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Scaling Limits in Models of Statistical Mechanics

Organised by
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Abstract. The workshop brought together researchers interested in spatial random processes and their connection to statistical mechanics. The principal subjects of interest were scaling limits and, in general, limit laws for various two-dimensional critical models, percolation, random walks in random environment, polymer models, random fields and hierarchical diffusions. The workshop fostered interactions between groups of researchers in these areas and led to interesting and fruitful exchanges of ideas.

Mathematics Subject Classification (2000): 82xx, 60xx.

Introduction by the Organisers

The workshop *Scaling Limits in Models of Statistical Mechanics* was organized by Kenneth Alexander (USC), Marek Biskup (UCLA), Remco van der Hofstad (Eindhoven) and Vladas Sidoravicius (CWI Amsterdam/IMPA Rio de Janeiro). Nearly fifty participants attended which included senior researchers as well as mid-career and junior scientists, including a number of graduate students.

The aim of the meeting was to allow leading figures in the field, including graduate students, to report on their progress in the study of limit laws of various spatial random processes. We proceed by summarizing the highlights of the meeting; more detailed descriptions can be found in the abstracts that follow.

Two-dimensional critical models: A great deal of attention was paid to the rapidly developing area of two-dimensional critical models. Here, P. Nolin reported on his progress in analyzing monochromatic crossing exponents in 2D critical percolation,

M. Damron discussed the relation between the incipient infinite cluster and invasion percolation in 2D. C. Hongler explained a rigorous calculation of the energy density in 2D critical Ising model, H. Duminil-Copin announced a proof of an RSW theorem for the same system. H. Lei outlined an axiomatic approach to a proof of the full scaling limit in critical 2D percolation. G. Grimmett elucidated the role of the sharp threshold phenomenon in a number of statistical mechanical models. Finally, J. van den Berg showed how to establish sharpness of a percolation transition in 2D contact process. These recent results strengthen rigorously the relation between two-dimensional statistical physical models and the Schramm-Loewner Evolution. It can be expected that, in the coming years, considerable important progress will be made in this exciting area.

Random walk in random environment: Another quite active area where results were reported was that of random walks in random environment. Here M. Holmes studied various foundational facts concerning degenerate random environments, where the usual ellipticity condition fails and therefore, little is rigorously known. N. Gantert outlined her recent proof of the Einstein relation for symmetric diffusions in random environment relating the variance in the central limit theorem to the derivative of the speed with respect to a variable measuring the tilting of the transition probabilities. N. Berger explained an approach to a quenched local CLT for space-time random walks in random environment. T. Kumagai discussed convergence of discrete Markov chains (related to RWRE) to jump processes. Finally, A. Ramirez talked about conditions for ballisticity for RWRE. While tremendous progress is being made in the area of RWRE, the question what the precise conditions are to ensure ballisticity remains open. Also, RWREs away from the usual ellipticity conditions are still rather poorly understood, and we hope that progress will be made in this area.

Random fields and renormalization: A third area that received a lot of attention was random fields and their scaling limits. Here J.-D. Deuschel outlined a number of results that can be proved for gradient models with non-convex potentials at sufficiently high temperatures, while R. Kotecký addressed a similar problem at very low temperatures by means of a multiscale analysis related to the renormalization group approach. Ideas of renormalization appeared in a talk of F. den Hollander who used it to describe a scaling limit of a system of interacting diffusions on a hierarchical lattice. Another talk that belongs both to the category of random fields and to 2D critical models is that by C. Newman who studied a scaling limit of the magnetization-profile random field in the critical 2D Ising model.

High-dimensional problems: A fourth area represented at the meeting was that of high-dimensional and mean-field models. Here D. Ioffe outlined a proof of a very sharp metastability phenomenon for Glauber dynamics in disordered complete graph Ising model. A. Sakai showed how to control the gyration radius for high-dimensional self-avoiding random walk and oriented percolation. M. Heydenreich explained how the scaling behavior of the critical percolation on high-dimensional tori is related to percolation on a random graph in the absence of geometry.

Statistical mechanics and interacting particle systems: The remainder of the talks dealt with a diverse selection of problems in statistical mechanics and interacting particle systems. Here O. Angel discussed the speed process in the asymmetric simple exclusion model. F. Redig presented a beautiful connection between a (quantum) heat-conduction model and an inclusion process. T. Bodineau discussed large deviations for stochastic particle systems. L. Rolla presented results on activated random walks. N. Zygouras discussed a phase transition in a randomly pinned polymer. Finally, T. Sasamoto talked about superdiffusivity for the one-dimensional KPZ problem.

General comments on the workshop: We have specifically chosen for each of the above topics to hold the talks on the same day of the workshop. We also placed an emphasis on the younger generation of researchers who presented exciting developments in our active field of research. With organisers from both Europe and the U.S. the aim was further to exchange information between these active research communities, while also giving the chance to researchers from Japan and South America to present their recent work. The atmosphere during the meeting was very positive indeed, and the participants used all opportunities to extensively discuss recent approaches and ideas, both during and after the lectures. We have specifically left room in the schedule for the meeting to allow for these informal discussions, and we are quite happy with how this worked out.

The organizers wish to thank the 'Mathematisches Forschungsinstitut Oberwolfach' for help in running the workshop, and for providing us with a friendly and encouraging environment throughout the entire meeting. Encouraged by the positive feedback from participants, and the lively atmosphere in the area of statistical mechanics, the organisers have taken the opportunity to discuss the possibility of a follow-up (i.e., a third) meeting in this area in about three years.

Workshop: Scaling Limits in Models of Statistical Mechanics

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Abstracts

Monochromatic Arm Exponents for 2D Percolation

Pierre Nolin
(joint work with V. Beffara)

We investigate the so-called *monochromatic arm exponents* for two-dimensional critical percolation [1]. Arm events are exceptional events that turned out to be instrumental to describe critical and near-critical percolation. They refer to the existence in annuli of a fixed number j of monochromatic crossings, with colors (black or white) prescribed in cyclic order.

A simple and nice combinatorial trick shows that the particular sequence of colors does not affect the asymptotic behavior of their probabilities – as the modulus of the annulus gets larger and larger – as long as the sequence is non-constant (*ie* both colors are present). In this case, they can be described by exponents $\alpha_j = (j^2 - 1)/12$, known as the *polychromatic arm exponents*: for any non-constant sequence of colors σ,

$$\mathbb{P}(j \text{ arms of colors } \sigma \text{ up to distance } n) = n^{-\alpha_j + o(1)}$$

as $n \to \infty$. This result was proved by Smirnov and Werner [7], using the precise knowledge of interfaces in the scaling limit, based on previous work by Smirnov [6] and Lawler-Schramm-Werner [4, 5]. The exponent for 1 arm is also known to be equal to $\alpha'_1 = 5/48$ (note that it does not fit in the previous family). By using Kesten's scaling relations [3], it then allows to derive most critical exponents usually associated with standard percolation $(\beta, \nu, \gamma \ldots)$.

However, the previously-mentioned arguments do not seem to apply in the monochromatic case, and a natural question is whether the corresponding probabilities still follow power laws, and then, if this is the case, how the associated exponents are related to the polychromatic ones, in particular if they constitute a different set of exponents.

We first prove the existence of monochromatic exponents: there exist exponents α'_j such that for each $j \geq 1$,

$$\mathbb{P}(j \text{ arms of colors } \sigma \text{ up to distance } n) = n^{-\alpha'_j + o(1)}.$$

Our main tool to establish this result is the *full scaling limit of percolation*, that was constructed by Camia and Newman [2].

We then relate these new exponents to the polychromatic exponents α_j: we show that for each j,

$$\alpha_j < \alpha'_j < \alpha_{j+1}.$$

The right-hand inequality is essentially trivial, while the left-hand one is more intricate and can be obtained by using an "entropy vs energy" argument. Roughly speaking, the number of arm configurations that one can choose on a "typical" configuration with j arms is much larger in the monochromatic case than in the polychromatic case, while the expected number of such configurations is the same in the two cases. However, making this remark rigorous is not completely direct and we use the winding angles of the arms as a way of distinguishing arm configurations. We also use a particular correlation inequality, that was an intermediate step in the proof of the Van den Berg-Kesten-Reimer (BKR) inequality. We finally present a "suggestion" on the value of the monochromatic exponents, based on numerical simulations.

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Invasion Percolation and Incipient Infinite Clusters in 2D

Michael Damron

(joint work with A. Sapozhnikov and B. Vágvölgyi)

The first part of this thesis centers on the two-dimensional invasion percolation model. It is a great example of self-organized criticality; in other words, it requires no parameter, yet some characteristics of the model resemble those at criticality of a parametric model with a phase transition. The work described here [2, 3] on invasion percolation concerns comparing its properties to those of other critical objects, such as critical percolation and the incipient infinite cluster.

The definition of the invasion model is as follows. We consider the lattice $\mathbb{Z}^d$ and associate with each nearest-neighbor edge e a random number $\tau(e)$ (a weight). The weights are i.i.d. from edge to edge and are uniformly distributed in $[0,1]$. The invasion graph of the origin is defined as the limit of a recursively-defined sequence of graphs: we first let

$$G_0 = (V_0, E_0) \text{ with } V_0 = \{(0,0)\} \text{ and } E_0 = \{\}.$$

For any $n \geq 1$, we define

$$G_n = (V_n, E_n) \text{ with } V_n = V_{n-1} \cup \{v_n\} \text{ and } E_n = E_{n-1} \cup \{e_n\},$$

where e_n is the least-weight edge on the boundary of G_{n-1} (that is, it has at least one vertex in G_{n-1} but is not in E_{n-1}) and v_n is the vertex not in V_{n-1} which is an endpoint of e_n (if one such exists). Last, we define

$$\mathcal{S} = \lim_{n\to\infty} G_n$$

to be the **invasion percolation cluster (IPC) of the origin**.

Much of the work will serve to to compare the IPC to independent Bernoulli percolation; therefore, we use a definition of Bernoulli percolation that makes the coupling with invasion percolation immediate. We consider the same i.i.d. weights $\{\tau_e\}$ on the edges of $\mathbb{E}^d$ that were introduced in the previous paragraph. For any $p \in [0,1]$ we say that an edge $e \in \mathbb{E}^d$ is *p-open* if $\tau_e < p$ and *p-closed* otherwise. It is obvious that the resulting random graph of p-open edges has the same distribution as the one obtained by declaring each edge of $\mathbb{E}^d$ open with probability p and closed with probability $1-p$, independently of the states of all other edges. The percolation probability $\theta(p)$ is the probability that the origin is in the (unique) infinite cluster of p-open edges. There is a critical probability

$$p_c = \inf\{p \ : \ \theta(p) > 0\} \in (0,1).$$

From here, *we restrict to two dimensions*. We study the so-called *pond* construction of the invasion, introduced first in [6], and which we now describe. It was shown in [5], under the assumption that the critical parameter for slab percolation in $\mathbb{Z}^d$ equals that for all of $\mathbb{Z}^d$, that for all $p > p_c$ the invasion intersects the infinite p-open cluster with probability one. Furthermore, the definition of the invasion mechanism implies that if the invasion reaches the p-open infinite cluster for some p, then if will never leave this cluster. Combining these facts yields that if e_i is the edge added at time i then $\limsup_{i\to\infty} \tau_{e_i} = p_c$. It is well-known in two dimensions that $\theta(p_c) = 0$, which implies that for every $t > 0$ there is an edge $e(t)$ such that $e(t)$ is invaded after step t and $\tau_{e(t)} > p_c$. The last two results give that $\hat{\tau}_1 = \max\{\tau_e \ : \ e \in E_\infty\}$ exists and is greater than p_c. Let $\hat{e}_1$ denote the edge at which the maximum value of τ is taken and assume that $\hat{e}_1$ is invaded at step $i_1 + 1$. Following the terminology of [6], we call the graph G_{i_1} the *first pond* of the invasion, and we denote it $\hat{V}_1$. The edge $\hat{e}_1$ is called the *first outlet*. The second pond of the invasion is defined similarly. Note that the same argument as above implies that $\hat{\tau}_2 = \max\{\tau_{e_i} \ : \ e_i \in E_\infty, i > i_1\}$ exists and is greater than p_c. If we assume that $\hat{\tau}_2$ is taken on the edge $\hat{e}_2$ (the second outlet) at step $i_2 + 1$, we call the graph $G_{i_2} \setminus G_{i_1}$ the *second pond* of the invasion, and we denote it $\hat{V}_2$. The further ponds $\hat{V}_k$ can be defined analogously.

In [2], we compute the exact decay rate of the distribution of $\hat{R}_k$, the radius of the k^{th} pond. Unlike the decay rate of the distibution of the radius of the first pond [1], it is strictly different from that of the radius of the critical cluster of the origin. Furthermore, in [3], we compute the exact decay rate of $\hat{V}_k$, the volume of the k^{th} pond. This result can be also seen as a statement about the sequence of steps i_k at which $\hat{e}_k$ are invaded. Specifically, the results are the following:

Theorem 1. *Let $k \geq 1$. There exist constants $c_1 - c_4 > 0$ such that for all n,*

$$c_1 \, \mathbb{P}(\hat{R}_k > n) \leq (\log n)^{k-1} \mathbb{P}_{cr}(0 \leftrightarrow \partial B(n)) \leq c_2 \, \mathbb{P}(\hat{R}_k > n),$$

$$c_3 \, \mathbb{P}(\hat{V}_k > n) \leq \mathbb{P}_{cr}(|C(0)| > n) \leq c_4 \, \mathbb{P}(\hat{V}_k > n).$$

We compare invasion percolation to the incipient infinite cluster (IIC). For this, we need to define the latter object. Using the notation that $\mathbb{P}_{cr}$ is the critical percolation measure, it was shown in [4] that the limit

$$\nu(E) = \lim_{N \to \infty} \mathbb{P}_{cr}(E \mid 0 \leftrightarrow \partial B(N))$$

exists for any event E that depends on the state of finitely many edges in $\mathbb{E}^2$. The unique extension of ν to a probability measure on configurations of open and closed edges exists. Under this measure, the open cluster of the origin is a.s. infinite. It is called the *incipient infinite cluster* (IIC). The following result is in [2].

Theorem 2. *The laws of IPC and IIC are mutually singular.*

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The Energy Density in the 2D Ising Model

Clément Hongler

(joint work with S. Smirnov)

We study the Ising model in two dimensions at criticality. Let us consider a Jordan domain Ω and discretize it by taking Ω^δ, the largest connected component of its intersection with $\delta\mathbb{Z}^2$, the square grid of mesh size $\delta > 0$. The Ising model on Ω^δ is a random assignment of ± 1 spins to the vertices $\mathcal{V}\left(\Omega^\delta\right)$ of Ω^δ, with the probability of each spin configuration $\sigma \in \{\pm 1\}^{\mathcal{V}\left(\Omega^\delta\right)}$ given by $\mathbb{P}\{\sigma\} := \exp\left(-\beta H\left(\sigma\right)\right)/\mathcal{Z}_\beta$, where $\beta > 0$ is the so-called inverse temperature, where the energy H of σ is equal to minus the sum over all edges of Ω^δ of the product of the spins at the edpoints of each edge (i.e. $H\left(\sigma\right) := -\sum_{\langle x,y\rangle \in \mathcal{E}(\Omega^\delta)} \sigma_x \sigma_y$) and the so-called partition function $\mathcal{Z}_\beta$ is defined by $\mathcal{Z}_\beta := \sum_{\sigma \in \{\pm 1\}^{\mathcal{V}(\Omega^\delta)}} \exp\left(-\beta H(\sigma)\right)$. In our setup, we will consider the Ising model with $+$ boundary conditions, that is, we add $+1$ spins on the boundary of Ω^δ.

We are interested in describing the $\delta \to 0$ limit at the so-called critical value of β, which is equal to $\ln\left(\sqrt{2}+1\right)/2$. More precisely, we would like to describe the effect of the shape of the domain Ω on local quantities. We want to understand here how the energy of the system distributes across the domain: the contributions to the energy are given by the edges, so we want to compute how much each edge contributes to the energy of the system.

Let us fix an edge a in the bulk of the domain, with endvertices x and y. What is the probability that the product of the spins at the endvertices of e is equal to 1 or -1? We have the following result:

Theorem 1. *As $\delta \to 0$, we have*

$$\frac{\mathbb{P}\left\{\sigma_x = \sigma_y\right\} - \frac{\sqrt{2}+2}{2}}{\delta} \to \frac{1}{2\pi} l_\Omega\left(a\right),$$

where $l_\Omega(a)$ is the element of the hyperbolic metric of Ω at a (a shrinks to a point as $\delta \to 0$), defined in the following way: for $\psi : \Omega \to D(0,1)$ a conformal mapping to the unit disc $D(0,1)$ mapping a to 0, we have $l_\Omega(a) = |\psi'(a)|$.

We now give a sketch of the proof of this result.

(1) We first express the probability $\mathbb{P}\left\{\sigma_x = \sigma_y\right\}$ in terms of a statistics over families of contours, using the so-called low-temperature expansion of the Ising model.
(2) We then perform a discrete holomorphic deformation of this statistics. More precisely, we define an observable, which is a function $f_a^\delta : \Omega^\delta \to \mathbb{C}$ with $f_a^\delta(a) = \mathbb{P}\left\{\sigma_x = \sigma_y\right\}$. Using combinatorial techniques, we show the following properties:
   (a) We show that f_a^δ is discrete holomorphic in a certain sense on $\Omega^\delta \setminus \{a\}$.
   (b) We show that f_a^δ has a discrete singularity at a.
   (c) We study the values of f_a^δ on the boundary of the domain Ω: f_a^δ solves a so-called Riemann-Hilbert boundary value problem.
(3) We pass then to the $\delta \to 0$ limit, and show that we obtain a continuous holomorphic deformation in the limit.
   (a) We first show precompactness of the family of functions $\left(f_a^\delta\right)_{\delta>0}$.
   (b) We then identify uniquely the limit f_a, which is given by $f_a(z) = \frac{1}{2\pi}\frac{\psi_a(z)+1}{\psi_a(z)}\sqrt{\psi_a'(z)}\sqrt{\psi_a'(a)}$ where $\psi_a : \Omega \to D(0,1)$ is the unique conformal mapping from Ω to $D(0,1)$ that maps a to 0 and has positive derivative at a.
   (c) We finally recover $\lim_{\delta\to 0}\left(\mathbb{P}\left\{\sigma_x = \sigma_y\right\} - \frac{\sqrt{2}+2}{2}\right)/\delta$ as the constant term in the Laurent expansion of f_a at a.

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Sharp-thresholds and Box-crossings

Geoffrey Grimmett
(joint work with B. Graham)

Let $X_1, X_2, \dots, X_N$ be independent Bernoulli random variables with parameter $\frac{1}{2}$. Let A be an event in the sample space $\Omega = \{0,1\}^N$, and define the absolute influence of the index e on the event A by

$$I_A(e) = P\big(1_A(\omega^e) \neq 1_A(\omega_e)\big),$$

where 1_A is the indicator function of A, and ω^e (respectively, ω_e) is the configuration ω with $\omega(e)$ set to 1 (respectively, 0).

It has effectively been proved by Kahn, Kalai, and Linial [5] and Talagrand [6] that there exists an absolute positive constant c such that

$$\sum_e I_A(e) \geq cP(A)P(\overline{A}) \log[1/M]$$

where $M = \max\{I_A(e) : 1 \leq e \leq N\}$. Note the absence of any assumption of symmetry in the inequality.

When the X_j have parameter p, Russo's formula gives that

$$\frac{d}{dp} P_p(A) \geq cP_p(A)P_p(\overline{A}) \log[1/M_p],$$

a formula that has found several applications.

The above two inequalities have been extended to families generated by including an external-field term within a measure satisfying the FKG lattice condition, see [2, 3].

Four applications are given, as reported in [3]. Firstly, it is shown how to shorten the proof of [1] that $p_c = \frac{1}{2}$ for bond percolation on the square lattice. The method is to bound M_p from above with A the event of a left–right crossing of a box with dimensions n by $n+1$.

The same approach is effective with the box-crossing probability in a random-cluster process on $\mathbb{Z}^2$. This results in a proof that the box-crossing probability increases steeply from near 0 to near 1 over a short interval containing the self-dual point $p_{\mathrm{sd}} = \sqrt{q}/(1+\sqrt{q})$. This falls short of a proof of the conjecture that p_{sd} equals the critical point, since no RSW-type argument is yet known for this system.

The third application is to box-crossings by + paths in the Ising model on $\mathbb{Z}^2$ with external field. It turns out that such probabilities have sharp thresholds at the critical value $h_c(\beta)$ discussed by Higuchi [4]. This was of course known already, but the current proof is very short once one has the influence inequality.

The final application is to the 'coloured random-cluster model', obtained from a random-cluster measure by assigning one of two possible colours to each cluster, and then perturbing the measure with an external field. The proof is more complicated in this case, since the spin measure is not generally a nearest-neighbour Gibbs state.

The lecture ended with a statement of two open problems for the two-dimensional Ising model.

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Convergence to SLE_6 for Percolation Models

Helen K. Lei

(joint work with I. Binder and L. Chayes)

In this talk we explain one way to obtain convergence to SLE_6 for percolation models satisfying fairly general assumptions, provided Cardy's Formula for crossing probability can be established. We start with some domain $\Omega \subset \mathbf{R}^2$ with boundary Minkowski dimension less than two, with two boundary prime ends a and c. We then tile Ω with some lattice at scale ε and perform percolation at the critical value. Given any percolation configuration, there is an interface running from a to c which separates the largest occupied cluster connected to the boundary $[a, c]$ from the largest vacant cluster connected to the boundary $[c, a]$. We equip the space of curves with a weighted version of the supremum norm; and we let μ_ε denote the probability measure on random curves generated by the percolation interface described. We then prove, under fairly general conditions, that μ_ε converges in law to SLE_6.

Our proof of convergence works for percolation models with the following properties:

(1) Russo–Seymour–Welsh (RSW) theory: Uniform estimates for probabilities of crossings (of either type) on all scales plus the ability to stitch smaller crossings together without substantial degradation of the estimates – FKG–type inequalities.

(2) BK–type inequalities whereby probabilities of separated path type events can be estimated in terms of the individual probabilities.

(3) Explicit ("superuniversal") "bounds" on full–space multiple colored five–arm events and half–space multiple colored three–arm events: The probability of observing disjoint crossings of an annulus with aspect ratio a is, on all scales, bounded above by a constant times a^{-2}.

(4) A self–replicating definition of an Exploration Process and a class of *admissible* domains with the property that this class is preserved under the operation of deleting the beginning of a typical explorer path in an admissible domain.
(5) The validity of Cardy's Formula for the above–mentioned admissible domains, i.e., the crossing probability converges to the so–called Cardy's Formula as the lattice spacing tends to zero.

Let us now elaborate on item 4. What we really mean by Exploration Process is that there exists a way to parametrize all possible interfaces so that a Markov identity is valid for crossing probabilities. More precisely, let us add two boundary points b and d so that we now have a conformal rectangle. We then let C_ε denote the probability of an occupied crossing from $[a, b]$ to $[c, d]$. What we require then is that the following display is valid:

$$C_\varepsilon\left(\Omega \setminus \mathbb{X}^\varepsilon_{[0,t]}, \mathbb{X}^\varepsilon_t, b, c, d\right) = C_\varepsilon\left(\Omega, a, b, c, d \mid \mathbb{X}^\varepsilon_{[0,t]}\right). \tag{1}$$

In the case of hexagonal tiling (equivalently, site percolation on the triangular lattice), this parametrization is realizable as a simple Exploration Process where a random sample of the interface is drawn by starting a path at a (aiming towards c) with each subsequent step(s) indicated by flipping a fair coin. For other percolation models, the precise way in which such an Exploration Process should be defined will depend on lattice mechanics and the definition of the model; the details for the triangular type model studied in [3] are spelled out in [4]. We should also note that as implied by the display, we have to consider domains which are formed by deleting a portion of the interface, which *a posteriori* we know converges to SLE_6, and hence we actually need to consider a fairly general class of domains (i.e., with boundary dimension in excess of one); this is the source of one set of technical difficulties in [4]. Further, Cardy's Formula needs to be established for such domains (this we accomplish in [4] for the triangular type models introduced in [3], where Cardy's Formula was only established for piecewise smooth domains).

The reason for requiring (1) is because thanks to Schramm's Principle ([7, 10]), random curves are described by SLE if and only if the law is conformally invariant and satisfies the Domain Markov Property, i.e.

$$\mu(\Omega, a, c) \mid_{\gamma'} = \mu(\Omega \setminus \gamma', a', c)$$

(here a' is the tip of γ'), and therefore to prove convergence to SLE, we must retrieve both properties in the continuum limit. For percolation, conformal invariance is encoded by Cardy's Formula and the Domain Markov Property should follow by taking the $\varepsilon \to 0$ limit of

$$\mathbb{E}_{\mu_\varepsilon}\left[C_\varepsilon\left(\Omega \setminus \mathbb{X}^\varepsilon_{[0,t]}, \mathbb{X}^\varepsilon_t, b, c, d\right) \mid \mathbb{X}^\varepsilon_{[0,s]}\right] = C_\varepsilon\left(\Omega \setminus \mathbb{X}^\varepsilon_{[0,s]}, \mathbb{X}^\varepsilon_s, b, c, d\right), \tag{2}$$

where $t > s > 0$ are two fixed times ([10]).

The general scheme for proving convergence consist of roughly a three step process ([7, 10]). The first step is to obtain a weak limit of μ_ε which is supported on Loewner curves. This requires the result of [1], which takes as input estimates on the probability of multiple crossings of an annulus by the interface (more precisely, the probability of n such crossings tends to zero as n tends to infinity; here is where BK–inequality would come in handy). Further, appropriate upper bounds on the probability of having three long arms in the half space and five long arms in the full space are required to show that in fact the limit is supported on *Loewner* curves (some of this requires results and arguments of [5, 6]).

The second step is to take the ε tends to zero limit of (2), after which a Taylor expansion "at infinity" using the precise form of Cardy's Formula shows that we have SLE_6. To accomplish this we establish some equicontinuity of the C_ε's, which can be loosely stated as follows

Proposition 1. *Given $\theta > 0$, $\exists \eta > 0$ such that for all ε small enough ($\varepsilon \ll \eta$) and for all curves γ_1, γ_2 (starting from a and aimed at c) outside a set of small (uniform in ε) measure, for T not too large (so that the curves are not too close to c) and assuming b, c, d are all in the same connected component in the domains $\Omega \setminus \gamma_1[0,T]$ and $\Omega \setminus \gamma_2[0,T]$*

$$|C_\varepsilon(\Omega \setminus \gamma_1[0,T], \gamma_1(T), b, c, d) - C_\varepsilon(\Omega \setminus \gamma_2[0,T], \gamma_2(T), b, c, d)| < \theta,$$

provided that γ_1 and γ_2 are η–close in the weighted sup–norm distance.

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LLN in ASEP and the ASEP Speed Process

Omer Angel

I present several results from recent works with Holroyd-Romik, Amir-Valkó and Balázs-Seppäläinen concerning the asymptotic speeds of particles in the TASEP and ASEP.

1. Results

The (T)ASEP dynamics. The multi-type TASEP on $\mathbb{Z}$ is a Markov process $\{Y_t\}$ with state space $\mathbb{R}^{\mathbb{Z}}$. This is interpreted as having a particle at every site of $\mathbb{Z}$, with a real-valued class: $Y_t(n)$ is the state of the particle at n at time t. For each n, if $Y_t(n) < Y_t(n+1)$ then the two swap at rate 1. (Holes may be implemented as particles of class ∞.) In the ASEP dynamics the same holds, and additionally, if $Y_t(n+1) > Y_t(n)$ then the two swap at rate $q \in (0,1)$. It is straightforward to write the generator.

The relation to the classical (T)ASEP is as follows: Let $\xi_t(n) = 1_{\{Y_t(n) \le k\}}$. Then ξ_t evolves as (T)ASEP, i.e., the set of particles with labels at most k perform biased random walks with exclusion: any jump to an occupied site is forbidden.

It is well known that there is a unique ergodic distribution on $\mathbb{R}^{\mathbb{Z}}$ that is stationary to the ASEP dynamics and has marginals uniform on $[-1,1]$. We denote this distribution by μ (depending implicitly on q). There is nothing special about the uniform distribution.

The permutation process and speeds. Consider the ASEP with $Y_0(n) = n$, i.e. Y_0 is the identity permutation. Clearly Y_t is a permutation on $\mathbb{Z}$. Denote its inverse by $X_t = Y_t^{-1}$, so that $X_t(n)$ is the location of particle n at time t. Let $\gamma = 1 - q$ be the maximal possible speed of a particle.

Theorem 1 (A.-Balázs-Seppäläinen)**.** *There are uniform $[-1,1]$ variables U_n so that*

$$\frac{X_t(n)}{\gamma t} \xrightarrow[t\to\infty]{a.s.} U_n.$$

In the case of the TASEP this was proved by Mountford and Guiol [6]. Strengthening weaker forms of this convergence shown by Rost [7] and by Ferrari and Kipnis [5].

Theorem 2 (Amir-A.-Valko, [1])**.** *The joint law of the limiting U_n above is μ.*

The following is a key tool for proving Theorem 2

Theorem 3. *Fix t. With the notations above, X_t has the same law as Y_t.*

This is proved in [2] in the case of the TASEP, and in [1] for the ASEP.

Hydrodynamic limits. Let $\xi_t(n) = 1_{\{Y_t(n)\le 0\}}$, so that Then ξ_t evolves as a TASEP. Let $\xi' = 1_{\{Y_t<0\}}$. It follows that $\xi_t - \xi_t' = \delta_{X_t(0)}$. However, in distribution ξ_t' is a translation of ξ_t, and therefore $\mathbb{P}(X_t(0) = m) = \mathbb{E}\xi_t(m) - xi_t(m+1)$.

What is $\mathbb{E}\xi_t(m)$? The hydrodynamic limit roughly states it is close to $f_t(m)$, a function that decays linearly from 1 to 0 over $[-t, t]$.

Theorem 4 (Rost [7]). *With high probability*

$$\sum_{at}^{bt} \xi_t(i) = t \int_{at}^{bt} f_t(x)dx + o(t).$$

This is enough to deduce convergence in distribution of $X_t(n)/t$. However, it does not imply joint convergence in distribution for several n's.

Strong hydrodynamics. A key element in the proof of Theorem 1 is the following result: Start an ASEP with particles at $n < 0$ and holes at $n \ge 0$. Let $L_m(t)$ be the location of the m'th rightmost particle at time t.

Lemma 1. *For any $\varepsilon > 0$ there is a C so that for $0 \le m \le t$,*

$$\mathbb{P}\left(\left|X_m(t/\gamma) - t + \sqrt{4mt}\right| > t^{1/3+\varepsilon}\right) \le Ct^{-1/3}.$$

In comparison, the regular hydrodynamic limit for ASEP implies only that with high probability $X_m(t/\gamma) = t - \sqrt{4mt} + o(t)$. The above strengthens the hydrodynamic limit in two ways. First, it improves the fluctuation bound from $o(t)$ to $t^{1/3+\varepsilon}$. Secondly, this gives an estimate on the rate of decay of the failure probability.

Lemma 1 is (nearly) optimal in terms of the exponent $t^{1/3+\varepsilon}$, due to the $t^{1/3}$ fluctuations of $X_m(t)$ (with Tracy-Widom F_2 limit distribution. If the Tracy-Widom limit holds also for large deviations this would imply a stretched exponential bound on the probability above.

Corollary 1. *Let I be an interval of length $|I| > t^{1/3+\varepsilon}$, then the number of particles in I at time t has probability at most $O(t^{-1/3})$ of deviating from its expectation by more than $t^{1/3+\varepsilon}$.*

This is a third strengthening of classical hydrodynamic limits, which only implies an estimate for intervals of length $\Theta(t)$. It is very plausible that such estimates hold already for much smaller intervals, since locally sites are close to independent.

2. Proofs

Symmetry and Stationary. Theorem 2 has a particularly elegant proof in the case of the TASEP. Hammersley's graphical construction of the TASEP has a Poisson point process on $\mathbb{Z} \times \mathbb{R}^+$ as a source of randomness. Extend this to a Poisson process on $\mathbb{Z} \times \mathbb{R}$, which is clearly invariant to shifting by any s (along the $\mathbb{R}$-direction). Let $U_s(n)$ be the resulting speed of particle n after this shift. Direct observation (or manipulation of generators) shows that U_s evolves as a TASEP, but on the other hand it is clearly stationary.

In the general case, Theorem 2 follows from Theorem 3, with the idea being as follows: For some large t, the speed process $\{U_n\}_n$ is close in distribution to $\{X_t(n)/t\}_n$, and by symmetry also to $\{Y_t(n)/t\}_n$. Applying ASEP dynamics for some small time s results in $\{Y_{t+s}(n)/t\}_n$ with the same law as $\{X_{t+s}(n)/t\}_n \approx \{X_{t+s}(n)/(t+s)\}_n$. This in turn is again close to the disrtibution of the speed process, and therefore it's law is μ.

Theorem 3 has a elementary combinatorial/algebraic proof. At its core is the fact that applying a sequence of transpositions in reverse order yields the reverse permutation. This must be comined with Matsumoto's lemma, and in the case of the ASEP some algebraic relations between involved operators (see [1])

LLN. Theorem 1 is proved by careful analysis of a coupling of three TASEP processes, following [3, 4]. Consider the standard coupling of three particle systems with initial condition given by $\xi_0^-(n) = 1_{n<0}$, $\xi_0^+(n) = 1_{n\leq 0}$, and η_0 i.i.d. Bernoulli(ρ). At any given time t at any position, the possible values of the three lines (ξ^-, ξ^+, η) are

$$\text{hole} = (0,0,0) \qquad \uparrow = (1,1,0) \qquad Q^\uparrow = (1,0,1)$$
$$\text{particle} = (1,1,1) \qquad \downarrow = (0,0,1) \qquad Q^\downarrow = (1,0,0)$$

These line evolve roughly as an ASEP with class order hole $<\uparrow< Q <\downarrow<$ particle, (both types of Q are the same) except that an $\uparrow$ and a $\downarrow$ interact and transform to a particle and hole. Of these, there is always a single Q, and its position evolves as $X_t(0)$, in which we are interested. Moreover, all $\uparrow$'s are always to the left of all $\downarrow$'s.

If $X_t(0)/t < 2\rho - 1$, (where ρ is the density of η) then with high probability the second class particle (Q) is to the left of many $\uparrow$'s. From this we deduce that the same is likely to hold at tme $2t$, so that the speed does not increase too much. To quantify both of these claims we use the strong hydrodynamic estimate.

Strong hydrodynamics are proved using recent formulae discovered by Tracy and Widom [8].

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Metastability via Coupling in Potential Wells

DMITRY IOFFE
(joint work with A. Bianchi and A. Bovier)

Roughly speaking, metastable systems are characterized by the fact that the state space can be decomposed into several disjoint subsets, with the property that transition times between subspaces are long compared to characteristic mixing times within each subspace. In the sequel N will be a large parameter. We consider (families of) Markov processes, $\sigma(t)$, with finite state spaces, $S_N \equiv \{-1,1\}^N$, and transition probabilities p_N that are reversible w.r.t. a (Gibbs) measure, μ_N. Transition probabilities p_N always have the following structure: At each step a site $x \in \Lambda$ is chosen with uniform probability $1/N$. Then the spin at x is set to ± 1 with probabilities $p_x^{\pm}(\sigma)$; $p_x^+(\sigma) + p_x^-(\sigma) \equiv 1$. In the sequel we shall assume that there exists $\alpha \in (0,1)$ such that

$$\max_{x,\sigma,\pm} p_x^{\pm}(\sigma) \leq \alpha.$$

A key hypothesis is the existence of a family of "good" mesoscopic approximations of our processes. By this we mean the following: There is a sequence of disjoint partitions, $\{\Lambda_1, \dots, \Lambda_n\}$, of $\Lambda \equiv \{1, \dots, N\}$, and maps family of maps, $\underline{m}^{(n)} : S_N \to \Gamma_n \subset \mathbb{R}^n$, satisfying, where

$$m_i^n(\sigma) = \frac{1}{N} \sum_{x \in \Lambda_i} \sigma_x$$

It will be convenient to introduce the notation $S_N^n[\underline{m}] \subset S_N$ for the set-valued inverse images of $\underline{m}^n$. We think of the maps $\underline{m}^n$ as some block averages of our "microscopic variables σ_i over blocks of decreasing (in n) "mesoscopic" sizes.

As it is well known, the image process, $\underline{m}^n(\sigma(t))$, is in general not Markovian. However, there is a canonical Markov process, $\underline{m}^n(t)$, with state space Γ_n and the very same reversible measure μ_N. For all $\underline{m}, \underline{m}' \in \Gamma_n$, the transition probabilities of this chain are given by

$$r_N(\underline{m}, \underline{m}') \equiv \frac{1}{\mu_N(\underline{m})} \sum_{\sigma \in S_N^n[\underline{m}]} \sum_{\sigma' \in S_N^n[\underline{m}']} \mu_N(\sigma) p_N(\sigma, \sigma').$$

By "good approximations" we will mean that the following two conditions hold:

(A.1) the sequence of chains $\underline{m}^n(t)$ approximates $\underline{m}^n(\sigma(t))$ in the strong sense that that there exists $\epsilon(n) \downarrow 0$, as $n \uparrow \infty$, such that for any $m, m' \in \Gamma_n$,

$$\max_{\sigma \in S^n[\underline{m}], \sigma' \in S^n[\underline{m}']} \left| \frac{p_N(\sigma, \sigma') |S_N^n[\underline{m}']|}{r_N(\underline{m}, \underline{m}')} - 1 \right| \leq \epsilon(n).$$

(A.2) The microscopic flip rates satisfy: If $\underline{m}(\sigma) = \underline{m}(\eta)$ and $\sigma_x = \eta_x$, then $p_x^{\pm}(\sigma) = p_x^{\pm}(\eta)$.

Finally, we need to place us in a "metastable" situation. Specifically, we will assume that there are two disjoint sets $A = \{\sigma \in S_N : \underline{m}^{n_0}(\sigma) \in \mathbf{A}\}$ and $B =$

$\{\sigma \in S_N : \underline{m}^{n_0}(\sigma) \in \mathbf{B}\}$, for some n_0 and sets $\mathbf{A}, \mathbf{B} \in \Gamma_{n_0}$, there exists a constant $C > 0$ such that, for all $n \geq n_0$ large enough and for all $\sigma, \sigma' \in A$,

$$\mathbb{P}_{\sigma'}\left[\tau_B < \tau_{\underline{m}^n(\sigma)}\right] \leq e^{-CN}, \tag{1}$$

where, with a little abuse of notation, we denoted by $\tau_{\underline{m}^n(\sigma)}$ the first hitting time of the set $S_N^n[\underline{m}^n(\sigma)]$.

In this setting we will prove the following theorem.

Theorem 1. *Consider a Markov process as described above, and let A, B be such that* (1) *holds. Then,*

$$\max_{\sigma,\sigma' \in A} \left| \frac{\mathbb{E}_\sigma \tau_B}{\mathbb{E}_{\sigma'} \tau_B} - 1 \right| \leq e^{-CN/2}.$$

The proof of Theorem 1 is based on a rather far reaching adaptation of coupling ideas introduced in [2]. One application we had in mind was the Random Field Curie-Weiss Model, with the *random Hamiltonian*, H_N, being defined as

$$H_N(\sigma) \equiv -\frac{N}{2}\left(\frac{1}{N}\sum_{x\in\Lambda}\sigma_x\right)^2 - \sum_{x\in\Lambda} h_x \sigma_x,$$

where as before $\Lambda \equiv \{1, \ldots, N\}$ and h_x, $x \in \Lambda$, are i.i.d. random variables. Accordingly, the *random* Gibbs measure μ_N is given by

$$\mu_N(\sigma) = Z_N^{-1} \exp\left(-\beta H_N(\sigma)\right),$$

The corresponding macroscopic landscape has a multi-well structure for a variety of distributions of h and inverse temperatures β. In [1] we have studied asymptotics of metastable hitting times for processes which start from equilibrium distributions. Theorem 1 enables to upgrade these asymptotics to microscopic starting points. Furthermore,

Theorem 2. *In the random field Curie-Weiss model, under the hypothesis of Theorem 1,*

$$\mathbb{P}_\sigma\left(\tau_B / \mathbb{E}_\sigma \tau_B > t\right) \to \mathrm{e}^{-t}, \text{ as } N \uparrow \infty,$$

for all $\sigma \in A$ and for all $t \in \mathbb{R}_+$, almost surely with respect to the environment.

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Duality and Exact Correlations in a Model of Heat Conduction

Frank Redig
(joint work with C. Giardina, J.Kurchan, and K. Vafayi)

In this talk we introduce an interacting diffusion model of heat conduction. We show that the model has a particle system dual which we call the "inclusion process". The inclusion process itself is self-dual, and the duality functions can be derived from and underlying $SU(1,1)$ symmetry.

We first define the bulk diffusion process in the heat conduction model, directly on a general countable set. Let S be a countable set and $p(i,j)$ a symmetric irreducible random walk kernel on S. The Brownian momentum process (BMP) associated to $p(i,j)$ is the process on $\mathbb{R}^S$ with generator

$$\mathcal{L} = \sum_{i,j\in S} p(i,j)\left(x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}\right) \tag{1}$$

To define the dual process, we introduce particle configurations $\xi : S \to \mathbb{N}$. A particle configuration is finite if $\sum_{i\in S} \xi_i < \infty$. For $x \in \mathbb{R}^{\mathbb{Z}^d}$ and ξ a finite particle configuration we define the duality function

$$\mathcal{D}(\xi, x) = \prod_{i\in S} D(\xi_i, x_i) \tag{2}$$

where for $k \in \mathbb{N}$, $a \in \mathbb{R}$,

$$D(k,a) = \frac{a^{2k}}{(2k-1)!!}$$

The dual process is a process on $\mathbb{N}^S$ with generator

$$Lf(\xi) = \sum_{i,j\in S} p(i,j) 2\xi_i(1+2\xi_j)(f(\xi^{i,j}) - f(\xi)) \tag{3}$$

where $\xi^{i,j}$ denotes the particle configuration that arises by removing a particle in ξ from the site i and putting it at site j.

The process ξ_t with generator L is called the symmetric inclusion process (SIP) associated to the kernel $p(i,j)$.

The duality relation between the SIP and the BMP then reads

$$\mathbb{E}_x(\mathcal{D}(\xi, x_t)) = \mathbb{E}_\xi(\mathcal{D}(\xi_t, x)) \tag{4}$$

The SIP is a process of its own interest, and can be generalized easily by introducing a parameter $m \in [0,\infty)$, and introducing the generator

$$L_m f(\xi) = \sum_{i,j\in S} p(i,j)\xi_i\left(\frac{m}{2} + \xi_j\right)(f(\xi^{i,j}) - f(\xi)) \tag{5}$$

The process with generator L_m is called the $SIP(m)$.

For $m = 1$ this corresponds to the SIP (up to a global factor of 4 in the rates), and for general integer m, this process is the dual of a system of interacting diffusions, generalizing the BMP, where at every site one considers m momenta. We refer to [2] for precise definitions and more details.

The SIP is a natural “bosonic” analogue of the symmetric exclusion process (which is of fermionic nature). The interpretation of the rates $\xi_i\left(\frac{m}{2}+\xi_j\right)$ is as follows. Every site i has a random walk clock of rate m, and an “inclusion” clock of rate 1, different clocks are independent. When the random walk clock runs at i, a particle at i makes a random walk jump according to $p(i,j)$. When the inclusion clock runs at i, a particle from j is selected with probability $p(j,i)=p(i,j)$ and that particle moves to i. The inclusion jumps induce a “attractive” interaction between particles (sites with many particles attract new particles at high rate), contrary to the exclusion process where particles have an effective repulsion (due to exclusion).

The $SIP(m)$ is self-dual in the following sense. Define for $k,n\in\mathbb{N}$, $k\le n$

$$d(k,n)=\frac{n!}{(n-k)!}\frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}+k\right)}$$

and $d(k,n)=1$ for $k>n$. For $\xi\in\mathbb{N}^S$ a finite particle configuration, and $\eta\in\mathbb{N}^S$ we define

$$\mathbb{D}(\xi,\eta)=\prod_{i\in S} d(\xi_i,\eta_i) \tag{6}$$

The self-duality of the $SIP(m)$ then reads

$$\mathbb{E}_\eta\mathbb{D}(\xi,\eta_t)=\mathbb{E}_\xi\mathbb{D}(\xi_t,\eta) \tag{7}$$

The expectation in the lhs of (7) is over the infinite particle system η_t (if the set S is infinite), whereas the expectation in the rhs of (7) is over the finite particle system ξ_t. Therefore, the duality relation (7) represents a serious simplification: all relevant expectations in the infinite $SIP(m)$ reduce to expectation in the finite $SIP(m)$.

The duality between the SIP and the BMP, as well as the self-duality of the $SIP(m)$ can be easily obtained by rewriting the generator of the BMP in terms of generators of the $SU(1,1)$ algebra.

Going from the BMP to the SIP corresponds to a change of representation, whereas the self-duality of the $SIP(m)$ corresponds to using a symmetry. We refer to [2] for more details on the $SU(1,1)$ symmetry and more generally, the one-to-one correspondence between symmetries (operators that commute with the generator) and duality functions.

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Multiscale Analysis of Gradient Fields

Roman Kotecký
(joint work with S. Adams and S. Müller)

Consider a random field $\varphi : \Lambda \to \mathbb{R}$, on a finite lattice $\Lambda \subset \mathbb{Z}^d$ with fixed boundary conditions ψ and with the gradient Gibbs distribution $\mu_{\Lambda,\psi}$ defined as

$$\mu_{\Lambda,\psi}(d\varphi) = \frac{1}{Z_{\Lambda,\psi}} \exp\bigl(-\beta H_\Lambda(\nabla\varphi \,|\, \psi)\bigr)\nu_\Lambda(d\varphi).$$

Here, $H_\Lambda(\nabla\varphi \,|\, \psi) = \sum_{\substack{(x,x+e)\\ x,x+e\in\Lambda\cup\partial\Lambda}} U(\nabla_e\varphi(x))$ with $\varphi(x) = \psi(x)$ for $x \in \partial\Lambda$, $\nabla_e\varphi(x)$ is the discrete derivative in the direction of the unit coordinate vector e, $Z_{\Lambda,\psi} = \int_{\mathbb{R}^\Lambda/\mathbb{R}} \exp\bigl(-\beta H_\Lambda(\nabla\varphi \,|\, \psi)\bigr)\nu_\Lambda(d\varphi)$, and $\nu_\Lambda(d\varphi)$ is the *a priori* measure on gradient fields generated by Lebesgue measure $\prod_{x\in\Lambda} d\varphi(x)$. One is particularly interested in tilted boundary conditions $\Psi_u(x) = \langle x, u\rangle$ for a tilt $u \in \mathbb{R}^d$. An object of basic relevance in this context is the surface energy (or free energy) defined by the limit

$$\sigma(u) = -\lim_{\Lambda\uparrow\mathbb{Z}^d} \frac{1}{\beta|\Lambda|} \log Z_{\Lambda,\Psi_u}.$$

The properties of the corresponding random field are well understood in the case of a strictly convex potential U. In particular, Funaki and Spohn [1] have shown that σ is convex as function of the tilt (strict convexity of the surface tension was then proven in [2]) and that, for each $u \in \mathbb{R}^d$, there is a unique ergodic Gibbs state with tilt u.

More realistic non-convex potentials allow phase transitions (coexistence of several Gibbs states with the same value of the tilt) [3]. Nevertheless, for high temperatures (in the small β regime), the proof of strict convexity of the surface tension was recently extended [4] to a class of non-convex potentials.

Here, we address the strict convexity of the surface tension for large β (low temperatures) and sufficiently small tilt. The simplest case to consider is a non-convex perturbation of a gradient Gaussian field, $U(z) = \frac{1}{2}z^2 + V(z)$. Instead of enforcing the tilt by boundary conditions ψ_u, we follow Funaki and Spohn by considering a lattice torus (well adapted for our multiscale analysis is the torus $\mathbb{T}_N = \bigl(\mathbb{Z}/L^N\mathbb{Z}\bigr)^d$ with a fixed $L \in \mathbb{N}$), and replace the gradient $\nabla_e\varphi(x)$ in all definitions above by $\nabla_e\varphi(x) - \langle u, e\rangle$.

Mayer expanding the perturbative terms $\prod_{x,e} \exp\{V(\nabla_e\varphi(x))\}$, the integral for the partition function can be rewritten in terms of a polymer representation

$$Z_{\mathbb{T}_N,u} = \sum_{X\subset\mathbb{T}_N} \int \exp\{-H_0(\mathbb{T}_N \setminus X, \varphi)\} K_0(X,\varphi)\mu(d\varphi)$$

with suitable quadratic functions $H_0(Y,\varphi)$ and a Gaussian measure μ. (The functions H_0, K_0, as well as the measure μ depend on β and u). Even though the "polymer weights" $K_0(X,\varphi)$ can be split into the product $\prod_j K_0(X_j,\varphi)$ over the components X_j of X, and the weights $K_0(X_j,\varphi)$ are suppressed with the size of

X_j, they are strongly correlated in the underlying Gaussian measure μ and the standard cluster expansion cannot be used.

Instead, we rely on multiscale techniques based on a finite range decomposition of the Gaussian measure $\mu = \mu_1 * \cdots * \mu_N * \mu_{N+1}$ (as proposed in [6] and [5] and extended and modified for our needs in [7]) and the renormalisation group approach as introduced by Brydges and his collaborators (see [8] for the original paper and [9] for a recent exposition). Eventually, we get an expanding sequence of quadratic functions H_k and contracting sequence of polymer weights K_k, $k = 0, \ldots, N$, both of them "living" on sets $X \subset \mathbb{T}_N$ consisting of blocks of size L^k, so that for each $k = 0, \ldots, N-1$ one has

$$\sum_{X \subset \mathbb{T}_N} \int \exp\{-H_k(\mathbb{T}_N \setminus X, \varphi + \xi)\} K_k(X, \varphi + \xi) \mu_{k+1}(d\varphi) = \\ = \sum_{X \subset \mathbb{T}_N} \exp\{-H_{k+1}(\mathbb{T}_N \setminus X, \varphi)\} K_{k+1}(X, \varphi)$$

and thus, finally,

$$Z_{\mathbb{T}_N, u} = \int \big(\exp\{-H_N(\mathbb{T}_N, \varphi)\} + K_N(\mathbb{T}_N, \varphi)\big) \mu_{N+1}(d\varphi).$$

Using the flexibility in the choice of H_0 (tradeoff between H_0 and μ) and the (difficult) fact that all the construction depends smoothly on H_0, we can tune it in such a way that the resulting H_N vanishes.

The final expression $Z_{\mathbb{T}_N, u} = \int (1 + K_N(\mathbb{T}_N, \varphi)) \mu_{N+1}(d\varphi)$ then allows to read off the strict convexity (for small u) in a rather straightforward way once the perturbation V is sufficiently small; an assumption that is stated in terms of an appropriate norm of the function $f_{V,\beta,u}(z) = \exp\big\{-\beta \sum_{i=1}^d V\big(\frac{z_i}{\sqrt{\beta}} - u_i\big)\big\} - 1$.

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Large Deviations for Stochastic Particle Systems

Thierry Bodineau
(joint work with B. Derrida, J. Lebowitz, V. Lecomte, and F. van Wijland)

In the scaling limit, stochastic dynamics can be described by hydrodynamic equations which record the evolution of the density [8, 10]. The large deviations of the density from the typical hydrodynamic scaling have been analyzed in [9, 7]. In this talk, we report on generalization to the joint large deviations of the current and the density [1]-[6].

We consider the Symmetric Simple Exclusion Process (SSEP), in a periodic domain with N sites. After rescaling the time by N^2 and the space by N, the typical density trajectory $\bar\rho(x,t)$ follows a hydrodynamic equation of the form

$$(1)\quad \forall x \in [0,1], \qquad \partial_t \bar\rho(x,t) = -\partial_x q(x,t), \qquad \text{with,} \qquad \bar q(x,t) = -\partial_x \bar\rho(x,t)$$

where $\bar q(x,t)$ is the local current (at time t and position x).

The probability of observing during the time interval $[0,T]$ an atypical trajectory $(\rho(x,t), q(x,t))$ satisfying the relation $\partial_t \rho(x,t) = -\partial_x q(x,t)$ has an exponentially small cost given by the functional

$$(2)\qquad F(\rho,q) = \int_0^T dt \int_0^1 dx \frac{\big(q(x,t) + \partial_x \rho(x,t)\big)^2}{2\sigma(\rho(x,t))}\,,$$

where $\sigma(u) = 2u(1-u)$. The formula can be understood heuristically as the summation of the local Gaussian deviations of the current $q(x,t)$ from the typical local current given by $-\partial_x \rho(x,t)$ (see (1)) with a variance determined by the conductivity $\sigma(\rho(x,t))$. Using (2), one can predict [1]-[3] the cost of observing an atypical current $\mathcal{J} = \frac{1}{T}\int_0^T dt \int_0^1 dx q(x,t)$ over a very long time T and it is exponentially decaying with a rate

$$(3)\qquad T \inf_\rho \left\{ \int_0^1 dx \frac{\big(\mathcal{J} + \partial_x \rho(x)\big)^2}{2\sigma(\rho(x))} \right\},$$

where one has to optimize over the densities ρ which are only space dependent.

The same hydrodynamic approach applies also to currents in higher dimension. However this approach does not always catch the correct scaling of the large deviations or of the cumulants of the current in higher dimensions. To illustrate this, we consider the SSEP on a square lattice of size L, with periodic boundary conditions and study the current flowing through a vertical slit of length $\ell < L$. One reason for considering the fluctuations of this partial current is that in experiments it is often only possible to measure the fluctuations of local quantities and not of global quantities.

In two dimensions, when $\ell = L$, i.e. when one considers the total current flowing through the system, the large deviation function derived from the hydrodynamic theory satisfies a scaling similar to the one of the one-dimensional case (3). If one keeps the ratio $h = \ell/L$ fixed, then for all $0 < h < 1$ the large deviation cost is 0 for the hydrodynamic scaling considered in (3). This can be understood by the occurrence of vortices at the edge of the slit which carry the excess current. This shows that the current large fluctuations through a slit must have a different scaling from the current through a section of the system. In particular, this can be seen by an anomalously large variance of the current flowing through a slit. For the SSEP at mean density $\bar{\rho}$ on a periodic square domain, an explicit calculation shows a logarithmic divergence of the second cumulant for a slit of size $\ell = Lh$ and $0 < h < 1$

$$\lim_{\tau \to \infty} \frac{\langle Q^{(h)}(\tau)^2 \rangle_c}{\tau} \sim \frac{2\bar{\rho}(1-\bar{\rho})}{\pi} \log L \quad \text{as} \quad L \to \infty,$$

where $Q^{(h)}(\tau)$ is the flux of particles through the slit during time τ.

Another interesting aspect is the occurrence of phase transitions for the current large deviations [1]-[5]. The divergence of the two-point correlations when the system approaches the critical point has been analyzed in [6].

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Asymptotic Behavior of the Gyration Radius for Long-range Self-avoiding Walk and Long-range Oriented Percolation

Akira Sakai

(joint work with L.-C. Chen)

Let $\alpha > 0$ and suppose that the 1-step distribution D for random walk on $\mathbb{Z}^d$ decays as $D(x) \approx |x|^{-d-\alpha}$ such that its Fourier transform $\hat{D}(k) = \sum_{x \in \mathbb{Z}^d} e^{ik \cdot x} D(x)$

satisfies

$$(1) \qquad 1-\hat{D}(k)=v_\alpha|k|^{\alpha\wedge 2}\times\begin{cases}1+O(|k|^\epsilon) & (\alpha\neq 2),\\ \log\frac{1}{|k|}+O(1) & (\alpha=2),\end{cases}$$

for some $v_\alpha \in (0,\infty)$ and $\epsilon > 0$. The following long-range Kac potential, for any $L \in [1,\infty)$, satisfies the above property [3]:

$$(2) \qquad D(x)=\frac{h(y/L)}{\sum_{y\in\mathbb{Z}^d}h(y/L)} \quad (x\in\mathbb{Z}^d),$$

where $h(x) \equiv |x|^{-d-\alpha}(1+O(|x|^\epsilon))$ is a rotation-invariant function on $\mathbb{R}^d$.

Let $\varphi_t(x)$ denote the two-point functions for random walk and self-avoiding walk whose 1-step distribution is given by the above D and for oriented percolation on $\mathbb{Z}^d \times \mathbb{Z}_+$ whose bond-occupation probability for each bond $((u,s),(v,s+1))$ is given by $pD(v-u)$, independently of $s \in \mathbb{Z}_+$. More precisely,

$$(3) \qquad \varphi_t(x)=\begin{cases}\displaystyle\sum_{\substack{\omega:o\to x\\|\omega|=t}}\prod_{s=1}^t D(\omega_s-\omega_{s-1}) & \text{(RW)},\\ \displaystyle\sum_{\substack{\omega:o\to x\\|\omega|=t}}\prod_{s=1}^t D(\omega_s-\omega_{s-1})\prod_{0\le i<j\le t}(1-\delta_{\omega_i,\omega_j}) & \text{(SAW)},\\ \mathbb{P}_p\big((o,0)\longrightarrow(x,t)\big) & \text{(OP)}.\end{cases}$$

The order-r gyration radius $\xi_t^{(r)}$, defined as

$$(4) \qquad \xi_t^{(r)}=\bigg(\frac{\sum_{x\in\mathbb{Z}^d}|x|^r\varphi_t(x)}{\sum_{x\in\mathbb{Z}^d}\varphi_t(x)}\bigg)^{1/r},$$

represents a typical spatial size of a linear polymer or a cluster at time t. It has been expected (and is certainly true for random walk for any dimension) that, above the common upper-critical dimension $d_c = 2(\alpha \wedge 2)$ for self-avoiding walk and oriented percolation, for every $r \in (0,\alpha)$,

$$(5) \qquad \xi_t^{(r)}=\begin{cases}O(t^{\frac{1}{\alpha\wedge 2}}) & (\alpha\neq 2),\\ O(\sqrt{t\log t}) & (\alpha=2).\end{cases}$$

The conjecture was proved to be affirmative for self-avoiding walk, but only for small $r < \alpha \wedge 2$ [4].

In my recent joint work with L.-C. Chen [3], we have proved the following sharp asymptotics:

Theorem 1 ([3])**.** *Consider the above three models defined by the long-range Kac potential. For random walk in any dimension with any L, and for self-avoiding walk and critical/subcritical oriented percolation for $d > 2(\alpha \wedge 2)$ with $L \gg 1$, the following holds for every $r \in (0,\alpha)$: there are constants $C_1, C_2 = 1 + O(L^{-d})$*

($C_1 = C_2 = 1$ for random walk) and $\epsilon > 0$ such that, as $m \nearrow m_c$,

$$(6)\qquad \sum_{t=0}^{\infty}\sum_{x\in\mathbb{Z}^d}|x_1|^r\varphi_t(x)\,m^n = \frac{2\sin\frac{r\pi}{\alpha\vee 2}}{(\alpha\wedge 2)\sin\frac{r\pi}{\alpha}}\,\Gamma(r+1)\,\frac{C_1(C_2v_\alpha)^{\frac{r}{\alpha\wedge 2}}}{(1-\frac{m}{m_c})^{1+\frac{r}{\alpha\wedge 2}}} \times\begin{cases}1+O\big((1-\frac{m}{m_c})^{\epsilon}\big) & (\alpha\neq 2),\\ \Big(\log\frac{1}{\sqrt{1-\frac{m}{m_c}}}\Big)^{r/2}+O(1) & (\alpha=2).\end{cases}$$

where m_c is the radius of convergence for the sequence $\sum_{x\in\mathbb{Z}^d}\varphi_t(x)$.

In fact, the above C_1, C_2 are the following model-dependent constants [1, 2, 4]:

$$(7)\qquad \sum_{x\in\mathbb{Z}^d}\varphi_t(x)\underset{t\uparrow\infty}{\sim}C_1m_c^{-t},\qquad\qquad \frac{\sum_{x\in\mathbb{Z}^d}e^{ik_t\cdot x}\varphi_t(x)}{\sum_{x\in\mathbb{Z}^d}\varphi_t(x)}\underset{t\uparrow\infty}{\sim}e^{-C_2|k|^{\alpha\wedge 2}}.$$

Theorem 2 ([3])**.** *Under the same condition as in Theorem 1,*

$$(8)\qquad \frac{\sum_{x\in\mathbb{Z}^d}|x_1|^r\varphi_t(x)}{\sum_{x\in\mathbb{Z}^d}\varphi_t(x)}\underset{t\uparrow\infty}{\sim}\frac{2\sin\frac{r\pi}{\alpha\vee 2}}{(\alpha\wedge 2)\sin\frac{r\pi}{\alpha}}\,\frac{\Gamma(r+1)}{\Gamma(\frac{r}{\alpha\wedge 2}+1)}\,(C_2v_\alpha)^{\frac{r}{\alpha\wedge 2}} \times\begin{cases}t^{\frac{r}{\alpha\wedge 2}} & (\alpha\neq 2),\\ (t\log\sqrt{t})^{r/2} & (\alpha=2).\end{cases}$$

This immediately proves the conjecture (5) for all $r\in(0,\alpha)$.

As far as we notice, the sharp asymptotics for random walk in the above two theorems are new.

The proof is based on the derivatives of the lace expansion and the new fractional-moment analysis for the derivatives of the lace-expansion coefficients, initiated in [2]. It is worth emphasizing that the same proof applies to finite-range models, for which α is regarded as infinity.

In the talk, I explain the general framework to treat all three models simultaneously and show some complex analysis for the derivation of the right constants in the asymptotics.

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Degenerate Random Environments

Mark Holmes
(joint work with T. Salisbury)

For fixed $d \geq 2$ let $\mathcal{E} = \{\pm e_i : i = 1, \dots, d\}$ be the set of unit vectors in $\mathbb{Z}^d$, and let $\mathcal{P}$ denote the power set of $\mathcal{E}$. For any set A, let $|A|$ denote the cardinality of A. Let μ be a probability measure on $\mathcal{P}$. A *degenerate random environment* is an element $\mathcal{G} = \{\mathcal{G}_x\}_{x \in \mathbb{Z}^d}$ of $\mathcal{P}^{\mathbb{Z}^d}$, equipped with the product σ-algebra and the product measure $\nu = \mu^{\otimes \mathbb{Z}^d}$. Site percolation, oriented site percolation and oriented bond percolation are examples of previously-studied models which fall into this more general class.

We say that the environment is 2-*valued* when μ charges exactly two points, i.e. there exist $A_1, A_2 \subset \mathcal{P}$ and $p \in (0,1)$ such that $\mu(\{A_1\}) = p$, $\mu(\{A_2\}) = 1 - p$. In [1] we study the connectivity structure of the resulting random directed graphs, with particular emphasis on 2-dimensional 2-valued environments.

Given an environment $\mathcal{G}$:

- x is *connected* to y, $(x \to y)$ if: $\exists\ n \geq 0$, $x = x_0\, x_1, \dots, x_n = y$ such that $x_{i+1} - x_i \in \mathcal{G}_{x_i}$ for $i = 0, \dots, n-1$
- $\mathcal{C}_x = \{y \in \mathbb{Z}^d : x \to y\}$
- $\mathcal{B}_y = \{x \in \mathbb{Z}^d : x \to y\}$
- x and y are *mutually connected*, or *communicate*, $(x \leftrightarrow y)$ if $x \to y$ and $y \to x$
- $\mathcal{M}_x = \{y \in \mathbb{Z}^d : x \leftrightarrow y\} = \mathcal{B}_x \cap \mathcal{C}_x$.

Note that these models in general do not have the usual monotonicity properties present in percolation theory (where switching an unoccupied site to an occupied site cannot break connections). There is however a model to model monotonicity property of the following form: if a connection exists with probability ρ in some degenerate random environment model, then it occurs with probability at least ρ for any degenerate random environment model that can be obtained from the original one (under some natural coupling) by adding arrows.

Since our interest is primarily in asymptotic properties of random walks in these random environments, as in [2], we want to rule out environments where the walk gets stuck on a finite set of sites. This is equivalent to the restriction that

$$\boxed{\nu(|\mathcal{C}_o| = \infty) = 1}.$$

Remark 1. If $\exists$ an orthogonal set $V \subset \mathcal{E}$ such that $\mu(\{A : A \cap V \neq \emptyset\}) = 1$ then $|\mathcal{C}_o| = \infty$, ν-almost surely. This implies that, among many other models, the 2-dimensional 2-valued environments defined by

(*) $\mu(\{\uparrow, \rightarrow\}) = p = 1 - \mu(\{\downarrow, \leftarrow\})$ and
(**) $\mu(\{\uparrow, \downarrow\}) = p = 1 - \mu(\{\leftarrow, \rightarrow\})$

are among those satsifying the above condition.

Remark 2. By translation invariance we have that $\nu(o \to x) = \nu(-x \to o)$ so that $\nu(x \in \mathcal{C}_o) = \nu(-x \in \mathcal{B}_o)$. There is also a simple proof of the fact that

$\nu(|\mathcal{C}_o| = \infty) = p\nu(|\mathcal{B}_o| = \infty)$ in the "percolation" setting $\mu(A) = p = 1 - \mu(\{\emptyset\})$. This will not be true in most of the situations that we are interested in, where $\nu(|\mathcal{C}_o| = \infty) = 1$ and $\nu(|\mathcal{B}_o| = \infty) < 1$. However, based on the $x \mapsto -x$ symmetry apparent in many pictures, we are investigating some more general kind of duality between the shapes of $\mathcal{C}_o$ and $\mathcal{B}_o$.

Many of the models that we are interested in, e.g. (**), fall into the class of those environments discussed in the following proposition (or a variant of it, in the case of (*)).

Proposition 1. *Fix $d = 2$. If ν-almost surely every site has [$\uparrow$ or $\rightarrow$] and every site has [$\uparrow$ or $\leftarrow$], and each arrow [$\uparrow, \rightarrow, \leftarrow, \downarrow$] is possible, then the following ν-a.s. exhaust the possibilities for $\mathcal{B}_o$:*

(i) *$\mathcal{B}_o$ is finite;*
(ii) *$\mathcal{B}_o = \mathbb{Z}^2$;*
(iii) *$\exists\, W : \mathbb{Z} \to \mathbb{Z}$ such that $\mathcal{B}_o = \{y : y^{[2]} \leq W(y^{[1]})\}$;*
(iv) *$\exists\, W : \mathbb{Z} \to \mathbb{Z}$ such that $\mathcal{B}_o = \{y : y^{[2]} \geq W(y^{[1]})\}$.*

Only one of (ii), (iii), (iv) can have positive probability.

A consequence of the above, using the symmetry in the model (**) we see that either $\mathcal{B}_o$ is finite (this has positive probability for any p) or it is all of $\mathbb{Z}^2$ in this case. It turns out that there are interesting phase transitions in some models, such as $(*)$, where (ii) has positive probability for p sufficiently close (determined by the critical p_c for some oriented site percolation model on a particular lattice) to 0.5, whereas (iii) or (iv) have positive probability otherwise.

Turning to the mutually connected cluster $\mathcal{M}_o$, we have the following result, which implies ν-almost sure finiteness of this cluster under rather strong assumptions:

Proposition 2. *For each $d \geq 2$ there exists ϵ_d such that the following holds: If there exists an orthogonal set V of unit vectors such that $\mu(\{A : \emptyset \neq A \subset V\}) > 1 - \epsilon_d$, then $\mathbb{E}[|\mathcal{M}_o|] < \infty$.*

The proof gives explicit values (certainly not sharp) for ϵ_d in terms of self-avoiding walk connective constants. The proposition applies for example to the model (*). We give additional constructive proofs in some cases, including model (**) for all p and model (*) for p sufficiently close to 0.5 that $\nu(|\mathcal{M}_o| = \infty) > 0$, and moreover that the cluster $\mathcal{M}_o$ is giant when this occurs.

Members of the audience at this workshop gave suggestions for references (to work on related but different models). Although not including in this abstract, some of these may appear in the paper.

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Random Graph Asymptotics on High-dimensional Tori

Markus Heydenreich
(joint work with R. van der Hofstad)

It is well-known that Bernoulli bond percolation on the hypercubic lattice $\mathbb{Z}^d$ in dimension $d \geq 2$ obeys a phase transition whose transition point can be characterized as

$$p_c = \inf \big\{p \colon \mathbb{P}_{\mathbb{Z},p}(|\mathcal{C}(0)| = \infty) > 0\big\} = \sup \big\{p \colon \mathbb{E}_{\mathbb{Z},p}|\mathcal{C}(0)| < \infty\big\},$$

where

- every bond $\{x, y\}$ with $x, y \in \mathbb{Z}^d$, $|x - y| = 1$ is *occupied* with probability $p \in [0, 1]$, independently of each other, and the corresponding product measure and expectation are denoted $\mathbb{P}_{\mathbb{Z},p}$ and $\mathbb{E}_{\mathbb{Z},p}$;
- $\mathcal{C}(x)$ denotes the (unique) connected component of the random subgraph of occupied bonds containing vertex x, and $|\mathcal{C}(x)|$ is the number of vertices in $\mathcal{C}(x)$.

The basic question to be addressed in this work is the following: What is the behavior of critical percolation (i.e., $p = p_c$) on a *finite* box? We give an answer for the high-dimensional case, where $\mathbb{Z}^d$-percolation is fairly well understood.

More precisely, we consider bond percolation on the d-dimensional torus $\mathbb{T}_r^d = (\mathbb{Z}/r\mathbb{Z})^d$, that is, percolation on a d-dimensional box of side length r with periodic boundary conditions. The corresponding probability measure is denoted $\mathbb{P}_{\mathbb{T},p}$, and we write V for the number of vertices in the torus, $V = r^d$.

Our first result concerns the size of the largest cluster on the torus, $|\mathcal{C}_{\max}|$.

Theorem 1 ([7, 8])**.** *There is $d_0 \geq 6$, such that for any $d > d_0$ the following holds for percolation on the d-dimensional torus $\mathbb{T}_r^d$. There exists a constant $b > 0$, such that for all $\omega \geq 1$,*

$$\mathbb{P}_{\mathbb{T},p_c}\Big(\omega^{-1}V^{2/3} \leq |\mathcal{C}_{\max}| \leq \omega V^{2/3}\Big) \geq 1 - \frac{b}{\omega} \qquad \text{uniformly in } r. \tag{1}$$

Moreover, $|\mathcal{C}_{\max}|V^{-2/3}$ is not concentrated as $V \to \infty$.

It should be noted that periodic boundary conditions are essential for the result. Indeed, the combined results of Aizenman [1] and Hara [5] show that in dimension $d \geq 19$ under *bulk* boundary conditions, $|\mathcal{C}_{\max}|$ is of the order r^4, which is much smaller than $V^{2/3}$. The case of free boundary conditions is an open problem.

The $V^{2/3}$-asymptotic of $|\mathcal{C}_{\max}|$ is known as *random graph asymptotic* since the same is observed for critical Erdős-Rényi random graphs, that is, percolation on the *complete* graph with V vertices and percolation probability $p = V^{-1}$. In other words, the high dimensional *spatial* model shows the same behavior as the *non-spatial* model. This, on the other hand, is expected to be false when $d < 6$.

The proof of Theorem 1 is based on the one hand on the work of Borgs et al. [2, 3], which develops lace expansion for various high-dimensional tori but for a different notion of criticality. On the other hand, a number of results for the high-dimensional infinite lattice are needed, e.g. [4, 5, 6]. Finally, in [7] we construct a coupling of $\mathbb{Z}^d$- and $\mathbb{T}_r^d$-percolation that provides a link between the two regimes.

We shall now present further results where critical percolation on the high-dimensional torus and on the complete graph show the same qualitative behavior. To this end, we denote by $\mathcal{C}_{(i)}$ the ith largest cluster of the torus (i.e., $\mathcal{C}_{(1)} = \mathcal{C}_{\max}$).

Theorem 2 ([8])**.** *There is $d_0 \geq 6$, such that for any $d > d_0$ the following holds for percolation on the d-dimensional torus $\mathbb{T}_r^d$. For every $m = 1, 2, \dots$ there exist constants $b_1, \dots, b_m > 0$, such that for all $\omega \geq 1$, and all $i = 1, \dots, m$,*

$$\mathbb{P}_{\mathbb{T},p_c}\Big(\omega^{-1} V^{2/3} \leq |\mathcal{C}_{(i)}| \leq \omega V^{2/3}\Big) \geq 1 - \frac{b_i}{\omega} \qquad \text{uniformly in } r. \tag{2}$$

Finally, it is possible to identify asymptotics for the *diameter* of the cluster $\mathcal{C}_{(i)}$, $\operatorname{diam}(\mathcal{C}_{(i)})$, and for the *mixing time* of lazy simple random walk, $T_{\mathrm{mix}}(\mathcal{C}_{(i)})$. Lazy simple random walk on a (finite) graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a Markov chain with transition probability $\mathrm{p}(x, x) = 1/2$, $x \in \mathcal{V}$, and $\mathrm{p}(x, y) = (2\,\mathrm{degree}(x))^{-1}$ whenever $\{x, y\} \in \mathcal{E}$. The stationary distribution $\pi(x)$ is proportional to the degree of x, and the mixing time of lazy simple random walk on $\mathcal{G}$ is defined as

$$T_{\mathrm{mix}}(\mathcal{G}) = \min\big\{n \colon \|\mathrm{p}^n(x,\cdot) - \pi(\cdot)\|_{\mathrm{TV}} \leq 1/4 \text{ for all } x \in \mathcal{V}\big\},$$

with $\|\cdot\|_{\mathrm{TV}}$ being the total variation norm.

Theorem 3 ([8, 9, 10])**.** *There is $d_0 \geq 6$, such that for any $d > d_0$ the following holds for percolation on the d-dimensional torus $\mathbb{T}_r^d$. For every $m = 1, 2, \dots$ there exist constants $c_1, \dots, c_m > 0$, such that for all $\omega \geq 1$, and all $i = 1, \dots, m$,*

$$\mathbb{P}_{\mathbb{T},p_c}\Big(\omega^{-1} V^{1/3} \leq \operatorname{diam}(\mathcal{C}_{(i)}) \leq \omega V^{1/3}\Big) \geq 1 - \frac{c_i}{\omega^{1/3}}, \tag{3}$$

$$\mathbb{P}_{\mathbb{T},p_c}\Big(\omega^{-1} V \leq T_{\mathrm{mix}}(\mathcal{C}_{(i)}) \leq \omega V\Big) \geq 1 - \frac{c_i}{\omega^{1/34}} \qquad \text{uniformly in } r. \tag{4}$$

The core of Theorem 3 is a criterion developed by Nachmias and Peres [10] in the context of Erdős-Rényi random graphs. The prerequisites for the criterion are related to the one-arm exponent in the intrinsic metric, and these are recently proved by Kozma and Nachmias [9] for the high-dimensional infinite lattice $\mathbb{Z}^d$. In [8] we adapt the latter argument to the case of finite tori.

It is expected that all our theorems hold with $d_0 = 6$, however the technique of proving these results, the lace expansion, does not show this. Nevertheless, there is strong support for the conjecture that $d_0 = 6$ by considering a spread-out version of the model where any two vertices $x, y \in \mathbb{T}_r^d$ are linked by a bond whenever the distance between x and y (modulo r) is less than L. Indeed, (1)–(4) hold with $d_0 = 6$ whenever L is sufficiently large.

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Einstein Relation for Symmetric Diffusions in Random Environment

Nina Gantert
(joint work with P. Mathieu and A. Piatnitski)

We shall be dealing with diffusion processes in $\mathbb{R}^d$ whose generators are of the form

$$\mathcal{L}^\omega f(x) = \frac{1}{2} e^{2V^\omega(x)} \operatorname{div}(e^{-2V^\omega} a^\omega \nabla f)(x), \tag{1}$$

where a^ω and V^ω are realizations of a random environment with finite range of dependence.

More precisely, our assumptions are as follows.

Let $(\Omega, \mathcal{A}, \mathcal{Q})$ be a probability space equipped with a group action of $\mathbb{R}^d$ through measure preserving maps that we denote with $(x, \omega) \to x.\omega$. We also assume that the map $(x, \omega) \to x.\omega$ is $(\mathcal{B}_d \times \mathcal{A}, \mathcal{A})$-measurable, where $\mathcal{B}_d$ is the Borel σ-field on $\mathbb{R}^d$.

Assumption 1: the action $(x, \omega) \to x.\omega$ is ergodic.

Let V be a measurable real-valued function on Ω and let σ be a measurable function on Ω taking its values in the set of real $d \times d$ symmetric matrices. Define

$$V^\omega(x) = V(x.\omega)\,,\ \sigma^\omega(x) = \sigma(x.\omega)\,.$$

We also introduce the notation

$$a^\omega = (\sigma^\omega)^2 \ \text{ and } \ b^\omega = \frac{1}{2}\operatorname{div} a^\omega - a^\omega \nabla V^\omega\,.$$

Assumption 2: for any environment ω, the functions $x \to V^\omega(x)$ and $x \to \sigma^\omega(x)$ are smooth. To avoid triviality, we also assume that at least one of them is not constant.

Assumption 3: V is bounded and a^ω is uniformly elliptic, namely there exists a constant κ such that, for all ω, x and y,

$$\kappa|y|^2 \leq |\sigma^\omega(x)y|^2 \leq \kappa^{-1}|y|^2 \,. \tag{2}$$

For a Borel subset $F \subset \mathbb{R}^d$, we define the σ-field

$$\mathcal{H}_F = \sigma\{V(x.\omega), \sigma(x.\omega) \;:\; x \in F\}$$

and we assume the following independence condition:

Assumption 4: there exists R such that for any Borel subsets F and G such that $d(F,G) > R$ (where $d(F,G) = \inf\{|x-y| \;:\; x \in F, y \in G\}$ is the distance between F and G) then

$$\mathcal{H}_F \text{ and } \mathcal{H}_G \text{ are independent}\,. \tag{3}$$

Let $(W_t \;:\; t \geq 0)$ be a Brownian motion defined on some probability space $(\mathcal{W}, \mathcal{F}, P)$. We denote expectation with respect to P by E. By *diffusion in the environment* ω we mean the solution of the stochastic differential equation

$$dX^\omega(t) = b^\omega(X^\omega(t))\,dt + \sigma^\omega(X^\omega(t))\,dW_t\,;\; X^\omega(0) = x\,. \tag{4}$$

Then X^ω is indeed the Markov process generated by the operator $\mathcal{L}^\omega$ in equation (1). We shall denote with P_x^ω the law of X^ω on the path space $C(\mathbb{R}_+, \mathbb{R}^d)$. It is usually refered to as the *quenched* law of the diffusion in a random environment. We will also need the so-called *averaged* law:

$$\mathbb{P}_x[A] := \int d\mathbb{Q}(\omega) \int dP_x^\omega(w)\mathbf{1}_A(\omega, w)\,, \tag{5}$$

for any measurable subset $A \subset \Omega \times C(\mathbb{R}_+, \mathbb{R}^d)$. Expectation with respect to P_x^ω will be denoted with E_x^ω and expectation with respect to $\mathbb{P}_x$ will be denoted with $\mathbb{E}_x$.

We use the notation $X(t)$ for the coordinate process on path space $C(\mathbb{R}_+, \mathbb{R}^d)$.

Definition 1. Let Σ be the *effective diffusivity* matrix defined by

$$e \cdot \Sigma\, e := \lim_{t \to +\infty} \frac{1}{t}\mathbf{E}_0[(e \cdot X(t))^2]\,, \tag{6}$$

where e is any vector in $\mathbb{R}^d$ and $x \cdot y$ denotes the scalar product of the two vectors x and y.

The fact that the limit in (6) exists is (almost) a consequence of the Central Limit Theorem for the process X under $\mathbb{P}_0$. More is actually known: X satisfies a full invariance principle. Namely: for almost any realization of the environment ω, the laws of the sequence of rescaled processes $(X^\varepsilon(t) = \varepsilon X(t/\varepsilon^2)\,;\, t \geq 0)$ under P_0^ω weakly converge as ε goes to 0 to the law of a Brownian motion with covariance matrix Σ. References on this Theorem include [3], [10], [13], [16], [17] among others. The convergence of the variance of the process to Σ is explicitely stated in [3] formula (2.44).

The invariance principle also has a PDE counterpart in terms of homogenization theory, see for instance the book [9]. The generator of the process X^ε under P_x^ω is the rescaled operator

$$\frac{1}{2}a^\omega(\frac{\cdot}{\varepsilon})\Delta + \frac{1}{2\varepsilon}b^\omega(\frac{\cdot}{\varepsilon})\nabla \,.$$

As one can see this is an elliptic operator with rapidly oscillating coefficients. Its limit, in the sense of homogenization theory, is the elliptic operator with constant coefficient

$$\frac{1}{2}\mathrm{div}(\Sigma\nabla)\,,$$

where Σ is the same matrix as in (6).

The effective diffusivity Σ is a symmetric matrix. As a consequence of Assumption 1 on ergodicity, Σ is deterministic (i.e. Σ does not depend on ω). Furthermore, due to the ellipticity Assumption 3, Σ is also known to be positive definite.

We shall now consider perturbations of the process X obtained by inserting a drift in equation (4).

We use the following notation. Let e_1 be a non-zero vector and $\lambda > 0$. We define $\hat\lambda$ to be the vector $\hat\lambda = \lambda e_1$. We think of e_1 as being fixed while λ is due to tend to 0.

Let us consider the perturbed stochastic differential equation:

(7)
$$dX^{\lambda,\,\omega}(t) = b^\omega(X^{\lambda,\,\omega}(t))\,dt + \,a^\omega(X^{\lambda,\,\omega}(t))\hat\lambda\,dt + \sigma^\omega(X^{\lambda,\,\omega}(t))\,dW_t\,;\,X^{\lambda,\,\omega}(0) = x\,.$$

The process $X^{\lambda,\,\omega}$ is now a Markov process with generator

$$\begin{aligned}\mathcal{L}^{\lambda,\,\omega}f(x) &= \frac{1}{2}e^{2V^\omega(x)}\,\mathrm{div}(e^{-2V^\omega}\,a^\omega\,\nabla f)(x) + \,a^\omega(x)\hat\lambda\,\nabla f(x)\\ &= \frac{1}{2}e^{2V^{\lambda,\,\omega}(x)}\,\mathrm{div}(e^{-2V^{\lambda,\,\omega}}\,a^\omega\,\nabla f)(x)\,,\end{aligned} \tag{8}$$

where $V^{\lambda,\,\omega}(x) = V^\omega(x) - \hat\lambda\cdot x$. We shall use the notation $P_x^{\lambda,\,\omega}$ for the law of $X^{\lambda,\,\omega}$, $E_x^{\lambda,\,\omega}$ for the corresponding expectation as well as $\mathbb{P}_x^\lambda$ and $\mathbf{E}_x^\lambda$ for the averaged probability and expectation.

Our model is a special case of diffusions with drifts considered by L. Shen in [19] and for which the author proved a law of large numbers: for almost any environment ω, the ratio $X^{\lambda,\,\omega}(t)/t$ has an almost sure limit, say $\ell(\lambda)$. The convergence also holds in $L^1(\mathbb{P}_0^\lambda)$. Moreover $\ell(\lambda)$ is deterministic and $\hat\lambda\cdot\ell(\lambda)\neq 0$. Note that the proof strongly relies on the independence property Assumption 4. We thus define the velocity:

Definition 2. Let $\lambda > 0$. Let $\ell(\lambda)$ be the *effective drift* vector defined by

$$\ell(\lambda) = \lim_{t\to+\infty}\frac{1}{t}\mathbf{E}_0^\lambda[X(t)]\,. \tag{9}$$

By convention $\ell(0) = 0$.

Definition 3. Call *mobility* of the process X^ω in the direction e_2 the derivative at $\lambda = 0$ of the velocity $e_2\cdot\ell(\lambda)$ (if it exists).

This definition is justified by the following version of the Einstein relation, which is our main result.

Theorem 1. *For any vector* e_2*, the function* $\lambda \to e_2 \cdot \ell(\lambda)$ *has a derivative at* $\lambda = 0$ *and the mobility satisfies*

$$\lim_{\lambda \to 0} \frac{1}{\lambda} e_2 \cdot \ell(\lambda) = e_2 \cdot \Sigma \, e_1 \, . \tag{10}$$

The proof combines different ingredients: homogenization arguments and Girsanov transforms, PDE estimates and a-priori bounds on hitting times for perturbed diffusions and renewal arguments.

We hope that our strategy can eventually be applied to other models of diffusions and random walks in random environments.

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Quenched Local Limit Theorem for Space-time Random Walk in Random Environment

Noam Berger

Space time random walk in random environment is a model of random environment for which after each step, the entire environment is resampled. It is equivalent to a regular RWRE such that in each step the first coordinate has to increase by 1.

This model is easier to handle than other RWRE models since, in the averaged sense, each step is independent of any other step. Indeed, the averaged measure of the space-time RWRE is that of a (possibly biased) simple random walk.

Thus the main challenge in this model is to understand its quenched behavior. The space-time model may also serve as a toy model for the more complicated ballistic case.

We introduce the following notation: $\mathbb{P}$ is the averaged distribution, and for every environment ω, we use P_ω for the quenched distribution. $\mathbf{P}$ is the distribution of environments.

We prove the following result:

Theorem 1. *Consider a space-time RWRE which is i.i.d. nearest neighbor and elliptic, in dimension $1+d$ with $d \geq 2$. Let $\{M_N\}_{N=0}^{\infty}$ be a sequence going to infinity, and let $\epsilon > 0$. For every N, let $\{Q_i^{(N)}\}_{i=1}^{\infty}$ be a partition of $\mathbb{Z}$ into cubes of side-length M_N. Then*

$$\mathbf{P}\left(\omega : \sum_{i=1}^{\infty} \left|\mathbb{P}\big(X_N \in Q_i^{(N)}\big) - P_\omega\big(X_N \in Q_i^{(N)}\big)\right| > \epsilon\right)$$

goes to zero faster than any polynomial in N.

Conjecture 1. *The same does not hold for $d = 1$.*

Remark 1. Using the same techniques, one can prove the same result for any RWRE satisfying Sznitman's condition (T'), under some quantitative requirements for the rate of growth of $\{M_N\}$.

The proof technique involves several ideas. the main idea is an adaptation of Bolthausen and Sznitman's exposing neighborhood technique [1], coupled with a bootstrap argument. The second moment method from [1] is replaced by using the following variant of the Azuma-Hoefding inequality:

Lemma 1. *Let* $\{M_k\}_{k=1}^n$ *be a zero mean martingale with respect to a filtration* $\{\mathcal{F}_k\}_{k=1}^n$ *on the probability space* $(\Omega, \mathcal{B}, \mu)$*. Assume that* $M_0 = 0$ *and* $\mathcal{F}_0 = \{\emptyset, \Omega\}$*. for* $k = 1, \ldots, n$*, let* $D_k = M_k - M_{k-1}$*. Define*

$$U_k = esssup(|D_k| \mid \mathcal{F}_{k-1}) = \lim_{p\to\infty} \left[E(|D_k|^p|\mathcal{F}_{k-1})\right]^{\frac{1}{p}}$$

and we define the essential variance *of the martingale to be*

$$U := esssup\left(\sum_{k=1}^n U_k^2\right).$$

Then for every K*,*

$$\mu(|M_n| > K) \leq 2e^{-\frac{K^2}{2U}}.$$

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Russo-Seymour-Welsh Theory for the FK-Ising Model in Dimension 2

Hugo Duminil-Copin
(joint work with C. Hongler and P. Nolin)

Arguments coming from statistical physics show that 2D critical lattice models should be conformally invariant at criticality. Great progresses have been made in this field during the last decade: in the specific case of the Ising model, Stanislav Smirnov proved the conformal invariance of the random cluster representation of the Ising model while the convergence of the so-called exploration path to SLE(16/3) has been achieved by S. Smirnov and A. Kemppainen very recently. These results open new possibilities of computations for the 2D Ising model by identifying the continuum scaling limit. Our theorem is instrumental for coming back to the discrete case since it mainly deal with boundary conditions. It also gives a very short proof of the critical temperature for the Ising model.

The probability measure $\mathbb{P}^b_{p,q,G}$ denotes the random cluster model with parameters p and q on the graph G with boundary conditions b. Consider the random cluster model with parameters $q = 2$ and $p = p_c$ on the two-dimensional square lattice. For a rectangle R, we define $\mathcal{C}_v(R)$ to be the event that there is an open path linking the bottom edge to the top edge (an open path is a sequence of open edges such that two adjacent edges share a vertex of the graph). With these notations, the theorem can be stated as follows:

Theorem 1. ***(H. D-C, C. Hongler, P. Nolin)*** *For any $\alpha > 0$, there exist $0 < c(\alpha) \leq d(\alpha) < 1$ such that for any $n \geq 1$ and any boundary condition b,*

$$c_\alpha < \mathbb{P}^b_{2,p_c,R_n}(\mathcal{C}_v(R_n)) < d_\alpha$$

where R_n is a rectangle with dimensions $(\alpha n, n)$.

The important fact is that the result does not depend on boundary conditions. The proof strongly relies on the so-called **parafermionic observable**. The goal of this talk is to give a precise definition of the parafermionic observable for 2-dimensional random cluster models. This random variable is employed in order to prove estimates on crossing probabilities of rectangles with various boundary conditions.

Roughly speaking, consider the random cluster model on a finite graph with boundary conditions wired on one part of the boundary and free on the other part. It is possible to define the interface between the primal open cluster linked to the wired arc and the dual open cluster linked to the free arc; this interface is called the **exploration path** γ. The parafermionic observable is a modification of the probability that one edge of the medial lattice belongs to γ.

In the end of the talk, we will briefly show how the parafermionic observable can be employed in other contexts to find new proofs of already known results and new strategies for old conjectures. This is partly joint work with S. Smirnov and V. Beffara.

Convergence of Discrete Markov Chains to Jump Processes and Applications to Random Conductance Models

Takashi Kumagai

(joint work with P. Kim and Z.-Q. Chen)

We give general criteria concerning weak convergence of Markov chains to jump processes and we apply the results to some random conductance models. We first give an example that can be deduced from our main results (Theorem 1 and 2).

Example: For $d \geq 2$, let $\{\xi_{xy}\}_{x,y\in\mathbb{Z}^d}$ be i.i.d. on $(\Omega, \mathcal{F}, \mathbf{P})$ such that $0 \leq \xi_{xy}$, $M := E[\xi_{xy}] \in (0, \infty)$ and Var $[\xi_{xy}] < \infty$. Define random conductance $C(\cdot, \cdot)$ as

$$C(x, y) := \frac{\xi_{xy}}{|x-y|^{d+\alpha}}, \qquad x, y \in \mathbb{Z}^d,\ x \neq y,\ \ 0 < \alpha < 2.$$

Let $X^{(1)}$ be the corresponding continuous time Markov chain on $\mathbb{Z}^d$, namely $X^{(1)}$ stays at x for an exponential length of time with parameter $\sum_{z\neq x} C(x, z)$ and then jumps to y with probability $C(x, y)/\sum_{z\neq x} C(x, z)$. Then $X^{(k)}_t = k^{-1}X^{(1)}_{k^\alpha t}$ converges to Z^α_{Mt} in the f.d.d. sense $\mathbf{P}$-a.s. (quenched), where Z^α_t is the rotationally invariant α-stable process. Further, if $\xi_{xy} \leq C_1$, $\{(X^{(k)}, \mathbb{P}^{(k)}_\varphi);\ k \geq 1\}$ converges weakly to $(Z^\alpha_{M\cdot}, \mathbb{P}_\varphi)$ $\mathbf{P}$-a.s., where $\mathbb{P}^{(k)}_\varphi, \mathbb{P}_\varphi$ are defined in (2).

1. Background and known results

For a Hunt process X on $\mathbb{R}^d$, consider the following question:

(Q1) Can we approximate X by a family of Markov chains $X^{(n)}$ on $n^{-1}\mathbb{Z}^d$?

A closely related question is the following. Let $X^{(n)}$ be a sequence of Markov chains on $n^{-1}\mathbb{Z}^d$.

(Q2) When does $X^{(n)}$ converge weakly to a 'nice' Hunt process X as $n \to \infty$?

We discuss these questions when X is a symmetric pure jump process.

Let us briefly mention some works on these problems when X is a diffusion. When X is a diffusion corresponding to an operator in nondivergence form, these problems were considered for example in the book of Stroock-Varadhan ([6, Chapter 11]) by solving the corresponding martingale problem. When X is a symmetric diffusion that corresponding to a uniform elliptic divergence form, (Q1) was solved completely by Stroock-Zheng [7]. Let $X_t^{(n)}$ be a continuous time symmetric Markov chain on $n^{-1}\mathbb{Z}^d$ and conductances $C^{(n)}(x,y)$. In [7], they also answered (Q2) when $C^{(n)}(\cdot,\cdot)$ was of finite range (i.e. $C^{(n)}(x,y) = 0$ if $|x-y| \ge R_0/n$ for some $R_0 > 0$) and it had some uniform regularity. The core of the paper was to establish the discrete version of the De Giorgi-Moser-Nash theory. Recently, in [3] the theorem in [7] was extended in two ways: chains with unbounded range was allowed and the strong uniform regularity conditions in [7] was weakened. This was further extended in [4] so that the limiting process X had a continuous part and a jump part. For both [3, 4], one of the key part was to obtain apriori estimates of the solution of the heat equation, which was enabled thanks to the recent developments of the De Giorgi-Moser-Nash theory for jump processes.

Now consider the case where X is a pure jump symmetric Hunt process. We will consider the corresponding Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(\mathbb{R}^d;m)$ where

$$\mathcal{E}(u,v) := \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d\setminus d}(u(x)-u(y))(v(x)-v(y))j(x,y)m(dx)m(dy), \tag{1}$$

for $u,v \in \mathcal{F} := \{u \in L^2(\mathbb{R}^d,m) : \mathcal{E}(u,u) < \infty\}$ (m is the Lebesgue measure). Here $j(\cdot,\cdot)$ is a symmetric non-negative function on $\mathbb{R}\times\mathbb{R}^d \setminus d$, and we put a suitable assumption to make $(\mathcal{E},\mathcal{F})$ regular. The paper [5] considered (Q1)–(Q2) when $j(x,y) \asymp |x-y|^{-d-\alpha}$ for some $0 < \alpha < 2$. This was considerably extended in [2] to some class of the Dirichlet form $(\mathcal{E},\mathcal{F})$. Again, for both [5, 2], the crucial point was to obtain apriori Hölder estimates of the solution of the heat equation, However, the approach of obtaining apriori estimates of the heat kernel does not work in general. Indeed, even in the case $c_1|x-y|^{-d-\alpha_1} \le j(x,y) \le c_2|x-y|^{-d-\alpha_2}$ for $|x-y| < 1$ where $\alpha_1 < \alpha_2$, one can construct an example where there is a bounded harmonic function that is not continuous (see [1, Theorem 1.9]).

We will answer (Q1) affirmatively for a very general Dirichlet form of the shape of (1) (Theorem 3), and give answer to (Q2) when $X^{(n)}$, X satisfy conditions **(A1)**–**(A4)** below (Theorem 2). We do not rely on the apriori estimates of the heat kernel, instead use various techniques from the theory of Dirichlet forms such as the Lyons-Zheng decomposition and the Mosco convergence. The drawback is we can only obtain tightness when the initial distribution is in a sense ‘smooth’.

2. The model and tightness

Let $V_k = k^{-1}\mathbb{Z}^d$, $m_k(x) = k^{-d}$ for $x \in V_k$ and $B_j = B(0,j)$. Let $\{\mathcal{C}^{(k)}(x,y),\ x,y \in V_k\}$ be a family of conductance so that $\mathcal{C}^{(k)}(x,y) = \mathcal{C}^{(k)}(y,x) \geq 0$. We assume $\mathcal{C}^{(k)}(x,x) = 0$. Define a Dirichlet form $(\mathcal{E}^{(k)}, \mathcal{F}^{(k)})$ on $L^2(V_k, m_k)$ by

$$\mathcal{E}^{(k)}(u,v) := \frac{1}{2}\sum_{x,y\in V_k}(u(x)-u(y))(v(x)-v(y))\mathcal{C}^{(k)}(x,y)m_k(x)m_k(y),$$

for all $u, v \in \mathcal{F}^{(k)} := \{u \in L^2(V_k; m_k)\,;\ \mathcal{E}^{(k)}(u,u) < \infty\}$. We assume the following.
(A1). There exists $k_0 \geq 1$ such that for all $j \geq 1$,

$$\sup_{k\geq k_0}\ \sup_{x\in \overline{B}_j\cap V_k}\sum_{y\in V_k}\mathcal{C}^{(k)}(x,y)\Big(|x-y|^2\wedge 1\Big)m_k(y) < \infty,$$
$$\sup_{k\geq k_0}\ \sup_{x\in B_{j+2}^c\cap V_k}\sum_{y\in B_j\cap V_k}\mathcal{C}^{(k)}(x,y)m_k(y) < \infty.$$

Note that under **(A1)**, $(\mathcal{E}^{(k)}, \mathcal{F}^{(k)})$ is regular. Define $\{\{X_t^{(k)}\}_{t\geq 0}, \{\mathbb{P}_x^{(k)}\}_{x\in V_k}\}$ as the corresponding symmetric Markov chain. Now for all $\varphi \in C_c^+(\mathbb{R}^d)$, define

$$(2)\quad \mathbb{P}_\varphi^{(k)}(\,\cdot\,) := \sum_{x\in V_k}\mathbb{P}_x^{(k)}(\,\cdot\,)\varphi(x)m_k(x) \quad\text{and}\quad \mathbb{P}_\varphi(\,\cdot\,) := \int_{\mathbb{R}^d}\mathbb{P}_x(\,\cdot\,)\varphi(x)m(dx).$$

Let $\zeta^{(k)}$ be the lifetime of the process $X^{(k)}$. Then the following holds.

Proposition 1. *Assume* **(A1)**. *Then for all* $\varphi \in C_c^+(\mathbb{R}^d)$ *the laws of* $\{X_t^{(k)}\}_{t\in[0,T]}$ *on* $\{\zeta^{(k)} > T\}$ *with initial distribution* $\varphi(x)m_k(dx)$ *is tight in* $\mathbb{D}([0,\infty), \mathbb{R}^d)$.

We don’t know tightness when the initial distribution is concentrated on a point.

3. Main theorem: weak convergence and discrete approximation

(A2). Let $j : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ be the symmetric measurable function such that

$$(3)\qquad \sup_{x\in K}\int_{\mathbb{R}^d}(|x-y|\wedge 1)^2 j(x,y)m(dy) < \infty,\ \forall K:\ \text{compact}.$$

Let $(\mathcal{E}, \mathcal{F})$ be the Dirichlet form on $L^2(\mathbb{R}^d, m)$ defined by (1). Note that from (3), we have $\mathrm{Lip}_c(\mathbb{R}^d) \subset \mathcal{F}$.

(A3). (i) $\mathrm{Lip}_c(\mathbb{R}^d)$ is dense in $(\mathcal{F}, \mathcal{E}(\cdot,\cdot) + \|\cdot\|_2^2)$.
(ii) $\mathcal{L}_{j,\delta}f$ is continuous for all $f \in \mathrm{Lip}_c(\mathbb{R}^d)$ where $\mathcal{L}_{j,\delta}$ is defined below.

Note that under **(A2)** and **(A3)**(i), $(\mathcal{E}, \mathcal{F})$ is regular. Let $\{\{X_t\}_{t\geq 0}, \{\mathbb{P}_x\}_{x\in\mathbb{R}^d}\}$ be the corresponding symmetric Hunt process.

Below, we extend $\{\mathcal{C}^{(k)}(x,y) : x,y \in V_k\}$ to $\{\mathcal{C}^{(k)}(x,y) : x,y \in \mathbb{R}^d\}$ in a natural way. Set

$$\mathcal{L}_{j,\delta}u(x) = \int_{B_j} (u(y)-u(x))1_{\{|x-y|>\delta\}} j(x,y)m(dy) \qquad \forall x \in B_j,$$

and define $\overline{\mathcal{L}}^{(k)}_{j,\delta}u$ similarly by changing $j(x,y)$ to $C^{(k)}(x,y)$.
(A4). (i) For any compact subset $K \subset \mathbb{R}^d$,

$$\lim_{\eta\to 0}\limsup_{k\to\infty} \int\int_{\{(x,y)\in K\times K : |x-y|\le\eta\}} |x-y|^2 \mathcal{C}^{(k)}(x,y)m(dx)m(dy) = 0,$$

$$\lim_{j\to\infty}\limsup_{k\to\infty}\int_K\int_{B_j^c} \mathcal{C}^{(k)}(x,y)m(dx)m(dy) = \lim_{j\to\infty}\sup_{x\in K} j(x,B_j^c) = 0.$$

(ii) $\lim_{k\to\infty}\|\overline{\mathcal{L}}^{(k)}_{j,\delta}f\|^2_{2,B_j} = \|\mathcal{L}_{j,\delta}f\|^2_{2,B_j}$, $\forall f \in \mathrm{Lip}_c(\mathbb{R}^d)$, $\forall\delta > 0$ and $\forall j \in \mathbb{N}$.
(iii) $\mathcal{C}^{(k)}(x,y)m(dx)m(dy) \overset{\text{weakly}}{\longrightarrow} j(x,y)m(dx)m(dy)$ on $B_j \times B_j \setminus \{|x-y| > \delta\}$.

Define $\pi_k : L^2(\mathbb{R}^d, m) \to L^2(V_k, m_k)$ and $E_k : L^2(V_k, m_k) \to L^2(\mathbb{R}^d, m)$ as follows:

$$\pi_k f(x) = \frac{1}{m_k(x)}\int_{U_k(x)} f(y)m(dy), \qquad E_k g(z) = g(x) \text{ for } z \in U_k(x),\ x \in V_k.$$

Let $P_t f(x) := \mathbb{E}_x[f(X_t)]$ and $P^{(k)}_t g(x) := \mathbb{E}^{(k)}_x[g(X^{(k)}_t)]$. Then we have the following main theorems.

Theorem 1. *Assume that* **(A2)**-**(A4)** *hold.*
Then $E_k P^{(k)}_t \pi_k$ converges to P_t strongly (and uniformly for $t \le T$) in $L^2(\mathbb{R}^d, m)$.

Theorem 2. *Assume* **(A1)**–**(A4)** *and that X is conservative. Then, for all $\varphi \in C^+_c(\mathbb{R}^d)$, $\{(X^{(k)}, \mathbb{P}^{(k)}_\varphi); k \ge 1\}$ converges weakly to $(X, \mathbb{P}_\varphi)$.*

Remark 1. i) All the above results hold in a class of metric measure space with volume doubling condition.
ii) We have another version of the results that does not require **(A3)**(ii).

The above example can be obtained by checking **(A1)**–**(A4)** and applying the above theorems.

For $x = (x_1, \cdots, x_d)$, let $U_k(x) = \Pi^d_{i=1}[x_i - (2k)^{-1}, x_i + (2k)^{-1}]$. We finally state our theorem on the discrete approximation.

Theorem 3. *Let $j(x,y) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ be such that*

$$j(x,y)1_{\{|x-y|\ge 1\}} \le M_0 < \infty\ \forall x,y \in \mathbb{R}^d,\ \lim_{j\to\infty}\sup_{x\in K} j(x,B_j^c) = 0\ \forall K \subset\subset \mathbb{R}^d.$$

Assume that the Dirichlet form $(\mathcal{E},\mathcal{F})$ determined by $j(x,y)$ satisfies **(A2)**–**(A3)** *(i) and it is conservative. For $x,y \in V_k$, define*

$$\mathcal{C}^{(k)}(x,y) := \mathbf{1}_{\{|x-y|\ge c_1/k\}} \frac{1}{m_k(x)m_k(y)}\int_{U_k(x)}\int_{U_k(y)} j(\xi,\eta)m(d\xi)m(d\eta).$$

Then, the corresponding $(\mathcal{E}^{(k)}, \mathcal{F}^{(k)})$ *is a regular Dirichlet form on* $L^2(V_k, m_k)$. *Let* $X^{(k)}$ *be the associated Markov chain. Then,* $\{(X^{(2^k)}, \mathbb{P}_\varphi^{(2^k)});\, k \geq 1\}$ *converges weakly to* $(X, \mathbb{P}_\varphi)$ *for all* $\varphi \in C_c^+(\mathbb{R}^d)$.

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Absorbing-State Phase Transition for Stochastic Sandpiles and Activated Random Walks

Leonardo Trivellato Rolla
(joint work with V. Sidoravicius)

We study the long-time behavior of conservative interacting particle systems in $\mathbb{Z}$: The Activated Random Walk Model for reaction-diffusion systems and the Stochastic Sandpile. Our main result states that both systems locally fixate when the initial density of particles is small enough, establishing the existence of a non-trivial phase transition in the density parameter. This fact is predicted by theoretical physics arguments and supported by numerical analysis.

1. Introduction

Modern Statistical Mechanics offers large and important class of driven dissipative systems that naturally evolve to a critical state, characterized by power-law distributions of relaxation events. Among concepts and theories which attempt to explain long-ranged spatio-temporal correlations the physical paradigm called 'self-organized criticality' takes its particular place. It refers to systems that are attracted to a stationary critical state without being tuned to a critical point. This phenomenon is believed to be behind random fluctuations at the macroscopic scale, self-similar shapes, and huge avalanches caused by small perturbations.

When it refers to non-equilibrium steady states, now it is understood at theoretical level that self-organized criticality (further SOC) is related to a conventional phase transition. It concerns a system whose natural dynamics drives it towards, and then maintains it, at the *edge* of stability. The known examples are related to underlying non-equilibrium systems which actually do have a parameter and exhibit critical phenomena, the so-called *absorbing-state phase transitions.* Such phase transitions arise from a conflict between spread of activity, and a tendency for this activity to die out, or between creation and annihilation processes. The transition point separates an active phase and an absorbing phase in which the dynamics gets eventually frozen.

The Stochastic Sandpile Model (SSM) is a continuous-time evolution corresponding to a sandpile model. In this evolution, sites are stable when they bare 0 or 1 grain and unstable if at least 2 grains are present. Each unstable site topples at rate 1, sending 2 grains to neighbors chosen independently at random.

The Activated Random Walks (ARW) is a reaction-diffusion model given by the following conservative particle dynamics in $\mathbb{Z}^d$. Each particle can be in one of two states: an active A-state, and a passive (or sleepy) S-state. A-particles perform continuous-time random walk with jump rate $D_A = 1$ without interacting. S-particles do not move, that is, $D_S = 0$. Each particle changes its state $A \to S$ at some halting rate $\lambda > 0$ and the reaction $A + S \to 2A$ happens immediately. The catalyzed transition $A + S \to 2A$ and the spontaneous transition $A \to S$ represent the spread of activity versus a tendency of this activity to die out.

We prove phase transition for both the ARW and the SSM in the one-dimensional case. The core of our approach is to use the Diaconis-Fulton representation for each model and state local fixation in terms of total number of jumps.

2. The models and results

The Stochastic Sandpile Model evolves as follows. When a site x has at least 2 particles, it is called unstable and topples at rate 1. When it topples it sends 2 grains to neighboring sites chosen independently at random, that is, according to the distribution $p(y-x)$, where $p(z) = \frac{1}{2d}$ if $\|z\| = 1$ and 0 otherwise.

The state of the SSM at each time $t \geq 0$ is given by $\eta_t \in (\mathbb{N}_0)^{\mathbb{Z}^d}$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\eta_t(x)$ denotes the number of particles found at site x at time t. For each site $x \in \mathbb{Z}^d$, the transitions $\eta \to \tau_{xy}\tau_{xw}\eta$ happen at rate $A\big(\eta_t(x)\big)p(y-x)p(w-x)$, where

$$\tau_{xy}\eta(z) = \begin{cases} \eta(x) - 1, & z = x \\ \eta(y) + 1, & z = y \\ \eta(z), & \text{otherwise,} \end{cases}$$

and $A(k) = \mathbf{1}_{k \geq 2}$ indicates whether site x is unstable. If ν denotes the distribution of η_0, let $\mathbb{P}^\nu$ denote the law of $(\eta_t)_{t \geq 0}$. This evolution is well defined because the jump rates are bounded.

We say that the system *locally fixates* if $\eta_t(x)$ is eventually constant for each x, otherwise we say that the system *stays active.*

Theorem 1. *Consider the Stochastic Sandpile Model in the one-dimensional lattice $\mathbb{Z}$, with initial distribution ν given by i.i.d. Poisson random variables with parameter μ. There exists $\mu_c \in \left[\frac{1}{4}, 1\right]$, such that the system locally fixates a.s. if $\mu < \mu_c$, and stays active a.s. if $\mu > \mu_c$.*

We now turn to the description of the Activated Random Walk model. Each particle in the A-state performs a continuous-time random walk with jump rate $D_A = 1$. The jumps are distributed as $p(\cdot)$, and we assume that $p(0) = 0$, $p(z) \geq 0$, and $\sum_{z \in \mathbb{Z}^d} p(z) = 1$. Independently of anything else, each A-particle turns to the S-state at a halting rate $\lambda > 0$. Once a particle is in the S-state, it stops moving, i.e., its jump rate is $D_S = 0$, and it remains in the S-state until the instant when another particle is present at the same vertex. At such an instant the particle which is in S-state flips to the A-state, giving the transition $A + S \to 2A$. An S-particle stays still forever if no other particle ever visits the vertex where it is located. According to these rules, the transition $A \to S$ *effectively* occurs if and only if, at the instant of such transition, the particle *does not share* the vertex with another particle (the innocuous instantaneous transition $2A \to A + S \to 2A$ is not observed). A-particles do not interact among themselves.

The state of the ARW at time $t \geq 0$ is given by $\eta_t \in \Sigma = (\mathbb{N}_{0\varrho})^{\mathbb{Z}^d}$, where $\mathbb{N}_{0\varrho} = \mathbb{N}_0 \cup \{\varrho\}$. In this setting $\eta_t(x)$ denotes the number of particles found at site x at time t, and ϱ means one passive particle. We make $\mathbb{N}_{0\varrho}$ be an ordered set by setting $0 < \varrho < 1 < 2 < \cdots$, and let $|\varrho| = 1$. We define the addition by setting $\varrho + 0 = 0 + \varrho = \varrho$ and $\varrho + n = n + \varrho = 1 + n$ for $n > 0$, that is, the addition already contains the $A + S \to 2A$ transition. We also define $\varrho \cdot 1 = \varrho$ and $\varrho \cdot n = n$ for $n > 1$, the $A \to S$ transition.

The process evolves as follows. For each site x, denoting by $A(k) = k\mathbf{1}_{k \geq 1}$, so $A\big(\eta_t(x)\big)$ is the number of active particles at site x at time t, we have the transitions $\eta \to \tau_{xy}\eta$ at rate $A\big(\eta_t(x)\big)p(y - x)$ and $\eta \to \tau_{x\varrho}\eta$ at rate $\lambda A\big(\eta_t(x)\big)$, where

$$\tau_{x\varrho}\eta(z) = \begin{cases} \varrho \cdot \eta(x), & z = x \\ \eta(z), & \text{otherwise.} \end{cases}$$

Theorem 2. *Consider the Activated Random Walk Model with nearest-neighbor jumps in the one-dimensional lattice $\mathbb{Z}$ with fixed halting rate λ. Suppose the $\eta_0(x)$ are i.i.d. random variables in $\mathbb{N}_0$, having a Poisson distribution with parameter μ. There exists $\mu_c \in \big[\frac{\lambda}{1+\lambda}, 1\big]$ such that the system locally fixates a.s. if $\mu < \mu_c$ and stays active a.s. if $\mu > \mu_c$. In the particular case of $\lambda = +\infty$ we have $\mu_c = 1$.*

3. Comment on the proof

For the proof we use an equivalence between fixation for the dynamic stochastic evolution in infinite volume and stabilizability of configurations in the discrete, commutative Diaconis-Fulton setting. We introduce semi-legal operations, as well as some replacements in the Diaconis-Fulton instructions, which as we show give upper bounds for the amount of activity. Later we explore the instructions, looking ahead in the future, but in a careful way in order not to explore too much, so that the subsequent steps would always find fresh randomness. After the exploration, we performe some semi-legal operations, or replace some of the instructions, in meticulously chosen locations, so that the particles were settled at convenient locations, at one hand packed together as much as possible, in order to leave free space for the next steps, and on the other hand making sure they would neither distrub previously settled partilces nor move into regions with explored-but-unused instructions.

In one dimension, the volume of the convex envelope of the sites where the particles are settled in our construction, together with the sites where instructions are explored but not used, is proportional to the number of such particles, and this does not seem to have a straightforward analogous in higher dimensions. So, even though this general method is promising for other settings, the proof of fixation is so far restricted to this case.

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Phase Transitions in the Random Pinning Polymer.

Nikos Zygouras
(joint work with K. S. Alexander)

A polymer pinning model is described by a Markov chain $(X_n)_{n\geq 0}$ on a state state space Σ, containing a special point 0 where the polymer interacts with a potential. The space-time trajectory of the Markov chain represents the physical configuration of the polymer, with the nth monomer of the polymer chain located at (n, X_n). We denote the distribution of the Markov chain in the absence of the potential, started from 0, by P^X and we assume that it is recurrent and has an excursion length distribution (from the 0 state) with power-law decay:

$$P^X(\mathcal{E} = n) = \frac{\varphi(n)}{n^c}, \qquad n \geq 1. \tag{1}$$

Here $\mathcal{E}$ denotes the length of an excursion from 0, $c \geq 1$, and $\varphi(\cdot)$ is a slowly varying function, that is, a function satisfying $\varphi(\kappa n)/\varphi(n) \to 1$ as n tends to infinity, for all $\kappa > 0$.

When the chain visits 0 at some time n, it encounters a potential of form $u + V_n$, with the values V_n typically modeling variation in monomer species. This (quenched) pinning model is described by the Gibbs measure

$$d\mu^{\beta,u,\mathbf{V}_N}(\mathbf{x}) = \frac{1}{Z_N} e^{\beta H_N^u(\mathbf{x},\mathbf{V})\, dP^X(\mathbf{x})} \tag{2}$$

where $\mathbf{x} = (x_n)_{n\geq 0}$ is a path, $\mathbf{V} = (V_n)_{n\geq 0}$ is a realization of the disorder, and

$$H_N^u(\mathbf{x}, \mathbf{V}) = \sum_{n=0}^{N} (u + V_n)\delta_0(x_n) \tag{3}$$

and the normalization

$$Z_N = Z_N(\beta, u, \mathbf{V}) = E^X\left[e^{\beta H_N^u(\mathbf{x},\mathbf{V})}\right]$$

is the partition function. The disorder $\mathbf{V}$ is a sequence of i.i.d. random variables with mean zero, variance one and finite exponential moments. Often it is considered to be standard Gaussian. We denote the distribution of the sequence $\mathbf{V}$ by P^V. The parameter $u \in \mathbb{R}$ is thus the mean value of the potential, and $\beta > 0$ is the inverse temperature.

One would like to understand how the presence of the random potential affects the path properties of the Markov chain, and in particular how the case with disorder differs from the homogeneous case $V_n \equiv 0$, or the annealed case, i.e. when an average over the disorder is performed. These effects can be quantified via the free energy $f_q(\beta, u)$ given by

$$\beta f_q(\beta, u) = \lim_{N\to\infty} \frac{1}{N} \log Z_N(\beta, u, \mathbf{V}); \tag{4}$$

the fact that the free energy exists and is nonrandom (off a null set of disorders) is proved in [2]. The free energy is 0 if $u < u_c^q(\beta)$ and strictly positive if $u > u_c^q(\beta)$, where $u > u_c^q(\beta)$ is the critical point of the phase transition.

There is a rich phenomenology in this model related to different phase diagrams and path behaviors in the case where the exponent $c < 3/2$ or $c > 3/2$. The critical case $c = 3/2$ is particularly interesting since the phase diagram depends on the slowly varying function $\varphi(\cdot)$.

In this talk we will review the study of the phase diagram that has been obtained recently in a series of papers [1], [2], [3], [4], [5] etc.

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Superdiffusivity of 1D Lattice KPZ Equation

Tomohiro Sasamoto

(joint work with H. Spohn)

To describe one-dimensional stochastic surface growth phenomena, Kardar, Parisi and Zhang introduced in 1986 a nonlinear stochastic differential equation,

$$\partial_t h(x,t) = \tfrac{1}{2}\lambda_0(\partial_x h(x,t))^2 + \nu_0 \partial_x^2 h(x,t) + \sqrt{D_0}\xi(x,t), \tag{1}$$

where $\xi(x,t)$ is white noise with covariance $\langle \xi(x,t)\xi(x',t')\rangle = \delta(x-x')\delta(t-t')$. Applying the dynamical renormalization group analysis, they showed that the exponent β for the height fluctuations is $1/3$. This is characteristic of non-Gaussian nature of the fluctuations of the phenomena. The equation (1) is called the KPZ equation and the universality class related to the KPZ equation is refered to as the KPZ universality class.

Recently there have been a lot of new information available for the KPZ universality class due to the explicit computations using the techniques of random matrices, Bethe ansatz and so on. They are obtained for other growth models like PNG model and ASEP. There have been only few analysis of the KPZ equation itself. One reason is the singular behavior of the noise term in the KPZ equation. In this talk we avoid this difficulty by discretizing space. The equation we study is

$$\begin{aligned}\frac{d}{dt}u_j = {}& \tfrac{1}{6}\lambda_0(u_{j+1}^2 + u_j u_{j+1} - u_{j-1}u_j - u_{j-1}^2)\\ & + \nu_0(u_{j+1} - 2u_j + u_{j-1}) + D_0^{1/2}(\xi_j - \xi_{j-1})\end{aligned}$$

where $j \in \mathbb{Z}$ and $\{\xi_j\}$ is independent white noise. There is an ambiguity for the discretization but our special choise allows us to obtain explictly the translationay invariant stationary measures, which is jsut the product of independent Gaussians. The quantity of our interest is the stationary two-point function defined by

$$S(j,t) = \langle u_j(t)u_0(0)\rangle - \langle u_0(0)\rangle^2.$$

Here $\langle\cdots\rangle$ is the average wrt stationary measure. We prove $\frac{1}{4} \le \beta \le \frac{1}{2}$ using a method due to Landim et al applied to ASEP. We also show the "relaxation time approximation" gives $\beta = 1/3$.

After a ground state transformation one can rewrite the generator (which we call the Hamiltonian) in the form,

$$H = H_0 + \lambda(A^* - A)$$

with

$$\begin{aligned}H_0 &= \sum_{j\in\mathbb{Z}}(a_{j+1} - a_j)^*(a_{j+1} - a_j),\\ A &= \sum_{j\in\mathbb{Z}}(a_j a_{j+1}^* a_{j+1} + a_j a_j a_{j+1}^* - a_j^* a_{j+1} a_{j+1} - a_j^* a_j a_{j+1}).\end{aligned}$$

Here a_j, a_j^* satisfy $[a_i, a_j^*] = \delta_{ij}$.

After some computations one finds that the main part of the Laplace transform of the second moment of the two point function is given by

$$\langle \hat{g}^0, (\zeta + H)^{-1}\hat{g}^0\rangle^0$$

where $\hat{g}^0$ is a certain vector in Fock space. From the KPZ scaling it is expected this quantity behaves like

$$\langle \hat{g}^0, (\zeta + H)^{-1}\hat{g}^0\rangle^0 \overset{\zeta\to 0}{\cong} 3^{-2/3}\Gamma(7/3)(\lambda^2\zeta)^{-1/3}\int dx x^2 f_{\mathrm{KPZ}}(x)$$

when $\zeta \to 0$. Here f_{KPZ} is the scaling function of the two point function in the KPZ universality class.

To get bounds of this quantity we employ the method of Landim, Quastel, Salmhofer and Yau applied to ASEP. We truncate the resolvent equation associated with H up to the n-particle subspace of the Fock space. By taking $n = 2, 3$ we prove

Theorem. In the limit $\zeta \to 0$, the following bounds are valid

$$\lambda^{-1} 2^{-5/4} 3^{-3/2} \zeta^{-1/4} \leq \langle \hat{g}^0, (\zeta + H)^{-1} \hat{g}^0 \rangle^0 \leq 2^{-3/2} \zeta^{-1/2}$$

By using the *relaxation time approximation* applied to this scheme we also obtain the true KPZ exponent 1/3. More details can be found in arXiv: 0908.2096.

Ballisticity Conditions for Random Walk in Random Environment

Alejandro F. Ramírez
(joint work with A. Drewitz)

For $x \in \mathbb{Z}^d$ define $w(x) := \{w(e,x) : |e| = 1\}$ where $w(e,x) \in [0,1]$ and $\sum_e w(e,x) = 1$. The quantity $w := \{w(x) : x \in \mathbb{Z}^d\}$ is called the *environment* and it takes values on the space $\Omega := \mathcal{P}^{\mathbb{Z}^d}$, where $\mathcal{P} = \{\{p(e) : |e| = 1\} : p(e) \geq 0, \sum_e p(e) = 1\}$. We define a random walk $\{X_n : n \geq 0\}$ on the environment w as the Markov chain which jumps form site y to the nearest neighbor $y + e$ with probability $w(e,y)$. We denote by $P_{x,w}$ its law on $(\mathbb{Z}^d)^{\mathbb{N}}$. Let μ be a probability measure on the space Ω with its Borel σ-algebra so that the environment w is random. If $X_0 = x$, $P_{w,x}$ is then called the *quenched law* and $P_x := \int P_{w,x} d\mu$ the *averaged* or *annealed law* of the *random walk in random environment* (RWRE) $\{X_n : n \geq 0\}$. When for every x and e, $\mu(w(e,x) > 0) = 1$, one says that the RWRE is *elliptic*, while if there is a constant $\kappa > 0$ such that $\mu(w(e,x) \geq \kappa) = 1$ one says that it is *uniformly elliptic*. We will consider the case in which μ is a product measure, so that $\{w(x) : x \in \mathbb{Z}^d\}$ are i.i.d. This model, has origins in phenomena from biology, crystallography and metal physics. Chernov in [1], introduced it as simplified model for the replication of DNA chain (see also [14]). Some reviews on the state of the art of the subject have been given in [12], [17], [18] and [16].

In dimension $d = 1$, Solomon established [9] well known recurrence-transience and ballisticity criteria for an RWRE. Furthermore, in the one-dimensional recurrent case, known as Sinai's random walk [7], the walk moves anomalously slowly and some refined results have been obtained (see [5] and [3] for a description of the limiting law; [2] for results related to the local time asymptotics and [8] for a review). Later, with a series of works during the 2000's, the one-dimensional RWRE acquired a high level of mathematical maturity.

Several fundamental problems remain open for the RWRE in dimensions $d \geq 2$. A basic one is to understand under which conditions on the environment, is the walk transient or ballistic in a given direction given by a non-zero vector $l \in \mathbb{S}^d$: one says that the RWRE is *transient in the direction* l, if P_0-a.s.

$$\lim_{n\to\infty} X_n \cdot l = \infty;$$

one says that it is *ballistic in the direction* l if

$$\varliminf_{n\to\infty} \frac{X_n \cdot l}{n} > 0.$$

Using regeneration times, it can be shown that ballisticity on a given direction implies a law of large numbers $\lim_{n\to\infty} X_n/n = v$, P_0-a.s., where v is a deterministic speed. Recently, examples of RWRE on elliptic i.i.d. environments which are transient but not ballistic in a given direction have been given (see for example [6] and [15]). The following is one of the basic fundamental open problems in the field.

Conjecture 1. (Transience implies ballisticity). *Consider an RWRE on a uniformly elliptic i.i.d. environment in dimensions $d \geq 2$, which is transient in the direction l. Then it is ballistic in the direction l.*

In 1981, Kalikow [4] studied sufficient conditions which imply transience, proving in particular the so called *Kalikow's zero-one law*, which states that for every $l \in \mathbb{S}^d$, the probability $P_0(A_l \cup A_{-l})$ can have only the values 0 or 1. In 1999, Sznitman and Zerner [13], proved that any RWRE on dimensions $d \geq 2$, on a uniformly elliptic i.i.d. environment satisfying Kalikow's condition (see [4]), is ballistic. Sznitman introduced the ballisticity conditions (T) in [10] and (T') in [11], weaker than Kalikow's condition, which also imply ballistic behavior and an averaged functional central limit theorem so that

$$n^{-1/2}(X_{[nt]} - [nt]v)$$

converges in P_0-law to a Brownian motion with positive variance. Let $\gamma \in (0,1)$ and $l \in \mathbb{S}^d$. We say that condition $(T)_\gamma$ *is satisfied relative to the direction* l (written in shorthand as $(T)_\gamma|l)$ if for every l' in a neighborhood of l one has that

$$\varlimsup_{n\to\infty} L^{-\gamma} \log P_0(X_{T_{U_{l',b,L}}} \cdot l' > 0) < 0,$$

for all $b > 0$ and $U_{l',b,L} := \{x \in \mathbb{Z}^d : -bL < x \cdot l' < L\}$ with $T_{U_{l',b,L}}$ denoting the first exit time from $U_{l',b,L}$. One says that condition (T') *is satisfied relative to* l (written as $(T')|l)$ if for every $\gamma \in (0,1)$, $(T)_\gamma|l$ holds. The following is conjectured.

Conjecture 2. (Equivalence of ballisticity conditions). *Consider an RWRE in a uniformly elliptic i.i.d. environment in dimensions $d \geq 2$. Then, for every $\gamma \in (0,1)$, $(T)_\gamma|l$ and $(T')|l$ are equivalent.*

Sznitman proved in [11] the above equivalence for each $\gamma \in (0.5,1)$. Here we present the following result.

Theorem 1. (Drewitz-Ramírez). *Let $d \geq 2$ and*

$$\gamma_d := \frac{\sqrt{3d^2 - d} - d}{2d - 1}.$$

Then, for each $\gamma \in (\gamma_d, 1)$ and $l \in \mathbb{S}^{d-1}$, $(T)_\gamma|l$ is equivalent to $(T')|l$.

It can be observed that γ_d is monotonically decreasing in d. Therefore, $\gamma_\infty := \lim_{d\to\infty}\gamma_d = \frac{\sqrt{3}-1}{2}$ and

$$0.366 \approx \gamma_\infty < \gamma_d \le \gamma_2 \approx 0.387.$$

The proof of Theorem 1 follows [11] and is based on showing that $(T)_\gamma|l$ and $(T')|l$ are equivalent to the so called *effective criterion*, introduced by Sznitman in [11]. The effective criterion in direction l, is a condition which somehow mimics Solomon's one-dimensional ballisticity condition [9]. A crucial ingredient in the proof of Theorem 1, is the introduction of control estimates for the occurrence of atypical quenched exit distributions through boxes B of side L as $L \to \infty$:

$$P_0\left(P_{0,w}(X_{T_B}\cdot l \ge L) \le e^{-L^\beta}\right),$$

Here T_B is the first exit time of the random walk from this box and β is an appropriately chosen parameter. These controls are obtained via renormalization methods. These estimates have then to be used carefully to obtain a good upper bound for the probability that the random walk exits large boxes oriented towards the direction l, starting from itse center, through the side in the direction $-l$.

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Sharpness of the Percolation Transition in the Two-Dimensional Contact Process

Jacob van den Berg

The contact process was introduced as a stochastic model for the spread of an infection in a population with a geometric structure, usually represented by the d-dimensional cubic lattice. (See [14] for background, many results, and references).

Each vertex x of this lattice represents a location (or individual), of which the state, 1 (occupied, infected) or 0, (vacant, healthy) at time t is denoted by $\sigma_x(t)$. The dynamics in this model is as follows: A vertex in state 0 goes to state 1 at a rate equal to λ times the number of neighbours of that vertex that are in state 1. A vertex in state 1 goes to state 0 at rate 1. Here λ is the parameter of the model, called the infection rate. In this talk we restrict to the case $d = 2$.

The configuration at time t is denoted by $\sigma(t) := (\sigma_x(t), x \in \mathbb{Z}^2)$. Let μ_t denote the distribution of σ_t when we start at time 0 with all vertices occupied. It is well-known that μ_t converges weakly as $t \to \infty$. We denote the limit distribution by ν $(=\nu_\lambda)$. This measure ν is called the upper invariant measure. The occupied cluster of a vertex x (that is, the maximal connected component which contains x and of which every vertex is occupied) is denoted by $\mathcal{C}_x$. (If x is the origin, $\mathbf{0}$, we omit the subscript).

We study the sizes of occupied clusters under the measure ν. Motivation comes from work by Liggett and Steif ([15]) who showed that for λ sufficiently large percolation occurs (that is, $\nu_\lambda(|\mathcal{C}| = \infty) > 0$), and from work by biologists and agricultural researchers. In this latter work (see [12]), limit distributions of contact-like processes (more complicated than the 'basic process' described above) were used to model vegetation patterns in arid regions in Spain and North-Africa. For some of these models it was claimed in [12] that simulations suggest power law behaviour of the cluster size distribution in an interval of some parameter.

In ordinary percolation models it is known that below the percolation threshold the distribution of the cluster size has exponential decay, and that power-law behaviour can only occur at the percolation threshold. Triggered by the above mentioned claim by biologists and agricultural researchers concerning very different behaviour in 'their' contact-like processes, we study this question for ν_λ.

The proof of exponential decay for ordinary (independent) two-dimensional percolation goes back to the celebrated paper [13] by Kesten. A crucial step in that paper is, somewhat informally and in 'modern' terminology, that if the probability of the event A that there is an occupied crossing of a given, large, box (square) is neither close to 0 nor close to 1, the expected number of so-called pivotal vertices (or, for bond percolation, pivotal edges) is large. (These are vertices with the property that flipping the state of the vertex, flips the occurrence/non-occurrence of the event A). This step was proved in a 'constructive' way, with a 'geometric' flavour. The above mentioned large expectation of pivotal vertices implies that the derivative (w.r.t. the parameter p) of the probability of A is large. Hence, once the probability of A is not very small, a small increase of p makes it close to 1. (This property would now be called a 'sharp-threshold' phenomenon').

Moreover, by separate arguments, so-called finite-size criteria hold: if the probability of A is smaller than some absolute constant ϵ, the cluster size is finite a.s. (and its distribution has exponential decay), while if it is larger than $1 - \epsilon$ the system percolates. Combining these things gives exponential decay of the cluster size for all p smaller than p_c.

Russo ([17]) proved a very general 'approximate zero-one law' and showed that the above mentioned sharp-threshold phenomenon can be obtained from this more general law, using only a minimum of percolation arguments. In particular, in this way Kesten's 'constructive, geometric' arguments could be avoided, which is very useful because carrying out such arguments turns out to be (too) hard in many dependent models. (We should note, however, that for independent percolation the 'constructive' argument still gives the shortest self-contained proof, and that in *some* dependent models, see [2], it gives the only currently known proof).

Unfortunately, the above-mentioned finite-size criteria involved a so-called RSW result of which no ('reasonably general') extension to dependent models was known. This explains why, for a long time, Russo's approximate zero-one law did not receive much attention in the percolation community. In the meantime, results related to Russo's approximate zero-one law, but considerably sharper and more explicit, were obtained (in other areas of probability and mathematics in general) by Kahn, Kalai and Linial ([11]), Talagrand ([18]) and Friedgut and Kalai ([8]).

The importance for percolation of threse sharp-theshold results became clear much later, when Bollobás and Riordan ([5]) proved a more robust version of the RSW theorem which, combined with a clever use of the sharp-theshold results, led to the proof of the long-standing conjecture that the critical probability for random Voronoi percolation in the plane is 1/2 (and that below 1/2 this model has exponential decay). The robustness of these arguments led to similar results for several other two-dimensional percolation models (see [6], [7] and [3]).

The last mentioned paper proved, for 2D lattice models, exponential decay below the percolation threshold under the quite general condition that, informally speaking, the model has a 'nice finitary representation' (in a well-defined sense) in terms of finite-valued independent random variables. It turned out that under that condition only a weak (not explicitly quantitative) form, close to that of Russo's ([17]), of the sharp-threshold results was needed. As an example it was shown that the Ising model (with fixed $\beta < \beta_c$ and external field parameter h) belongs to this class (thus giving an alternative, more streamlined, proof of the main result in Higuchi's paper [10]). Here the role of finite-valued independent random variables was played by the 'independent updates' in a suitable discrete-time dynamics.

Such a dynamics was possible by (among other things) the nearest-neighbour Gibbs property of the Ising model. This is a big difference with the contact process, for which we don't know a suitable discrete-time dynamics. Therefore, we were not able to derive exponential decay for this model from Theorem 2.2 in [3], but instead exploited the full quantitative nature of the sharp-threshold results from [11] and [18] and followed more closely the route used in [5] and [7] for the Voronoi model and the Johnson-Mehl model (which, like the Voronoi model, is a model of planar tessellations, but more complicated than te Voronoi model). (Yet another route, namely by using results in [9], might work if ν would satisfy the strong FKG condition, which however (as has been shown by Liggett) it does not. We should also note here that the exponential-decay arguments in [1] and [16], which for ordinary percolation work in all dimensions, so far have (even in 2D) no suitable analog for dependent percolation).

Our main result is the following:

Theorem 1. *Let λ be such that*

$$\nu_\lambda(|\mathcal{C}| = \infty) = 0.$$

Then, for every $\lambda' < \lambda$ there exist $C_1, C_2 > 0$ such that for all $n \geq 1$

$$\nu_{\lambda'}(|\mathcal{C}| \geq n) \leq C_1 \exp(-C_2 n). \tag{1}$$

The proof follows the main strategy of [5] and [7]. However, the model-specific properties of the contact process lead to many non-trivial differences in the steps. Therefore, and because the contact process is one of the main random spatial models, the proof (in [4]) is given in detail. During the talk we highlight some of the steps.

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Renormalization of Interacting Diffusions

Frank den Hollander
(joint work with J.-B. Baillon, Ph. Clément, D. Dawson, A. Greven, R. Sun, J. Swart, and L. Xu)

Systems of *hierarchically interacting diffusions* allow for a rigorous *renormalization analysis.* By bringing into play the powerful machineries of stochastic analysis and functional analysis, it is possible to arrive at a complete classification of the *large space-time behavior* of these systems into *universality classes.*

In this talk we describe a renormalization program that was put forward by D. Dawson and A. Greven in 1993, and present an overview of what has been achieved so far. The object of interest is a system of coupled SDE's labelled by the hierarchical lattice Ω_N of order N. The components of this system take values in a non-empty convex state space $S \subset \mathbb{R}^d$, $d \geq 1$, each perform a diffusion according to a local diffusion function $g\colon S \to [0,\infty)$, drawn from a suitable class $\mathcal{H}$, and mutually interact according to a random walk transition kernel on Ω_N that is *critically recurrent*, with weights that are functions of the hierarchical distance only. The initial condition is taken to be constant. This system arises as the continuum limit of discrete evolution equations from *population dynamics*, where individuals of different types are organized into *colonies* and are subject to *resampling* within a colony and *migration* between colonies.

Choices of S that are of interest are:

(1) $S = d$-dimensional simplex (describing the total *fractions* of individuals of $d+1$ different types in the colony);
(2) $S = [0,\infty)^d$ (describing the total *masses* of individuals of d different types in the colony).

The main focus is on *block averages* on space-time scale k, i.e., sums of components over blocks of hierarchical radius k whose time is multiplied by N^k, where $k \in \mathbb{N} \cup \{0\}$. The goal is to show that:

- For each k and in the limit as $N \to \infty$ (the so-called hierarchical mean-field limit), a block average of order k evolves according to an *autonomous* SDE whose diffusion function is given by $F^k g$, where F is a *renormalization transformation* acting on g within the class $\mathcal{H}$.
- As $k \to \infty$, $F^k g$ converges after appropriate scaling to a limiting diffusion function g^* in $\mathcal{H}$.

For $d = 1$ the renormalization program has been fully carried through. In example (1), the *Wright-Fisher diffusion* with $g^*(x) = x(1-x)$, $x \in [0,1]$, is the global attractor with a scaling that is independent of g. In example (2), the *Feller diffusion* with $g^*(x) = x$, $x \in [0, \infty)$, is the global attractor with a scaling that depends on the asymptotic behavior of g at infinity.

For $d \geq 2$ the renormalization program has been partially carried through. In example (1), the global attractor g^* is the solution of the equation $\Delta g^* = -2$ on $\mathrm{int}(S)$ and $g^* = 0$ on ∂S, with a scaling that is independent of g. In example (2), there is a 3-parameter family of attracting fixed points g^*, corresponding to the *branching*, *catalytic branching*, respectively, *mutually catalytic branching* diffusion, with a scaling that depends on the asymptotic behavior of g at infinity.

Euclidean Field Scaling Limits of Ising Models and Measure Ensembles

Charles M. Newman
(joint work with F. Camia)

We discuss here joint work with Federico Camia [1], and at the end joint work in progress with Federico Camia and Christophe Garban, which provides a representation for the scaling limit of the critical Ising magnetization field in dimension $d = 2$ as a (conformal) random field Φ^0, by using Fortuin-Kasteleyn (FK) clusters and their rescaled area measures.

Φ^0 is a limit as $a \to 0$ of $\Theta_a \sum_{x \in \mathbb{Z}^2} S_x \delta(z - ax)$, where S_x ($x \in \mathbb{Z}^2$) are the ± 1 spin variables in the critical Ising model and ${\Theta_a}^{-1}$ is the standard deviation of $\hat{M}_a$, the sum of S_x with ax in the unit square $\Lambda_1 = [0,1]^2$. The representation is

$$\Phi^0 = \sum_j \eta_j \mu_j^0(dz), \tag{1}$$

where the η_j's are uniformly random plus or minus signs and $\{\mu_j^0\}$ is an ensemble of finite positive measures with bounded support on the plane such that $\sum_j [\mu_j^0(\Lambda)]^2 < \infty$ for bounded Λ. Convergence of the sum in (1) is in L^2 as a cutoff parameter ε, corresponding to deleting those μ_j^0's whose support has diameter less than ε, tends to zero.

An elementary consequence of the FK representation and a key feature in our analysis is the identity,

$$E(\hat{M}_a^2) = E(\sum_i |\hat{C}_i^a|^2), \tag{2}$$

where $\hat{C}_i^a$ are the critical FK clusters (in $a\mathbb{Z}^2$) intersected with the unit square Λ_1, and $|\hat{C}_i^a|$ are their areas (numbers of sites). Our published results [1] are primarily tightness and the representation (1) for any subsequence limit Φ^0. Two key parts of the proofs are showing that (i) there are $O(1)$ (as $a \to 0$) clusters touching Λ_1 whose macroscopic diameter is greater than any fixed $\varepsilon > 0$ and (ii) other clusters contribute negligibly to $E(\sum_i |\hat{C}_i^a|^2)$ for small ε. An important technical tool in the proofs is the validity for the critical FK model of Russo-Seymour-Welsh type inequalities, as were recently proved by H. Duminil-Copin, C. Hongler and P. Nolin.

Work in progress includes uniqueness and conformal covariance of $\{\mu_j^0\}$ and showing that this measure ensemble is a functional of the conformal loop ensemble $\mathrm{CLE}_{16/3}$. This loop ensemble, related to the Schramm-Loewner Evolution (SLE) [2] with the same parameter 16/3, is the scaling limit of the collection of critical FK cluster boundary loops, as announced by Smirnov [3]. Other work in progress or planned includes near-critical scaling limit representations for the massive (i.e., with exponentially decaying correlations) Euclidean fields that should accompany the massless critical field Φ^0. An interesting aspect of measure ensemble representations is that in principle they should be valid also for Ising model scaling limits in dimension $d = 3$ and for q-state Potts models in $d = 2$ with $q = 3$ or 4 (so that the critical point corresponds to a second order phase transition).

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Decay of Covariances, Uniqueness of Ergodic Component and Scaling Limit for a Class of Gradient Systems with Non-convex Potential

Jean-Dominique Deuschel

(joint work with C. Cotar)

Phase separation in $\mathbb{R}^{d+1}$ can be described by effective interface models. In this setting we ignore overhangs and for $x \in \mathbb{Z}^d$, we denote by $\phi(x) \in \mathbb{R}$ the height of the interface above or below the site x. Let Λ be a finite set in with boundary

$$(1)\quad \partial\Lambda := \{x \notin \Lambda,\ ||x-y|| = 1 \text{ for some } y \in \Lambda\},\ \text{where } \|x-y\|=\sum_{i=1}^{d} |x_i - y_i|$$

and with given boundary condition ψ such that $\phi(x) = \psi(x)$ for $x \in \partial\Lambda$. Let $\bar{\Lambda} := \Lambda \cup \partial\Lambda$ and let $d\phi_\Lambda = \prod_{x\in\Lambda} d\phi(x)$ be the Lebesgue measure over $\mathbb{R}^\Lambda$. For a finite region $\Lambda \subset \mathbb{Z}^d$, *the finite Gibbs measure* ν_Λ^ψ *on* $\mathbb{R}^{\mathbb{Z}^d}$ *with boundary condition* ψ for the field of height variables $(\phi(x))_{x\in\mathbb{Z}^d}$ over Λ is defined by

$$(2)\qquad \nu_\Lambda^\psi(d\phi) = \frac{1}{Z_\Lambda^\psi} \exp\left\{-\beta H_\Lambda^\psi(\phi)\right\} d\phi_\Lambda \delta_\psi(d\phi_{\mathbb{Z}^d\setminus\Lambda})$$

with

$$Z_\Lambda^\psi = \int_{\mathbb{R}^{\mathbb{Z}^d}} \exp\left\{-\beta H_\Lambda^\psi(\phi)\right\} d\phi_\Lambda \delta_\psi(d\phi_{\mathbb{Z}^d\setminus\Lambda}),$$

where $\delta_\psi(d\phi_{\mathbb{Z}^d\setminus\Lambda}) = \prod_{x\in\mathbb{Z}^d\setminus\Lambda} \delta_{\psi(x)}(d\phi(x))$; ν_Λ^ψ is characterized by the inverse temperature $\beta > 0$ and the Hamiltonian H_Λ^ψ on Λ, which we assume to be of gradient type:

$$(3)\qquad H_\Lambda^\psi(\phi) = \sum_{x\in\Lambda, y\in\Lambda\cup\partial\Lambda} U(\phi(x) - \phi(y)),$$

that is, the interaction between two neighboring sites x, y depends only on the discrete gradient $\nabla\phi(x,y) = \phi(x) - \phi(y)$. We thus have a massless model with a continuous symmetry. $U \in C^2(\mathbb{R})$ is a function with quadratic growth at infinity:

$$(4)\qquad U(\eta) \geq A|\eta|^2 - B, \qquad \eta \in \mathbb{R}$$

for some $A > 0, B \in \mathbb{R}$. Our state space $\mathbb{R}^{\mathbb{Z}^d}$ being unbounded, such models are facing delocalization in lower dimensions $d = 1, 2$, and no infinite Gibbs state exists in these dimensions. Instead of looking at the Gibbs measures of the $(\phi(x))_{x\in\mathbb{Z}^d}$, Funaki and Spohn proposed to consider the distribution of the gradients $(\nabla\phi(x,y))_{x,y\in\mathbb{Z}^d}$ under ν in the so-called **gradient Gibbs** measures, which in view of the Hamiltonian (3), can also be given in terms of a Dobrushin-Landford-Ruelle description.

Assuming strict convexity of U:

$$(5)\qquad 0 < C_1 \leq U'' \leq C_2 < \infty$$

Funaki and Spohn showed in [6], the existence and uniqueness of ergodic gradient Gibbs measures for every tilt $u \in \mathbb{R}^d$. Moreover, they also proved that the corresponding free energy, or surface tension, $\sigma \in C^1(\mathbb{R}^d)$ is convex. Both results are essential for the derivation of the hydrodynamical limit of the Ginzburg Landau model.

In fact under the strict convexity assumption (5) of U, much is known for the gradient field. At large scales it behaves much like the harmonic crystal or gradient free fields which is a Gaussian field with quadratic U. In particular Naddaf and

Spencer [8] showed that the rescaled gradient field converges weakly as $\epsilon \searrow 0$ to a continuous homogeneous Gaussian field, that is for $f \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^d)$,

$$S_\epsilon(f) = \epsilon^{d/2} \sum_{x \in \mathbb{Z}^d} \sum_{i=1}^{d} (\nabla\phi(x, x+e_i) - u_i) f_i(\epsilon x) \to N(0, \sigma_u^2(f)) \text{ as } \epsilon \to 0.$$

where the convergence takes place under ergodic ν with tilt u (see also Giacomin et al. [7] and Biskup and Spohn [3] for similar results). This scaling limit theorem derived at standard scaling $\epsilon^{d/2}$, is far from trivial, since, as shown in Delmotte and Deuschel [5], the gradient field has slowly decaying, non absolutely summable covariances, of the algebraic order

$$(6) \qquad |cov_\nu(\nabla\phi(x, x+e_i), \nabla\phi(y, y+e_j))| \sim \frac{C}{1 + \|x-y\|^d}, i, j \in \{1, \ldots d\}.$$

The aim of this paper is to relax the strict convexity assumption (5). Our potential is of the form

$$U(\nabla\phi(x,y)) = V(\nabla\phi(x,y)) + g(\nabla\phi(x,y))$$

where $V, g \in C^2(\mathbb{R})$ are such that

$$(7) \qquad C_1 \leq V'' \leq C_2,\ 0 < C_1 < C_2 \text{ and } -C_0 \leq g'' \leq 0, \text{ with } C_0 > C_2$$

and

$$(8) \qquad \|g''\|_{L^1(\mathbb{R})} < \infty \text{ or } \|g''\|_{L^2(\mathbb{R})} < \infty \text{ or } \|g'\|_{L^1(\mathbb{R})} < \infty.$$

Our main result shows that if the inverse temperature β is sufficiently small, that is if:

$$(9) \qquad \sqrt{\frac{\beta}{C_1}} \|g''\|_{L^1(\mathbb{R})} \leq \frac{C_1}{2C_2\sqrt{d}},$$

or

$$(10) \qquad (\beta)^{1/4} ||g''||_{L^2(\mathbb{R})} < \frac{(C_1)^{3/2}}{2(C_2)^{3/4} d^{1/4}}$$

or

$$(11) \qquad (\beta)^{3/4} ||g'||_{L^2(\mathbb{R})} \leq \frac{(C_1)^{3/2}}{2(C_2)^{5/4}} \frac{1}{(2d)^{3/4}},$$

then the results known in the strict convex case hold. In particular we have uniqueness of the ergodic component at every tilt $u \in \mathbb{R}^d$, strict convexity of the surface tension, scaling limit theorem and decay of covariances. As stated above, the hydrodynamical limit for the corresponding Ginzburg-Landau model, should then essentially follow from these results.

Note that uniqueness of the ergodic measures is not true at any β for this type of models: Biskup and Kotecky give an example of non convex U which can be described as the mixture of two Gaussians with two different variances, where two ergodic gradient Gibbs measures coexists at $u = 0$ tilt, cf. Biskup and Kotecky [2]. The situation at lower temperature (i.e. large β) is again quite different: using renormalization group techniques, Adams et al. show the strict convexity for small tilt u, cf. [1].

In a previous paper with S. Mueller, cf. [4], we have proved strict convexity of the surface tension for moderate β in a regime similar to (9). The method used in [4], based on two scale decomposition of the free field, gives less sharp estimates for the temperature, however it is more general and could be applied to non bipartite graphs. In this paper we use a different technique, which relies on the bipartite property of our model. We consider the distribution of the even gradient (that is of $\phi(y) - \phi(x)$ where both x, y are even): which is again a gradient field and show that under the condition (9), that the resulting Hamiltonian is strictly convex. The main idea, similar to [4], is that convexity can be gained via integration; In fact we show more: the Hamiltonian associated to the even variables admits an random walk representation, which is the key tool in deriving covariance estimates such as (6) and scaling limit theorems. The other ingredient is the fact, that given the even gradients, the conditional law of the odd variables is simply a product law. Of course this is a special feature of our bipartite model, in particular it would be quite challenging to iterate the procedure, a scheme which could possibly lower the temperature towards the transition β_c.

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