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Mini-Workshop: Surface Modeling and Syzygies

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Abstract. The problem of determining the implicit equation of the image of a rational map $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ is of theoretical interest in algebraic geometry, and of practical importance in geometric modeling. There are essentially three methods which can be applied to the problem: Gröbner bases, resultants, and syzygies. Elimination via Gröbner basis methods tends to be computationally intensive and, being a general tool, is not adapted to the geometry of specific problems. Thus, it is primarily the latter two techniques which are used in practice. This is an extremely active area of research where many different perspectives come into play. The mini-workshop brought together a diverse group of researchers with different areas of expertise.

Mathematics Subject Classification (2000): 14Q10 (Primary) 13D02, 14Q05, 65D17 (Secondary).

Introduction by the Organisers

A central problem in geometric modeling is to find the implicit equation for a curve or surface defined by a rational map. For surfaces, the two most common situations are the images of parameterizations $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^3$ or $\mathbb{P}^2 \dashrightarrow \mathbb{P}^3$.

The implicitization problem involves interesting commutative algebra. Implicitization is a problem in elimination theory, which allows one to use standard tools such as Gröbner bases or resultants. The surprise is that more sophisticated tools from commutative algebra are also being used, and syzygies play a leading role. We now describe some aspects of this, referring to [C1] for a detailed survey and to [BJ] for a more general algebraic point of view.

In [SC], Sederberg and Chen introduced the method of moving curves and surfaces. For a curve parametrization (a, b, c), a *moving line* that follows the parametrization is a element of the syzygy module on the generators of the ideal $I = \langle a, b, c\rangle$. The syzygy module of I is free of rank two, and a Hilbert function computation shows that if a, b, c are homogeneous of degree n without common factors, then there is an n-dimensional vector space of moving lines of degree $n-1$. Write each moving line as

$$A_i(s,t)x + B_i(s,t)y + C_i(s,t)z$$

where the x, y, z are placeholders, representing the fact that $A_i \cdot a + B_i \cdot b + C_i \cdot c = 0$. By collecting coefficients, we can write

$$A_i(s,t)x + B_i(s,t)y + C_i(s,t)z = \sum_{j=0}^{n-1} L_{ij}(x,y,z) s^j t^{n-1-j}.$$

A main theorem of [CSC] is that the determinant of the $n \times n$ matrix of the L_{ij} is a power of the implicit equation for the image.

For a surface parametrization given by (a, b, c, d), a moving plane that follows the parametrization is an element of the syzygy module on the generators of the ideal $I = \langle a, b, c, d \rangle$, and a moving quadric that follows the parametrization is an element of the syzygy module on the generators of I^2. The moving surface method of [CGZ] requires knowing that a syzygy of the form

$$(c_1 a + c_2 b + c_3 c + c_4 d)a + (c_5 b + c_6 c + c_7 d)b + (c_8 c + c_9 d)c = 0$$

comes from the Koszul complex when a, b, c, d have no common zeros. In the case of $\mathbb{P}^2$ (i.e., when a, b, c, d are homogeneous polynomials), this is proved by observing that a, b, c form a regular sequence, so that every syzygy comes from the Koszul complex. For $\mathbb{P}^1 \times \mathbb{P}^1$ (i.e., when a, b, c, d are bihomogeneous polynomials), the Koszul complex is not exact in all bidegrees, but by vanishing theorems for cohomology, it can be seen that the sequence is exact in the bidegree of interest.

The main result of [CGZ] involves parameterizations without base points. When base points are allowed, [BCD] uses results of [CS] to show that for $\mathbb{P}^2$, the moving surface method of [CGZ] applies when the base points are local complete intersections. This is also true for $\mathbb{P}^1 \times \mathbb{P}^1$, by [AHW]. The proofs in [BCD] use results about the regularity of I and I^2; in a similar way, the proofs in [AHW] use results about bigraded regularity. In [CCL] a special case of the Serre conjecture is used to conclude that syzygy modules are always free for affine surface parameterizations.

Interestingly, the abstract general setting for this application of syzygies for the implicitization problem is given by the approximation complexes defined by W. Vasconcelos and coauthors [HSV, V]. The application of this method to compute the implicit equation of a parameterized hypersurface has been developed recently in [BJ, BCh, Ch]. It has been implemented in the case of finite nice base points by Busé [B]. The implicit equation is obtained from a double complex which provides a resolution of the blow up algebra of the ideal of base points, in case this ideal is of linear type, i.e. in case the associated Rees algebra and symmetric algebra coincide. The article [BChJ] studies optimal degree estimates and the extraneous factors that appear when the base points are almost local complete intersections, as well as a link with some particular resultant computations.

As noted in [BJ], the set of all moving hypersurfaces that follow a given hypersurface parametrization form an ideal that is the ideal of relations defining the

Rees algebra associated with the parametrization. The structure of this ideal is investigated in some special cases in [C2] using local cohomology, local duality and the *Sylvester forms* introduced by Jouanolou [J]. The paper [C2] uses results from the commutative algebra literature (the paper [MU] of Morey and Ulrich) and from the geometric modeling literature (the paper [SGD] of Sederberg, Goldman and Du). Understanding the general case for curves is an interesting open problem in both commutative algebra and geometric modeling. The surface case is also completely open.

The parametrization is also not an intrinsic property of the parametrized variety, in contrast to the implicit equation. In some cases, it seems worthwile to replace the given parametrization by a simpler one before one attempts to implicitize. In the curve case and in the surface case, we can simplify the parametrization (if possible) without implicitizing, as shown in [Schi3]. The smallest possible parametric degree has been studied in [Schi1].

Another line of research developed in [EK, STY] aims to determine a priori the Newton polytope of the resulting equation, to translate then the implicitization problem to an interpolation linear algebra problem. These articles are based in the theory of sparse resultants and the use of tropical geometry. Numerical issues in the implementation of the theoretical results are also relevant [Schi2].

1. The Workshop Program

The mini-workshop brought together a diverse group of researchers with different areas of expertise. On Monday and Tuesday, there were survey talks during the morning timeframe. In the afternoons, and on the remaining days of the conference, we combined specialized talks with informal working sessions.

Monday

Time	Speaker
900–1030	D. Cox
1100–1230	R. Goldman
1400–1500	B. Mourrain
1600–1700	J. Schicho
1700–1800	F.O. Schreyer

Tuesday

Time	Speaker
900–1030	M. Chardin
1100–1230	L. Buśe
1600–1630	H. Wang
1630–1700	H. Schenck
1700–1730	R. Goldman
1730–1800	D. Cox

Wedensday

Time	Speaker
900–1000	A. Galligo
1030–1130	H. Schenck
1130–1230	R. Goldman

Thursday

Time	Speaker
900–1030	A. Dickenstein
1030–1130	I. Emiris
1130–1230	B. Mourrain, J. Schicho
1600–1630	L. Buśe
1630–1700	C. D'Andrea
1700–1730	F.O. Schreyer

Friday

Time	Speaker
900–1000	N. Botbol
1030–1130	M. Dohm
1130–1230	H. Wang

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Mini-Workshop: Surface Modeling and Syzygies

Table of Contents

Abstracts

The implicitization problem for the case $\phi : \mathbb{P}^n \dashrightarrow (\mathbb{P}^1)^{n+1}$

Nicolás S. Botbol

In this work we study the implicitization problem for a given rational map $\phi : \mathbb{P}^n \dashrightarrow (\mathbb{P}^1)^{n+1}$, defining a hypersurface in $(\mathbb{P}^1)^{n+1}$, given by couples of homogeneous polynomials of the same degree d_i for $i = 0, \ldots, n$.

We will show that the classical study of Macaulay Resultants and Koszul complexes coincides with the new approach introduced by L. Busé and J.-P. Jouanolou in [1] and developed by them and M. Chardin in [2, 3, 4, 5], using approximation complexes.

The procedure consists is computing the implicit equation by means of the classical methods of elimination theory, adapted for this case. More precisely, we will consider the multigraded k-algebra $\mathcal{B}$, that corresponds to the incidence variety associated to the given rational map ϕ. This algebra can be presented as a quotient of the polynomial ring R in all the groups of variables, by some linear equations $L_0, \ldots, L_n$. Consequently we as a resolution for $\mathcal{B}$, the Koszul complex $\mathcal{K}^R_\bullet(L_0, \ldots, L_n)$, denoted by $\mathcal{K}_\bullet$, and we study an give a geometric interpretation of its acyclicity conditions.

In this case, we obtain the implicit equation (up to a power) by taking the determinant of some suitable strand of a multigraded resolution, this is:

$$H^{\deg(\phi)} = \mathrm{Res}(L_0, \ldots, L_n) = \det((\mathcal{K}_\bullet)_\nu), \qquad \text{for } \nu \gg 0.$$

Later we analyze the geometrical meaning of the results studied before. For instance, we give algebraic and geometric conditions for knowing when the computed equation defines the scheme theoretic image of ϕ. And, when it is not, what are the extra objects that appear.

Finally, we give some applications to the problem of computing discriminants, or A-discriminants (see [6]), by means of implicitization techniques.

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Implicitization of algebraic surfaces and linear syzygies

Laurent Busé

Surface implicitization, i.e. finding the implicit equation of an algebraic surface defined parametrically, is a classical problem and there are numerous approaches to its solution. This talk reported on two approaches that allow to represent surfaces in terms of generically full rank matrices whose rank drops exactly on the surface. Such a representation is not only more compact than the implicit equation, it has also the advantage that it makes it possible to solve geometric problems by applying linear algebra techniques. To give an example, let us suppose that we want to decide if a given point lies on the surface. It suffices to evaluate the rank of matrix representation in this point: it lies on the surface if and only if the rank drops.

The first part of the talk reported on a method that introduced in [4] and [1]. It relies on the use of linear syzygies of the parametrization and represents the implicit equation of a surface parametrized over $\mathbb{P}^2$ by the maximal minors of a non-square matrix. It is based on the theory of approximation complexes. Recently, this method has been extended to surfaces parametrized over $\mathbb{P}^1 \times \mathbb{P}^1$ in [2].

The second part of the talk reported on resultant computations to solve the implicitization problem. Thes eliminant polynomials have the advantage to provide universal formulas for the implicitization problem, formulas which are attached to a particular class of parametrizations. Each formula is related to a particular construction of a resultant corresponding to a certain compactification of a dense open subset of the sapce of parameters. The construction of residual resultants [3] has been described in details.

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The geometry of syzygies

Marc Chardin

We presented in this lecture a method to compute the image of a rational map from $\mathbf{P}^{n-1}$ to $\mathbf{P}^n$, under suitable hypotheses on the base locus and on the image.

The formalism we use is due to Jean-Pierre Jouanolou, who gave a course on this approach in the University of Strasbourg during the academic year 2000–2001. In his joint article with Laurent Busé [1], this formalism is explained in detail and applications to the implicitization problem are given. The idea of using a matrix of syzygies for the implicitization problem goes back to the work of Sederberg and Chen [4] and was at the origin of several important contributions to this approach (see for instance [2], [3],[5]). Most of this lecture was dedicated to presenting the method, the geometric ideas behind it and the tools from commutative algebra that are needed. We then presented some of the most advanced results we know related to this approach. Let us point out the geometrical reason why syzygies are involved. Suppose given

$$\phi\ :\mathbf{P}^{n-1}\cdots\to\mathbf{P}^{n},$$

a rational map defined by $f := (f_0, \ldots, f_n)$, $f_i \in R := k[X_1, \ldots, X_n]$ homogeneous of degree $d \geq 1$, such that the closure of its image is a hypersurface $\mathcal{H}$. Let :

- $I := (f_0, \ldots, f_n) \subset R$ be the ideal generated by the f_i's,
- $X := \mathrm{Proj}(R/I) \subset \mathbf{P}^{n-1}$ be the base loci scheme defined by I.

If $\Gamma_0 \subset \mathbf{P}^{n-1} \times \mathbf{P}^n$ is the graph of $\phi : (\mathbf{P}^{n-1} - X) \longrightarrow \mathbf{P}^n$ and Γ the Zariski closure of Γ_0, one has :

$$\mathcal{H} = \overline{\pi(\Gamma_0)} = \pi(\Gamma)$$

where $\pi : \mathbf{P}^{n-1} \times \mathbf{P}^n \longrightarrow \mathbf{P}^n$ is the projection, and the bar denotes the Zariski closure (or equivalently the closure for the usual topology in the case $k = \mathbf{C}$), and

$$\Gamma = \mathrm{Proj}(\mathcal{R}_I)$$

with $\mathcal{R}_I := R \oplus I \oplus I^2 \oplus \cdots$. The embedding $\Gamma \subset \mathbf{P}^{n-1} \times \mathbf{P}^n$ corresponds to the natural graded map :

$$\begin{array}{ccc} S := R[T_0, \ldots, T_n] & \stackrel{s}{\longrightarrow} & \mathcal{R}_I \\ T_i & \longmapsto & f_i \in I = (\mathcal{R}_I)_1. \end{array}$$

If $\mathfrak{P} := \ker(s)$, $\mathfrak{P}_1$ (the degree 1 part of $\mathfrak{P}$) is the module of syzygies of the f_i's

$$a_0T_0 + \cdots + a_nT_n \in \mathfrak{P}_1 \iff a_0f_0 + \cdots + a_nf_n = 0.$$

Then setting $\mathcal{S}_I := \mathrm{Sym}_R(I)$ and $V := \mathrm{Proj}(\mathcal{S}_I)$, we have natural onto maps

$$S \longrightarrow S/(\mathfrak{P}_1) \qquad \text{and} \qquad \mathcal{S}_I \simeq S/(\mathfrak{P}_1) \longrightarrow S/\mathfrak{P} \simeq \mathcal{R}_I$$

which correspond to the embeddings

$$\Gamma \subseteq V \subset \mathbf{P}^{n-1} \times \mathbf{P}^n.$$

$\mathcal{R}_I$ is the bigraded domain defining Γ and

Theorem. $\Gamma = V$ *if* X *is an l.c.i, and the converse holds if* $\dim X = 0$.

This theorem explains the key role of syzygies in computing H: they are equations of definition of the graph Γ when X is locally a complete intersection. In other words, if X is an l.c.i., the image of the rational map is the projection on the second factor of the subscheme of $\mathbf{P}^{n-1} \times \mathbf{P}^n$ defined by the syzygies.

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Questions about moving curves, moving surfaces, and polynomials of degree (2,1)

David A. Cox
(joint work with many people)

Given an ideal $I = \langle a_1, \dots, a_n \rangle$ in a noetherian ring R, the *Rees algebra* is the graded R-algebra

$$R[I] = \bigoplus_{i=1}^{\infty} I^i t^i \subseteq R[t].$$

The map $R[x_1, \dots, x_n] \to R[I]$ defined by $x_i \mapsto a_i t$ is a surjective R-algebra homomorphism. The minimal generators of the kernel of this map are the *defining equations of the Rees algebra*.

When $R = k[x_0, \dots, x_n]$ and $I = \langle f_0, \dots, f_{n+1} \rangle$, where the f_i are homogeneous of the same degree, we can think of the f_i as giving a rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}^{n+1}$ parametrizing a hypersurface. As noticed by Busé and Jouanolou [2], the defining equations of the Rees algebra can be interpreted in terms of the moving hypersurfaces introduced by Sederberg and Chen [14]. For this reason, the ideal generated by the defining equations is called the *moving curve ideal* when $n = 1$ and the *moving surface ideal* when $n = 2$.

My lecture discussed three questions related to curve and surface implicitization.

Moving Curves. The moving curve ideal has a lot of structure, some of which is mysterious even in fairly simple cases. I will first discuss the case $n = 4, \mu = 2$, where the minimal generators can be constructed from the moving lines (the μ-*basis* in the terminology introduced by Cox, Sederberg and Chen [9]) using iterated Sylvester forms (see Cox [6], Cox, Hoffman and Wang [8] and Hong, Simis and Vasconcelos [12]). But there are other structures as well, such as the way the resultant of the moving lines gives the implicit equation, and suitable resultants of moving conics and the implicit equation give the original parametrization. For more general curves, there are also interesting questions about singularties. Song, Chen and Goldman [15] have results about how singularities relate to the moving lines. Chen, Wang and Liu [4] and Pérez Díaz [13] have also written about the singularities of parametrized curves. But what about the rest of the moving curve ideal? This is related to a conjecture of Cox [6] and work-in-progess by Busé [1] on adjoint linear systems.

Moving Surfaces. The moving surface ideal is harder to study, in part because when one works homogeneously, the module of moving planes is rarely free. But when we work affinely, the moving plane module is always free of rank three (the μ-*basis* in the terminology of by Chen, Cox and Liu [5]). I will report on two questions about moving surfaces. The first involves results of Chen, Cox and Liu [5] about an affine version of the moving surface ideal, and the second involves a mysterious implicitization method of Zheng, Sederberg, Chionh and Cox [16] that works in the presense of really bad base points.

Polynomials of degree (2,1). A tensor product surface of degree $(2,1)$ involves four bihomogeneous polynomials of degree 2 in x, y and degree 1 in z, w. These surfaces have been studied by Elkadi, Galligo and Lê [11]. They are examples of ruled parametric surfaces, where μ-bases have been studied by Chen and Wang [3] and Dohm [10]. Another question is what can be said about the moving surface ideal in this case.

If instead we have three such polynomials that don't vanish simultaneously on $\mathbb{P}^1 \times \mathbb{P}^1$, then we ask about the free resolution of the ideal they generate. This was worked out by Cox, Dickenstein and Schenck [7], with the surprising result that there are two possibilities for the free resolution. Sylvester forms make an appearance in the early part of the resolution, and make a mysterious reappearance in the Tate resolutions associated with these polynomials.

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The Newton polytope of a rational curve via tropical tools

Alicia Dickenstein
(joint work with Eva M. Feichthner, Bernd Sturmfels)

We consider a rational map $f = (f_1, f_2) : \mathbb{P}^1(k) \dashrightarrow k^2$, where f_1, f_2 are non constant rational functions of degree 0 with coefficients in a field k of characteristic zero. Let H be a defining equation of the closure of the image of f. Here is a quick way of computing the Newton polytope $P = N(H)$ before actually computing H.

We can factorize f_1, f_2 over the algebraic closure $\overline{k}$ of k, and so there exist (two by two linearly independent) homogeneous linear forms $\ell_1, \dots, \ell_r$, $\ell_j(t) = \alpha_{j0}t_0 + \alpha_{j1}t_1, j = 1, \dots, r$, which we encode in a matrix $U \in \overline{k}^{r\times 2}$, and a matrix $V \in \mathbb{Z}^{2\times r}$ such that $f_i = \prod_{j=1}^r \ell_j^{v_{ij}}$ (up to a non zero multiplicative constant). Then, our map f factorizes as the composition of the linear map $t \to (\ell_1(t), \dots, \ell_r(t))$ read from the matrix U and the monomial map $(y_1, \dots, y_r) \to (y^{v_1}, y^{v_2})$ read from the matrix V.

This is a simple instance of Theorem 3.1 in [3] or Proposition 3.1 in [7], which describes the tropicalization $\tau(H)$ of the ideal (H) as the image under the linear map $\mathbb{R}^r \to \mathbb{R}^2$ specified by the matrix V, of the Bergman fan $\tau(\mathrm{im}(U))$. Important particular cases of varieties rationally parametrized by monomials in linear forms are the sparse discriminants and resultants, and their dehomogenizations, which we study in detail in [3]. The multiplicities attached to the different cones as in [3, Section 2] are further clarified in [5, Theorem 6.4]. One of the corollaries of this approach is that the tropicalization $\tau(H)$ only depends on the matroid of U (and on V).

We can regard the tropicalization $\tau(\mathrm{im}(U))$ associated to our regular matroid, inside the tropical quotient $\mathbb{R}^r/\mathbb{R}(1, \dots, 1)$, where it equals the union of the coordinate rays $\mathbb{R}_{\geq 0}\, e_j$, $j = 1, \dots, r$. Therefore, its image under V is the fan with rays equal to $\mathbb{R}_{\geq 0}\,(v_{1j}, v_{2j})$, for some j, with respective multiplicity equal to the sum of the lattice lengths of all the vectors $v_k = (v_{1k}, v_{2k})$ which lie in the ray, divided by the degree of the map f.

Now, $\tau(H)$ is just the codimension one skeleton of the normal fan of P, together with the information of the lenghts of the edges, which in principle defines P up to translates, but since H is an irreducible polynomial, this determines P uniquely by the condition that it must lie in the first orthant and intersect both positive coordinate axes. Then, P can be determined as in [4, Corollary 4.6] and [2, Theorem 1.1]: concatenate the vectors v_j in counterclockwise cyclic order starting from any of them (this will give a closed polytope since the sum of the columns of V equals 0), then rotate the resulting polytope 90 degrees clockwise and shift it until it lies in the first orthant intersecting both axes.

This result was more recently stated and proved with different methods in [2], where the authors remark that the matrix V can be computed symbolically with operations over k, without factorizing f_1, f_2 over $\overline{k}$. The case of generic rational functions (i.e where the vectors $v_k = \pm(1,0)$ or $\pm(0,1)$) was also recently studied by computing the Newton polytopes of specialized resultants in [1]. For generic coefficients, higher dimensional cases can be computed using the free software TrIm by J. Yu and B. Sturmfels, explained in [6].

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The implicit equation of a canal surface

Marc Dohm

(joint work with Severinas Zube)

A canal surface is an envelope of a one parameter family of spheres. In this paper we present an efficient algorithm for computing the implicit equation of a canal surface generated by a rational family of spheres. By using Laguerre and Lie geometries, we relate the equation of the canal surface to an equation of a dual variety of a certain curve in 5-dimensional projective space. We define a kind of μ-basis for the dual variety to the curve and present a simple algorithm for its computation. The implicit equation of the dual variety and the canal surface are obtained by means of the resultant associated with the μ-basis. In surface design, the user often needs to perform rounding or filleting between two intersecting surfaces. Mathematically, the surface used in making the rounding is defined as the envelope of a family of spheres which are tangent to both surfaces. This envelope of spheres centered at $c(t) \in \mathbb{R}^3$ with radius $r(t)$ is called a canal surface with spine curve $e = (c(t), r(t))$. If the radius $r(t)$ is constant the surface is called a pipe surface. Moreover, if additionally we reduce the dimension (take $c(t)$ in a plane and consider circles instead of spheres) we obtain the offset to the curve. Canal surfaces are very popular in Geometric Modelling, as they can be used as a blending surface between two surfaces, see [Kazakeviciute (2005)] for examples of blending with canal surfaces.

Here we study the implicit equation of a canal surface and its implicit degree. The implicit equation of a canal surface can be obtained after elimination of the family variable t from the system of two equations $F : f_1(y,t) = f_2(y,t) = 0$ (here f_1, f_2 are quadratic in $y \in \mathbb{R}^4$). By using Lie and Laguerre geometry, we see that the system of equations F is related to the system $h_1(\bar{y}, t) = h_2(\bar{y}, t) = Q(\bar{y}) = 0, \bar{y} \in \mathbb{P}^5$, where h_1, h_2 are linear in $\bar{y}$ and $Q(\bar{y})$ is the so-called Lie quadric. It turns out that the variety defined by the system of equations $h_1(\bar{y}, t) = h_2(\bar{y}, t) = 0$ is a dual variety to the curve $\bar{e} \in \mathbb{P}^5$, where $\bar{e}$ is a curve on the Lie quadric determined by the spine curve e. The dual variety $V(\bar{e}^\vee)$ can be seen as a generalized ruled surface, for which we define a μ-basis in a similar way as for usual ruled surfaces, compare [Chen et al. (2001)]. It consists of two polynomials $p_1(\bar{y}, t), p_2(\bar{y}, t)$ which are linear in $\bar{y}$, and of degree d_1, d_2 in t are such that $d_1 + d_2$ is minimal. It turns out that the resultant of p_1 and p_2 with respect to t gives the implicit equation of the variety $V(\bar{e}^\vee)$. If the parametrization of e is not birational, we can use the approach of [Dohm (2006)] to prove that the resultant is a power of the implicit equation.

The talk is organized as follows. First we define the canal surface and show how solving the system F from above, leads to the appearance of extraneous geometric components. In the next section we develop some algebraic formalism about modules with two quasi-generators. We define the μ-basis for these modules, present an algorithm for its computation and proceed to prove that the resultant of the μ-basis gives the canal surface without extraneous components. We conclude with two applications of these results: finding a parametrization of the canal surface of minimal degree (see [Krasauskas (2007)]) and the computation of offsets to the surface.

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Computing the Newton Polygon of the Implicit Equation

Ioannis Z. Emiris
(joint work with Christos Konaxis, Leonidas Palios)

We consider the implicitization problem for hypersurfaces, namely the question of switching from a parametric representation to an algebraic one. Let $h_0, \ldots, h_n \in \mathbb{C}(t_1, \ldots, t_n)$. The problem is to compute the prime principal ideal

$$\langle \phi \rangle \subset \mathbb{C}[x_0, \ldots, x_n]$$

such that $\phi(h_i) \equiv 0$ in $\mathbb{C}[t_i]$. Here we consider planar curves ($n = 1$), where all polynomials in its parameterization have fixed supports. We determine (see [3] for details) the vertices of the Newton polygon $N(\phi)$ of the implicit equation, or *implicit polygon*, without computing $\phi(x_i)$, under the assumption of *generic* coefficients in the parametric polynomials, relative to their supports.

An algorithm for this problem was proposed in [4] by means of the Newton polytope of the toric resultant, whereas in [5, 2] tropical geometry is applied to solve the problem. In [1] the Philippon-Sombra estimate on the number of roots of a polynomial system is used to compute the implicit polygon with no genericity condition on the coefficients.

Let us start with two examples. First consider the following curve:

$$x = \frac{t^6 + 2t^2}{t^7 + 1}, y = \frac{t^4 - t^3}{t^7 + 1},$$

Our methods yield vertices $(7, 0), (0, 7), (0, 3), (3, 1), (6, 0)$, which define the actual implicit polygon. An instance where the implicit polygon has 6 vertices is:

$$x = \frac{t^3 + 2t^2 + t}{t^2 + t - 2},\ y = \frac{t^3 - t^2}{t - 2}.$$

Our results yield implicit vertices $\{(0, 1), (0, 3), (3, 1), (1, 3), (2, 0), (3, 2)\}$ which define the actual polygon. See fig. 1.

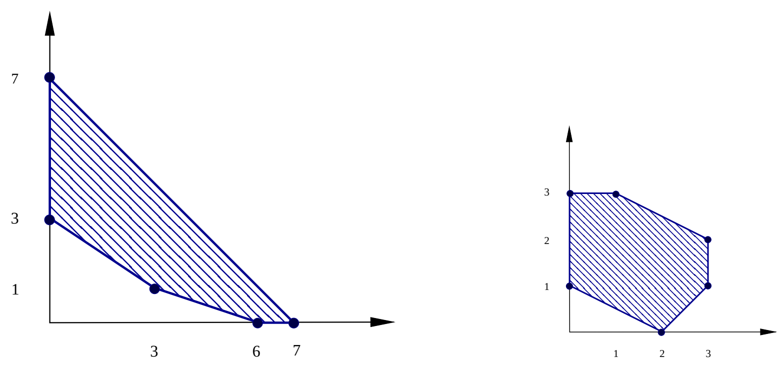

Figure 1. The implicit polygons of the example.

We now present our main results. Let us consider rationally parameterized curves, when both denominators are the same:

$$x = \frac{P_0(t)}{Q(t)},\ y = \frac{P_1(t)}{Q(t)},\ \gcd(P_i(t), Q(t)) = 1,\ P_i, Q \in \mathbb{C}[t],\ i = 0, 1.$$

Let the input Newton segments be

$$B_i = N(P_i), i = 0, 1,\ B_2 = N(Q),\ \text{where } B_i = \{b_{iL}, \ldots, b_{iR}\}, i = 0, 1, 2,$$

where $b_{iL},\ b_{iR}$ are the endpoints of segment B_i. We prove that one needs to examine only subsegments defined by endpoints $b_{iL}, b_{iR} \in B_i$.

Theorem. Let $u = \max\{b_{0R}, b_{1R}, b_{2R}\}$: If all $\mathrm{CH}(B_i \cup B_j) = [0, u], i \neq j$, then $N(\phi)$ is a triangle with vertices $(0,0), (0,u), (u,0)$. If exactly one $\mathrm{CH}(B_k) = [0, u], k \in \{0, 1, 2\}$, then $N(\phi)$ has up to 5 vertices in the following set of vectors:

$$\{(u, 0), (0, u), (0, u - b_{iR} + b_{iL}), (b_{jL}, u - b_{iR}), (u - b_{jR} + b_{jL}, 0)\},$$

where $\{i, j, k\} = \{0, 1, 2\}$, assuming $b_{iL}(u - b_{jR}) \geq b_{jL}(u - b_{iR})$. If none of the B_t's is equal to $[0, u]$, then we may choose $\{i, j, k\} = \{0, 1, 2\}$ such that:

$$0 < b_{iL} \leq b_{iR} = u, 0 = b_{jL} \leq b_{jR} < u, 0 \leq b_{kL} \leq b_{kR} < u.$$

Then, $N(\phi)$ has at most 5 or 4 vertices, depending on whether b_{kL} is positive or 0. In the former case, the vertices lie in

$$\{(b_{jR}, 0), (b_{kR}, u - b_{kR}), (b_{kL}, u - b_{kL}), (0, u - b_{0L}), (0, 0), \}$$

and, in the latter case, the third and fourth vertices are replaced by $(0, u)$.

As an example, take the folium of Descartes:

$$x = \frac{3t^2}{t^3 + 1}, y = \frac{3t}{t^3 + 1} \Rightarrow \phi = x^3 + y^3 - 3xy = 0.$$

Our results above yield x^3, y^3, xy, hence an optimal support.

Now consder polynomial parameterizations:

$$f_0 = x - P_0(t),\ f_1 = y - P_1(t) \in (\mathbb{C}[x, y])[t].$$

The supports of f_0, f_1 are $A_0 = \{a_{00}, a_{01}, \ldots, a_{0n}\}$ and $A_1 = \{a_{10}, a_{11}, \ldots, a_{1m}\}$.

Theorem. If P_0 or P_1 (or both) contain a constant term, then the implicit polygon is the triangle with vertices $(0,0), (b_m, 0), (0, a_n)$. Otherwise, P_0, P_1 contain no constant terms, and the implicit polygon is the quadrilateral with vertices $(a_{11}, 0), (a_{1m}, 0)$, $(0, a_{0n}), (0, a_{01})$, see fig. 2.

For the Fröberg-Dickenstein example:

$$x = t^{48} - t^{56} - t^{60} - t^{62} - t^{63}, y = t^{32},$$

our method yields vertices $(32, 0), (0, 48), (0, 63)$, which define the optimal polygon.

Consider, lastly, rationally parameterized curves, with different denominators:

$$f_0(t) = xQ_0(t) - P_0(t),\ f_1(t) = yQ_1(t) - P_1(t)\ \in (\mathbb{C}[x, y])[t], \gcd(P_i, Q_i) = 1.$$

The supports of the f_k are $A_k = \{a_{k0}, \ldots, a_{kn}\}$. We say that a_{ki} is selected iff $a_{ki} \in \mathrm{supp}(Q_k(t)), k = 0, 1$. In order to denote that a_{ki} is selected (non-selected,

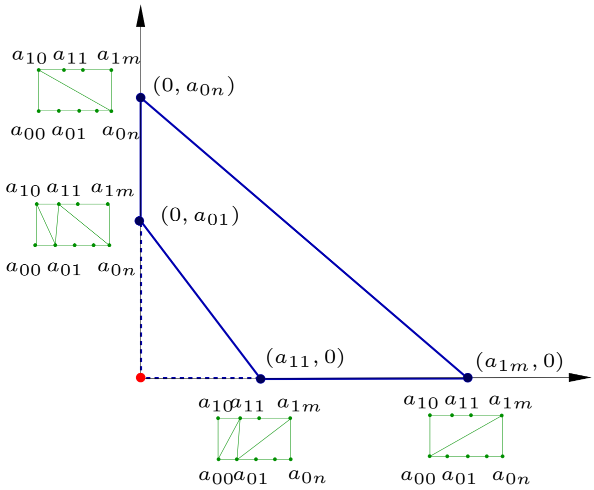

FIGURE 2. The Cayley triangulations corresponding to the implicit vertices of a polynomial curve.

resp.), we write a^+_{ki} (a^-_{ki} resp.). Let a^+_{kR}, a^+_{kL} stand for the rightmost and leftmost selected points (not necessarily distinct) in $A_k, k = 0, 1$.

We are now ready to state a representative result for this case; for a full description of $N(\phi)$ see [3]. Let e_0, e_1 denote the exponent of x, y respectively.

Theorem. Suppose that $e_1^{max}|_{e_0^{max}} \neq a_{0n} - a_{00}$ and let $\delta = (a_{0n} - a^+_{0R}) \cdot (a^+_{1L} - a_{10}) - (a^+_{0L} - a_{00}) \cdot (a_{1m} - a^+_{1R})$. Then, the upper right corner of $N(\phi)$ consists of a single edge, unless none of the $a_{00}, a_{10}, a_{0n}, a_{1m}$ is selected and $\delta \neq 0$, in which case there is vertex $p = (a_{1m} - a^+_{1L}, a^+_{0R} - a_{00})$ if $\delta < 0$, or $p = (a^+_{1R} - a_{10}, a_{0n} - a^+_{0L})$ if $\delta > 0$.

Corollary. Take a polygon at the origin with two edges lying on the positive axes: Polynomial parameterizations have $N(\phi)$ defined by a right triangle with at most one corner cut, which excludes the origin. Rational parameterizations with equal denominators have $N(\phi)$ defined by a right triangle with at most two cuts, on the same or different corners. Rational parameterizations with different denominators have $N(\phi)$ defined by a quadrilateral with at most two cuts, on the same or different corners.

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Mu-Bases for Polynomials Systems in One Variable: Or What About Non-Planar Curves?

Ron Goldman
(joint work with Ning Song, Xiaohong Jia, Falai Chen)

The syzygy module of a collection of univariate polynomials $f_1(t), \ldots, f_k(t)$ with real coefficients is the collection of k-tuples

$$\mathrm{syz}(f_1(t), \ldots, f_k(t)) = \{\big(p_1(t), \ldots, p_k(t)\big) | p_1(t)f_1(t) + \cdot s + p_k(t)f_k(t) \equiv 0\}.$$

If $\mathrm{GCD}(f_1(t), \ldots, f_k(t)) = 1$, then $\mathrm{syz}(f_1(t), \ldots, f_k(t))$ is known to be a free module with $k-1$ generators [4, 3]. A μ-basis for $\mathrm{syz}(f_1(t), \ldots, f_k(t))$ is a special basis $u_1(t), \ldots, u_{k-1}(t)$ for $\mathrm{syz}(f_1 t, \ldots, f_k(t))$ such that

$$\sum_{j=1}^{k-1} \deg(u_j(t)) = \max_{1 \le i \le k} \deg(f_i(t)).$$

Though syzygy modules are standard objects in Algebraic Geometry, this notion was recently rediscovered independently in the Geometric Modeling community by Sederberg and his collaborators [7] during their investigation of rational planar curves. Sederberg called a parametrized implicit equation of a line in the plane

$$A(t)x + B(t)y + C(t)z = 0, \tag{1}$$

a *moving line*,where $A(t), B(t), C(t)$ are polynomials in t. He said that a moving line *follows* a rational planar curve

$$P(t) = (x(t), y(t), w(t)) \tag{2}$$

if

$$A(t)x(t) + B(t)y(t) + C(t)w(t) \equiv 0.$$

Thus a moving line (1) follows a rational curve (2) if

$$(A(t), B(t), C(t)) \in \mathrm{syz}(x(t), y(t), w(t)).$$

Sederberg and his collaborators used moving lines to generate a compact representation for the implicit equation of a rational planar curve [5]. The notion of μ-bases in Algebraic Geometry can be traced back to Hilbert [4], but μ-bases for rational curves were introduced into Geometric Modeling by Cox and Sederberg [3] to help in the investigation of moving lines that follow rational planar curves.

Though the focus in Geometric Modeling has been on μ-bases for rational planar curves – that is, on μ-bases for the syzygy module for three polynomials in one variable – in this talk we discuss the theory and applications of μ-bases for the syzygy module of an arbitrary number of polynomials in one variable. We provide a general algorithm for finding a μ-bases for any collection of univariate polynomials based solely on linear algebra (essentially Gaussian elimination) [8, 2, 10].

Our initial focus is, as usual, on applications of μ-bases to the investigation of rational planar curves. Here the main geometric applications include compact representations for the implicit equation and robust intersection algorithms [5, 6]. We also discuss some recent work on the detection and analysis of singularities using

μ-bases [1, 9]. We show that there is a 1-1 correspondence between singularities of order k and axial moving lines (moving lines that go through a common point) of degree $n-k$ that follow the curve. We then explain how to apply μ-bases to find all the axial moving lines that follow a curve.

Next we shift our attention to consider some recent new work on applications of μ-bases to rational space curves [10]. Here we develop novel techniques based on μ-bases for determining if a point lies on a rational space curve along with methods for finding the parameter value(s) corresponding to a point on a rational space curve (inversion). In this context we show how to apply our μ-basis algorithm to find the GCD of an arbitrary collection of univariate polynomials. Finally, we consider intersection algorithms for rational space curves. Intersection algorithms naturally raise the question of implicitization procedures for rational space curves – that is, procedures for finding low degree algebraic surfaces whose intersection is the given space curve. We present some new results on implicitization for rational space curves along with some conjectures suggesting some new open problems for future research.

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Syzygies and geometry of a basepoint free subspace of $H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,1))$

Hal Schenck

(joint work with David Cox, Alicia Dickenstein)

Let $R = k[x, y, z, w]$ be a bigraded ring over an algebraically closed field, with x, y of degree $(1, 0)$ and z, w of degree $(0, 1)$. Let $R_{m,n}$ denote the graded piece in degree (m, n). Note that $R_{m,n} \simeq H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(m, n))$. Suppose $I = \langle p_0, p_1, p_2\rangle \subseteq R$

is generated by three elements of degree $(2,1)$, such that the base locus of I in $\mathbb{P}\times\mathbb{P}$ is empty. The paper [1] describes the possible bigraded minimal free resolutions of such an ideal. View I as a projective plane in $\mathbb{P}^5$, via:

$$\mathbb{P}^2 = \mathbb{P}(I_{2,1}) \subseteq \mathbb{P}(R_{2,1}) = \mathbb{P}^5.$$

Let $\Sigma_{2,1}$ be the smooth cubic threefold in $\mathbb{P}^5$ which is the image of the Segre map

$$\mathbb{P}^2 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^5.$$

A main result of [1] is that there are two possible types of resolution for I, which are determined by the geometry $X_I = \mathbb{P}(I_{2,1}) \cap \Sigma_{2,1}$. Generically, X_I consists of three points.

Theorem 1. If X_I is a smooth conic, then the minimal free resolution of I is:

$$0 \to R(-6,-3) \to \begin{matrix} R(-4,-3)^2 \\ \oplus \\ R(-6,-2)^2 \end{matrix} \to \begin{matrix} R(-6,-1) \\ \oplus \\ R(-4,-2)^3 \\ \oplus \\ R(-2,-3) \end{matrix} \to R(-2,-1)^3 \to I \to 0.$$

If X_I is not a smooth conic, then the minimal free resolution of I is:

$$0 \to R(-6,-3) \to \begin{matrix} R(-4,-3)^3 \\ \oplus \\ R(-6,-2)^2 \end{matrix} \to \begin{matrix} R(-6,-1) \\ \oplus \\ R(-4,-2)^3 \\ \oplus \\ R(-3,-3)^2 \end{matrix} \to R(-2,-1)^3 \to I \to 0.$$

The differentials in these complexes are determined quite explicitly, from iterated mapping cones [3]. The three first syzygies of degree $(-4,-2)$ which occur in both resolutions are Koszul syzygies. To understand the other first syzygies, consider the second case of the theorem. There are two minimal first syzygies of degree $(-3,-3)$. Suppose g, h are bihomogeneous polynomials such that

$$p_i = A_i\, g + B_i\, h, \quad i \in \{0,1,2\}.$$

Let $C_{ij} = A_iB_j - A_jB_i$. Since

$$\det \begin{bmatrix} p_0 & A_0 & B_0 \\ p_1 & A_1 & B_1 \\ p_2 & A_2 & B_2 \end{bmatrix} = 0,$$

(C_{12}, C_{20}, C_{01}) is a first syzygy, and [1] shows all first syzygies are of this type.

When we consider the generalization of this result to the case that I is four dimensional, the situation becomes much more complex. In [4], Elkadi-Galligo-Lê give a geometric description of the image and singular locus of a map $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^3$ defined by four elements of $H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,1))$ satisfying certain genericity conditions. They show that the implicit equation has a compact form, and that the singular locus, generically, consists of a twisted cubic curve, together with four isolated double points.

When I is not generic, Lê [5] has recently enumerated all possible singular loci for the image of a such a map. In work in progress [2], we are investigating the connection between the geometry of the singular locus of the implicit surface, and the syzygies of the defining equations. In particular, when I contains an ideal $J = \langle p_0, p_1, p_2 \rangle$ generated in degree $(2,1)$ such that $\mathbb{P}(J_{2,1}) \cap \Sigma_{2,1}$ is a smooth conic, then the singular locus of the implicit surface degenerates in several different ways. We close with an example.

Example 1. Let $J = \langle x^2z, y^2w, x^2w + y^2z \rangle$, and $I = J + \langle xy(z + 17w) \rangle$. The variety X_J is a smooth conic. The implicit equation of the surface is:

$$256a_1^3a_2 - 8704a_1^2a_2^2 + 73984a_1a_2^3 - 544a_1^2a_2a_3 + 9248a_1a_2^2a_3 + 289a_1a_2a_3^2 + \\ a_1^2a_4^2 + 222a_1a_2a_4^2 + 289a_2^2a_4^2 - a_1a_3a_4^2 - 289a_2a_3a_4^2 + a_4^4$$

The singular locus has two components:

$$V(a_1 - 289a_2, 73984a_2^2 - 289a_2a_3 + a_4^2) \bigcup V(a_4, 16a_1 - 272a_2 - 17a_3).$$

The two components (a line and plane conic) meet at the point

$$V(a_4, 256a_2 - a_3, 256a_1 - 289a_3).$$

A computation shows that for this example, there is a minimal first syzygy in bidegree $(0,2)$, whereas in the generic case, there is no such syzygy. The syzygy is involves only the generators of J, and is simply the sygygy which is guaranteed to exist by Theorem 1.

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Families of Rational Curves on Surfaces

Josef Schicho

(joint work with Niels Lubbes)

There are two types of algebraic surfaces with a family of rational curves: for the birationally ruled surfaces, there exists exactly one such family, constituting the fibers of a birational map to a curve pf positive genus; for rational surfaces, there exist infinitely many such families, some being fibrations and some not.

In this talk we consider the following problem: given is a rational surface, not necessarily regular, embedded in projective space of arbitrary dimension. We want to find a family of rational curves on it with degree as small as possible. For instance, in the case of a nonsingular cubic, the smallest degree is 2, and there are exactly 27 families of conics, corresponding to the 27 lines.

The problem is related to the problem of finding a parametrization with of smallest possible degree, see [2, 3].

It turns out that there is a combinatorial counterpart in lattice geometry (see [1]), namely the problem to find the width of a given lattice polygon and the directions in which the width is achieved as miminal length of a projected one-dimensional set of lattice points.

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Problem: Numerical Implicitization

Josef Schicho

Most existing algorithms for implicitization are not numerical: when the input parametrization has coefficients given by floats and are subject to some limits of precision, the computed answer is either meaningless or does not give an answer which is best possible under these limits. This is partially due to the fact that implicitization is numerically ill-posed, i.e. the answer to the problem does not depend continuously on the input parameters. On the other hand, there exist standard techniques for treating this type of problem: one identifies the most special structural case which is compatible with the given data, and computes the exact solution with minimal backward error within this case (see [4] and the references therein).

Here is the state of the art on the problem: There exist algorithms which seem to be numerically applicable; we mention [1]. An attempt of an error analysis, based on the algorithm [2], has been given in [3]. However, in this analysis it is assumed that the degree of the implicit equation is known, and the result clearly indicate that something goes wrong from the numerical point of view if the guessed degree was not correct.

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Syzygies and the Rees Algebra

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(joint work with David Cox, J. William Hoffman)

Let a, b, c be linearly independent homogeneous polynomials in the standard $\mathbb{Z}$-graded ring $R := k[s, t]$ with the same degree d and no common divisors. This defines a morphism $\mathbb{P}^1 \to \mathbb{P}^2$. The Rees algebra $\mathrm{Rees}(I) = R \oplus I \oplus I^2 \oplus \cdots$ of the ideal $I = \langle a, b, c\rangle$ is the graded R-algebra which can be described as the image of an R-algebra homomorphism h: $R[x, y, z] \to \mathrm{Rees}(I)$. This talk discusses one result concerning the structure of the kernel of the map h and its relation to the problem of finding the implicit equation of the image of the map given by a, b, c. In particular, we prove a conjecture of Hong, Simis and Vasconcelos [2]. We also relate our results to the theory of adjoint curves and prove a special case of a conjecture of Cox [1].

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