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Teichmüller Theory

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Abstract. This is a report on the workshop on Teichmüller theory held in Oberwolfach, from November 28 to December 4, 2010. The workshop brought together people working in various aspects of the field, with a focus on recent developments. The topics discussed included higher Teichmüller theory, moduli spaces of flat connections, cluster algebras, quantization of Teichmüller spaces, the dynamical aspects of the Teichmüller and Weil-Petersson geodesic flows, the metric and the boundary theory of Teichmüller space including the new developments on Thurston's asymmetric metric, string topology, geometric analysis on moduli spaces, and relations with three-manifold topology and with minimal surface theory were also highlighted. The mapping class group was also discussed in detail, from various points of view, including its actions on simplicial complexes and on infinite-dimensional Teichmüller spaces, its asymptotic dimension, the relation with the arc operad, the generalizations of the Johnson homomorphisms to the monoid of homology cylinders, making contact with knot theory and with the Casson invariant and other 3-manifolds invariants. There was an open problem session, which is also reported on here.

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Introduction by the Organisers

Teichmüller theory is a subject that has multiple facets, and it is developing in several directions with interesting interactions in low-dimensional topology, algebraic topology, hyperbolic geometry, representations of discrete groups in Lie groups, topological quantum field theory, string theory, arithmetic groups, and in many other fields.

A workshop on this subject, organised by Shygeyuki Morita (Tokyo), Athanase Papadopoulos (Strasbourg) and Robert Penner (Aarhus), was held November 28–December 4, 2010 at Oberwolfach. The topics discussed were Teichmüller theory in a broad sense, including geometric structures, quantization, cluster algebras, higher Teichmüller theory, Kleinian groups, the relation with arithmetic groups, the study of the mapping class group with its various actions and algebraic properties and the relation with 3-manifolds invariants.

The fact that Teichmüller theory makes connections with several areas in mathematics follows in part from the diversity and the richness of the structures that Teichmüller space itself carries, and classical material regarding these structures (Weil-Petersson and Teichmüller metrics, the boundary structure, etc.) was also highlighted.

This workshop was the second one held on the subject at Oberwolfach. The first one took place on May 28–June 3, 2006, with the same organizers. The number of people in the world working in the field has become relatively large, and not more than the third of the participants of the second workshop were already present of the first one. It was the intention of the organizers to include many new participants in the second workshop. The comparison between the two workshops shows that the subject is growing at an exceptional rate, and that many new ideas connections between several domains of mathematics have emerged recently.

The workshop was well attended by 53 participants with broad geographic representation from Europe, Asia and America. Many of the world specialists of the subject were present, and the organisers also kept a balance between young researchers (graduate students, post-docs) and newcomers to the field on the one hand, and senior researchers on the other hand.

The abstract of the lectures that follow provide an excellent image of the developments in the field. There was an open problem session, which is also reported on here.

Workshop: Teichmüller Theory

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Abstracts

Noncommutative Teichmüller spaces and deformation varieties of knot complements

Rinat Kashaev

Let $\Sigma = \Sigma_{g,s}$ be an oriented surface of finite type (of genus g and s punctures) satisfying the condition $(2g-2+s)s > 0$. Penner's λ-coordinates on the decorated Teichmüller space $\widetilde{\mathcal{T}}(\Sigma)$ [5] is a convenient starting point for various algebraic constructions leading to representations of the mapping class group of Σ. We describe one such construction which generalizes the "ratio" coordinates of [1].

A group G is called *group with addition* if it is provided with an associative and commutative binary operation called *addition* with respect to which the group multiplication is distributive.

One can show that no finite group can be a group with addition. The set of positive real numbers $\mathbb{R}_{>0}$ is naturally a group with addition as well as its subgroup of positive rationals $\mathbb{Q}_{>0}$. The group of integers $\mathbb{Z}$ is also a group with addition where the addition is the maximum operation $\max(m,n)$. An example of a non Abelian group with addition is given by the group of upper-triangular real two-by-two matrices with positive reals on the diagonal. The addition here is given by the usual matrix addition.

Theorem 1. *Let G be a group with addition and $c \in G$ a central element (for example, the identity element 1). Then, there exists a canonical homomorphism of the mapping class group of the surface Σ into the group of set-theoretical bijections of the set $G^{4(2g-2+s)}$ such that there exists a Dehn twist whose image is conjugated to the transformation $T\colon G^4 \to G^4$ given by the formula*

$$T(x_1, x_2, y_1, y_2) = (x_1y_1, x_1y_2 + x_2, (1 + y_2x_2^{-1}x_1)^{-1}y_1, (x_2y_2^{-1} + x_1)^{-1}).$$

In the case with $G = \mathbb{R}_{>0}$ and $c = 1$, the corresponding homomorphism is essentially the action of the mapping class group in the Teichmüller space described in terms of the ratio coordinates of [1].

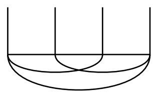
There is a three-dimensional aspect of this homomorphism, where one interprets the surface Σ as the boundary of a three-manifold M with a tangle, embeds a group with addition into a ring R, and chooses $c = -1$. As the combinatorial object underlying the λ-coordinates is an ideal triangulation of Σ, we choose as the starting combinatorial object for M a *Hamiltonian triangulation* to be referred later as H-triangulation. This is a cell decomposition of M where all cells are simplexes, and the tangle is realized as a Hamiltonian sub-complex of it. By extending above theorem to such setting, one can associate to any H-triangulation of M and any ring R, a set of homomorphisms of $\pi_1(M)$ into the group $GL(2, R)$ which eventually can be interpreted as the set of ring homomorphisms from a certain universal ring (associated to the H-triangulation of M) into R. In general, this universal ring is not a topological invariant of M, but one can show that two such universal rings associated to two H-triangulations admit a common localization. By using, the notion of the Δ-groupoid of [2, 3, 4], the universal ring just described is the B'-ring associated to the Δ-groupoid obtained from the underlying H-triangulation, where the Hamiltonian subcomplexe is thrown away. The construction makes sense also in the case where the boundary of M is empty and thus the tangle is a link. Explicit calculations for the trefoil and the figure-eight knots in S^3 show that the universal ring is a (noncommutative) generalization of the deformation variety of the knot complement [6] associated to its ideal triangulation obtained from the H-triangulation by collapsing the knot. To describe these results, let us first give the explicit presentation of the B'-ring in the case of knots in S^3 and H-triangulations with only one vertex.

Given an H-triangulation τ of S^3 with a knot K (given by one edge of τ). Let V be the only vertex of τ and $\tilde{\tau}$ the cell complex of the exterior of V: $X_V = S^3 \setminus N(V)$, where $N(V)$ is a small open ball centered at V. There are two types of 2-cells in $\tilde{\tau}$: triangular and hexagonal faces, the latter being remnants of the faces of τ. There are also two types of edges in $\tilde{\tau}$: *short* edges which bound the triangular faces and *long* edges which are remnants of the edges of τ. The ring $B'(\tau)$ is presented by a set of generators given by associating a pair of generators (u_e, v_e) with each oriented short edge e not intersecting the knot K, and a set of relations:

- if $\overline{e}$ is the short edge e taken with opposite orientation, then $u_{\overline{e}} = u_e^{-1}$, $v_{\overline{e}} = -u_e^{-1} v_e$;
- if $\check{e}$ is the unique oriented short edge such that it belongs to the same hexagonal face as e, and the terminal points of e and $\check{e}$ form the boundary of a long edge, then $u_{\check{e}} = v_e$ and $v_{\check{e}} = u_e$;
- if e_1, e_2, e_3 are cyclically oriented short edges constituting the boundary of a triangular face, then $u_{e_1} u_{e_2} u_{e_3} = 1$ and $u_{e_1} u_{e_2} v_{e_3} + u_{e_1} v_{e_2} + v_{e_1} = 0$.

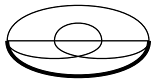
This presentation is such that there exists a canonical homomorphism of the knot group into the group $GL(2, B'(\tau))$ given by associating the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ to each (oriented) long edge not coming from K, and the matrix $\begin{pmatrix} u_e & v_e \\ 0 & 1 \end{pmatrix}$ to each oriented short edge e not intersecting K.

In what follows, the following graphical notation is used for an oriented tetrahedron:



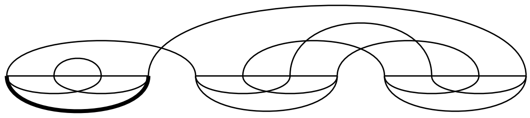
where the lower part of the graph is the tetrahedron itself with the vertices ordered linearly from left to right, while the vertical segments correspond to the faces of the tetrahedron with the order induced from that of the vertices through the bijective correspondence between the faces and the vertices obtained by associating to each face the vertex opposite to it.

The trefoil knot. There exists an H-triangulation τ with one vertex, two edges, and one tetrahedron given by the graph



where the knot is drawn by the thick line. The ring $B'(\tau)$ is isomorphic to the ring $\mathbb{Z}[t, t^{-1}]$. It is a commutative ring isomorphic to the ring of regular functions of the deformation variety.

The figure-eight knot. There exists an H-triangulation τ with one vertex, four edges, and three tetrahedra given by the graph



where the knot is drawn by the thick line. One can show that the ring $B'(\tau)$ is given by the presentation:

$$\mathbb{Z}\langle a, b, c^{\pm 1} |\ a(a+1) = c,\ b(b+1) = c^{-1}\rangle$$

which is four-dimensional over its center. Its abelianization is isomorphic to the ring of regular functions of the deformation variety, with the complete hyperbolic structure corresponding to $c = -1$.

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Various ways of compactifying Teichmüller spaces and end invariants of Kleinian groups

Ken'ichi Ohshika

There are several ways of compactifying Teichmüller spaces, among which are the Thurston compactification, the Teichmüller compactification, the Gardiner-Masur compactification, and the Bers compactification. The former three of these are defined using only two-dimensional structures of Riemann surfaces, whereas the deformation spaces of Kleinian groups are involved in the third. We are interested in relationships between the former kind of compactification to the latter one. More concretely, we should like to determine the Kleinian group appearing as a limit in the Bers compactification for a given sequence in the Teichmüller space whose asymptotic behaviour is expressed in terms of the former kind of compactification. To do this, as we shall see, it will be necessary to invoke the notion of geometric limits, to understand their properties, and to introduce new kinds of compactification.

1. Definition of several kinds of compactification

Let us recall the definition of three kinds of compactification of Teichmüller space. We begin with the Thurston compactification. Regard the points in the Teichmüller space $\mathcal{T}(S)$ as marked hyperbolic structures on S. Then, we can embed $\mathcal{T}(S)$ into the projective space of functions on the set of essential simple closed curves on S, which we denote by $P\mathbb{R}^{\mathcal{S}}$, by setting the value of $\iota(m)$ at $s \in \mathcal{S}$ to be the class of the hyperbolic translation length of s with respect to m. The closure of $\iota(\mathcal{T}(S))$ in $P\mathbb{R}^{\mathcal{S}}$ is compact and is said to be the Thurston compactification of $\mathcal{T}(S)$. Thurston showed the boundary coincides with the projective measured lamination space which is embedded in $PR^{\mathcal{S}}$ by the intersection number. See Fathi-Laudenbach-Poénaru [3] for details.

Replacing the hyperbolic length above with the extremal length, and regarding the points in $\mathcal{T}(S)$ as marked conformal structures, we get the Gardiner-Masur compactification of $\mathcal{T}(S)$. This was first defined by Gardiner-Masur in [4]. The naming is due to Miyachi [8].

Next, let us explain the third one, the Bers compactification. By the simultaneous uniformisation theorem proved by Ahlfors and Bers, the space of quasi-Fuchsian groups (modulo conformal conjugacy) with the topology induced from the representation space is parametrised by $\mathcal{T}(S) \times \mathcal{T}(S)$ using the Ahlfors-Bers map qf. It was shown by Bers [1] that if we fix a point $m_0 \in \mathcal{T}(S)$, then $qf(\{m_0\} \times \mathcal{T}(S))$ is relatively compact in the space of all faithful discrete type-preserving representations of S, which we denote by $AH(S)$. The closure of $qf(\{m_0\} \times \mathcal{T}(S))$ in $AH(S)$ is called the Bers compactification of $\mathcal{T}(S)$. This depends on the choice of m_0 as we shall explain below.

2. Geometric limits and the reduced Bers boundary

For a sequence of Kleinian group $\{G_i\}$ its geometric limit Γ is defined to be a Kleinian group such that any $\gamma \in \Gamma$ is a limit of some sequence $\{g_i \in G_i\}$, and any convergent sequence $\gamma_{i_j} \in G_{i_j}$ has limit in Γ. In our situation, if we consider $G_i = qf(m_0, m_i)$ converging in $AH(S)$, then the sequence $\{G_i\}$ always has a geometrically convergent subsequence, whose limit is a Kleinian group containing the algebraic limit in $AH(S)$ as a subgroup. Therefore, the geometric limits give rise to a new way of compactifying $\mathcal{T}(S)$.

Using geometric limits, Kerckhoff-Thurston showed in [5] that the Bers compactification depends on the basepoint m_0. In particular, they showed that the action of the mapping class group on $\mathcal{T}(S)$ does not extend continuously to the Bers boundary. What causes this dependence on basepoints is the existence of quasi-conformal deformation spaces within the Bers boundary. Therefore it is natural to consider the quotient of the Bers boundary obtained by collapsing quasi-conformal deformation spaces to points.

Definition 1. Let $\partial^B_{m_0}\mathcal{T}(S)$ be the Bers boundary of $\mathcal{T}(S)$ with base point at $m_0 \in \mathcal{T}(S)$. We collapse each quasi-conformal deformation space in $\partial^b_{m_0}\mathcal{T}(S)$ to a point and get its quotient space $\partial^{RB}_{m_0}\mathcal{T}(S)$, which we call the reduced Bers boundary based at m_0.

Theorem 1. *The reduced Bers slice does not depend on the basepoint m_0. In particular the action of the mapping class group on $\mathcal{T}(S)$ extends continuously on its reduced Bers boundary.*

Therefore, we may denote the reduced Bers boundary by $\partial^{RB}(S)$ dropping the basepoint.

After giving a talk in Oberwolfach, I found that Thurston already considered this space, probably back in the 1980's! (See McMullen [6].) Actually, the theorem above has turned out to be an affirmative answer to his conjecture.

3. Maps from some new kinds of compactification

As the result of Kerckhoff-Thurston mentioned above shows, there is no natural map from the Thurston or Masur-Gardiner compactification to the Bers compactification. Still, thanks to recent progress in the theory of Kleinian groups, we have a tool to determine what is its limit in the reduced Bers boundary for a given sequence in the Teichmüller space.

Theorem 2 (A small refinement of a result in [9])**.** *Let $\{m_i\}$ be a divergent sequence in $\mathcal{T}(S)$. Let P_i be a shortest pants decomposition with respect to the hyperbolic structure compatible with m_i, and M_i a shortest clean marking. Regarding P_i as a union of closed geodesics in a fixed hyperbolic surface (S, m_0) and suppose that P_i and M_i converge to geodesic laminations λ and μ in the Hausdorff topology respectively. Suppose that $\{qf(m_0, m_i)\}$ converges to a Kleinian group G' in $AH(S)$. Then, the following hold.*

(1) *Every minimal component of λ that is not a simple closed curve is an ending lamination of $\mathbb{H}^3/G'$.*
(2) *For each component c of λ that is a simple closed curve, if the length of c with respect m_i goes to 0, then c represents a parabolic element of G'.*
(3) *Otherwise c represents a parabolic element if and only if it is also contained in μ.*

Using this theorem, we can define a map as below.

Definition 2. We consider the compactification of $\mathcal{T}(S)$ by defining the limit of $\{m_i\}$ as above to be (λ, μ) for geodesic laminations λ and μ defined above, where a component of λ has weight coming from the limit of the length if there is no leaves spiralling around it. Then we have a compactification $\mathcal{T}(S) \cup (\mathcal{GL}(S)^+ \times \mathcal{GL}(S))$, where $\mathcal{GL}(S)$ is the space of geodesic laminations with the Hausdorff topology and $\mathcal{GL}(S)^+$ is obtained from $\mathcal{GL}(S)$ by attaching to each compact leaf a weight in

$[0, K)$ for some universal constant K. The topology of $\mathcal{GL}^+(S)$ is defined so that if some simple closed curves c_i converge to another one c_∞, the weight of c_∞ is 0 unless $c_i = c_\infty$ for all large i and the limit of weights is positive.

Theorem 3. *There is a continuous map from this compactification*

$$\mathcal{T}(S) \cup (\mathcal{GL}^+(S) \times \mathcal{GL}(S)) \to \mathcal{T}(S) \cup \partial^{RB}\mathcal{T}(S).$$

In contrast to the Hausdorff limit of shortest pants decompositions, the Thurston compactification (or the Masur-Gardiner compactification) ignores the part where the divergence has lower order than the fastest diverging part. To capture the part which diverges in lower orders, we need to consider a refinement of the Thurston compactification, as was done by Morgan-Shalen and Chiswell ([7], [2]). (The same kind of refinement is also possible for the Masur-Gardiner compactification.) We need to refine their construction further to take into consideration the divergence whose logs are not comparable to the fastest diverging part. We denote such a compactification by $\mathcal{MSC}(S)$.

Theorem 4. *There is a well-defined map from $\mathcal{MSC}(S)$ to $\partial^{RB}\mathcal{T}(S)$ which is an extension of the identity on $\mathcal{T}(S)$.*

We note that the topology on the unmeasured lamination space and that of $\partial^{RB}\mathcal{T}(S)$ are different unless $\dim \mathcal{T}(S) = 2$ although we can naturally identify them as sets. I am grateful to Ursula Hamenstädt whose question after my talk made me aware of this fact.

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Fatgraph Nielsen Reduction

Alex James Bene

Nielsen reduction is an algorithm which decomposes any automorphism of a free group into a product of elementary Nielsen transformations. While this may be applied to a mapping class of a surface $S_{g,1}$ with one boundary component, the resulting decomposition in general will not have a topological interpretation. In this talk, we discuss a variation called fatgraph Nielsen reduction which decomposes such a mapping class into elementary Nielsen transformations interpreted as rearrangements of polygon domains for $S_{g,1}$ described by systems of arcs in $S_{g,1}$. These elementary moves generate a the chord slide groupoid of $S_{g,1}$, very closely related to the well-known Ptolemy groupoid, and explicit relations of this groupoid are known.

This talk essentially summarizes the results of [2] and [3], as well as some results of [1]. However, the perspective taken here is different in that we do not emphasize the use of fatgraphs and chord diagrams, but rather choose to focus on the dual notions of triangulations and polygon domains of $S_{g,1}$. In some ways, this perspective is the most natural and classical, and hopefully this will allow for these results to be accessible to a wider audience.

Let us briefly recall Nielsen reduction for an automorphism of a free group F_n on n free generators $\{x_i\}_{i=1}^n$. An *elementary Nielsen transformation* with respect to these generators is an automorphism of F_n given by either permuting two generators: $x_i \mapsto x_j$, inverting some generator: $x_i \mapsto \bar{x}_i$, or multiplying some generator by another: $x_i \mapsto x_i x_j$ for $j \neq i$. *Nielsen reduction* is an algorithm which applies basic cancellation theory to decomposes every $f \in \mathrm{Aut}(F_n)$ into a product of these elementary transformations. Roughly, the algorithm proceeds by continuously applying elementary transformations which reduce the total word length of the generating set $\{f(x_i)\}_{i=1}^n$ with respect to the x_i's whenever possible, with the final goal of obtaining $\{x_i\}_{i=1}^n$. In the event that no length-reducing transformation is available, a lexicographical ordering of F_n is used to ensure progress is made towards this final goal.

Let $S_{g,1}$ be a surface of genus g with one boundary component, and let $\pi = \pi_1(S_{g,1}, p)$ be its fundamental group with respect to a basepoint $p \in \partial S_{g,1}$ on the boundary. The mapping class group $\mathcal{M}_{g,1}$ of $S_{g,1}$ acts on π and can be identified with the subgroup of $\mathrm{Aut}(\pi)$ which preserves the element $\partial S_{g,1} \in \pi$ representing the boundary. Since $\pi \cong F_{2g}$ is a free group, every mapping class $\varphi \in \mathcal{M}_{g,1}$ can be decomposed via Nielsen reduction; however, this decomposition has no obvious topological interpretation.

Instead of considering all free generating sets for π, consider only those which are representable by a system of $2g$ disjointly embedded arcs based at p. (Equivalence classes of) such generating sets can be identified with so-called "polygon domains" for $S_{g,1}$. These polygon domains have $4g+1$ edges, one of which corresponds to the (oriented) boundary $\partial S_{g,1}$ of $S_{g,1}$, the others each corresponding to a one of the $2g$ generators or its inverse. By identifying sides according to their identifications with generators of π, the surface $S_{g,1}$ is reconstructed. Moreover, the natural ordering of the edges of polygon domain P can be used to determine a canonical free generating set of π corresponding to P. Thus, given two polygon domains P_1 and P_2 for $S_{g,1}$, we obtain an element of $\mathrm{Aut}(\pi)$ as the automorphism of π which takes the set of generators corresponding to P_1 to the set corresponding to P_2.

There is a natural operation on polygon domains, called a CS-move (for cut-slide or chord-slide), where we cut off a triangle of the polygon and reattach it to another side according to the identifications with π. It is not hard to see that the corresponding element of $\mathrm{Aut}(\pi)$ is an elementary Nielsen transformation in a natural way. While not all Nielsen transformations are realized in this way, we show in this talk that there are enough of these moves to adapt the Nielsen reduction algorithm to this context. In particular, we show that any two polygon domains can be related by a sequence of CS-moves according to a variation of Nielsen reduction algorithm, which we call *fatgraph Nielsen reduction* (the presence of the term fatgraph in the name arises from a dual perspective, and is not discussed in such detail in this talk).

The proof that a Nielsen reduction algorithm exists for polygon domains follows much along the lines of the proof of the usual Nielsen reduction. In particular, the algorithm proceeds by repeatedly attempting to apply a CS-move which reduces the total word length of a free set of generators for π with respect to some target generating set. As in the classical case, this is not always possible, and when no such length-reducing move exits, the algorithm relies on an *energy* function which is essentially a numeric version of the standard lexicographical ordering of π extending the usual word length function. The heart of the proof lies in showing that always either a word length-reducing move or an energy-reducing move exists. This relies heavily on cancellation theory and the "combinatorics" of the word representing $\partial S_{g,1}$ in the generators corresponding to a polygon domain.

Motivated by this application to Nielsen reduction, we introduce the *chord slide groupoid* $\mathfrak{CS}_{g,1}$, the groupoid naturally generated by CS moves, which can be thought of as a groupoid laying somewhere "between" the mapping class group $\mathcal{M}_{g,1}$ and the full automorphism group $\mathrm{Aut}(\pi)$. This groupoid is a close cousin to the well-known Ptolemy groupoid. More precisely, while the Ptolemy groupoid corresponds to ideal triangulations of $S_{g,1}$ and their mutations via diagonal flips, the chord slide groupoid corresponds to polygon domains of $S_{g,1}$ and their mutation via CS-moves. In the dual perspective, this can be rephrased as saying that the Ptolemy groupoid is generated by Whitehead moves on trivalent fatgraphs, while the chord slide groupoid is generated by chord-slides on linear chord diagrams. Please see the references [2], [3], and [1] for more details.

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Uniformization of minimal singular surfaces

Sumio Yamada

The classical Plateau Problem is the problem of finding a surface minimizing the area among all surfaces which are images of a map from a disk and spanning a given Jordan curve. We can formulate this problem more precisely as follows. Let $\triangle$ be the unit disk in $\mathbf{R}^2$. The area of a map $\alpha : \triangle \to \mathbf{R}^n$ is

$$A(\alpha) = \int_\triangle \sqrt{\left|\frac{\partial\alpha}{\partial x}\right|^2 \left|\frac{\partial\alpha}{\partial y}\right|^2 - \Big(\frac{\partial\alpha}{\partial x}\cdot\frac{\partial\alpha}{\partial y}\Big)^2}\, dxdy.$$

The classical Plateau Problem. *Given a Jordan curve Υ, let*

$$\mathcal{F} = \{\alpha : \overline{\triangle} \to \mathbf{R}^n : \alpha \in W^{1,2}(\triangle) \cap C^0(\overline{\triangle}) \text{ and } \alpha\big|_{\partial\triangle} : \partial\triangle \to \Upsilon \text{ is a homeomorphism}\}.$$

Find $\alpha^* \in \mathcal{F}$ so that $A(\alpha^*) \le A(\alpha)$ for all $\alpha \in \mathcal{F}$.

The classical Plateau Problem can be solved by finding a map α^* which is conformal and harmonic. The study of classical minimal surfaces via harmonic maps is now well developed, starting with the solution of this problem by J.Douglas in the 1930's. (T. Rado also solved the Plateau problem independently.)

Our interest is a singular version of this problem. In [MY1], we investigated properties of 1-dimensional (2-dimensional resp.) subsets of a Euclidean space which minimize length (area resp.) among all the sets sharing a certain boundary set. Due to the shape of the boundary set, the minimizer is not a curve (surface resp.) but a union of curves (surfaces resp.). A variational formulation of this problem was presented in [MY1] as a minimization problem for a certain energy functional of maps from a suitable polyhedral domain (or a finite complex). We refer to a singular surface as a union of the images of three maps, each from a disk, joined along a curve γ and bounded by a curvilinear graph Γ as pictured below. The boundary set Γ in this picture is the graph consisting of three edges meeting at two vertices, and the curve γ is the set where three disk-type surfaces are meeting. In the singular Plateau Problem, we look for a singular surface which minimizes area among all other singular surfaces bounding the given graph Γ.

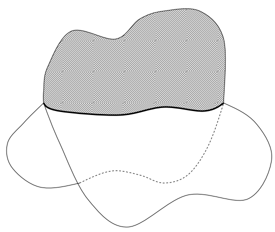

Figure 1. Singular Surface

In order to capture an area-minimizer as an energy-minimizer in our singular setting where several (three) energy functionals are being minimized concurrently, we introduced in [MY1] a variational problem involving a weighted l^2-energy. As the name suggests, the weighted l^2-energy functional is a quadratic combination of Dirichlet energies with respect to some weights. The minimizer in the spaces of maps, weights, and conformal structures/glueing maps is shown to be an area minimizer. Furthermore, this area minimizing map is weighted l^2-energy minimizing and weakly conformal.

A well-known soap-film model giving a singular surface minimizing the area is the $(M, 0, \delta)$-minimal sets in $\mathbf{R}^3$ (cf. [Alm], [T]). J.Taylor [T] showed that in the interior of a $(M, 0, \delta)$-minimal set, there are only two types of singularities. One is the **Y**-type, characterized by three minimal surfaces meeting at $120°$ degrees along a curve (which we will call a free boundary). Each free boundary is shown in [T] to be $C^{1,\alpha}$. The other is the **T**-type, characterized by four of those free boundary curves meeting at a vertex, whose tangent cone is the cone over the regular tetrahedron and each free boundary is $C^{1,\alpha}$ up to the vertex point $(0 < \alpha < 1)$.

We study a construction of the singular configuration of three surfaces meeting along a free boundary as an image of a weighted l^2-energy minimizing map from a singular domain which is homeomorphic to three standard 2-simplices glued along a 1-simplex. The key technical aspect in this approach is that there is an infinite dimensional Teichmüller space $\mathcal{P}$ of conformal structures defined on the domain. Here the infinite dimensionality is caused by the different ways of glueing the three copies of the 2-simplices. A known fact that the free boundary is real analytic in the $(M, 0, \delta)$-minimal setting suggests that the class of glueing maps associated with area minimizers should consist of smooth elements. In particular we consider $C^{2,\alpha}$ glueing maps. We proved [MY2] a regularity result for a minimizing map for a weighted l^1-energy functional where this functional is a linear combination of Dirichlet energies with respect to some weights. More specifically, given a $C^{2,\alpha}$ glueing map of 2-simplices, the associated energy minimizing map is real analytic up to the common 1-simplex. In particular, if an area minimizers is the energy minimizing element associated to a $C^{2,\alpha}$ glueing map, then the resulting minimal singular surface has real analytic free boundary.

Real analyticity appearing in the minimal surface theory is often related to the theory of the single variable complex analysis. One of the basic theorems of two-dimensional surfaces is the existence of isothermal coordinates, which says that a Riemannian surface is a Riemann surface. The isothermal coordinate system around each point is obtained by solving the Beltrami equation. We call this parameterization a local uniformization. When a planar domain Ω is minimally embedded in an Euclidean space by $x(u, v) = (x^1(u, v), x^2(u, v), \cdots, x^n(u, v))$, it is well known that the coordinate functions $\{x^i(u, v)\}$ are harmonic functions. Hence the map $x : \Omega \to \mathbf{R}^n$ is conformal and harmonic.

In our singular setting [MY2], we can show that an analogy to the classical surface theory holds. Recall that a map $f = (f_1, f_2, f_3) : Y_0 \to \mathbf{R}$ is said harmonic if it is minimizing with respect to l^1-energy function with the same weights for

$i = 1, 2, 3$. We demonstrate that if three surfaces Σ_1, Σ_2 and Σ_3 meet non-tangentially along a singular curve γ and each surface is real analytic up to γ, then there is an isothermal coordinate system around each point on γ.

When we additionally assume the singular surface $S = (\cup\Sigma_i) \cup \gamma$ is minimal (i.e. each surface Σ_i is minimal and the three minimal surfaces meet along a curve at 120^o angles), then the isothermal coordinate system is harmonic [MY2].

An important difference between the classical and singular Plateau problems is that in the former case, the harmonic conformal parameterization is of the entire minimal disk while in the latter case, we only assert the existence of a neighborhood of a **Y**-type singular point conformally equivalent to Y_0, which is harmonically parameterized. This is because any two disk-type surfaces are *globally* conformally equivalent by the uniformization theorem which has no analog in the singular setting. From this viewpoint, the key observation we have learned from [MY1] and [MY2] is that *locally there is only one conformal structure* Y_0 to the singular surfaces while *globally there are many*, parameterized by the infinite dimensional Teichmüller space $\mathcal{P}$. We further note that the conformal structure Y_0 is conformally equivalent to the trivial element X_{Id} (where the glueing is by the identity map) of $\mathcal{P}$.

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Coarse Schottky problem and geometric analysis on moduli spaces of Riemann surfaces

Lizhen Ji

In this talk, we discussed several problems related to the geometry and spectral theory of the moduli spaces of Riemann surfaces.

(1) The coarse Schottky problem, i.e., the description of the Jacobian varieties in the space of abelian varieties in the sense of coarse geometry of Gromov.
(2) The vanishing of simplicial volume of the moduli spaces of Riemann surfaces.
(3) The spectral theory of the Weil-Petersson metric of the moduli spaces of Riemann surfaces.

In some sense, these results deal with properties of the moduli space of Riemann surfaces from a perspective different from the extensive results on the moduli spaces in algebraic geometry and topology.

Schottky problem and the coarse Schottky problem.

Let Σ_g be a compact Riemann surface of genus g, and $\mathcal{M}_g$ be the moduli space of Σ_g. Let $\mathfrak{h}_g$ be the Siegel upper space of degree g, $\mathfrak{h}_g = \{X + iY \mid X, Y \text{ real symmetric } n \times n\text{-matrices}, Y > 0\}$. Then the symplectic group $\mathrm{Sp}(2g, \mathbb{R})$ acts holomorphically and transitively on $\mathfrak{h}_g$, and the stabilizer of iI_g is isomorphic to $\mathrm{U}(g)$. Therefore, $\mathfrak{h}_g = \mathrm{Sp}(2g, \mathbb{R})/\mathrm{U}(g)$. The Siegel modular group $\mathrm{Sp}(n, \mathbb{Z})$ acts properly on $\mathfrak{h}_g$, and the quotient $\mathrm{Sp}(2g, Z)\backslash\mathfrak{h}_g$ is called the Siegel modular variety. It can be identified with the moduli space of principally polarized abelian varieties of dimension g and is hence denoted by $\mathcal{A}_g$.

For each compact Riemann surface Σ_g, its Jacobian variety $J(\Sigma_g)$ is a principally polarized abelian variety of dimension g. This defines the Jacobian map

$$J : \mathcal{M}_g \to \mathcal{A}_g.$$

It can also be defined in terms of the periods of Σ_g with respect to sympletic bases of $H_1(\Sigma_g, \mathbb{Z})$. Specifically, a basis $a_1, b_1; \cdots ; a_g, b_g$ of $H_1(\Sigma_g, \mathbb{Z})$ is called a symplectic basis if

$$a_i \cdot a_j = 0, b_i \cdot b_j = 0, a_j \cdot b_j = \delta_{ij}.$$

For each symplectic basis, there exists a unique basis $\omega_1, \cdots, \omega_g$ of $H^0(\Sigma_g, \Omega^1)$ satisfying $\int_{a_i} \omega_j = \delta_{ij}$. Let $\Omega_{ij} = \int_{b_i} \omega_j$. Then $\Omega = (\Omega_{ij})$ is called the period of Σ_g. The Riemann relations imply that $\Omega \in \mathfrak{h}_g$. Since the symplectic basis of $H_1(\Sigma_g, \mathbb{Z})$ is well-defined up to $\mathrm{Sp}(2g, \mathbb{Z})$, this gives a well-defined *period map*

$$\mathcal{M}_g \to \mathrm{Sp}(2g, \mathbb{Z})\backslash\mathfrak{h}_g,$$

which coincides with the Jacobian map above.

The Torelli theorem says that the Jacobian map $J : \mathcal{M}_g \to \mathcal{A}_g$ is injective. The image $J(\mathcal{M}_g)$ is called the Jacobian locus. When $g = 1$, the Jacobian variety of Σ_g is isomorphic to itself, and the map $J : \mathcal{M}_1 \to \mathcal{A}_1$ is an isomorphism. For general g, it is known that $J(\mathcal{M}_g)$ is an algebraic subvareity of A_g.

Since $\dim_{\mathbb{C}} \mathcal{M}_g = 3g - 3$ and $\dim_{\mathbb{C}} \mathcal{A}_g = \frac{g(g+1)}{2}$, the Jacobian locus $J(\mathcal{M}_g)$ is a subvariety of strictly smaller dimension than $\mathcal{A}_g$ if and only if $g \geq 4$.

The celebrated Schottky problem is to characterize $J(\mathcal{M}_g)$ inside $\mathcal{A}_g$, or equivariantly to characterize Jacobian varieties among abelian varieties. It was raised formally by Schottky in 19th century, though Riemann asked this question before.

After a lot of work, it was finally solved by T. Shiota in the 1980s via a solution of the Novikov conjecture which states that an abelian variety is a Jacobian variety if and only if its Riemann theta function satisfies a specific nonlinear differential equation, a so-called KP-equation.

But given an explicit abelian variety, it is difficult to check whether the condition is satisfied and to decide whether the abelian variety is a Jacobian variety. Motivated by the problem of constructing explicit examples of abelian varieties that are not Jacobian varieties, in 1994, Buser and Sarnak showed that with respect to the metric d of $\mathcal{A}_g$, which is induced from the invariant metric of $\mathfrak{h}_g$, the Jacobian locus $J(\mathcal{M}_g)$ is contained in an exponentially small neighborhood of the boundary $\partial\mathcal{A}_g$ as $g \to +\infty$.

Motivated by the result of Buser and Sarnak, Benson Farb raised the **Coarse Schottky problem**: *understand the size of $J(\mathcal{M}_g)$ inside $\mathcal{A}_g$ in the sense of the coarse geometry, or more specifically, the induced image of $J(\mathcal{M}_g)$ in the tangent cone at infinity of $\mathcal{A}_g$.*

In a joint work with Leuzinger [JL], we proved

Theorem 1. *For every g, there exists a positive constant c_g such that $J(\mathcal{M}_g)$ is c_g-dense in $\mathcal{A}_g$ with respect to the metric d: for any $x \in \mathcal{A}_g$, there exists some $y \in J(\mathcal{M}_g)$ such that $d(x, y) \leq c_g$. Therefore, the induced image of $J(\mathcal{M}_g)$ in the tangent cone of $\mathcal{A}_g$ is equal to the whole cone.*

In view of the result of Buser-Sarnak, $c_g \to +\infty$, when $g \to +\infty$. The precise asymptotic of c_g is not known.

Simplicial volume of the moduli spaces $\mathcal{M}_g$.

The simplicial volume of manifolds was introduced by Gromov in the 1980s to give new homotopy invariants. For a connected oriented compact manifold, it is defined to be the infimum of the ℓ^1-norm of all cycles with coefficients in $\mathbb{R}$ that represent the fundamental class. For a connected oriented noncompact manifold, it can be similarly defined using the fundamental class in the locally finite homology group.

Given a hyperbolic manifold of finite volume, its simplicial volume is proportional to the hyperbolic volume by a universal positive constant. This was used by Gromov to give a new proof of the Mostow strong rigidity for hyperbolic manifolds.

For a general manifold, the simplicial volume gives a lower bound for the minimal volume of the manifold up to a universal positive constant. Therefore, given a manifold, a natural problem is to determine whether its simplicial volume vanishes or not.

It is known that for a compact locally symmetric space of noncompact type, its simplicial volume is positive [LS] [BK] and the simplicial volume of an arithmetic locally symmetric space of $\mathbb{Q}$-rank at least 3 vanishes [LoS].

Motivated by this result, we proved in [J] the following vanishing result.

Theorem 2. *The simplicial volume of $\mathcal{M}_g$ is equal to 0 for all $g \geq 1$.*

With a few exceptions, the simplicial volume of $\mathcal{M}_{g,n}$ vanishes as well.

Spectral theory of the Weil-Petersson metric of the moduli spaces $\mathcal{M}_{g,n}$.

The Weil-Petersson metric of $\mathcal{M}_{g,n}$ is incomplete. A basic result in spectral geometry is that for any complete Riemannian metric, its Laplacian is essentially self-adjoint, i.e., its Laplacian admits a unique self-adjoint extension. If the manifold is compact, then the extended Laplacian has only a discrete spectrum, whose counting function satisfies the Weyl law.

In a joint work with R. Mazzeo, W. Müller and A. Vasy, we proved the following result.

Theorem 3. *The Laplacian of $\mathcal{M}_{g,n}$ with respect to the Weil-Petersson metric is essentially self-adjoint, and the extended Laplacian has only a discrete spectrum, and its counting function satisfies the Weyl law.*

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Asymptotic dimension of mapping class groups is finite

Koji Fujiwara

(joint work with Mladen Bestvina and Ken Bromberg)

1. Asymptotic dimension

The *asymptotic dimension* $\text{asdim}(\mathcal{X})$ of a metric space $\mathcal{X}$ is said to be $\leq n$ if for every $R > 0$ there is a covering of $\mathcal{X}$ by sets U_i such that every metric R-ball in $\mathcal{X}$ intersects at most $n+1$ of the U_i's, and $\sup \text{diam}\, U_i < \infty$. This definition is due to Gromov [4] and it is invariant under quasi-isometries (or even coarse isometries). In particular, asymptotic dimension of a finitely generated group is well-defined. It is not hard to see that $\text{asdim}(\mathbb{R}^2) \leq 2$ by considering the usual "brick decomposition" of $\mathbb{R}^2$ (with large bricks), and more generally, $\text{asdim}(\mathbb{R}^n) \leq n$. This inequality is also easily seen using the product formula $\text{asdim}(\mathcal{X} \times \mathcal{Y}) \leq \text{asdim}(\mathcal{X}) + \text{asdim}(\mathcal{Y})$.

Gromov proved that δ-hyperbolic groups have finite asymptotic dimension. Here is a proof. Assume that $R \gg \delta$ is an integer. For every vertex v in the Cayley graph of the group at distance $5kR$ from 1, $k = 1, 2, 3, \cdots$, consider the set

$$U_v = \{x \in \Gamma \mid d(1,x) \in [5(k+1)R, 5(k+2)R] \text{ and } v \text{ lies on some geodesic } [1,x]\}$$

An easy thin triangle argument shows that if v, w are two vertices at distance $5kR$ such that both U_v and U_w intersect the same R-ball, then $d(v,w) \leq 2\delta$. This gives a bound on the number of U_v's that can intersect the same R-ball, and this bound is independent of R; thus $\text{asdim}(\Gamma) < \infty$. We can also apply this argument to a tree T to show that $\text{asdim}(T) \leq 1$.

2. Main results

Bell-Fujiwara [2] modified the argument by Gromov to show that curve complexes of surfaces of finite type have finite asymptotic dimension. They are hyperbolic by the celebrated work of Masur-Minsky [5], but not locally finite, resulting in an infinite bound. The trick is to use *tight* geodesics in place of arbitrary geodesics. Finiteness properties of tight geodesics proved by Bowditch [3] imply that asymptotic dimension is finite.

Let Σ be a possibly punctured closed surface, $MCG(\Sigma)$ its mapping class group, and $\mathcal{T}(\Sigma)$ its Teichmüller space equipped with the Teichmüller metric. The following is the main result in [1].

Theorem 1. $\mathrm{asdim}(MCG(\Sigma)) \leq \mathrm{asdim}(\mathcal{T}(\Sigma)) < \infty$.

We do not know any estimates from above. A reasonable guess would be that $\mathrm{asdim}(MCG(\Sigma)) = v.c.d.(MCG(\Sigma))$ and $\mathrm{asdim}(\mathcal{T}(\Sigma)) = \dim(\mathcal{T}(\Sigma))$, where v.c.d. is the virtual cohomological dimension and dim is the topological dimension.

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String topology and compactified moduli spaces

Kate Poirier

Our goal is to understand the algebraic topology of manifolds. We would like to determine algebraic structure on $H_* = H_*^{S^1}(LM, M)$, the S^1-equivariant homology of the free loop space LM of a closed, compact, complete Riemannian manifold M, relative to its subspace of constant loops $M \subset LM$. We may do so by finding a solution to the quantum master equation $\partial X = X * X$ where $X \in \oplus_{k,\ell} Hom(P_*^{\otimes k}, P_*^{\otimes \ell})$, P_* is a chain complex whose homology is H_* and $*$ is properadic composition. Homotopical algebra tells us that the solution X passes to a solution on homology, again called $X \in \oplus_{k,\ell} Hom(H_*^{\otimes k}, H_*^{\otimes \ell})$. The master equation here becomes $0 = X * X$. The solution X is composed of an infinite number of k-to-ℓ operations and the master equation gives a concise way to package an infinite number of quadratic relations among these operations. (There is, for example, a 2-to-1 operation on H_* satisfying the Jacobi relation.) Presently, using string diagrams and a compactification of the moduli space of Riemann surfaces, we can build a solution to a related equation $\partial X = X * X + A$ where A can be eliminated in many examples and likely in general.

Roughly, a string diagram of type (g, k, ℓ) is a graph constructed from a disjoint union of k parametrized *input* circles and chords whose endpoints are attached to the circles. There is a cyclic order of the half-edges adjacent to each vertex such that the resulting ribbon surface has genus g and $k+\ell$ boundary components, k of which are isotopic to the k input circles. The remaining ℓ boundary components are called *output* circles. The space $\overline{SD}(g, k, \ell)$ turns out to be a cell complex and we construct a map

$$ST : C_*(\overline{SD}(g, k, \ell)) \to Hom(P_*^{\otimes k}, P_*^{\otimes \ell})$$

called the *string topology construction* from the complex of cellular chains of $\overline{SD}(g, k, \ell)$ to the homomorphism complex between k-th and ℓ-th tensor powers of the chain complex P_*. The image of a cellular chain is called a *string topology operation.* It is important to note that for all (g, k, ℓ), $\overline{SD}(g, k, \ell)$ is a compact space and that each string topology operation is a k-to-ℓ operation on the chain complex P_*, not on its homology H_*; compare with [2, 3, 4, 5].

To build the solution X, we enlarge the space of graphs under consideration and extend the string topology construction ST to the larger space. Roughly, a string diagram with levels of type (g, k, ℓ) is a graph whose construction is similar to that of a string diagram of type (g, k, ℓ), but now we remember at what stage, or level, a chord is attached and we assign a spacing parameter $s_i \in (0, 1]$ between the ith and $i+1$st levels. We denote the space of such diagrams by $\overline{LD}(g, k, \ell)$. We also introduce an equivalence relation $\sim$ on $\overline{SD}(g, k, \ell)$ and a corresponding one on $\overline{LD}(g, k, \ell)$ to obtain the following.

Proposition 1. (1) *The space $\overline{LD}(g, k, \ell)/\sim$ is an orientable pseudomanifold with boundary. Its dimension is $6g - 6 + 3k + 3\ell - 1$.*

(2) *String diagrams with levels on the boundary of $\overline{LD}(g, k, \ell)/\sim$ either have an output consisting only of chords or have a spacing parameter $s_i = 1$ for some i. (We call this "good boundary.")*

(3) *$\overline{LD}(g, k, \ell)/\sim$ contains $\overline{SD}(g, k, \ell)/\sim$ as a deformation retract.*

(4) *Let $\mathfrak{M}(g, k, \ell)$ be the moduli space of Riemann surfaces of genus g and $k+\ell$ punctures (which have each been decorated in some way). Then there is an injective continuous map $\mathfrak{M}(g, k, \ell) \to \overline{LD}(g, k, \ell)/\sim$ whose image is a union of open cells and is dense.*

The map $\mathfrak{M}(g, k, \ell) \to \overline{LD}(g, k, \ell)/\sim$ is similar to one found in [1]. Note that if we define $\overline{LD}(g, k, \ell)/\sim$ as a compactification $\overline{\mathfrak{M}}(g, k, \ell)$ of $\mathfrak{M}(g, k, \ell)$, then $\overline{\mathfrak{M}}(g, k, \ell)$ contains $\overline{SD}(g, k, \ell)/\sim$ as a deformation retract.

We would like to extend the string topology construction ST to $\overline{LD}(g, k, \ell)/\sim$. The equivalence relations $\sim$ on $\overline{SD}(g, k, \ell)/\sim$ and $\overline{LD}(g, k, \ell)/\sim$ are cellular and so it makes sense to consider equivalent cellular chains. If the string topology construction ST were the same for cellular chains of $\overline{SD}(g, k, \ell)$, then we could extend it to $C_*(\overline{LD}(g, k, \ell)/\sim)$ by assigning compositions of string topology operations to $s = 1$ boundary components and then, using a homotopy, by interpolating the construction defined on $C_*(\overline{SD}(g,k,\ell)/\sim)$ and cellular chains on the good boundary. If this were the case, we could then define $X(g,k,\ell)$ as the image of the fundamental chain of $\overline{LD}(g,k,\ell)/\sim$ under ST. As ST is a chain map, we would then have the relations $\partial X(g,k,\ell) = \sum X(g',k',\ell') * X(g'',k'',\ell'')$, where each $s = 1$ boundary component of $\overline{LD}(g,k,\ell)/\sim$ gives rise to one term of the sum and each term of the sum corresponds to a particular composition. Then, letting $X = \sum X(g,k,\ell)$ we would obtain $\partial X = X * X$.

The string topology operations arising from equivalent chains of $\overline{SD}(g,k,\ell)/\sim$ need not agree, however, they differ by a boundary in $Hom(P_*^{\otimes k}, P_*^{\otimes \ell})$. This again allows us to again enlarge the space of graphs and extend the string topology construction over the new extension. We denote the enlarged space of graphs by $\overline{\mathbb{LD}}(g,k,\ell)$. Again, it is a pseudomanifold with boundary and its boundary exactly describes terms in the master equation. However, $\overline{\mathbb{LD}}(g,k,\ell)$ may have boundary that is not good, that is, a region of the boundary such that the corresponding string topology operation is not a composition. The consequence is an unwanted term in the master equation. Our goal is then to eliminate such boundary from $\overline{\mathbb{LD}}(g,k,\ell)$. We have in the following cases and expect to be able to eliminate such terms in general.

Theorem 1. *Let $X(g,k,\ell)$ be the image of the fundamental chain of $\overline{\mathbb{LD}}(g,k,\ell)$ under the extended string topology construction. Then*

(1) $\partial X(g,k,\ell) = \sum X(g',k',\ell') * X(g'',k'',\ell'') + A(g,k,\ell)$
(2) $A(g,k,\ell) = 0$ *if* $(g,k,\ell) = (0,k,1), (0,1,\ell), (0,2,2)$ *and* $(1,1,1)$.

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Non-varying sum of Lyapunov exponents for the Teichmüller geodesic flow

Martin Möller
(joint work with Dawei Chen)

The Teichmüller geodesic flow acts on the tangent bundle to the moduli space of curves, which may be identified with the vector bundle of quadratic differentials on curves of genus g. This flow preserves differentials that are global squares of abelian differentials. The dynamics there is interesting enough and we restrict to this subspace. Riemann surfaces together with an abelian differential (X, ω) are also called *flat surfaces*, since the abelian differentials provides a flat metric on the surface. The bundle of flat surfaces is naturally stratified by the number and multiplicity of the zeros of ω. The strata have up to three components, distinguished by a parity of a spin structure and by consisting exclusively of hyperelliptic curves ([KZ03]). On flat surfaces, there is an action of $\mathrm{SL}_2(\mathbb{R})$ extending the obvious action on planar polygons. The Teichmüller geodesic flow is the action of the diagonal subgroup.

Lyapunov exponents measure the growth rate of the Hodge norm of cohomology classes under parallel transport along the Teichmüller geodesic flow. More precisely, given a measure μ on the space of flat surfaces, there is a decreasing filtration $E_i$of the bundle with fiber $H^1(X, \mathbb{R})$ over (X, ω) and a decreasing set of numbers λ_i, called Lyapunov exponents, such that for μ-almost every (X, ω) the logarithmic growth rate of a vector in $E_i \setminus E_{i+1}$ is precisely λ_i.

Motivations for studying Lyapunov exponents in this particular setting stem from deviations of ergodic averages on individual flat surfaces and from counting problems for closed geodesics on flat surfaces, see [Zo06], [EKZ] and the reference in there.

Lyapunov exponents of dynamical systems are often hard to calculate and the Teichmüller geodesic flow is no exception to this rule. However, the sum of Lyapunov exponents is more accessible. For the natural measure μ_{gen} supported on connected components of strata this sum can be calculated using [EKZ] together with [EMZ03].

But the knowledge of the (sum of) Lyapunov exponents for μ_{gen}, i.e. for a *generic* flat surface is of little use, if one wants to calculate this quantity for a *particular* flat surface. This applies in particular for the *Veech surfaces*, surfaces whose $\mathrm{SL}_2(\mathbb{R})$-orbit is closed and thus supports its own natural measure that stems from the Haar measure. These orbits are called Teichmüller curves.

More than ten years ago, Kontsevich and Zorich observed by computer experiments that for hyperelliptic strata and for strata in low genus the sum of Lyapunov exponents is *non-varying*, i.e. it is the same for all Teichmüller curves as for the ambient connected component of the stratum. This conjecture has recently been proved almost completely.

Theorem 1 ([EKZ])**.** *For all hyperelliptic strata the sum of Lyapunov exponents in non-varying.*

Theorem 2 ([CM]). *For the strata* $(4), (3,1), (2,2), (2,1,1)$ *in genus* $g = 3$*, the strata* $(6), (5,1), (3,2,1), (3,3), (2,2,2)^{\text{odd}}$ *in genus* $g = 4$ *and the strata* $(8), (5,3)$ *in genus* $g = 5$ *the sum of Lyapunov exponents in non-varying. With the exception possibly of* $(4,2)^{\text{even}}$*, of* $(4,2)^{\text{odd}}$ *and of* $(6,2)^{\text{odd}}$*, for all the remaining non-hyperelliptic strata in genus* $g \leq 5$ *the sum of Lyapunov exponents is varying.*

For the strata $(4,2)^{\text{even}}$, $(4,2)^{\text{odd}}$ and $(6,2)^{\text{odd}}$ there is strong numerical evidence that they are indeed non-varying. Work on these strata is in progress.

The strategy of proof of both theorems is very different. The main result of [EKZ] compares the sum of Lyapunov exponents with Siegel-Veech constants. Since Siegel-Veech constants behave in a controlled way under double coverings, the result on hyperelliptic curves is a corollary.

The result of [CM] starts with the observation that a sum of Lyapunov exponents can be calculated comparing the intersection number of a Teichmüller curve with the Hodge class to the intersection number with the boundary divisor in the moduli space of curves. To do so, it suffices, for each stratum, to find a divisor that Teichmüller curves in the given stratum cannot intersect and whose class in the Picard group can be calculated by other techniques, i.e. by appropriate test curves. It is for most cases not possible to find such a divisor in the moduli space of curves, but only in the moduli space of curves with one or two marked points.

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Teichmüller spaces and holomorphic maps

Hiroshige Shiga

1. Introduction

Let X be a hyperbolic Riemann surface and $T(X)$ the Teichmüller space of X. We are interested in holomorphic maps φ from a complex manifold B to $T(X)$ or to the moduli space $M(X) := T(X)/Mod(X)$, where $Mod(X)$ is the mapping class group of X. We are interested in those maps with the following topics:

Holomorphic family of Riemann surfaces: The map φ assigns to each point $p \in B$ a Riemann surface $\varphi(p)$. Since φ is holomorphic, φ defines a holomorphic family of Riemann surface over B;

Holomorphic motion: When X is a Riemann surface of genus 0 with $n(> 3)$ punctures, the movement of each puncture by φ defines a holomorphic map from B to the Riemann sphere $\hat{\mathbb{C}}$. Those maps give a *holomorphic motion* of the punctures over B.

In the following sections, we will explain our recent research on these topics.

2. Holomorphic family of Riemann surfaces

Let B be a hyperbolic Riemann surface of finite type. We may take a torsion free finitely generated Fuchsian group Γ_B acting on the unit disk Δ so that $B = \Delta/\Gamma_B$. A triple (Φ, θ, B) is said to define a holomorphic family of Riemann surfaces over B if it satisfies the following conditions:

(1) Φ is a non-constant holomorphic map from Δ to $T(X)$;
(2) θ is a homomorphism from Γ_B to $Mod(X)$;
(3) the map Φ and the homomorphism θ satisfy

$$\Phi \circ \gamma(z) = \theta(\gamma) \circ \Phi(z)$$

for any $\gamma \in \Gamma_B$ and for any $z \in \Delta$.

It is known that the solutions of the Diophantine problem in function fields arise from holomorphic families of Riemann surfaces. Hence, the number of holomorphic families is one of the main interests. In our research, we estimate the number of holomorphic families of Riemann surfaces over B in terms of the hyperbolic structure of B;

Theorem 1. *Let B be a Riemann surface of type (p, k). Suppose that the injectivity radius of any point in B_{thick} is greater than $r > 0$. Then the number of holomorphic families of Riemann surfaces of type (g, n) over B is less than*

$$\alpha e^{\beta r^{-\delta}|\log r|},$$

where $\alpha, \beta, \delta > 0$ are constants depending only on g, n, p and k.

3. Holomorphic motions

Definition 1. Let V be a connected complex manifold with a basepoint x_0 and let E be a closed subset of the Riemann sphere $\hat{\mathbb{C}}$ containing $0, 1$ and ∞. A *holomorphic motion of E over V* is a map $\phi\colon V \times E \to \hat{\mathbb{C}}$ that has the following three properties:

(1) $\phi(x_0, z) = z$ for all z in E,
(2) the map $\phi(x, \cdot)\colon E \to \hat{\mathbb{C}}$ is injective for each x in V, and
(3) the map $\phi(\cdot, z)\colon V \to \hat{\mathbb{C}}$ is holomorphic for each z in E.

A holomorphic motion ϕ is called *normalized* if $\phi(x,\cdot)$ fixes $0, 1$ and ∞ for any $x \in V$. Without loss of generality, we may assume that a holomorphic motion is always normalized.

Holomorphic motion plays an important role in complex dynamics as well as in the theory of Kleinian groups. One of the main problems in holomorphic motion theory is the extendability, that is, we consider when a holomorphic motion of E over V etends to a holomorphic motion of $\hat{\mathbb{C}}$ over V. A celebrated theorem of Slodkowski asserts that if V is the unit disk Δ, then every holomorphic motion of E over Δ is extended to a holomorphic motion of $\hat{\mathbb{C}}$ over Δ. However, if V is a higher dimensional space, the same statement as Slodkowski's theorem does not hold in general even if V is simply connected.

On the other hand, it is not hard to see that the extendability of a holomorphic motion ϕ of a closed set E is reduced to the extendability of holomorphic motions restricted to finite subsets of E. Therefore, in considering the extendability, we may consider only holomorphic motions of finite subsets of $\hat{\mathbb{C}}$.

Let $E = \{0, 1, \infty, a_1, \ldots, a_n\}$ and ϕ be a normalized holomorphic motion of E over V. Then, $\phi_j(x) := \phi(x, a_j)$ are holomorphic functions on V $(j = 1, 2, \ldots, n)$ and a map :

$$V \ni x \mapsto R_x := \hat{\mathbb{C}} \setminus \{0, 1, \infty, \phi_1(x), \ldots, \phi_n(x)\}$$

gives a holomorphic family of Riemann surfaces of type $(0, n+3)$ over V. Hence, holomorphic maps on Teichmüller spaces and monodromies also appear as in §2.

Let $K \subset \Delta$ be an AB-removable compact subset of Δ and $E \supset \{0, 1, \infty\}$ a closed subset of $\hat{\mathbb{C}}$. We write $\Delta_K := \Delta \setminus K$. By using Teichmüller theory, we have the following theorems([1]);

Theorem 2. *Let $\phi : \Delta_K \times E \to \hat{\mathbb{C}}$ be a holomorphic motion. Suppose that the monodromy of the restriction of ϕ to $\Delta_K \times E'$ is trivial for any finite subset E' of E, containing* $0, 1$ *and* ∞. *Then, ϕ can be extended to a holomorphic motion of $\hat{\mathbb{C}}$.*

Theorem 3. *Suppose that every connected component of $\hat{\mathbb{C}} \setminus E$ is neither simply connected nor conformally equivalent to the punctured disk. Then, every holomorphic motion of E over Δ_K can be extended to a holomorphic motion of $\hat{\mathbb{C}}$ over Δ_K.*

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Asymptotic Nielsen realization problem and stable quasiconformal mapping class group

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(joint work with Katsuhiko Matsuzaki)

Throughout this report, we assume that a Riemann surface R admits a hyperbolic structure. We consider the group $\mathrm{QC}(R)$ of all quasiconformal automorphisms of R and the quasiconformal mapping class group $\mathrm{MCG}(R)$ of all homotopy equivalence classes of elements of $\mathrm{QC}(R)$. Then we have a surjective homomorphism $q : \mathrm{QC}(R) \to \mathrm{MCG}(R)$. The *realization problem* for the quasiconformal mapping class group $\mathrm{MCG}(R)$ asks whether there exists a homomorphism $\mathcal{E} : \Gamma \to \mathrm{QC}(R)$ such that $q \circ \mathcal{E} = \mathrm{id}|_\Gamma$ for a given subgroup Γ of $\mathrm{MCG}(R)$. Furthermore, the *Nielsen realization problem* is the realization problem for a finite subgroup of $\mathrm{MCG}(R)$.

The quasiconformal mapping class group $\mathrm{MCG}(R)$ acts on the Teichmüller space $T(R)$ of R biholomorphically. Then we have a homomorphism ι_T from $\mathrm{MCG}(R)$ to the biholomorphic automorphism group $\mathrm{Aut}(T(R))$ of $T(R)$, and define the *Teichmüller modular group* for R by $\mathrm{Mod}(R) = \iota_T(\mathrm{MCG}(R))$. The homomorphism ι_T is bijective for all Riemann surfaces R of non-exceptional type. In particular, $\mathrm{Mod}(R) = \mathrm{Aut}(T(R))$.

For an analytically finite Riemann surface, Kerckhoff [4] proved the following fixed point theorem.

Proposition 1. *Let R be an analytically finite Riemann surface. Then a subgroup of* $\mathrm{Mod}(R)$ *is finite if and only if it has a common fixed point in $T(R)$.*

Since an element of $\mathrm{Mod}(R)$ having a fixed point $p \in T(R)$ is realized as a conformal automorphism of the Riemann surface R_p corresponding to the fixed point, Proposition 1 is equivalent to the statement that every finite subgroup of $\mathrm{MCG}(R)$ can be realized as a group of conformal automorphisms of R_p. Since the groups $\mathrm{QC}(R)$ and $\mathrm{QC}(R_p)$ are quasiconformally conjugate, this gives an affirmative answer to the Nielsen realization problem.

The Nielsen realization problem is also true for an analytically infinite Riemann surface. Indeed, a generalization of the fixed point theorem to analytically infinite Riemann surfaces has been given by Markovic [5].

Proposition 2. *Let R be a Riemann surface in general. For a subgroup G of* $\mathrm{Mod}(R)$*, the orbit $G(p)$ is bounded for some $p \in T(R)$ if and only if G has a common fixed point in $T(R)$.*

We consider an asymptotic version of the Nielsen realization problem for the asymptotic Teichmüller modular group acting on the asymptotic Teichmüller space. The asymptotic Teichmüller space $AT(R)$ is defined in a similar manner to the Teichmüller space by replacing conformal equivalence with asymptotically conformal equivalence. The quasiconformal mapping class group $\mathrm{MCG}(R)$ acts on $AT(R)$ biholomorphically. Then we have a homomorphism ι_{AT} from $\mathrm{MCG}(R)$ to the biholomorphic automorphism group $\mathrm{Aut}(AT(R))$ of $AT(R)$, and define the *asymptotic Teichmüller modular group* for R by

$$\mathrm{Mod}_{AT}(R) = \iota_{AT}(\mathrm{MCG}(R)).$$

It is different from the case of the representation ι_T that the homomorphism ι_{AT} is not injective, namely, $\mathrm{Ker}\,\iota_{AT} \neq \{[\mathrm{id}]\}$ unless R is either the unit disc or the once-punctured disc. We call $\mathrm{Ker}\,\iota_{AT}$ the *asymptotically trivial mapping class group*.

For our statements, we need the following geometric condition on Riemann surfaces.

Definition 1. We say that a Riemann surface R satisfies the *bounded geometry condition* if the injectivity radii at all points in R are uniformly bounded from above and below.

Now we state our main theorem, which can be regarded as an asymptotic version of the fixed point theorem.

Theorem 1 ([3]). *Let R be a Riemann surface satisfying the bounded geometry condition. Then every finite subgroup of* $\mathrm{Mod}_{AT}(R)$ *has a common fixed point in* $AT(R)$.

This gives the Nielsen realization for the asymptotic Teichmüller modular group as in Theorem 2 below. For its statement, we use the following definition.

Definition 2. We say that two quasiconformal automorphisms of a Riemann surface R are *end equivalent* if they coincide outside some topologically finite geodesic subsurface of R. The *end quasiconformal automorphism group* $\mathrm{QC}_e(R)$ is the group of all end equivalence classes of quasiconformal automorphisms of R. Furthermore, the *end conformal automorphism group* $\mathrm{Conf}_e(R)$ is the subgroup of $\mathrm{QC}_e(R)$ consisting of all end equivalence classes that have representatives conformal outside some topologically finite geodesic subsurfaces of R.

Let $e : \mathrm{QC}(R) \to \mathrm{QC}_e(R)$ be the natural projection. Also we have a surjective homomorphism $q_e : \mathrm{QC}_e(R) \to \mathrm{Mod}_{AT}(R)$ and the following commutative diagram.

$$\begin{array}{ccc} \mathrm{QC}(R) & \xrightarrow{\ e\ } & \mathrm{QC}_e(R) \\ \Big\downarrow q & & \Big\downarrow q_e \\ \mathrm{MCG}(R) \cong \mathrm{Mod}(R) & \xrightarrow{\ \iota_{AT}\ } & \mathrm{Mod}_{AT}(R) \end{array}$$

Theorem 1 with its proof yields the following theorem.

Theorem 2 ([3]). *Let R be an analytically infinite Riemann surface satisfying the bounded geometry condition. Then every finite subgroup $\hat{\Gamma}$ of* $\mathrm{Mod}_{AT}(R)$ *can be realized in the end conformal automorphism group* $\mathrm{Conf}_e(R_p)$ *of a Riemann surface R_p which is quasiconformally equivalent to R. In particular, there exists a homomorphism $\mathcal{E}_e : \hat{\Gamma} \to \mathrm{QC}_e(R)$ such that $q_e \circ \mathcal{E}_e = \mathrm{id}|_{\hat{\Gamma}}$.*

The proof of Theorem 1 is based on a topological characterization of the asymptotically trivial mapping class group as in Theorem 3 below.

Definition 3. The *stable quasiconformal mapping class group* $G_\infty(R)$ is a subgroup of $\mathrm{MCG}(R)$ consisting of all essentially trivial mapping classes. Here a quasiconformal mapping class $[g] \in \mathrm{MCG}(R)$ is said to be *essentially trivial* if there exists a topologically finite geodesic subsurface V_g of R such that, for each connected component W of $R - V_g$, the restriction $g|_W : W \to R$ is homotopic to the inclusion map $\mathrm{id}|_W : W \hookrightarrow R$ relative to the ideal boundary at infinity.

It is clear that $G_\infty(R) \subset \mathrm{Ker}\, \iota_{AT}$. The inclusion is not necessarily an equality. However, under the bounded geometry condition of Riemann surfaces, we have a complete characterization of $\mathrm{Ker}\, \iota_{AT}$.

Theorem 3 ([2])**.** *Let R be a Riemann surface satisfying the bounded geometry condition. Then* $\mathrm{Ker}\, \iota_{AT} = G_\infty(R)$.

In the last part of this report, we determine the asymptotic Teichmüller modular group as the biholomorphic automorphism group of a certain Teichmüller space by using the characterization of the asymptotically trivial mapping class group. Note that the homomorphism ι_{AT} is not surjective.

Definition 4. The *intermediate Teichmüller space* of a Riemann surface R is defined by

$$IT(R) = T(R)/\mathrm{Ker}\, \iota_{AT}.$$

If R is an analytically finite Riemann surface, then $IT(R)$ coincides with the moduli space of R. If R is the unit disc, then $T(\mathbb{D}) = IT(\mathbb{D})$. Hereafter, we assume that R is a topologically infinite Riemann surface satisfying the bounded geometry condition. Then $IT(R) = T(R)/G_\infty(R)$ by Theorem 3. Since $G_\infty(R)$ acts on $T(R)$ discontinuously and freely by [1, Theorem 5.1], the intermediate Teichmüller space has a complex manifold structure induced from that of the Teichmüller space. We consider the biholomorphic automorphism group $\mathrm{Aut}(IT(R))$ of $IT(R)$. Then we have the following characterization of the asymptotic Teichmüller modular group.

Theorem 4 ([2])**.** *Let R be a Riemann surface satisfying the bounded geometry condition. Then* $\mathrm{Mod}_{AT}(R)$ *is geometrically isomorphic to* $\mathrm{Aut}(IT(R))$.

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On Teichmüller spaces for surfaces of infinite topological type

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(joint work with Lixin Liu, Athanase Papadopoulos, Weixu Su, Zongliang Sun)

In this work we studied Teichmüller spaces for surfaces of infinite topological type using geometric techniques, e.g. hyperbolic structures, pairs of pants decompositions and Fenchel-Nielsen coordinates. For the details you can see the original papers [1], [2], [3]. Here, by Teichmüller spaces we mean reduced Teichmüller spaces, a definition that is more suited to be studied using such techniques (as opposed to the techniques of complex analysis). (For the definition of the non-reduced spaces, see [6]).

In several ways, surfaces of infinite topological type are much more complicated than surfaces of finite type. They can't be classified by the genus and the number of punctures only, but there is a nice classification theorem, see [8]. The most important topological property we need is that every orientable surface of infinite type can be decomposed into pairs of pants, i.e. there exists a system of simple closed curves $(C_i) \subset S$ such that $S \setminus \bigcup_i C_i$ is a disjoint union of spheres minus three holes. This can be proved easily using the mentioned classification theorem.

To define Fenchel-Nielsen coordinates on Teichmüller spaces we have to show that every complex structure on S can be constructed by gluing hyperbolic pairs of pants. First we need to associate a hyperbolic metric to every complex structure, and for this we will use the **intrinsic metric**, defined by Bers; see [1] for details. Then, we need to characterize the hyperbolic metrics that can be constructed by gluing hyperbolic pairs of pants. Note that on surfaces of infinite type, it is not true that any topological pair of pants decomposition gives rise to a hyperbolic pairs of pants decomposition. The following theorem gives such a characterization.

Theorem 1. ([1]). *Let (S, h) be an orientable surface with a hyperbolic metric. The following are equivalent.*

1. *(S, h) can be constructed by gluing hyperbolic pairs of pants.*
2. *(S, h) is a convex core hyperbolic metric.*
3. *For every topological pairs of pants decomposition $(C_i) \subset S$, there exists a pairs of pants decomposition $(\gamma_i) \subset S$ such that for all i, γ_i is a geodesic homotopic to C_i.*

Note that the intrinsic metric on every complex structure is always a convex core hyperbolic metric, hence this theorem can be applied to it. The implication (2) $\Rightarrow$ (1) was proved in [4], but here we need (2) $\Rightarrow$ (3), in order to define Fenchel-Nielsen coordinates on Teichmüller spaces.

Now let's discuss the definition of Teichmüller spaces in this context. Let Σ be a fixed orientable surface of infinite type. One would like to define the Teichmüller space of Σ in the following way:

$$\mathcal{T}(\Sigma) = \{(f, X) \mid X \text{ is a Riemann surface and } f : \Sigma \to X \text{ is a diffeo } \}/\sim$$

where $(f, X) \sim (f', X')$ if and only if there exists a biholomorphism $h : X \to X'$ such that $h \circ f$ is isotopic to f'. Note that this equivalence relation is the one giving the reduced theory of Teichmüller spaces; it is the most natural definition, but it is not the most widely used.

The set $\mathcal{T}(\Sigma)$ defined in this way parametrizes the complex structures on Σ, but it is not easy to find interesting structures on this set. To define distances, we need to consider some subsets of this set containing only comparable complex structures. To compare complex structures, we will use some functionals R defined over the diffeomorphisms $h : X \to Y$, and satisfying the following properties: $0 \leq R(h) \leq \infty$, $R(h) = 0 \iff h$ is a biholomorphism, and the triangle inequality $R(h \circ h') \leq R(h) + R(h')$. Given a functional h with these properties, one can define a distance in the following way:

$$d_R((f, X), (f', X')) = \inf_h R(h) \leq \infty$$

where the infimum is taken over all the h such that $h \circ f$ is isotopic to f'.

Then we need to choose a base point $\mathbb{X}_0 = (f_0, X_0)$, and we can define the following subset of $\mathcal{T}(\Sigma)$:

$$\mathcal{T}_R(\mathbb{X}_0) \quad = \quad \{(f, X) \mid d_R(\mathbb{X}_0, (f, X)) < \infty\}/\sim \quad \subset \quad \mathcal{T}(\Sigma)$$

The pair $(\mathcal{T}_R(\mathbb{X}_0), d_R)$ is a metric space.

We will discuss different definitions of Teichmüller spaces, for different choices of R. The most important one is when R is the quasiconformal dilatation of h, $R(h) = qc(h) = \log(K(h))$, where K is the quasiconformal constant of h. This gives rise to what we call the **quasiconformal Teichmüller space**, denoted by $(\mathcal{T}_{qc}(\mathbb{X}_0), d_{qc})$, a complete metric space. Another possibility is to use the length-spectrum dilatation, $R(h) = ls(h)$, defined by the following formula, for a diffeomorphism $h : X \to Y$:

$$ls(h) = \sup_\alpha \left\{ \left| \log \frac{\ell_Y(h(\alpha))}{\ell_X(\alpha)} \right| \right\}$$

where the sup is over the set of all simple closed curves α. This gives rise to what we call the **length-spectrum Teichmüller space**, denoted by $(\mathcal{T}_{ls}(\mathbb{X}_0), d_{ls})$. It is also possible to use Fenchel-Nielsen coordinates to define R. If $h : X \to Y$ is a diffeomorphism, and $(C_i) \subset X$ is a pairs of pants decomposition, also $(h(C_i)) \subset Y$ is a pairs of pants decomposition, and we can compare the Fenchel-Nielsen coordinates in the following way:

$$R(h) = FN(h) = \sup_i \max \left(\left| \log \frac{\ell_Y(h(C_i))}{\ell_X(C_i)} \right|, |\tau_Y(h(C_i)) - \tau_X(C_i)| \right)$$

After having chosen a fixed pairs of pants decomposition $(C_i) \subset \Sigma$ of our base topological surface, this gives rise to what we call the **Fenchel-Nielsen Teichmüller space**, denoted by $(\mathcal{T}_{FN}(\mathbb{X}_0), d_{FN})$. This space depends on the chosen pairs of pants decomposition of Σ, but it has a clear structure, with explicit coordinates, and it is isometric to the sequence space ℓ^∞. We will use this space to describe the others.

To do this we need some hypotheses on the base point of the space. A Riemann surface X is **upper bounded** with reference to a pairs of pants decomposition

$(C_i) \subset X$, if there exists a constant M such that for all i we have $\ell(C_i) \leq M$. We have the following:

Theorem 2. (See [1]). *If* $\mathbb{X}_0$ *is upper bounded, then* $\mathcal{T}_{qc}(\mathbb{X}_0) = \mathcal{T}_{FN}(\mathbb{X}_0)$ *and the identity map* $id : (\mathcal{T}_{qc}(\mathbb{X}_0), d_{qc}) \to (\mathcal{T}_{FN}(\mathbb{X}_0), d_{FN})$ *is locally bi-Lipschitz. In particular* $\mathcal{T}_{qc}(\mathbb{X}_0)$ *is locally bi-Lipschitz equivalent to the sequence space* ℓ^∞.

The last remark should be compared with a recent result of A. Fletcher ([5]) giving a similar property for non-reduced Teichmüller spaces.

Now let's see some properties of the length-spectrum Teichmüller space.

Theorem 3. (See [2]). *The metric space* $(\mathcal{T}_{ls}(\mathbb{X}_0), d_{ls})$ *is complete.*

A very important fact is an inequality due to Wolpert (see [11]) that we can state as $d_{ls} \leq d_{qc}$. This also implies that $\mathcal{T}_{qc}(\mathbb{X}_0) \subset \mathcal{T}_{ls}(\mathbb{X}_0)$.

Shiga studied the length spectrum metric on the quasiconformal Teichmüller space and he introduced the following condition that we name after him: a Riemann surface X satisfies the **Shiga's condition** with reference to a pairs of pants decomposition $(C_i) \subset X$, if there exists a constant M such that for all i we have $\frac{1}{M} \leq \ell(C_i) \leq M$. Under this condition on the basepoint, he proved that d_{ls} and d_{qc} induce the same topology on $\mathcal{T}_{qc}(\mathbb{X}_0)$ (see [9]).

Later, Liu and Papadopoulos introduced the length-spectrum Teichmüller space, and they proved that under the same condition, the two spaces are the same set (see [7]). We refined these results as follows:

Theorem 4. (See [3]). *If* $\mathbb{X}_0$ *satisfies Shiga's condition, then* $\mathcal{T}_{ls}(\mathbb{X}_0) = \mathcal{T}_{qc}(\mathbb{X}_0) = \mathcal{T}_{FN}(\mathbb{X}_0)$ *and the identity maps between any two of these spaces are locally bi-Lipschitz, with reference to the respective distances* d_{ls}, d_{qc}, d_{FN}. *In particular* $\mathcal{T}_{ls}(\mathbb{X}_0)$ *is locally bi-Lipschitz equivalent to the sequence space* ℓ^∞.

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Domains of Discontinuity for Anosov Representations

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(joint work with Olivier Guichard)

1. Motivation

The concept of *Anosov representations* has been introduced by F. Labourie [8] in his study of Hitchin representations of surface groups. Anosov representations $\rho : \Gamma \to G$ can be defined for any word-hyperbolic group Γ into any semisimple (real) Lie group (see Definition 1 below). When Γ is a free group or a surface group, Anosov representations should be thought of providing generalizations of quasi-Fuchsian representations. The goal of this talk is to describe a geometric picture for Anosov representation similar to the following classical examples.

(1) **Teichmüller space:** Let S be a closed surface. The Teichmüller space $\mathcal{T}(S)$ can be realized as a connected component in the representation variety $\mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$ consisting of discrete embeddings. Any such representation gives rise to an action of $\pi_1(S)$ on the hyperbolic plane $\mathbb{H}^2$, which is properly discontinuous, free and with compact quotient. The quotient is the surface S endowed with a hyperbolic structure.
(2) **Quasi-Fuchsian space:** Embedding $\mathrm{PSL}(2,\mathbb{R})$ into $\mathrm{PSL}(2,\mathbb{C})$ a neighborhood of $\mathcal{T}(S)$ is given by the space of Quasi-Fuchsian representations $\mathcal{QF}(S) \subset \mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2,\mathbb{C}))/\mathrm{PSL}(2,\mathbb{C})$. Every Quasi-Fuchsian representation $\rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ admits a ρ- equivariant embedding $\xi : S^1 \to \mathbb{CP}^1$. On the complement $\Omega = \mathbb{CP}^1 \backslash \xi(S^1)$, the action of $\pi_1(S)$ (via ρ) is properly discontinuous, free and with compact quotient. The quotient consists of two connected components, which are both surfaces homeomorphic to S, naturally endowed with a $\mathbb{CP}^1$-structure.

Examples of Anosov representations include so called higher Teichmüller spaces, i.e. Hitchin representations or positive representations into split real Lie groups (e.g. $\mathrm{SL}(n,\mathbb{R})$) and maximal representations into Lie groups of Hermitian type (e.g. $\mathrm{Sp}(2n,\mathbb{R})$), as well as their "Quasi-Fuchsian" deformations into complex Lie groups, [7, 4, 8, 3, 2].

The main result discussed here, is a construction of domains of discontinuity with compact quotient for all Anosov representations. As a consequence, we associate deformation space of geometric structures on compact manifolds to all higher Teichmüller spaces.

2. Definition and Results

Let Γ be a word-hyperbolic group and denote by $\partial\Gamma$ its boundary. Further let G be a semisimple (real) Lie group.

Definition 1 (with a little bit of cheating)**.** Let $\rho : \Gamma \to G$ be a representation, let $B < G$ be a Borel subgroup. Then ρ is said to be an Anosov representation if and only if there exists a continuous ρ-equivariant *transverse* map $\xi : \partial\Gamma \to G/B$.

Definition 2. A map $\xi : \partial\Gamma \to G/B$ is said to be *transverse* if for all $t \neq t' \in \partial\Gamma$ the pair $(\xi(t), \xi(t'))$ is contained in the unique open G-orbit in $G/B \times G/B$.

Remark 1.
(1) In general one would fix a parabolic subgroup $P < G$ and define the class of P-Anosov representations. For simplicity we restrict here to the class of B-Anosov representations.
(2) Definition 1 is almost equivalent to the original definition of Anosov representations, it is equivalent for representations with Zariski-dense image. (See [6] for a detailed discussion.)
(3) The set of Anosov representations is open in the representations variety. Anosov representations have finite kernel and discrete image. [8, 6].

In order to state the results, let us recall the Iwasawa decomposition $G = KAN$ and the corresponding decomposition of the Borel group as $B = MAN$.

Theorem 1. *Let $\mathbb{F}_n$ be the free group on $n \geq 2$ letters. Let $\rho : \mathbb{F}_n \to G$ be an Anosov representation. Then there exists a non-empty open set $\Omega \subset G/AN$ such that the action of $\mathbb{F}_n$ on Ω is properly discontinuous, free and with compact quotient.*

Let $\pi_1(S)$ be the fundamental group of a closed surface of genus ≥ 2. Let G be a semisimple Lie group. Assume that G does not have a compact factor and that no factor is locally isomorphic to $\mathrm{PSL}(2, \mathbb{R})$*. Then there exists a non-empty open set $\Omega \subset G/AN$ such that the action of $\pi_1(S)$ on Ω is properly discontinuous, free and with compact quotient.*

Corollary 1. *Higher Teichmüller spaces parametrize geometric structures on compact manifolds.*

3. Examples

(1) Let $\rho : \pi_1(S) \to \mathrm{Sp}(2n, \mathbb{R})$ be a maximal representation, and denote by $\mathrm{Lag}(\mathbb{R}^{2n})$ the space of Lagrangians. Then there exists a continuous ρ-equivariant transverse map $\xi : S^1 \to \mathrm{Lag}(\mathbb{R}^{2n})$ (see [1]). In this situation the domain of discontinuity can actually be constructed in $\mathbb{RP}^{2n-1}$. Namely, let $K_\xi = \bigcup_{t \in S^1} \mathbb{P}(\xi(t))$ and set $\Omega = \mathbb{RP}^{2n-1} \backslash K_\xi$. Then $\pi_1(S)$ acts on Ω properly discontinuously, freely and with compact quotient. The quotient manifold M is homeomorphic to the total space of an $\mathrm{O}(n)/\mathrm{O}(n-2)$-bundle over S.
(2) The previous construction with the same conclusion works also for Hitchin representations $\rho : \pi_1(S) \to \mathrm{SL}(2n, \mathbb{R})$.

(3) For Hitchin representation $\rho : \pi_1(S) \to \mathrm{SL}(2n+1, \mathbb{R})$ the situation is more complicated. Several non-equivalent domains of discontinuity can be constructed in the space of full flags $\mathcal{F}$ of $\mathbb{R}^{2n+1}$. The simplest domain of discontinuity can be constructed in the space $\mathcal{F}_{1,2n}$, consisting of a line and an incident hyperplane. For this let $\xi : S^1 \to \mathcal{F}$ be the continuous ρ-equivariant transverse map. Let

$$K_\xi = \bigcup_{t \in S^1} \{(l, H) \in \mathcal{F}_{1,2n} \mid \exists 1 \leq k < l \leq 2n+1 \text{ such that } l \subset \xi^k(t) \text{ and } \xi^{l-1}(t) \subset H\},$$

and set $\Omega = \mathcal{F}_{1,2n} \backslash K_\xi$. Then $\pi_1(S)$ acts on Ω properly discontinuously, freely and with compact quotient. We have not yet determined the homeomorphism type of the quotient for $n \geq 2$.

(4) Let $\mathbb{F}_n$ be realized as the fundamental group of a non-compact surface S. Let $\rho : \mathbb{F}_n \to \mathrm{PSL}(2, \mathbb{C})$ be a Quasi-Fuchsian representation, then the quotient of Ω by $\mathbb{F}_n$ is the double of the surface S.

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The horofunction boundary of Thurston's Lipschitz metric

Cormac Walsh

The horofunction boundary of a metric space was introduced by Gromov in the late 1970s. To construct it for a metric space (X, d), one assigns to each point $z \in X$ the function $\psi_z : X \to \mathbb{R}$,

$$\psi_z(x) := d(x, z) - d(b, z),$$

where b is some basepoint. If X is a proper geodesic metric space, then the map $\psi : X \to C(X)$, $z \mapsto \psi_z$ defines an embedding of X into $C(X)$, the space of continuous real-valued functions on X endowed with the topology of uniform convergence on compact sets. The horofunction boundary is defined to be $X(\infty) := \mathrm{cl}\{\psi_z \mid z \in X\} \backslash \{\psi_z \mid z \in X\}$, and its elements are called horofunctions.

Applications of the horofunction boundary include the study of isometric group actions, non-expansive maps, random walks, and quantum groups. It also provides a good general setting for Patterson-Sullivan measures.

The definition can be extended without too much difficulty to non-symmetric metrics, such as the Thurston's Lipschitz metric on Teichmüller space. When the metric is non-symmetric there are actually two horofunction boundaries, one in the forward direction and one in the backward direction.

Recently, I have determined the horofunction boundary (in the forward direction) of Teichmüller space with Thurston's Lipschitz metric [5]. This metric is defined to be

$$L(x,y) := \log \inf_{\phi \approx \mathrm{Id}} \sup_{p \neq q} \frac{d_y(\phi(p), \phi(q))}{d_x(p,q)}, \qquad \text{for } x, y \in \mathcal{T}(S).$$

In other words, the distance from x to y is the logarithm of the smallest Lipschitz constant over all homeomorphisms from x to y that are isotopic to the identity. Thurston [4] showed that this is indeed a metric, although a non-symmetric one. He also showed that this distance can be written

$$L(x,y) = \log \sup_{\alpha \in \mathcal{S}} \frac{\ell_y(\alpha)}{\ell_x(\alpha)},$$

where $\mathcal{S}$ is the set of isotopy classes of non-peripheral simple closed curves on S, and $\ell_x(\alpha)$ denotes the shortest length in the metric x of a curve isotopic to α.

The Lipschitz metric has not been as intensively studied as Teichmüller's metric. Most of the literature on it is very recent. See [2] for a survey.

In the preprint [5], I show that the horofunction boundary of Thurston's Lipschitz metric is just the usual Thurston boundary.

An immediate consequence is that all geodesics in this metric space converge in the forward direction to a point in the Thurston boundary. This had only been known previously for a special class of geodesics.

An important application of the horofunction boundary is to study isometric group actions. It is useful for this purpose because the action of the isometry group of a metric space extends continuously to an action by homeomorphisms on the horofunction compactification.

I used this action to determine the isometry group of Teichmüller space with Thurston's Lipschitz metric, except in certain special cases of low genus. It is easy to see that the elements of the mapping class group Mod_S are isometries of the Lipschitz metric. I showed [5] that, if S is not a sphere with four or fewer punctures, nor a torus with two or fewer punctures, then every isometry of $\mathcal{T}(S)$ with Thurston's Lipschitz metric arises as an element of Mod_S.

This theorem is an analogue of Royden's theorem concerning the Teichmüller metric, which was proved by Royden [3] in the case of compact surfaces and analytic automorphisms of $\mathcal{T}(S)$, and extended to the general case by Earle and Kra [1]. My proof is inspired by Ivanov's proof of Royden's theorem, which was global and geometric in nature, as opposed to the original, which was local and analytic.

Using similar techniques, I was also able to show that distinct surfaces give rise to distinct Teichmüller spaces, again with some possible exceptions in low genus. Denote by $S_{g,n}$ a surface of genus g with n punctures. I showed that the only possible isometric equivalences between surfaces of different genus or with a different number of punctures are $T(S_{1,1}) \cong T(S_{0,4})$, $T(S_{1,2}) \cong T(S_{0,5})$, and $T(S_{2,0}) \cong T(S_{0,6})$. I do not know whether these isometric equivalences actually hold.

If one takes instead the Teichmüller metric, the same theorem holds and it is known that the three exceptional isometric equivalences actually hold. This is a theorem of Patterson.

It would also be interesting to work out the horofunction boundary of the reversed Lipschitz metric, that is, the metric $L^*(x, y) := L(y, x)$. Since L is not symmetric, L^* differs from L, and one would expect their horofunction boundaries to also differ.

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Mapping Class Groups do not have Kazhdan's Property (T)

Jørgen Ellegaard Andersen

We construct a counter example to Kazhdan's property (T) [Kazh] for the mapping class groups of closed oriented surfaces of genus at least two by using the Reshetikhin-Turaev Topological Quantum Field Theory constructed in [RT1], [RT2] and [Tu]. These TQFT-constructions by Reshetikhin and Turaev was given on the basis of the suggestions by Witten in [W], which gave a quantum field theory description of the Jones polynomial [J].

Indeed we shall need the geometric constructions of these TQFT's as proposed by Witten in that paper and by Atiyah in [At]. That the geometric construction gives the same representations as the Reshetikhin-Turaev TQFT representations follow from combining the results of [L] and [AU1], [AU2], [AU3] and [AU4]. In fact this identifies the geometrically constructed representations with the TQFT representations constructed by Blanchet, Habegger, Masbaum and Vogel in [BHMV1] and [BHMV2], which is the skein theory construction of the RT-TQFT's.

Let us briefly recall the geometric construction of these representations of the mapping class group.

Let Σ be a closed oriented surface of genus at least two. Let p be a point on Σ. Let M be the moduli space of flat $SU(2)$ connections on $\Sigma - p$ with holonomy around p equal $-\mathrm{Id} \in SU(2)$. This moduli space is smooth and has a natural symplectic structure ω. There is natural smooth symplectic action of the mapping class group Γ of Σ on M. More over there is a unique prequantum line bundle $(\mathcal{L}, \nabla, (\cdot,\cdot))$ over (M,ω). The Teichmüller space $\mathcal{T}$ of complex structures on Σ naturally Γ-equivariantly parametrizes Kähler structures on (M,ω). For $\sigma \in \mathcal{T}$, we denote (M,ω) with is corresponding Kähler structure M_σ.

By applying geometric quantization to the moduli space M, one gets a certain finite rank bundle over Teichmüller space $\mathcal{T}$, which we will call the *Verlinde* bundle $\mathcal{V}_k$ at level k, where k is any positive integer. The fiber of this bundle over a point $\sigma \in \mathcal{T}$ is $\mathcal{V}_{k,\sigma} = H^0(M_\sigma, \mathcal{L}^k)$.

The main result pertaining to this bundle $\mathcal{V}_k$ is that its projectivization $\mathcal{V}_k$ supports a natural flat Γ-invariant connection $\hat\nabla$. This is a result proved independently by Axelrod, Della Pietra and Witten [ADW] and by Hitchin [H]. This flat connection $\hat\nabla$ induces a *flat* connection $\hat\nabla^{\mathrm{e}}$ in $\mathrm{End}(\mathcal{V}_k)$. Let $\mathrm{End}_0(\mathcal{V}_k)$ be the subbundle consisting of traceless endomorphisms. The connection $\hat\nabla^{\mathrm{e}}$ also induces a connection in $\mathrm{End}_0(\mathcal{V}_k)$, which is invariant under the action of Γ.

We get this way for each k a finite dimensional representation of Γ, namely on the covariant constant sections, say $\tilde{\mathcal{H}}_k$, of $\mathrm{End}_0(\mathcal{V}_k)$ over $\mathcal{T}$. Let

$$\tilde{\mathcal{H}} = \bigoplus_{k+2 \text{ prime}}^{\infty} \tilde{\mathcal{H}}_k$$

on which Γ acts. From the proof of the asymptotic faithfulness in [A1], one see that this representation of Γ is faithful.

Each of the vector spaces $\tilde{\mathcal{H}}_k$ has a natural positive definite Hermitian structure $[\cdot,\cdot]$, which is preserved by the action of Γ. This Hermitian structure is clear from the skein theory construction of $\tilde{\mathcal{H}}_k$ following [BHMV2]:

Consider the BHMV-TQFT (as defined in [BHMV2]) at $A = \exp(2\pi i/4(k+2))$. The label set for this theory is then $L_k = \{0, 1, \dots, k\}$. We denote by Z_k the vector space this theory associates to $\Sigma \sqcup \overline{\Sigma}$ with $p \in \Sigma$ labeled by the last color k in both copies of Σ.

Since the vector space Z_k is part of a TQFT, there is in particular an action of the mapping class group of $\Sigma \sqcup \overline{\Sigma}$ on Z_k. There is a natural homomorphism of Γ into the mapping class group of $\Sigma \sqcup \overline{\Sigma}$ given by mapping $\phi \in \Gamma$ to $\phi \sqcup \phi$.

In [BHMV2] a Hermitian structure $\{\cdot,\cdot\}$ is constructed on Z_k, which is invariant under the action of the mapping class group of $\Sigma \sqcup \overline{\Sigma}$ and therefore also invariant under the action of Γ. For the choice of A made above, it is proved in [BHMV2], that the Hermitian structure $\{\cdot,\cdot\}$ is positive definite.

By the work of Andersen and Ueno [AU1], [AU2], [AU3] and [AU4] combined with the work of Laszlo [L], we have an identification of the two constructions.

Theorem 1 (Andersen & Ueno). *There is a natural* Γ*-equivariant isomorphism*

$$I_k : Z_k \to \tilde{\mathcal{H}}_k$$

Using the isomorphism I_k, we define the the positive definite Hermitian structure $[\cdot,\cdot]$ on $\tilde{\mathcal{H}}_k$ by the formula

$$[\cdot,\cdot] = \{I_k^{-1}(\cdot), I_k^{-1}(\cdot)\}.$$

The norm associated to $[\cdot,\cdot]$ is denoted $[\cdot]$. The Hermitian structure $[\cdot,\cdot]$ on $\tilde{\mathcal{H}}_k$ induces a Hermitian structure on $\mathrm{End}(\mathcal{V}_k)$, which is parallel with respect to $\hat{\nabla}^{\mathrm{e}}$ and which is Γ-invariant.

Definition 1. We define $\tilde{\mathcal{H}}$ to be the Hilbert space completion of $\tilde{\mathcal{H}}$ with respect to the norm $[\cdot]$.

This is an infinite dimensional Hilbert space, on which Γ acts isometrically. This representation provides us with the needed counter example to Kashdan's property (T) for Γ. Let us discuss the proof of this.

Theorem 2 (Roberts). *The only* Γ *invariant vector in* $\tilde{\mathcal{H}}$ *is* 0*.*

This theorem follows from the fact that the representations $\tilde{\mathcal{H}}_k$ are irreducible as Γ-representation for k, such that $k+2$ is prime. This results was established in the un-twisted case by Roberts in [Ro] and his proof can be applied word for word also to this case.

The basic idea behind building the required almost fixed vector for $\tilde{\mathcal{H}}$ is to consider coherent states on M_σ, $\sigma \in \mathcal{T}$.

Fix a point $x \in M$. Evaluation at x gives a section of $\mathcal{V}_k^*$ up to scale. Using $[\cdot,\cdot]$ we get induced a section $e_x^{(k)}$ of $\mathcal{V}_k$ up to scale. For each $\sigma \in \mathcal{T}$, $e_x^{(k)}(\sigma)$ is the coherent state associated to x on M_σ. Let $E_x^{(k)}$ be the section of $\mathrm{End}(\mathcal{V}_k)$ obtained as the orthogonal projection (with respect to $[\cdot,\cdot]$) onto the one dimensional subspace spanned by $e_x^{(k)}$. We observe that $E_x^{(k)}$ only depends on x.

Theorem 3. *The sections* $E_x^{(k)}$ *of* $\mathrm{End}(\mathcal{V}_k)$ *over* $\mathcal{T}$ *are asymptotically covariant constant. I.e. for any pair of points* $\sigma_0, \sigma_1 \in \mathcal{T}$ *there exists a constant* C *such that*

$$\left[P^e_{\sigma_0,\sigma_1}\left(E_x^{(k)}(\sigma_0)\right) - E_x^{(k)}(\sigma_1)\right] \leq \frac{C}{k},$$

where $P^e_{\sigma_0,\sigma_1}$ *is the parallel transport from* σ_0 *to* σ_1 *in* $\mathrm{End}(\mathcal{V}_k)$*.*

To produce the almost fixed vectors, we now pick a finite subgroup Λ of $SU(2)$ which contains $-1 \in SU(2)$ and we consider the finite subset X of M, consisting of connections which reduces to Λ. We get a non empty finite subset of M this way which is invariant under the action of the mapping class group and such that $|X| > 1$. Let now $E_X^{(k)}$ be the section of $\mathrm{End}(\mathcal{V}_k)$ given by

$$E_X^{(k)} = \sum_{x \in X} E_x^{(k)}.$$

Let $E^{(k)}_{X,0}$ be the traceless part of $E^{(k)}_X$. For large enough k we have a unique vector in $\mathcal{E}^{(k)}_{X,0} \in \tilde{\mathcal{H}}_k$, which at σ_0 agrees with $E^{(k)}_{X,0}(\sigma_0)/[E^{(k)}_{X,0}(\sigma_0)]$.

Theorem 4. *The sequence* $\{\mathcal{E}^{(k)}_{X,0}\}$ *is an almost fixed vector for the action of* Γ *on* $\tilde{\mathcal{H}}$.

It is a result of F. Taherkhani that the mapping class group of a closed oriented surface of genus two does not have Kazhdan's property (T) [Ta].

We would like to thank Vaughan Jones for suggesting the use of these representations to settle this property for these mapping class groups. Jones has also posted this problem on the CTQM problem list at www.ctqm.au.dk (Problem 14). At the time of this writing it remains an open problem to what extend the mapping class groups has the Haagerup property. Further we would like to thank Gregor Masbaum for valuable discussions.

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Moduli Spaces, Foliations and Algebraic Structures

Ralph Kaufmann

1. Introduction

There is a wonderful story which involves alternating between algebraic and geometric aspects of basic structures. The tool for this is given by operads. In this setting one can associate geometric meaning to algebraic equations and derive algebraic relations from geometries and vice–versa. The most fruitful and ubiquitous geometries for this correspondence are given by Moduli Spaces and their Teichmüller theory.

2. Algebraic Structures from Moduli Spaces/Foliations

2.1. Classical results.

2.1.1. *Classical Paradigm: Geometry.* A classical paradigm for the relationship we will study is given by the following moduli space. Consider the moduli space $\hat{M}_{n+1}$ of maps from $k+1$ copies of the discs mapping holomorphically to S^2,

$$[\amalg_{n+1} D \to S^2]/PSL_2$$

such that their images do not intersect and the maps are biholomorphic on their image.

By gluing the complement of the 0th into the $i-th$ disc, there are compositions

$$\circ_i : \hat{M}_{n+1} \times \hat{M}_{m+1} \to \hat{M}_{m+n-2}$$

Theorem 1 (Getzler). *$H_*(\hat{M}_{n+1}, k) = bv(k)$ that is the homology is the operad for BV algebras.*

These words are explained in a minute.

2.1.2. *Classical Paradigm: Algebra.* Consider an algebra A which is unital and associative over k. Set $CH^n(A,A) = Hom(A^{\otimes n}, A)$. Then by substituting the function g in there are compositions

$$\circ_i : CH^n(A,A) \times CH^m(A,A) \to CH^{n+m-1}$$

For $f \in CH^n(A,A), g \in CH^m(A,A)$

$$(1)\quad f \circ g := \sum_{i=1}^{n} (-1)^{(i-1)(m-1)} f \circ_i g \text{ and } \{f \bullet g\} := f \circ g - (-1)^{(n-1)(m-1)} g \circ f$$

There is an associative multiplication $\cup$ by multiplying the functions and a differential $d : CH^n(A,A) \to CH^{n+1}(A,A)$ called the Hochschild differential. Its cohomology is called the Hochschild cohomology and denoted by $HH^*(A,A)$. The differential uses $\cup$ in its definition. Finally there is an $\mathbb{S}_n$ action on $CH^n(A,A)$ given by permuting the variables.

Theorem 2 (Gerstenhaber)**.** *$(CH^*(A,A), \circ)$ is pre–Lie and $(HH^*(A,A), \cup, \{\ \bullet\ \})$ is a Gerstenhaber algebra.*

2.1.3. *Operads.* Generalizing the above structure we get a sequence of objects $\mathcal{O}(n)$[1]. together with $\mathbb{S}_n$ actions on $\mathcal{O}(n)$ and operations $\circ_i$ that are $\mathbb{S}_n$ equivariant. A typical example is $End(V)(n) = Hom(V^{\otimes n}, n)$ with $\mathbb{S}_n$ actions and substitution compositions as above. An algebra V over $\mathcal{O}$ is then a morphism of operads $\rho : \mathcal{O} \to End(V)$ [2].

In this context Gerstenhaber's theorem can be rephrased: in a (linear) operad the direct sum $\bigoplus \mathcal{O}(n)$ is an odd Lie algebra. This can be found in Gerstenhaber/Voronov and Kaufmann. Moreover as Kapranov and Manin showed, this structure descends to the coinvariants $\bigoplus \mathcal{O}(n)_{\mathbb{S}_n}$.

2.2. **Moduli spaces/Foliations and the Arc operad.** Together with Muriel Livernet and Bob Penner, we defined an operad which yields cellular models for the operad considered by Getzler and Moduli space itself. This operad is given by considering mapping class group orbits of projectively weighted arcs which run from boundary to boundary on a surface with one marked point on each boundary component. These points and hence the components are considered labeled by 0 to n. With their weights these arcs can be thought of as a band of parallel leaves of a partially measured foliation.

Using a suboperad given by surfaces of genus zero with the restriction that arcs run only from 0 to $i \neq 0$, we were able to prove

Theorem 3 (Kaufmann)**.** *The Hochschild cochains of a Frobenius algebra carry a chain level action of a chain model of the little discs operad.*

[1]in a fixed (closed) symmetric monoidal category

[2]In a general *closed* monoidal category V is just an object and $End(V)$ is defined in the same way.

which I called the cyclic Deligne conjecture. Unknown to me it was actually conjectured by Tamarkin and Tsygan. The action is via the Cacti operad, which was invented by Voronov. I had previously shown that this operad is equivalent to the framed little discs, which is another version of Getzler's operad. A consequence is that $HH^*(A, A)$ is a BV algebra — a fact previously proven by Menichi.

This theorem has applications to string topology, since for a simply connected compact manifold $H_*(LM) = HH^*(S^*(M), S_*(M))$.

3. Geometry from algebra

3.1. **New algebraic structures within operads.** In joint work with Ben Ward and Javier Zuniga we were able to show that

Theorem 4. *If a generalized operad has odd non–self gluings, then the coinvariants carry an odd Lie bracket. If a generalized operad has odd self–gluings, its coinvariants carry a differential Δ. It the generalized operad has horizontal compositions given by the tensor product, then Δ is BV for the tensor multiplication.*

Without giving the details of our novel monoidal category framework defining generalized operads, there are several examples we can discuss.

3.1.1. *Cyclic case.* If $\mathcal{O}$ is a anti–cyclic operad, there is a new Lie cyclic bracket on the cyclic coinvariants. A cyclic operad has an extended $\mathbb{S}_{n+1}$ action on $\mathcal{O}(n)$ and the cyclic coinvariants are $\bigoplus \mathcal{O}(n)_{\mathbb{S}_{n+1}}$. The above example of Getzler, Moduli spaces and the Arc operad are all cyclic (even on the topological level). Another new result we show is that this bracket lifts to the cyclic invariants $\bigoplus \mathcal{O}(n)_{\mathbb{Z}/(n+1)\mathbb{Z}}$. The relation between odd and anti–cyclic is that one is a suspension of the other. The most prominent example of an anti–cyclic operad is that of $End(V)(n) = Hom(V^{\otimes n}, V)$ with V a symplectic vector space.

Famous examples and applications of our theorem are the three Lie families of algebras used by Kontsevich and Conant–Vogtman to give complexes computing the cohomology of Moduli and Outer space.

3.1.2. *$\mathfrak{K}$–modular Operads.* A prime example of both odd self and non–self gluing is that of a $\mathfrak{K}$ modular operad. These arise in the bar construction for modular operads which is called the Feynman transform.

Barannikov showed that the algebras on a vector space V over the Feynman transform of a modular operad $\mathcal{O}$ are given by the solution to the master equation $dS + \Delta S + \frac{1}{2}\{S \bullet S\} = 0$ with $S \in \bigoplus (\mathcal{O}(n) \otimes V^{\otimes n+1})^{\mathbb{S}_{n+1}}$ of degree 0.

A good example of a modular operad for Teichmüller theorists is $H_*(\bar{M}_{g,n})$ where the non–self gluings come from gluing together two stable surfaces at the marked points creating a new node and the self gluings, as the name suggests, glue two marked points on the same surface together.

3.1.3. *Topological Master Equation drives the compactification.* The motivation for our new theorem is the Harrelson–Voronov–Zuniga interpretation of Zwiebach's string field theory. Their work and the previous work of Kimura–Stasheff–Voronov (KSV) gives a topological example of the structures we are discussing. Consider

$\bar{M}^{KSV}_{g,n}$ which are real blow ups or $\bar{M}_{g,n}$. An element in this space is a possibly nodal curve with a unit tangent vector at each node[3]. In this spaces there is no modular operad structure, since we also have to specify a unit tangent vector at each newly created node. There is however a twist gluing which produces the S^1 family of all such choices. In the chain and homology level this gives the odd gluings, since this family is of dimension one[4]. Their result is that the sum of fundamental classes $S = \sum[\bar{M}^{KSV}_{gn}]$ is a solution of the master equation.

3.2. **Outlook.** In further work we wish to analyze the master equation in the following new contexts.

(1) The arc operad. Here there are two versions. The full operad or moduli space. This should be related to the work of McShane and Penner on screens and the Fulton MacPherson compactification. A smaller version just like cacti. Here the compactification should be the one used by myself and R. Schwell in our cellular solution to the A_∞ conjecture.
(2) String Topology. Here we can interpret the master equation in our framework of string topology operations as put forth in our series of papers on Hochschild actions that appeared in the Journal of Noncommutative geometry.

Appendix Algebras and operads

For the readers conveniece, we list the definition of the algebras we talk about. Let A be a graded vector space over k and let $|a|$ be the degree of an element a. Let's fix char $k = 0$ or at least $\neq 2$.

(1) Pre–Lie algebra. $(A, \circ : A \times A \to A)$ s.t.

$$a \circ (b \circ c) - (a \circ b) \circ c = (-1)^{|a||b|}[a \circ (c \circ b) - (a \circ c) \circ b]$$

(2) Odd Lie. $(A, \{\,\bullet\,\} : A \otimes A \to A)$

$\{a \bullet b\} = (-1)^{|a|-1)(|b|-1)}\{b \bullet a\}$ and Jacobi with appropriate signs

(3) Odd Poisson or Gerstenhaber. $(A, \{\,\bullet\,\}, \cdot)$ Odd Lie plus another associative multiplication for which the bracket is a derivation with the appropriate signs.
(4) (dg)BV. $(A, \cdot, \Delta)$. $(A, \cdot)$ associative (differential graded), Δ a differential of degree 1: $\Delta^2 = 0$ and

$$\{a \bullet b\} := (-1)^{|a|}\Delta(ab) - a\Delta(b) - (-1)^{|a|}\Delta(a)b$$

is a Gerstenhaber bracket

Reference table for ty[es of operads. Typical operads:

[3]More precisely an element of $(S^1 \times S^1)/S^1$

[4]Secretly the notation $\bullet$ refers to this S^1 family

operad	$End(V)$
odd operad	naïve suspension of operad
cyclic operad	$End(V)$, V with symmetric non–degenerate bi–linear form
anti–cyclic operad	$End(V)$, V with anti–symmetric non–degenerate bilinear form
modular operad	$H_*(\bar{M}_{gn})$
$\mathfrak{K}$ modular	Feynman transform of modular operad or $H_*(\bar{M}^{KSV}_{g,n})$

Some topics on the Takhtajan-Zograf metric

Kunio Obitsu

(joint work with Wing-Keung To, Lin Weng and Scott A. Wolpert)

Consider the asymptotic behavior of the Weil-Petersson and the Takhtajan-Zograf metrics near the boundary of $\mathcal{M}_{g,n}$ which is the moduli space of Riemann surfaces of genus g with n punctures $(2g-2+n>0)$. Let $\overline{\mathcal{M}}_{g,n}$ denote the Deligne-Mumford compactification. Set $D=\overline{\mathcal{M}}_{g,n}\backslash\mathcal{M}_{g,n}$ the compactification divisor. Pick $R_0\in D$ a stable curve of genus g with n punctures and k nodes.

Each node q_i $(i=1,2,\dots,k)$ has a neighborhood $N_i=\{(z_i,w_i)\in\mathbf{C}^2 \mid |z_i|,|w_i|<1, z_iw_i=0\}$. Let R_t be the smooth surface gotten from R_0 after cutting and pasting N_i under the relation $z_iw_i=t_i, |t_i|$ small. Then, D is locally described as $\{t_1\cdots t_k=0\}$. D has locally the pinching coordinate around $(0,0)=[R_0]$, $(t,s)=(t_1,\dots,t_k,s_{k+1},\dots,s_{3g-3+n})$. Set $\alpha_i=\partial/\partial t_i, \beta_\mu=\partial/\partial s_\mu\in T_{(t,s)}(\mathcal{M}_{g,n})$. We put

$$g_{i\bar{j}}(t,s):=\langle\alpha_i,\alpha_j\rangle_{WP}(t,s), g_{i\bar{\mu}}(t,s):=\langle\alpha_i,\beta_\mu\rangle_{WP}(t,s), g_{\mu\bar{\nu}}(t,s):=\langle\beta_\mu,\beta_\nu\rangle_{WP}(t,s),$$

$(i,j=1,2,\dots,k,\ \mu,\nu=k+1,\dots,3g-3+n)$ the Riemannian tensors for the Weil-Petersson metric, and

$$h_{i\bar{j}}(t,s):=\langle\alpha_i,\alpha_j\rangle_{TZ}(t,s), h_{i\bar{\mu}}(t,s):=\langle\alpha_i,\beta_\mu\rangle_{TZ}(t,s), h_{\mu\bar{\nu}}(t,s):=\langle\beta_\mu,\beta_\nu\rangle_{TZ}(t,s),$$

$(i,j=1,2,\dots,k,\ \mu,\nu=k+1,\dots,3g-3+n)$ the Riemannian tensors for the Takhtajan-Zograf metric.

The following theorem is a pioneering result for the asymptotic behavior of the Weil-Petersson metric near the boundary of the moduli space.

Theorem 1 (Masur [3]). *As $t_i,s_\mu\to 0$,*

$$i)\quad g_{i\bar{i}}(t,s)\approx\frac{1}{|t_i|^2(-\log|t_i|)^3}\qquad \text{for } i\le k,$$

$$ii)\quad g_{i\bar{j}}(t,s)=O\left(\frac{1}{|t_i||t_j|(\log|t_i|)^3(\log|t_j|)^3}\right)\qquad \text{for } i,j\le k, i\neq j,$$

$$iii)\quad g_{i\bar{\mu}}(t,s)=O\left(\frac{1}{|t_i|(-\log|t_i|)^3}\right)\qquad \text{for } i\le k,\mu\ge k+1,$$

$$iv)\quad g_{\mu\bar{\nu}}(t,s)\longrightarrow g_{\mu\bar{\nu}}(0,0)\qquad \text{for } \mu,\nu\ge k+1.$$

Yamada ([9]) improved Masur's formula to investigate the Weil-Petersson completion of the Teichmüller space. Recently, we updated Masur's and Yamada's results by improving Wolpert's formula ([7]) for the asymptotic of the hyperbolic metric for degenerating Riemann surfaces.

Theorem 2 (O. and Wolpert [6]). *As* $t \to 0$,

$$iv)' \; g_{\mu\overline{\nu}}(t,s) = g_{\mu\overline{\nu}}(0,s) + \frac{4\pi^4}{3}\sum_{i=1}^{k}(\log|t_i|)^{-2}\Big\langle \beta_\mu, (E_{i,1}+E_{i,2})\beta_\nu \Big\rangle_{WP}(0,s)$$
$$+ O\Big(\sum_{i=1}^{k}(\log|t_i|)^{-3}\Big) \text{ for } \mu,\nu \geq k+1.$$

Here, $E_{i,1}, E_{i,2}$ *denote a pair of the Eisenstein series with index* 2 *associated with the* i*-th node of the limit surface* R_0.

On the other hand, we have a result for asymptotics of the Takhtajan-Zograf metric near the boundary of the moduli space.

Definition Let R_0 be a Riemann surface with n punctures $p_1, \cdots, p_n$ and k nodes $q_1, \cdots, q_k$. A node q_i is said to be **adjacent to punctures** if the component of $R_0 \backslash \{q_1, \cdots, q_{i-1}, q_{i+1}, \cdots, q_k\}$ containing q_i also contains at least one of the p_j's. Otherwise, it is said to be **non-adjacent to punctures**.

Theorem 3 (O.-To-Weng [5]). *As* $(t,s) \to 0$, *we observe the followings:*
i) For any $\varepsilon > 0$, *there exists a positive constant* $C_{1,\varepsilon}$ *such that*

$$h_{i\overline{i}}(t,s) \leq \frac{C_{1,\varepsilon}}{|t_i|^2(-\log|t_i|)^{4-\varepsilon}} \quad \text{for } i \leq k;$$

For any $\varepsilon > 0$, *there exists a positive constant* $C_{2,\varepsilon}$ *such that*

$$h_{i\overline{i}}(t,s) \geq \frac{C_{2,\varepsilon}}{|t_i|^2(-\log|t_i|)^{4+\varepsilon}} \quad \text{for } i \leq k$$

and the node q_i *adjacent to punctures*;

$$ii) \; h_{i\overline{j}}(t,s) = O\left(\frac{1}{|t_i||t_j|(\log|t_i|)^3(\log|t_j|)^3}\right) \quad \text{for } i,j \leq k, i \neq j;$$

$$iii) \; h_{i\overline{\mu}}(t,s) = O\left(\frac{1}{|t_i|(-\log|t_i|)^3}\right) \quad \text{for } i \leq k, \mu \geq k+1;$$

$$iv) \; h_{\mu\overline{\nu}}(t,s) \longrightarrow h_{\mu\overline{\nu}}(0,0) \quad \text{for } \mu,\nu \geq k+1.$$

Conjecture ([5])

$$h_{i\overline{i}}(t,s) \approx \frac{1}{|t_i|^2(-\log|t_i|)^4} \; (t \to 0) \text{ if } i \leq k \text{ with the node } q_i \text{ adjacent to punctures.}$$

Very recently, with To and Weng, the author improved *i)* in Theorem 3 which will appear elsewhere. Other topics related to the Takhtajan-Zograf metric are discussed in [8].

Open problems

1. Determine $H^*_{(2)}(\mathcal{M}_{g,n}, \omega_{TZ})$ for general (g, n), originally asked by To and Weng. For that, we need more informations on precise asymptotics of degenerating Eisenstein series. (See [4].)

2. Is it possible that the index theorem for punctured surfaces could be derived from the one for compact surfaces through degeneration?
−Bismut-Bost ([1]) studied a related problem.

3. Is the curvature of the Takhtajan-Zograf metric negative?

4. If the answer to the question 3. is YES, study $-\mathrm{Ric}\,\omega_{TZ}$.
− Recently, K. Liu, X. Sun & S.-T. Yau (2004, 2005, 2008-) find good geometry of the moduli of curves using $-\mathrm{Ric}\,\omega_{WP}$. (see [2].)

5. Does the Takhtajan-Zograf Kähler form have a global representation formula?
− The Weil-Petersson Kähler form has a global representation formula in terms of the Fenchel-Nielsen global coordinates, which reveals the symplectic nature of the Teichmüller space. (S.A. Wolpert (1982, 1983, 1985), See [8].)

6. Does the Laplace operator Δ_{TZ} of the Takhtajan-Zograf metric have a self-adjoint extension?
−Lizhen Ji announced in this workshop that the Laplace operator Δ_{WP} of the Weil-Petersson metric has a self-adjoint extension.

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Automorphisms of the complex of separating curves and the Torelli complex

Yoshikata Kida
(joint work with Saeko Yamagata)

Let $S = S_{g,p}$ be a connected, compact and orientable surface of genus g with p boundary components. Unless otherwise stated, we assume that a surface satisfies these conditions. We define $\mathrm{Mod}^*(S)$ as the *extended mapping class group* for S, i.e., the group of isotopy classes of homeomorphisms from S onto itself, where isotopy may move points in the boundary of S. The complex of curves for S, denoted by $\mathcal{C}(S)$, plays an important role in the study of various aspects of $\mathrm{Mod}^*(S)$. For example, understanding of automorphisms of $\mathcal{C}(S)$ leads to a description of any isomorphism between finite index subgroups of $\mathrm{Mod}^*(S)$ as discussed by Ivanov [4] and Korkmaz [8] (see also [9] for automorphisms of $\mathcal{C}(S)$). We report an analogous result on the Johnson kernel and the Torelli group.

A simple closed curve in S is said to be *essential* in S if it is neither homotopic to a point of S nor isotopic to a boundary component of S. Let $V(S)$ denote the set of isotopy classes of essential simple closed curves in S. For each $\alpha \in V(S)$, we denote by $t_\alpha \in \mathrm{Mod}^*(S)$ the Dehn twist about α. The *complex of curves* for S, denoted by $\mathcal{C}(S)$, is defined as the abstract simplicial complex such that the set of vertices is $V(S)$, and a non-empty finite subset σ of $V(S)$ is a simplex of $\mathcal{C}(S)$ if and only if any two elements of σ have disjoint representatives.

A simple closed curve a in S is said to be *separating* in S if $S \setminus a$ is not connected. Otherwise, a is said to be *non-separating* in S. Let $V_s(S)$ denote the subset of $V(S)$ consisting of all isotopy classes of separating curves in S. The *complex of separating curves* for S, denoted by $\mathcal{C}_s(S)$, is defined as the full subcomplex of $\mathcal{C}(S)$ spanned by $V_s(S)$. The natural action of $\mathrm{Mod}^*(S)$ on $\mathcal{C}_s(S)$ induces the homomorphism

$$\pi_s \colon \mathrm{Mod}^*(S) \to \mathrm{Aut}(\mathcal{C}_s(S)),$$

where $\mathrm{Aut}(\mathcal{C}_s(S))$ denotes the automorphism group of $\mathcal{C}_s(S)$. The *Johnson kernel* $\mathcal{K}(S)$ is defined as the subgroup of $\mathrm{Mod}^*(S)$ generated by all t_α with $\alpha \in V_s(S)$. The following theorem for closed surfaces is due to Brendle-Margalit [1], [2]. We refer to [6] for the other cases.

Theorem 1. *Let $S = S_{g,p}$ be a surface and assume one of the following three conditions: $g = 1$ and $p \geq 3$; $g = 2$ and $p \geq 2$; or $g \geq 3$ and $p \geq 0$. Then*

(i) *π_s is an isomorphism.*
(ii) *if Γ_1 and Γ_2 are finite index subgroups of $\mathcal{K}(S)$ and if $f \colon \Gamma_1 \to \Gamma_2$ is an isomorphism, then there exists a unique element γ_0 of $\mathrm{Mod}^*(S)$ with the equality $f(\gamma) = \gamma_0 \gamma \gamma_0^{-1}$ for any $\gamma \in \Gamma_1$.*

We note that $\mathcal{C}_s(S_{2,1})$ consists of countably infinitely many $\aleph_0$-regular trees. This is a direct consequence of a result due to Kent-Leininger-Schleimer [5].

A pair of non-separating curves in S, $\{a, b\}$, is called a *bounding pair (BP)* in S if a and b are disjoint and non-isotopic and if $S \setminus (a \cup b)$ is not connected. Let $V_{bp}(S)$ denote the set of isotopy classes of BPs in S. The *Torelli complex* $\mathcal{T}(S)$ for

S is defined as the abstract simplicial complex such that the set of vertices is the disjoint union $V_s(S) \sqcup V_{bp}(S)$, and a non-empty finite subset σ of $V_s(S) \sqcup V_{bp}(S)$ is a simplex of $\mathcal{T}(S)$ if and only if any two elements of σ have disjoint representatives. We have the natural homomorphism

$$\pi_t \colon \mathrm{Mod}^*(S) \to \mathrm{Aut}(\mathcal{T}(S)),$$

where $\mathrm{Aut}(\mathcal{T}(S))$ denotes the automorphism group of $\mathcal{T}(S)$. The *Torelli group* $\mathcal{I}(S)$ for S is defined as the subgroup of $\mathrm{Mod}^*(S)$ generated by all t_α with $\alpha \in V_s(S)$ and all $t_\beta t_\gamma^{-1}$ with $\{\beta, \gamma\} \in V_{bp}(S)$. An analogous result on $\mathcal{T}(S)$ and $\mathcal{I}(S)$ is the following:

Theorem 2. *Let $S = S_{g,p}$ be a surface and assume one of the following three conditions: $g = 1$ and $p \geq 3$; $g = 2$ and $p \geq 1$; or $g \geq 3$ and $p \geq 0$. Then*

(i) *π_t is an isomorphism.*
(ii) *if Λ_1 and Λ_2 are finite index subgroups of $\mathcal{I}(S)$ and if $h \colon \Lambda_1 \to \Lambda_2$ is an isomorphism, then there exists a unique element λ_0 of $\mathrm{Mod}^*(S)$ with the equality $h(\lambda) = \lambda_0 \lambda \lambda_0^{-1}$ for any $\lambda \in \Lambda_1$.*

We note that Farb-Ivanov [3] announce the computation of the Torelli geometry for a closed surface, which is the Torelli complex with a certain marking, and also announce Theorem 2 (ii) for $S = S_{g,0}$ with $g \geq 5$. McCarthy-Vautaw [10] compute automorphisms of $\mathcal{I}(S)$ for $S = S_{g,0}$ with $g \geq 3$. Brendle-Margalit [1], [2] show Theorem 2 for $S = S_{g,0}$ with $g \geq 3$. Theorem 2 for $S = S_{g,p}$ with $(g, p) \neq (2, 1)$ is proved in [6]. Finally, the case $S = S_{2,1}$ is discussed in [7].

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Geodesics of the Weil-Petersson metric

Ursula Hamenstädt

The Weil-Petersson metric on Teichmüller space $\mathcal{T}(S)$ of a Riemann surface S of finite type and higher complexity is a Kähler metric of negative sectional curvature which is invariant under the action of the mapping class group. However, this metric is incomplete. Nevertheless, any two points can be connected by a unique geodesic. As a consequence, its completion $\overline{\mathcal{T}(S)}$ is a CAT(0)-space. The completion locus can be described as follows.

A *surface with nodes* is obtained from S by pinching one or more simple closed curves on S to punctures. The Teichmüller space of the surfaces with a node obtained from S by pinching the components of a multi-curve c admits itself a Weil-Petersson metric provided that this surface is of higher complexity. The completion locus then consists of strata, one for each multi-curve, equipped with the Weil-Petersson metric. Any two points in the completion locus can be connected by a unique geodesic. The boundary strata are strictly convex.

The mapping class group $\mathrm{Mod}(S)$ acts on $\mathcal{T}(S)$ as a group of isometries. Each mapping class is *semi-simple*. A mapping class ϕ is elliptic if and only for some $k > 0$, ϕ^k is a (possibly trivial) Dehn multitwist. Pseudo-Anosov mapping classes are precisely those mapping classes which have in axis in $\mathcal{T}(S)$.

The *Weil-Petersson geodesic flow* Φ^t acts on the quotient $T^1\mathcal{M}(S)$ of the unit tangent bundle of $\mathcal{T}(S)$ under the action of the mapping class group. This flow is not everywhere defined. However, as was shown by Wolpert, there is a dense G_δ-set A of points so that for every $x \in A$ the flow line through x is defined for all times. Moreover, this set has full Lebesgue measure. Indeed, the Lebesgue measure is finite and ergodic under the action of Φ^t [3].

On the other hand, the *Teichmüller flow* Ψ^t is defined on the moduli space $\mathcal{Q}$ of unit area quadratic differentials. We discuss the following result [4].

Theorem 1. *There is a Ψ^t-invariant Borel subset $B \subset \mathcal{Q}$ and a Borel map $F : B \to T^1\mathcal{M}(S)$ with the following properties.*

(1) *For every Ψ^t-invariant Borel measure ν we have $\nu(B) = 1$.*
(2) *If $x \in \mathcal{Q}$ is such that the Ψ^t-orbit of x is contained in some compact subset of $\mathcal{Q}$ then $x \in B$. In particular, every periodic point for Ψ^t is contained in B.*
(3) *There is a bounded Borel cocycle $\sigma : B \times \mathbb{R} \to \mathbb{R}$ with $\sigma(x, s+t) = \sigma(\Phi^s x, t) + \sigma(x, s)$ such that $F(\Psi^t x) = \Psi^{\sigma(x,t)} F(x)$ for all $x \in B, s \in \mathbb{R}$.*

This theorem is related to recent work of Brock, Masur, Minsky [1, 2]. It implies that there is an injective map of the space of Ψ^t-invariant Borel probability measures into the space of Φ^t-invariant Borel probability measures. We do not know whether this map is surjective.

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The logarithms of Dehn twists

Nariya Kawazumi
(joint work with Yusuke Kuno [6])

Let Σ be an oriented connected compact surface of genus $g\,(\geq 1)$ with 1 boundary component. Choose a basepoint $* \in \partial\Sigma$. We denote $\pi := \pi_1(\Sigma, *)$ and $H := H_1(\Sigma; \mathbb{Q})$. The simple loop going around the boundary in the opposite direction defines an element $\zeta \in \pi$.

Any simple closed curve $C \subset \Sigma$ defines the right handed Dehn twist t_C along C as an element of the mapping class group of the surface Σ relative to the boundary $\partial\Sigma$. The classical formula says the action $|t_C|$ of the Dehn twist t_C on the homology group H is given by

$$|t_C| = 1_H - [C] \otimes [C] \in \mathrm{Hom}(H, H),$$

where $[C] \in H$ is the homology class of C with a fixed orientation, and we identify $H \otimes H = \mathrm{Hom}(H, H)$, $Y \otimes Z \mapsto (X \mapsto (X \cdot Y)Z)$, by the Poincaré duality. Our result generalizes this formula to the action of t_C on the completed group ring $\widehat{\mathbb{Q}\pi}$, where the completion is induced by the augmentation ideal $I\pi \subset \mathbb{Q}\pi$.

Massuyeau [10] introduced the notion of a symplectic expansion of the group π, which provides an isomorphism of pairs of complete Hopf algebras

$$\theta : (\widehat{\mathbb{Q}\pi}, \widehat{\mathbb{Q}\langle\zeta\rangle}) \cong (\widehat{T}, \mathbb{Q}[[\omega]]).$$

Here $\widehat{T} := \prod_{m=0}^{\infty} H^{\otimes m}$ is the completed tensor algebra generated by H, and $\omega \in H^{\otimes 2}$ is the symplectic form. Explicit symplectic expansions have been constructed by Kawazumi [5] (over $\mathbb{R}$), Massuyeau [10], Kuno [9] and Bene-Kawazumi-Kuno-Penner [1].

Any mapping class φ on the surface Σ relative to the boundary $\partial\Sigma$ defines an automorphism of the complete Hopf algebra $\widehat{\mathbb{Q}\pi}$. By using the isomorphism θ, we may regard it as an automorphism $T^\theta(\varphi)$ of the complete Hopf algebra $\widehat{T}$, which we call *the total Johnson map* [4]. Our result describes the map $T^\theta(t_C)$ in an explicit way.

We introduce a linear map $N : \widehat{T} \to H \otimes \widehat{T} \subset \widehat{T}$ by $N|_{H^{\otimes 0}} = 0$ and $N(X_1 X_2 \cdots X_m) = \sum_{i=1}^{m} X_i \cdots X_m X_1 \cdots X_{i-1}$, $X_j \in H$. Here and for the rest of this report, we often drop the symbol $\otimes$. Take the logarithm of the symplectic expansion θ, $\ell^\theta := \log\theta : \pi \to H \otimes \widehat{T}$. Then we define a map $L^\theta : \pi \to \mathrm{Hom}(H, \widehat{T})$ by

$$L^\theta(x) := \frac{1}{2} N(\ell^\theta(x)\ell^\theta(x)) = N\theta(\frac{1}{2}(\log x)^2) \in H \otimes \widehat{T} = \mathrm{Hom}(H, \widehat{T})$$

for $x \in \pi$. It is easy to show $L^\theta(x)$ is an invariant of unoriented loops on the surface Σ. Here we identify $H \otimes \widehat{T} = \mathrm{Hom}(H, \widehat{T})$ by the Poincaré duality. Further the space $\mathrm{Hom}(H, \widehat{T})$ is naturally identified with $\mathrm{Der}(\widehat{T})$, the Lie algebra of derivations of the algebra $\widehat{T}$. In particular, $L^\theta(x)$ is regarded as a derivation of $\widehat{T}$, so that we may define an algebra automorphism of $\widehat{T}$ by taking the exponential $e^{-L^\theta(x)}$.

Theorem 1. *For any symplectic expansion θ and a simple closed curve $C \subset \Sigma$, we have*

$$T^\theta(t_C) = e^{-L^\theta(C)}$$

as algebra automorphisms of the algebra $\widehat{T}$. In other words, the invariant $-L^\theta(C)$ is **the logarithm of the Dehn twist** t_C.

The degree 0 part of this formula is exactly the classical formula stated above. As a corollary of the theorem, the action of the Dehn twist t_C on N_k, the k-th nilpotent quotient of π, depends only on the conjugacy class of a based loop representing C in N_k. Moreover, using the exponential $e^{-L^\theta(C)}$, we can define the *Dehn twist along a not-necessarily-simple closed curve* C as an automorphism of the completed group ring $\widehat{\mathbb{Q}\pi}$. It would be interesting if one could realize this automorphism in a geometric context.

The key to the proof of the theorem is a geometric interpretation of symplectic derivations of the algebra $\widehat{T}$. Under the identification $\widehat{T_1} = \mathrm{Der}(\widehat{T})$, the subspace $N(\widehat{T_1})$ is exactly equal to the Lie algebra of symplectic derivations, $\mathrm{Der}_\omega(\widehat{T}) = \{D \in \mathrm{Der}(\widehat{T}); D\omega = 0\}$. Kontsevich's "associative" Lie algebra [8] is a Lie subalgebra of $\mathrm{Der}_\omega(\widehat{T})$. Let $\mathbb{Q}\hat{\pi}$ be the Goldman Lie algebra of the surface Σ [2]. We have a natural homomorphism of Lie algebras

$$\sigma : \mathbb{Q}\hat{\pi} \to \mathrm{Der}(\mathbb{Q}\pi).$$

In general, let M be a d-dimensional oriented C^∞ manifold, and $*$ a basepoint on M. Then we can construct a natural map $H_i(L(M \setminus \{*\})) \otimes H_j(\Omega(M,*)) \to H_{i+j+2-d}(\Omega(M,*))$ in a similar way to [3], where $\Omega(M,*) = \mathrm{Map}((S^1,0),(M,*))$ and $L(M \setminus \{*\}) = \mathrm{Map}(S^1, M \setminus \{*\})$.

For any symplectic expansion θ, we define a map

$$-\lambda_\theta : \mathbb{Q}\hat{\pi} \to N(\widehat{T_1}) = \mathrm{Der}_\omega(\widehat{T}), \quad x \mapsto -N\theta(x).$$

Theorem 2. *The diagram*

$$\begin{array}{ccc} \mathbb{Q}\hat{\pi} \times \mathbb{Q}\pi & \xrightarrow{\sigma} & \mathbb{Q}\pi \\ {\scriptstyle -\lambda_\theta \times \theta}\downarrow & & \downarrow{\scriptstyle \theta} \\ \mathrm{Der}_\omega(\widehat{T}) \times \widehat{T} & \longrightarrow & \widehat{T}, \end{array}$$

where the bottom horizontal arrow means the derivation, commutes.

Let $\{\alpha_i, \beta_i\} \subset \pi$ be a symplectic generating system. The Dehn twist along α_1 satisfies $t_{\alpha_1}(\alpha_1) = \alpha_1$ and $t_{\alpha_1}(\beta_1) = \beta_1\alpha_1$. Hence the "logarithm" $\log(t_{\alpha_1})$ should satisfy $\log(t_{\alpha_1})(\beta_1) = \beta_1 \log \alpha_1$. On the other hand, we have $\sigma({\alpha_1}^n)(\beta_1) = n\beta_1{\alpha_1}^n$

for any $n \geq 0$, so that $\sigma(f(\alpha_1))(\beta_1) = \beta_1\alpha_1 f'(\alpha_1)$ for any formal power series $f(x)$ in $x-1$. If $f(x)$ satisfies $xf'(x) = \log(x)$ and $f(1) = 0$, then it must be $\frac{1}{2}(\log x)^2$. This is the reason why the logarithm of $T^\theta(t_C)$ equals $-L^\theta(C) = -N\theta(\frac{1}{2}(\log x)^2)$, where C is represented by $x \in \pi$.

The map $-\lambda_\theta : \mathbb{Q}\hat{\pi} \to \mathrm{Der}_\omega(\widehat{T})$ is a Lie algebra homomorphism. Using this homomorphism, we can compute the center of the Goldman Lie algebra of an oriented surface of infinite genus [7].

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Circle homeomorphisms and shears

Dragomir Šarić

Denote by $Homeo(S^1)$ the group of orientation preserving homeomorphisms of the unit circle S^1. An orientation preserving homeomorphism $h : S^1 \to S^1$ is said to be *quasisymmetric* if there exists $M \geq 1$ such that

$$\frac{1}{M} \leq \frac{|h(I)|}{|h(J)|} \leq M$$

where $I, J \subset S^1$ are adjacent intervals with disjoint interiors of equal length $|I| = |J|$. Denote by $QS(S^1)$ the group of all quasisymmetric maps of S^1. An orientation preserving homeomorphism $h : S^1 \to S^1$ is said to be *symmetric* if

$$\sup_{|I|=|J|\leq\epsilon} \frac{|h(I)|}{|h(J)|} \to 1$$

as $\epsilon \to 0$. The group of symmetric maps is denoted by $Symm(S^1)$. Let $M\ddot{o}b(S^1)$ be the group of orientation preserving fractional linear transformations which preserve the unit circle S^1.

The hyperbolic plane $\mathbf{H}$ has the unit circle S^1 as the ideal boundary and $M\ddot{o}b(S^1)$ is identified with the isometry group of $\mathbf{H}$. Farey tesselation $\mathcal{F}$ of the hyperbolic plane $\mathbf{H}$ is an ideal triangulation of $\mathbf{H}$ which is invariant under the group of hyperbolic inversions in the edges of $\mathcal{F}$. More precisely, $\mathcal{F}$ is the set of edges of the above tesselation. If one component of the complement of $\mathcal{F}$ is the ideal hyperbolic triangle with vertices $1, i, -1 \in S^1$ then $\mathcal{F}$ is uniquely determined by the above property.

The set $\mathcal{G}(\mathbf{H})$ of oriented geodesics of $\mathbf{H}$ is identified with $S^1 \times S^1 \setminus diag$ by assigning to each oriented geodesic of $\mathbf{H}$ the pair of its ideal endpoints on S^1. A homeomorphism $h : S^1 \to S^1$ extends to a homeomorphism $h : \mathcal{G}(\mathbf{H}) \to \mathcal{G}(\mathbf{H})$ via this identification. Consequently, $h \in Homeo(S^1)$ maps Farey tesselation $\mathcal{F}$ onto another ideal geodesic triangulation $h(\mathcal{F})$ of $\mathbf{H}$.

Let Δ_1 and Δ_2 be two ideal geodesic triangles in $\mathbf{H}$ with ideal vertices a, b, d and b, c, d, disjoint interiors and common boundary side $(b, d) \in \mathcal{G}(\mathbf{H})$. The *shear* $s(\Delta_1, \Delta_2)$ of the two triangles is the signed hyperbolic distance between the orthogonal projections of a and d onto the common boundary side (b, c). For example, all shears for the adjacent complementary triangles of $\mathcal{F}$ are zero. An edge $e \in \mathcal{F}$ is common to exactly two complementary triangles Δ_1^e, Δ_2^e. Each $h \in Homeo(S^1)$ induces a *shear function*

$$s : \mathcal{F} \to \mathbf{R}$$

by the rule

$$s(e) = s(\Delta_1^e, \Delta_2^e).$$

Two maps $h_1, h_2 \in Homeo(S^1)$ differ by a post-composition with some $\gamma \in M\ddot{o}b(S^1)$ if and only if their induced shear functions s_1, s_2 are equal. In other words, the assignment of shear functions is bijective on the sets $M\ddot{o}b(S^1)/Homeo(S^1)$, $M\ddot{o}b(S^1)/QS(S^1)$ and $M\ddot{o}b(S^1)/Symm(S^1)$. R. Penner [2] posed the problem of finding characterizations of shear functions which arise from homeomorphisms, quasisymmetric maps or symmetric maps. R. Penner and D. Sullivan [2] gave a sufficient condition in terms of lambda lengths such that the induced map is quasisymmetric. The space $M\ddot{o}b(S^1)/QS(S^1)$ is the *universal Teichmüller space* $T(\mathbf{H})$ which is the Teichmüller space of the hyperbolic plane $\mathbf{H}$. Thus the posed problem is to find a parametrization of the universal Teichmüller space $T(\mathbf{H})$ in terms of shear functions.

In the case of finite area hyperbolic punctured surface S, Thurston and Penner [3] gave parametrization of the Teichmüller space $T(S)$ in terms of shear coordinates on locally finite ideal triangulations of S. For closed hyperbolic surface S, Thurston and Bonahon [1] gave parametrization of $T(S)$ in terms of shears on locally infinite ideal geodesic triangulations of S.

We give a parametrization of $T(\mathbf{H})$ in terms of shears on $\mathcal{F}$. A *fan of geodesics* $\mathcal{F}_p$ with *tip* $p \in \mathcal{F}^0$ in $\mathcal{F}$ consists of all geodesics in $\mathcal{F}$ which have one ideal endpoint p, where $\mathcal{F}^0$ consists of vertices of $\mathcal{F}$ which are the endpoints of edges of $\mathcal{F}$. Let C_p be a horocycle with center p oriented such that the corresponding horoball is to the left. Then, for $e, e' \in \mathcal{F}_p$, we define ordering by $e < e'$ if $C \cap e$ is before $C \cap e'$ for the orientation on C, and $e' < e$ otherwise. The ordering on $\mathcal{F}_p$ puts it in a one-one correspondence with the integers $\{e_n\}_{n \in \mathbf{Z}} = \mathcal{F}_p$, where $e_n < e_{n+1}$ for all $n \in \mathbf{Z}$.

Let $s : \mathcal{F} \to \mathbf{R}$ be an arbitrary function. In general, the function s is not necessarily induced by a homeomorphism. However, there exists a developing map $h_s : \mathcal{F}^0 \to S^1$ which realizes $s : \mathcal{F} \to \mathbf{R}$. The map h_s does not necessarily extend to a homeomorphism of S^1. Let $h_s(\mathcal{F}_p)$ be the image of the fan $\mathcal{F}_p$ of $\mathcal{F}$ and let C' be a horocycle with center $h_s(p)$. For $m \in \mathbf{Z}$ and $k \in \mathbf{N} \cup \{0\}$, let $s(p; m, k)$ be the ratio of the length of the arc of C' between $C' \cap h_s(e_m)$ and $C' \cap h_s(e_{m+k+1})$ to the length of the arc of C' between $C' \cap h_s(e_m)$ and $C' \cap h_s(e_{m-k-1})$.

Theorem 1. [4] *Let $s : \mathcal{F} \to \mathbf{R}$ be an arbitrary function. Then s is induced by a quasisymmetric map of the unit circle S^1 if and only if there exists $M \geq 1$ such that*

$$\frac{1}{M} \leq s(p; m, k) \leq M$$

for all $p \in \mathcal{F}^0$, $m \in \mathbf{Z}$ and $k \in \mathbf{N} \cup \{0\}$.

Moreover, the function s is induced by a symmetric map if and only if

$$s(p; m, k) \rightrightarrows 1$$

as Farey generations of e_{m+k+1} and e_{m-k-1} converge to infinity independently of the tip p.

The above theorem establishes parametrization of the universal Teichmüller space $T(\mathbf{H})$ and $M\ddot{o}b(S^1)/Symm(S^1)$. One should note that the formula for $s(p; m, k)$ involves only shears in the fan $\mathcal{F}_p$. Thus to guarantee quasisymmetry it is enough to consider relations between shears in fans and there is no need to combine shears from different fans.

A *chain of geodesics* in $\mathcal{F}$ is a sequence $\{e_n\}_{n\in\mathbf{N}}$ of geodesics of $\mathcal{F}$ such that e_n and e_{n+1} are adjacent and no repeating is allowed. Note that the geodesics of a chain do not necessarily all belong to a single fan. In fact, geodesics of most chains belong to infinitely many fans. The set of geodesics of a fan accumulate to a unique point of S^1 and each point of S^1 is the accumulation point of chains of geodesics.

Theorem 1. [4] *A function $s : \mathcal{F} \to \mathbf{R}$ is induced by a homeomorphism of S^1 if and only if for each chain of geodesics $\{e_n\}_{n\in\mathbf{N}}$ we have*

$$\sum_{n=1}^{\infty} \exp(s_{1,n} + s_{2,n} + \cdots + s_{n,n}) = \infty$$

where $s_{i,n} = \pm s(e_i)$. In more details, $s_{n,n} = s(e_n)$ if $e_n < e_{n+1}$, otherwise $s_{n,n} = -s(e_n)$. If $n > 1$ and $i < n$, then $s_{i,n} = s(e_i)$ if $e_i < e_{i+1}$ and the number of times we change fans from e_i to e_{n+1} is even, or if $e_{i+1} < e_i$ and the number of times we change fans from e_i to e_{n+1} is odd. Otherwise $s_{i,n} = -s(e_i)$.

Let $t \mapsto h_t$ be a differentiable path of quasisymmetric maps which passes through the identity when $t = 0$. Then the derivative $\frac{d}{dt}h_t|_{t=0} = V$ is a Zygmund vector field, and each Zygmund vector field arises in this fashion. The corresponding path of shear functions $s_t : \mathcal{F} \to \mathbf{R}$ is differentiable, and $\frac{d}{dt}s_t|_{t=0} = \dot{s} : \mathcal{F} \to \mathbf{R}$ determines the vector field V up to infinitesimally trivial vector field.

Theorem 2. [5] *A shear function* $\dot{s} : \mathcal{F} \to \mathbf{R}$ *is induced by a Zygmund vector field if and only if there exists* $C > 0$ *such that for all* $m \in \mathbf{Z}$, $k \in \mathbf{N} \cup \{0\}$ *and all fans* $\mathcal{F}_p = \{e_n\}_{n\in\mathbf{Z}}$ *we have*

$$\left|\dot{s}(e_m) + \frac{k}{k+1}[\dot{s}(e_{m+1}) + \dot{s}(e_{m-1})] + \cdots + \frac{1}{k+1}[\dot{s}(e_{m+k}) + \dot{s}(e_{m-k})]\right| \leq C$$

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Polyhedral realization of the Thurston compactification

Yohei Komori

(joint work with Matthieu Gendulphe)

1. Introduction

Let $\Sigma = \Sigma_{g,n}$ (resp. $\Sigma^-_{g,n}$) be an orientable (resp. a non-orientable) compact surface of genus g with n boundary components, whose Euler-Poincaré characteristic $\chi(\Sigma)$ is negative. We denote $\mathscr{S}$ the set of free homotopy classes of simple closed curves on Σ, which do not retract into a point or a boundary component. The *Teichmüller space* $\mathsf{Teich}(\Sigma)$ is the space of isotopy classes of hyperbolic metrics on Σ with totally geodesic boundary and boundary components of fixed lengths. It admits a natural smooth structure, for which it embeds in the projective space $\mathbf{P}(\mathbf{R}^{\mathscr{S}})$ *via* length functions of geodesics. The image of $\mathsf{Teich}(\Sigma)$ in $\mathbf{P}(\mathbf{R}^{\mathscr{S}})$ is an open ball of dimension $-3\chi(\Sigma) - n$, where n denotes the number of boundary components.

The *Thurston boundary* of the Teichmüller space is the boundary of $\mathsf{Teich}(\Sigma)$ in $\mathbf{P}(\mathbf{R}^{\mathscr{S}})$. It is a topological sphere of dimension $\dim(\mathsf{Teich}(\Sigma)) - 1$ denoted $\partial\mathsf{Teich}(\Sigma)$. *Via* intersection functions, the Thurston boundary is canonically identified with the projectivized space of measured foliations $\mathsf{PMF}(\Sigma)$, and with the projectivised space of measured geodesic laminations $\mathsf{PML}(\Sigma)$. The *Thurston compactification* of the Teichmüller space is the adherence $\overline{\mathsf{Teich}}(\Sigma)$ of $\mathsf{Teich}(\Sigma)$ in $\mathbf{P}(\mathbf{R}^{\mathscr{S}})$.

In this talk we will study the following problem: whether the Thurston compactification $\overline{\mathsf{Teich}}(\Sigma)$ can be realized as a polyhedron in the finite dimensional projective space $\mathbf{P}(\mathbf{R}^N)$ if we chose N elements of $\mathscr{S}$ properly (c.f. [1, 2, 4]).

In practice, we will treat the case for Σ^-_3, connected sum of three projective planes, and consider this problem for two cases of curve systems.

2. Geometry and topology of the non-orientable surface of genus 3

Let X be a hyperbolic connected sum of three projective planes.

Proposition 1. *There is a unique simple closed geodesic σ in X which produces a one holed torus $\mathbb{T}_X$ after cutting.*

This induces a canonical bijection between $\mathsf{Teich}(\Sigma_3^-)$ and $\cup_{b>0}\mathsf{Teich}_b(\Sigma_{1,1})$.

Proposition 2. *Let γ be a simple closed geodesic of X distinct from σ.*

(1) *If γ is orientable, then γ is disjoint from σ.*
(2) *If γ is non-orientable, then γ intersect σ in exactly one point.*
(3) *There exists a unique simple closed geodesic $\gamma' \neq \sigma$ disjoint from γ which has opposite orientability. We say that γ and γ' are* duals.

The duality defines an involution of $\mathscr{S}$ which has γ_X as unique fixed point. The lengths of an orientable simple closed geodesic γ and its dual γ' are related by

$$\cosh\frac{\ell(\gamma)}{2} \quad = \quad \sinh\frac{\ell(\gamma')}{2}\ \sinh\frac{\ell(\sigma)}{2}. \tag{1}$$

The corresponding identity on the intersection numbers is

$$\mathsf{i}(\gamma,\cdot) \quad = \quad \mathsf{i}(\gamma',\cdot)+\mathsf{i}(\sigma,\cdot) \text{ on } \mathscr{S}\setminus\{\gamma',\sigma\}. \tag{2}$$

Definition 1. A *triangle* is a triple (α,β,γ) of orientable simple closed geodesics with all intersection numbers equal to one.

Remark 1. A triple (α,β,γ) is a triangle if and only if its dual triple (α',β',γ') consists of three disjoint simple closed geodesics.

Proposition 3. *The length functions with respect to a triangle (α,β,γ) of Σ_3^- define the injective map from the Teichmüller space $T(\Sigma_3^-)$ into $\mathbb{R}^3$.*

3. Two polyhedral realizations of $T(\Sigma_3^-)$ in $P(\mathbb{R}^4)$

Let (α,β,γ) a triple of Σ_3^-.

Theorem 1. *The following maps are embeddings of $T(\Sigma_3^-)$ into $P(\mathbb{R}^4)$:*

$$\begin{aligned} L_1 : T(\Sigma_3^-) &\to P(\mathbb{R}^4)\\ x &\mapsto (\ell_\alpha(x) : \ell_\beta(x) : \ell_\gamma(x) : \ell_\sigma(x)) \end{aligned}$$

$$\begin{aligned} L_2 : T(\Sigma_3^-) &\to P(\mathbb{R}^4)\\ x &\mapsto (\ell_\alpha(x) : \ell_\beta(x) : \ell_\gamma(x) : \ell_{\alpha'+\beta'+\gamma'}(x)) \end{aligned}$$

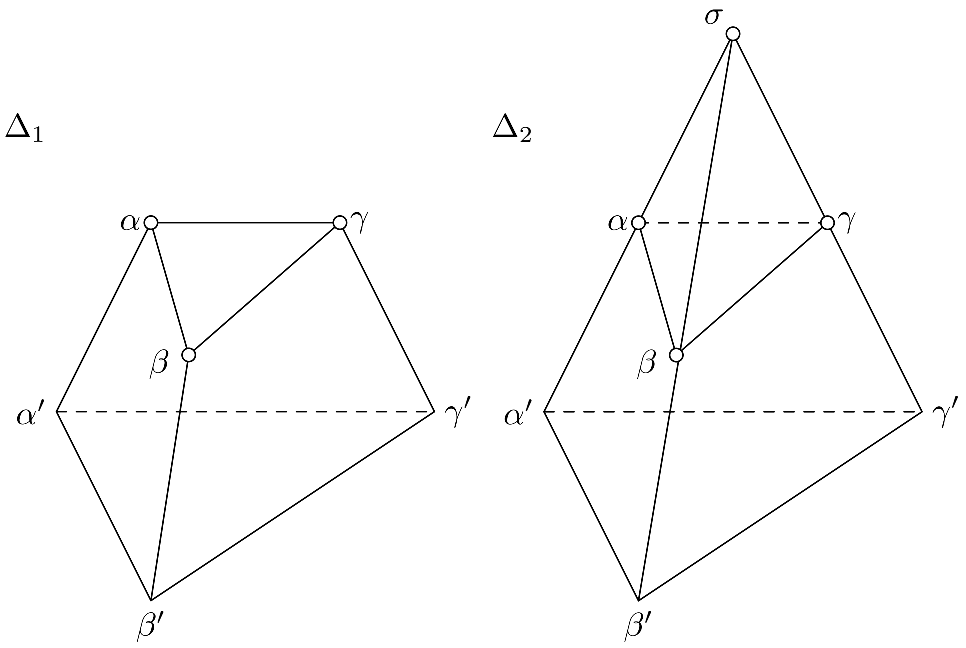
Corollary 2. *The images of L_1 and L_2 are convex polyhedra in $P(\mathbb{R}^4)$:*

- *The image $L_1(T(\Sigma_3^-))$ is the truncated simplex defined by*

$$\Delta_1 := \{(a:b:c:d)\in\mathcal{P}(\mathbb{R}^4) \ \mid \ a+b>c,\ b+c>a,\ c+a>b,\ d>0 \text{ and } a+b+c>2d\}.$$

- *The image* $L_2(T(\Sigma_3^-))$ *is the simplex defined by*

$$\Delta_2 := \{(a:b:c:d) \in \mathcal{P}(\mathbb{R}^4) \mid a+b>c,\ b+c>a,\ c+a>b \text{ and } d>0\}.$$



4. Main results

Theorem 2. (1) *The truncated simplex* Δ_1 *is a compactification of* $T(\Sigma_3^-) = \cup_{b>0}\mathsf{Teich}_b(\Sigma_{1,1})$. *Its boundary decomposes into two pieces: One piece corresponds to the set of projective measured foliations of a one-holed torus, where leaves transverse to the boundary are allowed. The other piece corresponds to the Teichmüller space of once-punctured tori. The frontier between these two pieces is* $\mathsf{PMF}(\Sigma_{1,1})$.

(2) *The simplex* Δ_2 *is a convex polyhedral realization of the Thurston compactification of* $T(\Sigma_3^-)$. *The map* $(\ell_\alpha, \ell_\beta, \ell_\gamma, \ell_{\alpha'+\beta'+\gamma'})$ *is an embedding of* $T(\Sigma_3^-)$ *onto the interior of* Δ_2. *It has a continuous extension which induces an isomorphism given by* $(i_\alpha, i_\beta, i_\gamma, i_{\alpha'+\beta'+\gamma'})$ *between* $\mathsf{PMF}(\Sigma_3^-)$ *and the boundary of* Δ_2.

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On the Johnson homomorphisms of the automorphism group of a free group

Takao Satoh

Let F_n be a free group of rank $n \geq 2$, H the abelianization of F_n and $\mathrm{Aut}\, F_n$ the automorphism group of F_n. The kernel of a homomorphism $\mathrm{Aut}\, F_n \to \mathrm{GL}(n, \mathbb{Z})$ induced from the action of $\mathrm{Aut}\, F_n$ on H is called the IA-automorphism group of F_n, denoted by IA_n. The subgroup IA_n reflects much of the richness and complexity of the structure of $\mathrm{Aut}\, F_n$, and plays important roles on various studies of $\mathrm{Aut}\, F_n$.

Let $\Gamma_n(k)$ be the lower central series of F_n. For each $k \geq 1$, we denote by $\mathcal{A}_n(k)$ the kernel of a natural homomorphism $\mathrm{Aut}\, F_n \to \mathrm{Aut}(F_n/\Gamma_n(k+1))$ induced from the action of $\mathrm{Aut}\, F_n$ on $F_n/\Gamma_n(k+1)$. The groups $\mathcal{A}_n(k)$ define a descending central filtration

$$\mathrm{IA}_n = \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots .$$

This is called the Johnson filtration of $\mathrm{Aut}\, F_n$. Each graded quotient $\mathrm{gr}^k(\mathcal{A}_n) := \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$ naturally has a $\mathrm{GL}(n, \mathbb{Z})$-module structure, and from it we can extract some valuable information for IA_n. For example, $\mathrm{gr}^1(\mathcal{A}_n)$ is just the abelianization of IA_n due to Andreadakis [1], and $\mathrm{gr}^2(\mathcal{A}_n)$ is applied to determine the image of the cup product $\cup_{\mathbb{Q}} : \Lambda^2 H^1(\mathrm{IA}_n, \mathbb{Q}) \to H^2(\mathrm{IA}_n, \mathbb{Q})$ by Pettet [5].

In order to study the $\mathrm{GL}(n, \mathbb{Z})$-module structure of $\mathrm{gr}^k(\mathcal{A}_n)$, we consider a $\mathrm{GL}(n, \mathbb{Z})$-equivariant injective homomorphism

$$\tau_k : \mathrm{gr}^k(\mathcal{A}_n) \to \mathrm{Hom}_{\mathbb{Z}}(H, \mathcal{L}_n(k+1)) = H^* \otimes_{\mathbb{Z}} \mathcal{L}_n(k+1)$$

defined by

$$\sigma \;\mapsto\; (x \mapsto x^{-1}x^{\sigma}), \quad x \in H$$

where $\mathcal{L}_n(k) = \Gamma_n(k)/\Gamma_n(k+1)$. The map τ_k is called the k-th Johnson homomorphism of $\mathrm{Aut}\, F_n$. Historically, the study of the Johnson homomorphisms was originally begun in 1980 by D. Johnson who determined the abelianization of the Torelli subgroup of the mapping class group of a surface.

Set $\tau_{k,\mathbb{Q}} := \tau_k \otimes \mathrm{id}_{\mathbb{Q}}$. Then we have a natural question to ask:

Problem 1. Determine the $\mathrm{GL}(n, \mathbb{Z})$-module structure of the image, or equivalently the cokernel of $\tau_{k,\mathbb{Q}}$.

For $1 \leq k \leq 3$, $\mathrm{Coker}(\tau_{k,\mathbb{Q}})$ is determined. Furthermore, by a recent remarkable work of Morita [4], it is known that there appears the symmetric tensor product $S^k H_{\mathbb{Q}}$ of $H_{\mathbb{Q}} := H \otimes_{\mathbb{Z}} \mathbb{Q}$ in $\mathrm{Coker}(\tau_{k,\mathbb{Q}})$.

Here, we consider the lower central series $\mathcal{A}'_n(1) = \mathrm{IA}_n, \mathcal{A}'_n(2), \ldots$ of IA_n. Since the Johnson filtration is central, $\mathcal{A}'_n(k) \subset \mathcal{A}_n(k)$ for any $k \geq 1$. It is conjectured that $\mathcal{A}'_n(k) = \mathcal{A}_n(k)$ for each $k \geq 1$ by Andreadakis who showed $\mathcal{A}'_2(k) = \mathcal{A}_2(k)$ for each $k \geq 1$. Now, it is known that $\mathcal{A}'_n(2) = \mathcal{A}_n(2)$ due to Bachmuth [2], and that $\mathcal{A}'_n(3)$ has at most finite index in $\mathcal{A}_n(3)$ due to Pettet [5].

For each $k \geq 1$, set $\mathrm{gr}^k(\mathcal{A}'_n) := \mathcal{A}'_n(k)/\mathcal{A}'_n(k+1)$. Then we can define a $\mathrm{GL}(n,\mathbb{Z})$-equivariant homomorphism

$$\tau'_k : \mathrm{gr}^k(\mathcal{A}'_n) \to H^* \otimes_{\mathbb{Z}} \mathcal{L}_n(k+1)$$

by the same way as τ_k. We also call τ'_k the Johnson homomorphism of $\mathrm{Aut}\, F_n$. We are interested in the study of the cokernel of τ'_k for the following reasons:

(1) We can directly compute $\mathrm{Coker}(\tau'_k)$ using finitely many generators of $\mathrm{gr}^k(\mathcal{A}'_n)$.
(2) By determining $\mathrm{Coker}(\tau'_{k,\mathbb{Q}})$, we obtain an "upper bound" on $\mathrm{Coker}(\tau_{k,\mathbb{Q}})$ as a $\mathrm{GL}(n,\mathbb{Z})$-module.

We remark that if the Andreadakis's conjecture is true, we have $\mathrm{Coker}(\tau_{k,\mathbb{Q}}) = \mathrm{Coker}(\tau'_{k,\mathbb{Q}})$.

In this note, we show that the cokernel of the rational Johnson homomorphism $\tau'_{k,\mathbb{Q}} := \tau'_k \otimes \mathrm{id}_{\mathbb{Q}}$ is determined in stable range. Namely, we have

Theorem 1. *For any $k \geq 2$ and $n \geq k+2$,*

$$\mathrm{Coker}(\tau'_{k,\mathbb{Q}}) = \mathcal{C}_n^{\mathbb{Q}}(k).$$

Here $\mathcal{C}_n(k)$ be a quotient module of $H^{\otimes k}$ by the action of cyclic group of order k on the components:

$$\mathcal{C}_n(k) := H^{\otimes k} \big/ \langle a_1 \otimes \cdots \otimes a_k - a_2 \otimes \cdots \otimes a_k \otimes a_1 \mid a_i \in H \rangle,$$

and $\mathcal{C}_n^{\mathbb{Q}}(k) := \mathcal{C}_n(k) \otimes_{\mathbb{Z}} \mathbb{Q}$.

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Johnson homomorphisms in knot theory

Takuya Sakasai
(joint work with Hiroshi Goda, [2, 3], see also [10])

1. Homology cylinders and homologically fibered knots

Let $\Sigma_{g,1}$ be a compact connected oriented surface of genus $g \geq 0$ with one boundary component, and let $\mathcal{M}_{g,1}$ be the mapping class group of $\Sigma_{g,1}$. We study $\mathcal{M}_{g,1}$ from a view point of 3-dimensional topology by considering the following *enlargement* of $\mathcal{M}_{g,1}$:

Definition 1. A *homology cylinder* over $\Sigma_{g,1}$ consists of a compact oriented 3-manifold M with two embeddings $i_+, i_- : \Sigma_{g,1} \hookrightarrow \partial M$, called the *markings*, such that:

(i) i_+ is orientation-preserving and i_- is orientation-reversing;
(ii) $\partial M = i_+(\Sigma_{g,1}) \cup i_-(\Sigma_{g,n})$ and $i_+(\Sigma_{g,1}) \cap i_-(\Sigma_{g,1}) = i_+(\partial\Sigma_{g,1}) = i_-(\partial\Sigma_{g,1})$;
(iii) $i_+|_{\partial\Sigma_{g,1}} = i_-|_{\partial\Sigma_{g,1}}$;
(iv) $i_+, i_- : H_*(\Sigma_{g,1}) \to H_*(M)$ are isomorphisms (i.e. M is a homology product).

We denote by $\mathcal{C}_{g,1}$ the set of all marking-preserving diffeomorphism classes of homology cylinders over $\Sigma_{g,1}$. It has a natural monoid structure given by stacking.

Example 1. For each diffeomorphism φ of $\Sigma_{g,1}$ which fixes $\partial\Sigma_{g,1}$ pointwise, we can construct a homology cylinder by setting

$$(\Sigma_{g,1} \times [0,1], \mathrm{id} \times 1, \varphi \times 0).$$

This construction gives a monoid homomorphism from $\mathcal{M}_{g,1}$ to $\mathcal{C}_{g,1}$. In fact, it is an injective homomorphism.

The structure of the monoid $\mathcal{C}_{g,1}$ was systematically studied by Goussarov [4] and Habiro [5] in their theory of finite-type invariants of 3-manifolds using *clasper* (or *clover*) *surgeries*. See also Garoufalidis-Levine [1] and Levine [7].

In [2], we started to study homology cylinders from knot theory. For a given knot K in the 3-sphere S^3 with a Seifert surface R, a sutured manifold (M_R, K) called the *complementary sutured manifold* is obtained by cutting the knot complement along R. The boundary of M_R is divided into two copies of R along K, so that we may regard M_R as a cobordism between them. A natural question is: *When a complementary sutured manifold becomes a homology cylinder* (*after fixing a pair of markings of boundary*)? We can show that such a case occurs exactly when a knot K is in the following class:

Definition 2 ([2])**.** A knot K in S^3 is called a *homologically fibered knot* of genus g if it has the following properties which are equivalent to each other:

(a) The Alexander polynomial $\Delta_K(t)$ of K is monic (i.e. the leading coefficient is ± 1) and its degree is equal to twice the genus $g = g(K)$ of K;

(b) For any minimal genus Seifert surface R of K, its Seifert matrix S is invertible over $\mathbb{Z}$;
(c) The sutured manifold (M_R, K) for any minimal genus Seifert surface R is a homology product over $\mathbb{Z}$.

It is well known that fibered knots satisfy the above conditions. They define homology cylinders with the product cobordism $\Sigma_{g,1} \times [0,1]$ whose markings are induced from *monodromies* of fibered knots. Roughly speaking,

$$\text{“}\mathcal{C}_{g,1} : \mathcal{M}_{g,1} = \text{(Homologically fibered knots) : (Fibered knots)”}$$

holds. Note that our construction gives *visible* homology cylinders.

2. Johnson homomorphisms

Johnson homomorphisms for mapping class groups were originally defined by Johnson [6] and generalized by Morita [8]. They form a sequence of homomorphisms

$$\tau_k : \mathcal{M}_{g,1}[k+1] \longrightarrow \mathfrak{h}_g(k)$$

for $k \geq 1$, where $\mathcal{M}_{g,1}[k]$ is the k-th term of a filtration, called the *Johnson filtration*, of $\mathcal{M}_{g,1} = \mathcal{M}_{g,1}[1]$ determined by its action on the fundamental group of $\Sigma_{g,1}$ and $\mathfrak{h}_g(k)$ is a certain free abelian group obtained from tensor products of $H := H_1(\Sigma_{g,1})$. The module $\mathfrak{h}_g(k)$ (resp. $\mathfrak{h}_g(k) \otimes \mathbb{Q}$) has a natural action of the Siegel modular group $Sp(2g, \mathbb{Z})$ (resp. $Sp(2g, \mathbb{Q})$), which yields a strong connection to the theory of finite-type invariants.

In Habiro [5] and Garoufalidis-Levine [1], the Johnson filtration and Johnson homomorphisms were extended to $\mathcal{C}_{g,1}$. That is, for each $k \geq 1$, we have a homomorphism

$$\tau_k : \mathcal{C}_{g,1}[k+1] \longrightarrow \mathfrak{h}_g(k).$$

Moreover, Garoufalidis-Levine showed that τ_k is surjective for every k. On the other hand, it was shown by Morita [9] that τ_k is generally *not* surjective for $\mathcal{M}_{g,1}[k]$. It has been an important problem to know the structure of the cokernels $\mathfrak{h}_g(k)/\tau_k(\mathcal{M}_{g,1}[k+1])$ and their *topological meaning*. Our recent research is intended to understand them in the most direct way: We use them as fibering obstructions of homologically fibered knots.

Let K be a homologically fibered knot with a minimal genus (say g) Seifert surface R and let $M_R = (M_R, K)$ be its complementary sutured manifold, which is a homology cylinder over $\Sigma_{g,1}$. If K is a fibered knot, then there exists a mapping class $[\varphi] \in \mathcal{M}_{g,1}$ such that $M_R = [\varphi] \in \mathcal{M}_{g,1}$. In particular, $M_R \cdot [\varphi]^{-1}$ is in the kernel of all τ_k. In the case where K is not fibered, there *may* exist some $k \geq 2$ such that no mapping classes $[\psi] \in \mathcal{M}_{g,1}$ satisfy $M_R \cdot [\psi]^{-1} \in \operatorname{Ker} \tau_k$. We now define

$$J(K,R) = \min\{k \mid \nexists [\psi] \in \mathcal{M}_{g,1} \text{ s.t. } M_R \cdot [\psi]^{-1} \in \operatorname{Ker} \tau_k\},$$
$$J(K) = \min_R J(K,R).$$

By definition, $J(K) = \infty$ holds for any fibered knot K. By using these numbers, our main results are given as follows:

Theorem 1. (i) *For any $g \geq 2$, there exist homologically fibered knots K with $J(K) = 2$. In fact, we can check that all the non-fibered homologically fibered knots of* 12*-crossings, which consist of* 13 *knots, are such knots.*

(ii) *When $g = 1$, there exists a homologically fibered knot K with $J(K) \geq 3$.*

(iii) *For any $g \geq 2$ and $k \geq 2$, there exists a homologically fibered knot K with a minimal genus Seifert surface R with $J(K, R) = k$.*

(iv) *For any $g \geq 1$, there exists a non-fibered homologically fibered knot K with a minimal genus Seifert surface R with $J(K, R) = \infty$. Moreover, such a knot and a Seifert surface can be taken as those which are concordant to a fibered knot and its fiber surface.*

An example for (iii) is obtained from the fiber surface of a fibered knot by a clasper surgery. For the detection of the non-fiberedness of a knot of (iv), we may use Reidemeister torsion invariants for sutured manifolds.

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Equivalence relations on three-dimensional manifolds defined by subgroups of the Torelli group & the core of the Casson invariant

Gwénaël Massuyeau
(joint work with Jean–Baptiste Meilhan)

Let Σ be a compact connected oriented surface of genus g with one boundary component. A *homology cylinder* over Σ is a compact oriented 3-manifold M with an orientation-preserving homeomorphism $m : \partial\left(\Sigma \times [-1,1]\right) \to \partial M$ such that

$$\begin{array}{ccc} H_*\left(\Sigma \times [-1,1];\mathbb{Z}\right) & \overset{\exists}{\underset{\simeq}{\dashrightarrow}} & H_*(M;\mathbb{Z}) \\ {\scriptstyle \mathrm{incl}_*}\uparrow & & \uparrow{\scriptstyle \mathrm{incl}_*} \\ H_*\left(\partial(\Sigma \times [-1,1]);\mathbb{Z}\right) & \underset{m_*}{\overset{\simeq}{\longrightarrow}} & H_*(\partial M;\mathbb{Z}). \end{array}$$

Homology cylinders over Σ can be regarded as cobordisms (with corners) between two copies of Σ, namely from $m\left(\Sigma \times \{+1\}\right)$ to $m\left(\Sigma \times \{-1\}\right)$. Thus homology cylinders can be "composed" in the usual way so that, if we consider them up to homeomorphisms (that preserve orientations and boundary parametrizations), we get a monoid $\mathcal{IC}(\Sigma)$. For instance, $\mathcal{IC}(\Sigma)$ is in genus $g = 0$ isomorphic to the monoid of homology 3-spheres. In genus $g > 0$, the mapping cylinder construction

$$\mathbf{c} : \mathcal{I}(\Sigma) \longrightarrow \mathcal{IC}(\Sigma),\ s \longmapsto \big(\Sigma \times [-1,1], (\mathrm{Id} \times \{-1\}) \cup (\partial\Sigma \times \mathrm{Id}) \cup (s \times \{1\})\big)$$

defines an embedding of the Torelli group of the surface Σ into the monoid $\mathcal{IC}(\Sigma)$.

Two homology cylinders M and M' over Σ are said to be Y_k*-equivalent* if M' can be obtained from M by "twisting" an arbitrary embedded surface E in the interior of M with an element of the k-th term $\Gamma_k\mathcal{I}(E)$ of the lower central series of the Torelli group $\mathcal{I}(E)$ of E. (The surface E has an arbitrary position in M, but it is assumed to be compact connected oriented with one boundary component.) The J_k*-equivalence* relation on $\mathcal{IC}(\Sigma)$ is defined in a similar way using the k-th term of the Johnson filtration of $\mathcal{I}(E)$ instead of its lower central series: in other words, the "twisting" homeomorphism is required to act trivially at the level of the k-th nilpotent quotient $\pi_1(E)/\Gamma_{k+1}\pi_1(E)$ of the fundamental group $\pi_1(E)$. All these equivalence relations are organized as follows:

$$\begin{array}{ccccccccccc} Y_1 & \Longleftarrow & Y_2 & \Longleftarrow & Y_3 & \Longleftarrow & \cdots & Y_k & \Longleftarrow & Y_{k+1} & \Longleftarrow & \cdots \\ \| & & \Downarrow & & \Downarrow & & & \Downarrow & & \Downarrow & & \\ J_1 & \Longleftarrow & J_2 & \Longleftarrow & J_3 & \Longleftarrow & \cdots & J_k & \Longleftarrow & J_{k+1} & \Longleftarrow & \cdots \end{array}$$

The Y_k-equivalence relations have been introduced by Goussarov and Habiro in the context of finite-type invariants [1, 4]. They have developed a surgery calculus in dimension three, which is kind of a topological analogue of the commutator calculus in groups and is called "clasper calculus" [2, 4]. The Y_k-equivalence relations can be reformulated and studied using this clasper calculus. Having this strong tool at one's disposal is a big advantage of the Y_k-equivalence relations with respect to the J_k-equivalence relations.

The Y_1-equivalence relation is trivial on $\mathcal{IC}(\Sigma)$ [4, 3], whereas the Y_2-equivalence is a non-trivial relation whose classification is known [4, 8]. This talk reported on a work in progress [9], where we give a characterization of the Y_3-equivalence in terms of three classical invariants. The first invariant is the action of $M \in \mathcal{IC}(\Sigma)$ on the third nilpotent quotient of $\pi_1(\Sigma)$:

$$\rho_3(M) \in \mathrm{Aut}\big(\pi_1(\Sigma)/\Gamma_4\pi_1(\Sigma)\big).$$

The second invariant is, in some sense, the quadratic part of the Alexander polynomial of $M \in \mathcal{IC}(\Sigma)$ relative to its bottom boundary $m(\Sigma \times \{-1\})$, which we interpret as a degree 2 symmetric tensor over $H_1(\Sigma;\mathbb{Z})$:

$$\alpha(M) \in S^2\big(H_1(\Sigma;\mathbb{Z})\big).$$

To define the third and last invariant, we need to *choose* an embedding $j : \Sigma \hookrightarrow S^3$ such that $j(\Sigma)$ union with a disk splits S^3 into two handlebodies of genus g. Then, the Casson invariant of the homology 3-sphere obtained by "inserting" M into S^3 in a neighborhood of $j(S^3)$ is denoted by

$$\lambda_j(M) \in \mathbb{Z}.$$

Theorem A. *Two homology cylinders M and M' are Y_3-equivalent if, and only if, we have $\rho_3(M) = \rho_3(M')$, $\alpha(M) = \alpha(M')$ and $\lambda_j(M) = \lambda_j(M')$.*

In genus $g = 0$, Theorem A asserts that two homology 3-spheres are Y_3-equivalent if and only if they have the same Casson invariant, which is due to Habiro [4]. The theorem is proved by means of the LMO homomorphism introduced in [5], which is a generalization of the LMO invariant of homology 3-spheres [7]. We show that the degree ≤ 2 part of the LMO homomorphism classifies the Y_3-equivalence and we analyse how ρ_3, α and λ_j are encoded in this universal invariant.

In contrast with the J_1-equivalence, the J_2-equivalence is not trivial but classified by the action on the second nilpotent quotient of $\pi_1(\Sigma)$. This can be deduced from the characterization of the Y_2-equivalence given in [8] with a little bit of clasper calculus. Similarly, the following can be deduced from Theorem A and the existence, proved by Morita [10], of a homology 3-sphere whose Casson invariant is equal to ± 1 and which is J_3-equivalent to S^3.

Theorem B. *Two homology cylinders M and M' are J_3-equivalent if, and only if, we have $\rho_3(M) = \rho_3(M')$ and $\alpha(M) = \alpha(M')$.*

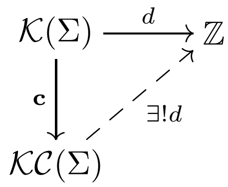
In genus $g = 0$, Theorem B asserts that any homology 3-sphere is J_3-equivalent to S^3. This fact was expected by Morita [10] and has been proved by Pitsch [12].

Although the invariant λ_j is easy to compute by surgery techniques, it is not completely satisfactory in that it depends on j. This phenomenon already appears at the level of the Torelli group, i.e. for the composition $\lambda_j \circ \mathbf{c} : \mathcal{I}(\Sigma) \to \mathbb{Z}$ which has been studied by Morita [10, 11]. More precisely, he has shown that its restriction to the Johnson subgroup $\mathcal{K}(\Sigma)$, i.e. to the second term of the Johnson filtration, is a group homomorphism which decomposes as

$$-\lambda_j \circ \mathbf{c}|_{\mathcal{K}(\Sigma)} = q_j + \frac{1}{24} d. \tag{1}$$

Here the homomorphism $q_j : \mathcal{K}(\Sigma) \to \mathbb{Q}$ is explicitly determined by the action on $\pi_1(\Sigma)/\Gamma_4\pi_1(\Sigma)$ in a way which involves j, while the homomorphism $d : \mathcal{K}(\Sigma) \to \mathbb{Z}$ does not depend on j. The J_3-equivalence relation being trivial for homology 3-spheres [12], formula (1) shows that all the information on homology 3-spheres carried by the Casson invariant is contained in this map d: thus Morita calls it the *core of the Casson invariant.* Let $\mathcal{KC}(\Sigma)$ be the submonoid of $\mathcal{IC}(\Sigma)$ that acts trivially on $\pi_1(\Sigma)/\Gamma_3\pi_1(\Sigma)$.

Theorem C. *There exists a unique extension of d to a monoid homomorphism $d : \mathcal{KC}(\Sigma) \to \mathbb{Z}$ which is invariant by Y_3-equivalence, by the mapping class group action and by stabilization of the surface Σ.*



The unicity of the extension of d is justified by comparing the decomposition of $\frac{\Gamma_2\mathcal{I}(\Sigma)}{\Gamma_3\mathcal{I}(\Sigma)} \otimes \mathbb{Q}$ into irreducible $\mathrm{Sp}(2g;\mathbb{Q})$-modules [6] to that of $\frac{Y_2\mathcal{IC}(\Sigma)}{Y_3} \otimes \mathbb{Q}$ [5], where $Y_2\mathcal{IC}(\Sigma)$ denotes the submonoid of homology cylinders M that are Y_2-equivalent to $\Sigma \times [-1,1]$. The existence can be proved by means of the LMO homomorphism. The extension of d to the monoid $\mathcal{KC}(\Sigma)$ takes the form

$$d = -24\,(\lambda_j + q_j) + (\text{something derived from } \alpha \text{ using } j).$$

This generalizes Morita's formula (1) since α is trivial on $\mathcal{K}(\Sigma)$.

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Problem session

(Problems compiled by Norbert A'Campo, with the help of Anna Wienhard)

Problem 1. (Athanase Papadopoulos) Study the length-spectrum-metric d_{ls} on Teichmüller space $T(S)$ for closed surfaces S of genus > 1. Here d_{ls} is the symmetrization of the non-symmetric Thurston metric d^{Thurs}.

Question: Does $\mathbf{Isom}(T(S), d_{ls})$ contain the extended mapping class group $\mathbf{Mod}^*(S)$ strictly?

Remark: $\mathbf{Isom}(T(S), d_{Thurs})$ coincides with $\mathbf{Mod}^*(S)$ by a Theorem of C. Walsh. After symmetrization the isometry group may become larger.

Study the length-spectrum-metric for surfaces of inffinite type.

Question: (Anna Wienhard) What is the horofunction boundary of $(T(S), d_{ls})$?

Remark: d_{ls} is a Finsler metric because d^{Thurs} is Finsler. Note that d_{ls} is not quasi-isometric to the Teichmüller metric d^{Teich}.

Question: (Lizhen Ji) Is λ_1 for the metric d_{ls} or for the metric d^{Thurs} positive?

The number λ_1 is defined as the infimum over all functions with compact support of the Raleigh quotient: $\frac{\int |df|^2}{\int |f|^2}$.

Remark: McMullen proved that $\lambda_1 > 0$ for d^{Teich}.

Problem 2. (Lizhen Ji) Study the volume growth of balls $B^{T(S)}_{p,d_{ls}}(R)$. Compute or estimate the exponential growth rate exponent and compare with known results for the Weil-Petersson metric.

Problem 3. (Anna Wienhard) Is there a Finsler metric on $T(S)$ whose horofunction compactification is the Bers compactification?

Problem 4. What are central problems? Compute homology of moduli spaces $M_{g,n}$. More precisely, find unstable classes. Study homology of finite index subgroups of mapping class groups $Mod_{g,n}$.

Problem 5. (Ursula Hamenstädt) Can you find a subgroup of finite index Γ in $MCG(S)$ with $H_1(\Gamma, \mathbb{Q}) \neq 0$?

Remark: If Γ contains the Torelli subgroup or even the Johnson subgroup of level 3, such a Γ does not exist by a result of Richard Hain.

Problem 6. Classification of finite dimensional representations of the mapping class groups.

Question: (Jorgen Andersen) What is the condition for the representation to be a part of a TQFT in dimension $2 + 1$?

Question: (Robert Penner) Are there faithful linear representations?

Question: (Norbert A'Campo) Find new finite index subgroups in mapping class groups which are not commensurable with kernels after reduction modulo n of representations on TQFT's $V_k(S)$ of some level k or with kernels of representations on homology $H_1(S, k)$, k finite.

Remark: (Vladimir Fock) Stabilizers Γ_ρ of points $\rho \in \mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2, \mathrm{F_q})$ give finite index subgroups that do not contain the Torelli subgroup.

Problem 7. (Lizhen Ji) Count subgroups of finite index: $a_k := \#\{\Gamma \mid [MCG : \Gamma] = k\}$. Study the generating function $f(z) = \sum_k a_k z^k$, e.g. growth of coefficients and properties of $f(z)$.

Problem 8. (Norbert A'Campo) Study examples of "small" subgroups in MCG that do not have a Nielsen realization, for instance prove that a subgroup in MCG(S_g), $g \geq 2$, generated by two Dehn twist with core curves that intersect transversally in one point does not have a Nielsen realization.

(Shigeyuki Morita) There are many examples of Kodaira-Atiyah complex surfaces X that fiber over S_g with fiber $S_{g'}$ and with non-vanishing signature. The problem is to find an example such that the monodromy group $\pi_1(S_g) \to MCG(S_{g'})$ is Nielsen realizable.

Remark: Such a surface bundle will not admit a flat connexion, since otherwise the signature would vanish. Also it is known that if we make a suitable fibre sum of each Kodaira-Atiyah complex surface with a trivial $S_{g'}$-bundle over S_h for some h, then the resulting monodromy group $\pi_1(S_{g+h}) \to MCG(S_{g'})$ is Nielsen realizable (Kotschick-Morita).

Problem 9. (Robert Penner) Weil-Peterssson volume of moduli spaces.

Problem 10. (Ursula Hamenstädt) Find a cell complex C of dimension $4g-5(=$ virtual cohomological dimension of MCG(S_g)), contractible, with a proper cocompact action of $MCG(S_g)$. Does it exist? Are there obstructions?

Problem 11. (Sumio Yamada) Given a harmonic map $f : D \to \bar{T(S)}^{WP}$ from a domain D to the completion of $T(S)$ with respect to the Weyl-Petersson metric.

Question: Does the image belong to one stratum?

Remark: The answer is yes for Dim$(D) = 1$ by Daskalopoulos-Wentworth-Wolpert.

Problem 12. (Vladimir Fock) Define basic length-convex sets as sublevel sets of the length functions $l_\gamma : T(S) \to \mathbb{R}$ of closed (simple?) curves on S. Define length-convex sets as intersections of basic length-convex sets. Length-convex sets are special, but what are they? What is for instance the length-convex hull of a two point set $\{p, q\}$? Same questions for the extremal length functions.

Problem 13. (Athanase Papadopoulos) Realize Teichmüller space as a bounded convex set somewhere and study the Hilbert metric on it.

Problem 14. (Lizhen Ji) The curve complex $C(S_g)$ is homotopy equivalent to a bouquet of spheres.

Construct geometrically finite subcomplexes, for which geometric and homological dimension coincide, that represent essential spheres of the bouquet or that are homotopically non trivial.

Remark: Penner has analyzed the arc complex modulo mapping class group.

Problem 15. (Ralph Kaufmann) Question: What is the cohomology theory corresponding to the loop spectrum constructed by stabilizing the arc operad? Answer by calculating the homotopy groups of this spectrum.

Question: Which classes of the Deligne-Mumford compactification come from the Penner compactification?

Question: Is the Sullivan space homotopic to a moduli space with some compactification? True for $g = 0, n, m = 0$.

Problem 16. (Gabriele Mondello) Let $T^{>\epsilon}(S_g)$, $g > 1$, be the thick part of Teichmüller space where all closed essential curves on S have length $> \epsilon$.

Question: Is $T^{>\epsilon}(S)$ a Stein manifold? If yes, construct a plurisubharmonic exhaustion.

Remark: The space $T^{>\epsilon}(S)$ is contractible. The length functions have the wrong convexity by a Theorem of Wolpert. Compare with problems $13, 14$.

Question: What is the Dolbeault cohomology of $T^{>\epsilon}(S)$?

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