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Discrete Geometry

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Abstract. A number of important recent developments in various branches of discrete geometry were presented at the workshop. The presentations illustrated both the diversity of the area and its strong connections to other fields of mathematics such as topology, combinatorics or algebraic geometry. The open questions abound and many of the results presented were obtained by young researchers, confirming the great vitality of discrete geometry.

Mathematics Subject Classification (2010): 52Bxx, 52Cxx.

Introduction by the Organisers

Discrete Geometry deals with the structure and complexity of discrete geometric objects, from finite point sets in the plane to intersection patterns of convex sets in high dimensional spaces. It goes back to classical problems such as Kepler's conjecture on the density of packings of balls in space, and Hilbert's third problem on decomposing polyhedra, as well as works by Minkowski, Steinitz, Hadwiger, and Erdős form the heritage of this area. Over the past years, several outstanding problems were solved, for example: (1) Andrew Suk proved a near-tight bound for the *Happy-ending problem* open from 1935 about the largest point set in the plane without k points in convex position, (2) Florian Frick gave, completing an approach of Mabillard and Wagner, a counter-example to the *topological Tverberg conjecture* from the early 1980's, (3) Maryna Viazovska solved the *sphere packing problem* in 8 dimension (and later, with others, in 24 dimensions), (4) János Pach, Natan Rubin and Gábor Tardos proved the Richter-Thomassen conjecture from 1995, a breakthrough which was reported during the workshop, and (5) Jiří Matoušek, Aleksandar Nikolov, and Kunai Talwar proved a nearly tight lower bound for Tusnády's problem about the discrepancy of finite point sets with respect to axis-parallel boxes. This list illustrates the interdisciplinary nature of discrete geometry and its many relations to other fields of mathematics like algebra, topology, combinatorics, computational geometry, probability and discrepancy theory. It is also in the front line of applications like geographic information systems, mathematical programming, coding theory, solid modeling, computational structural biology, and crystallography.

The workshop gathered 51 participants. The outstanding contributions by young scholars include the lecture by Rubin on his proof (with Pach and Tardos) of the Richter-Thomassen conjecture from 1995: In the plane, n pairwise intersecting Jordan curves (no three intersecting in a point, but being allowed to touch) have not much less than n^2 intersection points. Another one was the lecture by Karim Adiprasito, who outlined his proof of a long-standing conjecture of Grünbaum, that the number of of triangles in a 2-dimensional simplicial complex embedded in $\mathbb{R}^4$ is at most 4 times the number of edges. There were 28 other, mostly short, lectures presenting new connections to classical topics in combinatorics (Holmsen, Padrol) as well as developments in classical topics such as the analysis of point configurations (Füredi, Swanepoel, Vogtenhuber), incidence geometry (Aronov, Lund, Raz, Zahl), polytope theory (Nevo, Pournin, Santos), topological methods (Blagojević, Paták, Akopyan), or algorithmic issues (Mustafa, Mulzer). József Solymosi gave a survey about the hypergraph container method of Balogh, Morris and Samotij and of Saxton and Thomason, which has recently lead to spectacular breakthroughs in extremal combinatorics, and outlined some of its geometric applications. Edgardo Roldán-Pensado (with Luis Montejano) presented their systematic method for constructing families of three-dimensional constant-width bodies (Meissner polyhedra) in a talk which was beside the main stream of topics but was nevertheless very much appreciated for the intuitive appeal of the result.

In addition, there was an open problem session chaired by János Pach on Tuesday evening; the collection of open problems resulting from this session can be found in this report. The program left ample time for research and discussions in the stimulating atmosphere of the Oberwolfach Institute. In particular, there were several special informal sessions, attended by smaller groups of the participants, on specific topics of common interest. On Wednesday afternoon, most participants joined the traditional outing to St. Roman with the black forest cherry cake, enjoying the beautiful spring air despite adverse weather predictions. *Acknowledgment:*

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Workshop: Discrete Geometry

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Abstracts

$T \leq 4E$

Karim Adiprasito

We prove that for a simplicial 2-complex X embedding into $\mathbb{R}^4$, we have

$$\#\{\text{triangles of } X\} \leq 4 \cdot \#\{\text{edges of } X\}.$$

Waists of balls in different spaces

Arseniy Akopyan
(joint work with Roman Karasev and Alfredo Hubard)

The Gromov waist of the sphere theorem [1, 4, 2] asserts that given a continuous map $f : \mathbb{S}^n \to Y$, with $\mathbb{S}^n$ the unit round sphere, Y an $(n-k)$-dimensional manifold, and map having zero degree if $k = 0$ (in what follows we only consider the case $k > 0$ and do not care about the degree), it is possible to find $y \in Y$ such that for every $t \geq 0$

$$\operatorname{vol} \nu_t(f^{-1}(y), \mathbb{S}^n) \geq \operatorname{vol} \nu_t(\mathbb{S}^k, \mathbb{S}^n),$$

where $\nu_t(X, M)$ denotes the t-neighborhood of X in the Rimannian manifold M.

Going to the limit $t \to 0$ this proves the result of Almgren, that for any sufficiently regular map $f : \mathbb{S}^n \to \mathbb{R}^{n-k}$ with all fibers piece-wise smooth manifolds, there exists $y \in \mathbb{R}^{n-k}$ such that the k-dimensional volume of the fiber $f^{-1}(y)$ is greater or equal to the volume of the sphere $\mathbb{S}^k$. For merely continuous maps the Gromov–Memarian theorem proves that $\underline{\mathcal{M}}_k\, f^{-1}(y) \geq \operatorname{vol}_k \mathbb{S}^k$, where $\underline{\mathcal{M}}_k$ denotes *the lower Minkowski content*, which is defined in the following way: Let M be a Riemannian manifold of dimension n and $X \subseteq M$ be its subset,

$$\underline{\mathcal{M}}_k(X, M) := \liminf_{t \to +0} \frac{\operatorname{vol} \nu_t(X, M)}{v_{n-k} t^{n-k}} \quad \text{and} \quad \overline{\mathcal{M}}_k(X, M) := \limsup_{t \to +0} \frac{\operatorname{vol} \nu_t(X, M)}{v_{n-k} t^{n-k}},$$

where v_m is the volume of the m-dimensional Euclidean unit ball. These values are called *lower and upper Minkowski k-dimensional content* and are normalized to coincide with the Riemannian k-dimensional volume of X in the case X is a smooth submanifold of M.

Gromov defines *k-waist* of a set X as the infimum of the numbers $w > 0$, for which X admits a continuous map $X \to \mathbb{R}^{n-k}$ such that the k-dimensional volume of the preimage of any point $y \in \mathbb{R}^{n-k}$ is not greater than w. So, the k-waist of the unit sphere $\mathbb{S}^n$ in terms of the lower Minkowski content equals the volume of $\mathbb{S}^k$. In fact, the question of how we define "k-volume" is rather subtle. One can consider k-dimensional Riemannian volume and ask for sufficiently regular maps, consider the lower Minkowski content for fibers of arbitrary continuous maps, or even consider the Hausdorff measure of the fibers.

Not much was known about waists of other Riemannian manifolds. But recently Klartag proved [3] that the k-waist of the unit cube equals 1. This generalizes the

Vaaler theorem stating that the volume of any section of the unit cube by a k-plane passing through the center of the cube has k-volume at least 1.

A simple result of ours shows that the 1-waist of a convex body in $\mathbb{R}^n$ is equal to its width.

Theorem 1. *For any convex body $K \subset \mathbb{R}^n$ and a continuous map $f : K \to \mathbb{R}^{n-1}$ there exists a fiber $f^{-1}(y)$ of 1-Hausdorff measure at least the width of K.*

For Euclidean balls we prove the tight estimate on the k-waist:

Theorem 2. *Any continuous map $f : B^n \to \mathbb{R}^{n-k}$ has a fiber $f^{-1}(y)$ with lower Minkowski content $\underline{\mathcal{M}}_k\, f^{-1}(y) \geq \operatorname{vol}_k(B^k)$.*

Sketch of the proof. The standard projection $P : \mathbb{S}^{n+1} \to B^n$ is 1-Lipschitz and, as was noted by Archimedes, transports the uniform measure in the sphere to the uniform measure in the Euclidean ball. Now apply Gromov's waist of the sphere theorem and not that the definition of the Minkowski content is suitable for 1-Lipschitz maps preserving the measure. □

Using the following theorem about the behavior of the Minkowski content under certain affine transformations we can find the waist of an ellipsoid:

Theorem 3. *The linear map $L(x_1, \dots, x_n) = (a_1x_1, a_2x_2, \dots, a_nx_n)$ with $0 < a_1 \leq a_2 \leq \dots \leq a_n$ has the property*

$$a_1 \dots a_k\, \underline{\mathcal{M}}_k\, X \leq \underline{\mathcal{M}}_k\, L(X) \leq a_{n-k+1} \dots a_n\, \underline{\mathcal{M}}_k\, X.$$

Corollary 4. *Consider an ellipsoid E with principal axes $a_1 \leq a_2 \leq \dots \leq a_n$ and a continuous map $f : E \to \mathbb{R}^{n-k}$. There exists a fiber $f^{-1}(y)$ with $\underline{\mathcal{M}}_k\, f^{-1}(y) \geq a_1a_2 \dots a_k \cdot \operatorname{vol}_k(B^k)$.*

We obtain analogues results about the waist of balls in the model constant curvature spaces:

Theorem 5. (The spherical case) *Let $B(R) \subset \mathbb{S}^n$ be a ball of radius R. Then for any continuous map $f : B(R) \to \mathbb{R}^{n-k}$ it is possible to find $y \in \mathbb{R}^{n-k}$ such that $\underline{\mathcal{M}}_k\, f^{-1}(y)$ is at least the volume of the k-dimensional ball of radius R in $\mathbb{S}^k$.*

(The hyperbolic case) *Let $B(R) \subset \mathbb{H}^n$ be a ball of radius R. Then for any continuous map $f : B(R) \to \mathbb{R}^{n-k}$ it is possible to find $y \in \mathbb{R}^{n-k}$ such that $\underline{\mathcal{M}}_k\, f^{-1}(y)$ is at least the volume of the k-dimensional ball of radius R in $\mathbb{H}^k$.*

The proof is based on a modification of Gromov's waist theorem that holds for maps discontinuous at one point. Making the stereographic projection from the sphere we obtain the waist inequality for the density $\rho = (1 + |x|^2)^{-k}$. After that we show that the required inequality is preserved under radial maps mapping the density ρ to densities obtained from the stereographic projection of spherical caps, or densities of hyperbolic balls taken in the Poincaré model.

Using the expanding maps we obtain the following corollaries:

Theorem 6. *Let M be an n-dimensional Riemannian manifold, which is* $\mathrm{CAT}(\kappa)$ *(complete, simply connected, and has sectional curvature $\leq \kappa$ everywhere); and let $B_M(R) \subset M$ be a ball of radius R there, $R < \pi/\sqrt{\kappa}$ in case $\kappa > 0$. Then for any continuous map $f : B_M(R) \to \mathbb{R}^{n-k}$, it is possible to find $y \in \mathbb{R}^{n-k}$ such that $\overline{\mathcal{M}}_k\, f^{-1}(y)$ is at least the volume of the k-dimensional ball in the model space $\mathbb{M}^k_\kappa$.*

Corollary 7. *Let M be an n-dimensional Riemannian manifold, which is* $\mathrm{CAT}(1)$. *Then for any continuous map $f : M \to \mathbb{R}^{n-k}$, it is possible to find $y \in \mathbb{R}^{n-k}$ such that*

$$\overline{\mathcal{M}}_k\, f^{-1}(y) \geq \mathrm{vol}_k\, \mathbb{S}^k.$$

For finite dimensional normed spaces, we obtain the improvement of the following recent result:

Theorem 8 (Klartag, 2016)**.** *Suppose $K \subset \mathbb{R}^n$ is a centrally-symmetric convex body and $f : K \to \mathbb{R}^{n-k}$ is a continuous map. Then there exists $y \in \mathbb{R}^{n-k}$ such that for any $t \in [0, 1]$*

$$\mathrm{vol}_n(f^{-1}(y) + tK) \geq \left(\frac{t}{2t+2}\right)^{n-k} \mathrm{vol}_n\, K.$$

Interchanging the quantifiers helps to improve the bound:

Theorem 9. *Suppose $K \subset \mathbb{R}^n$ is a convex body and $f : K \to \mathbb{R}^{n-k}$ is a continuous map. Then for any $t \in [0, 1]$ there exists $y \in \mathbb{R}^{n-k}$ such that*

$$\mathrm{vol}_n(f^{-1}(y) + tK) \geq t^{n-k}\, \mathrm{vol}_n\, K.$$

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Breaking depth-order cycles and other adventures in 3D

Boris Aronov

(joint work with Edward Y. Miller, Micha Sharir)

Given a collection of non-vertical lines in general position in 3D, there is a natural above/below relation defined on the lines. One line is *above* another if the unique vertical line that meets both meets the former at a point above the point where it meets the latter. One can similarly define the (partial) above/below relation on any set of reasonably well-behaved pairwise disjoint objects; a pair of objects is not related at all, if no vertical line meets both.

Motivated by a computer graphics problem, the following question was asked more than 35 years ago: What is the minimum number of pieces one must cut N lines into, in the worst case, to make sure that the resulting pieces have no cycles in their above/below relation? An N^2 upper bound is easy, but is the answer subquadratic? A lower bound of approximately $N^{3/2}$ was known, but there were no non-trivial upper bounds, in the general case. Restricted versions of the problem have been studied and are briefly discussed. We present a near-optimal near-$N^{3/2}$ upper bound.

We also sketch how to extend this to the original motivating question for computer graphics, which until now was unreachable: How many pieces does one have to cut N *triangles* into, to eliminate all cycles in the above/below relationship, as above? We obtain a near-$N^{3/2}$ bound in this case as well, though slightly weaker.

Joint work with Micha Sharir (Tel Aviv U), and also with Edward Y. Miller (NYU), for the extension to triangles.

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Thrifty approximations of convex bodies by polytopes

Alexander Barvinok

Given a convex body $B \subset \mathbb{R}^d$, containing the origin in its interior, and a real number $\tau > 1$, we are interested to construct a polytope P with as small number N of vertices as possible, such that $P \subset B \subset \tau P$. If B is symmetric, $B = -B$, the following result from [1] provides a close to optimal bound for a wide range of asymptotic regimes:

Suppose that

$$\left(\tau + \sqrt{\tau^2 - 1}\right)^k + \left(\tau - \sqrt{\tau^2 - 1}\right)^k \ \geq \ 6\binom{d+k}{k}^{1/2}$$

for some positive integer k. Then for any symmetric convex body $B \subset \mathbb{R}^d$ there exists a symmetric polytope P with

$$N \leq 8\binom{d+k}{k}$$

vertices such that $P \subset B \subset \tau P$. Varying k, we obtain a wide range of approximations, from coarse with

$$\tau = \sqrt{\epsilon d} \quad \text{and} \quad N = d^{O(1/\epsilon)}$$

to fine with

$$\tau = 1 + \epsilon \quad \text{and} \quad N \leq \left(\frac{\gamma}{\sqrt{\epsilon}} \ln \frac{1}{\epsilon} \right)^d,$$

where $\epsilon > 0$ is small and $\gamma > 0$ is an absolute constant.

A similar estimate, involving the coefficient of symmetry of B, can be obtained for general convex bodies. The situation becomes completely mysterious if B is not symmetric and we are allowed to choose the origin in B, so as to minimize N for a given τ.

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A counterexample to a connectivity conjecture of Bárány, Kalai and Meshulam

Pavle V. M. Blagojević

(joint work with Albert Haase and Günter M. Ziegler)

1. The conjecture

Let $d \geq 1$ and $k \geq 1$ be integers, and let Σ be non-trivial simplicial complex. For a continuous map $f\colon \Sigma \longrightarrow \mathbb{R}^d$ a *Tverberg k-partition* is a collection $\{\sigma_1, \ldots, \sigma_k\}$ of k pairwise disjoint faces of Σ such that $f(\sigma_1) \cap \cdots \cap f(\sigma_k) \neq \emptyset$. The *topological Tverberg number* $\mathrm{TT}(\Sigma, d)$ is the maximal integer $k \geq 1$ such that every continuous map $f\colon \Sigma \to \mathbb{R}^d$ has a Tverberg k-partition. The topological Tverberg theorem of Bárány, Shlosman, and Szűcs [4] implies that for k prime $\mathrm{TT}(\Delta_{(k-1)(d+1)}, d) = k$. It follows from the work of Özaydin [13] that this result remains true when k is also a power of a prime. Recently Frick [10], [7], using the "constraint method" [8] and building on the work by Mabillard and Wagner [11], showed that if $k \geq 6$ is not a prime-power and $d \geq 3k + 1$, then $\mathrm{TT}(\Delta_{(k-1)(d+1)}, d) < k$; see [2] for a recent survey.

In their recent work Bárány, Kalai, and Meshulam [3] gave lower bounds for the topological Tverberg number in the case of a matroid, regarded as the simplicial complex of its independent sets. Let M be a matroid of rank $d+1$ with b disjoint bases, then [3, Thm. 1] asserts that

$$\mathrm{TT}(M, d) \geq \sqrt{b}/4.$$

The results of [4], [13], and [3, Thm. 1] mentioned above are all obtained by using a "configuration space/test map scheme". This approach involves a space X related to the complex Σ, called the *configuration space*, and a space Y related to $\mathbb{R}^d$, called the *test space*, such that both spaces admit an action of a finite non-trivial group G. A continuous map *without* a Tverberg k-partition then defines a G-equivariant map $X \to Y$, called the *test map*. The method of proof is to show non-existence of such a G-equivariant map $X \to Y$. The configuration space test map scheme is a classical and frequently used tool to solve problems in combinatorics and discrete geometry; see Matoušek [12] for an introduction and a survey of applications.

In the configuration space/test maps scheme used in [14] and [3] the configuration space X is the k-fold deleted join Σ^{*k}_{Δ} of the complex Σ and the test space Y is a sphere $S^{(k-1)(d+1)-1}$ of dimension $(k-1)(d+1)-1$. If k is prime, then both spaces X and Y admit free actions by the cyclic group $\mathbb{Z}/k$.

In order to obtain best possible results using a configuration space/test map scheme it is necessary to determine proof strategies for the non-existence of an equivariant map from the configuration space X to the test space Y. One commonly used method is the *connectivity-based approach*, which can be applied if Y is a finite-dimensional CW-complex on which the group acts freely: If one establishes that the connectivity of the space X is at least as high as the dimension of the space Y, then Dold's theorem [9] implies that an equivariant map $X \to Y$ does not exist. For a more general version of Dold's theorem that is also applicable in this context see [15].

The connectivity-based approach (for k a prime power) yields tight bounds for the topological Tverberg number of $\Delta_{(k-1)(d+1)}$. The natural questions we are concerned in the case of more general simplical complex Σ are: What is the connectivity of the configuration spaces? Which results can/cannot be obtained via connectivity-based approaches? Having these questions in mind, Bárány, Kalai, and Meshulam [3, Conj. 4] formulated the following conjecture for the class of matroids.

Conjecture 1 (Bárány, Kalai, and Meshulam 2016)**.** *For any integer $k \geq 1$ there exists an integer $n_k \geq 1$ depending only on k such that for any matroid M of rank $r \geq 1$ with at least n_k disjoint bases, the k-fold deleted join M^{*k}_{Δ} of the matroid M is $(kr-1)$-dimensional and $(kr-2)$-connected.*

For $k = 1$ the conjecture is true, since a matroid of rank r is pure shellable and hence in particular $(r-2)$-connected [5, Thm. 4.1]. Using the connectivity-based approach the conjecture would imply that for a matroid M of rank $d+1$ with $b \geq n_k$ disjoint bases the topological Tverberg number satisfies $\mathrm{TT}(M, d) \geq k$.

2. A counterexample

We demonstrate that Conjecture 1 fails already in the case where $k = 2$ by exibiting a family of counterexample matroids. More precisely, the following theorem holds.

Theorem 1. *There is a family of matroids M_r $(r \in \mathbb{Z}, r \geq 3)$ such that each matroid M_r has rank r and r disjoint bases, while the 2-fold deleted join $(M_r)^{*2}_{\Delta}$ of M_r is is $(2r-1)$-dimensional and $(2r-3)$-connected, but not $(2r-2)$-connected.*

In the following we introduce the class of counterexample matroids, but first we recall necessary notions about matroids. A *matroid M* with *ground set E* is a simplicial complex with vertices in E such that for every $A \subseteq E$ the restriction

$M|A = \{\sigma \in M \colon \sigma \subseteq A\}$ is pure. We call a face of M an *independent set*, a facet of M a *basis*, and the cardinality of a (any) basis the *rank* of M. Let m and n be integers with $0 \leq m \leq n$. Given a ground set E of cardinality n, the *uniform matroid* $U_{m,n}(E)$ is given by the collection of all subsets of E of cardinality at most m. If $\Delta_{n-1}^{(m-1)}$ denotes the $(m-1)$-skeleton of the $(n-1)$-simplex of dimension $n-1$, then $\Delta_{n-1}^{(m-1)} = U_{m,n}(E)$. Given matroids $M_1, \dots, M_k$ with ground sets $E_1, \dots, E_k$, the *direct sum* $M_1 \oplus \cdots \oplus M_k$ of the family M_i is defined as the collection $\{I_1 \sqcup \cdots \sqcup I_k : I_i \in M_i\}$ and is a matroid with ground set $E_1 \sqcup \cdots \sqcup E_k$. The direct sum of a collection of matroids is equal to the join of the collection of matroids, viewed as simplicial complexes.

Definition 2 (The counterexample family M_r)**.** *Let $r \geq 3$ be an integer. Let E be a set of pairwise distinct elements v_i^j and w_j for $i = 1, \dots, r-1$ and $j = 1, \dots, r$. Denote* blocks *E_i by*

$$E_i = \{v_i^1, \dots, v_i^r\} \quad \textit{for} \quad i = 1, \dots, r-1, \quad \textit{and} \quad E_r = \{w_1, \dots, w_r\}.$$

Define a matroid $\widehat{M_r}$ by

$$\widehat{M_r} := U_{1,r}(E_1) \oplus \cdots \oplus U_{1,r}(E_{r-1}) \oplus U_{r,r}(E_r),$$

and the matroid M_r with ground set E as the $(r-1)$-skeleton of $\widehat{M_r}$, that is

$$M_r := \{I \in \widehat{M_r} : |I| \leq r\}.$$

The matroid M_r is of rank r and has r pairwise disjoint bases of the form $\{v_1^j, \dots, v_{r-1}^j, w_j\}$ for $j = 1, \dots, r$. Faces of M_r are given by choosing at most r vertices in total and at most 1 vertex in each of the first $r-1$ blocks.

Using the notion of shellability for non-pure complexes due to Björner and Wachs [5],[6], we show that the complex $(M_r)^{*2}_{\Delta}$ is shellable for $r \geq 3$, and use this to describe its homotopy type.

Theorem 3. *Let $r \geq 3$ be an integer. Then the deleted join $(M_r)^{*2}_{\Delta}$ is homotopy equivalent to a non-trivial wedge of spheres of dimensions $2r-1$ and $2r-2$.*

This theorem directly implies Theorem 1, and hence shows that the family M_r is a counterexample to Conjecture 1.

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One sided epsilon-approximants

Boris Bukh

(joint work with Gabriel Nivasch)

A common theme in mathematics is approximation of large, complicated objects by smaller, simpler objects. This paper proposes a new notion of approximation in combinatorial geometry, which we call one-sided ε-approximants. It is a notion of approximation that is in strength between ε-approximants and ε-nets. We recall these two notions first.

Let $P \subset \mathbb{R}^d$ be a finite set, and $\mathcal{F} \subset 2^{\mathbb{R}^d}$ a family of sets in $\mathbb{R}^d$. In applications, the family $\mathcal{F}$ is usually a geometrically natural family, such as the family of all halfspaces, the family of all simplices, or the family of all convex sets. A finite set $A \subset \mathbb{R}^d$ is called an *ε-approximant for P with respect to $\mathcal{F}$* if

$$\left| \frac{|C \cap P|}{|P|} - \frac{|C \cap A|}{|A|} \right| \leq \varepsilon \qquad \text{for all } C \in \mathcal{F}.$$

The notion of an ε-approximant was introduced by Vapnik and Chervonenkis [VČ71] in the context of statistical learning theory. They associated to each family $\mathcal{F}$ a number VC-dim$(\mathcal{F}) \in \{1, 2, 3, \ldots, \infty\}$, which has become known as *VC* *dimension*, and proved that if VC-dim($\mathcal{F}$) $< \infty$, then every set P admits an ε-approximant A of size $|A| \le C_{\text{VC-dim}(\mathcal{F})}\varepsilon^{-2}$, a bound which does not depend on the size of P. The ε-approximants that they constructed had the additional property that $A \subset P$. Following tradition, we say that A is a *strong ε-approximant* if $A \subset P$. When we wish to emphasize that our ε-approximants are not necessarily subsets of P, we call them *weak ε-approximants*. The bound has been improved to $|A| \le C_{\text{VC-dim}(\mathcal{F})}\varepsilon^{-2+2/(\text{VC-dim}(\mathcal{F})+1)}$ (see [Mat95, Theorem 1.2] and [Mat99, Exercise 5.2.7]) which is optimal [Ale90].

In a geometric context, Haussler and Welzl [HW87] introduced ε-nets. With P and $\mathcal{F}$ as above, a set N is called an *ε-net for P with respect to $\mathcal{F}$* if

$$\frac{|C \cap P|}{|P|} > \varepsilon \implies C \cap N \neq \emptyset \qquad \text{for all } C \in \mathcal{F}.$$

An ε-approximant is an ε-net, but not conversely. While an ε-net is a weaker notion of approximation, its advantage over an ε-approximant is that every set P admits an ε-net of size only $C_{\text{VC-dim}(\mathcal{F})}\varepsilon^{-1}\log\varepsilon^{-1}$, which is smaller than the bound for the ε-approximants. The ε-nets constructed by Haussler and Welzl are also strong, i.e., they satisfy $N \subset P$.

Most geometrically important families $\mathcal{F}$ have a bounded VC dimension. A notable exception is the family $\mathcal{F}_{\text{conv}}$ of all convex sets. Indeed, it is easy to see that a set of n points in convex position does not admit any strong ε-net of size smaller than $(1-\varepsilon)n$ with respect to $\mathcal{F}_{\text{conv}}$. Alon, Bárány Füredi, and Kleitman [ABFK92] showed that for every $P \subset \mathbb{R}^d$ there exists a (weak) ε-net of size bounded solely by a function of ε and d. No extension of their result to ε-approximants is possible.

Proposition 1. *If $P \subset \mathbb{R}^2$ is a set of n points in convex position, then every ε-approximant with respect to $\mathcal{F}_{\text{conv}}$ has size at least $n(\frac{1}{4} - \varepsilon/2)$.*

Proof. Let $p_1, p_2, \ldots, p_n$ be the enumeration of the vertices of P in clockwise order along the convex hull of P. For $i = 1, \ldots, \lfloor (n-1)/2 \rfloor$ write T_i for the triangle $p_{2i-1}, p_{2i}, p_{2i+1}$. Suppose $A \subset \mathbb{R}^2$ is an ε-approximant for P. Let $I \stackrel{\text{def}}{=} \{i : T_i \cap A = \emptyset\}$. Note that $|I| \ge n/2 - 2|A| - 1$ since each point of A lies in at most two triangles. Define $S \stackrel{\text{def}}{=} \{p_1, p_3, p_5, \ldots\}$ to be the odd-numbered points, and let $S' \stackrel{\text{def}}{=} S \cup \{p_{2i} : i \in I\}$. Let $C \stackrel{\text{def}}{=} \operatorname{conv} S$ and $C' \stackrel{\text{def}}{=} \operatorname{conv} S'$. Then $C \cap A = C' \cap A$, but $|C' \cap P|/|P| - |C \cap P|/|P| = |I|/|P| > \varepsilon$ if $|A| < |P|(\frac{1}{4} - \varepsilon/2)$. □

In light of Proposition 1, we introduce a new concept. A multiset $A \subset \mathbb{R}^d$ is a *one-sided ε-approximant for P with respect to the family $\mathcal{F}$* if

$$\frac{|C \cap P|}{|P|} - \frac{|C \cap A|}{|A|} \le \varepsilon \qquad \text{for all } C \in \mathcal{F}.$$

In other words, if $C \in \mathcal{F}$, then C might contain many more points of A than expected, but never much fewer. It is clear that an ε-approximant is a one-sided ε-approximant, and that a one-sided ε-approximant is an ε-net.

Our main result shows that allowing one-sided errors is enough to sidestep the pessimistic Proposition 1.

Theorem 2. *Let $P \subset \mathbb{R}^d$ be a finite set, and let $\varepsilon \in (0,1]$ be a real number. Then P admits a one-sided ε-approximant with respect to $\mathcal{F}_{\text{conv}}$ of size at most $g(\varepsilon, d)$, for some g that depends only on ε and on d.*

Unfortunately, due to the use of a geometric Ramsey theorem, our bound on g is very weak:

$$g(\varepsilon, d) \leq \operatorname{tw}_d\big(\varepsilon^{-c}\big)$$

for some constant $c > 1$ that depends only on d, where the tower function is given by $\operatorname{tw}_1(x) \stackrel{\text{def}}{=} x$ and $\operatorname{tw}_{i+1}(x) \stackrel{\text{def}}{=} 2^{\operatorname{tw}_i(x)}$. We believe this bound to be very far from sharp.

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Stretching weighted pseudoline arrangements

Michael Gene Dobbins

The (k,n)-Grassmannian is the quotient of vector configurations $(v_1, \ldots, v_n)$ spanning $\mathbb{R}^k$ by the general linear group. This space arises naturally as the classifying space of vector bundles and in the proof of the center-transversal theorem and various other places. We would like to have a discrete model for the Grassmannian, and one proposed model is the (k,n)-MacPhersonian, the poset of rank k oriented matroids on index set $\{1, \ldots, n\}$ ordered by degeneracy (the weak map order) [1, 4].

Oriented matroids provide a discrete representation of vector configurations that is defined by short purely combinatorial axioms for all the ways a vector configuration can be partitioned by a hyperplane. These axioms are necessary, but are not sufficient, for a set of sign vectors to arise from such a partition. Even for rank $k = 3$, deciding if a given oriented matroid is realized by a vector configuration is as hard as deciding if a polynomal system has a real solution.

Moreover, the subset of the Grassmannian realizing a given oriented matroid can have the homotopy type of any primary semialgebraic set [2].

Here I introduce the Pseudolinear Grassmannian as another analog of the Grassmannian, which is nicer from the point of view or oriented matroids, and avoids the difficulties of real algebraic geometry. The pseudolinear Grassmannian contains a homeomorphic copy of the Grassmannian as a subspace, and every oriented matroid is realized in this space. In rank 3, the pseudolinear Grassmannian is homotopic to the Grassmannian, and the subset of the pseudolinear Grassmannian realizing any rank 3 oriented matroid is contractible.

To this end, we represent a vector $v \in \mathbb{R}^k$ in what is effectively polar coordinates by $v \equiv (r, \theta)$ where $r = \|v\| \in \mathbb{R}_{\geq 0}$ and $\theta : \mathbb{S}^{k-1} \to \{+, 0, -\}$ by $\theta(x) = \operatorname{sign}\langle v, x\rangle$. We then extend $\mathbb{R}^k$ to a larger space of weighted pseudospheres by allowing θ to be a pseudosphere. That is, θ may be any map $\mathbb{S}^{k-1} \to \{+, 0, -\}$ that factors into $x \mapsto \operatorname{sign}\langle e_1, x\rangle$ and a self-homeomorphism of the sphere, provided that $r > 0$. If $r = 0$ then $\theta(x) = 0$ is constant.

We will similarly extend the Grassmannian, but in order to have a metric on the Grassmannian that also extends, we use an alternative definition. We may define the Grassmannian as the quotient of the space of Parseval frames, vector configurations $(v_1, \ldots, v_n)$ satisfying $\sum_{i=1}^{n} \langle v_i, x\rangle^2 = \|x\|^2$, by orthogonal transformations. We then extend the Parseval frames to pseudolinear configurations $((r_1, \theta_1), \ldots, (r_n, \theta_n))$, which are defined as above with the further condition that $(\theta_1, \ldots, \theta_n)$ defines a pseudosphere arrangement as in the Topological Representation Theorem for oriented matroids. Specifically, each θ_i is either a pseudosphere or is identically 0, and for every $x \in \mathbb{S}^{k-1}$ there is some i such that $\theta_i(x) \neq 0$, and the following recursive condition holds. For every $I \subseteq \{1, \ldots, n\}$ the set $S_I = \bigcap_{i \in I}\{x \in \mathbb{S}^{k-1} : \theta_i(x) = 0\}$ is homeomorphic to $\mathbb{S}^j$ for some $j < k$ and the image of the restriction of $(\theta_1, \ldots, \theta_n)$ to S_I is again a pseudosphere arrangement. If we further restrict the pseudospheres to be antipodally symmetric, then a pseudosphere arrangement in rank 3 is a pseudoline arrangement in the projective plane.

The pseudolinear Grassmannian is the quotient of the space of pseudolinear configurations $((r_1, \theta_1), \ldots, (r_n, \theta_n))$ by the action of the orthogonal group by

$$((r_1, \theta_1), \ldots, (r_n, \theta_n)) * Q = ((r_1, \theta_1 \circ Q), \ldots, (r_n, \theta_n \circ Q)).$$

We define distance between weighted pseudospheres by a weighted analog of Fréchet distance. Specifically,

$$\operatorname{dist}((r_1, \theta_1), (r_2, \theta_2)) = \inf{}_{\varphi_1, \varphi_2} \sup{}_{x} \|r_1\varphi_1(x) - r_2\varphi_2(x)\|,$$

where $\varphi_i : \mathbb{S}^{k-2} \to S_i$ is a parameterization of the null set S_i of θ_i. This induces a metric on the pseudolinear Grassmannian since we quotient by a group of isometries.

For any rank 3 oriented matroid $\mathcal{M}$ on n elements, the subset $\mathcal{R}$ of the pseudolinear $(3, n)$-Grassmannian realizing $\mathcal{M}$ is contractible. The proof has two main ingredients. For any pair of pseudolinear configurations A, B, we can define a self-homeomorphism φ of the sphere that sends A to B and depends continuously on the pair A, B. This can be done by using conformal maps from 2-cells of A to 2-cells of B, and then adjusting these appropriately to agree on the boundary between cells. By a theorem of Radó, these maps depend continuously on the boundary of each cell [7](see also [5, Section II.5 Theorem 2]). A theorem of Kneser says that the space of self-homeomorphisms of the 2-sphere strongly deformation retracts to the orthogonal group $O(3)$ [6](see also [3]). To define a deformation retraction from $\mathcal{R}$ to a point, fix a reference pseudolinear configuration A and express each B as the image of A by φ. Then deform B back to an orthogonal copy of A by deforming φ to an orthogonal transformation.

We next briefly describe a strong deformation retraction from the pseudolinear $(3, n)$-Grassmannian to the $(3, n)$-Grassmannian by a sequence of deformations defined in terms of the null sets S_i of θ_i as follows. We first deform the pseudocircles (or pseudolines in the symmetric case) to piecewise geodesic pseudocircles one at a time. Assume that for $j < m$, the 2-cells of $S_{i_1}, \ldots, S_{i_{j-1}}$ are convex, and that S_{i_j} is geodesic in each cell. We deform S_{i_m} in each 2-cell $S_{i_1}, \ldots, S_{i_{m-1}}$ to a segment in that 2-cell with the same end points. Since this does not change the combinatorial cell structure of $S_{i_1}, \ldots, S_{i_m}$, this may be done in the same way as the deformation retraction of $\mathcal{R}$ to a point as above, so that the rest of the pseudocircles are deformed as the self-homeomorphism φ of the sphere is deformed. The overall cell structure of the pseudocircles does not change, and after each deformation the cells of $S_{i_1}, \ldots, S_{i_m}$ are convex.

We then coarsen the subdivision in which the pseudocircles S_i are geodesic as follows. Assume that for $j \notin \{i_1, \ldots, i_m\}$, S_j is geodesic in the subdivision by $S_{i_1}, \ldots, S_{i_m}$. We then linearly interpolate S_j in each 2-cell of the subdivision by $S_{i_1}, \ldots, S_{i_{m-1}}$ to a segment in that 2-cell with the same end points as S_j. While the combinatorial cell structure may change as a result of such a deformation, no crossings are added or removed in each 2-cell of $S_{i_1}, \ldots, S_{i_{m-1}}$ by this deformation. Note that special care is needed when $m = 3$ to ensure the deformation depends continuously on the initial pseudocircles, but that case will not be a addressed here.

The sequence of deformations described only work under the assumption that S_{i_1}, S_{i_2}, S_{i_3} form a basis, and that if $r_i = 0$ then $r_j = 0$ for $j > i$. In order to accommodate these requirements, we recursively perform a partial deformation for every new index i_m we could potentially add to a sequence $i_1, \ldots, i_{m-1}$ up to some stopping time s_{i_m}. If the assumption fails, the stopping times is $s_{i_m} = 0$ and the deformation is not preformed at all. Otherwise, the stopping time is defined so that the "best" index i_m has stopping time $s_{i_m} = 1$ according to some heuristic indicating how far the assumptions are from failing for each weighted pseudocircle.

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On lines and points and the (p,q)-property

Vladimir Dol'nikov
(joint work with Maxim Didin and Mikhail Grigorev)

We discuss a problem of Ramsey type for configurations of lines and points and its connections to the (p,q)-problem of Hadviger and Debrunner.

Definition 1. *Let V be a finite set of points in the plane, let m, n be integers with $m,n \geq 3$. Denote by $RL(m,n)$ the minimal number N such that if a point set V in the plane has at least N elements, then V contains either an m-element subset in one line or an n-element subset in general position (every three points of the the subset do not belong to one line).*

Let us note that $R(m,n;3) \geq RL(m,n)$, where $R(m,n;3)$ is the Ramsey number for triples.

The notion of a (p,q)-property was initially introduced by Hugo Hadwiger and Hans Debrunner (1957) for families of convex subsets in $\mathbb{R}^d$ in connection with the investigation of the Helly and Helly–Gallai numbers of these families.

Definition 2 (Standard (p,q)-property)**.** *Let p and q be integers such that $p \geq q \geq 2$. We say that a family of sets $\mathcal{F}$ has the (p,q)-property provided $\mathcal{F}$ has at least p members, and among every p members some q have a common point.*

It is possible to give a dual definition for points and lines:

Definition 3 ((p,q)-property for collinearity)**.** *Let p and q be integers such that $p \geq q \geq 3$. We say that a point set V has a (p,q)-property provided V has at least p points, and among every p points some q points belong to a single line.*

Let us note that a point set V with $|V| \geq p$ does not contain p points in a general position if and only if V has the $(p,3)$–property.

Definition 4. *Call a hypergraph $P = (V,E)$ a* combinatorial plane *if for every two vertices (points) of V there exists precisely one hyperedge ("line") containing these two vertices.*

It is clear that the Euclidean plane is a combinatorial plane. We are going to transfer some results to any combinatorial plane. Of course, the results for the Euclidean plane are better in many cases.

We proved the following Theorem.

Theorem 5. *Suppose $p \geq 3$ is an integer and a family of points V has the $(p,3)$–property, then*

- *If $p = 4$ then we can delete one point so that the remaining points are on one line.*
- *If $p = 5$ and $|V| \geq 17$ then all points of V belong to some two lines.*
- *If $p = 6$ and $|V| \geq 56$ then we can delete one point so that the remaining points belong to some two lines.*

Theorem 6 (Dual theorem)**.** *Suppose $p \leq 6$ is a positive integer and a family of combinatorial lines $\mathcal{F}$ has the $(p,3)$–property. Then there exists a point set U such that $|U| \leq p-2$ and $\ell \cap U \neq \emptyset$ for all lines $\ell \in \mathcal{F}$.*

We have several consequences for the situation when the point set contains no 4 (or 5, or 6) points in general position:

Corollary 7.

- *$RL(m,4) = m+1$.*
- *If $m \geq 9$ then $RL(m,5) = 2m-1$.*
- *If $m \geq 28$ then $RL(m,6) = 2m$.*

We have several conjectures that seem plausible in view of the above results:

Conjecture 1. *Suppose $p \geq 3$ is an integer and a family of straight lines $\mathcal{F}$ has the $(p,3)$-property. Then there exists a point set U such that $|U| \leq p-2$ and $\ell \cap U$, for all lines $\ell \in \mathcal{F}$.*

Conjecture 2. *Suppose k is sufficiently large, $2k+1 \geq 3$, and a set of points V has the $(2k+1,3)$-property. Then V is contained in a union of some k lines.*

Conjecture 3. *Suppose k is sufficiently large, $2k \geq 3$, and a set of points V has the $(2k,3)$-property. Than all but one points of V are contained in a union of some $k-1$ lines.*

Theorem 8.

- *If $m \geq (2k-1)(2k^2-5k+2)$ then $RL(m,2k) = (k-1)(m-1)+2$.*
- *If $m \geq k(k-1)(2k-3)+3$ then $RL(m,2k+1) = k(m-1)+1$.*

Almost similar configurations

Zoltán Füredi
(joint work with Imre Bárány)

1. Abstract

This is a talk about how to use extremal (hyper)graph theory, i.e., Turán type results, to solve combinatorial geometry problems.

Let $A \subset \mathbb{R}^2$ be a fixed k-set. Two points sets $A = \{a_1, \dots, a_k\}$ and $B = \{b_1, \dots, b_k\}$ are ε-similar if $1 - \varepsilon < \frac{|a_i a_j|}{|b_i b_j|}, 1 + \varepsilon$ for all $i \neq j$. Let $h(n, A, \varepsilon)$ denote the maximum number of ε-similar copies of A in an n-element planar set. Conway, Croft, Erdős, and Guy (1979) studied the case when A is the regular triangle. Improving their result we show that for all triangles T that are close enough to a regular one there exist $\delta = \delta(T) > 0$ such that $h(n, T, \varepsilon) = (1 + o(1))n^3/24$ as $n \to \infty$ and $\varepsilon \in (0, \delta(T))$ is fixed. However, there are triangles with $h(n, T, \varepsilon) > n^3/15$.

2. Upper bounds on the number of almost similar triangles

A triangle almost T is called *almost regular* if all of its angles are between 59° and 61°. Conway, Croft, Erdős, Guy asked in 1979 [3] to determine $h(n)$, the maximal number of almost regular triangles in a planar n-set. They showed that $h(n) \leq (1/3 + o(n))\binom{n}{3}$. We are going to set up this question more generally.

Let T be a triangle with angles α, β, γ. Another triangle $\triangle$ is called *ε-similar* to T if one angle of $\triangle$ differs from α by less than ε, another angle differs from β by less than ε, and the third angle differs from γ by less than ε. Let $h(n, T, \varepsilon)$ denote the maximal number of triangles in a planar n-set that are ε-similar to T.

The following **construction** gives a lower bound on $h(n, T, \varepsilon)$: place the points in three groups of as equal sizes as possible, with each group very close to the vertices of T. This only gives the lower bound $n^3/27 - O(n)$. Iterating this yields a better bound: splitting each of the three groups into three further groups the same way, and continuing this way gives the recursion (with notation $f(n) = h(n, T, \varepsilon)$)

$$f(a + b + c) \geq abc + f(a) + f(b) + f(c)$$

where a, b, c are the sizes of the three groups. It follows from here that for every triangle T and for every $\varepsilon > 0$

$$h(n, T, \varepsilon) \geq \frac{n^3}{24} - n \log n. \tag{1}$$

The constructions in section 3 show that for some specific triangles better lower bounds are valid. However the following theorem shows that the bound in (1) is very precise for almost regular triangles.

Theorem 1. *For an almost regular triangle T there is an $\varepsilon > 0$ such that*

$$h(n, T, \varepsilon) \leq \frac{1}{24}(n^3 - n).$$

In particular, when n is a power for 3, $h(n, T, \varepsilon) = \frac{1}{24}(n^3 - n)$.

Theorem 1 shows that at most one quarter of the $\binom{n}{3}$ triangles is almost regular (and this is best possible by the construction). The next result uses extremal set theory, actually Turán theory of hypergraphs and flag algebra computations to give an upper bound for $h(n, T, \varepsilon)$ for almost every triangle T that is only 0.3% larger than the lower bound in (1).

Theorem 2. *For almost every triangle T there is $\varepsilon > 0$ such that*

$$h(n, T, \varepsilon) \leq 0.25072\binom{n}{3}(1 + o(1)).$$

3. Constructing many almost similar triangles

Example 1 when T is right angled. There is parallelogram P with right angles such that any three vertices of P form a triangle congruent to T. Place the points in four groups (as equal sized as possible) very close to the vertices of P, and continue iteratively. With notation $f(n) = h(n, T, \varepsilon)$ this gives the recursion

$$f(a + b + c + d) \geq abc + bcd + cda + dab + f(a) + f(b) + f(c) + f(d)$$

whose solution is

$$h(n, T, \varepsilon) \geq 4\left(\frac{n}{4}\right)^3 + 16\left(\frac{n}{16}\right)^3 + \ldots \geq \frac{n^3}{15} - O(n \log n),$$

a much larger upper bound than in (1). Most likely this type of triangles gives the largest value for $h(n, T, \varepsilon)$.

Example 2 when T has angles $120°, 30°, 30°$. Again we place the points in four groups very close to the three vertices and the centre of a regular triangle. The sizes of the groups are $a, a, a, n - 3a$ and the group of size $n - 3a$ is placed near the centre, the other groups near the vertices and iterate. The equation is $f(n) \geq 3a^2(n - 3a) + 3f(a) + f(n - 3a)$ whose solution is

$$h(n, T, \varepsilon) \geq \frac{n^3}{18.7979\ldots} + O(n^2),$$

Here the constant $18.7979\ldots$ is the solution of a maximization problem involving polynomials. This is another triangle shape with $h(n, T, \varepsilon)$ larger than for almost regular triangles (for instance).

Example 3 when T has angles $72°, 72°, 36°$. We place the points in five groups very close to the vertices of a regular pentagon, the sizes of the groups are again as equal as possible. The recursion gives

$$h(n, T, \varepsilon) \geq 5\left(\frac{n}{5}\right)^3 + 25\left(\frac{n}{25}\right)^3 + \ldots \geq \frac{n^3}{24} - O(n \log n).$$

The same construction works for the triangle angles $108°, 36°, 36°$.

Example 4 is for the triangle T with angles $\frac{4}{7}\pi$, $\frac{2}{7}\pi$, $\frac{1}{7}\pi$. The points are now in seven groups (of almost equal size again) placed close to the vertices of a regular 7-gon. The recursion is similar to the previous ones and gives again $h(n, T, \varepsilon) \geq \frac{n^3}{24} - O(n \log n)$.

4. Turán problems for hypergraphs

Turán's theory of extremal graphs and hypergraphs has several applications in geometry. Here we explain what we need for Theorem 2. Let $\mathcal{L}$ be a finite family of 3-uniform hypergraphs, the so-called *forbidden hypergraphs*, the question is to determine the maximal number of edges that a 3-uniform hypergaph $\mathcal{H}$ on n vertices can have if it does not contain any element of $\mathcal{L}$ as a subhypergraph. This maximal number is usually denoted by $\mathrm{ex}(n, \mathcal{L})$.

The strategy to prove Theorem 2 is to find a family $\mathcal{L}$ of 3-uniform hypergraphs so that $\mathrm{ex}(n, \mathcal{L})$ is close to $0.25\binom{n}{3}$ such that $\mathcal{H}(P, T, \varepsilon)$ does not contain any hypergraph in $\mathcal{L}$ for almost all triangle shapes for small enough ε.

Let $\mathcal{L}$ consist of the following 9 hypergraphs:

(1) $K_4^- = \{123, 124, 134\}$
(2) $C_5^- = \{123, 124, 135, 245\}$
(3) $C_5^+ = \{123, 124, 135, 146, 156\}$
(4) $L_2 = \{123, 124, 125, 136, 456\}$
(5) $L_3 = \{123, 124, 135, 256, 346\}$
(6) $L_4 = \{123, 124, 156, 256, 345\}$
(7) $L_5 = \{123, 124, 145, 346, 356\}$
(8) $L_6 = \{123, 124, 145, 346, 356\}$
(9) $P_7^- = \{123, 145, 167, 246, 257, 347\}$

For the proof of Theorem 2 we need the following fact.

Claim 3. $\lim_{n\to\infty} \mathrm{ex}(n, \mathcal{L})\binom{n}{3}^{-1} < 0.25072.$

The proof of this claim is based on the flag-algebra method due to Razbozov [7]. It requires computations by a computer: the "Flagmatic" package developed by Falgas-Ravry, Vaughan [5] (thanks to them) and the actual computation was carried out by Manfred Scheucher (thanks to him as well).

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High-dimensional Theta Numbers

Anna Gundert

(joint work with Christine Bachoc, Alberto Passuello)

The theta number $\vartheta(G)$ of a graph G was introduced by L. Lovász in his seminal paper [21], in order to provide spectral bounds of the independence number and of the chromatic number of G. In modern terms, $\vartheta(G)$ is the optimal value of a semidefinite program, and as such is computationally easy; in contrast, the independence number $\alpha(G)$ and the chromatic number $\chi(G)$ are difficult to compute. These three graph invariants satisfy the following inequalities, where $\overline{G}$ denotes the complement of G:

$$\alpha(G) \leq \vartheta(G) \leq \chi(\overline{G}).$$

The inequality $\alpha(G) \leq \vartheta(G)$ was one of the main ingredients in Lovász' proof of the Shannon conjecture on the capacity of the pentagon [21]. More generally, this inequality plays a central role in extremal combinatorics, sometimes in a disguised form: to cite a few, the Delsarte linear programming method in coding theory [6] and recent generalizations of Erdös-Ko-Rado theorems [5, 10, 11] can be interpreted as instances of this inequality. Analogs of the theta number in geometric settings have lead to many advances in packing problems (see [24] and references therein), in particular the very recent solutions to the sphere packing problems in dimensions 8 and 24 [3, 27].

In [1] we generalize this graph parameter to higher dimensions, in the framework of *simplicial complexes*. Let us recall that an (abstract) simplicial complex X on a finite set V is a family of subsets of V called *faces* that is closed under taking subsets. Thus, a graph can be considered as a simplicial complex of dimension 1. In recent years, considerable work has been devoted to generalizing the classical theory of graphs to this higher-dimensional setting. Much of the efforts have focused on the notion of expansion (see, e.g., [7, 13, 15, 19, 26]), but also other concepts such as random walks [25], trees [9, 18], planarity [23], girth [8, 22], independence and chromatic numbers [12, 14] have been extended to higher dimensions.

The familiar graph-theoretic notions of independence number and of chromatic number extend to this setting in a natural way: For a k-dimensional simplicial complex X, an independent set is a set of vertices that does not contain any maximal face of X, and the *independence number* $\alpha(X)$ is the maximal cardinality of an independent set. The *chromatic number*[1] $\chi(X)$ is the least number of colors needed to color the vertices so that no maximal face of X is monochromatic, in other words, it is the smallest number of parts of a partition of the vertices into independent sets.

In order to define the theta number $\vartheta_k(X)$ of a pure k-dimensional simplicial complex X, we follow an approach that leads to the inequality $\alpha(X) \leq \vartheta_k(X)$ in a natural way. The main idea is to associate a certain matrix to an independent

[1] In the study of hypergraphs, the chromatic number $\chi(X)$ is also known as the *weak chromatic number* while $\chi(X_1)$, the chromatic number of the 1-skeleton, is known as the *strong chromatic number*.

set S, and then to design a semidefinite program that captures as many properties of this matrix as possible. The matrix that we associate to an independent set is (up to a multiplicative factor) a submatrix of the *down-Laplacian of the complete complex*, which we denote by $L_{k-1}^{\downarrow}$. In the case of dimension 1, the down-Laplacian is simply the all-ones matrix, and we end up with one of the many formulations of the Lovász theta number.

A different upper bound for $\alpha(X)$ involving Laplacian eigenvalues was proved by Golubev in [14]. When every possible $(k-1)$-face is contained in at least one k-face, i.e., when X has a *complete* $(k-1)$*-skeleton*, Golubev's bound simplifies to

$$\alpha(X) \leq n\left(1 - \frac{d_{k-1}}{\mu_{k-1}}\right) \tag{1}$$

and can thus be seen as a natural generalization of the celebrated *ratio bound* for graphs attributed to Hoffman (see, e.g., [2, Theorem 3.5.2]). In this case, we can show that

$$\vartheta_k(X) \leq n\left(1 - \frac{d_{k-1}}{\mu_{k-1}}\right),$$

therefore $\vartheta_k(X)$ provides an upper bound of $\alpha(X)$ that is at least as good as (1). In the case of a non-complete $(k-1)$-skeleton, examples show that Golubev's bound and $\vartheta_k(X)$ are incomparable.

The theta number of a graph has many very nice properties; some of them, although unfortunately not all of them, can be generalized to higher dimensions. The relationship to the chromatic number generalizes only partially. Indeed, the inequality $\alpha(X) \leq \vartheta_k(X)$ immediately leads to the inequality $n/\vartheta_k(X) \leq \chi(X)$. However, in the case of graphs, the stronger inequality $\vartheta(\overline{G}) \leq \chi(G)$ holds. Its natural analog in the setting of k-complexes would be that $\vartheta_k(\overline{X}) \leq k\chi(X)$ and, unfortunately, this inequality does not hold in general. Instead, we introduce an ad hoc notion of *homomorphisms* between pure k-dimensional simplicial complexes and use this to define a chromatic number $\chi_k(X)$ for simplicial complexes. For this chromatic number, the inequality $\vartheta_k(\overline{X}) \leq \chi_k(X)$ holds. While $\chi(X)$ is defined using vertex colorings, the definition of $\chi_k(X)$ is based on colorings of $(k-1)$-faces respecting orientations.

We furthermore analyze the theta number of random simplicial complexes $X^k(n,p)$ for the model proposed by Linial and Meshulam in [20]. This model is a higher-dimensional analog of the Erdős-Rényi model $G(n,p)$ for random graphs and has gained increasing attention in recent years (see [17] for a survey).

We show that $\vartheta_k(X^k(n,p))$ is of the order of $\sqrt{(n-k)(1-p)/p}$ for probabilities p such that $c_0 \log(n)/n \leq p \leq 1 - c_0 \log(n)/n$ for some constant c_0. This result extends the known estimates for the value of the theta number of the random graph $G(n,p)$ [4, 16].

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Nerves and Minors

Andreas F. Holmsen
(joint work with Minki Kim, Seunghun Lee)

Background. Let $\mathcal{F}$ be a finite family of open connected sets in the plane such that any intersection of members in $\mathcal{F}$ is empty or connected. There is a well-known Helly theorem for such families: *If every four members have a point in common, then there is a point in common to every member of the* $\mathcal{F}$.

This Helly theorem also generalizes to higher dimensions if one assumes higher connectedness. Recently, Goaoc et al. [4] generalized this to families of quite general subsets of $\mathbb{R}^d$ building on a method developed by Matoušek [11]. All of these results are based on non-embeddability theorems, but the more "robust" Helly type theorems, such as the *colorful* Helly theorem [3], the *fractional* Helly theorem [10, 6], and the (p, q) theorem [2], seem out of reach by those methods (even in the planar situation described above).

One approach which is known to give such "robust" Helly type theorems is to consider the nerve of $\mathcal{F}$. This is the abstract simplicial complex $N(\mathcal{F})$ whose vertices are the members of $\mathcal{F}$ and whose faces are the subfamilies of $\mathcal{F}$ with non-empty intersection. Kalai, Meshulam, and others [1, 6, 7, 8, 9] have shown that if $N(\mathcal{F})$ is *d-Leray* (which means that every induced subcomplex has vanishing rational homology in dimensions d and greater), then the family $\mathcal{F}$ admits "robust" Helly theorems (with Helly number $d + 1$). Here we establish the following.

Theorem 1. *Let $\mathcal{F}$ be a finite family of open connected sets in the plane such that any intersection of members in $\mathcal{F}$ is empty or connected. Then the nerve $N(\mathcal{F})$ is 3-Leray.*

As a consequence, we get colorful and fractional versions of Helly's theorem, as well as a (p, q) theorem for such families of open connected sets. This answers a question due to Xavier Goaoc (personal communication).

Towards a nerve theorem for graphs. In order to prove Theorem 1 we find it more convenient to deal with families of graphs (rather than connected sets in the plane). Let $\mathcal{F} = \{G_1, \dots, G_n\}$ be a family of induced subgraphs of a fixed graph G. We say that $\mathcal{F}$ is a *connected family* in G provided that $\bigcap_{i\in\sigma} G_i$ is connected for every face $\sigma \in N(\mathcal{F})$. (In particular every G_i is a connected induced subgraph of G.) The question now becomes: *What can $N(\mathcal{F})$ tell us about the structure of G?* In general, not much, but one can show that if G contains K_{d+2} as a minor, then there *exists* a connected family in G which has non-vanishing homology in dimension d. This observation motivates us to introduce the following graph parameter.

For a finite graph G we define the *homological dimension* of G, denoted by $\gamma(G)$, to be the greatest integer d such that $H_d(N(\mathcal{F})) \neq 0$ for some connected family in G. (For the single vertex graph K_1 we define $\gamma(K_1) = -1$.)

It is not hard to show that the homological dimension is *minor-monotone* in the sense that if $H \prec G$, then $\gamma(H) \leq \gamma(G)$. It is also easy to see that $\gamma(K_{d+2}) = d$, and therefore if $K_{d+2} \prec G$, then $\gamma(G) \geq d$ (which was our motivating observation). One of our main results is that the converse holds for small values of d.

Theorem 2. *For any graph G we have the following.*

(1) $K_3 \prec G \iff \gamma(G) \geq 1$.
(2) $K_4 \prec G \iff \gamma(G) \geq 2$.
(3) $K_5 \prec G \iff \gamma(G) \geq 3$.

Note that Theorem 1 is a consequence of part (3) of Theorem 2. Given a finite family $\mathcal{F}$ of open connected sets in the plane such that the intersection of any members of $\mathcal{F}$ is empty or connected, then we can approximate $\mathcal{F}$ by a connected family of graphs $\mathcal{F}'$ in a *planar* graph G such that the nerves $N(\mathcal{F})$ and $N(\mathcal{F}')$ are isomorphic.

The basic properties of the homological dimension of a graph suggest a close relationship with other minor-monotone graph invariants [5, 12], in particular the invariant discussed in [5, section 5] and [12, section 18]. Our results show that K_3, K_4, and K_5 minors in a graph are perfectly detected by its homological dimension. Despite this limited evidence, we are tempted to conjecture that this holds for all complete minors.

Conjecture 3. *For every positive integer d and graph G,*

$$K_{d+2} \prec G \iff \gamma(G) \geq d.$$

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The number of flats spanned by a set of points in real space

Ben Lund

Let P be a set of n points in $\mathbb{R}^d$. We say that a k-flat Γ is spanned by P if Γ contains $k+1$ affinely independent points of P. For $0 \leq k \leq d$, let f_k be the number of k-flats spanned by P; in particular, $f_0 = n$ and $f_d = 1$.

Several classical results give bounds on f_k. For example, in 1946, de Bruijn and Erdős [2] showed that $f_1 \geq n$ unless the points of P are collinear. In 1983, Beck [1], proving a conjecture of Erdős, showed that $f_1 = \Omega(n(n-g_1))$, where g_1 is the greatest number of points of P contained in any line. In the same paper, Beck proved that there are constants $c_k > 0$ such that $f_k = \Omega(n^{k+1})$ unless $c_k n$ points of P are contained in a single k-flat.

In this talk, I stated upper and lower bounds on f_k (proved in [6]), and mentioned several applications.

The *essential dimension* $K = K(P)$ of P is the least t such that there exists a set $\mathcal{G}$ of flats such that

(1) P is contained in the union of the flats of $\mathcal{G}$,
(2) each flat $\Gamma \in \mathcal{G}$ has dimension $\dim(\Gamma) \geq 1$, and
(3) $\sum_{\Gamma \in \mathcal{G}} \dim(\Gamma) = t$.

For $i \geq 0$, let g_i be the maximum cardinality of a subset $P' \subseteq P$ such that $K(P') \leq i$; in particular, $g_0 = 0$ and $g_j = n$ for $j \geq K$.

The main theorem of [6] gives matching upper and lower bounds on each f_k in terms of the g_i for $0 \leq i \leq k$.

Theorem 1. *For $k < K$,*

$$f_k = \Theta\left(\prod_{i=0}^{k}(n-g_i)\right),$$

provided that $n - g_k \geq c_k$, for a constant c_k depending only on k.

For $k \geq K$,

$$f_k = O\left(\prod_{i=0}^{2(K-1)-k}(n-g_i)\right),$$

and either $f_{k-1} = f_k = 0$, or $f_{k-1} > f_k$.

Theorem 1 is tight except for the values of c_k and the constants implied in the asymptotic notation.

The proof of Theorem 1 relies on the Szemerédi-Trotter theorem [9] together with an induction on the dimension that relies only on the axioms of projective geometry and elementary combinatorics. Since the Szemerédi-Trotter theorem has been generalized to the complex numbers [10, 11], Theorem 1 is proved for point sets in complex space. On the other hand, it is not hard to construct counterexamples to the upper bound on f_k for $k < K$ in a vector space over a finite field; for example, take all of the points.

Theorem 1 has a number of applications. First, it provides a nearly complete answer to a question of Purdy [5, 7], who asked for conditions under which $f_k \leq f_{k-1}$.

Corollary 2. *For any $k > 1$, we have $f_k = \Omega((n - g_k)f_{k-1})$; in particular, there are constants c_k such that $n - g_k > c_k$ implies that $f_k > f_{k-1}$. On the other hand, $n - g_k = 0$ implies that $f_{k-1} > f_k$.*

Examples give in [6] show that c_k in Corollary 2 must increase at least linearly with k, and also provide counterexamples to a conjecture of Purdy [5] that Theorem 2 should hold for $k = 2$ with $c_2 = 1$.

Theorem 1 also immediately implies that the f_k are typically unimodal.

Corollary 3. *If $n - g_K > c_K$ then the sequence $f_0, f_1, \ldots, f_d$ is unimodal.*

Corollary 3 is a special case of a conjecture of Rota [8] that the sequence of f_k is unimodal for all matroids.

Theorem 1 implies that, for any $c < 1$, either $f_k = \Omega(n^{k+1})$, or there is a subset $P' \subset P$ contained in a set of flats $\mathcal{G}$ with $|P'| \geq cn$ and $\sum_{\Gamma \in \mathcal{G}} \dim(\Gamma) \leq k$. It is not hard to show that we can always partition $\mathcal{G}$ into two parts $\mathcal{G}_1$ and $\mathcal{G}_2$ so that the span of each of $\mathcal{G}_1$ and $\mathcal{G}_2$ has dimension at most k. From this, it is immediate that we can improve the constant c_k in Beck's theorem [1] to 1/2, as follows.

Corollary 4. *For each $k \geq 2$, either $(1 - o(1))n/2$ points are contained in a k-flat, or $f_k = \Omega(n^{k+1})$.*

We also obtain a generalization of an incidence bound of Elekes and Tóth [4]. A k-flat Γ is r-rich if $|\Gamma \cap P| \geq r$, and Γ is α-degenerate if at most $\alpha|\Gamma \cap P|$ points of $\Gamma \cap P$ are contained in any $(k-1)$-flat. Elekes and Tóth showed that, for any r and $\alpha < 1$, the number of r-rich, α-degenerate 2-flats is bounded above by $O(n^3 r^{-4} + n^2 r^{-2})$. They obtained a weak generalization to higher dimensions, showing that there are constants $\beta_k > 0$ such that, for any k, r and $\alpha < \beta_k$, the number of r-rich, α-degenerate k-flats is bounded above by $O(n^{k+1} r^{-k-2} + n^k r^{-k})$.

Do [3] proved a different generalization of this incidence bound, and the same result can be obtained as a corollary of Theorem 1 together with a result of Elekes and Tóth. In particular, a k-flat Γ is essentially-α-degenerate if the largest subset $P' \subseteq \Gamma \cap P$ having essential dimension $K(P') < k$ has size at most $\alpha|\Gamma \cap P|$. Note that an α-degenerate flat is necessarily essentially-α-degenerate, but not the other way around. Do's result is that, for any k, r and any $\alpha < 1$, the number of r-rich, essentially-α-degenerate k-flats is bounded above by $O(n^{k+1} r^{-k-2} + n^k r^{-k})$. It is possible to give a stronger upper bound on the number of essentially-α-degenerate

r-rich k-flats that are not also α-degenerate, and hence we have the following strong generalization of Elekes and Tóth's bound.

Corollary 5. *For any k, r, and any $\alpha < 1$, the number of k-rich r-flats is at most $n^{k+1}r^{-k-2} + n^k r^{-k}$).*

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Homological Sperner-type Theorems

Luis Montejano

Let K be a simplicial complex. Suppose the vertices of K are painted with $I = \{1, ..., m\}$ colours. A homological Sperner-type theorem concludes the existence of a rainbow simplex of K under the hypothesis that certain homology groups of certain subcomplexes of K are zero. We prove several homological Sperner-type theorem. A typical one is the following **Theorem 1**. Let K be a simplicial complex and suppose the vertices of K are painted with $I = \{1, ..., m\}$ colours. If $S \subset I$, let K_S be the subcomplex of K generated by vertices of colors in S.

Then K contains a rainbow simplex provided

$$\tilde{H}_{s-2}(K_S) = 0,$$

for every subset $S \subset I$ of s colours, $1 \leq s \leq m$.

The rainbow at the end of the line

Wolfgang Mulzer
(joint work with Fréderic Meunier, Pauline Sarrabezolles, Yannik Stein)

Let $C_1, \dots, C_{d+1}$ be $d+1$ point sets in $\mathbb{R}^d$, each containing the origin in its convex hull. A subset C of $\bigcup_{i=1}^{d+1} C_i$ is called a *colorful choice* (or *rainbow*) for $C_1, \dots, C_{d+1}$, if it contains exactly one point from each set C_i. The *colorful Carathéodory theorem* states that there always exists a colorful choice for $C_1, \dots, C_{d+1}$ that has the origin in its convex hull. This theorem is very general and can be used to prove several other existence theorems in high-dimensional discrete geometry, such as the centerpoint theorem or Tverberg's theorem. The colorful Carathéodory problem (Ccp) is the computational problem of finding such a colorful choice. Despite several efforts in the past, the computational complexity of Ccp in arbitrary dimension is still open.

We show that Ccp lies in the intersection of the complexity classes PPAD and PLS. This makes it one of the few geometric problems in PPAD and PLS that are not known to be solvable in polynomial time. Moreover, it implies that the problem of computing centerpoints, computing Tverberg partitions, and computing points with large simplicial depth is contained in PPAD $\cap$ PLS. This is the first nontrivial upper bound on the complexity of these problems.

Finally, we show that our PPAD formulation leads to a polynomial-time algorithm for a special case of Ccp in which we have only two color classes C_1 and C_2 in d dimensions, each with the origin in its convex hull, and we would like to find a set with half the points from each color class that contains the origin in its convex hull.

Local search for geometric optimization problems

Nabil H. Mustafa
(joint work with Norbert Bus, Shashwat Garg and Saurabh Ray)

Local-search is an intuitive approach towards solving combinatorial optimization problems: start with any feasible solution, and try to improve it by local changes. Like other greedy approaches, it can fail to find the global optimum by getting stuck on a locally optimal solution. In this talk I will survey the ideas and techniques behind the use of local-search in the design of provably good approximation algorithms for three types of geometric problems: combinatorial, metric and Euclidean. I will then focus on the use of local search for a basic combinatorial optimization problem, hitting set for disks, and present an improved analysis as well as an algorithm for it [1].

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Bounds for entries of γ-vectors of flag polytopes and spheres

Eran Nevo
(joint work with Jean-Philippe Labbé)

Facial enumeration of polytopes and simplicial spheres are of great interest since antiquity, and research on this topic revealed fascinating mathematics, notably in the celebrated g-theorem which characterizes the face numbers of simplicial d-polytopes [BL80, Sta80, Sta96].

In relation to the Charney–Davis conjecture, a subfamily of great importance is that of *flag* simplicial polytopes or more generally *flag* homology spheres. These objects have the property that their faces are exactly the cliques of their graphs, equivalently all their minimal non-faces have two elements. In the flag case the right analog of the g-vector (from the simplicial case) seems to be Gal's γ-vector: let $f_i(\Delta)$ denote the number of i-faces in a $(d-1)$-dimensional homology sphere Δ, then there are unique numbers $\gamma_i(\Delta)$ for $0 \le i \le \lfloor \frac{d}{2} \rfloor$ such that the following polynomial identity in variable t,

$$(1-t)^d \sum_{0 \le i \le d} f_{i-1}(\Delta)(\frac{t}{1-t})^i = \sum_{0 \le i \le \lfloor \frac{d}{2} \rfloor} \gamma_i(\Delta) t^i (1+t)^{d-2i},$$

holds. The γ-vector of Δ consists of the numbers $\gamma_i(\Delta)$. Gal conjectured that flag homology spheres have nonnegative γ-vectors [Gal05] which, by the same token, would validate the Charney–Davis conjecture. Later Nevo and Petersen strengthened Gal's conjecture by proposing the following combinatorial interpretation of the γ-vector.

Conjecture 0.1. [NP11] Let Δ be a flag homology sphere. The γ-vector of Δ is the f-vector of a flag simplicial complex.

Partial results toward this conjecture include [Gal05, NP11, NPT11, Ais12, Vol10, MN12, Zhe15, AH16]. The recent upper bound results of Adamaszek–Hladký and of Zheng [AH16, Zhe15] on the entries of the γ-vector require $\gamma_1 >> \dim(\Delta)$, or $\dim(\Delta) \in \{3, 5\}$ respectively. Conjecture 0.1 implies the following upper and lower bounds for the entries γ_i of the γ-vector in terms of γ_1 alone.

Conjecture 0.2. Let Δ be a flag homology sphere with $\gamma_1 = \ell$. For $i \ge 2$, the entries of the γ-vector of Δ satisfy

$$0 \le \gamma_i \le \binom{\ell}{i}.$$

Our first main result is a proof of parts of this conjecture.

Theorem 0.3. Let Δ be a flag homology sphere with $\gamma_1 = \ell$. The entries of the γ-vector of Δ satisfy the following properties:

i) $\gamma_2 \le \binom{\ell}{2}$,
ii) $\gamma_j = 0$ for all $j > \ell$,
iii) $\gamma_\ell \in \{0, 1\}$,

iv) $\gamma_{\ell-1} \in \{0, 1, 2, \ell\}$.

Further, we characterize the structure of extremal cases in the above theorem, omitted here for brevity.

A basic ingredient in the proof of Theorem 0.3 ii) to 0.3 iv) is to show that if the dimension of Δ is at least $2\gamma_1$ then Δ is a suspension over a lower dimensional flag homology sphere. Similarly to Perles' [Kal94, Theorem 1.1], we obtain the following analog as a corollary of the suspension result.

Theorem 0.4. Let $F_{b,k}(d)$ denote the number of combinatorial types of k-skeleta of flag homology $(d-1)$-spheres with at most $2d+b$ vertices. Given b and k, the set $\{F_{b,k}(i)\}_{i\geq 1}$ is bounded, hence finite.

Does the converse of Conjecture 0.1 hold? Namely, is it true that for any flag simplicial complex Γ, there exists a flag homology sphere Δ, of any dimension, such that γ^{Δ} equals the f-vector of Γ? We provide the first counterexamples.

Theorem 0.5. Let $k \geq 3$ and Γ_k be the flag balanced simplicial complex consisting of the disjoint union of a $(k-1)$-simplex and an isolated vertex. The f-vector of Γ_k is not the γ-vector of any flag homology sphere of any dimension.

Currently, a guess characterization of γ-vectors of flag homology spheres or flag simplicial polytopes seems to lack. In relation with the g-conjecture for homology spheres this poses the following problem.

Problem 1. Does there exist a vector which is the γ-vector of a flag homology sphere but not the γ-vector of a flag simplicial polytope?

Answering "No" to this question but "Yes" for the corresponding problem on f-vectors may be possible, as in Problem 1 we do not insist that the homology sphere and the polytope boundary have the same dimension. Given a flag homology sphere, the range of relevant dimensions for the flag polytopes is bounded though, due to the suspension result mentioned above. We remark that in the non-simplicial case, and for the finer flag-f-vector invariant, Brinkmann and Ziegler recently found a flag f-vector of a polyhedral 3-sphere which is not the flag f-vector of any 4-polytope [BZ16].

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On the structure of string graphs

János Pach

(joint work with Bruce Reed, Elena Yuditsky)

A *string graph* is an intersection graph of continuous arcs in the plane. The intersection graph of any finite family of plane convex sets is a string graph, but not all string graphs can be obtained in this way. We show that *almost all* string graphs on n vertices are intersection graphs of plane convex sets. This follows from the fact that the vertex set of almost all string graphs G on n vertices can be partitioned into *five* cliques such that two of them are not connected by any edge of G. The size of three of these cliques must be roughly $n/4$, while the size of the remaining two cliques (with no edge between them) must be about $n/8$.

This settles a conjecture of Janson and Uzzell [2], who established a weaker structural result was proved in terms of graphons. The proof is based on some ideas in [1] and [3].

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Colorful simplicial depth, Minkowski sums, and generalized Gale transforms

Arnau Padrol

(joint work with Karim Adiprasito, Philip Brinkmann, Pavel Paták, Zuzana Patáková, Raman Sanyal)

Let $C_0, \dots, C_d \subset \mathbb{R}^d$ be finite sets. They are thought of as color classes, and $C = \{C_0, \dots, C_d\}$ is called a *colorful configuration.* We call C *centered* if $0 \in \operatorname{relint}\operatorname{conv}(C_i)$ for all $0 \leq i \leq d$. A subset $S \subseteq \bigcup_i C_i$ is a *colorful simplex* if $|S \cap C_i| \leq 1$ for all i, and we call the simplex S *hitting* if $\dim S = |S| - 1 = d$ and $0 \in \operatorname{conv}(S)$.

Imre Bárány's Colorful Carathéodory Theorem [1] states that every centered colorful configuration contains at least one colorful hitting simplex. Deza, Huang, Stephen, Terlaky [3] introduced the *colorful simplicial depth*, $\operatorname{depth}(C)$, of a colorful configuration C as the number of hitting simplices of C. In particular, they initiated a systematic study of the extremal values of $\operatorname{depth}(C)$ as C ranges over all colorful configurations in $\mathbb{R}^d$.

For the case where $|C_i| = d + 1$ for all i, they conjectured that

$$1 + d^2 \;\leq\; \operatorname{depth}(C) \;\leq\; 1 + d^{d+1};$$

and showed that both bounds can be attained. The initial lower bound of $2d$ from [3] was improved in a series of papers [8, 4, 5, 7] culminating in the resolution of the conjectured lower bound by Sarrabezolles [6].

The first goal of the talk is to show that the upper bound conjecture is true in the following stronger form. We say that a colorful configuration C in $\mathbb{R}^d$ is in *relative general position* if no colorful simplex S of C of dimension $d - 1$ contains the origin in its convex hull.

Theorem 1. *Let $C = \{C_0, \dots, C_d\}$ be a centered colorful configuration in relative general position in $\mathbb{R}^d$ with $|C_i| \geq 2$ for all $0 \leq i \leq d$. Then*

$$\operatorname{depth}(C) \;\leq\; 1 + \prod_{i=0}^{d} (|C_i| - 1).$$

The proof of Theorem 1 uses tools from combinatorial topology. More precisely, we show that for centered colorful configurations in relative general position, the number of hitting simplices is related to the reduced Betti numbers of an associated simplicial complex, the *avoiding complex* $\mathcal{A}(C)$, which contains all the colorful simplices that are not hitting. In particular, Theorem 1 follows from the combination of the following two results.

Lemma 2. *Let $C = \{C_0, \dots, C_d\}$ be a centered colorful configuration in relative general position with $n_i := |C_i| \geq 2$, Then*

$$\operatorname{depth}(C) \;=\; \prod_{i=0}^{d} (n_i - 1) + \tilde{\beta}_{d-1}(\mathcal{A}) - \tilde{\beta}_d(\mathcal{A}).$$

Theorem 3. *Let $C = \{C_0, \ldots, C_d\}$ be a centered colorful configuration in relative general position with $n_i := |C_i| \geq 2$, then $\tilde{\beta}_{d-1}(\mathcal{A}(C)) = 1$.*

To prove this theorem, we introduce the notion of *flips* between colorful configurations, and we prove our result by 'flipping' any configuration to a fixed configuration.

The second goal of the talk is to highlight a connection between colorful configurations and faces of Minkowski sums. The *Minkowski sum* of convex polytopes $P_0, \ldots, P_s \subset \mathbb{R}^d$ is the polytope

$$P \;=\; P_0 + \cdots + P_s \;=\; \{p_0 + \cdots + p_s : p_i \in P_i \text{ for } 0 \leq i \leq s\}.$$

Considering the Gale transform of the Cayley embedding, we define *colorful Gale transforms* associated to a collection $P_0, \ldots, P_s$ that capture the facial structure of Minkowski sums in the combinatorics of colorful configurations.

A particularly interesting case is that of Minkowski sums of simplices. A face F of the Minkowski sum P is of the form $F = F_0 + \cdots + F_s$ for faces $F_i \subseteq P_i$. A face is *totally mixed* if each $F_i \subset P_i$ is an inclusion-maximal face, that is, F_i is a facet of P_i. If each P_i is a simplex, then totally mixed facets are exactly the hitting simplices of the associated colorful Gale transform. In particular, Theorem 1 implies the following.

Theorem 4. *For $d_0, \ldots, d_s \geq 1$ and $D = d_0 + \cdots + d_s - s$, let $P_i \subset \mathbb{R}^D$ for $0 \leq i \leq s$ be d_i-dimensional simplices whose Minkowski sum is of full dimension D. Then the number of totally mixed facets of $P_0 + \cdots + P_s$ is at most*

$$1 + d_0 d_1 \cdots d_s.$$

This dictionary between Minkowski sums and colorful configurations allows us to resolve a conjecture of Ben Burton [2] about the complexity of projective edge weight solution spaces in *normal surface* theory. The geometric formulation of the problem can be translated as an upper bound theorem for the number of totally mixed facets of a Minkowski sum of $d-1$ triangles in $\mathbb{R}^d$. Theorem 4 specialized to triangles provides the upper bound of $1 + 2^{d-1}$ originally conjectured by Burton.

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Proper coloring of geometric hypergraphs

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(joint work with Balázs Keszegh)

Given a finite set of points in the plane, S, we want to color the points of S with a small number of colors such that every member of some given geometric family that intersects S in many points will contain at least two different colors.

Pach conjectured in 1980 [4] that for every convex set D there is an m such that any finite set of points admits a 2-coloring such that any *translate* of D that contains at least m points contains both colors. This conjecture inspired a series of papers studying the problem and its variants - for a recent survey, see [7]. Eventually, the conjecture was shown to hold in the case when D is a convex polygon in a series of papers [5, 10, 9], but disproved in general [6]. In fact, the conjecture fails for any D with a smooth boundary, e.g., for a disk.

It follows from basic properties of generalized Delaunay triangulations and the Four Color Theorem that for any convex D it is possible to 4-color any finite set of points such that any *homothetic copy*[1] of D that contains at least two points will contain at least two colors. Therefore, the only case left open for homothets is when we have 3 colors. We conjecture that for 3 colors the following holds.

Conjecture 1. *For every plane convex set D there is an m such that any finite set of points admits a 3-coloring such that any homothetic copy of D that contains at least m points contains two points with different colors.*

The special case of Conjecture 1 when D is a disk has been posed earlier in [2], and is also still open. Our main result is the proof of Conjecture 1 for convex polygons.

Theorem 1. *For every convex n-gon D there is an m such that any finite set of points admits a 3-coloring such that any homothetic copy of D that contains at least m points contains two points with different colors.*

We would like to remark that the constructions from [6] do not exclude the possibility that for convex polygons the strengthening of Theorem 1 using only 2 colors instead of 3 might also hold; this statement is known to hold for triangles [3] and squares[2] [1].

[1] A homothetic copy or homothet of a set is a scaled and translated copy of it (rotations are *not* allowed).

[2] And since affine transformation have no effect on the question, also for parallelograms.

The constant m which we get from our proof depends not only on the number of sides, but also on the shape of the polygon. However, we conjecture that this dependence can be removed, and in fact the following stronger conjecture holds for any *pseudo-disk arrangement.*

Conjecture 2. *For any pseudo-disk arrangement any finite set of points admits a* 3*-coloring such that any pseudo-disk that contains at least* 4 *points contains two points with different colors.*

Remark. The fact that $m = 3$ is not sufficient is shown by a construction using pseudo-disks due to Géza Tóth (personal communication). For disks, $m = 3$ might be sufficient.

It follows from a construction of Pach and Tardos [8] that similar statements are false for any number of colors if instead of the plane we consider $\mathbb{R}^4$ or higher dimensional spaces, but in $\mathbb{R}^3$ the following might hold.

Conjecture 3. *For every convex set $D \subset \mathbb{R}^3$ there is an m such that any finite set of points admits a* 4*-coloring such that any homothetic copy of D that contains at least m points contains at least two colors.*

The reason why Conjecture 3 is stated with 4 colors is the following construction.

Theorem 2. *For every m there is a finite set of points $S \in \mathbb{R}^3$ such that for any* 3*-coloring of S there is a unit ball that contains exactly m points of S, all of the same color.*

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Embedding simplicial complexes to manifolds

Pavel Paták
(joint work with M. Tancer)

Since simplicial complexes may be used to model many real world objects, the natural question into which topological spaces can a simplicial complex be embedded is of crucial importance.

Unfortunately, it is also algorithmically undecidable. There are some trivial cases, namely, every k-dimensional complex embeds into any $(2k+1)$-dimensional manifold, and no $(k+1)$-dimensional complex embeds into a k-dimensional manifold. The first interesting regime is when we try to embed k-dimensional complex into a $2k$-dimensional manifold M.

Let us try to understand the extremal cases: What is the minimal number of vertices of a complex non-embeddable to M? What is the minimal number of k-faces of a complex non-embeddable to M? When answering the first question we may clearly assume that the minimal non-embeddable complex contains all its k-faces. Therefore the precise answer for $k = 1$ follows from Heawood inequality and Ringel-Youngs theorem [4] (The complete graph K_n can be embedded into a closed surface M different from Klein bottle if and only if $n \leq \frac{7+\sqrt{1+24\beta_1(M)}}{2}$, K_n can be embedded into Klein bottle if and only if $n \leq 6$.). If $M = \mathbb{R}^{2k}$, the precise answer follows from van Kampen-Flores theorem [2, 1]. (The k-skeleton of $(2k+2)$-dimensional simplex cannot be embedded into $\mathbb{R}^{2k}$.)

Kühnel proposed a common generalization of both of these inequalities [3, Conjecture B]: If the k-skeleton of n-dimensional simplex $\Delta_n^{(k)}$ embeds into a $(k-1)$-connected compact manifold M, then

$$\binom{n-k-1}{k+1} \leq \binom{2k+2}{k+1}\beta_k(M;\mathbb{Z}_2).$$

We present a method that may potentially yield stronger results. We say that a k-complex K is $\mathbb{Z}_2$-almost embeddable into M, if there exists a continuous map $f\colon K \to M$ with the following properties:

(1) restriction of f to the $(k-1)$-skeleton of K is an embedding, and
(2) any two disjoint faces of K cross even number of times

Let us observe that any two continous maps from K to M are connected by sequence of two types of moves: 1) homotopy, 2) homology changes (i.e. adding a homology cycle to the image of a k-face). It is not difficult to make these changes in such a way that we know how do they change the parity of crossings among k-faces. Changing an initial embedding into a $\mathbb{Z}_2$-almost embedding then corresponds to solving system of quadratic equations over $\mathbb{Z}_2$.

From the previous considerations we can derive our main results:

Theorem 1. *Let us assume that $\Delta_n^{(k)}$ $\mathbb{Z}_2$-almost embeds into M. Then:*

(1) $n \leq (2k+1) + (k+1)\beta_k(M;\mathbb{Z}_2)$,

(2) if the intersection form Ω on M, $\Omega\colon H_k(M;\mathbb{Z}_2)\times H_k(M;\mathbb{Z}_2)\to\mathbb{Z}_2$ is symplectic we may improve the bound to $n\le(2k+1)+(k+2)\beta_k(M;\mathbb{Z}_2)/2$.

Note that the first case agrees with Kühnel's conjecture if $\beta_k(M;\mathbb{Z}_2)\le 1$ and the second one matches the conjecture for $\beta_k(M;\mathbb{Z}_2)\le 2$.

Using a SAT solver for the quadratic equations, we obtain further results:

Theorem 2.

(1) For $k=1$ and $\beta_k(M;\mathbb{Z}_2)\le 5$ the upper bound matches Ringel-Youngs theorem.[1]

(2) For $k=2$, $\beta_2(M;\mathbb{Z}_2)=2$ and a non-symplectic intersection form we have the bound $n\le 3k+3$ (which is better than the conjectured value).

We note that we can also beat the proposed value in some other cases, most notably in the case $k=3$, $\beta_3(M;\mathbb{Z}_2)=1$.

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Meissner polyhedra

Edgardo Roldán-Pensado

(joint work with Luis Montejano)

A body of constant width is a closed convex subset of R^n that has the same width in every direction. Constant width bodies and their properties have been known for centuries. In 1875, Franz Reuleaux [4] published a book in which he mentioned constant width curves and gave some examples. He later gave the construction one of the simplest constant width curves which is not a circle, and which today bears his name. Although we know of many procedures to construct curves with constant width, the same is not true for their higher dimensional analogues.

We know by a theorem of Pál that every subset of R^n with diameter 1 is contained in a body of constant width [1]. Sallee [5], and Lachand and Outdet [2], among others, gave non-constructive procedures to find them, but besides the two Meissner Solids [3], and the obvious constant width bodies of revolution, there is no explicit example of a constant width body of dimension greater than 2 or a concrete finite procedure to construct one. In this talk we show concrete examples

[1] Larger values of β_k took too long to compute.

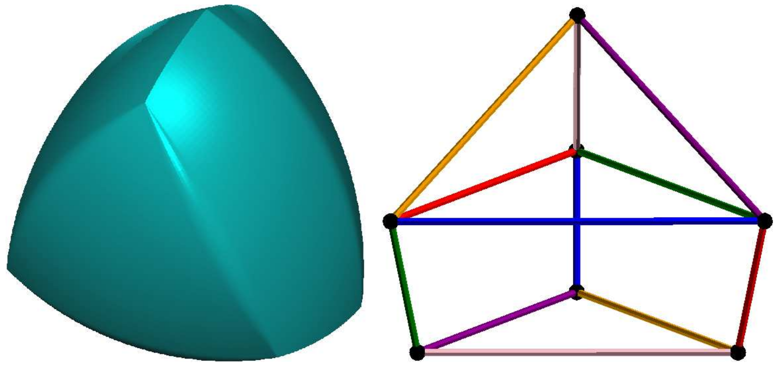

Figure 1. A Meissner polyhedra and its associated graph.

of constant width bodies in dimension three, which we call Meissner polyhedra. They are constructed from some special embedding of self-dual graphs (see Fig. 1). Our method gives bodies whose boundaries consist of spherical caps and circular arcs of revolution. This makes them candidates for the body of a given constant width minimal volume.

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A survey on the diameter of lattice polytopes

Lionel Pournin

(joint work with Antoine Deza)

The diameter of polyhedra has attracted much attention over the past decades. It is usually evaluated as a function of the dimension d of the polyhedron and the number n of its facets. In the case of the lattice polytopes contained in the hypercube $[0,k]^d$, another set of relevant parameters is d and k. We present new bounds on the maximal diameter of these polytopes as a function of d and k.

1. Earlier results

Various bounds on the maximal possible diameter of a polyhedron as a function of its dimension d and the number n of its facets have been obtained. Larman [10] gave an upper bound on this quantity that is linear as a function of n, but exponential as a function of d. Kalai and Kleitman [7] found another upper bound that is quasi-polynomial as a function of d and n. Lower bounds have also been obtained by Klee and Walkup [8] and by Santos [12], disproving the Hirsch conjecture for unbounded polyhedra and for polytopes, respectively.

In the case of lattice polytopes, a set of parameters alternative to n and d can be used. For some positive integer k, consider the polytopes contained in the d-dimensional hypercube $[0,k]^d$, and whose vertices have integer coordinates. Such polytopes are referred to as *lattice (d,k)-polytopes*, and the maximal possible value of their diameter is denoted by $\delta(d,k)$. When $k=1$, this quantity was investigated by Naddef [11] who proved that $\delta(d,1)\leq d$, and thus that lattice $(d,1)$-polytopes satisfy the Hirsch conjecture. This result was generalized to $\delta(d,k)\leq kd$ by Kleinschmidt and Onn [9]. When $d=2$, $\delta(d,k)$ can be deduced from the largest possible number of vertices of a lattice polygon contained in the square $[0,k]^2$, independently studied by Balog and Bárány [2], Thiele [13], and Acketa and Žunić [1]. In particular, $\delta(2,k)$ is known for all k.

2. Some new results

The upper bound of Kleinschmidt and Onn is sharp when $k=1$ as the diameter of the d-dimensional cube is equal to d. It turns out that it is only sharp when $k=1$. Indeed, Del Pia and Michini show the following.

Theorem 1 ([4]). $\delta(d,k)\leq kd-\lceil d/2\rceil$ when $k\geq 2$.

This bound is sharp when $k=2$, as shown in [4] using cartesian products of hexagons and line segments. The original proof for the above-mentioned upper bound by Kleinschmidt and Onn [9] relies on a codimension 1 argument. It consists in constructing a path between two vertices u and v of a lattice (d,k)-polytope P via a face wherein a coordinate is either minimal or maximal. Since P is a lattice (d,k)-polytope, the combined distance of u and v to one of these faces is at most k and the result is obtained by induction on d.

The proof of Theorem 1 is more involved as a number of cases have to be reviewed separately. However, the core of the proof is based on a codimension 2 argument similar to the codimension 1 argument of Kleinschmidt and Onn. It consists in constructing a path from u to v via a face F of P wherein the sum of two coordinates x_i and x_j is minimal or maximal. Assuming that $k\geq 2$, Del Pia and Michini prove that, when u and v are not vertices of $[0,k]^d$, one can choose i and j such that the first step of this path is a diagonal that affects both coordinates, thus saving at least one step every time the induction reduces the dimension by two. When F has dimension $d-1$, a portion of the path has to be built separately within F to some of its facets, and its length must be bounded above.

The result of Del Pia and Michini can be improved as follows when $k\geq 3$.

Theorem 2 ([6]). The following inequalities hold:

(i) $\delta(d,k)\leq kd-\lceil 2d/3\rceil-(k-3)$ when $k\geq 3$,
(ii) $\delta(d,k)\leq kd-\lceil 2d/3\rceil-(k-2)$ when $k\geq 4$,
(iii) $\delta(d,3)\leq\lfloor 7d/3\rfloor-1$ when $d\not\equiv 2 \mod 3$.

This theorem results from a codimension 3 argument: two vertices of a lattice (d, k)-polytope are related by a path via a face F wherein the sum of three coordinates is either minimal or maximal. Under the assumption that $k \geq 3$, it is shown that two diagonals can be found along such a path every time the induction reduces the dimension by three. When F has dimension $d-1$ or $d-2$, a portion of the path to a $(d-3)$-dimensional face of F has to be built separately, and its length must be bounded above. As observed in [6], generalizing this particular bit of the proof to a similar codimension 4 argument seems to be challenging.

It follows from Theorem 2 that $\delta(4,3) = 8$. Taking advantage of some of the properties stated in [6], Chadder and Deza [3] were further able to show using a computer-assisted proof that $\delta(3,4) = 7$ and $\delta(3,5) = 9$. The values of $\delta(d,k)$ known so far are reported in Table 1.

3. A conjecture

In [5] Deza, Manoussakis, and Onn build lattice (d, k)-polytopes of diameter $\lfloor (k+1)d/2 \rfloor$ when $k < 2d$. These polytopes are the Minkowski sums of sets of the shortest possible lattice vectors, no two of whose are collinear.

They conjecture the following.

Conjecture 1 ([5]). For any d and k, $\delta(d,k)$ is achieved up to translation by a Minkowski sum of lattice vectors. In particular, when $k < 2d$,

$$\delta(d,k) = \left\lfloor \frac{(k+1)d}{2} \right\rfloor .$$

Note that this conjecture holds for all the values of $\delta(d,k)$ reported in Table 1. The case when $k = 3$ is particularly interesting. In this case the upper bound from Theorem 2 reads $\delta(d,3) \leq \lfloor 7d/3 \rfloor$, while Conjecture 1 claims that $\delta(d,3) = 2d$.

	d \ k	1	2	3	4	5	6	7	8	9	…
	1	1	1	1	1	1	1	1	1	1	…
	2	2	3	4	4	5	6	6	7	8	…
	3	3	4	6	7	9					
d	4	4	6	8							
	⋮	⋮	⋮								
	d	d	$\lfloor \frac{3}{2}d \rfloor$								

Table 1. The largest possible diameter $\delta(d,k)$ of a lattice (d,k)-polytope.

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Distinct distances for points lying on two curves in $\mathbb{R}^d$

Orit E. Raz
(joint work with Micha Sharir)

Let γ_1, γ_2 be a pair of constant-degree irreducible algebraic curves in $\mathbb{R}^d$, neither of which is contained in a hyperplane in $\mathbb{R}^d$. We show that for every finite $P_1 \subset \gamma_1$ and $P_2 \subset \gamma_2$, each of size n, the number of distinct distances spanned by $P_1 \times P_2$ is at least $\Omega(n^{4/3})$, with a constant of proportionality that depends on $\deg \gamma_1, \deg \gamma_2, d$, unless each of γ_1, γ_2 is an *algebraic helix*. This extends earlier results of Charalambides [1], Pach and De Zeeuw [2], and Raz [3].

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The algebraic conspiracy

Günter Rote
(joint work with Mikkel Abrahamsen)

1. Problem Statement and Motivation

We consider the problem of sandwiching a polytope Δ with a given number k of vertices between two nested polytopes $P \subset Q \subset \mathbf{R}^d$: Find Δ such that $P \subseteq \Delta \subseteq Q$. The polytope P is not necessarily full-dimensional.

Besides the problem of computing Δ, we study the following question: Assuming that the given polytopes P and Q are rational polytopes (they have rational vertex coordinates), does it suffice to look for Δ among the rational polytopes?

This problem has several applications: (1) When Q is a dilation of P (or an offset of P), Δ can serve as a thrifty approximation of P. (2) The polytope nesting problem can model the nonnegative rank of a matrix, and thereby the extension complexity of polytopes, as well as other problems in statistics and communication complexity. It was in this context that question (b) was first asked [3].

2. Nested Polygons in the Plane

In the plane ($d = 2$), it has been shown in 1989 by Aggarwal, Booth, O'Rourke, Suri & Yap [2] that Δ can be computed in $O(n \log k)$ time, assuming unit-cost arithmetic operations. This algorithm computes in fact the smallest possible k for which Δ exists, while for $d \geq 3$, minimizing k is NP-hard [4, 5].

The approach of [2] is as follows: Choose a starting point x_0 on the boundary of Q and wind a polygonal path $x_1 = f_1(x_0)$, $x_2 = f_2(x_1)$, $\ldots$, $x_k = f_k(x_{k-1})$, around P by putting a sequence of tangents to P and intersecting them with the boundary of Q, see Figure 1a. If $x_k \geq x_0$, then a k-gon Δ can be found. We

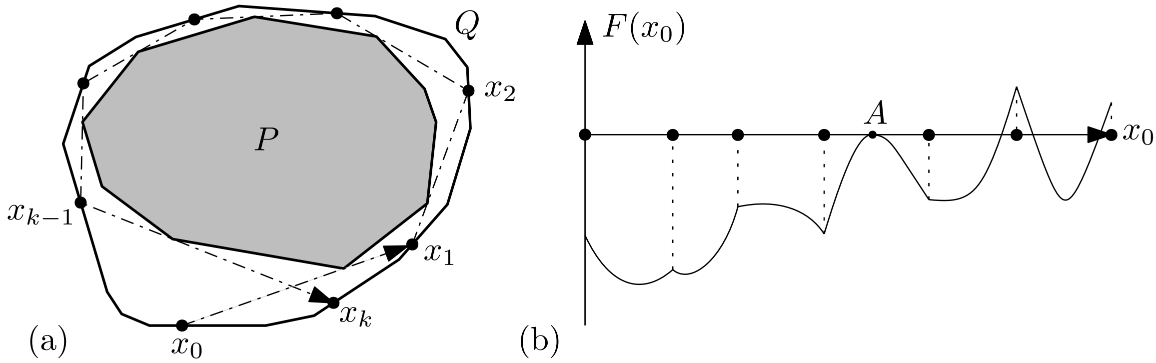

Figure 1. (a) the chain $x_0x_1x_2\ldots$ (b) a hypothetical function $F(x_0)$

parameterize the points x_0 by arc length along the boundary of Q from some fixed starting point. Now vary x_0 and follow the other points. As long as each point x_i moves on a fixed edge of Q and each segment $x_{i-1}x_i$ touches a fixed vertex of P, the function f_i is a rational linear function of the form $f_i(x) = \frac{ax+b}{cx+d}$. The composition of such functions is also of the same form. The function changes at the *breakpoints*, when an edge $x_{i-1}x_i$ of Δ lies flush with an edge P or a vertex x_i coincides with a vertex of Q. It follows that the function

$$F(x_0) := f_k(f_{k-1}(\cdots f_2(f_1(x_0))\cdots)) - x_0 \tag{1}$$

is piecewise rational, see Figure 1b. A solution of $F(x_0) \geq 0$ can be found by looking at the pieces and solving a quadratic equation for each piece.

Now, for some interval where the function f_i is smooth, the graph of the function is a hyperbola. It is easy to see that, for the range of the variable x_{i-1} that is of interest, the graph of $f_i(x_{i-1})$ lies on that branch of the hyperbola which is increasing and convex. The property of being increasing and convex is preserved under composition. Therefore, the function F in (1) is piecewise convex, unlike the function in Figure 1b. We obtain the following simplification of the algorithm.

Proposition 1. *To find the solutions of $F(x_0) \geq 0$, it is sufficient to look at the breakpoints of F.*

(For $k = 3$, this has been established before by Kubjas, Robeva, and Sturmfels [7], based on results from [8].) This implies in particular that the solution Δ can be found among the rational polygons. The existence of a rational solution has also been established in [9, Theorem 8] by observing that an isolated solution x_0 of $F(x_0) \geq 0$, like the point A in Figure 1b, would have to be rational for algebraic reasons, being a double zero of a quadratic equation. Our proof of Proposition 1 shows that such a situation cannot arise.

3. The Quest for an Irrational Solution in Higher Dimensions

A 3-dimensional example, in which the only polytope Δ with $k = 5$ vertices has irrational coordinates, has been constructed in [9], and it has been lifted to 4-dimensions (with a 3-dimensional polytope P) [10]. The case of a tetrahedron ($k = 4$) in 3 dimensions is open. It would also be interesting to have a 4-dimensional example where P is full-dimensional. (This corresponds to the *restricted* nonnegative rank [6].)

Figure 2 shows an attempt to construct a 3-dimensional instance which only has an irrational tetrahedron as a solution. Q has a horizontal bottom face Q_{bottom} and a horizontal top face Q_{top}. (The edges of Q are not fully shown.) P has six vertices and sits on Q_{bottom} with three vertices $P_1P_2P_3$. The tetrahedron Δ has an irrational vertex Δ_4 in the interior of Q_{top}. Figure 2b shows Q_{bottom} together with the projection $P_4'P_5'P_6'$ of the remaining vertices as seen from Δ_4, and it shows how the bottom face $\Delta_1\Delta_2\Delta_3$ of Δ is squeezed between $P_1P_2P_3 \cup P_4'P_5'P_6'$ and Q_{bottom}.

We have tried to construct such an example in reverse by building Q around Δ: After choosing a rational polytope $P = P_1P_2P_3P_4P_5P_6$ with $P_1P_2P_3$ on the horizontal plane of Q_{bottom}, we choose Δ_4 as an irrational point with coordinates in some quadratic extension field $\mathbb{Q}[\sqrt{r}]$. This leads to irrational projected points $P_4'P_5'P_6'$, and from this, the irrational points $\Delta_1\Delta_2\Delta_3$ can be constructed. Through each of these points, there is a unique rational line q_1, q_2, q_3, and these lines can be combined to form the boundary of Q_{bottom}. However, no matter how we try

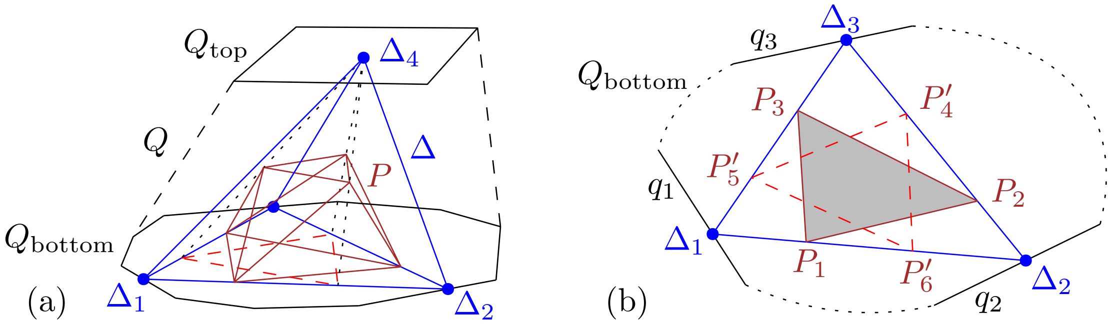

FIGURE 2. (a) $P \subset \Delta \subset Q$; (b) the situation on the bottom face Q_{bottom}

to choose the data, as if by some conspiracy, one of the lines q_1, q_2, q_3 always cuts into the triangle $\Delta_1\Delta_2\Delta_3$, making the completion of the construction impossible. Some experiments with dynamic geometry software suggest that this might be a systematic phenomenon: When we adjust the data so that one of the lines q_1, q_2, q_3 moves out of the triangle $\Delta_1\Delta_2\Delta_3$, another lines moves in precisely at the same time. If such an irrational example is indeed impossible, and examples of a different combinatorial type can also be excluded, it is conceivable that the solution for $k = 4$ is always rational if it exists. But this would so be for some deeper reason.

A similar "conspiracy" phenomenon has been observed in the construction of art gallery problems which require irrational guards [1]. The problem could be circumvented by modifying the construction and using more guards.

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An optimistic story of non-algebraic curves: tangencies, lenses, and a new crossing lemma

Natan Rubin
(joint work with János Pach, Gábor Tardos)

The study of low-degree algebraic curves (and their tangencies) has proven a huge success due to powerful partition methods from Algebraic Geometry (the partitioning polynomials) and Random Sampling (epsilon-nets). In contrast, the general arrangements of Jordan curves are still poorly understood, as any two of these curves can meet at arbitrary many points.

The celebrated Crossing Lemma of Ajtai, Chvátal, Newborn, Szemerédi and Leighton [ACNS82, Le83] gives a relation between (1) the number of edges in the graph, and (2) the minimum number of crossings in its planar embedding. Inspired by the celebrated Circle Packing Theorem of Koebe, Andreev, and Thurston [Koe36, An70, Thu97], we establish a suitable Crossing Lemma for contact graphs of arbitrary Jordan curves. This is achieved due to a novel use of lenses, which is rather different from that in Tamaki and Tokuyama [TT98], Agarwal et al. [ANPPSS04], or Ellenberg, Solymosi and Zahl [ESZ16].

This work stems from our recent proof [PRT16] of the Richter-Thomassen conjecure [RiT95] on the number of intersection points in a family of pairwise-intersecting Jordan curves.

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Towards a classification of 4-dimensional empty simplices

Francisco Santos
(joint work with M. Blanco, O. Iglesias-Valiño, C. Haase and J. Hofmann)

A *lattice d-simplex* is the convex hull of $d+1$ affinely independent integer points in $\mathbb{R}^d$. It is called *empty* if it contains no lattice point apart of the $d+1$ vertices. Empty simplices are the building blocks in the theory of lattice polytopes, in the sense that every lattice polytope P (i.e., polytope with integer vertices) can be triangulated into empty simplices. In algebraic geometry, empty simplices correspond to *terminal quotient singularities* and are fundamental objects in the so-called minimal model. In particular, both from the point of view of discrete geometry and of algebraic geometry there is interest in understanding empty simplices better and, hopefully, have full classifications of them [8, 7].

In dimension two it is an easy consequence of Pick's Theorem that every empty triangle is *unimodular* ; that is, it has volume 1/2 (or, equivalently, determinant 1). In dimension three there are infinitely many types of empty simplices, but yet their classification (modulo affine integer isomorphism, which we call *unimodular equivalence*) is quite simple (White 1964 [9]). A key step in it is the fact that all empty 3-simplices have *lattice width* equal to one, with the following definition: For each linear or affine functional $f : \mathbb{R}^d \to \mathbb{R}$ and any convex body $P \subset \mathbb{R}^d$ the width of P with respect tot f is the difference between the maximum and minimum values of f on P; that is, the length of the interval $f(P)$. The *lattice width* of P is the minimum width among all non-constant functionals with integer coefficients.

In dimension four the following facts are known:

Theorem (Properties of empty 4-simplices [3, 6, 7, 8])**.**

(1) There are infinitely many lattice emtpy 4*-simplices of width* 1 *(e.g., cones over empty tetrahedra) and of width* 2 [7, 8].
(2) Among the simplices of determinant ≤ 1000 *there are 178 of width three, one of width four, and none of larger width* [7].
(3) Every empty 4*-simplex is* cyclic [3]*. The same is not true in dimension five.*

The *determinant* or *normalized volume* of the d-simplex with vertices $v_0, \ldots, v_d$ is the Euclidean volume normalized to the lattice, so that unimodular simplices have volume one and every lattice polytope has integer volume. For a (perhaps not empty) lattice simplex P, it s determinant equals the order of the quotient group of the lattice $\mathbb{Z}^4$ by the sublattice generated by vertices of P. P is called *cyclic* if this quotient group is cyclic.

We here report on the proofs of the following theorems:

Theorem 1 (Blanco, Haase, Hoffman, Santos [6])**.** *For each fixed $n \in \mathbb{N}$ there are only finitely many lattice* 4*-polytopes of width larger than two with at most n lattice points.*

Letting $n = 5$ this implies:

Corollary 2. *There are only finitely many empty lattice 4-polytopes of width larger than two.*

Remark 3. Theorem 2 is the main result in Barile et al. [3], but we have found out that the proof given in their paper is wrong. In fact, the explicit classification that they give for the infinite families is incomplete. See more details in [6]

Theorem 4 (Santos and Iglesias [4])**.** *The only empty* 4*-simplices of width larger than two are the 179 found by Haase and Ziegler; 178 of width three and determinant ranging from* 41 *to* 179*. The other one of with four and determinant* 101*.*

The proof of Theorem 4 combines a theoretical and a computer part:

- Combining Minkowski theorems on successive minima, Kannan-Lovász theory of covering minima [5], and recent results of Averkov et al. on hollow lattice 4-polytopes [1, 2], we show that no empty 4-simplex of width larger than two can have determinant above 7600.
- Computationally, and relying on the fact that empty 4-simplices are cyclic [3] we exhaustively enumerate all empty 4-simplices of determinant up to 7600, and compute their width. These computations used about 10000 hours of CPU borrowed from the Altamira Supercomputer at the Institute of Physics of Cantabria (IFCA-CSIC).

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Decomposing arrangements of hyperplanes: VC-dimensions, combinatorial dimension, and point location

Micha Sharir
(joint work with Esther Ezra, Sariel Har-Peled, Haim Kaplan)

This work is motivated by several basic problems and techniques that rely on space decomposition of arrangements of hyperplanes in high-dimensional spaces, most notably Meiser's 1993 algorithm for point location in such arrangements [5]. A standard approach to these problems is via random sampling, in which one draws a random sample of the hyperplanes, constructs a suitable decomposition of its arrangement, and recurses within each cell of the decomposition. The efficiency of the resulting algorithm depends on the quality of the sample, which is controlled by various parameters.

One of these parameters is the classical *VC-dimension*, and its associated *primal shatter dimension*, of a suitably defined corresponding range space (see, e.g., [4]). Another parameter, which we refer to here as the *combinatorial dimension*, is the number of hyperplanes that are needed to define a cell that can arise in the decomposition of some sample of the input hyperplanes; this parameter arises in Clarkson's (and later Clarkson and Shor's) random sampling technique [1, 2].

We re-examine these parameters for the two main space decomposition techniques — *bottom-vertex triangulation*, and *vertical decomposition*, including their explicit dependence on the dimension d, and discover several unexpected phenomena, which show that, in both techniques, there are large gaps between the VC-dimension (and primal shatter dimension), and the combinatorial dimension.

For vertical decomposition, the combinatorial dimension is only $2d$ but the primal shatter dimension is at most $d(d+1)$, and the VC-dimension is at least $1 + d(d+1)/2$ and at most $O(d^3)$. For bottom-vertex triangulation, both the primal shatter dimension and the combinatorial dimension are $\Theta(d^2)$, but there is still a significant gap between them, as the combinatorial dimension is $\frac{1}{2}d(d+3)$, whereas the primal shatter dimension is $d(d+1)$ and the VC-dimension is between $d(d+1)$ and $O(d^2 \log d)$.

Our main application is to point location in an arrangement of n hyperplanes is $\Re^d$, in which we show that the query cost in Meiser's algorithm can be improved if one uses vertical decomposition instead of bottom-vertex triangulation, at the cost of increasing the preprocessing cost and storage. The best query time that we can obtain is $O(d^3 \log n)$ time, instead of $O(d^4 \log d \log n)$ in Meiser's algorithm. For these bounds to hold, the preprocessing and storage are rather large (super-exponential). We discuss the tradeoff between query cost and storage (in both approaches).

Another application yields an improved bound for the decision-tree complexity of k-SUM (determining whether there exist k elements of a set of n real numbers whose sum is 0). We show that $O(n^2 \log n)$ linear queries suffice, improving an earlier bound of $O(n^3 \log^3 n)$ of Cardinal et al. This part has appeared in [3].

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Geometric applications of the Hypergraph Container Method

József Solymosi

(joint work with József Balogh)

In this talk we studied some Erdős type problems in discrete geometry. Our main result was to show that there is a planar point set of n points such that no four are collinear but no matter how we choose a subset of size $n^{5/6+o(1)}$ it contains a collinear triple.

We proved the existence of some geometric constructions with a new tool, the so-called Hypergraph Container Method. This useful method was recently introduced independently by Balogh, Morris and Samotij [4], and by Saxton and Thomason [1]. Roughly speaking, it says that if a hypergraph $\mathcal{H}$ has a uniform edge distribution, then one can find a relatively small collection of sets, *containers*, covering all independent sets in $\mathcal{H}$. One can also require that the container sets span few edges only. In our applications this later condition guarantees that all container sets are small. The right geometric construction determines a hypergraph where all large subsets contain an edge (e.g. a collinear triple).

We refer the readers to [2, 3, 4, 1] for more details and applications on the container method.

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Almost-equidistant sets

Konrad Swanepoel
(joint work with Martin Balko, Attila Pór, Manfred Scheucher, Pavel Valtr)

A subset V of $\mathbb{R}^d$ is *almost equidistant* if for any three points in V, some two are at Euclidean distance 1. The problem of determining the largest number $f(d)$ of points in an almost-equidistant set in $\mathbb{R}^d$ comes from the work of Károly Bezdek and others [2, 1]. It was known that $f(2) = 7$ and $f(d) \geq 2d + 3$ for all $d \geq 3$ [2, 1]. We show that the only almost-equidistant set of 7 points in $\mathbb{R}^2$ is the Moser spindle. We show that $f(3) = 10$, again with a unique almost-equidistant set of 10 points in $\mathbb{R}^3$, and $12 \leq f(4) \leq 13$. The above results are based on a computer search, except for the lower bound of $f(4) \geq 12$, which is a generalization of the 10-point example in $\mathbb{R}^3$ and can be further generalized to give $f(d) \geq 2d+4$ for all $d \geq 3$. Our main result is to show that $f(d) = O(d^{3/2})$ by using a linear algebra method. The previous best known upper bound was given by the Ramsey number $f(d) < R(3, d + 2) = O(d^2/\log d)$.

The upper bound has since been improved to $O(d^{13/9})$ by Alexandr Polyanskii and, after the talk was given, to $O(d^{4/3})$ by Andrey Kupavskii, Nabil Mustafa and Konrad Swanepoel.

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New upper bounds for the density of translative packings of three-dimensional convex bodies

Frank Vallentin
(joint work with M. Dostert, C. Guzmán, D. de Laat, F.M. de Oliveira Filho)

Many geometric packing problems can be formulated as finding the independence number in some finite or infinite Cayley graph. Generally, a *Cayley graph* $\mathrm{Cayley}(G, \Sigma)$ is defined by a group G and a subset $\Sigma \subseteq G$. Here we shall focus on Abelian groups. The vertices of the Cayley graph are the elements of G and the neighborhood of the identity element 0 is Σ. Now this neighborhood is transported to all other group elements via the group action: x and y are adjacent whenever $x - y \in \Sigma$. We want to work with undirected graphs so we additionally require that $\Sigma = -\Sigma$ holds.

We shall consider two running examples: The 5-cycle graph C_5 defined by $C_5 = \mathrm{Cayley}(\mathbb{Z}/5\mathbb{Z}, \{1, 4\})$ as a simple model, and (our main point of concern) the graph $\mathrm{Cayley}(\mathbb{R}^n, \mathcal{K}^\circ)$ where $\mathcal{K}^\circ$ denotes the topological interior of a centrally symmetric convex body $\mathcal{K} \subseteq \mathbb{R}^n$.

A subset I of the vertex set G is called *independent* when every pair x, y of distinct elements in I is not adjacent. Independent sets determine packings; for example an independent set in $\mathrm{Cayley}(\mathbb{R}^n, \mathcal{K}^\circ)$ gives the translates of the body $\frac{1}{2}\mathcal{K}$ which form a packing, and vice versa. The graph parameter α—the independence number—measures the size of a largest independent set, e.g. we define

$$\alpha(\mathrm{Cayley}(\mathbb{R}^n, \mathcal{K}^\circ)) = \text{maximal density of packing of translates of } \frac{1}{2}\mathcal{K}.$$

We are interested in finding good upper bounds for α. A sequence of decreasing upper bounds converging to α is given by a generalization of Lasserre's hierarchy (sum-of-squares hierarchy) from the area of polynomial optimization [5]. Every step in the hierarchy gives a rigorous upper bound. The first step coincides with the ϑ-number of Lovász (more precisely the ϑ'-number of Schrijver, a variation of ϑ having additional inequality constraints). As an exercise we compute the value of $\vartheta(C_5)$ with the help of the discrete Fourier transform. This will give a formal derivation of the Cohn-Elkies linear programming bound [1]. We have

$$\begin{aligned} \alpha(G) \leq \vartheta(G) = \min \quad & M \\ & B - J && \text{is positive semidefinite,} \\ & B(x,x) = M && \text{for all } x \in V, \\ & B(x,y) = 0 && \text{for all } \{x,y\} \notin E \text{ where } x \neq y, \\ & M \in \mathbb{R},\ B \in \mathbb{R}^{V \times V} && \text{is symmetric,} \end{aligned}$$

where J is the matrix where all entries are equal to 1. We specialize to the 5-cycle and get for B

$$B = \begin{pmatrix} M & b_{01} & 0 & 0 & b_{04} \\ b_{01} & M & b_{12} & 0 & 0 \\ 0 & b_{12} & M & b_{23} 0 & \\ 0 & 0 & b_{23} & M & b_{34} \\ b_{04} & 0 & 0 & b_{34} & M \end{pmatrix}.$$

For the following cyclic permutation C we compute $C^\mathsf{T} B C$:

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad C^\mathsf{T} B C = \begin{pmatrix} M & b_{12} & 0 & 0 & b_{01} \\ b_{12} & M & b_{23} & 0 & 0 \\ 0 & b_{23} & M & b_{34} & 0 \\ 0 & 0 & b_{34} & M & b_{04} \\ b_{01} & 0 & 0 & b_{04} & M \end{pmatrix}.$$

Note that if B is a feasible solution for $\vartheta(C_5)$ then so is $C^\mathsf{T} B C$ because both matrices have the same eigenvalues and the same pattern of zeros. Furthermore, their objective values coincide. Now we take the group average and get

$$\frac{1}{5}\sum_{k=0}^{4} (C^k)^\mathsf{T} B C^k = \begin{pmatrix} M & b & 0 & 0 & b \\ b & M & b & 0 & 0 \\ 0 & b & M & b & 0 \\ 0 & 0 & b & M & b \\ b & 0 & 0 & b & M \end{pmatrix} \text{ with } b = \frac{1}{5}(b_{01} + b_{12} + b_{23} + b_{34} + b_{04}).$$

Hence, $\frac{1}{5}\sum_{k=0}^{4}(C^k)^{\mathsf{T}}(B-J)C^k$ is a *circulant matrix* $\mathrm{circ}(M-1, b-1, -1, -1, b-1)$ and

$$Y = \mathrm{circ}(y_0, y_1, \ldots, y_{n-1}) = \begin{pmatrix} y_0 & y_1 & \cdots & y_{n-1} \\ y_{n-1} & y_0 & \cdots & y_{n-2} \\ \ddots & \ddots & \ddots & \ddots \\ y_1 & y_2 & \cdots & y_0 \end{pmatrix} = (Y_{rs})_{r,s} = (y_{s-r})_{r,s} \in \mathbb{C}^{n\times n}.$$

Practically every matrix theoretic question for circulant matrices can be solved in closed form. Circulant matrices have a common system of eigenvectors. To see this we set $\omega_n = e^{2\pi i/n}$ and

$$\chi_a = (\omega_n^{-a\cdot 0}, \omega_n^{-a\cdot 1}, \ldots, \omega_n^{-a\cdot(n-1)})^{\mathsf{T}} \in \mathbb{C}^n, a = 0, \ldots, n-1.$$

Now the eigenvalues of Y are discrete Fourier coefficients of the vector $(y_0, \ldots, y_{n-1})$:

$$\begin{aligned}(Y\chi_a)_r = \sum_{s=0}^{n-1} y_{s-r}\omega_n^{-a\cdot s} &= \omega_n^{-a\cdot r}\sum_{s=0}^{n-1} y_{s-r}\omega_n^{-a\cdot(s-r)} \\ &= \omega_n^{-a\cdot r}\sum_{s=0}^{n-1} y_s\omega_n^{-a\cdot s} = \omega_n^{-a\cdot r}\widehat{y}(a),\end{aligned}$$

where $\widehat{y}(a) = \sum_{s=0}^{n-1} y_s\omega_n^{-a\cdot s}$.

Back to the computation of $\vartheta(C_5)$:

$$\begin{aligned}\vartheta(C_5) = \min \quad & M \\ & \mathrm{circ}(y_0, y_1, y_2, y_3, y_4) \quad \text{is positive semidefinite,} \\ & \text{with } y = (M-1, b-1, -1, -1, b-1).\end{aligned}$$

We compute the discrete Fourier coefficients and get *linear* constraints for the eigenvalues:

$$\begin{aligned}\widehat{y}(0) &= M + 2b - 5 \geq 0 \\ \widehat{y}(1) &= M + \omega_n^{-1\cdot 1}b + \omega_n^{-1\cdot 4}b + \textstyle\sum_{k=0}^{4}\omega_n^{-1\cdot k} \\ &= M + b(\cos(-2\pi/5) + i\sin(-2\pi/5) + \cos(-8\pi/5) + i(\sin -8\pi/5)) \\ &= M + 2b\cos(-2\pi/5) \geq 0 \\ \widehat{y}(2) &= M + 2b\cos(-4\pi/5) \geq 0 \\ \widehat{y}(3) &= \widehat{y}(2), \quad \widehat{y}(4) = \widehat{y}(1).\end{aligned}$$

Hence, computing $\vartheta(C_5)$ simplifies to a *linear program* with two variables M and b and three constraints. It is easy to see that the optimum is attained at $\widehat{y}(0) = \widehat{y}(2)$ which yields $M = \sqrt{5} = \vartheta(C_5)$.

This proof generalizes in many ways. For example for other cyclic graphs we get

$$\begin{aligned}\vartheta'(\mathrm{Cayley}(\mathbb{Z}/n\mathbb{Z}), \Sigma)) = \min \quad & y_0 \\ & \widehat{y}(0) \geq n, \;\; \widehat{y}(a) \geq 0, \; a = 1, \ldots, n-1, \\ & y_i \leq 0 \;\; \text{if } i \notin \Sigma.\end{aligned}$$

or formally for infinite Cayley graphs on Abelian groups:

$$\begin{aligned}\vartheta'(\mathrm{Cayley}(\mathbb{R}^n, \mathcal{K}^\circ)) = \min \quad & f(0) \\ & \widehat{f}(0) \geq 1, \;\; \widehat{f}(a) \geq 0, \; a \in \mathbb{R}^n \setminus \{0\}, \\ & f(x) \leq 0 \;\text{ if } x \notin \mathcal{K}^\circ,\end{aligned}$$

where $f \in L^1(\mathbb{R}^n)$ and continuous with Fourier transform

$$\widehat{f}(a) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot a} dx.$$

The optimization problem giving $\vartheta'(\mathrm{Cayley}(\mathbb{R}^n, \mathcal{K}^\circ))$ is the Cohn-Elkies linear programming bound.

Last year, in a breakthrough work, Viazovaska [7] applied this bound and constructed an optimal function f to solve the sphere packing problem in dimension 8. Shortly afterwards, in [2] the problem was also solved in dimension 24. See also [3] and [6].

In my talk I explained how one can use the Cohn-Elkies bound to find new upper bounds for the optimal density of translative packings of superballs ($\mathcal{K} = \{x \in \mathbb{R}^3 : \|x\|_p \leq 1\}$) and of translative packings of several polytopes which have tetrahedral symmetries. For this techniques from real algebraic geometry (sums of squares) and from the algebraic theory of finite reflection groups turned out to be crucial [4].

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On the Maximum Crossing Number

Pavel Valtr

(joint work with Markus Chimani, Stefan Felsner, Stephen Kobourov, Torsten Ueckerdt, and Alexander Wolff)

Alpert, Feder, and Harborth (2009) conjectured that any graph has a convex straight-line drawing, e.g., a drawing with vertices in convex position, that maximizes the number of edge crossings. In the talk an outline of a disproof of this conjecture was given by constructing a (planar) graph on twelve vertices that allows a non-convex drawing with more crossings than any convex one.

A superlinear lower bound on the number of 5-holes

Birgit Vogtenhuber
(joint work with Oswin Aichholzer, Martin Balko, Thomas Hackl, Jan Kynčl, Irene Parada, Manfred Scheucher, Pavel Valtr)

Let P be a finite set of points in the plane in *general position*, that is, no three points of P are on a common line. We say that P is in *convex position* if it is the vertex set of a convex polygon. We say that a set H of k points from P is a *k-hole in P* if H is the vertex set of a convex polygon containing no other points of P.

In the 1970s, Erdős [6] asked whether, for every positive integer k, there is a k-hole in every sufficiently large finite point set in general position in the plane. Harborth [8] proved that there is a 5-hole in every set of 10 points in general position in the plane and gave a construction of 9 points in general position with no 5-hole. After unsuccessful attempts of researchers to answer Erdős' question affirmatively for any fixed integer $k \geq 6$, Horton [9] constructed, for every positive integer n, a set of n points in general position in the plane with no 7-hole. The question whether there is a 6-hole in every sufficiently large finite planar point set remained open until 2007 when Gerken [7] and Nicolás [10] independently gave an affirmative answer.

For positive integers n and k, let $h_k(n)$ be the minimum number of k-holes in a set of n points in general position in the plane. Due to Horton's construction [9], $h_k(n) = 0$ for every n and every $k \geq 7$. The functions $h_3(n)$ and $h_4(n)$ are both known to be asymptotically quadratic [2, 4]. For $h_5(n)$ and $h_6(n)$, the best known asympotic bounds are $\Omega(n)$ and $O(n^2)$ [4, 7, 8, 10]. See, e.g., [2] for more details.

As our main result, we give the first superlinear lower bound on $h_5(n)$. This solves an open problem, which was explicitly stated, for example, in a book by Brass, Moser, and Pach [5, Chapter 8.4, Problem 5] and in the survey [1].

Theorem 1. *There is a fixed constant $c > 0$ such that for every integer $n \geq 10$ we have $h_5(n) \geq cn \log^{4/5} n$.*

Let P be a finite set of points in the plane in general position and let ℓ be a line that contains no point of P and that partitions P into two non-empty subsets A and B. We then say that $P = A \cup B$ is *ℓ-divided*.

The following result, which might be of independent interest, is a crucial step in the proof of Theorem 1.

Theorem 2. *Let $P = A \cup B$ be an ℓ-divided set with $|A|, |B| \geq 5$ and with neither A nor B in convex position. Then there is an ℓ-divided 5-hole in P.*

The proof of Theorem 2 is computer-assisted. We reduce the result to several statements about point sets of size at most 11 and then verify each of these statements by an exhaustive computer search. To verify the computer-aided proofs we have implemented two independent programs, which, in addition, are based on different abstractions of point sets. Both programs are available online [11, 3].

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New bounds for point-curve incidences in the plane

Joshua Zahl
(joint work with Micha Sharir)

A classical problem in combinatorial geometry is to determine the maximum number of incidences between a set of m points and n curves in the plane. If the curves are lines, then Szemerédi and Trotter proved that there could be at most $O(m^{2/3}n^{2/3}+m+n)$ incidences, and this bound is tight. For other classes of curves, very few tight bounds are known. Work in this area progressed in the 80s and 90s, culminating in an incidence bound by Pach and Sharir in 1998 that applies to a very general class of curves. Since then, there have only been improvements for a few specific types of curves.

In this talk I will discuss some new developments that improve upon Pach and Sharir's bound for a broad class of curves. A key innovation is the use of higher-dimensional incidence geometry, coupled with a new way of cutting collections of curves into segments so that the corresponding set of segments is better behaved than the original collection of curves.

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Matchings and covers in families of *d*-intervals and their duals.

Shira Zerbib

A classical theorem of Gallai is that in any family of closed intervals in the real line, the maximal number of disjoint intervals is equal to the minimal number of points piercing all intervals. Tardos and Kaiser extended this result (appropriately modified) to families of d-intervals, namely hypergraphs in which each edge is the union of d intervals. We prove an analogous result for dual d-interval hypergraphs, in which the roles of the points and the edges are reversed. The proof is topological. We also discuss a recent results on the piercing number of families of d-intervals having the (p, q) property. Join works with R. Aharoni, R. Holzman, and T. Kaiser.

Convex equipartitions of colored point sets by discretization

Günter M. Ziegler

(joint work with Pavle V. M. Blagojević and Johanna K. Steinmeyer)

1. The Holmsen–Kynčl–Valculescu Conjecture

In 1989, Alon and Akiyama obtained the following partition theorem for d-colored point sets in $\mathbb{R}^d$:

Theorem 1 (Alon & Akiyama [2]). *Let X be a d-colored set of $|X| = dn$ points in general position in $\mathbb{R}^d$, where each color class has size n.*

Then X can be partitioned into n disjoint "rainbow" $(d-1)$-simplices, that is, into n subsets of size d that capture each color exactly once, and such that the convex hulls are disjoint.

While for $d = 2$ this can be proved by analyzing a shortest perfect matching between red points and blue points, for general d Akiyama and Alon derived this from a topological result, the Ham Sandwich Theorem on equipartition of d measures in $\mathbb{R}^d$, which is famously equivalent to the Borsuk–Ulam Theorem.

The Alon–Akiyama result has inspired a lot of further work, some of it (in particular on planar cases) based on elementary arguments, some of it using topological tools; see, for example, Aichholzer et al. [1], Bespamyatnikh et al. [3], Ito et al. [7], Kano & Kynč [8], Kano et al. [9], and Sakai [11].

Most recently, Holmsen, Kynčl and Valculescu have formulated the following very general conjecture that would subsume most previous work.

Conjecture 1 (Holmsen, Kynčl & Valculescu [6]). *Let X be an m-colored set of $|X| = kn$ points in general position in $\mathbb{R}^d$, which has a partition into n d-colorful k-subsets, where "d-colorful" means that each of them contains points of at least d different colors. (This in particular implies that $k \geq d$ and $m \geq d$.)*

Then X has a partition into n d-colorful "k-islands," that is, k-subsets whose convex hulls are disjoint.

In their paper, Holmsen, Kynčl and Valculescu establish their conjecture for the cases $k \geq m = d = 2$ and $k - 1 = m = d \geq 2$, which then yields that the conjecture holds in general if $d = 2$.

An obvious approach to the conjecture in the special case $m = d$ is to try and derive this by "discretization" from Soberón's 2012 convex equipartition result for d finite measures in $\mathbb{R}^d$:

Theorem 2 (Soberón [12]). *Let $\mu_1, \dots, \mu_d$ be "nice" (absolutely continuous, finite) measures on $\mathbb{R}^d$, and $n \geq 1$, then there is a partition $\mathbb{R}^d = C_1 \cup \dots \cup C_n$ into n convex regions such that*

$$\mu_i(C_j) = \frac{1}{n}\mu_i(\mathbb{R}^d)$$

for $1 \leq i \leq d$ and $1 \leq j \leq n$.

(Stronger versions of this result were obtained by Blagojević & Ziegler [5] and Karasev, Hubard & Aronov [10].)

However, the discretization does not appear to be straightforward; Holmsen et al. note that

> "going from the continuous version to the discrete version seems to require, in many cases, a non-trivial approximation argument, and we do not see how the continuous results [...] could be used to settle our Conjecture 3 for the case $m = d$."

Nevertheless, in our lecture we announced that this can be done, and gave a very brief sketch of the proof.

2. Main Result and a Sketch of a Proof

We confirm Conjecture 1 for $m = d$, by a non-trivial approximation argument from a continuous result, as a direct corollary of our main theorem, which reads as follows:

Theorem 3. *Let X be an m-colored set of $|X| = kn$ points in general position in $\mathbb{R}^d$.*

Then X has a partition into n k-islands that "equipart" all color classes in the sense that each island contains either $\lfloor |X_i|/n \rfloor$ or $\lceil |X_i|/n \rceil$ points from each color class X_i.

Sketch of a proof.
(1) As usual (see e.g. [2]), we replace each of the points by an ε-ball, with $\varepsilon > 0$ small enough such that no hyperplane in $\mathbb{R}^d$ meets more than d balls, and from this obtain an absolutely continuous finite measure μ_i that is supported exactly on the balls centered at points of color i. Theorem 2 now yields a convex equipartition. If each of the little balls is contained in one single region, then we are done. But this will happen very rarely; in particular, it can happen only if the color classes have sizes that are multiples of n, that is, if $|X_i| = \kappa_i n$ for *integers* κ_i.

(2) Now we set up a bipartite graph with vertex sets $X \uplus W$ and Y. Here $X = X_1 \cup \dots \cup X_m$ is identified with the (m-colored) set of points, while $Y = Y_1 \cup$

$\cdots \cup Y_n$ has n parts that correspond to the regions, where Y_j contains $\lceil \kappa_i \rceil$ vertices that may be seen as "containers" for points of color i. This, however, altogether produces more than k containers in Y_j; to compensate for this, we create a set $W = W_1 \cup \cdots \cup W_n$, where each W_j contains $\sum_i \lceil \kappa_i \rceil - k$ "white points" that will eventually fill superfluous containers in region C_j.

Edges in this bipartite graph connect colored points in X to containers for points of that color in Y, but only if the ε-ball of the point intersects the region of the container. Edges connect white the points in W_j to specific containers in Y_j, where only one container for every "fractional" color (with non-integral κ_i) and region gets an edge to the white points in W_j.

Finally we assign positive weights to the edges, which for edges between X and Y are given by the amount of measure of an ε-ball that falls into a certain region, and for edges between Z and Y depend on the fractional parts of $\kappa_1, \ldots, \kappa_m$. We then verify that these weights form a fractional perfect matching.

(3) Now we use the (elementary) result that any bipartite graph that has a fractional perfect matching also has an integral perfect matching. This perfect matching tells us how to assign the points in X to the regions C_j, and hence to form the k-islands.

The proof is then completed by checking that (due to general position) the islands that are created from the interior points and the selected boundary points in a region C_j are disjoint. □

Details for this proof are documented in [13]. A simpler proof (using integer rounding of flows in a three-layer network) arose from the discussions with Günter Rote after the talk in Oberwolfach; it is presented in [3].

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Open problems

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"Given n slabs in $\mathbb{R}^d$ of total divergent width, can one cover the unit ball with their translates?"
In more details: is it true that there exists $C = C(d)$, such that for any $n_1, \ldots, n_s \in S^{d-1}$, $d > 2$, and any $\varepsilon_1, \ldots, \varepsilon_s \in \mathbb{R}_+$ with $\sum_{i=1}^{s} \varepsilon_i > C$, there exist $x_1, \ldots, x_s \in \mathbb{R}$ satisfying

$$\{x \in \mathbb{R}^d : \ |x| \leq 1\} \subseteq \{x \in \mathbb{R}^d : \ x_i \leq \langle x, n_i \rangle \leq x_i + \varepsilon_i\}?$$

Asked in [Makai–Pach, 1983].

E. Makai and J. Pach, Controlling function classes and covering Euclidean space, *Studia Sci. Math. Hungar.* **18** (1983), no. 2–4, 435–459.

2. Dömötör Pálvölgyi, dom@cs.elte.hu

Can we 3-color any (finite) set of points such that any disk with at least 3 points is non-monochromatic? Asked originally in [Keszegh, 2012].

B. Keszegh, Coloring half-planes and bottomless rectangles, *Computational Geometry: Theory and Applications*, **45(9)** (2012), 495–507.

3. Eran Nevo, nevo@math.huji.ac.il

Fix d even, and let $n \to \infty$:
Must d-polytopes with n vertices have only $o(n^{d/2})$ non-simplex facets? (The trivial upper bound is $O(n^{d/2})$.)

Jeff Erickson asked this in 1999, and conjectured that the answer is yes, also for $(d-1)$-polyhedral spheres.
For spheres the answer is no - as was proved in [Nevo–Santos–Wilson, 2016]

The case $d = 4$ of the above question is already very interesting. The lower bound obtained in Nevo *et al.* is $\Omega(n^{3/2})$.

E. Nevo, F. Santos and S. Wilson, Many triangulated odd-dimensional spheres, *Math. Ann.* **364** (2016), no. 3–4, 737–762.

4. Arseniy Akopyan, akopjan@gmail.com

Let P_1 and P_2 be two combinatorially equivalent convex polytopes in $\mathbb{R}^3$. Is it true that there exist corresponding edges t_1 of P_1 and t_2 of P_2, such that the dihedral angle of t_1 is not greater than the dihedral angle of t_2, or all the corresponded angles are equal? This problem is Conjecture 5.1 in the following preprint.

A. V. Akopyan and R. N. Karasev, *Bounding minimal solid angles of polytopes*, 2015, http://arxiv.org/abs/1505.05263.

5. Micha Sharir, michas@post.tau.ac.il

Danzer's problem. A finite set of pairwise intersecting disks in the plane can be stabbed by 4 points, and there exists a configuration of 10 pairwise intersecting disks that require 4 points [Danzer, 1986].

The problem:
(a) Understand Danzer's solution.
(b) Come up with a simpler solution.
(c) Make it constructive.

L. Danzer, Zur Lösung des Gallaischen Problems über Kreisscheiben in der Euklidischen Ebene [On the solution of the Gallai problem on circular disks in the Euclidean plane, in German], *Studia Sci. Math. Hungar.* **21** (1986), no. 1–2, 111–134.

6. Xavier Goaoc, goaoc@univ-mlv.fr

Fact: For any probability measure μ that charges no lines, there exist two order types $\omega_1(\mu)$ and $\omega_2(\mu)$ of size 6 such that if X is a set of 6 points $\sim \mu$ then

$$\mathbb{P}[X \text{ realizes } \omega_1(\mu)] > 1.8\ \mathbb{P}[X \text{ realizes } \omega_2(\mu)].$$

X. Goaoc, A. Hubard, R. de Joannis de Verclos, J-S. Sereni, and J. Volec, Limits of order types, *Proceedings of Symp. of Computational Geometry* (SOCG), vol 34, pp 300-314, 2015.

Question: Does there exist $c > 0$ such that $\forall \mu\ \exists\ \omega_1(\mu),\ \omega_2(\mu)$ with $|\omega_1(\mu)| = |\omega_2(\mu)| = n$ and

$$\mathbb{P}[X \simeq \omega_1] > c^n\ \mathbb{P}[X \simeq \omega_2(\mu)]?$$

7. Géza Tóth, geza@renyi.hu

Is the class of intersection graphs of lines in $\mathbb{R}^3$ (or $\mathbb{R}^d$) χ-bounded? Namely, is there a function f such that given n lines in the $\mathbb{R}^3$, no k of them pairwise crossing, the lines can be colored with $f(k)$ colors in such a way that crossing lines get different colors?

J. Pach, G. Tardos, and G. Tóth, Disjointness graphs of segments, *Proc. 33rd Annual Symposium on Computational Geometry (SoCG 2017)*, to appear.

8. Imre Bárány, barany@renyi.hu

k-crossing curves in $\mathbb{R}^d$. A curve γ in $\mathbb{R}^d$ is k-crossing if every hyperplane intersects it at most k times. Thus $k \geq d$. A d-crossing curve is called *convex*.

Theorem (Bárány, Matoušek, Pór)**.** *For every $d \geq 2$ there is $M(d)$ such that every $(d+1)$-crossing curve in $\mathbb{R}^d$ can be split into $M(d)$ convex curves.*

The proof gives $M(d) \leq 4^d$, $M(2) = 4$ and $M(3) \leq 22$.

Question: Give lower bounds for $M(d)$.

I. Bárány, J. Matoušek, A. Pór, Curves in $\mathbb{R}^d$ intersecting every hyperplane at most $d+1$ times, *J. Eur. Math. Soc. (JEMS)* **18** (2016), no. 11, 2469–2482.

9. Pavel Valtr, valtr@kam.mff.cuni.cz

Lines, line-point incidences, and crossing families in dense sets. Let P be a set of n points in $\mathbb{R}^2$ such that $\min \text{dist}(P) = 1$ and $\max \text{dist}(P) = O(\sqrt{m})$. Prove or disprove:

Conjecture 1. *P contains a crossing family of size $\Omega(n)$.*

Known: P contains a crossing family of size $\Omega(n^{1-\varepsilon})$.
Two lines are *essentially different* if either their direction differ by at least $1/n$, or their $\frac{1}{\sqrt{n}}$-neighborhoods do not intersect inside $\text{conv}(P)$.

Conjecture 2. *P determines $\Omega(n^2)$ pairwise essentially different lines.*

Known: P determines $\Omega(n^{2-\varepsilon})$ pairwise essentially different lines.
A point p and a line ℓ determine a *rough point-line incidence* if $\text{dist}(p, \ell) \leq \frac{1}{\sqrt{n}}$.

Conjecture 3. *Let P as before and L a set of n pairwise essentially different lines. Then the number of rough point-line incidences is at least $\Omega(n^{4/3})$.*

10. Luis Montejano, luis@matem.unam.mx

Let X be a polyhedron. Let $\mathcal{F} = \{A_1, \dots A_m\}$ be a polyhedral cover of X such that A_i is not empty but not necessarily connected. Let N be the nerve of $\mathcal{F}$.

Fact: Suppose that the following hold: (a) $H_1(X) = 0$, and (b) for every $i \neq j$, if $A_i \cap A_j \neq \emptyset$ then $A_i \cup A_j$ is connected. Then $H_1(N) = 0$.

Question: Suppose that the following hold: (a) $H_2(X) = 0$, (b) for every $i \neq j$, if $A_i \cap A_j \neq \emptyset$ then $A_i \cup A_j$ is connected, and (c) for every $i < j < k$, if $A_i \cap A_j \cap A_k \neq \emptyset$ then $H_1(A_i \cup A_j \cup A_k) = 0$. Is it true that $H_2(N) = 0$? The answer is yes if $m = 4$.

11. József Solymosi, solymosi@math.ubc.ca

Question 1: What is the minimum number of collinear triples in a subset of the integer grid $n \times n \times n$? If $|S| = n^{3-s}$, $S \subset n \times n \times n$, then S spans at least $\frac{n^{6-4s}}{c \log n}$ collinear triples. We (with Jozsi Balogh) do not think that this is sharp.

Question 2: Find a bipartite unit distance graph which is rigid.

12. Edgardo Roldán-Pensado, e.roldan@im.unam.mx

Centre of $BM(2)$. Let δ be the Banach-Mazur distance. Find the convex body $C \subset \mathbb{R}^2$ such that $\max\{\delta(C, D) : D \subset \mathbb{R}^2 \text{ a convex body}\}$ is minimized.

An update by Edgardo Roldán-Pensado: The answer to the problem is known. A solution appears in:
Stromquist, Walter. "The maximum distance between two-dimensional Banach spaces." Mathematica Scandinavica (1981): 205-225.

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