Mathematisches Forschungsinstitut Oberwolfach

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Mini-Workshop: Small Deviation Problems for Stochastic Processes and Related Topics

Octobers 12th – Octobers 18th, 2003

The present mini–workshop was organized by W. V. Li (Newark, Delaware), M. A. Lifshits (St. Petersburg) and W. Linde (Jena). It was the basic idea of the workshop to bring together mathematicians working on small deviation problems for stochastic processes with those investigating tightly related problems. To this end there were eight general survey talks during the first two days, while during the following days other participants presented their own recent research results. The remaining time was used for discussions and joint research.

Abstracts

Properties of Fractional Integration Operators

Eduard S. Belinsky

Approximative properties of two basic operators of fractional integration are discussed. It is shown how the close functional analytic properties of the Riemann–Liouville and the Weyl operator of fractional integration imply their "closeness" in approximative properties. A basic result due to E. S. Belinsky and W. Linde asserts that the Kolmogorov widths and entropy numbers of their difference decrease exponentially. This means that the Kolmogorov widths and entropy numbers of both types of operators are of the same order.

Approximation of SαS Lévy Processes in L_p–Norm

Jakob Creutzig

We study approximation numbers (average linear widths, average Kolmogorov widths and quantization numbers) of stable processes and connections to small ball probabilities. It is well–known that in the case of Gaussian processes, these quantities enjoy, if polynomial, weak asymptotic equivalences. In the symmetric stable case, we show that a "good approximability" leads to a "smooth" small ball behaviour. By studying the symmetric α–stable Lévy process in the space L_p, we find that in the case of $p < \alpha$ this domination improves to an equivalence in weak asymptotics, just like in the Gaussian case. On the other hand, for $\alpha < p \leq 2$, we find a different asymptotical behaviour, so the domination is not always improvable to weak asymptotic equivalence. In the case $p < \alpha$, the known results about small ball behaviour provide an extremely easy and convenient method for deriving lower estimates for approximation numbers, which is usually a more challenging problem.

Coding Theory and Small Ball Problems

Steffen Dereich

We study the high resolution coding problem for Gaussian measures on separable Banach spaces. The coding problem concerns the finding of a good representation of a random signal, the original, within a class of allowed representations. The class of allowed representations is defined through restrictions on the information content of these representations. Here, the information content allows different definitions. Given one of the information constraints (typically a support, entropy or Shannon mutual information constraint) we study the asymptotic quality of reconstructions based on the representations, the coding error.

In this talk the asymptotics of the coding errors are related to small ball probabilities. Moreover, in the case where the underlying space is a Hilbert space, we give explicit solutions for these problems under weak assumptions on the eigenvalues. Afterwards we relate a particular random coding strategy to the asymptotics of small ball probabilities around random centres. Motivated by this link, we consider general properties of random small ball probabilities. Finally we compare the efficiency of the optimal coding strategy with the random coding strategy in the Hilbert space setup.

Intrinsic Volumes of Spherical Polar Sets and Connections With Small Ball Probabilities

Fuchang Gao

We introduced spherical intrinsic volumes, discussed their connections with small deviation probabilities, and presented how in principle some geometric tools, such as solid angles, can be used to evaluate the small deviation probabilities, especially for the Gaussian processes with some simple covariance structure. In particular, we showed how the evaluation can be done in the case of Brownian motion.

Probabilities of Small Deviations for Stationary Gaussian Processes

Sebastien Gengembre

We find the logarithmic small ball asymptotics of stationary Gaussian processes, whose spectral density has some asymptotic behaviour. To this end, we begin with the well known small ball behaviour of the fractional Brownian motion $W^{(a)}$, $0 < a < 2$. If we define the Ornstein-Uhlenbeck process $U^{(a)}$, $0 < a < 2$, by $\forall t \in I\!R\ U^{(a)}(t) = e^{-at/2} W^{(a)}(e^t)$, we obtain easily the same kind of small ball results, which means:
Let I be a bounded interval in $I\!R$ and $1 \le q < \infty$. Then

$$\lim_{\varepsilon \to 0} \varepsilon^{2/a} \log P \left(\left\| U^{(a)} \right\|_{L_q(I)} \le \varepsilon \right) = -c(a,q) \, |I|^{1+\frac{2}{aq}} .$$

These processes are stationary, and their respective spectral densities g_a, $0 < a < 2$, have the following asymptotic behaviour: $g_a(u) \sim_{u \to \infty} \lambda_a u^{-a-1}$, where λ_a are known constants. Now that the Ornstein-Uhlenbeck process is well known, we can deduce the following theorem:
Let $0 < a < 2$, X be a centred, stationary Gaussian process with spectral density f defined for all u in $I\!R$ by $f(u) = L(|u|) \, |u|^{-a-1}$, where L is a slowly varying function. Then if I is a finite interval in $I\!R$ and $q \in [1; \infty]$, we have

$$\lim_{\varepsilon \to 0} \varepsilon^{2/a} \log P \left(\|X\|_{L_q(I)} \le \varepsilon \sqrt{L(\varepsilon^{-2/a})} \right) = -\lambda_a^{-1/a} c(a,q) \, |I|^{1+\frac{2}{aq}}$$

To prove this theorem, for each $\varepsilon > 0$ we cut the spectral density of X into three parts, which give three stationary processes. The middle process has a small ball behaviour similar to the Ornstein-Uhlenbeck process' one, although the two other processes are negligible for the small ball problem.

Laplace Transforms of L_2-Ball, Comparison Theorems and Integrated Brownian Motions

Jan Hannig

(joint work with F. Gao, F. Torcaso, and T.-Y. Lee)

We consider the Laplace transforms of L_2 norms of Gaussian stochastic processes. Except for some special cases, exact Laplace transforms are, in general, rarely obtained. It is the purpose of this talk to show that for many Gaussian random processes the Laplace transform can be expressed in terms of well understood functions using complex-analytic theorems on infinite products, in particular, the Hadamard Factorization Theorem.

The second part of the talk concerns the generalization of comparison theorem of Li (1992) to more sums of general random variables.

Finally, we will introduce a class of (m-times) integrated Brownian motions and discuss the explicit form of Laplace transforms of L_2 ball together with exact asymptotic expansions for the L_2 small ball probabilities will be discussed for members of this class.

Entropy Numbers and Related Quantities

THOMAS KÜHN

The aim of this talk is to give a survey on some important results and techniques related to entropy numbers of operators in Banach spaces. The concept of metric entropy was already introduced in the 1930s, and since then it has been successfully applied in many branches of mathematics, e.g. in analysis (eigenvalue distributions), approximation theory, coding theory, learning theory, and also in probability. The surprisingly close connection to small ball problems for Gaussian measures was established by Kuelbs and Li in the early 1990s.

First we give the definitions and some properties of (several variants of) entropy numbers of sets and of operators in Banach spaces. Then we discuss the following topics: entropy classes; entropy numbers and eigenvalues; interpolation of entropy numbers; duality of entropy numbers; relation to approximation, Gelfand and Kolmogorov numbers; relation to Gaussian measures. The methods of proof for these results include volume estimates, operator ideal techniques, combinatorial and probabilistic arguments. Moreover, a few examples are mentioned, in particular diagonal operators in sequence spaces, and finally some open problems are stated.

Weighted Occupation Measures and Escape Rates for Certain Processes

JIM KUELBS

A variety of occupation measure results and escape rates for a stochastic process $\{X(t) : t \geq 0\}$ follow when the maximal process $\{M(t) = sup_{0 \leq s \leq t}|X(s)|\}$ satisfies a Wichura functional law of the iterated logarithm. The proof of these results depends on three steps: they are to first establish a suitable functional LIL for the maximal process in question, then to maximize or minimize the occupational measure of interest over the limit set obtained for the functional LL, and finally, as far as is possible, to provide an explicit evaluation of this extremum. Results of this type have been obtained for three broad classes of processes. They are symmetric α-stable Levy processes, γ-fractional Brownian motions, and stochastic integrals of the form

$$X(t) = \int_0^t < AB(s), dB(s) >, t \geq 0,$$

where A is a d by d skew symmetric real matrix and $\{B(t) : t \geq 0\}$ is standard Brownian motion in R^d. Wichura's fundamental result in this direction was a functional LIL for $\{M(t) = sup_{0 \leq s \leq t}|X(s)|\}$ when $\{X(t) : t \geq 0\}$ was Brownian motion in R^1. His proof involved diffusion process methods whereas here they are quite different, and even vary considerably over three classes of processes considered. This is joint work with Wenbo Li and Xia Chen.

Small Value Problems in Mathematics

Wenbo V. Li

We present a collection of problems and techniques in various parts of mathematics from the point of view of small value probabilities. We believe a theory of small value probabilities should be developed and centred on:

- systematically studies of the existing techniques and applications.
- applications of the existing methods to a variety of fields.
- new techniques and problems motivated by current interests of advancing knowledge.

Slides of my talk can be found in my web site: http://www.math.udel.edu/˜wli.

Probabilistic Aspects of Small Deviations

Mikhail A. Lifshits

Consider a random vector X in a normed space $(E, \|\cdot\|)$. This survey talk is devoted to some new results on small ball (small deviation) probabilities: $\mathbf{P}\{\|X\| \le \varepsilon\}$, $\varepsilon \to 0$. The behaviour of small ball probabilities has important connections with entropy of compact operators, quantization of random vectors, approximation of random processes (average case analysis), Strassen-type laws etc.

We first briefly consider the case when E is a Hilbert space, since a lot of new results were obtained here quite recently. The two-term asymptotics of covariance operator's eigenvalues and, subsequently, the small ball behaviour, were found for Brownian motion W, Brownian bridge B, Ornstein-Uhlenbeck process U, m-times integrated W, B, and U, for conditioned and averaged bridges etc.

For fractional Brownian motion the small ball asymptotic was recently found by Bronski, while Nazarov and Nikitin extended this result for multi-parametric fractional random functions with sufficiently regular covariance.

Next, we consider the small balls for fractional processes in more or less arbitrary norms. The main result (by Th. Simon and the author) is as follows. For any continuous H-self-similar α-stable Riemann-Liouville process $R^{H,\alpha}$ there exists a limit

$$K = \lim_{\varepsilon\to 0} \varepsilon^{\gamma} \left|\log \mathbf{P}\left\{\|R^{H,\alpha}\| \le \varepsilon\right\}\right| \in (0,\infty],$$

where $\gamma = (H-\beta-1/p)^{-1}$ and β, p are the self-similarity and additivity rates of the norm, respectively.

Finally, we discuss the small ball results and open problems for tensor products of Gaussian processes.

Small Ball Problems and Compactness of Operators

Werner Linde

Given an operator u from a separable Hilbert space $\mathbb{H}$ into $C(T)$, the space of continuous functions over a compact metric space T, it generates a centred Gaussian process $X = (X_t)_{t\in T}$ by

$$X_t := \sum_{k=1}^{\infty} \xi_k (u f_k)(t)\,, \quad t \in T\,.$$

Here $(f_k)_{k\ge 1}$ denotes an ONB in the Hilbert space $\mathbb{H}$ and $(\xi_k)_{k\ge 1}$ is an i.i.d. sequence of $\mathcal{N}(0,1)$–distributed random variables. We discuss how certain analytic properties of the operator u are reflected by probabilistic properties of the process X and vice versa.

Small deviations in L_2-norm. Some recent results.

Yakov Nikitin

(joint work with A. I. Nazarov)

We present a survey of some recent new results obtained in St.Petersburg on the asymptotics of small deviations in the L_2-norm of some Gaussian random processes and random fields. These results can be divided in three parts.

The first part of our results deals with the description of exact small deviation asymptotics for a large class of m-times integrated Gaussian processes including the Brownian motion, Brownian bridge, Ornstein-Uhlenbeck process and their centred (by their mass) modifications. We write down explicitly the bondary-value problem (BVP) for the ordinary differential operator corresponding to the m-times integrated process under consideration. Then we apply known results on the spectral asymptotics of such BVP's. Using the self-adjointness of our operators we are able to obtain the two-term spectral asymptotics for the eigenvalues. Next step is the careful inverting of the Laplace transform using the recent paper by Dunker, Lifshits and Linde(1998), which gives us the exponential and polynomial term of the spectral asymptotics. The multiplicative "distortion" constant can be found using the methods of complex analysis. Independently similar results were obtained by Gao et al. (2001 - 2003) in a series of papers.

The second part addresses the logarithmic small ball asymptotics for fractional Gaussian processes. We use some known results of Birman and Solomyak (1970 -74) on the spectral asymptotics of weakly polar integral operators which enable us to get the desired asymptotics not only for the fractional Brownian motion, but for its weighted and integrated counterparts, as well as for the fractional Ornstein-Uhlenbeck process and its integrated and weighted variants. It is interesting to note that three different definitions of the fractional Ornstein-Uhlenbeck process have different covariances but the same small deviation asymptotics. We describe also the logarithmic small ball asymptotics for the fractional multiparameter Levy's Brownian motion and obtain some vector-valued generalizations.

In the third part we are interested in a broad class of Gaussian random fields which are tensor products of simpler Gaussian processes and fields. First we find the spectral asymptotics of the integral operators which are the tensor products of simpler integral operators in terms of the factors. These results are new and can be interesting for the theory of compact self-adjoint operators in the Hilbert space.Then we get the logarithmic small ball asymptotics in L_2-norm for many interesting Gaussian fields. We begin by the case when the factors have the same order of eigenvalues decrease like in the (fractional) Brownian sheet and then discuss the general case. It is worth to note that if the factors have different order of eigenvalues decrease, then the spectrum of one factor, (the "fast" one) should be known explicitly while we need only the one-term spectral asymptotics for the second (the "slow" factor). We outline also the vector-valued generalizations.

Average-Case Analysis

Klaus Ritter

In the average-case analysis of a numerical problem one studies average errors, average cost, etc. of algorithms w.r.t. a prior distribution on the class of all problem instances. The complexity of the problem is then characterized by the sequence of minimal average errors e_n over the class of all algorithms with average cost bounded by n.

In our survey lecture we first discuss integration of monotone, continuous functions $[0, 1] \to \mathbb{R}$ as a motivating example. The Ulam (or Dubins-Freedman) measure is a natural prior on this class of functions, and its turns out that worst-case and average-case results differ significantly.

Then we give an overview on average-case results for problems like nonlinear equations, integral equations, global optimization, and stochastic differential equations.

Finally we present some average-case results for integration and optimal recovery of functions $[0,1]^d \to \mathbb{R}$ (equivalently, random fields X_t, $t \in [0,1]^d$). The results for optimal recovery are linked to small ball probabilities via average linear n-widths.

Lower Tail Probabilities and Related Problems

Qi-Man Shao

Let $\{X(t), t \in T\}$ be a centred Gaussian process. The lower tail probability of X refers to $P(\sup_{t\in T}(X(t) - X(t_0)) \le x)$ as $x \to 0$, where t_0 is a fixed point in T. Examples include: the Brownian motion pursuit problem, most visited sites of symmetric stable processes and the probability of random polynomials having no real zeros. In this talk, we review recent developments in this area and related problems. Some open questions are also discussed.

Small Deviation for Subordinated Lévy Processes

Zhan Shi

Let $(A(t))_{0\le t\le 1}$ be a subordinator with Laplace exponent Φ, i.e. we have

$$\mathbb{E}\left[e^{-xA(t)}\right] = e^{-t\,\Phi(x)}$$

for all $x \ge 0$ and $0 \le t \le 1$. If W_H is a fractional Brownian motion on $[0,\infty)$ of Hurst index $H \in (0,1)$, independent of A, we define the process Y_H on $[0,1]$ by

$$Y_H(t) := W_H(A(t))\,, \quad 0 \le t \le 1\,.$$

The main assertion about Y_H is that for all $1 \le q \le \infty$, as $\varepsilon \to 0$, it follows that

$$\log \mathbb{P}(\|Y_H\|_q < \varepsilon) \approx -\Phi(\varepsilon^{-1/H})$$

as well as in the quenched as in the annealed case.

Small Ball Probabilities for Poisson Process and Strassen's Law of the Iterated Logarithm for Empirical Processes

Elena Yu. Shmileva

Let's consider $\Pi(t)$ — standard Poisson process and centred, normalized Poisson process of intensity ρ

$$Y_\rho(t) = \frac{\Pi(\rho t) - \rho t}{\sqrt{\rho}},\ t \in [0,1].$$

The small ball problem here studies the limiting behaviour of $P\{Y_\rho - \lambda f \in rU\}$ provided $\rho \to \infty,\ \lambda \to \infty,\ r \to 0$, where U is the unit ball in uniform norm and $f \in W_2^1([0,1])$, $f(0) = 0$.

The small ball probability for unshifted balls $P\{Y_\rho \in rU\}$ was studied by A. Mogulskii 1974. The approximation of $P\{Y_\rho - \lambda f \in rU\}$ provided $r = c/\lambda$, $c > 0$ was obtained by P. Deheuvels 2000 and then the conditions of this approximation were improved in the joined work of P. Deheuvels and M. Lifshits 2001.

We obtain the approximation of $P\{Y_\rho - \lambda f \in rU\}$ for any relation between r and λ.

There are two methods. The first one uses the strong invariance principle for Y_ρ (Y_ρ is close to a Wiener process) and gives us the same small ball behaviour as for Wiener process (Chung 1948).

The second method is based on the Skorokhod formula for the mutual density of the Poisson processes of variable intensity: making use of this formula one switches to the Poisson process with the expectation coinciding with the centre of the small ball, then applies the Mogulskii formula for unshifted small balls.

This method gives more complete results. Namely, we discovered that there is a zone of the parameters r, λ, ρ where the small ball behaviour differs from its Wiener counterpart.

We also consider the analogue of functional Strassen law for Y_ρ and for tail empirical processes. The approximation of $P\{Y_\rho - \lambda f \in rU\}$ is the key for the proof of these laws and for the estimation of the rates of convergence in these laws. It turns out that in some situations the rates in the Strassen law are not the same as for Wiener process.

Strong Local Nondeterminism of Gaussian Random Fields and Its Applications

Yimin Xiao

A sufficient condition for a Gaussian random field with stationary increments to be strongly locally nondeterministic is proven. As applications of this result, small ball probability estimates, Hausdorff measure of the sample paths, sharp Hölder conditions and tail probability estimates for the local times of Gaussian processes are established.

Edited by Werner Linde

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