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Recent Developments in Financial and Insurance Mathematics and the Interplay with the Industry

Organised by
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February 18th – February 24th, 2007

Abstract. The workshop brought together leading experts from all over the world to exchange and discuss the latest developments in mathematical finance and actuarial mathematics. Researchers from the industry had the opportunity to circulate their problems among mathematicians. The participants gained from a fruitful interaction between mathematical methods and practitioner's problems as well as from the interaction between finance and actuarial mathematics.

Mathematics Subject Classification (2000): 91Bxx, 60Gxx.

Introduction by the Organisers

The workshop *Recent Developments in Financial and Insurance Mathematics and the Interplay with the Industry*, organised by Søren Asmussen (Aarhus), Nicole Bäuerle (Karlsruhe) and Ralf Korn (Kaiserslautern) was held February 18th–February 24th, 2006. The participants came from all over the world, including Hong Kong and the US. The workshop was attended by 25 mathematicians, most of them were leading experts in mathematical finance and actuarial mathematics and a few researchers from the industry.

In total, there were 21 talks distributed over five days with a long lunch break between 12:30 and 16:00 which made it possible to discuss the latest results and open problems posed in the morning sessions. The major themes of the talks had been

- Optimization problems in finance and insurance
- Multidimensional modeling in finance and insurance
- Valuation of insurance products, credit risk and electricity markets
- Risk measures and distributions with heavy tails.

We proceed with a brief overview of the four subjects above and a summary of the principal results of those areas that were reported on in the workshop:

Optimization problems in finance and insurance: Quite a large number of the talks investigated stochastic dynamic portfolio problems in finance as well as in insurance. *Tomas Björk* who gave the first talk considered a general investment problem where the local rate of return is unobservable. Using non-linear filtering techniques and the martingale method he derived fairly explicit results. *Thaleia Zariphopoulou* introduced a new way to quantify performance measurement in asset allocation. Motivated by the martingale optimality principle she showed how optimal portfolios in this setup can be obtained by solving a fast diffusion equation posed inversely in time. *Ralf Korn* gave a review on old and new results for worst-case portfolio problems. He reported that he now found a HJB-system linking the indifference approach to classical stochastic control theory. *Martin Schweizer's* talk was on utility indifference valuation of contingent claims H in an incomplete market driven by two Brownian motions where the traded and non-traded assets are stochastically correlated. He showed explicit formulas for the indifference value of H in case of exponential utility. *Xin Guo* established a new theoretical connection between singular control of finite variation and optimal switching problems. This approach provides a novel method for solving explicitly high-dimensional singular control problems. *Jostein Paulsen* considered two optimal dividend problems with and without reinvestment (which could prevent from ruin) and where dividend payments and reinvestment are subject to fixed and proportional cost. He gave an explicit solution in case the expected discounted payout minus reinvestment has to be maximized. Finally *Mogens Steffensen* studied optimal consumption and insurance payment streams in a multistate Markovian framework seen from the individuals's perspective. He derived a general solution in case of power utility maximization.

Multidimensional modeling in finance and insurance: It is obvious from the workshop that multidimensional (correlated) stochastic processes have become a very important topic recently. In particular as far as Lévy processes are concerned.In his talk, *Filip Lindskog* extended (under some conditions) the classical Cramér-Wold results on projections and convergence of probability measures to measures with a singularity. This result can also be applied to Lévy processes. *Thomas Mikosch* showed how to combine functional regular variation with heavy-tailed large deviations for partial sums. As an application he derived asymptotic results for ruin probabilities of a multivariate random walk with regularly varying step sizes. *Claudia Klüppelberg* presented a multivariate model for operational risk processes. She discussed in particular the influence of the Lévy copula on the Value-at-risk of the summed process. Finally, *Nicole Bäuerle* showed how to characterize dependence properties and comparison results for Lévy processes with the help of the Lévy measure and the Lévy copula. Some applications in insurance and finance were reported.

Valuation of insurance products, credit risk and electricity markets: Several talks demonstrated that the mixture between financial and actuarial aspects of problems still is increasing. *Thomas Møller* told the audience that a new market for so-called mortality derivatives is now appearing due to the systematic risk in life insurance portfolios. He showed how insurers can reduce their risk by trading e.g. survivor swaps. *Andrew Cairns* discussed some new stochastic models for mortality, in particular the Olivier-Smith model which borrows ideas from interest rate modeling. *Hailiang Yang* investigated the valuation of insurance liabilities of equity-indexed annuities and participating life insurance policies where the equity price process is given by a Markov regime switching model. *Alfred Müller* reported on challenges in modeling electricity price processes and introduced a three-factor model for the spot market price which captures seasonal effects. *Uwe Schmock* presented a one-period model for dependent risks which generalizes both the standard collective risk model and CreditRisk$^+$ and where the portfolio loss distribution can be computed with a numerically stable algorithm. *Holger Kraft* introduced a unified framework for modeling credit risks with bankruptcy and contagion and showed how to compute prices of derivatives in the setup.
Risk measures and distributions with heavy tails: It is clear from the workshop that heavy tails still are an important topic as far as applications are concerned. *Christian Hipp* explained in his talk how third order asymptotic expansions for the tail probability of n-fold convolutions of claim sizes with regular varying tails can be computed. He ended with some conjectures about Weibull type and Lognormal type claim sizes. *Jens Perch Nielsen* talked about new approaches to regression which can determine the full distribution and where errors can have heavy tails. *Hansjörg Furrer* presented the evolution of the regulatory framework from Solvency 0 to Solvency II and posed some open questions concerning the risk measurement in a multi-period framework. *Dirk Tasche* investigated in his talk how kernel estimation can be combined with importance sampling to obtain efficient estimations of Value-at-risk contributions.

In addition to the excellent scientific program, there were two scheduled social activities: Due to the splendid weather, the traditional hike to St. Roman on Wednesday afternoon was a true pleasure despite the muddy short-cut at the beginning. On Thursday night there was a piano concert given by Mogens Steffensen with a lot of entertaining Elton John songs.

For some of the participants this was their first trip to Oberwolfach and they were very impressed by this experience. We, the organizers, would like to thank the "Mathematisches Forschungsinstitut Oberwolfach" for providing such an excellent environment and for the technical support. The participants encouraged the idea of organizing a similar workshop in about three years.

Søren Asmussen
Nicole Bäuerle
Ralf Korn

Recent Developments in Financial and Insurance Mathematics and the Interplay with the Industry

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Abstracts

Optimal Investments under Partial Information

Tomas Björk
(joint work with Mark Davis and Camilla Landen)

We consider a filtered probability space $(\Omega, \mathcal{F}, P, \mathbf{F})$. On this space we have a Wiener process W, and we study a price process with dynamics

$$dS_t = \alpha_t S_t dt + S_t \sigma_t dW_t, \tag{1}$$

where α and σ are adapted with $\sigma_t \geq \epsilon > 0$. The short rate is assumed to be zero. For any adapted self financing portfolio with weight u on the stock, and value X, the portfolio dynamics are

$$dX_t = u_t \alpha_t X_t dt + u_t X_t \sigma_t dW_t, \tag{2}$$

Now define the filtration $\mathbf{G}$ by $\mathcal{G}_t = \mathcal{F}_t^S$. The problem is to maximize

$$E^P \left[U(X_T) \right] \tag{3}$$

over the class of $\mathbf{G}$-adapted portfolios. Thus we will have a partially observed control problem. We solve this problem in two steps:

- We first consider the completely observable case, i.e. when $\mathcal{F}_t = \mathcal{F}_t^S$. This case is easily solved by the use of martingale methods.
- For the general case we project the dynamics of S onto the observable filtration, thus reducing the problem to the completely observable case.

For the cases of power. log, and exponential utility, and without any assumption of a Markovain structure, we manage to obtain surprisingly explicit expressions for the optimal portfolio and the optimal wealth process. This extends earlier work by Bäuerle and Rieder.

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Bankruptcy, counterparty risk, and contagion

Holger Kraft
(joint work with Mogens Steffensen)

Pricing defaultable bonds and credit derivatives has been one of the important topics in finance over the past decade. This development is fueled by a rapid growth in the demand for credit derivatives leading to a $12.4 trillion market by the end of 2005 (as reported by the Bank for International Settlements (BIS)). Credit default swaps (CDS) and more recently collateralized debt obligations (CDO) have become indispensable tools for the securitization of credit risks.[1] Furthermore, current developments in the automobile industry document the significance of so-called contagion effects: On October 8, 2005 the auto parts maker Delphi Corp. filed for Chapter 11 bankruptcy protection. On December 12, 2005 the rating agency S&P cut GM's corporate credit rating to B, five steps below investment grade. On March 3, 2006 the auto parts maker Dana Corp. filed for bankruptcy protection defaulting on $2.5 billion of debt. At the same time, it is argued that the market for credit derivatives is dominated by too few banks, making it vulnerable to a crisis if one of them fails to pay on contracts that insure creditors from companies defaulting. This kind of risk is usually referred to as counterparty risk. The goal of this paper is to offer a framework where all related pricing problems can be addressed in a unified way.

There are a few related papers dealing with the aforementioned issues. Jarrow and Turnbull (1995), Duffie and Singleton (1999a), and Lando (1994, 1998) demonstrate how default risk of a single entity can be handled. Lando (1998) also addresses the important question of default correlation: He puts forth a so-called Cox process framework where default correlation between entities occurs, since all default intensities of the entities depend on the same macroeconomic variables modeled as diffusions. There is an ongoing debate in the literature whether this kind of framework can produce enough default correlation. One of the reasons is that conditioned on the history of macroeconomic variables, defaults occur independently. Hence, there are no feedback effects from defaults of single firms on other firms. To include this kind of contagion effect, there are two suggestions in the literature: The first idea is to make the firm's default intensities dependent on past defaults of other firms in the economy. This idea goes back to Davis and Lo (1999) and Jarrow and Yu (2001). Therefore, conditioned on the history of the macroeconomic variables defaults are no longer independent. The second idea makes use of copulas. Li (2000), Schönbucher and Schubert (2001), as well as Rogge and Schönbucher (2003), among others, assume that conditioned on the macroeconomic variables, defaults are distributed according to some copula function. The first approach resembles the empirically observable fact that upon default of one entity spreads jump upwards, whereas in the second approach one of the key issues is to find a copula leading to the desired default correlation. In an insightful paper, Yu (2005a) demonstrates that models of the Jarrow-Yu type

[1] See, e.g., Longstaff (2006).

induce a certain copula and vice versa. Besides, a recursive procedure called "total hazard construction" is presented that overcomes the deficiencies of the primary-secondary approach by Jarrow and Yu (2001) and generalizes the construction of default times by Lando (1998). Nevertheless, in his examples he mostly needs to resort to Monte Carlo methods.

Building on these ideas by Jarrow and Yu, this paper provides a different view of the problems addressed in the aforementioned papers. Instead of starting by constructing default times, we demonstrate that a few relevant problems can be modeled as Markov chains. This includes models for the pricing of basket credit default swaps. Starting with the papers by Jarrow, Lando, and Turnbull (1997) and Lando (1998), so far Markov chains are almost exclusively used to model rating systems. The advantage of our approach is that it gives us access to many useful results on Markov chains simplifying the task of pricing credit risky assets significantly. By relating pricing problems with default risk to a system of partial differential equations, which without default risk collapses into a Black-Scholes-type equation, we demonstrate how these pricing problems are related to ordinary pricing problems without default risk. This is one of our main contributions. At the same time this paper provides a unified framework nesting various credit risk models. One of the main advantages of our approach is that closed-form solutions for several contingent claims exposed to various types of credit risk can easily be obtained. In particular, we also derive pricing formulas for three building blocks that are generalizations of contingent claims studied in Lando (1998).

Measuring risk concentration

Dirk Tasche

We consider the following stochastic credit portfolio loss model:

$$L = \sum_{i=1}^{n} L_i. \tag{1}$$

$L_1, \ldots, L_n \geq 0$ are random variables that represent the losses that a financial institution suffers on its exposures to borrowers $i = 1, \ldots, n$ within a fixed time-period, e.g. one year. The random variable L then expresses the portfolio-wide loss. We denote by $\mathrm{P}[\ldots]$ the real-world probability distribution that underlies model (1). In other words, $\mathrm{P}[\ldots]$ is calibrated in such a way that it reflects as close as possible observed loss frequencies.

It is common practice for financial institutions to measure the risk inherent in their portfolios in terms of economic capital (EC). As credit risk, for most institutions, is considered to be most important, this is in particular relevant for credit portfolios. EC is commonly understood as a capital buffer intended to cover the losses of the lending financial institution with a high probability. This interpretation makes appear very natural the definition

$$\mathrm{EC} = \mathrm{VaR}_{\mathrm{P},\alpha}(L) - \mathrm{E}_{\mathrm{P}}[L], \tag{2}$$

where the *Value-at-Risk* (VaR) is given as a high-level (e.g. $\alpha = 99.9\%$) quantile of the portfolio-wide loss:

$$\mathrm{VaR}_{\mathrm{P},\alpha}(L) \;=\; \min\{\ell : \mathrm{P}[L \leq \ell] \geq \alpha\}. \tag{3}$$

Hence, if a financial institutions holds EC according to (2) and charges the loans granted with upfront fees adding up to $\mathrm{E_P}[L]$, the probability that it will lose all its EC is not higher than $1 - \alpha$.

Active risk management involves more than just measuring portfolio-wide capital according to (2). Additionally, it is of interest to identify which parts of the portfolio bind the largest portions of EC. The corresponding process of determining a risk-sensitive decomposition of EC is called *capital allocation*. While for the expectation part $\mathrm{E_P}[L]$ of EC on the right-hand side of (2) there is the natural decomposition

$$\mathrm{E_P}[L] \;=\; \sum_{i=1}^{n} \mathrm{E_P}[L_i], \tag{4}$$

there is no such obvious decomposition

$$\mathrm{VaR}_{\mathrm{P},\alpha}(L) \;=\; \sum_{i=1}^{n} \mathrm{VaR}_{\mathrm{P},\alpha}(L_i \,|\, L) \tag{5}$$

for the VaR-part of EC into *risk contributions*. Interpreting risk sensitivity as compatibility with portfolio optimization, [5] proved that the risk contributions $\mathrm{VaR}_{\mathrm{P},\alpha}(L_i \,|\, L)$ on the right-hand side of (5) should be defined as directional derivatives, i.e.

$$\mathrm{VaR}_{\mathrm{P},\alpha}(L_i \,|\, L) \;=\; \frac{d\,\mathrm{VaR}_{\mathrm{P},\alpha}(L + h\,L_i)}{d\,h}\Big|_{h=0}. \tag{6}$$

As VaR is a positively homogeneous risk measure, by Euler's theorem, then (5) holds. Additionally, it turns out [2, 5] that, under fairly general conditions on the joint distribution of L and L_i, the derivative (6) coincides with an expectation of the loss related to borrower i conditional on the event of observing a portfolio-wide loss equal to VaR.

$$\frac{d\,\mathrm{VaR}_{\mathrm{P},\alpha}(L + h\,L_i)}{d\,h}\Big|_{h=0} \;=\; \mathrm{E_P}[L_i \,|\, L = \mathrm{VaR}_{\mathrm{P},\alpha}(L)] \tag{7}$$

As an important application of the risk contribution concept (6), [6] proposed to use it for measuring *risk concentration* and diversification, in the following sense:

Definition 1 *Let $L_1, \ldots, L_n$ be loss variables and let $L = \sum_{i=1}^n L_i$. Then*

$$\mathrm{DI}_{\mathrm{P},\alpha}(L) \;=\; \frac{\mathrm{VaR}_{\mathrm{P},\alpha}(L) - \mathrm{E_P}[L]}{\sum_{i=1}^n \mathrm{VaR}_{\mathrm{P},\alpha}(L_i) - \mathrm{E_P}[L]}$$

denotes the diversification index *of portfolio L with respect to EC based on* $\mathrm{VaR}_{\mathrm{P},\alpha}$. *The fraction*

$$\mathrm{DI}_{\mathrm{P},\alpha}(L_i \,|\, L) \;=\; \frac{\mathrm{VaR}_{\mathrm{P},\alpha}(L_i \,|\, L) - \mathrm{E_P}[L_i]}{\mathrm{VaR}_{\mathrm{P},\alpha}(L_i) - \mathrm{E_P}[L_i]}$$

denotes the marginal diversification index *of sub-portfolio* L_i *with respect to EC based on* $\mathrm{VaR}_{\mathrm{P},\alpha}$.

In general, $\mathrm{DI}_{\mathrm{P},\alpha}(L)$ assuming a value close to 1 will indicate that there is no significant diversification in the portfolio. Similarly, a value close to 1 of $\mathrm{DI}_{\mathrm{P},\alpha}(L_i \,|\, L)$ will indicate that there is almost no diversification effect with credit i. As the dependence – measured as degree of comonotonicity – in a portfolio is influenced both by idiosyncratic and systematic risk, the diversification indices according to Definition 1 capture name diversification as well as sectoral diversification.

In general, no closed-form representations of $\mathrm{VaR}_{\mathrm{P},\alpha}(L)$ and the risk contributions $\mathrm{VaR}_{\mathrm{P},\alpha}(L_i \,|\, L)$ are available. Therefore, often, these quantities can only be inferred from Monte-Carlo samples. This means essentially to generate a sample

$$(L^{(t)}, L_1^{(t)}, \ldots, L_n^{(t)}), \quad t = 1, \ldots, T, \tag{8}$$

and then to estimate the quantities under consideration on the basis of this sample. How to do this is quite obvious for VaR, but is much less clear for the risk contributions $\mathrm{VaR}_{\mathrm{P},\alpha}(L_i \,|\, L)$ as, in general, estimating derivatives of stochastic quantities without closed-form representation is a subtle issue. If $\mathrm{P}[L = \mathrm{VaR}_{\mathrm{P},\alpha}(L)]$ is positive, the conditional expectation on the right-hand side of (7) is given by

$$\mathrm{E}_{\mathrm{P}}[L_i \,|\, L = \mathrm{VaR}_{\mathrm{P},\alpha}(L)] \;=\; \frac{\mathrm{E}_{\mathrm{P}}[L_i \,\mathbf{1}_{\{L=\mathrm{VaR}_{\mathrm{P},\alpha}(L)\}}]}{\mathrm{P}[L = \mathrm{VaR}_{\mathrm{P},\alpha}(L)]}. \tag{9}$$

Even if $\mathrm{P}[L = \mathrm{VaR}_{\mathrm{P},\alpha}(L)]$ is positive, its magnitude will usually be very small, such as $1-\alpha$ or less. [1] showed how to apply *importance sampling* in such a situation in order to efficiently estimate $\mathrm{E}_{\mathrm{P}}[L_i \,|\, L = \mathrm{VaR}_{\mathrm{P},\alpha}(L)]$. [3] and [4] applied similar techniques to the problem of estimating contributions to Expected Shortfall.

However, a crucial condition for (7) to hold exactly is the existence of a density of the distribution of L. The probability $\mathrm{P}[L = \mathrm{VaR}_{\mathrm{P},\alpha}(L)]$ then equals zero, and consequently the right-hand side of (9) is undefined. In this situation, the conditional expectation $\mathrm{E}_{\mathrm{P}}[L_i \,|\, L = \mathrm{VaR}_{\mathrm{P},\alpha}(L)]$ is still well-defined by the Radon-Nikodym theorem, but its estimation from a sample like (8) requires more elaborated non-parametric methods. We follow here [2] who applied *kernel estimation* methods for VaR contributions when optimizing returns in a portfolio of stocks. The kernel estimation procedures, however, have to be adapted to the rare-event character of credit risk. Therefore, in [7] we modify the approach by [2] in a way that can be described as a combination of kernel estimation and importance sampling.

So far, the approach proposed in [7] is not yet fully satisfactory. In particular, the following issues are open:

- Finding an efficient way of choosing an optimal or nearly optimal tilting parameter for the exponential tilting procedure.
- Can the estimation performance be improved by using other kernel estimators than the Nadaraya-Watson estimator?
- How to adapt efficiently the approach by [3] to shifting the distribution of the systematic factors for estimating VaR contributions?

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A multifactor generalisation of the Olivier-Smith model for stochastic mortality

Andrew Cairns

Recent years have seen the development of a number of models for the future development of aggregate mortality rates. Amongst these the Olivier and Smith model (Olivier and Jeffery, 2004, and Smith, 2005) was developed within the forward-rate framework discussed by Cairns et al. (2006) and Miltersen and Persson (2005). This model has a numbe of useful properties that make it a very good model for use in the valuation of life insurance contracts that incorporate embedded options.

We discuss here a generalisation of the Olivier and Smith model. Dynamics of the model in its published form are driven by a sequence of *univariate* gamma random variables. We demonstrate that the model in this form does not adequately match historical data. We discuss a generalisation of the model that uses multivariate Gamma random variables as drivers. This approach potentially gives us much greater control over the term structure of volatility of spot survival probabilities and over the correlation term structure. We introduce a possible approach for simulation of multivariate gamma random variables that facilitates

Forward survival probabilities

We define the risk-neutral forward survival probability that an individual aged x at time 0, survives until time T conditional on the individual being alive at time t $(0 \leq t < T)$ and conditional on information about underlying mortality rates up until time s, to be: $p_Q(s, t, T, x)$. s might be any date greater than or equal to 0, and might be before or after s and before or after T.

The Olivier and Smith model proposes that

$$p_Q(t+1, t, T, x) = p_Q(t, t, T, x)^{g(t+1,T,x)G(t+1)}$$

where $G(1), G(2), \ldots$ is a sequence of i.i.d. Gamma(α, α) random variables, and the $g(t+1, T, x)$ are a set of M_t-measurable normalising constants that ensure that $E_Q[p(t+1, t, T, x)|M_t] = p_Q(t, t, T, x)$.

We consider a generalisation of this model

$$p_Q(t+1, t, T, x) = p_Q(t, t, T, x)^{g(t+1,T,x)G(t+1,T,x)}$$

where the $G(t+1, T, x) \sim$ Gamma$(\alpha(t+1, T, x), \alpha(t+1, T, x))$ are dependent Gamma random variables. Positive mortality rates require that the $p_Q(t+1, t, T, x)$ are strictly decreasing functions of T. This translates into a requirement that

$$G(t+1, T+1, x) > \phi(t+1, T, x)G(t+1, T, x)$$

for all $T > t$, for some M_t measurable constants $\phi(t+1, T, x)$.

In Cairns (2007) we discuss necessary conditions that must be satisfied by the joint distribution of the $G(t+1, T, x)$. However, a more detailed specification of the joint distribution remains an open problem. A partial solution has been proposed by A. Müller (personal communication), but a more flexible formulation is still sought.

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Efficient and Numerically Stable Aggregation of Dependent Risks

Uwe Schmock

(joint work with Richard Warnung)

We presented an actuarial one-period model for dependent risks. It extends the well-known collective risk model for portfolio losses used in actuarial science (cf. [7, Chap. 3]). It also generalizes CreditRisk$^+$, a credit risk model developed by Credit Suisse First Boston [2]. The presented model is suitable for the aggregation of certain dependent risks, which may be insurance, credit or operational risks. The guiding principle for the model extensions are analytic tractability and the possibility to calculate the portfolio loss distribution with an efficient and numerically stable algorithm without any Monte Carlo simulation.

Basically, the portfolio loss distribution is a compound Poisson distribution, where the claim size distribution is a mixture distribution consisting of the individual idiosyncratic random losses and the cumulative random losses caused by the K risk factors. For every risk factor, the losses come in clusters of random size, the losses themselves are again mixture distributions of individual losses, whose sizes are random multiples of a basic loss unit. If the cluster sizes are distributed according to a logarithmic, Poisson or negative binomial distribution, then the distribution of the cumulative losses due to the risk factors can be calculated in a numerically stable and efficient way using the Panjer algorithm. Here numerical stability refers to the provable fact that during the Panjer recursion for the above-mentioned cluster size distributions, only numbers of the same sign are added, hence cancellation does not occur. The final aggregation is done with the Panjer algorithm for the compound Poisson distribution. Therefore, for K risk factors, $K + 1$ numerically stable Panjer recursions are sufficient.

For the extended (truncated) negative binomial distribution, the Panjer recursion is numerically unstable due to cancellations, as can be shown by examples. However, when $\alpha \in (-1, 0)$, meaning that the support is $\mathbb{N}$, a modified algorithm, involving the stable Panjer recursion for the usual negative binomial distribution followed by a weighted convolution, is numerically stable (however, the number of operations is doubled), see [3].

The original CreditRisk$^+$ aggregation algorithm (corresponding to deterministic individual losses and logarithmically distributed cluster sizes) was reported to have numerical instabilities, cf. [4]. A modification proposed by G. Giese [4], for which Haaf, Reiß and Schoenmakers [6] proved the numerical stability, is a special case of the algorithm outlined above. We want to emphasis that our extension to random individual losses allows to model even dependent credit rating transition risk (which manifests itself as market risk as long as no default happens).

Our above algorithm allows to calculate quantiles and expected shortfall explicitly without any Monte Carlo simulation. For logarithmically distributed cluster sizes, the risk contributions can be calculated, too, adapting a lemma by Tasche [9, Section 3.4]. For certain coherent risk measures considered by Cherny and Madan [1], we can calculate tight upper and lower bounds numerically (without any Monte Carlo simulation).

More details can be found in [8], an application to operational risk and a comparison to the fast Fourier transform method is given in [10].

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On systematic mortality risk and risk-minimization with survivor swaps

Thomas Møller
(joint work with Mikkel Dahl, Martin Melchior)

A new market for so-called mortality derivatives is now appearing with survivor swaps (also called mortality swaps), longevity bonds and other specialized solutions. The development of these new financial instruments is triggered by the increased focus on the systematic mortality risk inherent in life insurance contracts, and their main focus is thus to allow the life insurance companies to hedge their systematic mortality risk. At the same time this new class of financial contracts is interesting from an investor's point of view since they increase the possibility for an investor to diversify the investment portfolio. The systematic mortality risk stems from the uncertainty related to the future development of the mortality intensities. Mathematically this uncertainty is described by modeling the underlying mortality intensities via stochastic processes.

We study a model with two mortality intensities and two underlying random processes, where the first mortality intensity represents the mortality of the insurance portfolio, and the second intensity represents the mortality of a population. We allow the two mortality intensities to be driven by the same underlying processes, where one process is taken to represent the general uncertainty and the other represents a more specific uncertainty for the given insurance portfolio. The model is inspired by the one proposed in [2] and [3] and uses the so-called CIR-processes known from the financial literature for the modeling of mortality intensities.

We consider different financial markets, which contain a zero coupon bond and possibly one or more survivor swaps, and study the possibilities of hedging in these markets. In all the markets we have more sources of risk (financial risk and mortality risks) than financial assets, so we apply theory from incomplete markets. More precisely, we use the criterion of risk-minimization introduced by [1] for contingent claims and extended to payment processes by [4] to determine risk-minimizing strategies. The strategies illustrate how the combined insurance and financial risk can be hedged partly with bonds and survivor swaps. This extends the work of [3]. The strategies are evaluated numerically.

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Decisions and Design in Life and Pension Insurance

Mogens Steffensen
(joint work with Holger Kraft)

Personal financial decision making plays an important role in modern finance. Decision problems about consumption and insurance are modelled in a continuous-time multi-state Markovian framework. The optimal solution is derived and studied. The model, the problem, and its solution are exemplified by two special cases: In one model the individual takes optimal positions against the risk of dying; in another model the individual takes optimal positions against the risk of losing income as a consequence of disability or unemployment. The solution leads to a discussion on optimal design of life insurance contracts.

Third order expansions for compound distributions and for ruin probabilities with claims having regularly varying tails

Christian Hipp

Consider compound sums $S = X_1 + ... + X_N$ with independent claim sizes X_i which have a distribution with regularly varying tails with index γ, i.e.

$$\mathbb{P}\{X_i > s\} = \overline{F}(s) = L(s)s^{-\gamma}$$

where $L(s)$ is slowly varying, and $F(s)$ admits a smooth density $f(s)$. Assume that the claim number N is independent of the claim sizes $X_i, i = 1, 2, ...$ and has a distribution belonging to the Panjer class. It is well known that the numerical computation of the total claims distribution $G(s) = \mathbb{P}\{S \leq s\}$ is time consuming because of the fat tail of the claim size distribution. This is true for both methods of computation: the integro-differential equation as well as Panjer's method with discretized claim sizes. For $\overline{G}(s)$ we have several suggestions for approximation which are theoretically valid for large s : the first order asymptotic formula of Embrechts and Veraverbeke $a_1(s)$ in [3] and the second order asymptotic formula of Baltrunas $a_2(s)$ from [2]. Further work on approximations of compound distributions in the heavy tailed case can be found in Grübel [4],Mikosch [5], and Willekens [6].

We derive a third order approximation $a_3(s)$ for $\overline{G}(s)$. Approximations for compound distributions lead to similar approximations for ruin probabilities in the classical Lundberg model. Under the net profit condition $c > \lambda\mu$ we can use the Pollaczek-Khintchine formula

$$\psi(s) = \sum_{n=1}^{\infty} p^n (1-p) H^{*n}(s, \infty),$$

where $p = \lambda\mu/c$ and H is the ladder height distribution with density

$$h(x) = \frac{1}{\mu}\overline{F}(x), x > 0.$$

We investigate the performance of all these approximations for the special case of Pareto claim sizes, for compound distributions as well as for ruin probabilities. For numerical calculations we discretize the distribution $F(s)$ to obtain arithmetic distributions F_1 and F_2 which are upper and lower bounds for F. If $\Delta > 0$ is a step size then the point probabilities $f_i(k\Delta)$ for F_i are given by

$$f_1(k\Delta) = \int_{k\Delta}^{(k+1)\Delta} f(x)dx, k = 0, 1, 2, ...$$

$$f_2(k\Delta) = \int_{(k-1)\Delta}^{k\Delta} f(x)dx, k = 1, 2, ...$$

are upper and lower bounds for $F(s)$ in the sense that for all $s \geq 0$

$$\overline{F}_1(s) \leq \overline{F}(s) \leq \overline{F}_2(s).$$

The correspondig approximations for $G(s)$ are upper and lower bounds:

$$G_1(s) \leq G(s) \leq G_2(s),$$

where for $i = 1, 2$

$$G_i(s) = \sum_{n=0}^{\infty} \mathbb{P}\{N = n\} F_i^{*n}(s), s \geq 0.$$

For the computation of the point probabilities $g_i(k\Delta)$ of $G_i(s)$ we use Panjer's recursion:

$$\begin{aligned} g_1(0) &= \sum_{n=0}^{\infty} \mathbb{P}\{N = n\} f_1(0)^n \\ g_1((k+1)\Delta) &= \sum_{j=1}^{k+1} \left(a + b\frac{j}{k+1}\right) \frac{f_1(j\Delta) g_1((k+1-j)\Delta)}{(1 - af_1(0))}, k = 0, 1, 2, ... \end{aligned}$$

and

$$\begin{aligned} g_2(0) &= \mathbb{P}\{N = 0\} \\ g_2((k+1)\Delta) &= \sum_{j=1}^{k+1} \left(a + b\frac{j}{k+1}\right) f_2(k\Delta) g_2((k+1-j)\Delta), k = 0, 1, 2, ... \end{aligned}$$

We need two different recursions because of $f_1\{0\} > 0$.

The first order approximation is classical: $\overline{G}(s) = a_1(s) + o(\overline{F}(s))$, with

$$a_1(s) = E[N]\overline{F}(s).$$

Our numerical computation reproduce the result of Abate et al. [1] that the first order asymptotic formula may underestimate $\overline{G}(s)$, i.e. the asymptotic formula is of little use in the range of finite s which are of interest in real world applications. Next, we consider the second order expansions for $\overline{G}(s)$ developed by Baltrunas (see [2], p. 132, Theorem 5.1.(iii)):

$$\overline{G}(s) = a_2(s) + o(f(s)),$$

where

$$a_2(s) = a_1(s) + E[N(N-1)]\mu f(s). \tag{1}$$

Also this approximation is not satisfactory.
We prove the following third order approximation for $\overline{G}(s)$:

$$\overline{G}(s) = a_3(s) + o(f'(s)),$$

where

$$a_3(s) = a_2(s) + \left(E[N(N-1)]\mu_2 - \frac{1}{2}E[N(N-1)(N-2)]\mu^2 \right) f'(s), \tag{2}$$

and μ_2 is the second moment of X_i. Our numerical comparisons show that the third order approximation reduces the error in the first and second approximation in such a way that it can be recommended for practical applications.

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Investment performance measurement under forward criteria

Thaleia Zariphopoulou
(joint work with Marek Musiela)

Traditionally, how well an investor does is assessed through expected utility criteria, typically formulated via a deterministic concave function of terminal wealth. A key element of this approach is the a priori specification of risk preferences at a fixed future time. As a result, optimal behavior is directly influenced by the horizon choice, a fact that not only limits the applicability of such criteria but also poses potential inconsistency problems in valuation and hedging across different maturities.

A new approach is proposed that alleviates the horizon dependence and maintains natural optimality properties. Specifically, one seeks for a process defined at all times, is a supermartingale at arbitrary controls and becomes a martingale at an optimum. Its martingality property is a natural consequence of the desire to maintain the current state, if an optimum has been achieved. The supermartingality property, on the other hand, expresses deviation from such a state and, thus, decreasing future expected performance. In contrast to the existing framework, however, the forward performance datum is determined today and not a (possibly remote) future time.

Several difficulties are encountered due to the fact that the associated stochastic optimization problems are posed "inversely in time" and, thus, existing techniques in portfolio choice have limited use, if any. A construction approach is proposed that applicable for a large class of models and produces a rich family of forward solutions. The method is based on two novel ingredients, namely, a stochastic time change (subordination) and the compilation of appropriately chosen differential and stochastic input. The differential input is determined by the investor's dynamic preferences while the stochastic input follows exclusively the changes in the market.

The initial datum is taken to be a concave function of wealth. The model is incomplete, non-Markovian and may include many securities. The approach is general enough so that it allows for measuring investment performance with regards to a benchmark as well as for cases in which the investor has different views about upcoming market behavior, or faces trading constraints.

Besides producing the forward solutions, the proposed method provides a direct way for closed form construction of the associated optimal allocations. Despite the non-Markovian nature of the model, optimal allocations turn out to be stochastic feedback functionals of current wealth. The stochastic time change is a key element for this local dependence. The optimal policies have also very pleasing form. Specifically, they consist of two funds that are, respectively, proportional to (benchmarked) wealth and the subordinated (benchmarked) risk tolerance process. The proportionality coefficients are processes depending only on the market parameters. This two-fund separation result holds for arbitrary initial data and provides a rather universal, and at the same time, intuitive structure of the optimal strategies. It is worth mentioning that in traditional expected utility models, the form of optimal portfolios is rather opaque and only implicitly deduced through martingale representation theorems in the dual domain.

Important role in the analysis plays the risk tolerance process. It is defined as the local risk tolerance function with its space and time arguments evaluated, respectively, at benchmarked wealth and at the time-subordinator process. The former function satisfies a fast-diffusion type differential constraint while its reciprocal, the investor's local risk aversion, solves a porous medium equation. These properties also enable us to construct a system of stochastic differential equations that is satisfied by the pair of wealth and risk tolerance processes, at the optimum. This autonomous system also comes as a surprise given the non-Markovian character of the model.

Dependence Properties and Comparison Results for Lévy Processes

Nicole Bäuerle
(joint work with Anja Blatter, Alfred Müller)

Whereas dependence properties and comparison results for random vectors are by now very well established, there still is need for research as far as stochastic processes are concerned. In this talk we investigate dependence properties and comparison results for multidimensional Lévy processes. The talk is based on [1].

1. Dependence Properties of Lévy Processes

Let $X = (X(t))_{t\geq 0}$ be a d-dimensional Lévy-process, i.e. a stochastically continuous process with independent and stationary increments. From the Lévy-Itô-decomposition we know that the distribution of a Lévy process is uniquely determined by a characteristic triplet (A, ν, γ), where A is a covariance matrix of a Brownian motion, γ is a drift parameter and ν is the Lévy-measure determining the frequency and size of jumps. Since the continuous part and the jump part of a Lévy process are independent ([6][Theorem 19.2]) it suffices to consider the dependence structure of the continuous and the discontinuous part of Lévy processes separately. In the following we will focus on the dependence structure of the jump part of a Lévy process only since the Brownian part is easy to handle. We will consider the following notions of positive dependence which have been defined for random vectors: *association, positive orthant dependence (POD)* and *positive supermodular dependence (PSMD)*. The last concept is in particular interesting for applications. These dependence concepts can now easily be extended to stochastic processes in the following way: The $\mathbb{R}^d$-valued stochastic process $X = (X(t))_{t\geq 0}$ is said to be associated (POD, PSMD) if and only if $(X(t_1), \ldots, X(t_n))$ is associated (POD, PSMD) for all $0 \leq t_1 < t_2 < \ldots < t_n$ and all $n \in \mathbb{N}$. Note that in the case of Lévy processes this is equivalent to $X(t)$ being associated (POD, PSMD) for all $t \geq 0$. Here we obtain the following result:

Theorem 1. *Let X be a d-dimensional Lévy process with Lévy measure ν.*

a) *X is associated if and only if ν is concentrated on*

$$\mathbb{R}^d_{++,--} = \{x \in \mathbb{R}^d \mid x_i \geq 0 \forall i \text{ or } x_i \leq 0 \forall i\}$$

b) *The following statements are equivalent:*
   (i) *X is associated.*
   (ii) *X is POD.*
   (iii) *X is PSMD.*

These notions of dependence can also be characterized by Lévy copulas, a concept which has been introduced recently by [4] and further refined by [5].

Theorem 2. *Let X be a d-dimensional Lévy process with Lévy copula F. X is associated (POD, PSMD) if and only if $F(u) = 0$ for $u \in \left(\mathbb{R}^d_{++,--}\right)^c \cap \prod_{i=1}^d \overline{RanU_i}$.*

However, note that there are other important concepts of dependence like *multivariate total positivity of order 2* (MTP_2) or *conditionally increasing in sequence* (CIS) which cannot be characterized by the Lévy copula.

2. Comparison Results for Lévy Processes

We will consider the supermodular $\leq_{sm}$ and the concordance order $\leq_c$ here. Analogously to the previous section these orderings can be extended to stochastic processes. Comparison results for semimartingales can be found in [2]. Among others we obtain the following results

Theorem 3. *For Lévy processes X, $\tilde{X}$ with Lévy measures $\nu, \tilde{\nu}$, the following conditions are equivalent:*
   (i) *$X \leq_{sm} \tilde{X}$.*
   (ii) *$\nu \leq_{sm} \tilde{\nu}$, i.e. $\int f d\nu \leq \int f d\tilde{\nu}$ for all supermodular $f \in \mathcal{B}_0$ where $\mathcal{B}_0 := \left\{f : \mathbb{R}^d \to \mathbb{R} \mid f \text{ is measurable and bounded and } \limsup_{x \to 0} \frac{f(x)}{\|x\|^2} < \infty\right\}$.*

Theorem 4. *Let $d = 2$. For Lévy processes X, $\tilde{X}$ with Lévy measures ν, $\tilde{\nu}$ and Lévy copulas F, $\tilde{F}$ the following conditions are equivalent:*
   (i) *$X \leq_c \tilde{X}$.*
   (ii) *$X \leq_{sm} \tilde{X}$.*
   (iii) *ν and $\tilde{\nu}$ have the same marginal tail integrals and $F \leq \tilde{F}$.*

3. Applications in Finance and Insurance

3.1. **Ruin Time Points.** Suppose the Lévy process $X = (X_1(t), \ldots, X_d(t))_{t \geq 0}$ represents the evolution of d risk reserve processes of different business lines. Denote by

$$\tau_j := \inf\{t \geq 0 \mid X_j(t) \leq 0\}$$

the ruin time of risk reserve $j = 1, \ldots, d$. If X is an $\mathbb{R}^d$-valued Lévy process and X is associated (or POD or PSMD) then the ruin time points $\tau = (\tau_1, \ldots, \tau_d)$ are associated (and thus also POD and PSMD). Analogously we obtain a comparison result: Let X and $\tilde{X}$ be two $\mathbb{R}^d$-valued Lévy processes. If $X \leq_{sm} \tilde{X}$ then the ruin time points are ordered:

$$\tau = (\tau_1, \dots, \tau_d) \leq_{sm} \tilde{\tau} = (\tilde{\tau}_1, \dots, \tilde{\tau}_d).$$

Now define by $X_t^+ := \sum_{i=1}^d X_i(t)$ the sum of these processes. Obviously the law of X^+ depends on the Lévy copula. By ψ_{X^+} we denote its probability of ruin

$$\psi_{X^+}(u) = P\left(\inf_{t\geq 0} X_t^+ < 0 \mid X_0^+ = u\right).$$

Suppose we have two portfolios of risk processes X and $\tilde{X}$ which are both $\mathbb{R}^d$-valued Lévy processes. If $X \leq_{sm} \tilde{X}$ then for all $u > 0$:

$$\int_u^\infty \psi_{X^+}(s)ds \leq \int_u^\infty \psi_{\tilde{X}^+}(s)ds$$

and thus also the adjustment coefficients are ordered whenever they exist, which extends results in [3].

3.2. **Stock and Option Prices.** Let X be an $\mathbb{R}^d$-valued Lévy process and let the price processes of d assets satisfy the following stochastic differential equation

$$\begin{aligned} dS_i(t) &= S_i(t-)\big[\mu_i(t)dt + \sigma_i(t-)dX_i(t)\big] \\ S_i(0) &= 1 \end{aligned}$$

where $\mu_i(t), \sigma_i(t)$ are bounded deterministic càdlàg functions. Further we assume for all $i = 1, \dots, d$ that $\sigma_i(t)\big(X_i(t) - X_i(t-)\big) \geq -1$ for all $t \geq 0$. This assumption guarantees that the stock prices stay positive. Now it can be shown that if the Lévy process X is associated (or POD or PSMD), then the price processes are associated (and thus also POD and PSMD). This result can be used to obtain bounds on option prices for options whose pay-offs depend on more than one stock price. For example take a contingent claim with pay-off $H = h(S_1(T), S_2(T))$. Its price is given by $\pi(H) = B_T^{-1} E_Q[h(S_1(T), S_2(T))]$ where Q is an adequately chosen pricing measure. If h is a supermodular function and S_1 and S_2 are associated under Q, then

$$\pi(H) \geq \pi(H^\perp)$$

where $\pi(H^\perp)$ is the price of the same option with independent price processes. Typical functions h which are supermodular are $h(x,y) = \big(\min(x,y) - K\big)^+$ and $h(x,y) = \big(x + y - K\big)^+$.

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Regular variation, large deviations and ruin

Thomas Mikosch
(joint work with Henrik Hult, Filip Lindskog, Gennady Samorodnitsky)

First we extend classical results by A.V. Nagaev (1969) on large deviations for sums of iid regularly rvarying andom variables to partial sum processes of iid regularly varying vectors. The results are stated in terms of a heavy-tailed large deviation principle on the space of càdlàg functions. The main result is analogous to Mogulski's theorem, i.e., it can be understood as a large deviation extension of Donsker's functional CLT. We illustrate how these results can be applied to functionals of the partial sum process, including ruin probabilities for multivariate random walks and long strange segments. These results make precise the idea of *heavy-tailed large deviation heuristics*: in an asymptotic sense, only the largest step contributes to the extremal behavior of a multivariate random walk.

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Modelling Dependence of Lévy Processes

Claudia Klüppelberg

Besides market risk and credit risk, the *risk of losses resulting from inadequate or failed internal processes, people and systems, or external events*, called operational risk has become a main focus of financial risk management. Simultaneous modelling of operational risks occurring in different event type/business line cells poses a great challenge for operational risk quantification. Financial risk is typically modelled by a high quantile, the so-called Value-at-Risk. Because of the similarities of operational risk severities to insurance claims, high quantile approximations for the single cell operational risk can be derived easily (cf. [2]).

To model the dependence structure of operational loss events we invoke in [3, 4] the concept of Lévy copulas. In particular, we derive approximations of similar quality and simplicity for multivariate operational VAR.

Lévy copulas have been suggested in [5] and separate the marginal Lévy processes from their dependence structure. For a spectrally positive Lévy process in $\mathbb{R}^d$, a Lévy copula is a measure on $\mathbb{R}_+^d$ with Lebesgue marginals. As standardisation of the marginals is rather arbitrary, other concepts are possible. To sharpen the understanding of Lévy measures a standardisation to a Lévy measure instead of Lebesgue measure seems favourable.

In [1] it was suggested to standardise the marginal processes to 1-stable Lévy processes. Any Lévy process can then be represented by its marginal standardisations and the Lévy copula representing the dependence structure. This new Lévy copula is then itself the Lévy measure of a Lévy process with 1-stable marginals. Relationships to arbitrary stable processes are obvious. The concept of spectral measure for regularly varying Lévy processes can also be embedded into this concept.

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Extreme value regression in practise, also when data are underreported, truncated or censored

Jens Perch Nielsen

This is summary of a longer paper. We only present the introduction and the new set of estimators. Asymptotic theory, the simulation study and the empirical study is left out of this summary that has to be at most eight pages long. Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be n independent identically distributed set of variables. We wish to estimate various functionals of the conditional distribution of Y_1 given X_1. We are in particular concerned about functionals emphasizing the importance of extreme high values of the dependent variable. We do not mind to be guided by some complexity reducing structure or prior knowledge on a useful parametric model. However, we do wish a nonparametric estimator to be available for our final judgement. In our case study below we apply the complexity reducing multiplicative structure:

$$Y_i = g(X_i)\varepsilon_i,$$

where the residuals $\varepsilon_1, \ldots, \varepsilon_n$ are independent identically distributed variables and we use the modified Champerknown distribution, see Buch-Larsen, Nielsen, Guillen, Bolance (2005), as our paramatric guidence.

Also, we do not observe $(X_1, Y_1), \ldots, (X_n, Y_n)$ directly, but only $(X_1, Y_1 \wedge C_1), \ldots, (X_n, Y_n \wedge C_n)$, where $(C_1, \ldots, C_n)$ are independent indetically distributed right censoring times. Also, our data are truncated to the left such that a pair (X, Y) is only observed when Y is above some threshold value T, where T is the time of left truncation. Had Y been below this T, we would not observe anything about the pair (X, Y); not even that it had been truncated. Let $(T_1, \ldots, T_n)$ be these truncation times. One prominent example where this statistical problem arises is in commercial insurance. The censoring applies when there is some upper limit on the insurance policy. Either as part of the actual contract or as a consequence of poor data collection where only the actual expense of the company is recorded disregarding amounts paid by the reinsurance company. Typically, an insurance company holds an excess of loss contract where the reinsurance company covers amounts above some threshold value exactly corresponding the right censoring mechanism described above. Left truncation exactly corresponds to the widely used deductibles. A loss below the deductible value is covered by the individual policy holder without even noticing the insurance company. Even in the simple one-dimensional case without any censoring or truncation our estimation problem is non-trivial and has given rise to an enormous amount of theory on the extreme value behaviour of distributions and its estimation. The so called EVT-theory, see Embrecht, Mikosch and Klüppelberg (1999) for a prominent textbook on this. However, most of this literature is based on the asymptotic behaviour of maximal values of stochastic variables arguing for the existence of a limit distribution - generalised pareto - to which all extreme values distributions can be approximated. In practise most EVT-methods are based on personal judgement; much like when a bandwidth is chosen by eye-ball in nonparametric smoothing problems. Automatised versions of EVT exist but have never convinced practioners and are not used very often. Also, there are surprisingly few simulation studies spelling out the actual benefits of EVT methods. This lead Bolance, Guillen and Nielsen (2003) and Buch-Larsen, Bolance, Guillen and Nielsen (2005) to view this this one-dimensional problem as a standard estimation problem attempting to improve estimation considering the classical trade off between variance and bias present in all problems of statistical inference. The extreme tail was accounted for by transformation methods inspired by the pioneering paper in the field of Ruppert, Marron and Wand (1991). In the working paper version of the first of these two papers, Bolance, Guillen and Nielsen (1999), a simulation study was carried out where it was shown that classical EVT models did not work very well for any of the distributions considered in the study and we have concluded that we do not use classical EVT methods for business decisions. Buch-Kromann, Guillen, Linton and Nielsen (2007) extended the approach of Buch-Larsen et al. (2005) to a multivariate setting, where the loss distribution is allowed to depend on covariates. This lead to various methods of multivariate density estimation and its adjustment guided by structured models. The widely available methodology of regression is not appropriate for this type of problems where we need a full model specification and not just mean functions or quantiles. In this paper we extend the approach of Buch-Kromann et al. (2007) to the more complicated setting where truncation and right censoring is present. From a theoretical point of view we have extend the methodology of Buch-Kromann et al. (2007) to a filtered data setting of this paper. Filtered data is short for data that might have been truncated or censored. We use counting process theory for this task and have to go through some theoretical steps that have recently been throdden within the field of nonparametric smoothing based on filtered data. Nielsen, Tanggaard and Jones (2007) notes that nonparametric smoothing of densities can be generalised in such a way that it in a filtered data context corresponds to local polynomial hazard estimation weighted with the classical Kaplan-Meier estimator. Without filtering this local constant estimator simply collapses to the standard kernel density estimator. Nielsen et al. (2007) also notice that they do not recommend this estimator in general for filtered data. The reason is what they call exposure robustness indicating that another weighting, the so called natural weighting combined with a smooth version of the Kaplan-Meier, works just as well as standard kernel density estimation when there is no filtering or when filtering is happening in a smooth an unsurprising way. However, when lack of robustness is present in the exposure pattern, the method with natural weighting and a smoothed Kaplan-Meier outperformances the other method enormously. Therefore, Nielsen et al. (2007) suggested always to use the latter approach since there was no pain, only gain. We generalise this latter approach to the multivariate setting. First we define a smoothed conditional Kaplan-Meier as a simple functional of the multivariate kernel hazard estimator of Nielsen and Linton (1995). Then we define our nonparemetric conditional density estimator as a weighted version of this very same local constant multivariate kernel hazard estimator, where the weight is the smoothed conditional Kaplan-Meier. Once a conditional density estimaor is available we can approximate this density to our complexity reducing structure. Finally we follow Buch-Kromann et. al (2007) and apply this structured density to guide a bias correcting leading to our final smooth nonparametric density estimator.

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Connecting Singular and Switching Controls, with Applications

Xin Guo
(joint work with Pascal Tomecek)

In [5], we establish a new theoretical connection between singular controls of finite variation and a class of switching controls. This correspondence provides a novel methodology for solving explicitly high-dimensional singular control problems, and links singular controls and Dynkin games through sequential optimal stopping problems.

Built on this correspondence, we analyze in [6] a class of singular control problems from (ir)reversible investment for which value functions are not necessarily smooth. Necessary and sufficient conditions for the well-known smooth fit principle, along with the regularity of the value functions, are given. Explicit solutions for the optimal policy and for the value functions are provided. In particular, when the payoff functions satisfy the usual Inada conditions, the boundaries between action and no-action regions are smooth and strictly monotonic as postulated and exploited in the existing literature ([4, 3, 7, 1, 9, 10, 2, 8]). Illustrative examples for both smooth and non-smooth cases are discussed, to highlight the pitfall of solving singular control problems with *a priori* smoothness assumptions.

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Electricity Markets: A Challenge for Financial Mathematics

Alfred Müller
(joint work with Markus Burger, Bernhard Klar, Gero Schindlmayr)

Contracts between electric utilities typically offer a substantial amount of flexibility in the form of complex embedded options. Demand for such optionalities arises naturally from the unpredictability of power consumption and from the optionalities inherent in power plants. In the past, there rarely was the necessity to precisely evaluate the value of these optional parts, because electricity was not a commodity which could easily be traded, and because supply of electric power was assured by utility companies under regulatory control. In fact, most counterparts did not use the flexibility of the delivery contracts in a market-orientated way. In recent years, these matters have changed dramatically. In many countries electric power markets have been liberalized and exchanges and online trading platforms for electricity contracts have been founded. Market participants now take advantage of the optionality in their contracts by optimizing against market prices and looking for arbitrage opportunities. Therefore, it has become an important task for utilities to develop new pricing models for the contracts they buy and sell and to quantify and manage the involved risks. This leads to new challenges for financial mathematics, as the stochastic dynamics of electricity prices are quite different from all other markets, which is mainly due to the non-storability of electricity.

In this talk we suggest a simulation model for spot market prices capturing the following stylized facts observed in the market:

- seasonal patterns of daily, weekly and yearly periodicities,
- extreme price spikes,
- mean reversion,
- price dependent volatilities,
- long-term non-stationarity.

The model is a three-factor model, where one factor describes the load process, whereas the second factor is a stationary process, describing the short-term mean-reverting behavior of the prices, and the third term describes the long-term non-stationary behavior as observed from futures prices. Moreover, the model takes into account for the nonlinear relation between load and price, and for the seasonality in the availability of power plants. For a detailed description of the model we refer to [1].

In the second part of the talk we suggest a simple heuristic for pricing swing options, based on a dynamic programming algorithm for a simplified model. By comparing this heuristic with explicitly computable upper and lower bounds, it is shown that this simple heuristic is a surprisingly good approximation of the optimal strategy. And the end of the talk we address several open problems. In particular it would be desirable to include the following features in a more realistic model:

- non-Gaussian dependence modelling of the risk factors, allowing for modelling tail dependence,

- modelling stochastic volatility of the risk factors (GARCH processes are not realistic, as the lead to infinite prices for many options!),
- better modelling of futures prices,
- explicit solutions for the exercising strategies and the prices of swing options and other complex derivatives.

Other important open problems concern the possibilities for hedging (e.g. with weather derivatives) and the question about the correct pricing measure, as the market is highly incomplete, and therefore there are many equivalent martingale measures, and it is not clear, which of these to choose as the right one for pricing derivatives.

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Risk-based solvency requirements

Hansjörg Furrer

A number of recent initiatives such as the preparation for Solvency II have led to an increase in the extent to which insurance companies manage their risk and capital. The aim of this talk is to give an overview of the current risk modeling initiatives in the context of prudential supervision. To begin with, the characteristics of Europe's first life and non-life solvency margin rules are discussed. These rules go back to the 1970s, and were established in 1973 under the First Non-Life Directive (73/239/EEC) and in 1979 under the first Life Directive (79/267/EEC). These two directives marked the first steps towards the establishment of the free market in insurance within the European Community. The rules for the calculation of the solvency margin requirements stem from the works of Campagne [5], and are based on simple ratios that are calculated as percentages of risk exposure measures such as technical provisions, premiums or claims. Their focus is mainly on insurance risks, though for with-profit (participating) business a factor of 4% based on technical provisions takes account of the investment risk. Campagne's life approach is based on some simplifying assumptions on the distribution function of the loss ratio, where the loss ratio is defined as the loss L in a year divided by the technical provisions R. The loss ratios were assumed to be iid for different years and companies, and distributed according to a Pearson-Type IV distribution. The Pearson-Type IV probability density function is as follows:

$$f(x) = k\biggl(1+\Bigl(\frac{x-\lambda}{a}\Bigr)^2\biggr)^{-m} \exp\Bigl\{-\nu \operatorname{artan}\Bigl(\frac{x-\lambda}{a}\Bigr)\Bigr\}, \qquad x \in \mathbb{R}, \tag{1}$$

where m, ν, a and λ are real-valued parameters and $k = k(m,\nu,a)$ is a normalizing constant that depends on m, ν and a. Observe that when $\nu = 0$, the Pearson-Type IV distribution is a version of Student's t-distribution. The minimum solvency margin MSM was defined as the Value-at-Risk of the loss ratio distribution at the 95% level. For the data that was used this led to a necessary minimum solvency margin of 4% of the technical reserves.

During the process with the third directives, the Council discussed the possibility to review the provisions concerning the solvency margins. In April 1994, the Insurance Committee agreed to ask the European supervisory authorities (now CEIOPS) to establish a working group to look into solvency issues in a broad sense. Helmut Müller from the Bundesaufsichtsamt für das Versicherungswesen (BAV) chaired that group. The report that the working group presented in 1997 is known as the "Müller Report", see [11]. It was considered in the Müller Report that the current solvency margin requirements had proved satisfactory. However, the Müller Report pointed out some specific cases of deficiencies that could have been avoided by more accurate solvency margin regimes. These deficiencies relate to investment and asset-liability mismatches, inadequate reinsurance etc. The preparatory work included in the Müller Report and a Commissions report essentially led to the proposal for a new life and non-life solvency margin Directive. The two Directives, known as Solvency I, focus on improving the existing rules for the calculation of the solvency margin requirement.

Solvency II is the continuation of the work already initiated in Solvency I. Whereas Solvency I aimed at revising and updating the then-current EU solvency regime, Solvency II has a much wider scope. It contains a fundamental and wide-ranging review of the current regime in the light of the current developments in insurance, risk management, financial reporting etc. One of the key objectives of Solvency II is the establishment of a solvency system that is matched to the true risks of an insurance undertaking. The Solvency II project is divided into three distinct phases. The first phase lasted from 2001 to 2003, and was designated to do the preparatory work. In particular, it consisted of gathering knowledge in relation to the general form of a solvency system. The second phase stretches from 2003 to 2007. It is more technical and is devoted to details such as the valuation of assets and liabilities and the determination of risk capital. Finally, the third phase is the implementing phase. This also includes the implementation in the national laws.

The different solvency assessment frameworks and proposals in the context of Solvency II have in common that their target capital requirements are based on four main risk categories: market risk, insurance risk, credit risk, and operational risk. Most of these main risk categories can be further subdivided into sub-risks. For example, insurance risk can be subdivided into u/w risk, mortality risk, sickness risk, surrender or lapse risk, and expense risk. Once the risk categories have been fixed, the procedure for the determination of the solvency requirement is as follows:

- quantify each risk category (subcategory) and equip it with a certain target capital requirement. The quantification is done by means of risk measures in connection with (simplifying) distibutional assumptions. Using Value-at-Risk as risk measure and the normal distribution as the first order approximation of the true distribution, this implies that the target capital requirement equals the standard deviation times a factor k, e.g. $k = 2.33$ or $k = 2.58$ depending on the level of confidence.

- combine the individual target capital requirements for each risk category to a single target capital, thereby taking the dependency structure into account. Most often, the dependency structure for the various risk categories is expressed via their (linear) correlations.
- compare the target capital requirement with the available solvency capital.

During spring 2003, the Federal Office of Private Insurance (FOPI) initiated the Swiss Solvency Test (SST) project. The SST approach is based on principles rather than on fixed rules. The assets and liabilities will be valued marked-to-market. The standard approach for calculating the target capital requirement is based on a variance-covariance type model. In addition to that, a number of plausible adverse scenarios must be evaluated and their impact on the available capital be determined. These scenario impacts must then be used to determine the ultimate distribution of the change in the available solvency capital over one year. The SST risk measure is the Tail-Value-at-Risk at the 99% confidence level. Contrary to the various European initiatives, operational risk must *not* be quantified within the SST. FOPI believes that simplified capital models such as the above described SST standard approach are not suited to assess the capital requirements for large companies, re-insurers, insurance groups and financial conglomerates. Also, the standard models are not designed to take into account the non-linearities emanting from embedded options and financial guarantees. Therefore, the FOPI sent out a letter to all Swiss life insurance companies asking them to abstain from simplified economic capital models such as variance-covariance type models. Rather, companies should develop and use internal models.

The presentation concludes with some remarks on multi-period risk measures. That final part is motivated by a decree of the Belgian supervising authority CBFA who requires Belgian life insurance companies to provide one, five, and ten year Value-at-Risks and Tail-Value-at-Risks, see [6], Article 21. It is shown that measuring risk over a multi-period time horizon is fundamentally different from measuring risk over a single period. The concept of *time-consistent* dynamic convex risk measures is introduced, and it is shown that Tail-Value-at-Risk is *not* time-consistent in the multi-period case. The material presented in this last part of the presentation is taken from [1], [7], [8], and [9].

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Optimal dividend (and reinvestment) policies when payments are subject to both fixed and proportional costs

Jostein Paulsen

The classical problem of maximizing the expected discounted value of dividends paid out from a diffusion income process is addressed. The income process without payments is given as

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dW_t.$$

Standing assumptions are

A1. $|\mu(y)| + |\sigma(y)| \leq K(1+y)$ for all $y \geq 0$ and some $K > 0$.
A2. μ and σ are continuously differentiable and the derivatives μ' and σ' are Lipschitz continuous for all $y \geq 0$.
A3. $\sigma^2(y) > 0$ for all $y \geq 0$.
A4. $\mu'(y) \leq r$ for all $y \geq 0$. Here r is the discount factor.

Let

$$Lg(y) = \frac{1}{2}\sigma^2(y)g''(y) + \mu(y)g'(y) - rg(y).$$

First the most common situation where business ends when the process hits zero is studied. Whenever dividends are paid a fixed cost d_0 pluss a proportional cost d_1 are incurred. This gives rise to a so called impulse control, and was studied in [1] when the income process is a Brownian motion. When there is no fixed costs connected with dividend payments we have a singular control problem, and this was solved in [2]. According to this paper we should look for a solution of $LV(y) = 0,\ y > 0$ and a y^* that satisfy

$$V(0) = 0, \quad V'(y^*) = \frac{1}{1+d_1} \quad \text{and} \quad V''(y^*) = 0.$$

On $y > y^*$ set

$$V^*(y) = V^*(y^*) + \frac{y - y^*}{1+d_1}.$$

Optimal solution is singular control at y^*, and V is the value function.

If there is no solution then there is no optimal control, but the value function is the limit of singular controls at barrier $\bar{y}$ for increasing $\bar{y}$.

Our problem is somewhat more complicated, but we are led to the following equations

E1:
$$\begin{aligned} LV(y) &= 0, \quad 0 < y < y^*, \\ V(0) &= 0, \\ V(y) &= V(y^*) + \tfrac{y-y^*}{1+d_1}, \qquad y > y^* \end{aligned}$$

E2:
$$\begin{aligned} V(y^*) &= V(y^* - \delta^*) + \tfrac{\delta^* - d_0}{1+d_1}, \quad \delta^* \in (0, y^*) \\ V'(y^*) &= \tfrac{1}{1+d_1}, \\ V'(y^* - \delta^*) &= \tfrac{1}{1+d_1}. \end{aligned}$$

E3:
$$\begin{aligned} V(y^*) &= \tfrac{y^* - d_0}{1+d_1}, \\ V'(y^*) &= \tfrac{1}{1+d_1}. \end{aligned}$$

The result is then:

Theorem 1

a) If E1+E2 has a solution for unknown V, y^* and δ^*, this solution is unique and the optimal strategy is to pay δ^* in dividends whenever $Y_{t-} = y^*$ and continue. The value function is $V(y)$.

b) If this is not the case, but instead, E1+E3 has a solution for unknown V and y^*, the optimal solution is to pay everything in dividends when $Y_{t-} = y^*$ and go bankrupt. The value function is again $V(y)$.

c) In all other cases there do not exist an optimal policy, but the value function is the limit of optimal value functions for given barriers $\bar{y}$ for increasing $\bar{y}$.

Assume that instead of bankruptcy when hitting zero it is possible to reinvest money, but with every reinvestment there is a fixed cost c_0 and a proportional cost c_1. The goal is to maximize expected value of discounted dividends minus reinvestments. Again if $c_0 = d_0 = 0$ the problem was solved in [2]. According to them we should look for a solution of $LV(y) = 0$, $y > 0$ and a y^* that satisfy

$$V'(0) = \frac{1}{1-c_1}, \quad V'(y^*) = \frac{1}{1+d_1} \quad \text{and} \quad V''(y^*) = 0.$$

On $y > y^*$ set

$$V^*(y) = V^*(y^*) + \frac{y - y^*}{1+d_1}.$$

Optimal solution is singular control at 0 and y^*, and V is the value function.

If there is no solution then there is no optimal control, but the value function is the limit of singular controls at barrier 0 and $\bar{y}$ for increasing $\bar{y}$.

Again the problem is somewhat more complicated when there are fixed costs, but we get the following result.

Theorem 2 Consider the following equations for unknown V, y^*, $\gamma^* \in (0, y^*)$ and $\delta^* \in (0, y^*)$,

$$\begin{aligned}
LV(y) &= 0, \quad 0 < y < y^*, \\
V(\gamma^*) &= V(0) + \tfrac{\gamma^* + c_0}{1 - c_1}, \\
V'(\gamma^*) &= \tfrac{1}{1 - c_1}, \\
V(y^*) &= V(y^* - \delta^*) + \tfrac{\delta^* - d_0}{1 + d_1}, \\
V'(y^* - \delta^*) &= \tfrac{1}{1 + d_1}, \\
V'(y^*) &= \tfrac{1}{1 + d_1}, \\
V(y) &= V(y^*) + \tfrac{y - y^*}{1 + d_1}, \quad y > y^*.
\end{aligned}$$

a) If this has a solution this solution is unique and the optimal solution is to pay γ^* whenever $Y_{t-} = 0$ and to pay δ^* whenever $Y_{t-} = y^*$. The value function is $V(y)$.
b) If this has no solution there is no optimal policy, but the value function is the limit of optimal value functions for given barriers $\bar{y}$ for increasing $\bar{y}$.

Numerical examples are given to illustrate the theorems.

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Cramér-Wold theorems for measures with a singularity

Filip Lindskog

(joint work with Jan Boman)

The well known Cramér-Wold theorem says that a probability measure on $\mathbf{R}^d$ is uniquely determined by the values it gives to halfspaces. A natural question is under which conditions this result can be extended to measures that are not necessarily non-negative and that may have infinite mass near the origin. This question is closely related to injectivity properties of the Radon transform. By using the close relation to Radon transforms it can be shown that Cramér-Wold theorem holds for more general measures that either dacay fast enough at infinity or have supports restricted to a proper cone. Moreover, without such an additional assumption the values the measure gives to halfspaces need not determine the measure. Similar results for convergence of sequences of measures can be formulated.

A consequence of the extended Cramér-Wold theorem is a characterization of regular variation for a random vector in terms of regular variation for linear combinations of the components of the vector.

Worst-Case Optimal Portfolios: Basics and Recent Work

Ralf Korn

(joint work with Olaf Menkens, Mogens Steffensen, Paul Wilmott)

The worst-case approach to portfolio optimization is a generalization of the classical stochastic control method of continuous-time portfolio optimization (such as developed by Merton in [5]). Its main intention is to explicitly capture the influence of possible crashes on the investment decision of an investor. We will present two variants of this new optimization method, one based on an indifference argument (as in [1],[2],[4]) and one that makes use of a system of inequalities and a complementarity condition which we call an HJB-System (see [3]).

For presenting the indifference approach, we look at a financial market where a bond with price given by

$$dP_0(t) = P_0(t)\, r dt, \quad P_0(0) = 1 \tag{1}$$

and a stock with price

$$dP_1(t) = P_1(t)\left(b dt + \sigma dW(t)\right), \quad P_1(0) = p_1 \tag{2}$$

with $b > r$ can be traded at each time instant $t \in [0, T]$. However, we assume that *at most* one crash can happen during this time, i.e. the stock price at the crash time τ jumps down by a factor $k \in [0, k^*]$ with $k^* < 1$. Thus, we have

$$P_1(\tau) = (1 - k)\, P_1(\tau-)\,. \tag{3}$$

We further assume that we have no probabilistic information on the distributions of the height and the time of the crash. If now the investor follows a portfolio process $\pi(.)$ then his corresponding wealth process $X^\pi(.)$ satisfies

$$X^\pi(\tau) = X^\pi(\tau-)\left(1 - \pi(\tau-)\, k\right) \tag{4}$$

which leads to a final wealth of

$$X^\pi(T) = \tilde{X}^\pi(T)\left(1 - \pi(\tau-)\, k\right) \tag{5}$$

with $\tilde{X}^\pi(t)$ denoting the wealth process in the usual crash-free setting.

In this setting, the aim of the investor is to solve the following *worst-case portfolio problem*:

$$\sup_{\pi(.)\in A(x)} \inf_{0\le t\le T, 0\le k\le k*} E\left(\ln\left(X^\pi(T)\right)\right). \tag{6}$$

To solve this problem, note that equation (5) yields a separation between the effects of investment (as given by $\tilde{X}^\pi(.)$) and of the crash (represented by the factor $(1 - \pi(\tau-)\, k)$). Considerably high values of the portfolio process lead to a high final wealth if no crash occurs but to big losses if a crash occurs. To balance these two effects, the indifference approach introduced in [4] for the choice of the log-utility function $U(x)$ determines that portfolio process $\hat{\pi}(.)$ such that the investor is indifferent between the worst possible crash of height k^* happening now and no crash happening at all in $[0,T]$. Indeed, it is shown in [4] that this portfolio is the optimal one in the above worst-case sense:

Theorem 5. *Indifference approach with log utility*
There exists a portfolio process $\hat{\pi}(.)$ such that the corresponding expected log-utility after an immediate crash equals the expected log-utility given no crash occurs at all. It is given as the unique solution $\hat{\pi}(.) \in [0, 1/k^*)$ of the differential equation

$$\pi'(t) = \frac{1}{k^*}\left(1 - \pi(t)\,k^*\right)\left(\pi(t)(b-r) - \frac{1}{2}\left(\pi(t)^2\sigma^2 + \left(\frac{b-r}{\sigma}\right)^2\right)\right), \tag{7}$$

$$\pi(T) = 0. \tag{8}$$

Further, this strategy yields the ***highest worst-case bound*** *for our problem (6). In particular, this bound is active at each future time point (uniformly optimal balancing). After the crash has happened the optimal strategy is given by*

$$\pi(t) \equiv \pi^* := \frac{b-r}{\sigma^2}. \tag{9}$$

The indifference approach can be generalized to yield similar theorems for the cases of (see [2],[1],[4])

- the possibility of multiple crashes,
- the presence of insurance risk (in the case of the exponential utility function),
- changing market coefficients after a crash.

However, for utility functions different from the log-utility one the indifference approach only yields the optimal portfolio process in the class of deterministic portfolios, a class which is much smaller than expected. To overcome this problem, a new approach that is much closer to classical stochastic control theory is developed in [3]. It is based on considering the usual partial differential operator occurring in the HJB-Equation of stochastic control only over a set of portfolio processes where the market can do the investor no harm (i.e. where the investor's situation (with respect to the above worst-case criterion) is not worse after a crash than before). This together with an inequality comparing the investor's value function before and after a crash and a complementarity condition form a system of inequalities and an equality (which we call an *HJB-System*) for which a classical verification result is given in [3]. Its presentation is beyond the s cope of this abstract.

Open problems for future research are:

- including a consumption process in the worst-case portfolio problem,
- the valuation of options in our crash model,
- generalization of the HJB-System approach,
- applying the worst-case approach to other areas of controlled systems with possibilities of catastrophes.

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Valuation of equity linked insurance products under regime switching models

Hailiang Yang

Nowadays, insurance business is changing; the changes are mainly due to the development of modern finance and mathematical finance, in particular the rapid growth of derivatives market. There are many products which exhibit option-embedded features sold by the insurance companies, and such kinds of insurance products, including variable annuities, participating life insurance contracts, equity-index annuities (EIAs) or equity-linked annuities (ELAs), guarantee annuity options (GAOs) and segregated funds, etc., become very popular recently. Earlier work on exploring the interplay between the option pricing theory and life insurance products can be dated back to Boyle and Schwartz (1977) and Brennan and Schwartz (1976, 1979). These works investigate the use of modern option pricing theory and its techniques for the valuation of equity-linked products and life insurance policies with a guarantee on asset value.

In this research, we shall consider the valuation of participating life insurance products and EIAs under regime switching models. For the EIA case, we use a Markovian regime switching Black-Scholes model to model the equity index dynamic, Esscher transform will be used to determine the martingale measure. The market is incomplete due to the jumps, but the Esscher transform still gives us a unique martingale measure. For some simple point to point designs, we can obtain closed form expression for the price of EIA. When we introduce mortality risk in the model, the problem becomes more complex, the price of the product can not be in the Black-Scholes no-arbitrage sense, it is a kind of risk measure.

In the participating life insurance product case, we use a generalized jump-diffusion model with a Markov-switching compensator. We suppose that the jump component is specified by the class of Markov-modulated kernel-biased completely random measures. The class of kernel-biased completely random measures is a wide class of jump-type processes. It has a very nice representation form, which is a generalized kernel-based mixture of Poisson random measures (or, in general, random measures). This provides a great deal of flexibility in modeling different types of finite and infinite jump activities compared with some existing models in the literature. We also provide additional flexibility to incorporate the impact of structural changes in macro-economic conditions and business cycles on the valuation of participating policies by introducing a continuous-time hidden Markov chain. For valuing participating products under the generalized jump-diffusion model, we shall the Esscher transform again to determine an equivalent martingale measure under the incomplete market setting. We consider various special cases of the Markov-modulated kernel-biased completely random measure for the jump component, namely, the Markov-modulated generalized Gamma (MGG) process, the scale-distorted version of the MGG process and the power-distorted version of the MGG process. The MGG process encompasses the Markov-modulated weighted Gamma (MWG) process and the Markov-modulated inverse Gaussian (MIG) process as special cases. We shall compare the fair values of the option embedded in the participating products implied by our generalized jump-diffusion models with those obtained from other existing models in the literature via simulation experiments and highlight some features of the qualitative behaviors of the fair values that can be obtained from different parametric specifications of our model. This research is based on papers with my coauthors, see Siu, Lau and Yang (2006) and Boyle, Tan and Yang (2007).

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Stochastic correlation in exponential utility indifference valuation

Martin Schweizer

(joint work with Christoph Frei)

We study the exponential utility indifference valuation of a contingent claim H in an incomplete market driven by two Brownian motions. The claim depends on an untraded asset which is stochastically correlated with the traded asset available for hedging. We use rigorous martingale arguments to provide upper and lower bounds, in terms of bounds on the correlation, for the dynamic value process V of the exponential utility maximization with the claim H as random endowment. This yields an explicit formula for the indifference value at any time, even with a fairly general stochastic correlation. Earlier results by Musiela/Zariphopoulou, Monoyios and Tehranchi, obtained in situations with constant correlation, are recovered and extended. The key to explaining why all this works is a new result which shows that the dynamic value process under a local martingale measure is monotonic in the correlation between traded and untraded asset.

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