- $\alpha_X \circ \alpha_Y = \mathrm{Ad}\, u_{X,Y} \circ \alpha_{X\otimes Y}$, $\alpha_{\mathbf{1}} = \mathrm{id}$,
- $u_{X,Y} u_{X\otimes Y,Z} = \alpha_X(u_{Y,Z}) u_{X,Y\otimes Z}$, $u_{X,\mathbf{1}} = 1 = u_{\mathbf{1},X}$,
- $T^\alpha \alpha_X(x) = \alpha_Y(x) T^\alpha$ for $x \in M$,
- $u_{X,Y}[S\otimes T]^\alpha = S^\alpha \alpha_U(T^\alpha) u_{U,V}$ for $S \in \mathscr{C}(U,X)$ and $T \in \mathscr{C}(V,Y)$.

The amenability of $\mathscr{C}$ in the sense of Popa means the Følner type condition on $\mathrm{Irr}(\mathscr{C})$ holds [1].

We will say that a cocycle action (α, c) of $\mathscr{C}$ on M is *centrally free* if for every $X \in \mathrm{Irr}(\mathscr{C})\backslash\{\mathbf{1}\}$, there exists no non-zero projection $p \in M$ such that $\alpha_X^\omega(x)p = xp$ for all $x \in M_\omega$. Our main result is the following [4].

Theorem 1. Let $\mathscr{C}$ be an amenable rigid C*-tensor category and M an infinite factor. Let (α, c^α) and (β, c^β) be a centrally free cocycle actions of $\mathscr{C}$ on M. If they are *approximately unitarily equivalent*, then they are cocycle conjugate. Namely, there exist $v_X \in M$ and $\theta \in \overline{\mathrm{Int}}(M)$ such that

- $\mathrm{Ad}\, v_X \circ \alpha_X = \theta \circ \beta_X \circ \theta^{-1}$ for $X \in \mathscr{C}$.
- $v_X \alpha_X(v_Y) c^\alpha_{X,Y} v^*_{X\otimes Y} = \theta(c^\beta_{X,Y})$.
- $v_Y T^\alpha v_X^* = \theta(T^\beta)$ for $T \in \mathscr{C}(X,Y)$.

If $\mathscr{C} = \mathrm{Hilb}_\Gamma$, the C*-tensor category of the finite dimensional Γ-graded Hilbert spaces (Γ denotes an amenable discrete group), then we have the following.

Corollary 2 (Connes, Jones, Ocneanu)**.** Two centrally free cocycle actions of an amenable discrete group on an AFD factor are cocycle conjugate when they are approximately unitarily equivalent.

For the representation category $\mathscr{C} = \mathrm{Rep}(G)$ of a coamenable compact quantum group of Kac type, we have the corollary obtained by Masuda and the author.

Corollary 3. Two centrally free cocycle actions of an amenable discrete Kac quantum group on an AFD factor are cocycle conjugate when they are approximately unitarily equivalent.

If M is the AFD type III_1 factor, then any two endomorphisms are approximately unitarily equivalent and a centrally trivial endomorphism is a direct sum of modular automorphisms. Hence we obtain the following result proved by Izumi for fusion categories and Masuda for strongly amenable categories.

Corollary 4 (Izumi, Masuda)**.** Any outer cocycle actions of amenable $\mathscr{C}$ on the AFD type III_1 factor M with no modular parts are cocycle conjugate.

Our result also has an application to subfactor theory. Let $\rho\colon N \to M$ be an inclusion $*$-homomorphism between infinite factors N and M. Suppose that ρ has a conjugate $*$-homomorphism $\overline{\rho}\colon M \to N$. Then ρ generates the full subcategory $\mathscr{C}^\rho = (\mathscr{C}^\rho_{ij})^1_{i,j=0}$ of the following C*-2-category:

$$\mathscr{D} := \begin{pmatrix} \mathscr{D}_{00} & \mathscr{D}_{01} \\ \mathscr{D}_{10} & \mathscr{D}_{11} \end{pmatrix} = \begin{pmatrix} \mathrm{Mor}(N,N)_0 & \mathrm{Mor}(M,N)_0 \\ \mathrm{Mor}(N,M)_0 & \mathrm{Mor}(M,M)_0 \end{pmatrix}.$$

Note $\rho \in \mathscr{C}^\rho_{10} \subset \mathscr{D}_{10}$ and $\overline{\rho} \in \mathscr{C}^\rho_{01} \subset \mathscr{D}_{01}$.
The standard invariant $\mathcal{G}(\mathscr{C}^\rho)$, which is introduced by Jones and axiomatized by Popa, is the data consisting of a certain commutative diagram with left inverses and the Jones projections. It is known that $\mathcal{G}(\mathscr{C}^\rho)$ is the complete isomorphism invariant for C^*-2-categories $\mathscr{C}^\rho$. Then we recover the following Popa's celebrated classification theorem of amenable subfactors ([2, Theorem 5.1], [3, Remark 7.2.1]).

Theorem 5. Let $N \overset{E}{\subset} M$ and $Q \overset{F}{\subset} P$ be inclusions of factors with separable preduals and finite indices. Suppose the following conditions hold:

- N and Q are isomorphic.
- The standard invariants of $N \overset{E}{\subset} M$ and $Q \overset{F}{\subset} P$ are amenable and isomorphic.
- $N \subset M$ and $Q \subset P$ are centrally free.
- $N \overset{E}{\subset} M$ and $Q \overset{F}{\subset} P$ are approximately inner.

Then $N \overset{E}{\subset} M$ is isomorphic to $Q \overset{F}{\subset} P$.

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Families of quadratic categories and their modular data

Pinhas Grossman
(joint work with Masaki Izumi)

Let $\mathcal{C}$ be a unitary fusion category. Let $\mathrm{Irr}(\mathcal{C})$ be the set of isomorphism classes of simple objects. Let $\mathrm{Inv}(\mathcal{C}) \subseteq \mathrm{Irr}(\mathcal{C})$ be the set of isomorphism classes of invertible objects, which forms a group. The group $\mathrm{Inv}(\mathcal{C})$ acts on $\mathrm{Irr}(\mathcal{C})$ by (left) tensor product. If $\mathrm{Inv}(\mathcal{C}) = \mathrm{Irr}(\mathcal{C})$, then the category $\mathcal{C}$ is called pointed. It is known that every pointed fusion category is of the form Vec_G^{ω}, the category of G-graded vector spaces with associator given by $\omega \in H^3(G, \mathbb{C}^*)$ (where $G = \mathrm{Inv}(\mathcal{C})$).

The category $\mathcal{C}$ is called quadratic if $\mathrm{Irr}(\mathcal{C})$ has a unique nontrivial orbit under the action of $\mathrm{Inv}(\mathcal{C})$. Near-group categories are quadratic categories such that $|\mathrm{Irr}(\mathcal{C}) \backslash \mathrm{Inv}(\mathcal{C})| = 1$. In this case, if we label the simple objects by $g \in G (= \mathrm{Inv}(\mathcal{C}))$ and label the unique non-invertible simple object by X, then the fusion rules are given by the formulas

$$g \otimes h \cong gh, \quad g \otimes X \cong X \otimes g \cong X \text{ and } X \otimes X \cong \bigoplus_{g \in G} g \oplus mX,$$

where m is a non-negative integer.

Another class of quadratic categories are Haagerup-Izumi categories. For a Haagerup-Izumi category $\mathcal{C}$, we have $|\mathrm{Irr}(\mathcal{C}) \backslash \mathrm{Inv}(\mathcal{C})| = |\mathrm{Inv}(\mathcal{C})|$ and the simple objects are labeled by g and $g \otimes X$ for $g \in \mathrm{Inv}(\mathcal{C})$ and a particular non-invertible simple object X. The fusion rules are given by

$$g \otimes h \cong gh, \quad g \otimes X \cong X \otimes g^{-1}, \text{ and } X \otimes X = 1 \oplus \bigoplus_{g \in G} g \otimes X.$$

(The definition given in [10] also has some cohomological assumptions, which are automatically satisfied for categories coming from 3^G subfactors, i.e. when $1 \oplus X$ admits a Q-system).

The principal even part of the Haagerup subfactor, which is the finite-depth subfactor with the smallest index above 4, is a Haagerup-Izumi category for the group $G = \mathbb{Z}_3$ [1]. Haagerup-Izumi categories are known to exist for all cyclic groups of order at most 10 [8, 3, 6], but have not been proven to exist for any groups with order greater than 10 (although numerical evidence for existence for odd cyclic groups up to order 19 is provided in [3]).

If $\mathcal{C}$ is a Haagerup-Izumi category for an odd group G, then the non-invertible simple objects of $\mathcal{C}$ lie in a single orbit under the action of the group of inner automorphisms $\mathrm{Inn}(\mathcal{C}) \cong \mathrm{Inv}(\mathcal{C}) = G$. One can then equivariantize by the action of $\mathrm{Inn}(\mathcal{C})$, and the resulting fusion category $\mathcal{C}^G$ is a near-group category with $\mathrm{Inv}(\mathcal{C}^G) = G \times G$ [9].

This is not the case for even groups, where conjugation by order-two elements fixes the non-invertible simple objects. However, it turns out that under a mild assumption, the number of order-two elements is necessarily small.

Theorem. [5] *Let $\mathcal{C}$ be a Haagerup-Izumi category for a finite Abelian group $G = Inv(\mathcal{C})$, and suppose that the object $1 \oplus (g \otimes X)$ admits a Q-system for every $g \in G$. Let $G_2 = \{g \in G : 2g = 0\}$. Then $|G_2| \leq 4$.*

In light of this result, we consider families of Haagerup-Izumi categories for groups $G = \mathbb{Z}_{2^m} \times \mathbb{Z}_{2^n} \times H$, with $|H|$ odd. The case $m = n = 0$ is the odd case, and is related to near-group categories by equivariantization, as noted above. There are no known examples with both $m, n > 1$.

For other values of m and n and trivial H, a number of prominent examples from the classification of small-index subfactors appear. The case $m = n = 1$ is realized by the even part $3^{\mathbb{Z}_2 \times \mathbb{Z}_2}$ subfactor with index $3 + \sqrt{5}$. A $\mathbb{Z}_3$ equivariantization of this category gives the even part of the 4442 subfactor [11, 10]. The case $m = 2$ and $n = 0$ gives the principal even part of the $3^{\mathbb{Z}_4}$ subfactor. A $\mathbb{Z}_2$ de-equivariantization of this category gives the even part of the $2D2$ subfactor [10]. The case $m = 2$ and $n = 1$ is realized by an equivariantization of a category Morita equivalent to the Asaeda-Haagerup categories [7].

We also compute the modular data for several examples of Haagerup-Izumi categories for small groups and their (de)-equivariantizations, and make some general conjectures. First we recall the situation for pointed categories.

A metric group is a finite Abelian group equipped with a non-degenerate quadratic form. There is an equivalence of categories between the category of metric groups (whose morphisms are group homomorphisms which preserve the quadratic forms) and the category of pointed modular tensor categories (whose morphisms are equivalence classes of braided monoidal functors) [2]. Under this correspondence, the quadratic form gives the T matrix of the modular tensor category.

Evans and Gannon conjectured a remarkably simple formula for the modular data of the Drinfeld center of a near-group category for an odd group H [4]. Their formula is expressed entirely in terms of two metric groups. One of these groups is H, with the quadratic form determined by the associativity structure of the near-group category. However, the other group is mysterious, and is not apparent in the near-group structure.

We generalize their conjecture to near-group categories for not-necessarily-odd groups, and formulate conjectures for the modular data of Drinfeld centers of various families of Haagerup-Izumi categories and their (de)-equivariantizations. In each case, the formula is expressed in terms of a pair of involutive metric groups, along with some conditions on their Gauss sums, fixed point subgroups, and relative sizes.

Ultimately, we describe five infinite families of potential modular data in [6]. For each family, the modular data for certain choices of groups and quadratic forms are conjectured to be realized by Drinfeld centers of quadratic categories. These conjectures have been verified for small examples, but the problem of existence of the corresponding infinite families of quadratic categories remains wide open. The families of potential modular data also includes many examples which do not correspond to Drinfeld centers of fusion categories, and the realization of these examples remains an interesting open question as well.

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Locally convex topological completions of modules for a vertex operator algebra

Yi-Zhi Huang

Vertex operator algebras are algebraic structures such that conformal field theories can be constructed and studied using modules and intertwining operators for them. See [H2], [H3], [H8] and [H11] for a program to construct conformal field theories using the representation theory of vertex operator algebras.

For a vertex operator algebra satisfying certain natural finiteness and reductivity conditions, the intertwining operators among irreducible modules satisfy commutativity and associativity (operator product expansion) (see [H1] and [H9]). An intertwining operator algebra is roughly speaking the direct sum of the (finitely many) irreducible modules for such a vertex operator algebra equipped with the (finite-dimensional) spaces of intertwining operators satisfying commutativity and associativity (see [H2] and [H6]). The notion of intertwining operator algebra can be viewed as a mathematical definition of chiral genus-zero rational conformal field theory (see [H4]).

Under stronger finiteness and reductivity conditions, intertwining operators have the modular invariance property, that is, the q-traces of products of intertwining operators form modular invariant vector spaces [H10]. These modular invariant vector spaces give genus-one chiral rational conformal field theories.

But to construct chiral weakly conformal field theories in the sense of G. Segal (see [S]), we need to construct locally convex topological completions of modules appearing in the intertwining operator algebras and prove that one can associate continuous and trace-class maps between tensor products of these completions to Riemann surfaces with parametrized boundaries and that Segal's axioms hold. We also expect that such completions should be closely related to the Hilbert spaces in the approach to conformal field theory using conformal nets.

In [H5] and [H7], the author constructed locally convex topological completions of finitely generated modules for a finitely generated vertex operator algebra using the correlation functions obtained from the products of vertex operators for modules and the exponentials of the Virasoro operators. But these completions are not large enough to allow the actions of intertwining operators and operators associated to genus-one elements.

For genus-zero chiral conformal field theories, correlation functions are given by analytic extensions of products of intertwining operators among modules. For genus-one chiral conformal field theories, correlations functions are given by analytic extensions of q-traces of genus-zero correlation functions. To construct locally convex topological completions of modules, we need to use all these correlation functions. But when we construct higher-genus correlation functions, we need to prove that multi-q-traces of genus-zero correlation functions are convergent. This convergence is still a conjecture now. In fact this higher-genus convergence conjecture is the main unsolved problem in the construction of higher-genus rational conformal field theories.

If we assume that this higher-genus convergence conjecture is true, then the idea in the construction in [H5] and [H7] still works. Consider a vertex operator algebra satisfying the finiteness and reductive conditions mentioned above so that the associativity and modular invariance of intertwining operators hold. Assume that the higher-genus convergence conjecture is true. Then using the correlation functions obtained from taking multi-q-traces of the genus-zero correlation functions obtained from the analytic extensions of products of intertwining operators, we can generalize the construction in [H5] and [H7] to obtain locally convex topological completions of modules for the vertex operator algebra. The correlation functions can be extended to maps between tensor products of these completions and we obtain a conformal field theory in the sense of G. Segal.

We now state a conjecture which should be related to the equivalence between the vertex operator algebra approach and the conformal nets approach to unitary rational conformal field theories.

Conjecture. If the chiral conformal field theory is unitary, then the Hilbert space completions and the locally convex topological completions of modules for the vertex operator algebra are the same.

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Semisimplicity for finite, non-zero index vertex operator subalgebras

Robert McRae

Subtheories of conformal field theories are rich sources of new conformal field theories, but then the question arises of whether the subtheory inherits nice properties, such as rationality, from the old one. For example, in the conformal nets approach to two-dimensional chiral conformal field theory, the following is known: if $B \subseteq A$ is a finite index inclusion of conformal nets, then B is rational if and only if A is. So far, it is still a conjecture that an equivalent result holds in the vertex operator algebra approach to conformal field theory. Indeed, it is not even completely clear what the "index" of a vertex operator algebra inclusion $V \subseteq A$ should be.

Here we present a rationality result for vertex operator subalgebras $V \subseteq A$ under the assumption that V has a braided ribbon tensor category $\mathcal{C}$ of modules that includes A, so that we can take the index of the inclusion to be the categorical dimension of A in $\mathcal{C}$. This means the index will be finite, but since A and V need not be unitary, we will need to assume the index is non-zero. Since rationality for vertex operator algebras is a semisimplicity property, the theorem we will present is a semisimplicity result for categories of grading-restricted modules, which have finite-dimensional homogeneous subspaces.

The starting point for studying extensions $V \subseteq A$ with A a module in a braided tensor category $\mathcal{C}$ of V-modules is [4, Theorem 3.2], which states that A is a commutative algebra in $\mathcal{C}$. Specifically, the vertex operator for A induces a multiplication homomorphism

$$\mu_A : A \boxtimes A \to A,$$

where $\boxtimes$ is the Huang-Lepowsky tensor product of V-modules. The multiplication μ_A is associative and commutative in the sense that

$$\mu_A \circ (\mathrm{Id}_A \boxtimes \mu_A) = \mu_A \circ (\mu_A \boxtimes \mathrm{Id}_A) \circ \mathcal{A}_{A,A,A}, \qquad \mu_{A,A} = \mu_{A,A} \circ \mathcal{R}_{A,A},$$

where $\mathcal{A}$ and $\mathcal{R}$ are the natural associativity and commutativity isomorphisms of $\mathcal{C}$, respectively. Moreover, the algebra is unital: the inclusion $\iota_A : V \to A$ satisfies

$$\mu_A \circ (\iota_A \boxtimes \mathrm{Id}_A) = l_A,$$

where l is the left unit isomorphism of $\mathcal{C}$.

For any algebra A in $\mathcal{C}$, we have the category $\mathcal{C}_A$ of A-modules in $\mathcal{C}$. Objects are pairs (X, μ_X) where X is an object of $\mathcal{C}$ and $\mu_X : A \boxtimes X \to X$ is an associative, unital A-action. Morphisms in $\mathcal{C}_A$ are $\mathcal{C}$-morphisms that intertwine A-actions. The category $\mathcal{C}_A$ is a tensor category but is not braided. However, the full subcategory $\mathcal{C}_A^0$ of "local" or "untwisted" modules, defined by the property

$$\mu_X \circ \mathcal{R}_{X,A} \circ \mathcal{R}_{A,X} = \mu_X,$$

is braided. For a vertex operator algebra extension $V \subseteq A$, $\mathcal{C}_A^0$ is the category of grading-restricted A-modules in $\mathcal{C}$ [4, Theorem 3.4].

We can now state the main result:

Theorem 1. *Suppose $V \subseteq A$ is a vertex operator algebra extension and A is an object of a braided ribbon category $\mathcal{C}$ of V-modules. In addition, assume that:*

- *There is a V-module homomorphism $\varepsilon_A : A \to V$ such that $\varepsilon_A \circ \iota_A = \mathrm{Id}_V$.*
- *The algebra A is a self-dual object in $\mathcal{C}$ with evaluation given by $\varepsilon_A \circ \mu_A : A \boxtimes A \to V$ and some coevaluation $i_A : V \to A \boxtimes A$.*
- *We have $\mu_A \circ i_A = [A : V]\iota_A$ for some non-zero scalar $[A : V]$.*

Then $\mathcal{C}$ is semisimple if and only if $\mathcal{C}_A^0$ is.

The "only if" direction of this theorem is [5, Theorems 3.2 and 3.3] and is a categorical generalization of Maschke's Theorem for finite groups. The index $[A : V]$ here is actually the categorical dimension of A in $\mathcal{C}$: although the definition of categorical dimension incorporates the ribbon structure on $\mathcal{C}$, that is, the natural isomorphism of objects with their double duals, this natural isomorphism on A will be the identity. Note also that by [5, Lemma 1.20], A will be self-dual with evaluation $\varepsilon_A \circ \iota_A$ automatically if A is a simple algebra.

For the "if" direction of the theorem, we need the index $[A : V]$ to construct a projection functor from $\mathcal{C}_A$ to $\mathcal{C}_A^0$: for any (X, μ_X) in $\mathcal{C}_A$, $[A : V]^{-1}$ times the composition

$$X \xrightarrow{l_X^{-1}} V \boxtimes X \xrightarrow{i_A \boxtimes \mathrm{Id}_X} (A \boxtimes A) \boxtimes X \xrightarrow{\mathcal{A}_{A,A,X}^{-1}} A \boxtimes (A \boxtimes X)$$
$$\xrightarrow{\mathrm{Id}_A \boxtimes \mathcal{R}_{A,X}} A \boxtimes (X \boxtimes A) \xrightarrow{\mathrm{Id}_A \boxtimes \mathcal{R}_{X,A}} A \boxtimes (A \boxtimes X) \xrightarrow{\mathrm{Id}_A \boxtimes \mu_X} A \boxtimes X \xrightarrow{\mu_X} X$$

is the projection from X onto its maximal $\mathcal{C}_A^0$-submodule. Using this projection, we can show that semisimplicity of $\mathcal{C}_A^0$ (a much weaker condition than semisimplicity of all $\mathcal{C}_A$) implies that the unit object V in $\mathcal{C}$ is projective. This plus the rigidity of $\mathcal{C}$ then implies $\mathcal{C}$ is semisimple.

The "if" direction of the theorem reduces the question of rationality for a finite, non-zero index subalgebra to C_2-cofiniteness of the subalgebra plus rigidity of its module category:

Corollary 1. Suppose $V \subseteq A$ is a vertex operator algebra extension such that A is strongly rational, V is C_2-cofinite, the braided tensor category of grading-restricted V-modules is rigid, and there is a V-module homomorphism $\varepsilon_A : A \to V$ such that $\varepsilon_A \circ \iota_A = \mathrm{Id}_V$. If the categorical dimension $[A : V] \neq 0$, then V is strongly rational.

The corollary applies for example to coset extensions: suppose $U \subseteq A$ is a strongly rational vertex subalgebra (with different conformal vector) that is equal to its double commutant and C is its commutant (or coset). Then we get an extension $U \otimes C \subseteq A$, and ε_A exists if A is semisimple as a $U \otimes C$-module. If the category of grading-restricted C-modules is rigid, then $[A : V] \neq 0$ by [2, Theorem 5.12]. Thus if C is C_2-cofinite with a rigid module category, the corollary applies to $V = U \otimes C$, so V, and hence also C, is strongly rational.

To see that the conditions in the corollary are necessary, consider the extension $W(p) \subseteq V_{\sqrt{2p}\mathbb{Z}}$, $p \in \mathbb{Z}_{\geq 2}$, where $W(p)$ is the triplet W-algebra and $V_{\sqrt{2p}\mathbb{Z}}$ is a lattice vertex operator algebra with modified conformal vector. This new Virasoro module structure on $V_{\sqrt{2p}\mathbb{Z}}$ is not semisimple, and $W(p)$ is the maximal semisimple Virasoro submodule. Although $V_{\sqrt{2p}\mathbb{Z}}$ is rational, $W(p)$ is C_2-cofinite [1], and the category of grading-restricted $W(p)$-modules is a rigid braided tensor category [3, 6], $W(p)$ is not rational because its representation category is not semisimple. The corollary fails because the inclusion $W(p) \hookrightarrow V_{\sqrt{2p}\mathbb{Z}}$ has no left inverse and $V_{\sqrt{2p}\mathbb{Z}}$ is not a self-dual $W(p)$-module. Thus the index cannot be defined as in the theorem, and it is natural to expect that the categorical dimension of $V_{\sqrt{2p}\mathbb{Z}}$ as a $W(p)$-module will be 0.

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Module categories and graph planar algebras

Noah Snyder
(joint work with Pinhas Grossman, Scott Morrison, David Penneys, Emily Peters)

A common question that appears throughout algebra is given an algebraic object described by generators and relations how do you show that the object this defines a nontrivial? In complete generality this problem is impossible to solve, in particular it is known that there's no algorithm which takes a presentation of a group and determines whether the group is nontrivial. A major technique used in group theory is to identify what G is by constructing a free and transitive action of G on a set X. In subfactor planar algebras, the main way to show that a subfactor planar algebra given by generators and relations is nonzero is to embed it inside the Jones Graph Planar Algebra of the principal graph Γ [6]. The goal of this talk, based on Section 3 of our paper [4], is to explain how the Jones-Penneys GPA embedding theorem [7] can be reinterpreted in terms of actions, and so is analogous to the above idea in group theory. In particular, this gives a generalization of the GPA embedding theorem to module categories, and in fact says that the classification of module categories is the same as the classification of GPA embeddings. More generally we expect that most constructions in subfactor theory can be generalized to incorporate a module category.

The starting point is the following theorem of Jones-Penneys which motivated constructing subfactor planar algebras via graph planar algebras following Peters [8] and used in Bigelow-Morrison-Peters-S. [1] to construct the Extended Haagerup subfactor.

Theorem 1 (Jones-Penneys)**.** The planar algebra $\text{PA}(N \subset M)$ of a subfactor embeds into the graph planar algebra of its principal graph $\text{GPA}(\Gamma)$

Question 2 (Jones)**.** What are all graphs Γ with embeddings $\text{PA}(N \subset M) \hookrightarrow \text{GPA}(\Gamma)$?

For the Haagerup subfactor a partial answer to this question was given by Peters.

Theorem 3 (Peters)**.** The planar algebra of the Haagerup subfactor embeds into the graph planar algebras of its principal graph, its dual principal graph, and one other graph (called the "broom"). Furthermore, any other graph with this property would have to be quite complicated.

The main result of our Section 3, based on related ideas in Etingof-Ostrik [2] and De Commer-Yamashita [3], is the following.

Theorem 4 (GMPPS)**.** If $\mathcal{C}$ is a pivotal unitary 2-shaded multi-fusion category, then embeddings $\mathcal{C} \hookrightarrow \mathrm{GPA}(\Gamma)$ are classified by pivotal unitary module categories whose fusion rules are given by Γ.

Warning 5. In order to make this statement precise one needs to be a bit careful about how it interacts with automorphisms of Γ. This is made precise in the paper, but it's a little tricky.

Warning 6. Note here that for subfactors $\mathcal{C}$ denotes the whole 2-shaded planar algebra $\mathrm{PA}(N \subset M)$, and *not* just the even part of that planar algebra. Relatedly this explains why both the principal graph and the dual principal graph occur. Namely $\mathcal{C}$ as a $\mathcal{C}$-module is not irreducible and splits into two summands: one of these summands corresponds to embedding into the GPA of principal graph and the other into the GPA of the dual principal graph.

The idea of the proof of our main theorem is given by the following outline:

- Module categories $\mathcal{M}$ correspond to embeddings of categories $\mathcal{C} \to \mathrm{End}(\mathcal{M})$. This is well-known and easy, and furthermore is exactly in analogy to group actions as discussed in the introductory paragraph, where a group action of G on X is the same thing as a map $G \to \mathrm{Aut}(X)$.
- Rewrite $\mathrm{End}(\mathcal{M})$ using graphs to get a version of a "graph planar algebra." This is essentially what is done already in Etingof-Ostrik [2] and De Commer-Yamashita [3].
- Add the adjectives pivotal and unitary. That is, we need to see that unitary pivotal maps $\mathcal{C} \to \mathrm{End}(\mathcal{M})$ correspond to unitary pivotal module categories. This is closely related to definitions of Schaumann in the pivotal case [9], and definitions and theorems of De Commer-Yamashita in the unitary case [3]. We also need to check that the graph-y version of $\mathrm{End}(\mathcal{M})$ agrees exactly with the Jones graph planar algebra (whereas with other adjectives one only gets something analogous to a GPA).

The speaker endeavored to explain that this whole approach could be summarized as thinking about $\mathcal{C} \to \mathrm{End}(\mathcal{M})$ until one achieved a state of Zen at which point it became clear that many theorems were just talking about $\mathcal{C} \to \mathrm{End}(\mathcal{M})$. The audience perhaps remained skeptical.

In particular, we have the following two consequences of our main theorem:

Theorem 7 (BMPPS)**.** The planar algebra of the Haagerup subfactor embeds into the graph planar algebras of its principal graph, its dual principal graph, the broom, and no other graphs.

Proof. Combine our main theorem with the classification of module categories over the Haagerup fusion categories by Grossman-S. [5]. (You need to be slightly careful about shadings in order to recover the correct bipartite graphs which are built by appropriately combining the graphs in GS.) □

Theorem 8 (BMPPS)**.** The Extended Haagerup subfactor has two additional module categories, this gives two new fusion categories in the higher Morita equivalence class of the Extended Haagerup fusion categories.

Proof. By our main theorem we need only construct an embedding of the Extended Haagerup planar algebra into the graph planar algebras of two new graphs. This can be done by exact computer calculation using the machinery developed in [1] □

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Generalised orbifolds of 3d TQFTs

Ingo Runkel

(joint work with Nils Carqueville, Gregor Schaumann, Vincentas Mulevičius)

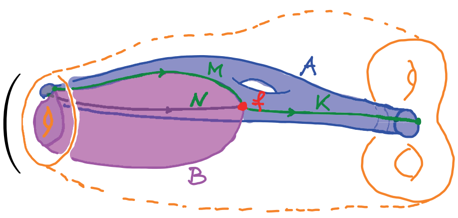
Let $\mathcal{C}$ be a modular fusion category over $\mathbb{C}$, that is, a semisimple $\mathbb{C}$-linear finite braided tensor category with a ribbon twist and whose braiding is non-degenerate.

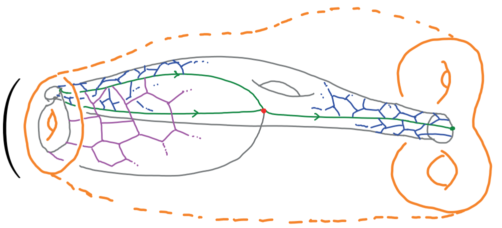
To $\mathcal{C}$, the Reshethikhin-Turaev construction [1] assigns a three-dimensional topological quantum field theory

$$Z^{\mathrm{RT}}_{\mathcal{C}} : \widehat{\mathrm{Bord}}_3(\mathcal{C}) \to \mathrm{Vec}$$

Here, $\widehat{\mathrm{Bord}}_3(\mathcal{C})$ denotes the category of three-dimensional bordisms with embedded $\mathcal{C}$-labelled ribbon graphs. The hat refers to additional geometric structure needed to cancel a glueing anomaly. Vec is the category of $\mathbb{C}$-vector spaces, and $Z^{\mathrm{RT}}_{\mathcal{C}}$ is the symmetric monoidal functor constructed in [1].

It is possible to extend the domain of $Z^{\mathrm{RT}}_{\mathcal{C}}$ to a larger bordism category $\widehat{\mathrm{Bord}}^{\mathrm{def}}_3(\mathcal{C})$ which contains oriented stratifications with strata of dimensions 0–3, labelled by certain data in $\mathcal{C}$. We denote the functor on this extended domain by $Z_{\mathcal{C}}$. Its definition is sketched in the following picture, whose ingredients we proceed to explain:

$Z_{\mathcal{C}}\Bigg($ 

$\Bigg) := Z_{\mathcal{C}}^{\mathrm{RT}}\Bigg($  $\Bigg)$

Let us start with the stratified bordism on the left hand side. The labelling is as follows:

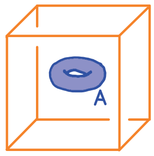
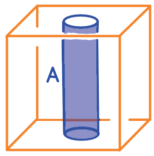
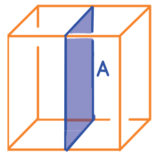
- *3-strata:* No label. Or, equivalently, all 3-strata are labelled by $\mathcal{C}$.
- *2-strata:* Special symmetric Frobenius algebras $A \in \mathcal{C}$, that is, symmetric Frobenius algebras such that product and coproduct compose to id_A and counit and unit compose to $\dim(A)$. (Labels A, B in the picture.)
- *1-strata:* (Bi)modules in $\mathcal{C}$ over the tensor product of algebras which label the 2-strata adjacent to a given 1-stratum. In the picture, M (where the 2-stratum labelled B ends on the 2-stratum labelled A) is an A - $A \otimes B$ bimodule, N an $A \otimes B$ - A bimodule, and K an A-A bimodule.
- *0-strata:* Morphisms in $\mathcal{C}$ with appropriate compatibility with the various algebra actions. In the picture, f is an A-A-bimodule morphism $M \otimes_{A\otimes B} N \to K$.

The value of $Z_{\mathcal{C}}$ on the left hand side is now defined in two steps. First, triangulate all 2-strata, pass to the dual of the triangulation and place ribbons labelled by the corresponding algebra on the edges and (co)product morphisms on the vertices [2, 3, 4]. This produces a bordism in $\widehat{\mathrm{Bord}_3}(\mathcal{C})$ as indicated on the right hand side. On that bordism evaluate the original $Z_{\mathcal{C}}^{\mathrm{RT}}$. To define $Z_{\mathcal{C}}$ on objects one in addition needs to pass to the image of an idempotent.

Theorem [4]**.** $Z_{\mathcal{C}} : \widehat{\mathrm{Bord}_3^{\mathrm{def}}}(\mathcal{C}) \to \mathrm{Vec}$ is a symmetric monoidal functor.

After defining $Z_{\mathcal{C}}$, one obvious question is if it can detect non-isotopic embeddings of surfaces into closed three-manifolds. For embedded spheres one quickly finds that the answer is "no", because the triangulation prescription above implies that an embedded sphere labelled A simply multiplies $Z_{\mathcal{C}}$ by an overall constant $\dim(A)$, independent of the embeddding.

However, $Z_{\mathcal{C}}$ can detect different embeddings of 2-tori, as the following example of three different embeddings into a 3-torus shows. The modular fusion category $\mathcal{C}$ here is that of integrable representations of $\widehat{su}(2)_{16}$, which allows for three Morita-classes of simple algebras [5], and we denote represetatives by A_{17}, D_{10} and E_7. (If one replaces a label A of a 2-stratum by a Morita-equivalent algebra A', and if both A and A' are simple as algebras, then $Z_{\mathcal{C}}$ just changes by an overall constant.) The value of $Z_{\mathcal{C}}$ in each case is:

A	$Z_{\mathcal{C}}$ ()	$Z_{\mathcal{C}}$ ()	$Z_{\mathcal{C}}$ ()
A_{17}	17	17	17
D_{10}	34	18	10
E_7	34	18	7

That the last column coincides with the rank is a consequence of the fact that the trace of the modular invariant matrix of the same designation has this property.

An important application of $Z_{\mathcal{C}}$ is the definition of *generalised orbifolds*. The idea is to "carry out a state-sum construction internal to a given TQFT", and generalised orbifolds can be defined in any dimension [6]. It is similar in spirit to the definition of higher idempotents [7].

Here, we will consider generalised orbifolds of Reshetikhin-Turaev TQFTs [8]. The orbifold theory is defined on $\widehat{\mathrm{Bord}}_3$, which is the same as $\widehat{\mathrm{Bord}}_3(\mathcal{C})$ but without the embedded $\mathcal{C}$-coloured graphs (the orbifold TQFT can be defined on stratified bordisms, too, but the details still have to be worked out). The definition of the orbifold TQFT depends on algebraic data $\mathbb{A}$ (see below) and is roughly as follows:

$$Z_{\mathcal{C}}^{\mathrm{orb},\mathbb{A}}(M) := Z_{\mathcal{C}}\Big(\quad\Big)$$

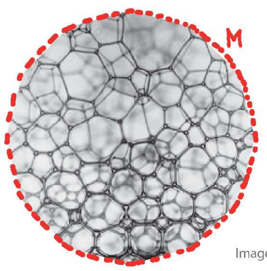


Image credit: ESA

Here, M is a bordism in $\widehat{\mathrm{Bord}}_3$. On the right hand side, we fill M with a "foam" (a stratification such that all cells are contractible). The labelling of the strata is fixed by an orbifold datum, which is a tuple

$$\mathbb{A} = (A, T, \alpha, \bar{\alpha}, \psi, \phi) \ .$$

The algebra A labels 2-strata, T is an A - $A\otimes A$-bimodule which labels 1-strata, and $\alpha, \bar{\alpha} : T \otimes T \to T \otimes T$ label the 0-strata (depending on the various orientations). $\psi \in \mathrm{End}_{AA}(A)^\times$ and $\phi \in \mathbb{C}^\times$ are normalisation constants. This data has to satisfy conditions as detailed in [8] which ensure that the right hand side above is independent of the choice of foam. One finds:

Theorem [8]**.** $Z_{\mathcal{C}}^{\mathrm{orb},\mathbb{A}} : \widehat{\mathrm{Bord}}_3 \to \mathrm{Vec}$ is a symmetric monoidal functor.

There are three key examples of orbifold data, the last of which justifies the name "generalised orbifold" [8].

(1) Given a spherical fusion category $\mathcal{S}$ one obtains an orbifold datum in $\mathcal{C} = \mathrm{Vec}$ with $A = \bigoplus_{U\in\mathrm{Irr}(\mathcal{S})} \mathrm{End}(U)$, $T = \bigoplus_{U,V,W\in\mathrm{Irr}(\mathcal{S})} \mathcal{S}(U \otimes V, W)$, and α, $\bar{\alpha}$ are defined via the associator of $\mathcal{S}$.

(2) Given a commutative special symmetric Frobenius algebra A in any modular fusion category $\mathcal{C}$ one can choose $T = A$, $\alpha = \bar{\alpha} = \Delta \circ \mu$ and $\psi = id_A$, $\phi = 1$.
(3) Let $\mathcal{B}$ be a G-crossed ribbon fusion category (for G a finite group), such that $\mathcal{C} = \mathcal{B}_e$ is modular. Choose a simple object $m_g \in \mathcal{B}_g$ for each $g \in G$. Then we can take $A = \bigoplus_{g\in G} m_g^* \otimes m_g$ and $T = \bigoplus_{g,h\in G} m_{gh}^* \otimes m_g \otimes m_h$.

In example 1, $Z_{\mathcal{C}}^{\text{orb},\mathbb{A}}$ is precisely the Turaev-Viro state sum construction and thus $Z_{\text{Vec}}^{\text{orb},\mathbb{A}}$ agrees with $Z_{\mathcal{Z}(\mathcal{S})}^{\text{RT}}$, where $\mathcal{Z}(\mathcal{S})$ denote the Drinfeld centre of $\mathcal{S}$. In example 2 one expects to obtain $Z_{\mathcal{D}}^{\text{RT}}$ for $\mathcal{D} = \mathcal{C}_A^{\text{loc}}$, the category of local A-modules, and in example 3 one expects $\mathcal{D} = \mathcal{B}^G$, the G-equivariantisation (or gauging) $\mathcal{B}^G$ of $\mathcal{B}$, but the details remain to be worked out.

Quite generally, we expect that $Z_{\mathcal{C}}^{\text{orb},\mathbb{A}}$ is equivalent to $Z_{\mathcal{D}}^{\text{RT}}$ for some modular fusion category $\mathcal{D}$ which lies in the same Witt-class as $\mathcal{C}$. To determine $\mathcal{D}$ from the pair $(\mathcal{C}, \mathbb{A})$ one can investigate what properties Wilson lines in the general orbifolds would have. This leads to the definition of a category [9]

$$\mathcal{C}_{\mathbb{A}} \qquad \text{for } \mathbb{A} = (A, T, \alpha, \bar{\alpha}, \psi, \phi) \text{ an orbifold datum.}$$

The objects of $\mathcal{C}_{\mathbb{A}}$ are triples (M, τ_1, τ_2) where M is an A-A-bimodule and $\tau_{1,2} : M \otimes T \to T \otimes M$ are certain intertwiners, subject to conditions. Morphisms are bimodule intertwiners $f : M \to N$ compatible with the τ's.

Theorem [9]**.** $\mathcal{C}_{\mathbb{A}}$ is a modular fusion category.

In examples 1 and 2 above, we have $\text{Vec}_{\mathbb{A}} \cong \mathcal{Z}(\mathcal{S})$ and $\mathcal{C}_{\mathbb{A}} \cong \mathcal{C}_A^{\text{loc}}$ as ribbon categories, respectively, and the third example (which should give $\mathcal{C}_{\mathbb{A}} \cong \mathcal{B}^G$) remains to be worked out. It is now natural to conjecture that $Z_{\mathcal{C}}^{\text{orb},\mathbb{A}}$ is equivalent to $Z_{\mathcal{C}_{\mathbb{A}}}^{\text{RT}}$.

After the talk, David Penneys and David Reutter explained to me that $\mathcal{C}_{\mathbb{A}}$ is most likely equivalent to an *enriched Drinfeld centre* as defined in [10] (it is in the three examples).

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Von Neumann equivalence and properly proximal groups

Lauren Ruth
(joint work with Ishan Ishan, Jesse Peterson)

Two countable groups Γ and Λ are measure equivalent if they have commuting measure-preserving actions on a σ-finite measure space (Ω, m) such that the actions of Γ and Λ individually admit a finite-measure fundamental domain. This notion was introduced by Gromov in [Gro93, 0.5.E] as an analogy to the topological notion of being quasi-isometric for finitely generated groups. The basic example of measure equivalent groups is when Γ and Λ are lattices in the same locally compact group G. In this case, Γ and Λ act on the left and right of G respectively, and these actions preserve the Haar measure on G. For certain classes of groups, measure equivalence can be quite a course equivalence relation. For instance, the class of countable amenable groups splits into two measure equivalence classes, those that are finite, and those that are countably infinite [Dye59, Dye63, OW80]. Amenability is preserved under measure equivalence, as are other (non)-approximation type properties such as the Haagerup property or property (T). Outside the realm of amenable groups there are a number of powerful invariants to distinguish measure equivalence classes (for example, Gaboriau's celebrated result that states that measure equivalent groups have proportional ℓ^2-Betti numbers [Gab00]) and there are a number of striking rigidity results, such as Furman's work in [Fur99a, Fur99b] where he builds on the superrigidity results of Margulis [Mar75] and Zimmer [Zim84], or Kida's works in [Kid10, Kid11] where he considers measure equivalence for mapping class groups, or for classes of amalgamated free product groups.

If $\Gamma \curvearrowright (X, \mu)$ is a free probability measure-preserving action on a standard measure space, then associated to the action is its orbit equivalence relation, where equivalence classes are defined to be the orbits of the action. If $\Lambda \curvearrowright (Y, \nu)$ is another free probability measure-preserving action, then the actions are orbit equivalent if there is an isomorphism of measure spaces that preserves the orbit equivalence relations, i.e., $\theta(\Gamma \cdot x) = \Lambda \cdot \theta(x)$, for each $x \in X$. If $E \subset X$ is a positive measure subset, then one can also consider the restriction of the orbit equivalence relation to E. The two actions are stably orbit equivalent if there exist positive measure subsets $E \subset X$ and $F \subset Y$ such that the restricted equivalence relations are measurably isomorphic. A fundamental result in the study of measure equivalence is that two groups are measure equivalent if and only if they admit free probability measure-preserving actions that are stably orbit equivalent [Fur99a, Section 3] [Gab05, $P_{\rm ME}5$]. Moreover, in this case one can take the actions to be ergodic. Also associated to each probability measure-preserving action $\Gamma \curvearrowright (X, \mu)$ is the Murray-von Neumann crossed product von Neumann algebra $L^\infty(X, \mu) \rtimes \Gamma$

[MvN37]. This is the von Neumann subalgebra of $\mathcal{B}(L^2(X,\mu)\,\overline{\otimes}\,\ell^2\Gamma)$ that is generated by a copy of $L^\infty(X,\mu)$ acting on $L^2(X,\mu)$ by pointwise multiplication, together with a copy of the group Γ acting diagonally by $\sigma_\gamma \otimes \lambda_\gamma$, where σ_γ is the Koopman representation $\sigma_\gamma(f) = f \circ \gamma^{-1}$ and λ_γ is the left regular representation. The crossed product $L^\infty(X,\mu) \rtimes \Gamma$ is a finite von Neumann algebra with a normal faithful trace given by the vector state corresponding to $1 \otimes \delta_e \in L^2(X,\mu)\,\overline{\otimes}\,\ell^2\Gamma$, and if the action is free then this will be a factor if and only if the action is also ergodic, in which case $L^\infty(X,\mu)$ is a Cartan subalgebra of the crossed product. Non-free actions are also of interest in this setting. In particular, the case when (X,μ) is trivial gives the group von Neumann algebra $L\Gamma$, which is a factor if and only if Γ is ICC, i.e., every non-trivial conjugacy class in Γ is infinite [MvN43]. A celebrated result of Singer shows that two free ergodic probability measure-preserving actions $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\nu)$ are stably orbit equivalent if and only if their von Neumann crossed products are stably isomorphic in a way that preserves the Cartan subalgebras [Sin55]. Singer's result demonstrates that the study of measure equivalence is closely connected to the study of finite von Neumann algebras, and there have been a number of instances where techniques from one field have been used to settle long-standing problems in the other. This exchange of ideas has especially thrived since the development of Popa's deformation/rigidity theory; see for instance [Pop06a, Pop06b, Pop06c, Pop07a, Pop08], or the survey papers [Pop07b, Vae07, Vae10, Ioa13, Ioa18], and the references therein.

Two groups Γ and Λ are W^*-equivalent if they have isomorphic group von Neumann algebras $L\Gamma \cong L\Lambda$. This is somewhat analogous to measure equivalence (although a closer analogy is made between measure equivalence and virtual W^*-equivalence, which for ICC groups asks for $L\Gamma$ and $L\Lambda$ to be virtually isomorphic in the sense that each factor is stably isomorphic to a finite index subfactor in the other factor [Pop86, Section 1.4]) and both equivalence relations preserve many of the same "approximation type" properties. These similarities led Shlyakhtenko to ask whether measure equivalence implied W^*-equivalence in the setting of ICC groups. It was shown in [CI11] that this is not the case, although the converse implication of whether W^*-equivalence implies measure equivalence is still open. As with measure equivalence, we have a single W^*-equivalence class of ICC countably infinite amenable groups [Con76], which shows that W^*-equivalence is quite coarse. Yet there do exist countable ICC groups that are not W^*-equivalent to any other non-isomorphic group [IPV13, BV14, Ber15, CI18].

Returning to discuss measure equivalence, if Γ and Λ have commuting actions on (Ω,μ) and if $F \subset \Omega$ is a Borel fundamental domain for the action of Γ, then on the level of function spaces, the characteristic function 1_F gives a projection in $L^\infty(\Omega,m)$ such that the collection $\{1_{\gamma F}\}_{\gamma\in\Gamma}$ forms a partition of unity, i.e., $\sum_{\gamma\in\Gamma} 1_{\gamma F} = 1$. This notion generalizes quite nicely to the non-commutative setting where we will say that a fundamental domain for an action on a von Neumann algebra $\Gamma \curvearrowright^\sigma \mathcal{M}$ consists of a projection $p \in \mathcal{M}$ such that $\sum_{\gamma\in\Gamma} \sigma_\gamma(p) = 1$, where the convergence is in the strong operator topology. Using this perspective for a fundamental domain we may then generalize the notion of measure equivalence by simply considering actions on non-commutative spaces.

Definition 1. [IPR19] Two groups Γ and Λ are von Neumann equivalent, written $\Gamma \sim_{vNE} \Lambda$, if there exists a von Neumann algebra $\mathcal{M}$ with a semi-finite normal faithful trace Tr and commuting, trace-preserving, actions of Γ and Λ on $\mathcal{M}$ such that the Γ and Λ-actions individually admit a finite-trace fundamental domain.

Von Neumann equivalence is clearly implied by measure equivalence, and, in fact, von Neumann equivalence is also implied by W^*-equivalence. Indeed, if $\theta : L\Gamma \to L\Lambda$ is a von Neumann algebra isomorphism then we may consider $\mathcal{M} = \mathcal{B}(\ell^2\Lambda)$ where we have a trace-preserving action $\sigma : \Gamma \times \Lambda \to \mathrm{Aut}(\mathcal{M})$ given by $\sigma_{(s,t)}(T) = \theta(\lambda_s)\rho_t T \rho_t^* \theta(\lambda_s^*)$, where $\rho : \Lambda \to \mathcal{U}(\ell^2\Lambda)$ is the right regular representation, which commutes with operators in $L\Lambda$. It's then not difficult to see that the rank one projection p onto the subspace $\mathbb{C}\delta_e$ is a common fundamental domain for the actions of both Γ and Λ. In fact, we show that virtual W^*-equivalence also implies von Neumann equivalence.

We introduce a general induction procedure for inducing representations via von Neumann equivalence from Λ to Γ, and using these induced representations we show that some of the properties that are preserved for measure equivalence and W^*-equivalence are also preserved for von Neumann equivalence.

Theorem 2. [IPR19] *Amenability, property (T), and the Haagerup property are all von Neumann equivalence invariants.*

A group Γ is properly proximal if there does not exist a left-invariant state on the C^*-algebra $(\ell^\infty\Gamma/c_0\Gamma)^{\Gamma_r}$ consisting of elements in $\ell^\infty\Gamma/c_0\Gamma$ that are invariant under the right action of the group. Properly proximal groups were introduced in [BIP18], where a number of classes of groups were shown to be properly proximal, including non-elementary hyperbolic groups, convergence groups, bi-exact groups, groups admitting proper 1-cocycles into non-amenable representations, and lattices in non-compact semi-simple Lie groups of arbitrary rank. It is also shown that the class of properly proximal groups is stable under commensurability up to finite kernels, and it was then asked if this class was also stable under measure equivalence [BIP18, Question 1(b)]. Proper proximality also has a dynamical formulation [BIP18, Theorem 4.3], and using this, together with our induction technique applied to isometric representations on dual Banach spaces, we show that the class of properly proximal groups is not only closed under measure equivalence but also under von Neumann equivalence.

Theorem 3. [IPR19] *If* $\Gamma \sim_{vNE} \Lambda$ *then* Γ *is properly proximal if and only if* Λ *is properly proximal.*

An example of Caprace, which appears in Section 5.C of [DTDW18], shows that the class of inner amenable groups is not closed under measure equivalence. Specifically, if p is a prime and F_p denotes the finite field with p elements, then the group $SL_3(F_p[t^{-1}]) \ltimes F_p[t,t^{-1}]^3$ is not inner amenable, although is measure equivalent to the inner amenable group $SL_3(F_p[t^{-1}]) \ltimes F_p[t^{-1}]^3) \times F_p[t]^3$. Using the previous theorem we then answer another question from [BIP18] by providing with $SL_3(F_p[t^{-1}]) \ltimes F_p[t,t^{-1}]^3$ an example of a non-inner amenable group that is also not properly proximal.

The notion of von Neumann equivalence also admits a generalization in the setting of finite von Neumann algebras.

Definition 4. [IPR19] Two finite von Neumann algebras M and N are von Neumann equivalent, written $M \sim_{vNE} N$ if there exists a semi-finite von Neumann algebra $\mathcal{M}$ containing commuting copies M and N^{op}, such that we have intermediate standard representations $M \subset \mathcal{B}(L^2(M)) \subset \mathcal{M}$ and $N^{\mathrm{op}} \subset \mathcal{B}(L^2(N)) \subset \mathcal{M}$ that satisfy the property that finite-rank projections in $\mathcal{B}(L^2(M))$ and $\mathcal{B}(L^2(N))$ are finite projections in $\mathcal{M}$.

We show that this does indeed give an equivalence relation, which is coarser than the equivalence relation given by virtual isomorphism. The connection to von Neumann equivalence for groups is given by the following theorem:

Theorem 5. [IPR19] *If Γ and Λ are countable groups, then $\Gamma \sim_{vNE} \Lambda$ if and only if $L\Gamma \sim_{vNE} L\Lambda$.*

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Geometric quantization via measured Gromov-Hausdorff convergence of metric measure spaces

Mayuko Yamashita
(joint work with Kota Hattori)

In this talk, I explain a new approach to problems in geometric quantizations, using the theory of convergence for metric measure spaces. This is a joint work with Kota Hattori (Keio University).

1. Geometric quantization

On a closed symplectic manifold (X, ω), the prequantum line bundle is a triple (L, ∇, h) of a complex line bundle $\pi\colon L \to X$ equipped with a hermitian metric h and a hermitian connection ∇ whose curvature form F^∇ is equal to $-\sqrt{-1}\omega$. Given a prequantized symplectic manifold $(X, \omega, L, \nabla, h)$, the geometric quantization is a procedure to give a representation of the Poisson angebra consisting of functions on (X, ω) on a Hilbert space $\mathcal{H}$, called the quantum Hilbert space.

There are several known ways to construct quantum Hilbert spaces. One fundamental problem in geometric quantization is to find relations among quantizations given by different methods. In this talk we consider two classes of quantizations, Kähler quantizations and real quantizations, as we now explain.

A Kähler quantization is given by choosing an ω-compatible complex structure J on $X = X_J$. In this case L becomes a holomorphic line bundle over X_J, and the quantum Hilbert space is defined by $\mathcal{H} = H^0(X_J, L)$, the space of holomorphic sections of L. On the other hand, a real quantization is given by choosing a Lagrangian fibration $\mu : X^{2n} \to B^n$. Given a Lagrangian fibration, a point $b \in B$ is called a Bohr-Sommerfeld point if the space of pararell sections on $(L, \nabla)|_{\pi^{-1}(b)}$, denoted by $H^0(\pi^{-1}(b); (L, \nabla))$, is nontrivial. The set of Bohr-Sommerfeld points, $BS \subset B$, is a discrete subset. In this case, the quantum Hilbert space is defined by $\mathcal{H} = \oplus_{b \in BS} H^0(\pi^{-1}(b); (L, \nabla))$.

The first natural problem is whether the dimensions of $\mathcal{H}$ coincide or not. Given a compatible complex structure J and a Lagrangian fibration μ, the equality $\dim H^0(X_J, L) = \#BS$ has been observed in many examples. This includes the case when the Lagarangian fibration is nonsingular and the Kodaira vanishing holds, and the case when μ is the moment map for a toric symplectic manifold, and the case for the moduli space of $SU(2)$-flat connections on a closed surfaces ([4]).

These interesting phenomena lead us to the next problem: Why they coincide? Can we provide a canonical isomorphism between the quantum Hilbert spaces obtained by two quantizations? One way to answer this problem is to construct a one-parameter family of ω-compatible complex structures $\{J_s\}_{s>0}$ on (X, ω) and show that the spaces $H^0(X_{J_s}, L)$ converge to the space $\oplus_{b \in BS} H^0(\pi^{-1}(b); (L, \nabla))$ in an appropriate sense. This has been worked out in several examples, including the case for the abelian varieties by Baier, Mourão and Nunes ([2]) and the case for toric symplectic manifolds by Baier, Florentino, Mourão and Nunes ([1]).

The purpose of this talk is to present a new approach to this problem using the theory of convergence of metric measure spaces. We investigate the behavior of the spectrum of $\bar{\partial}$-Laplacians, in particular that of the holomorphic sections, from the viewpoint of the spectral convergence of the Laplace operators on metric measure spaces.

2. Main results

If we have a ω-compatible complex structure J, it associates a Riemannian metric on X defined by $g_J := \omega(\cdot, J\cdot)$. The metric g_J, together with the hermitian connection ∇ on L, defines a Riemannian metric $\hat{g}_J$ on the frame bundle S of L. We have a canonical isomorphism

$$L^2(X, g_J; L) \simeq (L^2(S, \hat{g}_J) \otimes \mathbb{C})^\rho,$$

where ρ is the S^1 action given by principal S^1-action on $L^2(S, \hat{g}_J)$ and by the formula $e^{\sqrt{-1}t} \cdot z = e^{\sqrt{-1}t} z$ on $\mathbb{C}$. Under this isomorphism, we have an idetification of operators,

$$2\Delta_{\overline{\partial}_J} = \Delta^\rho_{\hat{g}_J} - (n+1),$$

where $\Delta^\rho_{\hat{g}_J}$ denotes the metric Laplacian on $(S, \hat{g}_J)$ restricted to the space $(L^2(S, \hat{g}_J) \otimes \mathbb{C})^\rho$. In this way, we reduce the problem to the analysis of the spectral structure given by $((L^2(S, \hat{g}_J) \otimes \mathbb{C})^\rho, \Delta^\rho_{\hat{g}_J})$.

From now on we fix a nonsingular Lagrangian fibration $\mu\colon X \to B$, and consider one parameter families of ω-compatible complex structures $\{J_s\}_{0<s<\delta}$ on (X, ω). First let us note that, the Lagrangian fibration μ defines a Lagrangian subbundle of $(TX \otimes \mathbb{C}, \omega)$ by the formula $\mathcal{P}_\mu := \ker d\mu \otimes \mathbb{C}$, and an ω-compatible complex structure J also defines a Lagrangian subbundle $\mathcal{P}_J := T^{1,0}_J X$. Let us denote by $\mathrm{Lag}(V, \alpha)$ the space of Lagrangian subspaces of a symplectic vector space (V, α) and a subspace $\mathrm{Lag}(V, \alpha)_+ \subset \mathrm{Lag}(V, \alpha)$ is defined in [3]. We assume the following condition ♠ for $\{J_s\}$. Let $\mathrm{pr}\colon X \times [0, \delta) \to X$ be the projection and $\mathrm{pr}^*\mathrm{Lag}_\omega$ be the pullback bundle.

♠ There is a smooth section $\mathcal{P}$ of $\mathrm{pr}^*\mathrm{Lag}_\omega \to X \times [0, \delta)$ such that $\mathcal{P}(\cdot, s) = \mathcal{P}_{J_s}|_U$ for $s > 0$, $\mathcal{P}(\cdot, 0) = \mathcal{P}_\mu|_U$ and

$$\frac{d}{ds}\mathcal{P}(x, s)\Big|_{s=0} \in T_{\mathcal{P}_\mu(x)}\mathrm{Lag}(T_x X \otimes \mathbb{C}, \omega_x)_+$$

for any $x \in X$.

So the basic strategy is to consider the family $\{(S, \hat{g}_{J_s})\}_{s>0}$ of Riemannian manifolds with isometric S^1-actions, analyze its Gromov-Hausdorff limit space and guarantee the spectral convergence to the operator on the limit space.

As for the convergence of spaces, we have the following. Let g_∞ and ν_∞ be a Riemannian metric and a measure on $\mathbb{R}^n \times S^1$ defined by

$$\begin{aligned} g_\infty &:= \frac{1}{(1+\|y\|^2)}(dt)^2 + \sum_{i=1}^n (dy_i)^2, \\ d\nu_\infty &:= dy_1 \cdots dy_n dt, \end{aligned}$$

where $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ and $e^{\sqrt{-1}t} \in S^1$. We define the isometric S^1-action on $(\mathbb{R}^n \times S^1, g_\infty, \nu_\infty)$ by $(y, e^{\sqrt{-1}t}) \cdot e^{\sqrt{-1}\tau} := (y, e^{\sqrt{-1}(t+\tau)})$ for $e^{\sqrt{-1}\tau} \in S^1$. The following is a part of the main results of [3].

Theorem 2.1 ([3, Theorem 7.16]). *Under the above assumptions, we further assume that there is $\kappa \in \mathbb{R}$ such that $\mathrm{Ric}_{g_s} \geq \kappa g_s$ holds for all $0 < s < \delta$. Let $b \in B$, and fix $u_b \in (\pi \circ \mu)^{-1}(b)$.*

(1) Assume that b is a Bohr-Sommerfeld point. Let $\nu_{\hat{g}_s}$ be the Riemannian measure of $\hat{g}_s$. Then for some positive constant $K > 0$, we have a pointed S^1-equivariant measured Gromov-Hausdorff convergence

$$\left\{\left(S, \hat{g}_s, \frac{\nu_{\hat{g}_s}}{K\sqrt{s}^n}, u_b\right)\right\}_s \xrightarrow{S^1\mathrm{pmGH}} \left(\mathbb{R}^n \times S^1, g_\infty, \nu_\infty, (0,1)\right)$$

as $s \to 0$. The spectrum of the limit Laplacian restricted to the subspace $(L^2(\mathbb{R}^n \times S^1; \nu_\infty) \otimes \mathbb{C})^\rho$ is given by

$$\mathrm{Spec}(\Delta^\rho_{g_\infty}) = 2\mathbb{Z}_{\geq 0} + n + 1,$$

and the multiplicity of the eigenvalue $2N + n + 1$ is $\frac{(N+n-1)!}{(n-1)!N!}$.

(2) Assume that b is not a Bohr-Sommerfeld point. Then, we have

$$\left\{\left(S, \hat{g}_s, \frac{\nu_{\hat{g}_s}}{K\sqrt{s}^n}, u_b\right)\right\}_s \xrightarrow{S^1\mathrm{pmGH}} \left(S^b_\infty, g^b_\infty, \nu^b_\infty, p^b_\infty\right)$$

as $s \to 0$. Here, the left hand side is some metric measure space with isometric S^1-action which satisfies

$$(L^2(S^b_\infty) \otimes \mathbb{C})^\rho = \{0\}.$$

However, the result above does not imply the desired spectral convergence directly, because we have $\mathrm{diam}(S, \hat{g}_{J_s}) \to \infty$ in our situation. By the localization argument of eigenfunctions, we were able to show the following spectral convergence result.

Theorem 2.2 (HY, in preperation). *Under the above assumptions, we have a compact convergence of spectral structures,*

$$((L^2(S, \hat{g}_{J_s}) \otimes \mathbb{C})^\rho, \Delta^\rho_{\hat{g}_{J_s}}) \to \oplus_{b \in BS}((L^2(\mathbb{R}^n \times S^1; \nu_\infty) \otimes \mathbb{C})^\rho, \Delta^\rho_{g_\infty}).$$

In particular, if we denote by λ^j_s the j-th eigenvalue ($j \geq 1$) of $\Delta_{\bar{\partial}_{J_s}}$ acting on $L^2(X; L)$, counted with multiplicity, we can express $\lim_{s \to 0} \lambda^j_s$ as follows. For

$j \geq 1$, let $N(j) \in \mathbb{Z}_{\geq 0}$ be such that the following inequality is satisfied.

$$\#BS \cdot \frac{(N(j) - 1 + n)!}{n!(N(j) - 1)!} < j \leq \#BS \cdot \frac{(N(j) + n)!}{n!(N(j))!}.$$

Then we have

$$\lim_{s \to 0} \lambda_s^j = N(j).$$

In particular, the number of eigenvalues converging to 0 is equal to $\#BS$.

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On Jones' actions of Thompson's groups

Arnaud Brothier

Richard Thompson's groups $F \subset T \subset V$ are countable discrete groups that have been extensively studied in geometric group theory. The group F is the collection of all piecewise linear bijections of the unit interval with slopes power of two and breakpoints dyadic rational. The larger groups T and V are defined similarly except that one allows to permute intervals in a cyclic way for T and in any way for V. In particular, F is a subgroup of the homeomorphisms of the unit interval and T a subgroup of the homeomorphisms of the unit circle. Those three groups follow uncommon behaviours and many questions regarding their analytical properties are open today. The most famous one is to know if F is amenable or not but even the same question regarding weak amenability in the sense of Cowling and Haagerup is still unknown.

Those groups recently appeared in Jones' subfactor theory somewhat surprisingly when Jones was trying to reconstruct a conformal field theory (CFT) directly from a subfactor. Thompson's group T emerged as a discrete replacement of the spatial diffeomorphism group. On the way to build CFT Jones found a beautiful formalism for constructing actions of the Thompson's groups and more generally actions of groups of fractions. This talk is about the use of these new machinery for proving that certain groups have the Haagerup property. The main result is the following:

Theorem: If G is any discrete group with the Haagerup property, then so does the (permutational) wreath product obtained from G and the usual action of Thompson's group V on the set of dyadic rational $\mathbb{Q}_2$ inside the unit interval.

Recall that a discrete group has the Haagerup property if it admits a net of positive definite functions vanishing at infinity and converging pointwise to one. It is a weakening of amenability that is essential in group theory but also in other fields of mathematics like operator algebras, ergodic theory and is linked to topology. Indeed, it is equivalent to Gromov A-(T)-meanability (existence of a proper isometric action on a Hilbert space), implies Baum-Connes conjecture and thus Novikov conjecture. It is also a key property used in Popa's deformation/rigidity theory. As Gromov's terminology suggests it is a strong negation of Kazhdan property (T): a discrete group having both property is necessarily finite. Farley proved that all three Thompson's groups have the Haagerup property using Gromov's characterization. However, this does not imply our theorem. Indeed, the class of groups with the Haagerup property is closed under taking subgroups, under taking direct product but unfortunately is not closed under taking extension and in particular semidirect products of groups having the Haagerup property needs not to satisfy the Haagerup property. Cornulier, Stalder and Valette have exhibit large classes of wreath products without the Haagerup property built from groups with the Haagerup property. Using actions on wall they were able to show that wreath product built from an homogenous action of a group on its quotient by a normal subgroup provides groups with the Haagerup property if the three groups involved have this property. Moreover, by defining wreath product for non-discrete groups and using Schlichting completions Cornulier was able to push this later result by replacing the normal subgroup by a commensurated subgroup. It turns out that the examples we provide using Jones' technology are very different from all previous examples produced. Moreover, thanks to a theorem of Cornulier concerning presentations of groups, they provide the first examples of wreath products with the Haagerup property that are finitely presented and don't have the Haagerup property for trivial reasons that is the group acting is not amenable (here V) and the base space on which V acts is not finite (here $\mathbb{Q}_2$).

Elements of F can be written as a pair of finite rooted binary trees with the same number of leaves. Formally, it is the group of fractions of the category of binary trees (at the object 1). Other nice categories give groups of fractions and for instance the braid groups arise in that way as well as the larger Thompson's groups T, V. Jones realized that given any functor starting from the category of forests to another category one can form an action of F that we call a Jones' action. In particular, if the target category is the category of Hilbert spaces one obtain a unitary representation of F that can be extended to V under mild assumptions on the functor. Our strategy to prove the theorem is then the following. First we produce a one parameter family of unitary representations of V using the deformation of a functor from the forests to Hilbert spaces. Second, we consider a particular functor from the forests to the category of groups. This provides a Jones' action of V on a group. We recognize that the semidirect product is the wreath product of the theorem. Third, we made the key observation that this later semidirect product can be interpreted as a group of fractions for a larger category of decorated forests. Fourth, we adapt the first point to the category of decorated forests and obtain the Haagerup property.

Dynamical characterization of categorical Morita equivalence

Sergey Neshveyev

(joint work with Makoto Yamashita)

Let $\mathbb{C}_1$ and $\mathbb{C}_2$ be essentially small rigid C*-tensor categories. Then the following conditions are equivalent:

(1) there is a rigid C*-bicategory (with two 0-cells) of the form

$$\begin{pmatrix} \mathbb{C}_1 & \mathbb{D} \\ \bar{\mathbb{D}} & \mathbb{C}_2 \end{pmatrix}$$

such that $\mathbb{D} \neq 0$;

(2) there is a C*-Frobenius algebra $B \in \mathbb{C}_2$ such that $\mathbb{C}_1 \cong \mathrm{Bimod}_{\mathbb{C}_2}(B)$;

(3) there is a nonzero $\mathbb{C}_1$-$\mathbb{C}_2$-module C*-category $\mathbb{D}$ such that
  (a) $\mathbb{D}$ is semisimple as a linear category,
  (b) the action of $\mathbb{C}_2$ on $\mathbb{D}$ is proper in the sense that if $(U_i)_i$ are representatives of the isomorphism classes of simple objects in $\mathbb{C}_2$ then, for any objects X and Y, we have $\mathbb{C}_2(X, YU_i) = 0$ for all but a finite number of i's,
  (c) the functor $\mathbb{C}_1 \to \mathrm{End}_{\mathbb{C}_2}(\mathbb{D})$ defined by the action of $\mathbb{C}_1$ on $\mathbb{D}$ is an equivalence of C*-tensor categories.

If these conditions are satisfied, then $\mathbb{C}_1$ and $\mathbb{C}_2$ are called Morita equivalent, and a $\mathbb{C}_1$-$\mathbb{C}_2$-module C*-category $\mathbb{D}$ as in (1) or (3) is called invertible. Such a bimodule category $\mathbb{D}$ is automatically indecomposable.

Assume now that $\mathbb{C}_i = \mathrm{Rep}\, G_i$ for a compact quantum group G_i, $i = 1, 2$. In this case it is known that any indecomposable $\mathbb{C}_1$-$\mathbb{C}_2$-module C*-category is equivalent to the category $\mathbb{D}_A$ of finitely generated G_1-G_2-equivariant C*-Hilbert A-modules for a unital C*-algebra A equipped with commuting actions $G_1 \curvearrowright A \curvearrowleft G_2$. A natural question is what invertibility of $\mathbb{D}_A$ means in terms of the actions of G_i on A.

Theorem. [1] *Assume that we are given commuting actions $G_1 \curvearrowright A \curvearrowleft G_2$ of compact quantum groups G_1 and G_2 on a unital C*-algebra A. Then the corresponding* $(\mathrm{Rep}\, G_1)$-$(\mathrm{Rep}\, G_2)$*-module category $\mathbb{D}_A$ is invertible if and only if the following conditions are satisfied:*

(1) for each $i = 1, 2$, the action of G_i on A is free, or equivalently, the subalgebra of regular elements of A with respect the action of G_i is a G_i-Galois extension of A^{G_i};

(2) the fixed point algebras A^{G_i} are finite dimensional;

(3) there is a G_1-G_2-equivariant isomorphism

$$A^{G_1} \otimes A \cong A^{G_2} \otimes A$$

of $A^{G_1} \otimes A^{G_2}$-A-modules.

Condition (3) in the theorem can be reformulated in the following less digestable, but more concrete and verifiable form:

(a) the quantum dimension $\dim_q A^{G_1}$ of A^{G_1} considered as a G_2-module coincides with the quantum dimension $\dim_q A^{G_2}$ of A^{G_2} considered as a G_1-module;

(b) if ψ_1 is the state on A^{G_1} obtained as the composition of the G_2-equivariant conditional expectation $A^{G_1} \to (A^{G_1})^{G_2}$ with the normalized categorical trace on $(A^{G_1})^{G_2} = \mathrm{End}_{\mathrm{Rep}\, G_2}(A^{G_1})$, $(x_i)_i$ is a basis in A^{G_1} and $(x^i)_i$ is the dual basis with respect to ψ_1, so that $\psi_1(x_i x^k) = \delta_{ik}$, and we similarly define a state ψ_2 on A^{G_2}, choose a basis $(y_j)_j$ in A^{G_2} and consider the dual basis $(y^j)_j$, then

$$\sum_i x^i y x_i = \lambda \psi_2(y) 1 \quad \text{and} \quad \sum_j y^j x y_j = \lambda \psi_1(x) 1$$

for all $x \in A^{G_1}$ and $y \in A^{G_2}$, where $\lambda = \dim_q A^{G_1} = \dim_q A^{G_2}$.

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A generating problem for subfactors

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Modern subfactor theory was initialed by Vaughan Jones by his remark index theorem [1]. Since then, there are many different understanding of the central object, namely, the *standard invariants* for subfactors [2, 3, 4]. Later on, Vaughan Jones introduced the subfactor planar algebras as a topological axiomatization of standard invariants [5]. A planar algebra $\mathscr{P}_\bullet$ consists of a sequence of finite-dimensional C^*-algebras $\mathscr{P}_{m,\pm}$ (which are called the m-box spaces) and a natural action of the operad of planar tangles. This perspective displays that the *standard invariants* is a representation of *fully labeled planar tangles* in the flavor of *topological quantum field theory.*

From the perspective of planar algebras, Bisch and Jones proposed the classification of subfactors by *simple generators and relations* [6, 7, 8]. The motivating examples are the Birman-Murakami-Wenzl (BMW) planar algebras which are known to satisfy the following conditions:

(I) The planar algebra is generated by its 2-boxes.

(II) $\dim \mathscr{P}_{2,\pm} = 3$.

(III) $\dim \mathscr{P}_{3,\pm} \leq 15$.

(IV) The planar algebra $\mathscr{P}_\bullet$ satisfies Yang-Baxter relations.

Such planar algebras are called Yang-Baxter planar algebras and they are completely classified in [9]. Significantly, a new family of subfactor planar algebras were discovered which has a deep connection to conformal field theory. It is worth to point out that Condition (IV) implies Condition (III) but not the other way around. Therefore, a natural question arises: does there exists subfactor planar algebra satisfying Condition (I), (II) and (III) but not (IV). In particular, Vaughan Jones asked the following question in the late nineties.

Question 0.1 (Jones, 1990s)**.** *Is the subfactor planar algebra for $R \rtimes (S_2 \times S_3) \subset R \rtimes S_5$, denoted by generated by its 2-boxes?*

It is not difficult to see that the subfactor planar algebra in Question 0.1 satisfies (II),(III) but not (IV). However, it is unknown whether (I) holds. This question is also closely related to the classification of spin models for Kauffman polynomial from self-dual strongly regular graphs by Jaeger. In particular, he discovered a new spin model based on the Higman-Sims graph. The spin model is described by a *spin model planar algebra* and the adjacency matrix is a 2-box. Therefore, Question 0.1 can be also asked for this particular spin model planar algebra associated to Higman-Sims graph. The answer was no provided by the fact the planar subalgebra generated by the adjacency matrix admits Yang-Baxter relations.

This question can be asked in a general setup: given a strongly regular graph Γ, the associated group-action model $\mathscr{P}_\bullet^\Gamma$ is defined to be fixed-point planar subalgebra of the spin model planar algebra. The adjacency matrix A_Γ is a 2-box in the planar algebra. Therefore, Question 0.1 can be asked for spin model planar algebras associated with strongly regular graphs in general, namely, whether the planar algebra $\mathscr{P}_\bullet^\Gamma$ is generated by A_Γ. Since the spin model planar algebra $\mathscr{P}_\bullet^\Gamma$ is defined by the combinatorial data of the graph Γ, the generating property in Question 0.1 is intrinsically determined by Γ.

Definition 0.2. *Let Γ be a strongly regular graph. We say Γ has property (G) if the associated planar algebra $\mathscr{P}^\Gamma$ has the generating property, namely, it is generated by its adjacency matrix A_Γ.*

In particular, the referred subfactor planar algebra in Question 0.1 can be obtained from the Kneser graph $KG_{5,2}$, also known as the Petersen graph. In [10], we provided an affirmative answer to Question 0.1, namely, the Kneser graph $KG_{5,2}$ has property (G). Later on, Jones imposed the same question with the subfactor planar algebras for $S_2 \times S_{n-2} \subset S_n$, namely,

Question 0.3 (Jones, 2017)**.** *Are the subfactor planar algebras for $S_2 \times S_{n-2} \subset S_n$ generated by their 2-boxes?*

By exploiting the universal skein theory for group-action models, we first give constructions of generators for the planar algebras $\mathscr{P}_\bullet^{KG_{n,2}}$ under the assumption that the transposition R is generated by 2-boxes, namely, $R \in \langle \mathscr{P}_2^{KG_{n,2}} \rangle$. Then we confirm the validity of the assumption provided with a universal construction, and thus we prove the main theorem, namely,

Theorem 0.4. *The subfactor planar algebra $\mathscr{P}_\bullet^{KG_{n,2}}$ has the generating property, namely, the Kneser graph $KG_{n,2}$ has property (G) for $n \geq 5$.*

We first remark that the relation between the transposition and the generating property was first revealed independently by Jones and Curtin. They showed that any planar subalgebra $\mathscr{Q}_\bullet$ of some spin model, $\mathscr{Q}$ has the generating property if and only if $R \in \langle \mathscr{Q}_2 \rangle$.We enhance the statement by dropping the assumption that $\mathscr{Q}$ is a planar subalgebra of some spin model. In this case, the transposition R is characterized by skein relations.

Theorem 0.5. *Suppose $\mathscr{Q}_\bullet$ is a planar algebra. Then the following are equivalent:*

(1) There exists a group action $G \curvearrowright X$ such that $\mathscr{Q}_\bullet$ is isomorphic to the associated group-action model $\mathscr{P}_\bullet^G$.

(2) There exists $S \in \mathscr{Q}_4$ and $W \in \mathscr{Q}_3$ such that they satisfy Reidemeister moves, flatness and Frobenius relations.

Secondly, it was pointed out by Snyder and Reutter during this workshop at Oberwolfach that the generating property in Definition 0.2 is also studied in the theory of quantum permutation groups. A graph Γ is said to have *no quantum symmetry* if its quantum automorphism group coincides with its automorphism group.

$$\Gamma \text{ has property } (G) \iff \Gamma \text{ has no quantum symmetry.}$$

An important task is to determine finite graphs with no quantum symmetry. Theorem 0.4 implies the following corollary.

Corollary 0.6. *The Kneser graph $KG_{n,2}$ has no quantum symmetry for $n \geq 5$.*

In the end, Theorem 0.4 confirms that the simplest generator for the planar algebra for $\mathscr{P}_\bullet^{KG_{n,2}}$ is a single 2-box A_Γ. However, Universal skein theory for group actions tells us that in the simplest skein theory, the generators are a 2-box and an n-box; and one of the relations appears in the $2n$-box space. This phenomenon gives us a hint that the complexity of skein theory might be more subtle than the sizes of generators and relations.

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Coideal Algebras and Subfactors

Hans Wenzl

Let H be a Hopf algebra with coproduct Δ. A subalgebra $K \subset H$ is a right coideal algebra if $\Delta(K) \subset K \otimes H$. This makes $Rep(K)$ into a right module category of the tensor category $Rep(H)$ of finite-dimensional representations of H.

An important class of fusion tensor categories were constructed using the Drinfeld-Jimbo quantum groups $U_q\mathfrak{g}$ by H.H. Andersen and coauthors (see e.g. [1]). They were obtained as a quotient of a special subcategory of $Rep(U_q\mathfrak{g})$. It was shown in [4] that for suitable roots of unity q these fusion categories are C^* tensor categories. They can be used to construct a sequence of examples of irreducible subfactors for each irreducible representation of a semisimple Lie algebra $\mathfrak{g}$. The main topic of this talk is the question whether we can find coideal subalgebras of $U_q\mathfrak{g}$ which would also yield nontrivial module categories for the associated fusion categories.

An interesting class of coideal subalgebras $U_q'\mathfrak{g}^\theta \subset U_q\mathfrak{g}$ were constructed by a number of authors about twenty years ago, see e.g. [2] for precise references. They are q-deformations of the fixed point algebra $U\mathfrak{g}^\theta \subset U\mathfrak{g}$, where θ is an order two automorphism of the Lie algebra $\mathfrak{g}$. Such order two automorphisms have all been determined by Cartan in his classification of symmetric spaces. Analogues of the corresponding module categories in the setting of fusion categories were constructed in [3] for special cases, namely for the embeddings $\mathfrak{so}_n \subset \mathfrak{sl}_n$ and $\mathfrak{sp}_n \subset \mathfrak{sl}_n$. However, this was done via categorical methods without explicitly using the coideal algebras. More precisely, for V the vector representation of $\mathfrak{sl}_n$, detailed knowledge of $\mathrm{End}_{U_q\mathfrak{sl}_n}(V^{\otimes m})$ and of $\mathrm{End}_{SO(n)}(V^{\otimes m})$ together with a compatibility condition for traces (essentially what is known in subfactor theory as the commuting square condition) determined the structure of an algebra which was shown to be isomorphic to $\mathrm{End}_{U_q'\mathfrak{so}_n}(V^{\otimes m})$. In particular, explicit formulas for the indices and descriptions for the principal graphs were determined in [3]. For the special cases treated in the paper, let $\mathfrak{g} = \mathfrak{g}^\theta \oplus \mathfrak{p}$ as a $\mathfrak{g}^\theta$ module. Then the index for the corresponding subfactor $N \subset M$ for $q = e^{\pi i/\ell}$ is given by

$$[M:N] = b(\mathfrak{g}^\theta)\ell^{n(\mathfrak{p})} \prod_{\omega>0} \frac{1}{4\sin^2(\omega,\check{\rho})\pi/\ell},$$

where the product goes over the weights $\omega > 0$ of $\mathfrak{p}$ coming from positive roots of $\mathfrak{g}$, $n(\mathfrak{p})$ is the multiplicity of the zero weight in $\mathfrak{p}$, $b(\mathfrak{g}^\theta)$ is a small integer depending on the special case $\mathfrak{g}^\theta \subset \mathfrak{g}$ and $\check{\rho}$ is half the sum of the positive roots of $\mathfrak{g}$. Moreover, it turned out that the indices of these subfactors are essentially the same as the ones constructed via conformal nets in [5] in the special case $SO(3) \subset SU(3)$. However, it is very hard to calculate indices of subfactors in the conformal net approach in general.

The evidence in the last paragraph suggests that similar constructions might be possible for other or quite possibly all embeddings $\mathfrak{g}^\theta \subset \mathfrak{g}$. To prove this, a general construction of module categories with a C^*-structure and subfactors from coideal algebras $U_q'\mathfrak{g}^\theta \subset U_q\mathfrak{g}$ would be desirable. This has proved to be surprisingly difficult so far (see also the talk by Makoto Yamashita which deals with a similar problem). Some evidence was given in the talk how a more careful study of the relative positions of the Cartan algebras of $\mathfrak{g}$ and $\mathfrak{g}^\theta$ can be used to get to the proper generalizations of dimension functions and index formulas for other embeddings $U_q'\mathfrak{g}^\theta \subset U_q\mathfrak{g}$. This is still work in progress.

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