

MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

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Funktionentheorie

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The meeting was organized by S. Hellerstein (Madison), St. Ruscheweyh (Würzburg) and N. Steinmetz (Dortmund)

The conference 'Funktionentheorie' is being held bi-annually at Oberwolfach. Each meeting is devoted to special areas within this broad field, but there is always space for new and important developments and methods from other parts of function theory. The present conference focused on

- Complex differential equations / Nevanlinna theory
- Complex dynamical systems / transcendental iteration
- Algebras of analytic functions.

Among the twenty-five lectures, which demonstrated the high activity and dynamical interaction in the field and beyond, the following were considered particular highlights:

- W. Bergweiler's proof of Ahlfors' famous Five-Island-Theorem via Zalcman's Lemma (Rescaling Lemma)
- The determination of the spherical Bloch's constant by M. Bonk & A. Eremenko
- A. Hinkkanen's proof of the Korenblum Maximum Principle (Hayman's Theorem)
- The (first rigorous) proof of the existence of the Painlevé transcendents I and II, due to A. Hinkkanen and I. Laine.

In two special ‘seminars’ several open problems were discussed.

The pleasant atmosphere at Oberwolfach and the friendly and patient assistance of the staff of the MFO contributed substantially to the success of the meeting.

Abstracts of the talks

V. Andrievskii *Variation of a quasidisk in problems of approximation theory*

Two results concerning the behaviour of the Green function under *small* variation of the boundary of a Jordan domain are presented. Their applications to the following four problems are discussed:

- behaviour of polynomials with uniform norm at most 1 and a prescribed zero on a quasiconformal curve;
- constructive characterization of Hölder classes of polyanalytic functions in a quasidisk;
- approximation of functions by harmonic polynomials;
- boundary behaviour of harmonic functions in a quasidisk.

I. N. Baker *Some recent results on Julia sets of entire functions*
(many of the results are joint work with P. Domínguez)

Although many results are known about the topological properties of the components of the Fatou set $F(f)$ which is the complement of the Julia set $J(f)$ of an entire or rational function, there has been relatively little study of topological properties of Julia sets for general entire functions.

For transcendental entire f it is useful to distinguish between $J(f)$, a subset of \mathbb{C} and $J_\infty(f)$, the closure of $J(f)$ in $\widehat{\mathbb{C}}$. The properties of $J_\infty(f)$, $J(f)$ may differ, for example the first may be connected and the second not.

Conditions are given for $J(f), J_\infty(f)$ to be connected or not. The nature of the components, in particular the existence of singleton components is discussed. Some questions of the accessibility of particular points (for example ∞ or periodic points) from $F(f)$ arise, as does the existence of 'buried' components K of $J(f)$ which do not meet the boundary of any component of $F(f)$. In some cases $J(f)$ fails to be locally connected at any point.

L. Baratchart *Inverse Problems for 2D-Laplacian and meromorphic approximation*

We consider a Dirichlet-Neumann problem on a domain in the complex plane whose outer boundary is a Jordan curve while the inner boundary is a slit inside the domain. The normal derivative on the slit is set to zero, and the temperature - or the flux - is imposed on the outer boundary. The slit models a crack in a metal piece that acts as a perfect isolator. The connection to meromorphic approximation was noticed in [1], where it is shown that the temperature plus its conjugate function times i is the sum of an analytic function in the domain plus slit and a Cauchy integral of the jump in the temperature across the slit, provided the temperature is in $H^{1/2}$ of the boundary and the crack is a piecewise $C^{1,\alpha}$ curve. If the crack is piecewise analytic we thus get a function with branch-points.

In this work we show that the poles of best meromorphic approximants of AAK-type tend (in the w^* -sense of the normalized counting measure) to the continuum of minimum condenser capacity (together with the outer boundary) joining the end-points of the crack, *provided the crack is analytic*.

We conjecture the result is true for piecewise analytic cracks and show numerical evidence of this.

These results lay hope for a new initialization technique to classically handle inverse problems for the 2-D Laplacian, as best AAK approximants are rather easily computed numerically.

[1] *How can meromorphic approximation help to solve inverse problems for the 2D Laplacian*, L. Baratchart, I. Leblond, F. Mandrea, and E. B. Saff, to appear in *Inverse Problems*.

D. Bargmann *Iteration of inner functions*

Dynamics of inner functions have turned out to be a very useful tool to study the structure of the boundary of unbounded invariant components of the Fatou set of an entire transcendental function f .

Let f be a transcendental entire function, and let D be an unbounded invariant component of the Fatou set of f . In a recent work I. N. Baker and P. Domínguez have obtained some results about the structure of ∂D by establishing a link between the boundary behaviour of the associated Riemann map $\phi : \mathbb{D} \rightarrow D$ and the dynamics of the inner function $g := \phi^{-1} \circ f \circ \phi$. Supposed that D is not a Baker domain and ∞ is an accessible point of ∂D , they prove that the set $\Theta := \{\theta \in \partial \mathbb{D} : \lim_{r \rightarrow 1} \phi(r\theta) = \infty\}$ is dense in $\partial \mathbb{D}$. We extend their result to a certain class of Baker domains by showing that the Julia set of the corresponding inner function is the whole unit circle. This is an easy consequence of a more general theorem which gives necessary and sufficient conditions for an inner function to be eventually conjugated to a linear transformation on the whole Fatou set of g .

W. Bergweiler *Normal families and the Ahlfors five islands theorem*

We discuss some applications of a rescaling lemma for (non-) normal families (due to Zalcman, the corresponding result for normal functions being due to Lohwater and Pommerenke). In particular, we show how this lemma can be used to obtain a new proof of the Ahlfors five islands theorem.

More specifically we show how the rescaling lemma, combined with the basic existence theorem for quasiconformal mappings, can be used to deduce the Ahlfors five islands theorem from the corresponding result of Nevanlinna concerning perfectly branched values. We also show how the rescaling lemma leads to a simple and elementary proof of Nevanlinna's result.

Finally we note that a refinement of the rescaling lemma (due to Pang) leads to a new proof of a theorem of Langley (conjectured by Hayman) which says that if f is meromorphic in the plane such that f and f'' have no zeros, then f has the form $f(z) = \exp(az + b)$ or the form $f(z) = (az + b)^{-n}$ with $a, b \in \mathbb{C}, a \neq 0, n \in \mathbb{N}$.

M. Bonk *Meromorphic functions and negative curvature*
(joint work with A. E. Eremenko)

The well-known Ahlfors Five Island Theorem says that given a non-constant meromorphic function f in the plane and given five Jordan subregions of the sphere with pairwise disjoint closures, there exists a branch of f^{-1} in one of the regions. In other words, f covers one of the five regions schlichtly. An immediate consequence is that f schlichtly covers spherical discs whose radius is arbitrarily close to $\pi/4 = 45^\circ$. This statement can be improved as follows.

Theorem: *Every non-constant meromorphic function in the plane schlichtly covers spherical discs of radius arbitrarily close to $\arctan \sqrt{8} \approx 70^\circ$. The number $\arctan \sqrt{8}$ is best possible.*

The theorem implies the Ahlfors Five Island Theorem.

The proof of the result is based on considering simply connected singular surfaces that arise as Riemann surfaces associated with meromorphic functions. These surfaces can be obtained by glueing together spherical triangles. If the circumscribed radius of all triangles is bounded away from $\arctan \sqrt{8}$, then a negatively curved metric can be introduced on the surface showing that it is of hyperbolic type.

F. W. Gehring *Quasiconformal mappings and their role in mathematics*

In this lecture I give some properties of quasiconformal mappings in euclidean n -space. I then describe connections between these mappings and the following areas of mathematics

1. Complex analysis: These consist of two important applications of the *measurable* Riemann mapping theorem and the Rodin-Sullivan circle packing proof of the Riemann mapping theorem.
2. Partial differential equations: This includes the reversed Hölder inequality used in regularity theory.
3. Functional analysis: These concern extension and invariance of the class BMO.

4. Geometry and elasticity: These involve the extension of quasi-isometries and the injectivity of local quasi-isometries.
5. Topology: These include the Mostow rigidity theorem and three applications of discrete convergence groups.

L. Geyer *Siegel discs, Herman rings and the Arnold family*

We provide new “prototypes” for two theorems of Yoccoz on analytic linearizability. The first theorem concerns fixed points, the second one circle diffeomorphisms. Let $f(z) = e^{2\pi i\alpha}z + O(z^2)$ be analytic with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. If α belongs to the set \mathcal{B} of *Brjuno numbers*, characterized by $\sum_n q_n^{-1} \log q_{n+1} < \infty$ for the convergents (p_n/q_n) of α , then f is always linearizable, i.e. there exists a local analytic conjugacy to $w \mapsto e^{2\pi i\alpha}w$. If $\alpha \notin \mathcal{B}$ then the quadratic polynomial $P_\alpha(z) = e^{2\pi i\alpha}z + z^2$ is not linearizable (Yoccoz).

Theorem 1: *If $\alpha \notin \mathcal{B}$ then $E_\alpha(z) = e^{2\pi i\alpha}ze^z$ is not linearizable.*

The Brjuno condition is also the sharp condition for analytic linearizability of circle diffeomorphisms, additionally assuming proximity to the rigid rotation. We show sharpness for the Arnold family:

Theorem 2: *Let $a, b \in \mathbb{R}$, $0 < b < (2\pi)^{-1}$ and let $f_{a,b}(x) = x + a + b \sin(2\pi x) \pmod{1}$ be analytically linearizable. Then the rotation number $\rho(f_{a,b})$ is a Brjuno number.*

Theorem 2 is proved by relating the families $f_{a,b}$ and E_α via quasiconformal surgery and using Theorem 1. One can push this analogy further and obtain geometric results on the maximal rotation domains (Siegel discs and Herman rings, resp.) of E_α and $f_{a,b}$ for some rotation numbers.

A. Goldberg *Approximation of subharmonic functions by logarithms of moduli of entire functions in integral metrics*
(joint work with M. Girnyk)

Wir beweisen folgenden Satz:

Satz: Sei u eine in \mathbb{C} subharmonische Funktion. Dann existiert eine ganze Funktion f , so daß

$$\|u(re^{i\varphi}) - \log |f(re^{i\varphi})|\|_{L^q(0,2\pi)} = Q(r, u)$$

gilt, wobei $Q(r, u) = O(\log r)$, $r \rightarrow \infty$, wenn u von endlicher Ordnung ist, und allgemein $Q(r, u) = O(\log r + \log n(r, u))$, $r \rightarrow \infty$, $r \notin E \subset (0, \infty)$, $\text{mes } E < \infty$; dabei ist $n(r, u) = \mu_u(\{z : |z| \leq r\})$ und μ_u das Rieszsche Maß von u .

Man kann in $Q(r, u)$ den Term $\log n(r, u)$ durch $\log T(r, u)$ oder durch $\log B(r, u)$ ersetzen.

We prove the following

Theorem: Let u be a subharmonic on \mathbb{C} . Then there exists an entire function f such that

$$\|u(re^{i\varphi}) - \log |f(re^{i\varphi})|\|_{L^q(0,2\pi)} = Q(r, u),$$

where $Q(r, u) = O(\log r)$, $r \rightarrow \infty$ holds if u has finite order, and in general $Q(r, u) = O(\log r + \log n(r, u))$, $r \rightarrow \infty$, $r \notin E \subset (0, \infty)$, $\text{mes } E < \infty$; here $n(r, u) = \mu_u(\{z : |z| \leq r\})$ and μ_u is the Riesz measure of u .

In the second definition of $Q(r, u)$ the term $\log n(r, u)$ may be replaced with $\log T(r, u)$ or $\log B(r, u)$.

P. Gorkin Division in the algebra $H^\infty + C$

Let \mathbb{D} denote the open unit disk and $\partial\mathbb{D}$ the unit circle. In this talk we look at division in the algebra $H^\infty + C = \{h + c : h \in H^\infty, c \in C(\partial\mathbb{D})\}$. While division in $H^\infty(\mathbb{D})$ is well-understood, questions remain about when one function divides another in $H^\infty(\mathbb{D}) + C$. We begin with a look at why $H^\infty + C$ is important, what is known about division in $H^\infty + C$, and we then look at a uniform algebra approach to division (joint work with R. Mortini and D. Suarez). We conclude with what remains to be studied in this area.

G. Gundersen *Meromorphic solutions of $f^6 + g^6 + h^6 \equiv 1$*

Hayman showed that for $n \geq 9$, there do not exist three non-constant meromorphic functions f , g , and h that satisfy $f^n + g^n + h^n \equiv 1$. In the other direction, we construct examples of three transcendental meromorphic functions f , g , and h that satisfy $f^6 + g^6 + h^6 \equiv 1$. Open questions will be discussed.

W. K. Hayman *Schlicht functions with gaps and large growth*
(joint work with A. E. Eremenko)

We construct schlicht functions in the unit disc, whose coefficient sequences have arbitrarily long intervals of zeros and at the same time arbitrarily long intervals where $|a_n| > \varepsilon_n n$ holds, $\varepsilon_n < 1$ being a preassigned sequence of positive numbers tending to zero. The initial interval of coefficients of such a function may be prescribed to be any interior point of the coefficient region.

S. M. Heinemann *Iteration theory for holomorphic endomorphism of \mathbb{C}^n*

We consider the problem to determine the Shilov boundary of compact subsets of \mathbb{C}^n . Instead of imposing conditions like pseudoconvexity or smoothness of the boundary we restrict our interest to sets which can be characterized by means of the iteration of polynomial endomorphisms of \mathbb{C}^n . For a certain class of hyperbolic maps $T : \mathbb{C}^n \rightarrow \mathbb{C}^n$ we show

Theorem A: *The Shilov boundary $\partial_{SH}K(T)$ of the set $K(T)$ of points with bounded forward orbit $\left\{ z : \limsup_{k \rightarrow \infty} \|T^k(z)\| < \infty \right\}$ equals the Julia set $J(T)$ of T . Furthermore, the set $\partial K(T) \setminus \partial_{SH}K(T)$ foliates into complex analytic sets.*

This is actually a consequence of the following theorem.

Theorem B: *The listed 5 sets coincide and are equivalent characterizations of the Julia set $J(T)$ of the (p, q) -regular map T :*

- i) $N(T) := \{z : \{T^k\} \text{ is not weakly normal in } z\}$;
- ii) $S(T) := \partial_{SH}K(T)$;
- iii) $R(T) := \overline{\{z : z \text{ repelling periodic point of } T\}}$;
- iv) $P(T) := \lim_{k \rightarrow \infty} T^{-k}(\partial_{SH}B_r^n(0))$, $r > R_T$;
- v) $M(T) :=$ support of the unique measure of maximal entropy
(= log (mapping degree of T)).

Moreover,

- I.) $J(T)$ is completely invariant under the action of T ;
- II.) the action of T on $J(T)$ is exact, i.e. for any open $U, U \cap J(T) \neq \emptyset$, there is $n_U < \infty$ such that $T^{n_U}(U) \supseteq J(T)$.

A. Hinkkanen *On Korenblum's maximum principle*

In 1991, Boris Korenblum conjectured that there is an absolute constant $c \in (0, 1)$ such that if f and g are analytic in the unit disk \mathbb{D} and if $|f(z)| \leq |g(z)|$ whenever $c < |z| < 1$, then $\|f\|_2 \leq \|g\|_2$, where

$$\|f\|_p^p = \int_{\mathbb{D}} |f(x + iy)|^p dx dy, \quad 0 < p < \infty.$$

After various partial results by Korenblum, Schwick, and others, W.K. Hayman proved this conjecture in 1998, with $c = 1/25$.

The speaker subsequently proved that for f and g as above, if $|f(z)| \leq |g(z)|$ whenever $0.15724 = c < |z| < 1$, then $\|f\|_p \leq \|g\|_p$ for all $p \geq 1$. In the proof, as in Hayman's proof, one first uses the fact that the integral mean of a suitably chosen subharmonic function over a circle is an increasing function of the radius, in order to estimate

$$\int_{|z|<c} (|f|^p - |g|^p) dx dy$$

in terms of

$$K \int_{c < |z| < 1} (|g|^p - |f|^p) dx dy,$$

where the positive constant K depends on c and on distortion properties of the analytic function f/g that maps the annulus $A = \{z : c < |z| < 1\}$ into \mathbb{D} . To estimate K , in order to get a better value for c , one can use the hyperbolic metric of A , or even better, Lehto's generalization of Schwarz's lemma applied to the universal covering map of \mathbb{D} onto A followed by f/g . The talk discussed the details of this proof.

K. Keller *On the combinatorial structure of complex quadratic dynamics*

The detailed structure of the Mandelbrot set M is extremely complicated. However, much of the structure can be described by different kinds of symmetry and self-similarity. For example, each neighborhood of a boundary point of M contains infinitely many topological copies of M itself, which is a consequence of the (unpublished) tuning results by Douady and Hubbard.

Whereas symmetry in the dynamic plane can mostly be explained by the action of the quadratic map, the situation is more complicated in parameter space. Often there is a correspondence between local structure in dynamic plane and in the parameter space which helps to understand a special symmetry in parameter space. Then one can follow Douady's philosophy 'to plough in dynamical plane and then to harvest in parameter space'.

Given a hyperbolic component W of period m , one may ask how similar two sub-limbs of W are from the combinatorial point of view. Clearly, the shape of the Mandelbrot set shows that one cannot expect complete similarities, but there are partial ones.

According to Lau and Schleicher, a given hyperbolic component 'behind' W is said to be *visible* if there is no hyperbolic component 'between' W and the given one having period less than the given one. We discuss special 'translation' symmetries between the trees of visible hyperbolic components in different sub-limbs of W .

The main idea is to relate the structure 'behind' W to the structure of the filled-in Julia set of $z^2 + c_0$, where c_0 denotes the center of W . Certain hy-

parabolic components ‘behind’ W correspond to certain Fatou components of $z^2 + c_0$, and this allows to transfer parts of the dynamics of $z^2 + c_0$ to the set of hyperbolic components considered.

B. Korenblum *On a Theorem of W. K. Hayman*

Denote $D_c = \{z \in \mathbb{C} : |z| < c\}$, $D_1 = \mathbb{D}$, $A_c = \mathbb{D} \setminus \overline{D}_c$ for $0 < c < 1$. For f analytic in \mathbb{D} and $p \geq 1$ set

$$\|f\|_p^p = \frac{1}{\pi} \int_{\mathbb{D}} |f(z)|^p dA_z$$

where dA_z is the Lebesgue area measure; similarly for f analytic in $\mathbb{D} \times \mathbb{D}$.

Theorem 1: (Hayman-Hinkkanen) *If f, g are analytic in \mathbb{D} and $|f(z)| \leq |g(z)|$ holds on A_c , with $c = 0.15724$, then for all $p \geq 1$ we have $\|f\|_p \leq \|g\|_p$.*

W. K. Hayman proved this result for $p = 2$ and $c = 1/25$ (June 1998).

A. Hinkkanen improved the constant c and extended the result to all $p \geq 1$. For $p = 2$ and some c the result was conjectured by Korenblum in 1991. Various partial results were obtained in 1991–1997.

Theorem 2: *There is a constant c , $0 < c < 1$, so that, whenever f, g are analytic in $\mathbb{D} \times \mathbb{D}$ and satisfy $|f| \leq |g|$ in the bi-annulus $A_c \times A_c$, then*

$$\|f\|_p \leq \|g\|_p \text{ for } p \geq 1 .$$

Some other results of similar nature are discussed.

I. Laine *Solutions of the first Painlevé equation are (in fact) meromorphic*

A time-honoured result is that all solutions of

$$w'' = z + 6w^2$$

are meromorphic in the complex plane. Originally, this is due to Painlevé (1900), repeated and modified by Ince, Golubew, Hille, Erugin, among others.

Unfortunately, these proofs are not rigorous in the present sense of the word. The idea of our work has been to fill the gaps. The original reasoning, as found in Hille's book, makes use of the auxiliary functions u , v , V defined as follows:

$$\begin{cases} w = v^{-2} \\ w' = -2v^{-3} - \frac{1}{2}zv - \frac{1}{2}v^2 + uv^3 \\ V = (w')^2 - 4w^3 - 2zw + \frac{w'}{w} + z. \end{cases}$$

In the indirect proof, the reasoning divides in four cases, depending on the boundedness of w , u and V . The original reasoning applies, up to some modifications, except for the final case where $|w|$ and $|V|$ are unbounded near a possible singularity z_0 while $\liminf_{z \rightarrow z_0} |w| = 0$ on a fixed path Γ .

What is needed here, and overlooked up to now, is to show that the path Γ may be modified to $\tilde{\Gamma}$ such that the inverse function of w is defined in a disc whose radius can be estimated to be larger than a positive constant as $z \rightarrow z_0$ along $\tilde{\Gamma}$. This needs a lengthy consideration.

Similarly, the second Painlevé equation

$$w'' = \alpha + zw + 2w^3, \quad \alpha \in \mathbb{C},$$

may be considered. We also believe that our method applies to the fourth Painlevé equation as well as to a modification of the third Painlevé equation.

J. K. Langley *Some new Bank-Laine functions*

A Bank-Laine function E is an entire function with the property that $E(z) = 0$ implies that $E'(z) = \pm 1$. Such functions E arise as the product $E = f_1 f_2$ of linearly independent solutions f_j of the equation

$$w'' + A(z)w = 0, \tag{1}$$

in which A is an entire function and the f_j are normalized so that the Wronskian $W(f_1, f_2)$ is 1. It has been conjectured that if A is a transcendental entire function and the order of A is finite but not a positive integer, then the zeros of the associated product function E always have infinite exponent of convergence.

There are relatively few examples in the literature of Bank-Laine functions of finite order associated with transcendental coefficient functions A . We will

describe two new ways to construct such Bank-Laine functions. The first uses quasiconformal modifications; the second is elementary.

J. Miles *On a conjecture of Fuchs*

It was conjectured by W. Fuchs some thirty-five years ago that if f is an entire function of order $p < 1/2$, then $\delta(0, f'/f) = 0$ where $\delta(0, f'/f)$ denotes the Nevanlinna deficiency of the value 0 for the logarithmic derivative of f . In 1994 Eremenko, Langley, and Rossi showed that $\delta(0, f'/f) \leq 1 - \cos \pi p$ for entire f order $p < 1/2$. Miles and Rossi now improve this estimate for small p by showing the existence of an absolute constant $c > 0$ such that $\delta(0, f'/f) < \exp(-c/\rho)$ for entire f of order ρ in $(0, 1/2)$. In fact a bit more is established, namely

$$\liminf_{r \rightarrow \infty} \frac{\left\| \log \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right| \right\|_2}{T(r, f'/f)} < \exp(-c/\rho) .$$

R. Mortini *Interpolating sequences in the spectrum of $H^\infty(\mathbb{D})$*

Let H^∞ be the uniform algebra of bounded analytic functions in the open unit disk \mathbb{D} and let $M(H^\infty)$ denote its spectrum, that is the set of all (nonzero) multiplicative linear functionals on H^∞ , endowed with the weak-* topology. By identifying each point $z_0 \in \mathbb{D}$ with the evaluation functional $\Phi_{z_0} : f \mapsto f(z_0)$, we can consider \mathbb{D} as a dense subset of $M(H^\infty)$ (Corona-Theorem).

A sequence (x_n) in $M(H^\infty)$ is said to be an *interpolating sequence* for $M(H^\infty)$ if for every bounded sequence (w_n) of complex numbers there exists a function $f \in H^\infty$ such that $f(x_n) = w_n$ for all $n \in \mathbb{N}$. Due to Carleson's interpolation theorem (1958-1962) we know that a sequence (a_n) in \mathbb{D} is interpolating if and only if it is uniformly separated, i.e. satisfies

$$\inf_{k \in \mathbb{N}} \prod_{j: j \neq k} \left| \frac{a_j - a_k}{1 - \bar{a}_j a_k} \right| \geq \delta > 0.$$

A point $x \in M(H^\infty)$ is called trivial, if its Gleason part reduces to $\{x\}$. We may view the Gleason parts as the connected components of $M(H^\infty)$ with respect to the operator norm topology. Recall that a sequence (x_n) in $M(H^\infty)$ is said to be *discrete* (in the weak-* topology), if for any n , x_n does not belong to the weak-* closure of $\{x_k : k \neq n\}$. Trivially, any interpolating sequence is discrete. What we have here, is the following characterization of the interpolating sequences contained in the set of trivial points:

Theorem: *A sequence (x_n) of trivial points is interpolating if and only if it is discrete.*

A proof of this result is sketched. Moreover, a sufficient topological condition for a sequence of nontrivial points to be interpolating is given.

O. Roth *Classes of analytic functions and control theory*

We discuss classes of analytic functions from a control theoretic point of view as the so-called reachable sets of a control system in $\mathcal{H}(\mathbb{D})$. This provides a unified approach to the study of a large number of classes of functions analytic in the unit disk such as the set of normalized univalent functions and the set of bounded non-vanishing functions.

We present compactness and approximation criteria for reachable sets and a general version of Pontryagin's maximum principle which contains the well-known Schiffer differential equation for the class of normalized univalent functions as a special case.

D. Schleicher *On iteration of exponential maps*

Our goal is to introduce successful methods from the well-known iteration theory of polynomials to certain entire maps. As a first case, (a „toy model“) we investigate the family of maps $z \rightarrow \lambda e^z$, $\lambda \in \mathbb{C}^*$. As a first step, we investigate „external rays“ developed by Douady and Hubbard for polynomials, and prove their existence also for our maps. They help for example to prove the conjecture of Eremenko/Lyubich and Baker/Rippon that boundaries of the sets in parameter space with attracting periodic cycles are connected.

Another corollary is the theorem that for every integer $n \geq 1$, the variety $\{(\lambda, z) \in \mathbb{C}^* \times \mathbb{C}^*, z \text{ has exact period } n \text{ under } z \mapsto \lambda e^z\}$ is connected. Our results justify the observation that the parameter space of these maps is a well-defined limit of the Mandelbrot set and its higher-degree cousins. Many of the methods apply to larger classes of iterated entire maps of finite type.

D. Shea *Sharp inequalities for norms of conjugate functions*

(joint work with M. Essén and C. Stanton)

Using a new method prompted by B. Cole's study of conjugate function problems, we prove inequalities like

$$\begin{aligned} \int_0^{2\pi} |\tilde{f}| \log^{\alpha-1}(e + |\tilde{f}|) \, d\theta &\leq \frac{2}{\pi\alpha} \int_0^{2\pi} |f| \log^\alpha(e + |f|) \, d\theta \\ &+ \frac{2}{\pi} \int_0^{2\pi} |f| \log(e + |f|) \log \log(e + |f|) \, d\theta \\ &+ A \int_0^{2\pi} |f| \log(e + |f|) \, d\theta \quad (1 < \alpha < \infty) \end{aligned}$$

for $f \in L \log^\alpha L$ and $\tilde{f} =$ conjugate function; here the constants $\frac{2}{\pi\alpha}$ and $\frac{2}{\pi}$ cannot be reduced.

H. Stahl *Rational best approximants in \mathbb{D}*

This talk is concerned with techniques from complex approximation theory (approximation theory in the complex plane) that has found applications in system theory, model fitting and/or the detection of cracks in solid materials. We shall address the problem of rational best approximation of functions from the Hardy spaces $H^2(D)$ and $H^\infty(D)$.

Let f be a function of the form $f(z) = \int (t - z)^{-1} \, d\mu(t)$, $\text{supp } (\mu) \subseteq (-1, 1)$, $\mu \geq 0$. We study rational best approximants $R_n^* \in H^2(\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}})$ in the L^2 -norm. Asymptotic error estimates are presented in the weak and strong sense, i.e., we consider expressions $\|f - R_n^*\|^{1/n}$ and $\|f - R_n^*\|$ as $n \rightarrow \infty$. Understanding the asymptotic error behaviour in the strong sense is essential for proving uniqueness of R_n^* .

E. Steinbart *Growth and oscillation theory of non-homogeneous linear differential equations*

For $n \geq 1$ consider the non-homogeneous linear differential equation

$$f^{(n)} + P_{n-1}(z)f^{(n-1)} + \dots + P_0(z)f = H(z), \quad (1)$$

where $P_0(z), P_1(z), \dots, P_{n-1}(z)$ are polynomials with $P_0(z) \not\equiv 0$, and $H(z) \not\equiv 0$ is an entire function of finite order. It is well known that every solution f of equation (1) is an entire function.

For an entire function $g \not\equiv 0$, let $\rho(g)$ denote the order of growth of g , and let $\lambda(g)$ denote the exponent of convergence of the sequence of zeros of g . For solutions f of equation (1), we investigate the possible values of $\rho(f)$, and the relationships between the values of $\rho(f)$, $\lambda(f)$, $\rho(H)$, and $\lambda(H)$. This research is joint work with Gary Gundersen and Shupeí Wang.

D. Suárez *Approximation by ratios of bounded analytic functions*

Let \mathbb{D} be the unit disk and E be a compact subset of the H^∞ maximal ideal space, $M(H^\infty)$. A germ on E is a continuous function f on an open neighborhood $U \subset M(H^\infty)$ of E such that $f|_{U \cap \mathbb{D}} \in H^\infty(U \cap \mathbb{D})$. We show that every germ can be uniformly approximated on E by ratios h/g , where $h, g \in H^\infty$ and g is zero free on E . As a consequence we show that every germ on E can be uniformly approximated by functions of H^∞ if and only if $E = \{x \in M(H^\infty) : |h(x)| \leq \sup_E |h|, h \in H^\infty\}$.

Problems and discussions session

K. Astala *A problem on the Beurling transform and the Beltrami equation*

Let $f \in W_{loc}^{1,q}(\mathbb{C})$ be a solution to the Beltrami equation

$$\partial_{\bar{z}}f = \mu \partial_z f, \quad (2)$$

where $\|\mu\|_\infty \leq k < 1$. The problem we are concerned with here is if f is then necessarily quasiregular, i.e. continuous, open and discrete and, in particular, representable as a composition of an analytic mapping and a (homeomorphic) quasiconformal mapping. For small q , that is for $q < 1 + k$, the answer is known to be “no” [IM]. On the other hand, from [A] it follows that the answer is “yes” for $q > 1 + k$.

Our first question is hence if the regularity still holds at the borderline case $q = 1 + k$, i.e. if the $W_{loc}^{1,1+k}$ -solutions of (2) are quasiregular.

Let T be the Beurling-operator

$$(Tg)(z) = -\frac{1}{2\pi i} \int_{\mathbb{C}} \frac{g(w)dw \wedge d\bar{w}}{(z-w)^2},$$

the unique Calderon-Zygmund operator with a holomorphic kernel. As is well known, $T(\bar{\partial}u) = (\partial u)$ for $u \in W^{1,2}(\mathbb{C})$. Moreover, T is bounded on $L^p(\mathbb{C})$ for $1 < p < \infty$, with $\|T\|_{L^2} = 1$.

In his problem A. Baernstein proposed methods to calculate the norm of T on L^p , $p \neq 2$. Here we have a related question: according to the theorem of Coifman and Fefferman T is bounded on the weighted spaces $L_w^p(\mathbb{C}) = \{f : \|f\|_{L_w^p} = (\int_{\mathbb{C}} |f|^p w \, dm)^{\frac{1}{p}} < \infty\}$ if the weight w belongs to A_p . This means that w satisfies the Muckenhoupt condition

$$\|w\|_{A_p} := \sup_B \left(\frac{1}{|B|} \int_B w \right) \left(\frac{1}{|B|} \int_B w^{-1/(p-1)} \right)^{p-1} < \infty, \quad (3)$$

where $1 < p < \infty$ and the supremum is taken over all disks $B \subset \mathbb{C}$.

Our second question asks how does the norm of T on L_w^p depend on the A_p -norm of w . We conjecture that the dependence is linear,

$$\|T\phi\|_{L_w^p} \leq C(p)\|w\|_{A_p}\|\phi\|_{L_w^p} \quad (4)$$

for all $w \in A_p$ and $\phi \in L_w^p$, with a constant $C(p)$ depending only on p , $1 < p < \infty$.

The two questions are closely related. In particular, [AIS] shows that if (4) is true then as a corollary we obtain the quasiregularity of the solutions of the Beltrami equation (2) at the borderline case $q = 1 + k$.

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A. Baernstein II *A conjecture about integrals of maximal and minimal stretches of functions in the plane.*

The conjecture below stems from my joint work with S.J. Montgomery-Smith. Let f be a complex-valued C^1 function on the closure of the unit disk \mathbb{D} in the complex plane. Let $M = |\partial f| + |\bar{\partial} f|$, $m = |\partial f| - |\bar{\partial} f|$, where ∂f and $\bar{\partial} f$ are the usual formal complex derivatives. Then $M(z)$ is the “maximal stretch” of f at a point z , and $m(z)$ is the signed minimal stretch.

Conjecture: $\frac{1}{\pi} \int_{\mathbb{D}} (M - 1)^+ (m - 1) dx dy \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} ((M(e^{i\theta}) - 1)^+)^2 d\theta$.

The conjecture is suggested by work of Burkholder on sharp inequalities for martingale transforms. If true, the conjecture would imply a conjecture of T. Iwaniec asserting that the norm in $L^p(\mathbb{C})$ of the “Ahlfors- Beurling” operator (the singular convolution operator with kernel $\frac{-1}{\pi z^2}$) is $\max(p - 1, 1/(p - 1))$, for $1 < p < \infty$. Iwaniec’s conjecture would imply, in turn, a stronger form of the sharp area distortion inequality for quasiconformal mappings proved by K.Astala in 1994.

Falsity of the conjecture for functions vanishing on the boundary of \mathbb{D} would provide confirmation of a conjecture of C.B. Morrey in the calculus of variations asserting that in dimension two, rank-one convex functions need not be quasiconvex.

The conjecture might be true in stronger form: The square of the L^2 -norm on the right hand side can perhaps be replaced by the square of the L^1 -norm.

W. Bergweiler *Relations between $n(r, a)$ and $\log M(r, f)$, f entire.*

Let f be an entire function whose order ρ satisfies $\frac{1}{2} \leq \rho \leq \infty$. Denote by $n(r, a)$ the number of zeros of $f(z) - a$ in $|z| \leq r$ and by $M(r, f)$ the maximum modulus of f . Is it true that

$$\limsup_{r \rightarrow \infty} \frac{n(r, a)}{\log M(r, f)} \geq \frac{1}{\pi}$$

for all $a \in \mathbb{C}$ with at most one exception? The answer is “yes” if $\frac{1}{\pi}$ is replaced by $\frac{1}{2\pi}$; see [W. Bergweiler, A quantitative version of Picard’s theorem, *Ark. Mat.* 34 (1996), 225-229] and [W. Bergweiler, Some remarks on Picard’s theorem and Nevanlinna theory, *Bull. Hong Kong Math. Soc.* 1 (1997), 219-224].

W. Bergweiler *Siegel disks of entire functions.*

Let f be an entire function with an irrationally indifferent fixed point at 0; that is, $f(z) = e^{2\pi i\alpha}z + O(z^2)$ as $z \rightarrow 0$, with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. A sufficient condition for linearizability of f at 0 is that α is a Brjuno number. This condition is also necessary if f is a quadratic polynomial. Cremer has given (weaker) necessary conditions for polynomials of higher degree and for transcendental entire functions satisfying growth restrictions. Is there a condition on α that implies non-linearizability for all entire functions f of the above form? For background and references see [R. Pérez-Marco, Solution complète au problème de Siegel de linéarisation d’une application holomorphe au voisinage d’un point fixe (d’après J.-C. Yoccoz), *Asterisque* 206 (1992), 273-310].

D. Drasin *Better solutions to the inverse problem.*

Nevanlinna’s defect relation

$$\sum_a \delta(a, f) \leq 2 \tag{5}$$

where f is a function meromorphic in the plane and the $\{a\}$ are constants, has been known to be sharp for over 20 years. Given a sequence $\{a_j\}$, chosen so that $\delta(a_j)$ is the desired sequence we wish to achieve, one is asked to construct a meromorphic function $f(z)$ such that $|f - a_j|$ is sufficiently small on a portion of the plane to ensure that $\int \log^+ \frac{1}{|f(e^{i\theta}) - a_j|} d\theta$ grows consistent with the defect relation (5). In the 1980’s, Steinmetz and Osgood showed that (5)

holds when constants a_j are replaced by functions $a_j(z)$ which are “small” compared to f . It seems substantially more difficult to present examples showing (5) sharp. Quasiconformal methods seem to fail: $(1 + o(1))a(z) \sim a(z)$ only when a is constant! Russakovskii (1996) has used the $\bar{\partial}$ -equation to get good partial results when f is entire, but I see at present no way to obtain a general meromorphic solution from that very appealing method.

S. Heinemann *Hyperbolicity and parameter space for holomorphic endomorphisms of \mathbb{C}^n .*

One-dimensional Background. Recall the standard setting in one-dimensional holomorphic iteration. One is particularly interested in the dynamical properties of the quadratic family, i.e. the maps

$$f_c : z \mapsto z^2 + c,$$

where $c \in \mathbb{C}$. A major problem is to identify *hyperbolic* maps, i.e. those whose restriction to their Julia set $J(f_c)$ is *expanding*. There are two types of these. If the parameter c is in the complement of the so-called Mandelbrot set \mathcal{M} (the set of all c such that the finite critical point 0 does not escape to infinity under iteration of f_c) then f_c is always hyperbolic and $J(f_c)$ is a Cantor set (the dynamics on $J(f_c)$ is equivalent to a two-sided 2-shift). A sufficient condition is, e.g. $|c| > 2$. One can also obtain hyperbolic dynamics for parameters in the Mandelbrot set. In this case the Julia set is connected. It might even be a Jordan curve. A sufficient condition is given by $|c| < 1/4$. Open sets in \mathcal{M} where the parameter yields hyperbolic dynamics are called *hyperbolic components*. There are infinitely many hyperbolic components.

Two-dimensional Setting. One way to generalize the above setting is to iterate quadratic maps with varying parameter. In \mathbb{C}^2 one can do this within the framework of holomorphic iteration theory of analytic endomorphisms. Namely, we fix two polynomials k and q , where the degree of q should be at least 2, and iterate the *skew product*

$$T : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x^2 + k(y) \\ q(y) \end{pmatrix} =: \begin{pmatrix} p_y(x) \\ q(y) \end{pmatrix}.$$

Unfortunately, the parameterspace might be of high dimension (depending on the degree of k). However, it makes perfect sense to restrict one’s interest

to subspaces of the following type. Let k be given in the form

$$k(y) := \sum_{j=0}^d \kappa_j y^j.$$

Then we define k_λ , for $\lambda \in \mathbb{C}$, by setting

$$\begin{aligned} \kappa_{\lambda 0} &:= \lambda \cdot (\lambda - 1 + \kappa_0), \\ \kappa_{\lambda j} &:= \lambda \cdot \kappa_j. \end{aligned}$$

We define $p_{\lambda y}$, T_λ in the obvious way. The Julia set $J(T_\lambda)$ is given by

$$J(T_\lambda) = \overline{\bigcup_{y \in J(q)} J(P_{\lambda y}) \times \{y\}}. \quad (6)$$

Here, $P_{\lambda y}$ denotes the family of one-dimensional maps

$$P_{\lambda y} := \{p_{\lambda y}, p_{\lambda q(y)} \circ p_{\lambda y}, p_{\lambda q^2(y)} \circ p_{\lambda q(y)} \circ p_{\lambda y}, \dots\}.$$

The ‘correct’ notion of hyperbolicity for skew products is *fibre hyperbolicity*, i.e. the question if the families $P_{\lambda y}$ are expanding on $J(P_{\lambda y})$.

From the form of our parametrisation one deduces that $T_1 = T$, furthermore, that if $|\lambda|$ is small enough then the $J(P_{\lambda y})$ are Jordan curves, if $|\lambda|$ is big enough, then the $J(P_{\lambda y})$ are Cantor sets. Moreover, in both cases one gets fibre hyperbolicity and the continuity of the map (using Hausdorff distance)

$$y \mapsto J(P_{\lambda y}),$$

which implies that one can skip the closure in (6). In all examples known so far even a stronger statement is true. One obtains a commuting diagram of the form

$$\begin{array}{ccc} J(T_\lambda) & \xrightarrow{T_\lambda} & J(T_\lambda) \\ \downarrow \Phi & & \downarrow \Phi \\ J(p) \times J(q) & \xrightarrow{p \times q} & J(p) \times J(q). \end{array} \quad (7)$$

Here p is a quadratic polynomial such that p restricted to $J(p)$ is hyperbolic, Φ is a homeomorphism which keeps the second coordinate ($= y$) fixed.

Conjecture 1: If the action of T_λ on $J(T_\lambda)$ is fibre hyperbolic, then this implies that T_λ acts on $J(T_\lambda)$ as a *pseudo product*, i.e. (7) holds with suitable

p and Φ .

One easily generalizes the notion of Mandelbrot set from the c - to the λ -plane (given the polynomial k). Namely, \mathcal{M}_k consists of all λ such that the *critical set* $\{0\} \times J(q)$ does not escape to infinity under iteration of T_λ . One can look for *hyperbolic components* in \mathcal{M}_k .

Conjecture 2: If k is not constant (in this case the T_λ are products and \mathcal{M}_k is the image of \mathcal{M} under a quadratic map) then \mathcal{M}_k contains only *finitely* many hyperbolic components.

Reference.

St.-M. Heinemann, *Julia Sets of Skew Products in \mathbb{C}^2* , Kyushu Journal of Mathematics, Vol. 52, No. 2, 1998, pp. 299-329.

G. Jank *Matrix Beltrami-Systems and Quaternions.*

We introduce a four dimensional matrix Beltrami-system which is related to the question of introducing holomorphic coordinates on almost complex manifolds. With methods from quaternionic analysis, in particular by using quaternionic integral operators of Ahlfors-Vekua type, we prove the existence of a global homeomorphism in \mathbb{R}^4 under the assumption of sufficient small coefficients. Although, the matrix Beltrami system is formally overdetermined we do not need any integrability conditions since we use an equivalent quaternionic system. (Joint work with H. Malonek).

G. Martin *Normal families problem.*

Let \mathcal{F} be a family of meromorphic functions defined in an open set U and valued in the Riemann sphere $\overline{\mathbb{C}}$. If $\varphi : U \rightarrow \overline{\mathbb{C}}$ is a continuous function, then we say the family \mathcal{F} *omits* φ if there are no solutions to the equation

$$f(z) = \varphi(z), \quad z \in U, \quad f \in \mathcal{F}. \quad (8)$$

Equivalently each $f \in \mathcal{F}$ has a graph disjoint from that of φ . The problem is the following.

Suppose that a family of meromorphic functions omits 3 continuous functions with pairwise disjoint graphs. Then is that family normal in U ?

We might ask for improved results under the assumption that \mathcal{F} contains only analytic functions and so forth.

Of course if the three continuous functions are constant, this is the usual Montel Theorem. If the 3 functions are themselves meromorphic, then it is fairly straightforward to see by considering an appropriate cross ratio that the result is true. Fatou seems to have first done this in his proof that repellers are dense in the Julia sets of rational functions. Indeed it is this proof that motivated the question since we are interested in generalising the “repellers are dense” theorem to higher dimensions for *uniformly quasiregular mappings* (quasiregular mappings which can be iterated without increasing the distortion beyond some fixed bound. These maps are rational with respect to some measurable Riemannian structure).

Using Ahlfors’ covering surface theory (and in particular the 5 Islands Theorem) A. Hinkkanen and I were able to solve the problem in the affirmative for 5 functions. M. Bonk showed us how to improve the argument to 3 functions and subsequently D. Bargmann gave a proof using Zalcman’s Lemma for 3 functions as well. This last approach can be generalised to quasiregular mappings in higher dimensions. This will all appear in joint work.

A subsequent question would be to ask to what extent the value distribution theory holds up if one considers the graphs of continuous functions as opposed to particular values (the graphs of constant functions). The question asks to formulate and prove the correct results as well as find interesting applications. It seems to me to be a “topological” or “homotopy” version of the value distribution theory and there might be important applications. Alternatively new interpretations of Ahlfors’ covering surface theory might arise.

G. Martin $3n + 1$ Conjecture.

This problem is not directly in Function Theory *per se* but can be reduced to a question about meromorphic functions in the disk which seems not to be well known to experts in Function Theory and so I raise the question here. The $3n + 1$ Conjecture has a nice pedigree and an interesting discussion of the question and its history can be found in *The dynamical system generated by the $3n + 1$ function*, Günther J. Wirsching, Lecture Notes in Math., #1681, Springer–Verlag. There are references in this book to the reduction I mention (Theorem 13.1), but there appears to be an error in a related reduction, (Theorem 13.2).

The $3n + 1$ Conjecture asserts the following algorithm always stops.

1. Choose a natural number n ;
2. If $n = 1$, then STOP; otherwise
3. If n is even, $n := n/2$; go to (2); otherwise
4. $n := 3n + 1$; go to (2)

So for instance if I choose the number 11, then I generate the sequence

$$11 \rightarrow 34 \rightarrow 17 \rightarrow 52 \rightarrow 26 \rightarrow 13 \rightarrow 40 \rightarrow 20 \\ \rightarrow 10 \rightarrow 5 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$$

and then I stop. Try $n = 27$ for fun! This conjecture has been verified numerically for very large numbers. The structure of the cycles and their length (if finite) is also of interest for a number of reasons. The conjecture has been shown to be equivalent to the following problem in Function Theory by G. Meinardus.

Let $\zeta = e^{2\pi i/3}$. Show that the only solutions of the functional equation

$$h(z^3) = h(z^6) + \frac{h(z^2) + \zeta h(\zeta z^2) + \zeta^2 h(\zeta^2 z^2)}{3z}$$

which are holomorphic in the unit disk have the form

$$h(z) = a + \frac{bz}{1-z}$$

for some constants $a, b \in \mathbb{C}$.

There is a wealth of other interesting things in the book I mentioned for those interested.

P. Poggi-Corradini *Boundary repelling fixed points.*

Let ϕ be an analytic function defined on the unit disk \mathbb{D} , with $\phi(\mathbb{D}) \subset \mathbb{D}$, $\phi(0) = 0$, and $\lambda = \phi'(0) \neq 0$. A classical theorem of Koenigs yields an analytic function σ on \mathbb{D} such that $\sigma \circ \phi = \lambda\sigma$. The growth of σ is related to the dynamics of ϕ near $\partial\mathbb{D}$. We call a point $\zeta \in \partial\mathbb{D}$ a boundary repelling fixed point for an analytic self-map ψ of the disk if ζ is fixed in the sense of non-tangential limits and the derivative ψ' has a finite non-tangential limit at

ζ . In [Angular derivatives at boundary fixed points for self-maps of the disk, Proc. Amer. Math. Soc. 126 (1998), 1697-1708] we showed that, when ϕ is one-to-one,

$$(\star) \quad \sup_{|z|=r} |\sigma(z)|^p (1-r) \rightarrow \infty, \quad \text{for some } p > 0,$$

if and only if some iterate ϕ_N of ϕ has a boundary repelling fixed point.

Question 1. Is this true in general? Without univalence.

Recently, P. Bourdon showed, in [Essential Angular derivatives and maximum growth of Kœnigs eigenfunctions, preprint], that (\star) is always equivalent to the fact that there exists a sequence $\zeta_n \in \partial\mathbb{D}$, for which $|\phi_n(\zeta_n)| = 1$ and $|\phi_n'(\zeta_n)| \leq M^n$ for some constant $M > 1$. It follows, for instance, that if ϕ is a finite Blaschke product, then (\star) holds. On the other hand, in this case the set of boundary repelling fixed points of the iterates is actually dense in $\partial\mathbb{D}$. So we pose the following.

Question 2. Suppose ϕ is inner. Is (\star) equivalent to the fact that the set of boundary repelling fixed points of the iterates of ϕ is dense in $\partial\mathbb{D}$?

J. Rossi *Filling circles, Normal families, Zalcman's Lemma.*

Problem 1 In how small a region does Picard's Theorem hold for f meromorphic in the plane?

We say f has a sequence of filling circles if there are circles $D_k = D(z_k, \varepsilon_k |z_k|)$, $|z_k| \rightarrow \infty$, $\varepsilon_k \rightarrow 0$, so that f takes on all but 2 values infinitely often in $\cup_{j=1}^{\infty} D_{k_j}$, for any subsequence $\{k_j\}$.

$$\text{VALIRON: } \limsup_{r \rightarrow \infty} \frac{T(r, f)}{(\log r)^2} = \infty \Rightarrow f \text{ has a seq. of filling circles (sharp!)}$$

Note that *any* transcendental entire function has a sequence of filling circles. Look at the growth of the spherical derivative:

$$\limsup_{r \rightarrow \infty} r \max_{|z|=r} f^\#(z) = \infty \Rightarrow \text{filling circles.}$$

This condition includes transcendental entire function (all done with normal family arguments.)

Problem 2 How big does f get in filling circles?

Let $D(z_k, \varepsilon_k |z_k|)$ be a set of filling circles for f . We say f grows transenden-

tally in $D_k = D(z_k, \varepsilon_k |z_k|)$ if given M, n , there exists N such that

$$\frac{|f(w_k)|}{|w_k^n|} > M, \quad k > N, \quad \text{for some } w_k \in D_k$$

If $\limsup_{r \rightarrow \infty} \frac{T(r, f)}{(\log r)^3} = \infty$, then such circles exist (sharp). All entire transcendental functions have such circles (Fenton & Rossi, Cercles de remplissage for entire functions, Bull. London Math. Soc. 31(1999), 59-66) (No normal families). For transcendental entire functions

$$\limsup_{r \rightarrow \infty} \frac{r \max_{|z|=r} f^\#(z)}{\log r} = \infty.$$

Is this the right condition? An incorrect use of Zalcman's Lemma indicates yes.

E. Saff *Discrete Minimal Energy Problems on the Riemann Sphere.*

General Problem: Let $S^2 := \{x \in \mathbb{R}^3 : |x| = 1\}$ denote the unit sphere, $|\cdot|$ the Euclidean norm in \mathbb{R}^3 , σ the Lebesgue area measure on S^2 , and $\omega_N = \{x_1, x_2, \dots, x_N\} \subset S^2$.

The general problem is to construct $\{\omega_N\}_1^\infty$ which are uniformly distributed on S^2 , i.e., in the weak-star topology of measures,

$$\frac{1}{N} \sum_{x \in \omega_N} \delta_x \longrightarrow \frac{1}{4\pi} \sigma \quad \text{as } N \rightarrow \infty,$$

where δ_x is the unit point mass at x . One approach to this problem is to use a **minimum energy criterion**.

Fix $s \geq 0$, $N \geq 2$ and define the s -energy by

$$E(s, \omega_N) := \sum_{1 \leq j < k \leq N} \frac{1}{|x_j - x_k|^s}, \quad \text{if } s > 0,$$

$$E(0, \omega_N) := \sum_{1 \leq j < k \leq N} \log \frac{1}{|x_j - x_k|}.$$

An extremal (or equilibrium) configuration $\omega_N^* = \{x_1^*, \dots, x_N^*\}$ is one that satisfies

$$E(s, \omega_N^*) = \min_{\omega_N \subset S^2} E(s, \omega_N) =: \mathcal{E}(s, N).$$

In particular, if

$s = 0$, then ω_N^* maximizes $\prod_{1 \leq j < k \leq N} |x_j - x_k|$;

$s = 1$, then ω_N^* are Fekete points or equilibrium points for the Coulomb potential;

$s \rightarrow \infty$, then ω_N^* tends to $\max_{\omega_N \subset S^2} \min_{1 \leq j < k \leq N} |x_j - x_k|$, the solution to the best-packing problem.

Problem 1 : Asymptotic Behavior of $\mathcal{E}(s, N)$.

For $0 < s < 2$,

$$I(s) := \frac{1}{(4\pi)^2} \iint_{S^2 \times S^2} \frac{1}{|x - y|^s} d\sigma(x) d\sigma(y) < \infty,$$

and it is known [SK] that

$$\mathcal{E}(s, N) = \frac{I(s)}{2} N^2 - R_{N,s},$$

where, for some positive constants c_1, c_2 ,

$$c_1 N^{1+s/2} \leq R_{N,s} \leq c_2 N^{1+s/2}, \quad N \geq 2.$$

Conjecture: There exists

$$\lim_{N \rightarrow \infty} \frac{R_{N,s}}{N^{1+s/2}}, \quad 0 < s < 2.$$

For $s = 1$, we further conjecture that this limit is given in terms of the Riemann zeta function by

$$-3 \left(\frac{\sqrt{3}}{8\pi} \right)^{1/2} \zeta \left(\frac{1}{2} \right) \sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{3n+1}} - \frac{1}{\sqrt{3n+2}} \right) \approx 0.553 \dots$$

For $s = 2$, it is known [KS] that

$$\lim_{N \rightarrow \infty} \frac{\mathcal{E}(2, N)}{N^2 \log N} = \frac{1}{8}.$$

For $s > 2$, the dominant term for $\mathcal{E}(s, N)$ is $\mathcal{O}(N^{1+s/2})$, but what is the coefficient of this term?

Problem 2 : (Smale, [S, Problem #7]) Find $\omega_N \subset S^2$ such that

$$E(0, \omega_N) - \mathcal{E}(0, N) \leq c \log N,$$

where c is a universal constant, and “find” means to determine an algorithm with halting time polynomial in N .

Problem 3 : (Separation of Points) For $s = 0$, $s = 1$, and $s > 2$ it is known [KS],[RSZ] that extremal points $\omega_N^* = \{x_1^*, \dots, x_N^*\}$ for the s -energy are well-separated in the sense that

$$|x_i^* - x_j^*| \geq \frac{c}{\sqrt{N}}, \quad \forall i \neq j,$$

where c is a constant independent of N . Find the best constant $c = c(s)$.

Problem 4 : Investigate these minimum energy problems on general Riemann surfaces.

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