Mathematisches Forschungsinstitut Oberwolfach

Report No. 11/2021

DOI: 10.4171/OWR/2021/11

# Hyperbolic Balance Laws: modeling, analysis, and numerics (hybrid meeting)

Organized by
Remi Abgrall, Zürich
Mauro Garavello, Milano
Mária Lukáčová-Medvid'ová, Mainz
Konstantina Trivisa, College Park

28 February – 6 March 2021

Abstract. This workshop brought together leading experts, as well as the most promising young researchers, working on nonlinear hyperbolic balance laws. The meeting focused on addressing new cutting-edge research in modeling, analysis, and numerics. Particular topics included ill-/well-posedness, randomness and multiscale modeling, flows in a moving domain, free boundary problems, games and control.

*Mathematics Subject Classification (2010):* 35L65, 35L40, 35L60, 35R60, 65M08, 60H15.

## Introduction by the Organizers

What are suitable admissibility criteria for a physically reasonable solution of multidimensional hyperbolic balance laws and the Euler system in particular? How this question affects complex models for multiphase flows, random systems or structure-preserving numerical schemes? Can we use hyperbolic balance laws to predict the evolution of a pandemic? These and many other challenging questions were discussed during the hybrid workshop *Hyperbolic Balance Laws: modeling, analysis, and numerics* that was organized by Rémi Abgrall (Zürich), Mauro Garavello (Milano), Mária Lukáčová (Mainz) and Konstantina Trivisa (College Park).

Due to the pandemic restrictions only a limited number of participants could attend the workshop at the institute. Altogether 54 participants attended, 45 of them virtually. The participants were from Europe, USA, Saudi Arabia and China. Due to time differences and in order to allow the participants to listen to all talks, the talks were recorded. This procedure was positively acknowledged

by the participants. Excellent video conference equipment at the MFO as well as the support of the IT staff in addition to the video conference assistants make the workshop realization very efficient and less affected by its hybrid form.

The program consisted of 23 lectures, supplemented by 6 shorter contributions by young scientists. It covered diverse areas of nonlinear hyperbolic equations: the Euler and shallow water equations, ill-posedness and turbulence, collective dynamics, stochastic hyperbolic problems and uncertainty quantification, structure-preserving numerical methods, active flux finite volume and residual distribution methods, multiphase, multiscale and plasticity problems. This report contains extended abstracts of all speakers illustrating variety of themes and exciting new developments in analysis and numerics. Stimulating discussions during the coffee breaks led to new collaborations.

The organizers wish to thank the MFO staff for comfortable, creative atmosphere and opportunity to realize hybrid workshop in this difficult time for collaborative research.

## Workshop (hybrid meeting): Hyperbolic Balance Laws: modeling, analysis, and numerics

### Table of Contents

# Abstracts

## Well Posedness and Control in Models Based on Conservation Laws

Rinaldo M. Colombo
(joint work with M. Garavello, F. Marcellini, V. Perrollaz and E. Rossi)

Conservation/Balance Laws are at the heart of a variety of models ranging, for instance, from fluid dynamics (that provided their original motivation), to vehicular traffic, to crowd dynamics and to a multitude of biological situations. The resulting fruitful interplay between general well posedness results and models' specific requests has been driving research in this area since several years. More recently, also control theory started developing along these two lines.

Below, we briefly present recent results, quite different in nature, but all rooted in questions concerning well posedness and control problems centered in conservation/balance laws.

**Backward Integration**. Consider the Cauchy Problem for a scalar convex flow:

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 \\ u(0,x) = u_o(x)\,. \end{cases} \tag{1}$$

Fix a positive time $T$ and a function $w \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$. Define the set $\mathcal{I}_T(w)$ of initial data $u_o$ whose corresponding solution to (1) attains the profile $w$ at time $T$, i.e.,

$$\mathcal{I}_T(w) = \{u_o \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})\colon u \text{ solves (1) and } u(T) = w\}\,. \tag{2}$$

The results in [5] provide a characterization of those $w$ such that $\mathcal{I}_T(w)$ is not empty, characterize the elements of $\mathcal{I}_T(w)$ in terms of integral inequalities and prove geometrical/topological properties of $\mathcal{I}_T(w)$. Instrumental in the proofs is the relation between (1) and the Hamilton–Jacobi Cauchy Problem (see also [8]), while the key tools are Dafermos theory of characteristics [7], Oleinik estimate [12] and Lax Formula [10]. Refer, for instance, to [1] for applications of these results.

**Biological Pest Control**. Some sort of *"pests"* infest a given cultivated region. A technique to reduce the damages caused by these pests relies on the careful introduction of specific predators feeding on them. We are thus lead to consider the following system where predators ($u$) feed on prey ($w$) and the former are inserted in the environment through a time and space dependent control $q = q(t,x)$:

$$\begin{cases} \partial_t u + \operatorname{div}(v(t,x,w)\,u) = f(t,x,w)\,u + q(t,x) \\ \partial_t w - \mu\,\Delta w = g(t,x,u,w)\,w \end{cases} \tag{3}$$

The vector $v$ describes the hunting strategy of the predators and typically is a *non local* function of the prey distribution $w$. $f$ and $g$ describe the effects of predation, mortality, natality, etc. The movement of prey is essentially diffusive. As stated in [6], one can prove the well posedness of the Cauchy Problem for (3) and the

stability of solutions w.r.t. the control $q$, under assumptions that comprise, for instance, the following speed $v$ and with $f$ and $g$ as in the Lotka-Volterra system:

$$v(t,x,w) = \kappa(t,x)\,\frac{\nabla(w*\eta)(x)}{\sqrt{1+\|\nabla(w*\eta)(x)\|^2}} \qquad \begin{array}{c} f(t,x,w) = \alpha\, w - \beta \\ g(t,x,u,w) = \gamma\, w - \delta\, u\, w\,. \end{array} \tag{4}$$

Numerical integrations show that different choices of $q$, while keeping the contribution $\int_{\mathbb{R}^2}\int_0^T q(t,x)\,dx\,dt$ fixed, provoke significant differences in pests' proliferation.

**Vaccination Strategies**. Since the famous paper [9], see also [11, 13], SIR type models have provided a standard framework to model the propagation of various diseases. Within this framework, vaccinations can be introduced and different vaccination strategies can be compared and tested. An attempt in this direction is described in [2], which presents the age structured SIR model

$$\begin{cases} \partial_t S + \partial_a S = -\, d_S(t,a)\, S - \int_0^{+\infty} \lambda(a,a')\, I(t,a')\, da'\, S \\ \partial_t I + \partial_a I = -\, d_I(t,a)\, I + \int_0^{+\infty} \lambda(a,a')\, I(t,a')\, da'\, S - r(t,a)\, I \\ \partial_t R + \partial_a R = -\, d_R(t,a)\, R + r(t,a)\, I\,. \end{cases} \tag{5}$$

Here, $d_S, d_I, d_R$ are the mortalities, $\lambda$ describes infection propagation and $r$ is the recovery rate. A vaccination campaign where at times $t_1, t_2, \ldots, t_N$ individual of all ages are dosed and immediately immunized introduces in (4) the conditions

$$\begin{cases} S(\bar t_j+, a) = (1-\nu(t))\; S(\bar t_j-, a) \\ I(\bar t_j+, a) = I(\bar t_j-, a) \\ R(\bar t_j+, a) = R(\bar t_j-, a) + \nu(t)\, S(\bar t_j-, a) \end{cases} \qquad j = 1, \ldots, N\,. \tag{6}$$

We refer to [2] for further models and for sample numerical integrations.

In the case of the Covid-19 pandemic, the above model needs to be modified. The introduction of quarantine is accomplished in [3], while a model considering where an age dependent time between the first dose and immunization is in [4].

## References


[1] E. Carlos, E. Zuazua: *The Inverse Problem for Hamilton–Jacobi Equations and Semiconcave Envelopes*, SIAM J. Math. Anal. **52** (2020), 5627-5657.

[2] R.M. Colombo, M. Garavello: *Optimizing Vaccination Strategies in an Age Structured SIR Model*, Mathematical Biosciences and Engineering, **17** (2020), 1074-1089.

[3] R.M. Colombo, M. Garavello, F. Marcellini, E. Rossi, *An Age and Space Structured SIR Model Describing the Covid-19 Pandemic*, J. of Mathematics in Industry **10** (2020), 22.

[4] R.M. Colombo, F. Marcellini, E. Rossi: In preparation, 2021.

[5] R.M. Colombo, V. Perrollaz: *Initial Data Identification in Conservation Laws and Hamilton-Jacobi Equations*, J. de Mathématiques Pures et Appliquées **9** (2020), 1-27.

[6] R.M. Colombo, E. Rossi: *A Modeling Framework for Biological Pest Control*, Mathematical Biosciences and Engineering, **17** (2020), 1413-1427.

[7] C.M. Dafermos: *Generalized characteristics and the structure of solutions of hyperbolic conservation laws*, Indiana Univ. Math. J. **26** (1977), 1097-1119.

[8] K.H. Karlsen, N.H. Risebro: *A note on front tracking and equivalence between viscosity solutions of Hamilton-Jacobi equations and entropy solutions of scalar conservation laws*, Nonlinear Anal. Ser. A: Theory Methods **4** (2002), 455-469.

[9] W.O. Kermack, A.G. McKendrick, A.G. Walker: *A contribution to the mathematical theory of epidemics*, Proc. Royal Society of London. Series A, **115** (1927), 200-721.

[10] P.D. Lax: *Hyperbolic systems of conservation laws II*, Comm. Pure Appl. Math., **10** (1957), 537-566.
[11] J.D. Murray: *Mathematical biology. I*, Springer-Verlag, 2002.
[12] O.A. Oleinik: *Discontinuous solutions of non-linear differential equations* Usp. Mat. Nauk, **12** (1957), 3-73.
[13] B. Perthame: *Transport equations in Biology*, Birkhäuser, 2007.

## Obstacle problem, Euler system and turbulence

Eduard Feireisl
(joint work with Martina Hofmanová)

Our aim is to clarify the commonly used statement in the physics literature: *"The effect of the boundary in the turbulence regime can be modeled in a statistically equivalent way by fluid equations driven by stochastic forcing"*, see e.g. Yakhot and Orszag [2]. We consider the problem of the compressible Navier–Stokes flow around a convex obstacle:

$$\partial_t \rho + \nabla \cdot (\rho \vec{u}) = 0$$

$$\partial_t (\rho \vec{u}) + \nabla \cdot (\rho \vec{u} \otimes \vec{u}) + \nabla p(\rho) = \nabla \cdot \mathbb{S}(\nabla \vec{u})$$

$$p(\rho) \approx a\rho^\gamma,\ \gamma > 1,\ \mathbb{S} = \mu \left( \nabla \vec{u} + \nabla^t \vec{u} - \frac{2}{d} \nabla \cdot \vec{u} \mathbb{I} \right) + \lambda \nabla \cdot \vec{u} \mathbb{I},\ \mu > 0, \lambda \geq 0$$

in $(0,T) \times Q$, where $Q = R^d \setminus B,\ d = 2, 3,\ B$ compact, convex, with the following boundary and far field conditions

$$\vec{u}|_{\partial Q} = 0,\ \rho \to \rho_\infty,\ \vec{u} \to \vec{u}_\infty \text{ as } |x| \to \infty.$$

We consider the high Reynolds number limit,

$$\epsilon_n \searrow 0,\ \mu_n = \epsilon_n \mu, \mu > 0,\ \lambda_n = \epsilon_n \lambda, \lambda \geq 0.$$

Introducing the (relative) energy

$$E\left(\rho, \vec{u} \,\middle|\, \rho_\infty, \vec{u}_\infty\right) = \frac{1}{2}\rho |\vec{u} - \vec{u}_\infty|^2 + P(\rho) - P'(\rho_\infty)(\rho - \rho_\infty) - P(\rho_\infty),$$

$$P'(\rho)\rho - P(\rho) = p(\rho),$$

we have a statistical sample of solutions satisfying

$$\frac{1}{N} \sum_{n=1}^{N} \left[ \sup_{0 \leq \tau \leq T} \int_Q E\left(\rho_n, \vec{m}_n \middle| \rho_\infty, \vec{u}_\infty\right)(\tau, \cdot) dx + \epsilon_n \int_0^T \int_Q \mathbb{S}(\nabla \vec{u}_n) : \nabla \vec{u}_n \, dx \, dt \right] \leq \overline{\mathcal{E}},$$

$$\vec{m}_n = \rho_n \vec{u}_n,$$

uniformly for $N \to \infty$.

The limit is shown to be a statistical *dissipative solution* of the compressible Euler system,

$$\partial_t \rho + \nabla \cdot \vec{m} = 0$$

$$\partial_t \vec{m} + \nabla \cdot \left( \frac{\vec{m} \otimes \vec{m}}{\rho} \right) + \nabla p(\rho) = -\nabla \cdot \mathfrak{R},$$

where

$$\mathfrak{R} \in L^{\infty}_{\text{weak}-(*)}(0,T;\mathcal{M}^{+}(Q;R^{d\times d}_{\text{sym}})).$$

On the other hand, we introduce the Euler system driven by stochastic forcing,

$$d\rho + \nabla\cdot\vec{m}\ dt = 0,\ d\vec{m} + \nabla\cdot\left(\frac{\vec{m}\otimes\vec{m}}{\rho}\right)dt + \nabla p(\rho)dt = \mathbf{F}dW,$$

where

$$W = (W_k)_{k\geq 1} \text{ is cylindrical Wiener process}$$
$$\mathbf{F} = (\mathbf{F}_k)_{k\geq 1} \ - \ \text{are diffusion coefficients.}$$

We introduce the following concept of *statistical equivalence*:

$$(\rho,\vec{m}) \text{ statistically equivalent to } (\widetilde{\rho},\widetilde{\vec{m}})$$

iff

- $$\mathbb{E}\left[\int_D \rho\right] = \mathbb{E}\left[\int_D \widetilde{\rho}\right],\ \mathbb{E}\left[\int_D \vec{m}\right] = \mathbb{E}\left[\int_D \widetilde{\vec{m}}\right]$$
- $$\mathbb{E}\left[\int_D \frac{|\vec{m}|^2}{\rho}\right] = \mathbb{E}\left[\int_D \frac{|\widetilde{\vec{m}}|^2}{\widetilde{\rho}}\right],\ \mathbb{E}\left[\int_D p(\rho)\right] = \mathbb{E}\left[\int_D p(\widetilde{\rho})\right]$$
- $$\mathbb{E}\left[\int_D \frac{1}{\rho}(\mathbb{J}_{x_0}\cdot\vec{m})\cdot\vec{m}\right] = \mathbb{E}\left[\int_D \frac{1}{\rho}(\mathbb{J}_{x_0}\cdot\widetilde{\vec{m}})\cdot\widetilde{\vec{m}}\right]$$

$$D\subset(0,T)\times Q,\ x_0\in R^d,\ \mathbb{J}_{x_0}(x) \equiv |x-x_0|^2\mathbb{I} - (x-x_0)\otimes(x-x_0)$$

**Conclusion**, see [1]:
If the statistical limit is equivalent to a solution of the stochastic Euler system then:

- **Noise inactive**
  $\mathfrak{R} = 0$, $(\rho,\vec{m})$ is a statistical solution to a *deterministic* Euler system
- **S-convergence (up to a subsequence) to the limit system**

$$\frac{1}{N}\sum_{n=1}^{N} b(\rho_n,\vec{m}_n) \to \mathbb{E}[b(\rho,\vec{m})] \text{ strongly in } L^1_{\text{loc}}((0,T)\times Q)$$

  for any $b\in C_c(R^{d+1})$, $\varphi\in C_c^{\infty}((0,T)\times Q)$
- **Conditional statistical convergence**

$$\text{barycenter } (\overline{\rho},\overline{\vec{m}}) \equiv \mathbb{E}[(\rho,\vec{m})] \text{ solves the Euler system}$$
$$\Rightarrow$$
$$\frac{1}{N}\#\left\{n\leq N \Big| \|\rho_n-\overline{\rho}\|_{L^{\gamma}(K)} + \|\vec{m}_n-\overline{\vec{m}}\|_{L^{\frac{2\gamma}{\gamma+1}}(K;R^d)} > \epsilon\right\} \to 0 \text{ as } N\to\infty$$

  for any $\epsilon>0$, and any compact $K\subset[0,T]\times Q$

References

[1] E. Feireisl and M. Hofmanová. Randomness in compressible fluid flows past an obstacle. **arxiv preprint No. 2101.08250**, 2021.

[2] V. Yakhot and S. A. Orszag. Renormalization group analysis of turbulence. I. Basic theory. *J. Sci. Comput.* **1** (1986), 3–51.

## Magnetohydrodynamic Turbulence: weak solutions and conserved quantities

László Székelyhidi Jr.

(joint work with Daniel Faraco and Sauli Lindberg)

In this talk we present recent work on the construction of weak solutions to the ideal MHD system in three space dimensions, which consists of the incompressible Euler equations coupled to the Maxwell system via Ohm's law. This system has a wealth of interesting structure, including three conserved quantities: the total energy, cross-helicity and magnetic helicity. Whilst the former two are analogous (and analytically comparable) to the total kinetic energy for the Euler system, magnetic helicity is known to be more robust and of a different nature. In particular, when studying weak solutions, Onsager-type conditions for all three quantities are known, and are basically on the same level of 1/3-differentiability as the kinetic energy in the ideal hydrodynamic case for the former two. In contrast, magnetic helicity does not require any differentiability, only $L^3$ integrability. From the physical point of view this difference lies at the heart of the Taylor-Woltjer relaxation theory. From the mathematical point of view it turns out to be closely related to the div-curl structure of the Maxwell system. In the talk we present and compare two recent constructions of weak solutions: one with solutions in the energy space $L^2$ which is supercritical with respect to magnetic helicity [1] and one with bounded solutions, i.e. subcritical with respect to magnetic helicity [2]. Along the way we highlight some of the hidden structures in the ideal MHD system.

References

[1] R. Beekie, T. Buckmaster, V. Vicol, *Weak Solutions of Ideal MHD Which Do Not Conserve Magnetic Helicity*, Ann. PDE **6** (2020), 1-40.

[2] D. Faraco, S. Lindberg, L. Székelyhidi Jr., *Bounded Solutions of Ideal MHD with Compact Support in Space-Time*, Arch.Rat.Mech.Analysis **239** (2021), 51-93.

## Hydrodynamics of Multi-Species

Eitan Tadmor

We study the hydrodynamics of multi-species driven by environmental averaging. What distinguishes one species from another is the way they interact with the environment: let $\phi_{\alpha\beta} \geq 0$ be the *communication kernel* between species $\alpha$ and $\beta$, then the is captured by the hydrodynamic description[1],

$$(1)\quad \begin{cases} \partial_t \rho_\alpha + \nabla \cdot (\mathbf{u}_\alpha \rho_\alpha) = 0; \\ \partial_t (\rho_\alpha \mathbf{u}_\alpha) + \nabla \cdot (\rho_\alpha \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha + \mathbb{P}_\alpha) = \\ \qquad\qquad \sum_{\beta \in \mathcal{I}} \int \phi_{\alpha\beta}(|\mathbf{x}-\mathbf{y}|)\big(\mathbf{u}_\beta(\mathbf{y}) - \mathbf{u}_\alpha(\mathbf{x})\big)\rho_\alpha(\mathbf{x})\rho_\beta(\mathbf{y})\mathrm{d}\mathbf{y}. \end{cases}$$

Each of the different species is identified by a pair of density/velocity $(\rho_\alpha, \mathbf{u}_\alpha)$, subject to initial condition $(\rho_\alpha, \mathbf{u}_\alpha)\big|_{t=0} = (\rho_{\alpha 0}, \mathbf{u}_{\alpha 0}) \in L^1_+(\mathbb{R}^d) \times W^{1,\infty}(\mathbb{R}^d)$, $\forall \alpha \in \mathcal{I}$. We report here on our results [1] for the mono-kinetic closure $\mathbb{P}_\alpha \equiv 0$; the extension for general pressure laws, assuming *uniformly bounded velocity fields* follows along the lines of [2].

There are two extreme cases: when $\phi_{\alpha\beta} \equiv \phi$ the crowd consists of a single species; when $\phi_{\alpha\beta} = \phi\delta_{\alpha\beta}$, the crowd of (1) splits into independent species driven by the same communication kernel. We study all the intermediate cases which involve a genuine *multi*-species dynamics, driven by *symmetric* communication array of radial decreasing kernels, $\Phi = \{\phi_{\alpha\beta}\}$,

$$(2)\qquad \phi_{\alpha\beta} = \phi_{\beta\alpha} \geq 0, \qquad \phi_{\alpha\beta} \text{ are radial and decreasing.}$$

0.1. **Smooth solutions must flock.** Let $\Phi(r) := \{\phi_{\alpha\beta}(r)\}_{\alpha,\beta\in\mathcal{I}}$ denote the array of communication kernels associated with (1). The main feature here is that flocking of multi-species dynamics does *not* require direct, global communication among all species — we allow $\phi_{\alpha\beta}(r)$ to vanish, indicating lack of communication between some species $\alpha$ and $\beta$. Instead, what matters is a minimal requirement that the communication among species forms a *connected network* in the sense that there is a connecting path which propagates the information of alignment between every pair of species. To this end, we introduce the *weighted* graph Laplacian associated with $\Phi(r)$,

$$(3)\qquad (\Delta_{\mathcal{M}}\Phi(r))_{\alpha\beta} := \begin{cases} -\phi_{\alpha\beta}(r)\sqrt{M_\alpha M_\beta}, & \alpha \neq \beta; \\ \sum_{\gamma\neq\alpha} \phi_{\alpha\gamma}(r) M_\gamma, & \alpha = \beta, \end{cases}$$

where the weights, $\mathcal{M} := \{M_\alpha\}_{\alpha\in\mathcal{I}}$, consist of the masses of the different species which are constant in time, $M_\alpha := \int \rho_{\alpha 0}(\mathbf{x})\mathrm{d}\mathbf{x} \equiv \int \rho_\alpha(t,\mathbf{x})\mathrm{d}\mathbf{x} > 0$. The communication array $\Phi(r)$ forms a connected graph as long as its second eigenvalue

[1] Unless otherwise stated, all integrals are taken over $\mathbb{R}^d$.

$\lambda_2\big(\Delta_{\mathcal{M}}\Phi(r)\big) > 0$. Our main result shows that inter-species connectivity implies the flocking behavior of the whole crowd.

**Theorem 1** (**Strong solutions must flock).**
*Let* $(\rho_\alpha(t,\cdot),\mathbf{u}_\alpha(t,\cdot)) \in (L^\infty \cap L^1_+(\mathbb{R}^d)) \times W^{1,\infty}(\mathbb{R}^d)$, $\alpha \in \mathcal{I}$ *be a strong solution of the multi-species dynamics* (1)*, with zero pressure* $\mathbb{P}_\alpha \equiv 0$ *and subject to compactly supported initial conditions* $(\rho_{\alpha 0},\mathbf{u}_{\alpha 0})$ *with finite velocity fluctuations* $\delta V_0 := \max\limits_{\alpha,\beta\in\mathcal{I}} \sup\limits_{\mathbf{x},\mathbf{y}\in S_0} |\mathbf{u}_{\alpha 0}(\mathbf{x}) - \mathbf{u}_{\beta 0}(\mathbf{y})| < \infty$, $\quad \mathcal{S}_0 := \cup_\alpha supp\{\rho_{\alpha 0}(\cdot)\}$. *Assume that the communication array* $\Phi(r) = \{\phi_{\alpha\beta}(r)\}_{\alpha,\beta\in\mathcal{I}}$ *satisfies a Pareto-type 'fat-tail' connectivity condition*

$$\lambda_2(\Delta_{\mathcal{M}}\Phi(r)) \gtrsim \frac{1}{(1+r)^\theta}, \quad \theta < 1. \tag{4}$$

*Then the support,* $\mathcal{S}(t) := \cup_\alpha supp\{\rho_\alpha(t,\cdot)\}$*, remains within a finite diameter* $D_\infty < \infty$ *(depending on* $1-\theta, M, \delta V_0$*), and the different species flock towards a limiting velocity* $\mathbf{u}_\infty$*,*

$$\sum_{\alpha\in\mathcal{I}} \int |\mathbf{u}_\alpha(t,\mathbf{x}) - \mathbf{u}_\infty|^2 \rho_\alpha(t,\mathbf{x})\mathrm{d}\mathbf{x} \le \sum_{\alpha\in\mathcal{I}} \int |\mathbf{u}_{\alpha 0}(\mathbf{x}) - \mathbf{u}_\infty|^2 \rho_{\alpha 0}(\mathbf{x})\mathrm{d}\mathbf{x} \cdot e^{-2\nu t}, \tag{5}$$

*at exponential rate,* $\nu = \dfrac{\zeta_{\mathcal{M}}}{(1+D_\infty)^\theta}$*, dictated by the spatial scale* $D_\infty$ *and* $\zeta_{\mathcal{M}} := 1 - \dfrac{\max_\alpha M_\alpha}{\sum_\alpha M_\alpha} > 0$.

The proof of theorem in [1, Theorem 1.1] is achieved by showing the decay of the *energy fluctuations*,

$$\delta E(t) = \sum_{\alpha,\beta\in\mathcal{I}} \iint |\mathbf{u}_\alpha(t,\mathbf{x}) - \mathbf{u}_\beta(t,\mathbf{y})|^2 \rho_\alpha(t,\mathbf{x})\rho_\beta(t,\mathbf{y})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y},$$

and the decay of *uniform fluctuations*,

$$\delta V(\mathbf{u}(t)) = \max_{\alpha,\beta\in\mathcal{I}} \sup_{\mathbf{x},\mathbf{y}\in\mathcal{S}(t)} |\mathbf{u}_\alpha(t,\mathbf{x}) - \mathbf{u}_\beta(t,\mathbf{y})|, \qquad \mathcal{S}(t) = \cup_\alpha \mathrm{supp}\{\rho_\alpha(t,\cdot)\},$$

imply that the whole crowd of different species remains within a uniformly bounded finite diameter, $D_\infty \le D_0 + C_\theta \cdot \delta V_0 < \infty$ (with $C_\theta \lesssim (1-\theta)^{\frac{\theta}{1-\theta}}$. It follows that the fluctuations, $\delta E(t), \delta V(t)$, decay at exponential rate and that all species 'aggregate' around an invariant limiting velocity $\mathbf{u}_\infty := \dfrac{\mathbf{m_0}}{M}$.

*Remark* 1 (**Why weighted Laplacian?)**. In case of equi-weighted species $M_\alpha \equiv 1$, the weighted Laplacian (3) amounts to the usual graph Laplacian $\Delta\Phi(r)$. Its Fiedler number, $\lambda_2(\Delta\Phi(r))$, quantifies the connectivity of the graph associated with the adjacency matrix $\Phi(r)$. The advantage of using the weighted $\lambda_2(\Delta_{\mathcal{M}}\Phi(r))$, however, is that it provides the right scaling for the decay rate of multi-species dynamics (5), (i) independent of the condition number, $\kappa := \dfrac{\max M_\alpha}{\min M_\alpha}$, $M = \sum\limits_{\alpha\in\mathcal{I}} M_\alpha$, and (ii) independent of the # of different species, $|\mathcal{I}|$.

*Remark* 2 (**Game of alignment**). The graph Laplacian of the communication array $\Phi(r)$ is independent of the self-interacting kernels $\{\phi_{\alpha\alpha} \,|\, \alpha \in \mathcal{I}\}$. Thus, according to theorem 1, flocking can be viewed as the outcome of a 'game' in which agents from one species interact with different species but are *independent* of the interaction with their own kind. A main feature in our multi-species alignment game (of two or more species) is that one can *ignore interactions with its own kind*, i.e., set $\phi_{\alpha\alpha} = 0$ in (1) and yet the information will eventually be reflected through interactions with the other connected species leading to overall flocking.

**Example 1.** *Consider the case of two species with* $2\times 2$ *symmetric communication array,*

$$\Phi = \begin{bmatrix} 0 & \phi_{12}(r) \\ \\ \phi_{21}(r) & 0 \end{bmatrix}, \qquad \phi_{12}(r) = \phi_{21}(r) \gtrsim \frac{1}{(1+r)^{\theta}}, \quad \theta < 1.$$

*In this case, agents in each of the two groups interact with the other group but not with their own kind* ($\phi_{11} = \phi_{22} \equiv 0$). *The large-time behavior of such 'game' leads to flocking.*

## References

[1] S. He and E. Tadmor. A game of alignment: collective behavior of multi-species, Annal. de l'institut Henri Poincare (c) Non Linear Analysis, in press (2020)

[2] E. Tadmor, On the mathematics of swarming: emergent behavior in alignment dynamics, Notices of the AMS **68** (2021), 493-503.

## Operator splitting based central-upwind schemes for shallow water equations with moving bottom topography

Alina Chertock

(joint work with A. Kurganov and T. Wu)

Shallow water models are widely used as a mathematical framework to study water flows in rivers and coastal areas as well as to investigate a variety of phenomena in atmospheric sciences and oceanography. One of the classical shallow water models is the Saint-Venant (SV) system [5], which in the two-dimensional (2-D) case can be written in the following form:

$$\begin{cases} h_t + (hu)_x + (hv)_y = 0, \\ (hu)_t + \left(hu^2 + \dfrac{g}{2}h^2\right)_x + (huv)_y = -ghB_x, \\ (hv)_t + (huv)_x + \left(hv^2 + \dfrac{g}{2}h^2\right)_y = -ghB_y. \end{cases} \tag{1}$$

Here, $h(x,y,t)$ is the fluid depth above the bottom, $u(x,y,t)$ and $v(x,y,t)$ are the $x$- and $y$-velocities, $g$ is the constant gravitational acceleration. We consider the case, which appears in many practical situations, when the bottom topography

$B = B(x, y, t)$ is time-dependent due to erosion, sediment transport, dam breaks, floods and submarine landslides, and is modeled by ([7, 8]):

$$B_t + A\left[u(u^2+v^2)^{(m-1)/2}\right]_x + A\left[v(u^2+v^2)^{(m-1)/2}\right]_y = 0, \tag{2}$$

where $A$ is a constant, which accounts for the porosity of the sediment layer and effects of sediment grain size and kinematic viscosity, and $m \in [1, 4]$ is a constant.

The system (1), (2) is a system of hyperbolic balance laws, which admits non-smooth solutions. Therefore, a numerical method for (1), (2) should be based on a shock-capturing scheme. Moreover, a good numerical method should be well-balanced, i.e., capable of accurately capturing both the steady states and their small perturbations (quasi-steady flows). This property ensures that the scheme suppresses the appearance of unphysical waves of magnitude proportional to the grid size, which are normally present when computing quasi-steady states. An additional difficulty difficulty in solving the coupled system (1), (2) numerically is associated with the fact that the speed of water surface gravity waves are typically much faster than the speed at which the changes in the bottom topography occur. This imposes a severe stability restriction on the size of time steps, which, in turn, leads to excessive numerical diffusion that affects the computed bottom structure; see, e.g., [9, 10, 2, 1, 6].

We propose to overcome the latter difficulty by developing an operator splitting method for the system (1), (2). To this end, we split the SV system (1) from the Exner equation (2). The size of splitting time steps will be made inversely proportional to the amplitude of a smaller eigenvalue of the Jacobians of the extended system (1), (2). We will then follow the approach that was utilized in the framework of the fast explicit operator splitting method [3, 4]: each SV splitting substep will consist of several smaller time evolution steps. This way we will ensure the stability of the SV substeps, while large Exner splitting substeps will prevent excessive numerical dissipation, which may severely affect the resolution of the bottom topography, especially in the case when $B$ is discontinuous. Each of the splitting substeps will be carried out using a second-order well- balanced central-upwind (CU) scheme proposed in [11]. Since this scheme uses a continuous piecewise linear (or bilinear) reconstruction of the bottom topography, the Exner equation will be solved on a staggered grid so that the point values of $B$ will be evolved in time at every finite-volume cell interface, while the cell averages of $h$, $hu$ and $hv$ will be evolved inside each cell.

A number of one- and two-dimensional numerical examples are presented to demonstrate the performance of the proposed method.

## References

[1] F. Benkhaldoun, S. Sahmim, and M. Seaïd, *A two-dimensional finite volume morphodynamic model on unstructured triangular grids*, Internat. J. Numer. Methods Fluids **63** (2010), no. 11, 1296-1327.

[2] M. J. Castro Díaz, E. D. Fernández-Nieto, A. M. Ferreiro, and C. Parés, *Two-dimensional sediment transport models in shallow water equations. A second order finite volume approach on unstructured meshes*, Comput. Methods Appl. Mech. Engrg. **198** (2009), no. 33-36, 2520-2538.
[3] A. Chertock and A. Kurganov, *On splitting-based numerical methods for convection-diffusion equations*, Numerical methods for balance laws, Quad. Mat., vol. 24, Dept. Math., Seconda Univ. Napoli, Caserta, (2009), pp. 303-343.
[4] A. Chertock, A. Kurganov, and G. Petrova, *Fast explicit operator splitting method for convection-diffusion equations*, Internat. J. Numer. Meth. Fluids **59** (2009), 309-332.
[5] A. J. C. de Saint-Venant, *Thèorie du mouvement non-permanent des eaux, avec application aux crues des rivière at à l'introduction des marèes dans leur lit.*, C.R. Acad. Sci. Paris **73** (1871), no. 147-154, 237-240.
[6] A. I. Delis and I. Papoglou, *Relaxation approximation to bed-load sediment transport*, J. Comput. Appl. Math. **213** (2008), no. 2, 521-546.
[7] F. M. Exner, *Zur physik der dünen*, Hölder, 1920.
[8] A. J. Grass, *Sediment transport by waves and currents*, University College, London, Dept. of Civil Engineering, 1981.
[9] J. Hudson and P. K. Sweby, *Formulations for numerically approximating hyperbolic systems governing sediment transport*, J. Sci. Comput. **19** (2003), no. 1-3, 225-252.
[10] ———, *A high-resolution scheme for the equations governing 2D bed-load sediment transport*, Internat. J. Numer. Methods Fluids **47** (2005), no. 10-11, 1085-1091.
[11] A. Kurganov and G. Petrova, *A second-order well-balanced positivity preserving central-upwind scheme for the Saint-Venant system*, Commun. Math. Sci. **5** (2007), no. 1, 133-160.

## Different Reformulations & IMEX Strategies for Asymptotic-Preserving IMEX-DG Methods: numerical stability and computational complexity

Fengyan Li

(joint work with Y. Cheng, J. Jang, Z. Peng, J.-M. Qiu, T. Xiong)

We consider a linear kinetic transport equation under a diffusive scaling,

$$\varepsilon\partial_t f + v\partial_x f = \frac{\sigma_s(x)}{\varepsilon}\left(\langle f\rangle - f\right) - \varepsilon\sigma_a(x) f + \varepsilon S(x), \tag{1}$$

where $f = f(x, v, t)$ is the density distribution function of particles, depending on position $x \in \Omega_x \subset \mathbb{R}$, velocity $v \in \Omega_v \subset [-v_m, v_m]$, and time $t$. The equation models particles propagating in, and interacting with, a background medium, with the operator on the left for free streaming, and the terms on the right for scattering, absorption, and external source, respectively. And $\sigma_s(x) > 0$, $\sigma_a(x) \geq 0$, $\langle f\rangle = \int_{\Omega_v} f d\mu$. As the Knudsen number $\varepsilon$ goes to 0, the model (1) becomes a diffusive type of equation,

$$\partial_t \rho = \langle v^2\rangle \partial_x(\partial_x\rho/\sigma_s) - \sigma_a\rho + S, \quad \text{with } \rho = \langle f\rangle, \tag{2}$$

at least away from the boundary of the spacetime domain. Though being simple, this 1D linear kinetic transport equation can serve as a prototype model to study many physical systems, including neutron transport and radiation transfer.

Our interest is to design and analyze accurate and efficient numerical methods that work well for the model (1) when $\varepsilon$ ranges from $O(1)$ to 0. We focus on AP

methods [5] that, as $\varepsilon \to 0$, give consistent and stable numerical methods for the limiting equation (2). Moreover, we hope to design AP methods that have *provable uniform* stability in $\varepsilon$, a property deemed important for the (potential) uniform convergence and accuracy, as well as for rigorous analysis of the AP property. Such effort will be pursued within the framework of discontinuous Galerkin (DG) methods, due to their flexibility in accuracy, adaptivity, parallel efficiency, as well as their applicability for various types of PDEs. Unlike the exploration in [8, 2] for the standard upwind DG method and its AP property for *stationary* linear kinetic transport models, we here follow a different route, that leads to different opportunities to achieve AP methods with uniform stability, and call for different linear solvers with different computational complexities for implementation.

Three AP methods are presented here, which were developed, analyzed and numerically tested in [3, 4, 9, 10, 11]. The three ingredients common in these methods are the following: (i) reformulation(s) of the model (1); (ii) globally stiffly accurate implicit-explicit (IMEX) Runge-Kutta (RK) methods in time [1]; (iii) DG methods in space. The differences lie in the reformulation, the IMEX strategy (i.e. which terms are treated implicitly/explicitly), and subsequently, the type of DG discretizations and the choice of numerical fluxes. The model reformulation is inspired by the micro-macro decomposition [7, 6], with one method also using the idea of adding & subtracting a weighted diffusive term [1]; the IMEX strategies ensure the numerical solutions, regardless from inner stages or after a full RK step, stay close to the local equilibrium in the diffusive regime, while dealing with the stiffness and rendering computational complexities reasonable with respect to the kind of numerical stability attained; DG methods contribute to high order accuracy in space for different regimes.

For each AP method, stabilization mechanisms are identified due to the scattering operator, the temporal & spatial discretizations, and uniform stability is proved with a judicially chosen discrete energy when the temporal accuracy is first order. It is no surprise that the discrete energy is scale dependent. All methods require hyperbolic-type time step condition, $\Delta t = O(\Delta x)$, in the kinetic regime ($\varepsilon = O(1)$), with two being unconditionally stable in the diffusive regime ($\varepsilon \ll 1$). Once energy-based uniform stability is available, error estimates follow naturally by using approximation theory and regularity of the solution. For the methods of second and third order accurate in time, similar uniform stability is confirmed by Fourier analysis, and it is yet to be proved following energy approaches. Other theoretical advancement includes mathematical understanding of the weight function and rigorous analysis for AP property. Special treatment is proposed to keep the AP property and the designed accuracy for non well-prepared initial data.

## References

[1] S. Boscarino, L. Pareschi, G. Russo. *Implicit-explicit Runge-Kutta schemes for hyperbolic systems and kinetic equations in the diffusion limit*, SIAM Journal on Scientific Computing, **35** (2013), A22-A51.

[2] J.-L. Guermond and G. Kanschat, *Asymptotic analysis of upwind discontinuous Galerkin approximation of the radiative transport equation in the diffusive limit*, SIAM Journal on Numerical Analysis, **48** (2010), 53-78.
[3] J. Jang, F. Li, J.-M. Qiu, T. Xiong, *Analysis of asymptotic preserving DG-IMEX schemes for linear kinetic transport equations in a diffusive scaling*, SIAM Journal on Numerical Analysis, **52** (2014), 1497-2206.
[4] J. Jang, F. Li, J.-M. Qiu, T. Xiong, *High order asymptotic preserving DG-IMEX schemes for discrete-velocity kinetic equations in a diffusive scaling*, Journal of Computational Physics, **281** (2015), 199-224.
[5] S. Jin, *Asymptotic preserving (AP) schemes for multiscale kinetic and hyperbolic equations: a review*, Lecture Notes for Summer School on Methods and Models of Kinetic Theory, (2000), 177-216.
[6] M. Lemou and L. Mieussens, *A new asymptotic preserving scheme based on micro-macro formulation for linear kinetic equations in the diffusion limit*, SIAM Journal on Scientific Computing, **31** (2008), 334-368.
[7] T. Liu and S.-H. Yu, *Boltzmann equation: micro-macro decompositions and positivity of shock profiles*, Communications in Mathematical Physics, **246** (2004), 133-179.
[8] E. Larsen, *On numerical solutions of transport problems in the diffusion limit*, Nuclear Science and Engineering, **83** (1983), 90-99.
[9] Z. Peng, Y. Cheng, J.-M. Qiu, F. Li, *Stability-enhanced AP IMEX-LDG schemes for linear kinetic transport equations under a diffusive scaling*, Journal of Computational Physics, **415** (2020), 109485.
[10] Z. Peng, Y. Cheng, J.-M. Qiu, F. Li, *Stability-enhanced AP IMEX1-LDG method: energy-based stability and rigorous AP property*, SIAM Journal on Numerical Analysis, to appear.
[11] Z. Peng and F. Li, *Asymptotic preserving IMEX-DG-S schemes for linear kinetic transport equations based on Schur complement*, SIAM Journal on Scientific Computing, to appear.

## On the modeling of non-mixing, compressible two-phase flow

Marco Petrella
(joint work with R. Abgrall, S. Mishra)

Compressible multiphase flow occur in many practical situations where two phases are separated by an interface. The modeling of such phenomena involve a strong coupling between physics and fluid-thermodynamics, making its accurate simulation a challenging task for applied scientists. Even under the stringent simplification of neglecting mass-transfer and heat conduction, *no unique model* is generally accepted for the description of two compressible constituents. Stemming from the seminal work of Stewart and Wendroff [7], many models have been proposed over the years; a very incomplete list include [2, 5, 3, 6]. Probably the most known to the community is the one put forward by Baer and Nunziato (BN) [2] for the description of the deflagration-to-detonation transition, for a mixture of a solid and a gas. Following the guideline of physical principles fulfillment, Saurel and Abgrall proposed a new non-equilibrium model [6], which constitutes a generalization of the classical BN model to general species. Such extension consists in the introduction of stiff source terms to enforce thermodynamical interactions between the various components. Indeed, this removes spurious mechanical phenomena and enforce the Abgrall criterion: "*phases under uniform mechanical conditions (unique velocity and unique pressure) should evolve preserving such conditions*".

The newly introduced source terms force the non-equilibrium mixture to converge to a single velocity and single pressure, as it was noticed during experiments [3]. Such mechanical interaction is typically referred as a *relaxation* term, and determines the rate at which velocities and pressures reach equilibrium.

Most of the known models to date can be reformulated into the following generalized Bear-Nunziato form: for each $k \in \{1, 2\}$,

$$\partial_t \alpha_k + u_I \partial_x \alpha_k = \mu(p_k - p_l)$$
$$\partial_t(\alpha_k \rho_k) + \partial_x(\alpha_k \rho_k u_k) = 0$$
$$\partial_t(\alpha_k \rho_k u_k) + \partial_x\Big(\alpha_k(\rho_k u_k^2 + p_k)\Big) = p_I \partial_x \alpha_k - \lambda(u_k - u_l)$$
$$\partial_t(\alpha_k \rho_k E_k) + \partial_x\Big(\alpha_k u_k(\rho_k E_k + p_k)\Big) = p_I u_I \partial_x \alpha_k - \mu p_I^{'}(p_k - p_l) - \lambda u_I^{'}(u_k - u_l)$$

The notation is classical, $\alpha, \rho, u, p$ denote the volume fraction, density, velocity and pressure of phase $k$, respectively. The total energy $E$ depends on the internal energy via $E = \frac{1}{2}u^2 + e$, where $e = e(\rho, p)$ is a specific equation of state for each phase $k$. The interfacial quantities $u_I, p_I, u_I^{'}, p_I^{'}$ are the distinctive parameters that lead to different models as the parameter $\lambda, \mu \to \infty$. Indeed, a major difficulty in the field of two-phase flow modeling is to capture the lost information at the microscopic scale, and to inject it in the expression of such interfacial quantities.

A novel approach to capture such lost information at the numerical level was put forward in [1]. This strategy aims at computing the statistical averages of the quantities of interest, by considering any possible topology of the flow. Inspired by the ensemble averaging of [4], the scheme has been widely used for applications leading to incredible results. Unfortunately, we proved that the algorithm is ill-posed: there exists infinitely many schemes reproducible following the same algorithmic procedure. In particular, the underlying non-uniqueness is related to the probabilities $\mathbb{P}_{i+\frac{1}{2}}[\Sigma_k, \Sigma_l]$ of finding phase $k$ and phase $l$ on the two sides of cell boundary $i + \frac{1}{2}$. We showed that such probability coefficients are indeed convex, namely there exists two extreme points and a unique parameter $r \in [0, 1]$ spanning all the possible probabilities.

Interesting enough, specific choices of such parameter $r^n_{i+\frac{1}{2}}$ lead to many well-known models in the limit of stiff mechanical relaxation.

This motivated us to refine the scale of description as to capture the underlying microstructure. By means of the Front-Tracking method and the Monte-Carlo approach we perform the first ab-initio simulation of two-phase flow models. Numerical experiments have been shown to define a convergent sequence under both physical mesh refinement and dispersed particle size refinement.

As a consequence we obtained the following generalization: for a given initial condition,

(1) If the macroscopic models do not depend of the choice of the parameter $r^n_{i+\frac{1}{2}}$, then the ab-initio framework is capturing the same (unique) solution. In particular, macroscopic models converge to the ab-initio framework under physical mesh refinement.

(2) If the macroscopic models show discrepancies depending on the choice of $r^n_{i+\frac{1}{2}}$, then there are test cases for which the ab-initio framework is yielding a solution outside of the capturing capabilities of the $r$-dependent models.

This suggests that the ab-initio framework constitutes a generalization of classical two-phase flow models, and that the inclusion is proper.

## References


[1] R. Abgrall, R. Saurel, *Discrete equations for physical and numerical compressible multiphase mixtures*. Journal of Computational Physics **186** (2003), 361–396.

[2] M. R. Baer, J. W. Nunziato, *A Two-Phase Mixture Theory for the Deflagration-To-Detonation Transition (DDT) in Reactive Granular Materials*. Journal of Multiphase Flow **12** (1986), No. 6, 861–889.

[3] J. B. Bdzil, R. Menikoff, S. F. Son, A. K. Kapila, D. S. Stewart, *Two-phase modeling of deflagration-to-detonation transition in granular materials: a critical examination of modeling issures*. Physics of Fluids **11** (1999), 378.

[4] D. A. Drew, S. L. Passman, *Thoery of Multicomponent Fluids*. Springer Verlag New York, Inc. 1998.

[5] A. Murrone, H. Guillard, *A five equation reduced model for compressible two phase flow problems*. Journal of Computational Physics **202** (2005), 664–698.

[6] R. Saurel, R. Abgrall, *A Multiphase Godunov Method for Compressible Multifluid and Multiphase Flows*. Jounral of Computational Physics **150** (1999), 450–467.

[7] H. B. Stewart, B. Wendroff, *Two-Phase Flow: Models and Methods*. Journal of Computational Physics **56** (1984), 363–409.


## Application and experimental results of the stochastic Galerkin method for cloud simulation


Bettina Wiebe

(joint work with Mária Lukáčová, Alina Chertock, Alexander Kurganov, Peter Spichtinger)


Clouds are one of the most uncertain components in numerical weather prediction models, since, unlike the atmospheric flows, they cannot be modeled using first principles of physics. Our goal is to quantify the uncertainties arising in cloud modeling and investigate the propagation of uncertainties, since clouds are one of the most important components in the Earth-atmosphere system. They influence the hydrological cycle and by interacting with radiation they control the energy budget of the system.

We introduce a stochastic Galerkin method for a random coupled Navier-Stokes-cloud system consisting of the weakly compressible Navier-Stokes equations and cloud evolution equations for water vapor, cloud water and rain that models dynamics of warm clouds, see [3]. In this model the Navier-Stokes equations describe weakly compressible flows with viscous and heat conductivity effects, while microscale cloud physics is modeled by a system of advection-diffusion-reaction equations. Our goal is to explicitly describe the evolution of uncertainties that arise due to unknown input data, such as model parameters and initial or boundary conditions and investigate the influence on atmospheric flows. The developed

stochastic Galerkin method combines the space-time approximation obtained by a suitable finite volume method with a spectral-type approximation based on the generalized polynomial chaos expansion in the stochastic space, see [1].

In [1] we restricted our consideration to the case in which the uncertainties are only in the cloud physics representation and the Navier-Stokes system is still deterministic. Here, we present an approach to extend our proposed method to the fully stochastic Navier-Stokes-cloud system and focus on the application of the stochastic Galerkin method to the Navier-Stokes equations. The Galerkin projection will yield to a system for the generalized polynomial chaos (gPC) expansion coefficients of each variable. The projection of the linear terms can be obtained in a straightforward way by using the orthogonal property of the stochastic expansion polynomials and for the nonlinear terms we additionally use discrete and inverse discrete transforms. The then used method for the space and time discretization of the deterministic system for the gPC coefficients is an extension of the approach proposed in [2]. This method is based on the operator splitting approach, in which the system is split into the macroscopic Navier-Stokes equations and microscopic cloud model. The Navier-Stokes equations are then solved by an asymptotic preserving implicit-explicit finite-volume method and the cloud equations are solved by a finite-volume method combined with an explicit Runge-Kutta method with an enlarged stability region. The resulting numerical scheme yields a second-order accurate approximation in both space and time and spectral convergence in the stochastic space with an experimentally determined exponential decay.

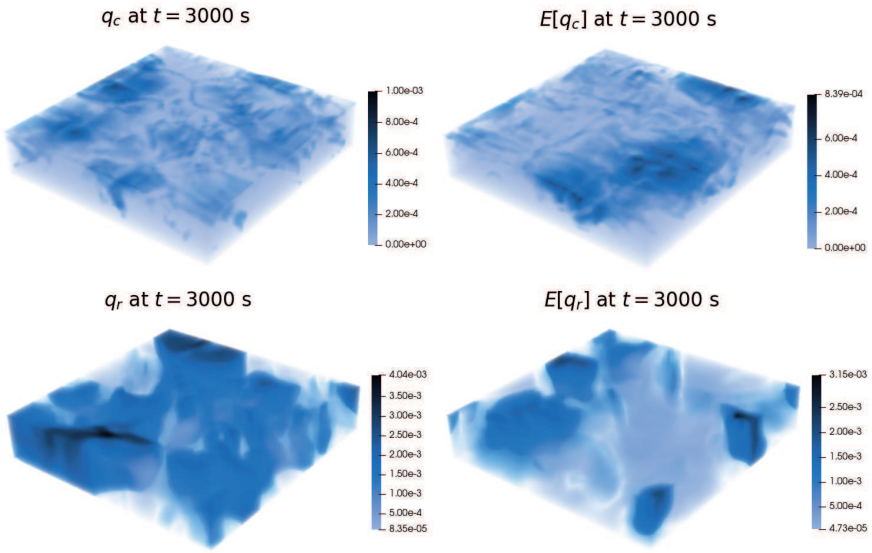


FIGURE 1. Cloud drops concentration $q_c$ and rain drops concentration $q_r$ using 0% (left column) and 20% (right column) perturbation of the initial data in $q_v$.

We compare the stochastic Galerkin method to a standard Monte Carlo approach and show that both methods converge experimentally against the same solution, whereby the stochastic Galerkin method converges faster and exhibits lower errors. We also use the proposed method to study the behavior of clouds in certain perturbed scenarios, for example, the moist Rayleigh-Bénard convection and show that already small perturbations can lead to qualitatively very different results and even to changes in macroscopic patterns, see Figure 1.

**Acknowledgement**
The research leading to these results has been done within the sub-project A2 of the Transregional Collaborative Research Center SFB/TRR 165 "Waves to Weather" funded by the German Science Foundation (DFG).

## References

[1] A. Chertock, A. Kurganov, M. Lukáčová-Medvid'ová, P. Spichtinger, B. Wiebe, *Stochastic Galerkin method for cloud simulation*, Math. Clim. Weather Forecast volume **5**, (2019), 65-106.
[2] M. Lukáčová-Medvid'ová, J. Rosemeier, P. Spichtinger, B. Wiebe, *IMEX finite volume methods for cloud simulation*, Finite volumes for complex applications VIII—hyperbolic, elliptic and parabolic problems, Springer Proc. Math. Stat. **200** (2017), 179-187.
[3] N. Porz, M. Hanke, M. Baumgartner, P. Spichtinger *A model for warm clouds with implicit droplet activation, avoiding saturation adjustment*, Math. Clim. Weather Forecast **4** (2018), 50-78.

## Space-times resonances and weakly dissipative hyperbolic systems

Roberta Bianchini
(joint work with Roberto Natalini)

We consider the framework of general hyperbolic multidimensional systems, to which many physical models belong [5]. Their common feature is the loss of regularity: solutions can develop shocks in finite time, no matter how much the initial data are small [5]. Nevertheless, there are some specific classes of hyperbolic systems whose solutions stay smooth for all times, at least for small initial data. The simplest physical mechanism preventing the formation of singularities is dissipation, one of the more relevant hyperbolic dissipative models with global smooth solutions being the compressible Euler system with damping. In [4, 10, 8], a rigorous framework was proposed to characterize a general class of partially dissipative hyperbolic systems whose smooth solutions are global in time. The simplest preliminary condition to intruduce is *entropy dissipation* [4]. However, it is very easy to see that *entropy dissipation* alone is not enough to prevent singularities in finite time. Hence, besides entropy dissipation, one requires the so-called Shizuta-Kawashima condition, which has been originally introduced in [8]. This condition can be stated in many different ways, see for instance [4]. In terms of stability, it provides the necessary coupling between conserved/non conserved quantities of a weakly dissipative system in such a way that each state variable is dissipated. Entropy dissipation together with [SK] is thus *sufficient* to ensure that the solutions

stay smooth for all times [4, 10]. However, there are many physical systems, especially in multiD, which are entropy dissipative but violate [SK]. A nice example is the model for gas dynamics in rotational and vibrational non-equilibrium,

$$(1)\qquad \begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0, \\ \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla P(\rho, e) = 0, \\ \partial_t \left(\rho(\frac{1}{2}|u|^2 + e + q)\right) + \nabla \cdot \left((\frac{1}{2}|u|^2 + e + q + P(\rho, e))u\right) = 0, \\ \partial_t (\rho q) + \nabla \cdot (\rho q u) = \frac{1}{\tau}\left(\rho(Q(e) - q)\right), \end{cases}$$

where $\rho$, $u$, $e$, and $q$ are respectively the density, the velocity, the internal energy and the internal vibrational energy of the gas, see [9]. This system does not satisfy [SK], see [4]. Nevertheless, Y. Zeng proved in [11] the global existence in 1D. There are many other physically interesting dissipative systems which violate [SK], see [2] for further examples. However, at the present time there is no general theory which allows to weaken [SK] and enlarge the class of systems with global in time smooth solutions for small data. We mention the attempt of [1], where a (linear) analysis of systems for which [SK] fails in small sets (of zero measure) is carried out. We want to investigate a more general setting, where some of the variables are not dissipated at all and this holds in any point of the frequency space. The strategy is to bypass the failure of the [SK] condition on the linearized part of the system, by taking advantage of some special features of the nonlinear terms. A first attempt to implement this approach in a very specific case is contained in [6], where, for a simple class of one-dimensional hyperbolic systems, *linear degeneracy counterbalances the lack of dissipation in preventing shocks.* Here we try to make a step forward in this sense. We consider a partially dissipative hyperbolic system violating [SK] in some directions, where the equations for the non-dissipative variables have a (nonlinear) source term with a special structure, i.e. a suitable generalization of linear degeneracy. That is, in the directions where [SK] fails, we consider the *nonresonant bilinear forms* introduced in [7] in the context of the *space-time resonance method* [3], as a natural generalization of Klainerman's *null forms.* The system reads

$$\begin{cases} \partial_t u + i\Lambda v = \mathcal{Q}_1(u, v), \\ \partial_t v + i\Lambda u + v = \mathcal{Q}_2(u, v), \\ \partial_t w + i\Lambda w = \mathcal{B}_{nonres}(w, w) + vw, \end{cases}$$

where $u, v, w : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}$ are the unknowns, $\lambda = Op(|\xi|)$, $\xi \in \mathbb{R}^3$, $\mathcal{Q}_i$ are quadratic terms and $\mathcal{B}_{nonres}$ is a quadratic nonresonant source. The result is a global in time existence of smooth solutions for small data, in a properly weighted functional space, which combines Green function estimates with the space-time resonance method. Our system contains most of the main features of the proposed framework, but it is a toy model. It shows that the cooperation of very weak dissipation violating Shizuta-Kawashima with linearly degenerate nonlinearities is successful for handling systems with quadratic terms for long times in a particular case. As a special structure of the nonlinearity is a common feature of many

physically relevant models violating [SK], our method should be applied to specific cases as the out-of-equilibrium gas-dynamics.

## References


[1] K. Beauchard, E. Zuazua, *Large time asymptotics for partially dissipative hyperbolic systems*, Arch. Ration. Mech. Anal. **199** (2011), 177–227.

[2] R. Bianchini, R. Natalini, *Nonresonant bilinear forms for partially dissipative hyperbolic systems violating the Shizuta-Kawashima condition*, preprint arXiv:1906.02767 (2019).

[3] P. Germain, N. Masmoudi, J. Shatah, *Global solutions for the gravity water waves equation in dimension 3*, Ann. of Math. **175** (2012), 691–754.

[4] S. Bianchini, B. Hanouzet, R. Natalini, *Asymptotic behavior of smooth solutions for partially dissipative hyperbolic systems with a convex entropy*, Comm. Pure Appl. Math.**60** (2007), 1559-1622.

[5] A. Majda, *Compressible fluid flow and systems of conservation laws in several space dimensions*, Appl. Math. Sci., vol. 53, Springer, New York 1978.

[6] C. Mascia, R. Natalini, *On relaxation hyperbolic systems violating the Shizuta-Kawashima condition*, Arch. Ration. Mech. Anal. **195** (2010), no. 3, 729–762.

[7] F. Pusateri, J. Shatah, *Space-Time Resonances and the Null Condition for First-Order Systems of Wave Equations*, Comm. Pure and Appl. Math.**66** (2013), 1495-1540.

[8] Y. Shizuta, S. Kawashima, *Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation*, Hokkaido Math. J. **14** (1985) n.2, 249-275.

[9] G.B. Whitham, *Linear and nonlinear waves*, Wiley & Sons, New York, 1974.

[10] W.-A. Yong, *Entropy and global existence for hyperbolic balance laws*, Arch. Ration. Mech. Anal. **172** (2004), 247–266.

[11] Y. Zeng, *Gas dynamics in thermal nonequilibrium and general hyperbolic systems with relaxation*, Arch. Rational. Mech. Anal. **150** (1999), 225–279.


## Extension of Entropy Correction Terms for Residual Distribution Schemes: Application to Structure Preserving Discretization


Philipp Öffner

(joint work with Rémi Abgrall, Hendrik Ranocha)


Numerical schemes should been constructed to preserve the structural properties of the underlying physical model. In the context of hyperbolic conservation laws (balance laws), one of such properties is that the solution fulfills an entropy (in)equality. By translating this into the discrete framework, one speaks about entropy **conservative (dissipative) schemes** meaning that also the numerical solutions fulfills such entropy (in)equality. Recently, a lot of attentions has been given in the development of entropy conservative (dissipative) schemes in the context of finite difference and discontinuous Galerkin schemes using summation-by-parts operators, cf. [3] and references therein. However, a slightly different approach was recently introduced by Abgrall [1]. The main idea is to add suitable entropy correction terms to the scheme not violating the conservation relation but to ensure the entropy condition. For the explanation of the correction approach, we shortly repeat the idea in the context of residual distributions (RD) schemes. Constructed in the FV framework, RD is nowadays interpreted in the FE context. The selection of the subresiduals $\Phi_\sigma^K$ and the approximation space specify

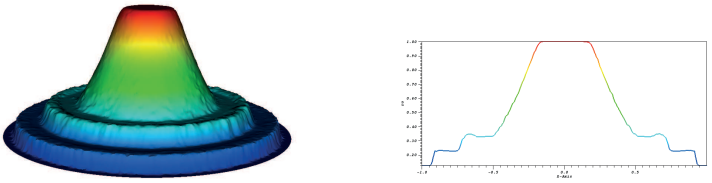

FIGURE 1. SOD, 4th-ord. Galerkin with entropy correction $r$ and MOOD

the method and RD unifies several high-order methods like CG, DG and FR in a common framework, cf. [1] for more details.

First, we focus on the steady-state equation $\operatorname{div} f(u) = 0$ in $\Omega$. In this context, an entropy conservative scheme fulfills $\sum_{\sigma \in K} < v_\sigma, \tilde{\Phi}_\sigma^K >= \oint_{\partial K} g^{num}\left(v_{|K}^h, v_{|K^-}^h\right) n \,\mathrm{d}\,\gamma$, where $v_\sigma$ denotes the approximation of the entropy variable and $g^{num}$ the numerical entropy flux. However, our selected residual $\Phi_\sigma^K$ introduced an entropy error $\mathcal{E} = \oint_{\partial K} g^{num}\left(v_{|K}^h, v_{|K^-}^h\right) \cdot n \,\mathrm{d}\,\gamma - \sum_{\sigma \in K} < v_\sigma, \Phi_\sigma^K >$. To obtain an entropy conservative scheme, we add a suitable correction term $r_\sigma^K$ to our residual. It is $\tilde{\Phi}_\sigma^K = \Phi_\sigma^K + r_\sigma^K$, where $r_\sigma^K = \frac{\mathcal{E}}{\sum_{\sigma \in K} (v_\sigma - \overline{v})^2}(v_\sigma - \overline{v})$, where $\overline{v}$ is the arithmetic mean in $K$. The correction term is constructed to ensure the entropy property not violating the conservation property of the scheme in the steady-state case.

For unsteady problems, one should avoid using the method of lines to keep the high-order of the RD scheme. Therefore, we apply the DeC approach resulting in a fully explicit FE-based scheme [2].

The correction term can also be applied to ensure fully discrete entropy conservation. Here, the update procedure is summarized as follows:

(1) Compute the entropy difference $\eta(U^{(k)}) - \eta(U^0)$ at every DOF.
(2) Calculate the entropy flux using $U^{(k)}$ at every degree of freedom.
(3) Calculate the differences in the entropy in every element $K$ using the space-time entropy residual $\Phi_{t,x}^{K,e}$.
(4) Use the correction term with the calculated entropy differences to correct the space-time residual. With this, we obtain a fully discrete conservative scheme.

However, switching back to the semidiscrete setting. We can apply the correction term in a classical DG framework which yields to a slightly different entropy correction term $\mathbf{r}$. Actually, as proven in [2], we have re-formulated both correction terms as solutions of certain optimization problem.

**Theorem 1.** *If the constraints do not contradict each other, the correction term* $\mathbf{r}$ *is the unique optimal correction, measured in the discrete norm induced by* $M$,

*such that conservation and entropy conservation are satisfied, i.e.*

$$\min_{\mathbf{r}} \frac{1}{2}||\mathbf{r}||_M^2 \quad 1^T M\mathbf{r} = 0,\ v^T M\mathbf{r} = \mathcal{E},$$

*where* $||\mathbf{r}||_M^2 = \mathbf{r}^T M\mathbf{r}$. *The classical correction term* $r_i$ *is the unique optimal correction, measured in the discrete norm induced by* $\mathbb{I}$, *i.e.* $||r||_{\mathbb{I}}^2 = r^T \mathbb{I} r$.

Using this reinterpretation, we can further extend this application of the correction terms to other constraints like the kinetic energy preservation for the Euler equations. A combination of both terms yields schemes which are kinetic energy preserving and entropy conservative. Therefore, the correction terms can be seen as a universal tool. It can be applied to most of the used numerical methods to ensure the desired properties. However, it comes also with some drawback. For example for the Euler equations, the switching to the entropy variables has to be possible. Hence, it has to be ensured that density and pressure have to remain positive during the calculation. A detailed discussion can be found in our corresponding article [2].

## References

[1] R. Abgrall, *A general framework to construct schemes satisfying additional conservation relations. Application to entropy conservative and entropy dissipative schemes*, Journal of Computational Physics **372** (2018), 640–666.

[2] R. Abgrall, P. Öffner, H. Ranocha, *Reinterpretation and Extension of Entropy Correction Terms for Residual Distribution and Discontinuous Galerkin Schemes: Application to Structure Preserving Discretization*, arXiv:1908.04556 (2019).

[3] H. Ranocha, *Generalised summation-by-parts operators and entropy stability of numerical methods for hyperbolic balance laws*, PhD Thesis, Cuvillier Verlag, (2018).

## DeC and ADER: arbitrarily high order methods for hyperbolic PDEs (and ODEs)

Davide Torlo

(joint work with Maria Han Veiga, Philipp Öffner)

In the field of numerical methods for hyperbolic partial differential equations (PDEs), more and more high order accurate schemes have become very popular and used for a huge variety of simulations. These methods allow to achieve sharper solutions that shows more structures with smaller computational costs with respect to low order methods. The high order methods for hyperbolic PDEs have to take into account a high order spatial discretization, e.g. finite element, discontinuous Galerkin, and, on the other side, a high order time marching procedure, typically Runge Kutta, with the use of the method of lines.

Lately, more methods which combine space and time discretization in a unique process are used, without the need of using the method of lines directly. Examples are the deferred correction (DeC) method defined as ODE solver in [3], but used in combination with different spatial discretizations in [2] for hyperbolic PDEs, and the arbitrary derivative (ADER) method, originally presented as an exploitation

of the Cauchy–Kovaleskaya theorem [4] and recently presented as a high order accurate space–time discretization, inter alia [5].

In [1] we try to decouple the spatial discretization and the time discretization of these two methods (explicit DeC and explicit ADER), in order to acquire more insights on the properties and the similarities of these schemes.

The DeC is based on a subdivision of each timestep into $M$ subtimesteps, then a reconstruction in time, based on Lagrangian polynomials is performed. The Picard–Lindelöf theorem applied to the ODE leads to the integral form, that, written for each subtimestep, leads to a nonlinear system of $M$ unknowns, which is a high order approximation of the ODE. The DeC introduces a first order operator that simplifies such formulation into an explicit method and combines the two formulation in an explicit iterative procedure that after $K$ iterations converges to the high order accurate solution with an error of order $K$. This result is given by a convergence theorem which requires the existence of a solution of the high order operator, the coercivity of the low order operator and the Lipschitz continuity of the difference of the two operators. Usually these properties are easily verified.

The ADER method has lately been presented as a time–space discontinuous Galerkin, where this formulation leads to a local (in space and time) nonlinear system that can be solve with the fixed point iteration method. The solution of such problem is a high order accurate approximation of the problem. A final reconstruction step which allows also communication between cells through numerical fluxes is also performed. If we focus on the time integration part, we notice that the formulation is given by a Galerkin projection on the time basis functions space.

We observed that, introducing a low order approximation of the high order formulation, we can rewrite the ADER fixed point iteration exactly with the same formalism that the DeC introduces. This leads to an interesting result. We can prove the convergence of the fixed point iteration method and we know the number of iterations needed in order to obtain the required accuracy (number of iterations must be equal to the aimed order).

On the other side, the DeC high order discretization can be rewritten as a finite element method in time, if one introduces as test functions the characteristic functions on the subtimesteps. This can be seen as an ADER approximation, where we use a Petrov-Galerkin projection instead of a Galerkin one, maintaining the spirit of the high order finite element discretization in time.

Numerical results show that the two methods behaves very similarly in terms of accuracy, stability, see Figure 1, and simulations. The differences between the two methods are more remarkable in the implicit context or when the spatial discretization plays a role in the definition of the high and low order approximation operator as well. For instance, the first order approximation can be further simplified with the lumping of the mass matrix (in a finite element context), without decreasing the order of accuracy [2].

In the future, we aim to study more in details these schemes, in particular in their implicit versions, where stability has not been issued yet, entropy/energy stable versions and with more combination of spatial discretizations.

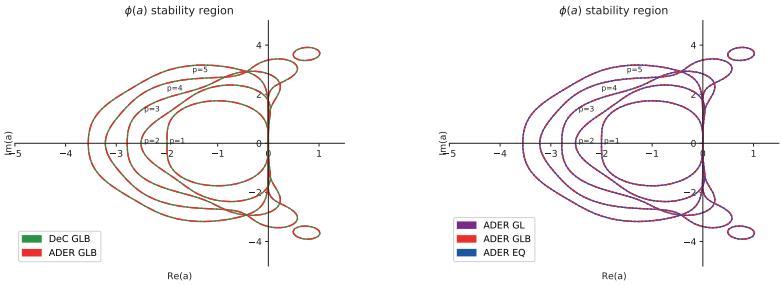


FIGURE 1. A-Stability regions for ADER and DeC methods (left) and different basis functions (right): order 2 to 6

## REFERENCES

[1] M. H. Veiga, P. Öffner, D. Torlo, *DeC and ADER: Similarities, Differences and a Unified Framework*, Journal of Scientific Computing **87** (2021), 2.
[2] R. Abgrall. *High order schemes for hyperbolic problems using globally continuous approximation and avoiding mass matrices,* Journal of Scientific Computing **73** (2017), 461–494.
[3] A. Dutt, L. Greengard, and V. Rokhlin. *Spectral Deferred Correction Methods for Ordinary Differential Equations*, BIT Numerical Mathematics **40** (2000), 241–266.
[4] V. A. Titarev and E. F. Toro. *Ader: Arbitrary high order Godunov approach*, Journal of Scientific Computing **17** (2002), 609–618.
[5] O. Zanotti, F. Fambri, M. Dumbser, and A. Hidalgo. *Space-time adaptive ADER discontinuous Galerkin finite element schemes with a posteriori sub-cell finite volume limiting*, Computers & Fluids **118** (2015), 204–224.

## A posteriori Error Estimates for Numerical Solutions to Hyperbolic Conservation Laws

MARIA TERESA CHIRI
(joint work with Alberto Bressan, Wen Shen)

Consider the Cauchy problem for a strictly hyperbolic system of conservation laws in one space dimension

$$u_t + f(u)_x = 0, \qquad u(0,x) = \bar{u}(x). \tag{1}$$

Assuming small total variation, it is well known that there exists a unique entropy-weak solution, depending Lipschitz continuously on the initial data $\bar{u}$ in the $\mathbf{L}^1$ norm [9].

A related question is the stability and convergence of various types of approximate solutions. Estimates on the convergence rate for a deterministic version of the Glimm scheme [17, 18] were derived in [14], and more recently in [2, 8] for a wider class of flux functions. For vanishing viscosity approximations, uniform BV bounds, stability and convergence as $\varepsilon \to 0$ were proved in [6], while convergence rates were later established in [13, 15]. Further convergence results were proved

by Bianchini for approximate solutions constructed by the semidiscrete (upwind) Godunov scheme [4], and by the Jin-Xin relaxation model [5].

A major remaining open problem is the convergence of fully discrete approximations, such as the Lax-Friedrichs or the Godunov scheme. Indeed, the convergence results known for these numerical algorithms rely on compensated compactness [16] and do not yield information about uniqueness or convergence rates.

Without an *a priori* bound on the total variation, it is not possible to compare an approximate solution with trajectories of the semigroup of exact solutions, and all the uniqueness arguments developed in [10, 11, 12] break down.

For general hyperbolic systems, however, it is known that the Godunov scheme is unstable w.r.t. the BV norm. Indeed in [3] an example was constructed, showing that the total variation of a numerical solution can become arbitrarily large as $t \to +\infty$.

For this reason, we here focus on *a posteriori* error estimates. Based on some additional information about the approximate solution, we give an estimate on the difference

$$\|u^{approx}(T,\cdot) - u^{exact}(T,\cdot)\|_{\mathbf{L}^1(\mathbb{R})}\,. \tag{2}$$

For any sufficiently small BV initial data $\bar{u}$, the unique entropy-admissible BV solution of (1) has two key properties [9]:

(i) The total variation of $u(t,\cdot)$ remains uniformly small, for all $t \geq 0$.

(ii) Given a threshold $\rho > 0$, one can identify a finite number of curves in the $t$-$x$ plane (shocks or contact discontinuities) such that, outside these curves, the solution has local oscillation $< \rho$.

However, for an approximation constructed by the Godunov scheme, the property (i) sometimes can fail as proved by the counterexample in [3].

The result we show is in brief the following. Let $u^{approx}$ be an approximate solution produced by a conservative scheme which dissipates entropy, and assume that

(i) the total variation of $u^{approx}(t,\cdot)$ is uniformly bounded,

(ii) outside a finite number of narrow strips in the domain $[0,T]\times\mathbb{R}$, the local oscillation of $u^{approx}$ remains small;

then the distance (2) is small. The error bound is derived by using a technique introduced in [7] which relies on two types of estimates:

- on regions where the oscillation is small, $u^{approx}$ is compared with the solution to a linear hyperbolic problem with constant coefficients,
- near a point where a large jump occurs, $u^{approx}$ is compared with the solution to a Riemann problem.

No regularity is required for the exact solution, apart from BV bounds.

For a numerically computed approximation, we also introduce a *post-processing algorithm*, which accomplishes three main tasks: check that the total variation remains bounded, trace the location of a finite number of large shocks, check that the oscillation of the solution remains small on a finite number of polygonal domains away from the large shocks.

The error estimate which we obtain can be applied to a wide class of approximation schemes, such as Godunov, Lax-Friedrichs, backward Euler approximations, and approximate solutions obtained by periodic mollifications.

## References


[1] A. Bressan, M.T. Chiri and W. Shen, A posteriori Error Estimates for Numerical Solutions to Hyperbolic Conservation Laws.(2020) *Submitted paper*

[2] F. Ancona and A. Marson, Sharp convergence rate of the Glimm scheme for general nonlinear hyperbolic systems. *Comm. Math. Phys.* **302** (2011), 581–630.

[3] P. Baiti, A. Bressan, and H. K. Jenssen, BV instability of the Godunov scheme, *Comm. Pure Appl. Math.* **59** (2006), 1604–1638.

[4] S. Bianchini, BV solutions of the semidiscrete upwind scheme. *Arch. Rational Mech. Anal.* **167** (2003), 1–81.

[5] S. Bianchini, Hyperbolic limit of the Jin-Xin relaxation model. *Comm. Pure Appl. Math.* **59** (2006), 688–753.

[6] S. Bianchini and A. Bressan, Vanishing viscosity solutions of nonlinear hyperbolic systems. *Annals Math.* **161** (2005), 223–342.

[7] A. Bressan, The unique limit of the Glimm scheme, *Arch. Rational Mech. Anal.* **130** (1995), 205–230.

[8] S. Bianchini and S. Modena, Convergence rate of the Glimm scheme. *Bull. Inst. Math. Acad. Sinica* **11** (2016), 235–300.

[9] A. Bressan, *Hyperbolic systems of conservation laws. The one-dimensional Cauchy problem.* Oxford University Press, Oxford, 2000.

[10] A. Bressan and P. Goatin, Oleinik type estimates and uniqueness for $n \times n$ conservation laws, *J. Differential Equations* **156** (1999), 26–49.

[11] A. Bressan and P. LeFloch, Uniqueness of weak solutions to systems of conservation laws, *Arch. Rational Mech. Anal.* **140** (1997), 301–317.

[12] A. Bressan and M. Lewicka, A uniqueness condition for hyperbolic systems of conservation laws, *Discr. Cont. Dyn. Syst.* **6** (2000), 673–682.

[13] A. Bressan, F. Huang, Y. Wang, and T. Yang, On the convergence rate of vanishing viscosity approximations for nonlinear hyperbolic systems, *SIAM J. Math. Analysis* **44** (2012), 3537–3563.

[14] A. Bressan and A. Marson, Error bounds for a deterministic version of the Glimm scheme, *Arch. Rational Mech. Anal.* **142** (1998), 155–176.

[15] A. Bressan and T. Yang, On the rate of convergence of vanishing viscosity approximations, *Comm. Pure Appl. Math.* **57** (2004), 1075–1109.

[16] X. Ding, G. Q. Chen, and P. Luo, Convergence of the fractional step Lax-Friedrichs scheme and Godunov scheme for the isentropic system of gas dynamics. *Comm. Math. Phys.* **121** (1989), 63–84.

[17] J. Glimm, Solutions in the large for nonlinear hyperbolic systems of equations. *Comm. Pure Appl. Math.* **18** (1965), 697–715.

[18] T. P. Liu, The deterministic version of the Glimm scheme, *Comm. Math. Phys.* **57** (1975), 135–148.

## A posteriori error analysis of finite volume approximations to scalar conservation laws using only one entropy

Jan Giesselmann
(joint work with Sam G. Krupa)

We are interested in the convergence analysis of numerical methods for systems of hyperbolic conservation laws in one spatial dimension. We focus on numerical schemes that combine Runge-Kutta discretisation in time with finite volume or discontinuous Galerkin schemes in space. Our goal is to derive *a posteriori* error estimates, i.e. computable upper bounds for the error (in some suitable norm) that can be computed from the numerical solution. A standard approach for deriving a posteriori error estimates is to insert the numerical solution into the PDE at hand, which it only satisfies up to some residual, and to use an appropriate stability theory to bound the difference between exact and numerical solution in terms of the residual. This asserts a tight relation between available stability theories for some PDE and a posteriori error estimates for corresponding numerical schemes, see e.g. [7, 6].

Existing a posteriori error estimates for schemes approximationg hyperbolic conservation laws can be classified into several groups accoring to the stability theory that has been used (for each group only one 'representative' reference will be provided): There are estimates based on Kruzkov's entropy solutions and the doubling of variables methods [4], these estimates are, by construction, limited to (multi-dimensional) scalar problems; estimates based on stability in the Lip+ norm of solutions to scalar conservation laws in one space dimension can be found in e.g. [8]. Estimates that can be extended to systems either use the theory of $L^1$ entropy semigroups, going back to Bressan and coworkers, [6, 1] or the relative entropy method [3].

Compared to the other methods, estimates based on the relative entropy method have some strength and weaknesses: The main weakness of the relative entropy method is that it can only be used if one of the solutions under consideration is Lipschitz continuous and its residual is in $L^2$. Hence, a Lipschitz continuous *reconstruction* of the numerical solution needs to be computed in order to evaluate the error estimator and, more importantly, the error estimator from [3] blows up under mesh refinement if the exact solution is discontinuous. An advantage of the error estimator in [3] is that (for Lipschitz continuous exact solutions) it decays with the same rate as the exact error for $h \searrow 0$. This is in contrast to the error estimators based on other stability theories.

Our aim is to extend the a posteriori error estimator from [3] in such a way that it converges (for reasonable schemes) for discontinuous exact solutions. Our strategy is based on the stability theory for scalar hyperbolic conservation laws [5] that uses only one entropy inequality and is, thus, expected to be extendable to systems of conservation laws. Indeed, an extension to $2 \times 2$ systems is available in [2].

For the time being, we restrict ourselves to scalar problems and numerical schemes that use piecewise constants in space and forward Euler in time. We believe that these restrictions can be removed in future work (at the expense of an increase in technical difficulty). The stability theory from [5] can handle (approximate) solutions that are piecewise Lipschitz continuous and increasing with downward jumps in between. Thus, we cannot apply the reconstructions from [3] directly but we need to modify the numerical scheme. We need to approximate the initial data by some function that is piecewise Lipschitz continuous and increasing. All the Lipschitz continuous and increasing pieces need to be extended to the whole computational domain. Then, the entropy solutions emanating from all these initial data can be approximated by some suitable numerical scheme. In addition, we need to keep track of the positions of discontinuities, i.e. the intervals where each of the pieces might be visible. Approximate positions of discontinuities can be computed using the Rankine-Hugoniot conditions and bounds for the position error can be obtained from the residuals of the different parts of the solution. Note that [5] provides control on the error in velocity of discontinuities.

All in all, we obtain computable error bounds for the error (at any given time) in the $L^1$ and $L^2$ norms in space and preliminary numerical experiments show that they scale with the same order as the exact error ($\mathcal{O}(h)$ for $L^1$ and $\mathcal{O}(h^{\frac{1}{2}})$ for $L^2$).

## References


[1] A. Bressan, M. T. Chiri, W. Shen , *A posteriori Error Estimates for Numerical Solutions to Hyperbolic Conservation Laws*, arxiv:2010.00428.

[2] G. Chen, S. G. Krupa, A. F. Vasseur *Uniqueness and weak-BV stability for* $2\times2$ *conservation laws*, arxiv:2010.04761.

[3] A. Dedner, J. Giesselmann, *A posteriori analysis of fully discrete method of lines discontinuous Galerkin schemes for systems of conservation laws.* SIAM J. Numer. Anal. **54** (2016), 3523–3549.

[4] A. Dedner, Ch. Makridakis, M. Ohlberger, *Error control for a class of Runge-Kutta discontinuous Galerkin methods for nonlinear conservation laws*, SIAM J. Numer. Anal. **45** (2007), 514–538.

[5] S. G. Krupa, *Criteria for the a-contraction and stability for the piecewise-smooth solutions to hyperbolic balance laws*, Commun. Math. Sci., **18** (2020), 1493–1537.

[6] M. Laforest, *A posteriori error estimate for front-tracking: Systems of conservation laws*, SIAM J. Math. Anal. **35** (2004) 1347–1370.

[7] Ch. Makridakis, *Space and time reconstructions in a posteriori analysis of evolution problems*, ESAIM, Proc. **21** (2007), 31–44.

[8] E. Tadmor, *Local error estimates for discontinuous solutions of nonlinear hyperbolic equations*, SIAM J. Numer. Anal. **28** 81991), 891–906.

## Measuring distorsive effects of finite volume schemes for conservation laws

Gabriella Puppo

(joint work with Isabella Cravero, Matteo Semplice, Giuseppe Visconti)

It is well known that to prevent the onset of spurious oscillations, numerical schemes for hyperbolic systems must be non linear, even on linear equations. This is due to the need of introducing non linear limiters in the reconstruction operators which are an essential part of these algorithms. Spurious effects induced by the reconstruction as artificial diffusion and artificial dispersion have been widely studied, as in [1]. But naturally, the non linearities in the schemes introduce also non linear distorting effects in the signals transported by the numerical solution, and these have drawn very little attention, except for [2], where we propose a quantitative notion of distortion and we introduce the notion of temperature of a numerical scheme, to measure the spread of spurious modes around the main mode.

We consider the effect of the numerical derivative on a single Fourier mode of the form $\hat{u}_k = e^{ikx}$. If $D_x$ denotes the discrete derivative, and the scheme is linear, we can write

$$D_x\big(e^{ikx}\big) = (ik + \omega_k)e^{ikx},$$

where $\omega_k$ is the error in the propagation of the $k$-th mode. In other words, for linear schemes, $e^{ikx}$ is an eigenfunction not only of the exact derivative, but also of the discrete derivative. If $\omega_k$ has a non-zero negative real part, than the scheme provides an artificial damping of the solution (artificial diffusion). If $\omega_k$ has a non-zero imaginary part, the scheme alters the propagation speed of the single mode, and one sees dispersive effects, which are responsible for the onset of spurious oscillations. Typically, $|\omega_k|$ grows fast with $k$, so that the spurious effects are much more noticeable on the high wave numbers.

If the scheme is non linear, the output of $D_x(e^{ikx})$ is no longer proportional to $e^{ikx}$. To study its effect, we compute the discrete Fourier transform of the output of the numerical derivative on each Fourier mode written in real form. If we consider a grid on $[-1, 1]$ with $2N + 1$ equidistant nodes, for $k = 1, \dots, N$, we find

$$D_x \begin{bmatrix} \sin(2\pi kx) \\ \cos(2\pi kx) \end{bmatrix} = \sum_{\ell=1}^{N} \begin{bmatrix} \omega_{2\ell,2k} & \omega_{2\ell,2k+1} \\ \omega_{2\ell+1,2k} & \omega_{2\ell+1,2k+1} \end{bmatrix} \begin{bmatrix} \sin(2\pi lx) \\ \cos(2\pi lx) \end{bmatrix}. \tag{1}$$

Let $\Omega$ be the matrix with elements $\omega_{ij}$. The exact derivative is the $2N$ by $2N$ block diagonal matrix

$$\mathbb{D} = \operatorname{diag}(E_k) = \operatorname{diag}\left(2\pi k \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right), \qquad k = 1, \dots, N,$$

where we do not consider the row and the column corresponding to the constant mode $k = 0$. For a linear scheme only the 2 by 2 blocks along the diagonal of $\Omega$ would be non zero. Thus, all terms in $\Omega$ off the two main diagonals are spurious distorting effects due to the non linearities of the reconstruction. The

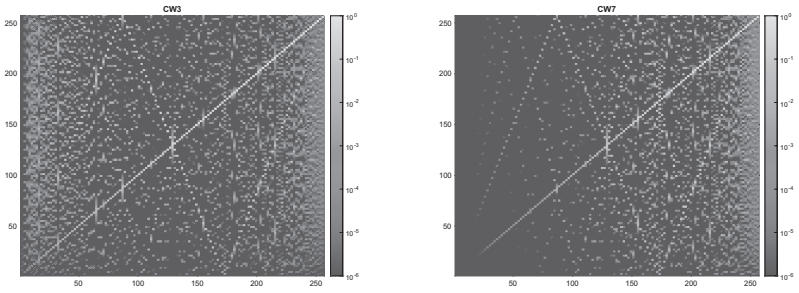


FIGURE 1. Error matrices for non linear reconstructions. From left to right third and seventh order CWENO reconstruction

error obtained on all modes can be represented with the matrix $\Omega - \mathbb{D}$. We define the *relative* error due to the non linear derivative as

$$E = |\Omega - \mathbb{D}| \operatorname{diag}\left(\frac{1}{2\pi k}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right). \tag{2}$$

In this way, the error on each mode is normalized with its frequency, so that the elements of $E$ represent the relative errors on each mode.

Here, we show results obtained with the CWENO reconstruction of [3] in Fig 1. The figure contains the magnitude of the elements of the error matrix. Along the abscissas we have the wave number of the input mode, while the ordinate shows the magnitude of each of the modes produced by the numerical derivative. One can see that the error along the diagonal, which corresponds to the already known notions of artificial diffusion and dispersion, is clearly distinguishable, even at relatively low wave numbers.

It is interesting to see that the spurious distorting error of the high order scheme is way smaller than the error induced by the third order scheme, especially considering that the right half of each figure corresponds to modes which cannot be properly resolved on the given grid. For this reason this work motivates even more the adoption of high order schemes.

Future research on this topic could include a study of distortion for other families of high order schemes, and techniques to control distortion, while preserving non oscillatory properties. More details can be found in [2].

## REFERENCES

[1] Pirozzoli, S., *On the spectral properties of shock capturing schemes*, J. Comput. Phys. **219** (2006), 489–497.

[2] Cravero, I. and Puppo, G. and Semplice, M. and Visconti, G., *Cool WENO schemes*, Computers and Fluids **169** (2018), 71–86.

[3] Cravero, I. and Puppo, G. and Semplice, M. and Visconti, G., *CWENO: uniformly accurate reconstructions for balance laws*, Math. Comp. **87** (2018), 1689–1719.

## On the Dynamics of Ferrofluids: Weak Solutions and Relaxation Limit for the Rosensweig Model

Franziska Weber
(joint work with Ricardo H. Nochetto, Konstantina Trivisa)

Ferrofluids are suspensions of nanometer-sized iron or other ferromagnetic particles. When subjected to a magnetic field, they become strongly magnetized, which has been useful for various applications in daily life, medicine and engineering. Mathematically, the dynamics of ferrofluids can be modeled using the system of partial differential equations derived by Rosensweig [4], which describes the flow of an incompressible ferrofluid subjected to an external magnetic field. In this model, the dynamics of the linear velocity $\mathbf{u}$, the pressure $p$, the angular momentum $\mathbf{w}$ and the magnetization $\mathbf{m}$ on a bounded simply connected domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, are governed by the conservation of linear momentum, the incompressibility condition, the conservation of angular momentum, the transport of the magnetization and the magnetostatics equations for the magnetic field $\mathbf{h}$:

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - (\nu+\nu_r)\Delta\mathbf{u} + \nabla p = 2\nu_r \mathrm{curl}\mathbf{w} + \mu_0(\mathbf{m}\cdot\nabla)\mathbf{h}, \tag{1a}$$

$$\mathrm{div}\mathbf{u} = 0, \tag{1b}$$

$$\mathbf{w}_t + (\mathbf{u}\cdot\nabla)\mathbf{w} - c_1\Delta\mathbf{w} - c_2\nabla\mathrm{div}\mathbf{w} + 4\nu_r = 2\nu_r\mathrm{curl}\mathbf{u} + \mu_0\mathbf{m}\times\mathbf{h}, \tag{1c}$$

$$\mathbf{m}_t + (\mathbf{u}\cdot\nabla)\mathbf{m} = \mathbf{w}\times\mathbf{m} - \frac{1}{\tau}(\mathbf{m} - \kappa_0\mathbf{h}), \tag{1d}$$

$$\mathrm{curl}\mathbf{h} = 0, \tag{1e}$$

$$\mathrm{div}(\mathbf{h}+\mathbf{m}) = 0, \tag{1f}$$

Here, the magnetic field $\mathbf{h}$ is given by

$$\mathbf{h} = \mathbf{h}_a + \mathbf{h}_d,$$

where $\mathbf{h}_a$ is the so-called *applied magnetic field* and $\mathbf{h}_d$ is the *demagnetizing field*. We assume that the former is smooth and rotation and divergence free in $\mathbb{R}^3$. The latter is rotation free in $\mathbb{R}^3$. Equation (1f) is the Maxwell equation $\mathrm{div}\mathbf{b} = 0$ in $\mathbb{R}^3$ for the magnetic induction given by $\mathbf{b} = \mu_0(\mathbf{h}+\mathbf{m})$ in $\Omega$ and $\mathbf{b} = \mu_0\mathbf{h}$ outside $\Omega$ where the magnetization $\mathbf{m}$ vanishes. Invoking the customary, although somewhat arbitrary, assumption that $\mathbf{h}_d = 0$ outside $\Omega$ along with the continuity of the normal components of $\mathbf{b}$ and $\mathbf{h}_a$ across the boundary $\partial\Omega$, we deduce

$$\mathbf{h}_d\cdot\mathbf{n} = -\mathbf{m}\cdot\mathbf{n} \quad \text{on } \partial\Omega,$$

where $\mathbf{n}$ is the unit outer normal on $\partial\Omega$. We also assume $\mathbf{u} = \mathbf{w} = 0$ on the boundary.

Whenever the magnetization $\mathbf{m}$ is small relative to $\mathbf{h}_a$, the demagnetizing field $\mathbf{h}_d$ is negligible and the effective magnetic field satisfies $\mathbf{h} \approx \mathbf{h}_a$. If $\mathbf{h}$ is a given field rather than the solution of the magnetostatics equations (1e)–(1f)), then the analysis of the reduced system (1a)–(1d) simplifies considerably. However, recent numerical simulations for a related two-phase flow model [1] indicate that the reduced system may not be able to capture the whole physical behavior of ferrofluids.

The famous Rosensweig instability, for example, can only be reproduced when $\mathbf{h}_d$ is present or equivalently when $\mathbf{h}$ solves the magnetostatics equation (1e)–(1f) (see [1, Figures 6 and 7]). We therefore analyze the full system (1). In particular, we prove that the system (1) has weak solutions. Letting $\mathcal{U} = (\mathbf{u}, \mathbf{w}, \mathbf{m}, \mathbf{h})$, these weak solutions satisfy the following energy inequality:

$$\int_\Omega E(\mathcal{U})(t)dx + \int_0^t \int_\Omega D(\mathcal{U})(s)\,dxds \leq \int_\Omega E(\mathcal{U})(0)dx + \mu_0 \int_0^t \int_\Omega \partial_t \mathbf{h}_a \cdot \mathbf{h}\,dxds,$$

where the energy $E$ is defined by

$$E(\mathcal{U}) = \frac{1}{2}\left(|\mathbf{u}|^2 + |\mathbf{w}|^2 + \frac{\mu_0}{\kappa_0}|\mathbf{m}|^2 + \mu_0|\mathbf{h}|^2\right),$$

and the dissipation functional $D$ is defined by

$$D(\mathcal{U}) = \left(\nu|\nabla\mathbf{u}|^2 + c_1|\nabla\mathbf{w}|^2 + c_2|\mathrm{div}\mathbf{w}|^2 + \nu_r|\mathrm{curl}\mathbf{u} - 2\mathbf{w}|^2 + \frac{\mu_0}{\tau\kappa_0}|\mathbf{m} - \kappa_0\mathbf{h}|^2\right).$$

In practical applications, the parameter $\tau > 0$, *the relaxation time*, is usually very small [3, 5]. We therefore also rigorously show that as $\tau \to 0$, weak solutions of the Rosensweig system converge in a suitable sense (see [2] for details) to solutions of the quasi-static equilibrium system

$$\text{(2a)} \quad \mathbf{U}_t + (\mathbf{U}\cdot\nabla)\mathbf{U} - (\nu + \nu_r)\Delta\mathbf{U} + \nabla P = 2\nu_r \mathrm{curl}\mathbf{W} + \frac{\mu_0\kappa_0}{2}\nabla|\mathbf{H}|^2,$$

$$\text{(2b)} \quad \mathrm{div}\mathbf{U} = 0,$$

$$\text{(2c)} \quad \mathbf{W}_t + (\mathbf{U}\cdot\nabla)\mathbf{W} + 4\nu_r\mathbf{W} = c_1\Delta\mathbf{W} + c_2\nabla\mathrm{div}\mathbf{W} + 2\nu_r\mathrm{curl}\mathbf{U},$$

$$\text{(2d)} \quad \Delta\Phi = 0, \quad \mathbf{H} = \nabla\Phi.$$

When solutions of this limiting system are smooth, one can show the convergence rate in $\tau$ (we use the subscript $\tau$ to denote the dependence of the solution of (1) on $\tau$):

$$\|\mathbf{u}_\tau - \mathbf{U}\|_{L^2(\Omega)}(t) + \|\mathbf{w}_\tau - \mathbf{W}\|_{L^2(\Omega)}(t) + \|\mathbf{m}_\tau - \kappa_0\mathbf{H}\|_{L^2(\Omega)}(t) + \|\mathbf{h}_\tau - \mathbf{H}\|_{L^2(\Omega)}(t) \leq C\sqrt{\tau}(1 + \exp(Ct)).$$

An open question remains whether there exists a stable numerical scheme whose approximations converge to solutions of the Rosensweig system (1).

## References


[1] R. H. Nochetto, A. J. Salgado, and I. Tomas. A diffuse interface model for two-phase ferrofluid flows. *Comput. Methods Appl. Mech. Engrg.* **309** (2016), 497-531.

[2] R. H. Nochetto, K. Trivisa, F. Weber. On the Dynamics of Ferrofluids: Global Weak Solutions to the Rosensweig System and Rigorous Convergence to Equilibrium. *SIAM J. Math. Anal.* **51** (2019), 4245-4286.

[3] C. Rinaldi. *Continuum modeling of polarizable systems.* PhD thesis, Massachusetts Institute of Technology, 2002.

[4] R. Rosenweig. *Ferrohydrodynamics.* Cambridge Monographs on Mechanics. Cambridge University Press, 1985.

[5] M. I. Shliomis. *Ferrohydrodynamics: Retrospective and Issues*, pages 85-111. Springer Berlin Heidelberg, Berlin, Heidelberg, 2002.

## Fifth-order A-WENO finite-difference schemes based on central-upwind numerical fluxes

Alexander Kurganov
(joint work with Alina Chertock, Shaoshuai Chu, Wai Sun Don, Naveen Kumar Garg, Yongle Liu, Bao-Shan Wang)

This talk consists of several parts. First, we briefly review the central-upwind schemes for hyperbolic systems of conservation laws. These schemes belong to the class of Riemann-problem-solver-free Godunov-type central schemes, but utilize a certain upwinding information in order to reduce the amount of excessive numerical dissipation typically present in non-oscillatory central schemes. The goal is achieved by evolving the solution in time using the space-time control volumes, whose spatial size is proportional to the local speeds of propagation, which, in turn, can be estimated using the smallest and largest eigenvalues of the flux Jacobian. For the detailed derivation of central-upwind schemes, we refer the reader to [6].

We then present a recent work [4], where we have introduced a new version of the central-upwind schemes—*central-upwind Rankine-Hugoniot* schemes—which contain a smaller amount of numerical dissipation, but at the same time are as robust as the original central-upwind schemes. The numerical dissipation is reduced using the following two mechanisms. First, we utilize the discrete Rankine-Hugoniot conditions in order to more accurately estimate the local propagation speeds. The second mechanism, which can be applied only in the multidimensional case, is related to the numerical dissipation switch recently proposed in [7]. The main idea of the switch is to reduce the local propagation speeds in the areas where the solution is near/at contact waves and shear layers. Such areas are detected by measuring of the relative contribution of the parts of the fluxes, which are supposed to be continuous at contact waves and shear layers, into the total flux jumps across each cell interface.

Equipped with the central-upwind Rankine-Hugoniot numerical fluxes, we incorporate them into the A-WENO finite-difference framework; see [8]. The developed A-WENO finite-difference schemes are very simple as they utilize the finite-volume central-upwind Rankine-Hugoniot numerical fluxes per se. A higher-order accuracy is achieved by:

- Computing the left and right point values at each cell interface using the fifth-order WENO-Z interpolant;
- Using these highly accurate point values for the evaluation of central-upwind Rankine-Hugoniot numerical fluxes;
- Including the finite-difference approximations of the partial derivatives of the fluxes required to be added to the second-order finite-volume numerical fluxes according to the sixth-order accurate Taylor expansion of the fluxes about the cell interface points.

Finally, we present the implementation of the fifth-order A-WENO schemes for designing a new hybrid multifluid algorithms based on the path-conservative

central-upwind scheme. In this work [2], we study compressible multifluids, which are supposed to be immiscible with the sharp interface separating different fluid components. The material interface is tracked using the level-set method and the pressure/velocity oscillations, which typically appear at the interfaces and then evolve in time and destroy the computed solutions if conservative numerical methods are used, are avoided using the hybrid approach originally proposed in [5]: we replace the total energy equation with the nonconservative pressure equation in the immediate vicinity of the material interfaces. Outside the interface, we solve the compressible Euler equations using either the finite-volume central-upwind or the finite-difference A-WENO scheme, while the nonconservative system, obtained after replacing the total energy equation with the pressure one, must be solved using an appropriate numerical method for nonconservative hyperbolic systems. We use the path-conservative central-upwind scheme from [1], which is designed to properly handle the contribution of the nonconservative product terms at each cell interface. The fifth-order accuracy of the path-conservative central-upwind scheme is obtained using the finite-difference A-WENO framework as proposed in [3].

## References


[1] M. J. Castro Díaz, A. Kurganov and T. Morales de Luna, *Path-conservative central-upwind schemes for nonconservative hyperbolic systems*, ESAIM Math. Model. Numer. Anal. **53** (2019), 959–985.

[2] A. Chertock, S. Chu and A. Kurganov, *Hybrid multifluid algorithms based on the path-conservative central-upwind scheme*. Preprint.

[3] S. Chu, A. Kurganov and M. Y. Na, *Fifth-order A-WENO finite-difference schemes based on the path-conservative central-upwind fluxes*. In preparation.

[4] N. K. Garg, A. Kurganov and Y. Liu, *Semi-discrete central-upwind Rankine-Hugoniot schemes for hyperbolic systems of conservation laws*, J. Comput. Phys. **428** (2021), 110078.

[5] S. Karni, *Hybrid multifluid algorithms*, SIAM J. Sci. Comput. **17** (1996), 1019–1039.

[6] A. Kurganov, *Central schemes: A powerful black-box solver for nonlinear hyperbolic PDEs*, Handbook of numerical methods for hyperbolic problems, 525-548, Handb. Numer. Anal., **17**, Elsevier/North-Holland, Amsterdam, 2016.

[7] A. Kurganov, Y. Liu and V. Zeitlin, *Numerical dissipation switch for two-dimensional central-upwind schemes*, ESAIM Math. Model. Numer. Anal. To appear.

[8] B.-S. Wang, W. S. Don, A. Kurganov and Y. Liu, *Fifth-order A-WENO finite-difference schemes based on the central-upwind Rankine-Hugoniot fluxes*. Preprint.


## On using Neural Networks to discretize transport equations with Lipschitz data

Bruno Després

On the one hand Neural Networks or Machine Learning (ML) [1] appear as a promising set of methods for interpolation of data and fitting of non linear functions. On the other hand finite volume (FV) schemes used in hyperbolic solvers are becoming more and more complex and non linear. So following [3, 4, 5], it is natural to ask if Neural Networks can help to make some progress on the design of efficient non linear FV solvers.

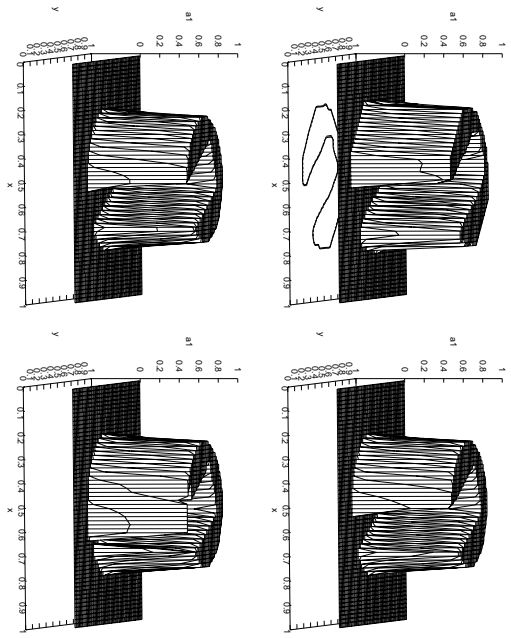

FIGURE 1. A convincing example from [7]. The value of volume fraction $a_1$ at time $t = 1$ is represented after one round advection. Top left, initial and final data. Top right, SLIC/Downwind scheme: the notched circle is transformed in a polygonal line as shown by the contour lines. Bottom left, blocks are $5 \times 5$, but the ML dataset does not have acute angles. Bottom right, blocks are $5 \times 5$, now the ML dataset has acute angles.
The new VoF-ML flux (bottom right) with optimal training and dataset is superior to any kind of traditional FV technique for this problem.

In [7] we evaluated this idea for a problem which is badly solved with traditional finite volume schemes. It is the design of solvers for indicatrix functions with corners which can be used to transport datas with Lipschitz regularity. This is deeply related to VoF methods [8, 9]. A model PDE is

$$\begin{cases} \partial_t \alpha + \mathbf{a} \cdot \nabla \alpha = 0, & x \in \mathbb{R}^2,\ t > 0, \\ \alpha(0, x) = \alpha_0(x), & x \in \mathbb{R}^2, \end{cases}$$

where $\alpha_0 \in \text{Lip}$ is the indicatrix function of a domain $\omega \subset \mathbb{R}^2$ where the perimeter of $\omega$ is bounded. If $\omega$ is a disk or a sphere, it models the transport of bubbles. If $\omega$ is a square, it models corners and triple points which may arise in some three-phases problems. A classical example is the steam+water+structure triple point

in nuclear engineering, or air+water+structure triple point for the modeling of boats.

The design of the FV+ML follows the classical avenue of learning methods, that is one assembles a dataset from a collection of numerical profiles as in [3]. Then the learning is performed with a standard software (Tensorflow in our case). What is remarkable is the accuracy of the result reported in [7] when everything runs correctly with the FV numerical simulation, see the Figure.

**An open problem.** It appears that it is relatively easy to enforce the VOF-ML (Volume-of-Fluid-Machine-Learning) scheme to satisfy the natural $L^\infty$ bounds associated to the PDE. On the contrary, it seems difficult to impose a rigorous control of oscillations (only strategy so far is try-or-fail). The situation can be reformulated purely in terms of convergence theory for non standard FV schemes: how a finite volume flux which does not respect the usual continuity condition [6] (nor any kind of TVD condition [2]) can be convergent? A positive answer to this open problem would be a strong impetus for the development of these FV solvers designed with ML techniques.

## References


[1] I. Goodfellow, Y. Bengio, A. Courville, Deep Learning, MIT Press, 2016, *http://www.deeplearningbook.org*.

[2] Goodman, Jonathan B. ; LeVeque, Randall J. On the accuracy of stable schemes for 2D scalar conservation laws. Math. Comp. **45** (1985), no. 171, 15-21.

[3] D. Ray and J. S. Hesthaven, An artificial neural network as a troubled-cell indicator, Journal of Computational Physics **367** (2018), 166-191.

[4] Mishra, Siddhartha. A machine learning framework for data driven acceleration of computations of differential equations. Math. Eng. **1** (2019), no. 1, 118-146.

[5] Bourriaud, Alexandre ; Loubère, Raphaël ; Turpault, Rodolphe. A priori neural networks versus a posteriori MOOD loop: a high accurate 1D FV scheme testing bed. J. Sci. Comput. **84** (2020), 31.

[6] Eymard, Robert ; Gallouët, Thierry ; Herbin, Raphaèle . Finite volume methods. Handbook of numerical analysis, Vol. VII, 713-1020, Handb. Numer. Anal., VII, North-Holland, Amsterdam, 2000.

[7] Després, Bruno ; Jourdren, Hervé. Machine learning design of volume of fluid schemes for compressible flows. J. Comput. Phys. **408** (2020), 109275.

[8] D.L. Youngs, Time-dependent multi-material flow with large fluid distortion, Numerical methods for fluid dynamics, Morton and Baines Editors (1982), 273-285.

[9] Y. Qi, J. Lu, R. Scardovelli, S. Zaleski and G. Tryggvason, Computing curvature for volume of fluid methods using machine learning, Journal of Computational Physics **377** (2019), 155-161.


## On the generalized polynomial chaos expansion for hyperbolic systems

Michael Herty

(joint work with S. Gerster and E. Iacomini)

We analyze properties of stochastic $2 \times 2$ hyperbolic systems using a Galerkin formulation, which reformulates the stochastic system as a deterministic one that describes the evolution of polynomial chaos (PC) modes. We investigate conditions

such that the resulting systems are hyperbolic in the case of the $p-$system as well as the second-order traffic flow model of Aw-Rascle and Zhang and here we elaborate in particular on this model. We are interested in uncertainty quantification in the sense of propagation of input uncertainty through traffic models. Several approaches are presented in the literature and can be classified in non-intrusive and intrusive methods. The main idea underlying the former approach is to solve the model for fixed number of samples using deterministic numerical algorithms. Then, the statistics of the quantities of interest are determined by numerical quadrature. Typical examples are Monte-Carlo and stochastic collocation methods [8]. In contrast, we consider the intrusive stochastic Galerkin method. Here, stochastic processes are represented as piecewise orthogonal functions, for instance Legendre polynomials or multiwavelets. These representations are known as generalized polynomial chaos (gPC) expansions [5, 2, 6, 7]. Expansions of the stochastic input are substituted into the governing equations and a Galerkin projection is used to obtain deterministic evolution equations for the coefficients of the series expansions.

Second-order models describe the velocity by an additional differential equation. In particular, we consider the **inhomogeneous Aw-Rascle-Zhang model** [1] with relaxation

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t\big(v + h(\rho)\big) + v\partial_x\big(v + h(\rho)\big) = \dfrac{1}{\tau + \xi}\big(V(\rho) - v\big). \end{cases} \tag{1}$$

Here, $h(\rho) : R^+ \to R^+$ is called hesitation or traffic pressure and we may consider e.g. $h(\rho) = \rho$. Also, $V$ is a given, desired velocity distribution. We already include the parametric uncertainty $\xi$ in the relaxation rate as a random variable $\xi$, defined on a probability space $\big(\Omega, \mathcal{F}(\Omega), \mathbb{P}\big)$. Then, the random solution $U(t, x, \xi) = (\rho(t, x, \xi), w(t, x, \xi))$ where $w = v + h(\rho)$ is expanded in terms of a truncated generalized polynomial chaos (gPC) series with base functions $\phi_k$ that is orthogonal to the probability density $p$ of the random variable $\xi$.

$$G_K[\hat{U}](t, x, \xi) \coloneqq \sum_{k=0}^{K} \hat{U}_k(t, x)\phi_k(\xi) \quad \text{with gPC modes} \quad \hat{U} \coloneqq \begin{pmatrix} \hat{\rho} \\ \hat{w} \end{pmatrix} \in \mathbb{R}^{2(K+1)},$$

It remains to derive the equations for the evolution of the gPC models $\hat{U}$ by Galerking projection of the flux onto the space $S_k = \{\phi_i : i = 0, \dots, K\}$. It turns out that in the formulation of $\hat{\rho}$ and $\hat{w}$ and for $h(\rho) = \rho$ this requires only to define the Galerkin product[3, 4]

$$\hat{\rho} * \hat{\alpha} \coloneqq P(\hat{\rho})\hat{\alpha} = \hat{z} \in \mathbb{R}^{K+1} \quad \text{for} \quad P(\hat{\rho}) \coloneqq \sum_{k=0}^{K} \hat{\rho}_k \mathcal{M}_k$$

where $\mathcal{M}_k \coloneqq \langle \phi_k, \phi_i\phi_j \rangle_{i,j=0,\dots,K}$. Several properties of the matrix $P(\hat{\rho})$ are known and can be exploited in particular for **wavelet-based** gPC expansions, see e.g. [4, Section 3.3]. Using the previously defined Galerkin product hyperbolicity properties as well as positivity properties of the car or gas density can be obtained.

The expansion in $(\rho, w)$ also allows to obtain also in particular cases the eigendecomposition of the gPC expanded system in closed form. Further, the expanded system can be studied in the small relaxation limit $\tau \to 0$ when the uncertainty is only in the initial data. Here, a stabilization result is obtained, when the system is relaxed to a first-order model.

An extension of the previous results towards general gPC expansions is still open as well as the corresponding properties. The question of suitable entropies in the case of second-order traffic flow model for the gPC expanded system is still open. Since the considered traffic flow model is an example of a Temple class system, one might also expect additional results for general Temple systems. Those points are still open.

Support by DFG HE5386/19-1 is acknowledged.

## References

[1] A. Aw and M. Rascle. Resurrection of "second order" models of traffic flow? *SIAM J. Appl. Math.* **60** (2000), 916–938.

[2] B. J. Debusschere, H. N. Najm, P. P. Pébay, O. M. Knio, R. G. Ghanem, and O. P. L. Maître. Numerical challenges in the use of polynomial chaos representations for stochastic processes. *SIAM Journal on Scientific Computing* **26** (2004), 698–719.

[3] S. Gerster; M. Herty; A. Sikstel, *Hyperbolic stochastic Galerkin formulation for the p-system*, J. Comput. Phys. **395** (2019), 186–204.

[4] S. Gerster; M. Herty; E. Iacomini, *Stability analysis of a hyperbolic stochastic Galerkin formulation for the Aw-Rascle-Zhang model with relaxation*, arXiv Preprint **arXiv:2102.09359** (2021).

[5] D. Gottlieb and J. S. Hesthaven. Spectral methods for hyperbolic problems. *Journal of Computational and Applied Mathematics* **128** (2001), 83–131.

[6] S. Jin and R. Shu. A study of hyperbolicity of kinetic stochastic Galerkin system for the isentropic Euler equations with uncertainty. *Chinese Annals of Mathematics, Series B* **40** (2019), 765–780.

[7] J. Kusch, J. Wolters, and M. Frank. Intrusive acceleration strategies for uncertainty quantification for hyperbolic systems of conservation laws. *Journal of Computational Physics* **419** (2020), 109698.

[8] O. P. L. Maître and O. M. Knio. *Spectral Methods for uncertainty quantification*. Springer Netherlands, 1 edition, 2010.

## Mathematical models of traffic flow

Nils Henrik Risebro

The early mathematical modeling of traffic flow dates back to at least the early fifties. Two distinct classes of models prevail — one where one tracks individual vehicles (Follow-the-Leader models) and one where traffic is sufficiently dense to justify a continuum approach where the density of vehicles is the fundamental quantity (traffic hydrodynamics). With the increased importance of traffic in modern society, the development of unprecedented computer power and tracking devices, as well as the advances in mathematical research, mathematical traffic modeling has become important. As with all mathematical modeling, it is essential to identify simple models that capture some important aspect of the phenomena

yet allow for a mathematical analysis. We will here describe a novel mathematical model that allows for the analysis of multilane traffic. However, before we come to that, let us start with the basics. Consider dense unidirectional traffic on a single lane where $\rho = \rho(x,t)$ describes the vehicular density. Let $v$ describe the velocity, and consider traffic on an interval $[x_1, x_2]$. The rate of change of the number of vehicles on this interval of the road, can be described by

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho(x,t)\,dx = -\Big|_{x_1}^{x_2} \rho v = -\int_{x_1}^{x_2} \frac{\partial}{\partial x}\rho v\,dx.$$

Re-ordering the expressions, taking the limit $x_2 - x_1 \to 0$, and assuming smoothness of all quantities, we obtain

$$\rho_t + (\rho v)_x = 0,$$

where subscripts denote partial derivatives. This is the celebrated Lighthill–Whitham–Richards (LWR) model, see [10, 11, 9]. In the simplest case we assume that the velocity depends on the density only, and mathematically we obtain a hyperbolic conservation law. If we scale the maximum density to unity, and assume a linear dependence in the velocity, that is, $v = 1 - \rho$ we obtain the (equivalent of the) inviscid Burgers equation

$$\rho_t + (\rho(1-\rho))_x = 0.$$

While innocent looking, solutions of the Burgers equation experience singularities, and in general are far from being smooth. Thus we need to develop the machinery of weak solutions and what is called entropy conditions to single out unique solutions. Entropy solutions are limits of solutions to the singularly perturbed equation

$$\rho^\varepsilon_t + \left(\rho^\varepsilon\,(1-\rho^\varepsilon)\right)_x = \varepsilon\rho^\varepsilon_{xx}$$

as $\varepsilon \to 0$, see [3].

There is actually a simple relation between the above macroscopic model and a simple microscopic follow-the-leader model. we assume that the speed of a vehicle $v$, is a function of the distance to the car in front. If $z_i$ is the position of the $i$th car on a single lane road, the model reads

$$\frac{d}{dt}z_i = v\Big(\frac{\ell}{z_{i+1} - z_i}\Big), \tag{1}$$

where $\ell$ is the length of a vehicle and $v$ is a decreasing function. For obvious reasons $z_{i+1} - z_i \geq \ell$. This simply says that the speed of each car is depending only on the distance to the car in front. That $v$ is decreasing means that the closer you are to the car in front of you, the slower you drive.

We define $\rho_i = \ell/(z_{i+1} - z_i)$, a straightforward calculation yields

$$\frac{d}{dt}\Big(\frac{1}{\rho_i}\Big) - \frac{1}{\ell}\Big(v(\rho_{i+1}) - v(\rho_i)\Big) = 0. \tag{2}$$

Now we can let $\ell \to 0$ with $z = i\ell$ fixed and obtain $\rho_i(t) \to \rho(t,z)$, where $\rho$ satisfies

$$\frac{\partial}{\partial t}\Big(\frac{1}{\rho}\Big) - \frac{\partial}{\partial z}v(\rho) = 0 \tag{3}$$

since (2) is a first order (in $\ell$) semi-discrete scheme for (3). To argue that the limit is the entropy solution we observe that (2) is actually a second order accurate scheme for the equation

$$\frac{\partial \xi}{\partial t} - \frac{\partial}{\partial z} V(\xi) = \frac{\ell}{2} \frac{\partial^2}{\partial z^2} \left( V(\xi) \right),$$

where

$$\xi = \frac{1}{\rho} \quad \text{and} \quad V(\xi) = v\Big(\frac{1}{\xi}\Big).$$

We can introduce a new coordinate $x = x(t, z)$ by

$$x_z = \frac{1}{\rho} \quad \text{and} \quad x_t = v\left(\rho\right),$$

to find that

$$\rho_t + (\rho v(\rho))_x = 0.$$

The above reasoning was formal and assumed differentiability of everything, but one can establish rigorously that the limit $\rho$ exists and is an entropy solution, [5, 6, 8].

Studying more general velocity functions, allowing these to depend on time and position, we see that the "hydrodynamic" approach to traffic on a single lane road is a rich source of interesting mathematical problems even in this very simple case.

**Two lanes.** Two lanes are modelled as two individual roads, where vehicles move according to the follow-the-leader model (1), and in addition cars are allowed to change lane. Our basic assumption is that the likelihood of a driver changing lane is zero if changing lanes would lead to a decrease in speed and is proportional to the potential gain in speed otherwise. This is a simple idea, but a bit awkward to describe mathematically.

Let the positions of the vehicles on the two lanes, lane $z$ and lane $y$, be denoted by $\{z_i\}$ and $\{y_j\}$ respectively, we assume that the drivers continuously monitor the speed (and thereby the position) they would have if they were in the other lane. Concretely, let $\Delta t$ be some small positive number, and let $\tilde{z}_i$ denote the position of an imaginary vehicle in lane $y$ in the interval $[t, t + \Delta t]$, and assume that $y_j$ is the position of the car in front of $\tilde{z}_i$ in the $y$ lane. Then $\tilde{z}_i$ satisfies

$$\tilde{z}_i'(\tau) = v\Big(\frac{\ell}{y_j(\tau) - \tilde{z}_i(\tau)}\Big), \ \tau \in (t, t + \Delta t], \quad \tilde{z}_i(t) = z_i(t).$$

The probability that the vehicle changes lane in the interval $[t, t+\Delta t]$ is then given by

$$\phi\left(\tilde{z}_i(t + \Delta t) - z_i(t + \Delta t)\right),$$

where $\phi$ is an increasing Lipschitz continuous function with $\phi(s) = 0$ for $s \leq 0$, $\lim_{s\to\infty} \phi(s) = 1$. Observe that in this model, drivers behave rather selfishly in that they do not care about the consequences for others (in particular for vehicle $y_{j-1}$!) of their lane changing. Nevertheless, we see others behaving like this all the time.

Taking the formal limit as $\Delta t \to 0$ and $\ell \to 0$, we get (as in the single lane case) that

$$\frac{\ell}{z_{i+1}-z_i} \to \rho_1(x,t), \quad \frac{\ell}{y_{j+1}-y_j} \to \rho_2(x,t),$$

so that $\rho_1$ is the density of vehicles in lane $z$ and $\rho_2$ the density of vehicles in lane $y$. The lane changing leads to a flux from lane $z$ to lane $y$ given by

$$S(\rho_1,\rho_2) = K\left[(v(\rho_2)-v(\rho_1))^+ \rho_1 - (v(\rho_2)-v(\rho_1))^- \rho_2\right],$$

where $a^\pm = (|a| \pm a)/2$, and $K$ is a constant. Therefore conservation of cars reads

$$\begin{aligned} &\frac{\partial \rho_1}{\partial t} + \frac{\partial}{\partial x}(\rho_1 v(\rho_1)) = -S(\rho_1,\rho_2), \\ &\frac{\partial \rho_2}{\partial t} + \frac{\partial}{\partial x}(\rho_2 v(\rho_2)) = S(\rho_1,\rho_2). \end{aligned} \tag{4}$$

This is a weakly coupled system of conservation laws, such systems are well posed in the sense of Hadamard. Furthermore, its special structure allows for the sharp estimate of the difference between $(\rho_1,\rho_2)$ and another solution $(\hat\rho_1,\hat\rho_2)$, see [7],

$$\sum_{k=1}^{2} \|\rho_k(\cdot,t) - \hat\rho_k(\cdot,t)\|_{L^1(\mathbb{R})} \le \sum_{k=1}^{2} \|\rho_k(\cdot,0) - \hat\rho_k(\cdot,0)|_{L^1(\mathbb{R})}.$$

We emphasise that in contrast to the single lane case, the limits leading to (4) are (for the moment) not rigorously established. An analogous estimate holds for models describing $N$ lanes. Mathematically, and probably outside the realm of traffic models, we can let $N \to \infty$. Then the "lane" is determined by a continuous coordinate $y \in (0,1)$ and it turns out that $\rho(x,t,y) \approx \rho^{Ny}(t,x)$ is an entropy solution to a degenerate convection diffusion equation

$$\begin{cases} \rho_t + (\rho v(\rho))_x = -\left(\rho v(\rho)_y\right)_y & t>0, \quad (x,y)\in\mathbb{R}\times(0,1), \\ v(\rho)_y = 0 \quad x\in\mathbb{R}, \quad y=0, \quad y=1, \\ \rho(0,x,y) = \rho_0(x,y) \quad (x,y)\in\mathbb{R}\times(0,1). \end{cases}$$

For precise results, see [1].

## References


[1] G.M. Coclite, H. Holden, and N. H. Risebro. Singular diffusion with Neumann boundary conditions *arXiv:2004.12428v1*.

[2] M. Garavello, K. Han, and B. Piccoli. *Models for Vehicular Traffic on Network.* AIMS Series on Applied Mathematics, vol. **9**. AIMS, Springfield, MO, 2016.

[3] H. Holden and N. H. Risebro. *Front Tracking for Hyperbolic Conservation Laws*, volume 152 of *Applied Mathematical Sciences*. Springer, Heidelberg, second edition, 2015.

[4] H. Holden and N. H. Risebro. A mathematical model of traffic flow on a network of unidirectional roads. *SIAM J. Math. Anal.* **26** (1995), 999–1017.

[5] H. Holden and N. H. Risebro. The continuum limit of Follow-the-Leader models—a short proof. *Discrete Contin. Dyn. Syst.* **38** (2018), 715–722.

[6] H. Holden and N. H. Risebro. Follow-the-leader models can be viewed as a numerical approximation to the Lighthill–Whitham–Richards model for traffic flow. *Netw. Heterog. Media* **13** (2018), 409–421.

[7] H. Holden and N. H. Risebro. Models for dense multilane vehicular traffic. *SIAM J. Math. Anal.* **51** (2019), 3694–3713.
[8] M. Di Francesco and M. D. Rosini. Rigorous derivation of nonlinear scalar conservation laws from follow-the-leader type models via many particle limit. *Arch. Ration. Mech. Anal.* **217** (2015), 831–871.
[9] V.L. Knoop. *Introduction to Traffic Flow Theory.* TU Delft, The Netherlands, 2nd ed., 2018.
[10] M. J. Lighthill and G. B. Whitham. Kinematic waves. II. A theory of traffic flow on long crowded roads. *Proc. Roy. Soc. (London), Series A* **229** (1955), 317–345.
[11] P. I. Richards. Shockwaves on the highway. *Operations Research* **4** (1956), 42–51.

## Topological states in collective dynamics

Pierre Degond
(joint work with Antoine Diez, Mingye Na)

The present abstract summarizes the submitted article [1]. This work lies at the intersection of two topics: topological states of matter on the one hand and collective dynamics on the other hand.

States of matter (such as solid, liquid, etc) are characterized by different types of order associated with local invariances under different transformation groups. Recently, a new notion of topological order, popularized by the 2016 physics nobel prize awarded to Haldane, Kosterlitz and Thouless, has emerged. It refers to the global rigidity of the system arising in some circumstances from topological constraints. Topologically ordered states are extremely robust i.e. "topologically protected" against localized perturbations.

Collective dynamics occurs when a system of self-propelled particles organizes itself into a coherent motion, such as a flock, a vortex, etc. Collective dynamics generate large-scale structures, much bigger than the typical interaction range between two agents. Lately, the mathematical understanding and numerical simulation of collective dynamics models has stimulated a large literature. In particular, understanding collective dynamics requires the joint use of microscopic agent-based (aka particle) models and macroscopic fluid-like models whose links must be rigorously justified. The present work relies on such earlier studies concerning a system of self-propelled solid bodies interacting through local full body alignment up to some noise developed in collaboration with A. Frouvelle (Paris-Dauphine), S. Merino-Aceituno (Vienna) and A. Trescases (Toulouse) [2, 3, 4].

Recently, the question of realizing topologically protected collective states has received increasing attention. In this work, we show that the above-mentioned body-orientation model gives rise to a system of hydrodynamic type equations in the large-scale limit. This macroscopic model exhibits topologically non-trivial explicit solutions characterized by non-vanishing appropriately defined winding numbers. These solutions have counterparts at the particle level. However, such particle solutions persist for a certain time but eventually decay towards a uniform flocking state, due to the stochastic nature of the particle system. We show numerically that the persistence time of these topologically non-trivial solutions

is far longer than for topologically trivial ones, showing a new kind of "topological protection" of a collective state. To our knowledge, it is the first time that a hydrodynamic model guides the design of topologically non-trivial states of a particle system and allows for their quantitative analysis and understanding.

### References

[1] P. Degond, A. Diez, M. Na, *Bulk topological states in a new collective dynamics model*, submitted. arXiv manuscript 2101.10864 (2021).

[2] P. Degond, A. Frouvelle, and S. Merino-Aceituno, *A new flocking model through body attitude coordination*, Math. Models Methods Appl. Sci., **27** (2017) 1005–1049.

[3] P. Degond, A. Frouvelle, S. Merino-Aceituno, and A. Trescases, *Quaternions in collective dynamics*, Multiscale Model. Simul. **16** (2018), 28–77.

[4] P. Degond, A. Frouvelle, S. Merino-Aceituno, and A. Trescases, *Alignment of self-propelled rigid bodies: from particle systems to macroscopic equations*, in G. Giacomin, S. Olla, E. Saada, H. Spohn, and G. Stoltz, editors, Stochastic Dynamics Out of Equilibrium, volume 282 of "Springer Proceedings in Mathematics and Statistics", pages 28–66. Institut Henri Poincaré, Paris, France, 2017, Springer International Publishing, 2019.

## A structure-preserving staggered semi-implicit scheme for continuum mechanics

Michael Dumbser

(joint work with I. Peshkov, E. Romenski, W. Boscheri, M. Ioriatti)

In this talk, we present a new class of structure-preserving semi-implicit schemes for the unified first order hyperbolic model of Newtonian continuum mechanics proposed by Godunov, Peshkov and Romenski (GPR), see [1, 2, 3, 4, 5, 6]. The GPR model is a geometric approach to continuum mechanics, which is able to describe the behavior of moving elasto-plastic solids as well as inviscid and viscous Newtonian and non-Newtonian fluids within one and the same hyperbolic governing PDE system. This is achieved via appropriate relaxation source terms in the evolution equations for the distortion field and the thermal impulse. In previous work it has already been shown that the GPR model reduces to the compressible Navier-Stokes equations in the stiff relaxation limit when the relaxation times tend to zero, see [6]. The governing PDE system belongs to the class of symmetric hyperbolic and thermodynamically compatible systems (SHTC), which have been studied for the first time by Godunov in 1961 and later in a series of papers by Godunov & Romenski. An important feature of the proposed model is that the propagation speeds of all physical processes, including dissipative processes, are finite. In the absence of source terms, the homogeneous part of the GPR model is endowed with some natural involutions, namely the distortion field A and the thermal impulse J need to remain curl-free.

In this talk we present a new structure-preserving scheme that is able to preserve the curl-free property of both fields exactly also on the discrete level. This is achieved via the definition of appropriate and compatible discrete gradient and curl operators on a judiciously chosen staggered grid, see [9]. Furthermore, the pressure terms are discretized implicitly in order to capture the low Mach number limit of

the equations properly, while all other terms are discretized explicitly [7, 8]. In this manner, the resulting pressure system is symmetric and positive definite and can be solved with efficient iterative solvers like the conjugate gradient method. Last but not least, the new staggered semi-implicit scheme is also able to reproduce the stiff relaxation limit of the governing PDE system properly, recovering an appropriate discretization of the compressible Navier-Stokes equations. To the best of our knowledge, this is the first pressure-based semi-implicit scheme for nonlinear continuum mechanics that is able to preserve all involutions and asymptotic limits of the original governing PDE system also on the discrete level [9].

Computational results for several test cases are presented in order to illustrate the performance of the new scheme.

## References


[1] S.K. Godunov, *An interesting class of quasilinear systems*, Dokl. Akad. Nauk SSSR **139** (1961), 521-523.

[2] S.K. Godunov and E.I. Romenski, *Nonstationary equations of nonlinear elasticity theory in Eulerian coordinates*, Journal of Applied Mechanics and Technical Physics **13** (1972), 868-884.

[3] E.I. Romenski, *Hyperbolic systems of thermodynamically compatible conservation laws in continuum mechanics*, Mathematical and Computer Modelling **28** (1998), 115–130.

[4] S.K. Godunov and E.I. Romenski, *Elements of continuum mechanics and conservation laws*, Kluwer Academic/Plenum Publishers (2003).

[5] I. Peshkov and E. Romenski, *A hyperbolic model for viscous Newtonian flows*, Continuum Mechanics and Thermodynamics **28** (2016), 85–104.

[6] M. Dumbser, I. Peshkov, E. Romenski and O. Zanotti, *High order ADER schemes for a unified first order hyperbolic formulation of continuum mechanics: viscous heat-conducting fluids and elastic solids*, Journal of Computational Physics **314** (2016), 824–862.

[7] M. Dumbser and V. Casulli, *A Conservative, Weakly Nonlinear Semi-Implicit Finite Volume Method for the Compressible Navier-Stokes Equations with General Equation of State*, Applied Mathematics and Computation **272** (2016), 479–497.

[8] M. Dumbser, D.S. Balsara, M. Tavelli and F. Fambri, *A divergence-free semi-implicit finite volume scheme for ideal, viscous and resistive magnetohydrodynamics*, International Journal for Numerical Methods in Fluids **89** (2019), 16–42.

[9] W. Boscheri, M. Dumbser, M. Ioriatti, I. Peshkov and E. Romenski, *A structure-preserving staggered semi-implicit finite volume scheme for continuum mechanics*, Journal of Computational Physics **424** (2021), 109866.


## Rectifiability of entropy defect measures for Burgers equation and applications to a variational problem

Elio Marconi

We consider bounded weak solutions to the inviscid Burgers equation

$$\partial_t u + \partial_x \left(u^2/2\right) = 0. \tag{1}$$

A pioneering result in the theory of conservation laws establishes the well-posedness of the associated Cauchy problem in the class of bounded entropy solutions, namely bounded functions $u$ satisfying

$$\mu_\eta := \partial_t \eta(u) + \partial_x q(u) \leq 0 \qquad \text{in } \mathcal{D}' \tag{2}$$

for every convex entropy $\eta : \mathbb{R} \to \mathbb{R}$ and corresponding flux $q : \mathbb{R} \to \mathbb{R}$ defined up to constants by $q'(v) = \eta'(v)v$. We are interested in the more general class of weak solutions with finite entropy production, where the distribution $\mu_\eta$ in (2) is only required to be a finite Radon measure (without constraints on its sign). Although weak solutions with finite entropy production are not locally of bounded variation, they share with BV functions most of their fine properties:

**Theorem 1** ([6])**.** *Let $u$ be a bounded weak solution to* (1) *with finite entropy production. Then there exists an $\mathcal{H}^1$-rectifiable set $J$ for which*

(1) *for $\mathcal{H}^1$-a.e. $x \in J$ there exist strong $L^1$-traces on both sides;*
(2) *every $x \in J^c$ is a vanishing mean oscillation point of $u$.*

The analogy with the structure of solutions with bounded variation is still not complete; the main result of this presentation is a contribution in this direction:

**Theorem 2.** *Let $u$ be a bounded weak solution to* (1) *with finite entropy production and $J$ as in Theorem 1. Then*

(1) *the set of non-Lebesgue points of $u$ has Hausdorff dimension at most 1* [7, 8]*;*
(2) *for every smooth entropy $\eta$ the measure $\mu_\eta$ is concentrated on $J$* [9].

The interest towards solutions with finite entropy production is motivated by the study of the asymptotic behavior as $\varepsilon \to 0^+$ of the following functionals introduced by Aviles and Giga [2]:

$$F_\varepsilon(u, \Omega) := \int_\Omega \left( \varepsilon |\nabla^2 u| + \frac{1}{\varepsilon} \left| 1 - |\nabla u|^2 \right|^2 \right) dx, \qquad \text{where } \Omega \subset \mathbb{R}^2.$$

They conjectured that limits as $\varepsilon \to 0$ of minimizers of $F_\varepsilon$ are solutions $u$ of the eikonal equation $|\nabla u| = 1$ which minimize

$$F_0(u, \Omega) := \frac{1}{3} \int_{J_{\nabla u}} |\nabla u^+ - \nabla u^-|^3 d\mathcal{H}^1,$$

where $J_{\nabla u}$ denotes the jump set of $\nabla u$ and $\nabla u^\pm$ the corresponding traces. A first difficulty is that we do not know if sequences of minimizers converge to solutions $u$ for which $\nabla u \in \mathrm{BV}(\Omega)$, hence it is already not trivial to give a meaning to the definition of $F_0$. If $u$ solves the eikonal equation, then $m := \nabla^\perp u =: (\cos\theta, \sin\theta)$ is divergence free, namely it solves the conservation law

$$\partial_{x_1} \cos\theta + \partial_{x_2} \sin\theta = 0. \tag{3}$$

The asymptotic domain of the functionals $F_\varepsilon$ is contained in the class of solutions to (3) with finite entropy production [5] and a result analogous to Theorem 1 holds also in this setting [3]. Moreover a functional $G$, coinciding with $F_0$ for solutions $u$ to the eikonal equation with $\nabla u \in \mathrm{BV}(\Omega)$, was proposed in [1] as candidate $\Gamma$-limit of $F_\varepsilon$ and the $\Gamma$-lim inf inequality was proved.

In order to complete the proof of the conjecture it would be sufficient to prove:

(1) $G(u,\Omega) = F_0(u,\Omega)$ for all $u$ such that $\nabla^\perp u$ is a weak solution to (3) with finite entropy production;
(2) $G(\cdot,\Omega) \geq \Gamma - \limsup_{\varepsilon\to 0} F_\varepsilon(\cdot,\Omega)$.

Both questions are still open and the missing part for proving (1) is the analogous of Property (2) in Theorem 2 for solutions to (3). This result is still not available, but it can be proven in some special cases of interest, in particular for minimizers of $G(\cdot,\Omega)$ when $\Omega$ is an ellipse. This allows to prove that $u = \mathrm{dist}(\cdot,\Omega)$ is the only minimizer of $G(\cdot,\Omega)$ with respect to its boundary conditions and it leads to

**Theorem 3** ([10])**.** *Let $\Omega \subset \mathbb{R}^2$ be an ellipse and for every $\varepsilon > 0$ let $u_\varepsilon$ be a minimizer of $F_\varepsilon(\cdot,\Omega)$ in the space*

$$\Lambda(\Omega) := \left\{ u \in W_0^{2,2}(\Omega) : \frac{\partial u}{\partial \nu} = -1 \text{ on } \partial\Omega \right\},$$

*where $\nu$ denotes the outer normal to $\Omega$. Then*

$$\lim_{\varepsilon\to 0} u_\varepsilon = \mathrm{dist}(\cdot,\Omega) \qquad \text{in } W^{1,1}(\Omega).$$

## References


[1] L. Ambrosio, C. De Lellis, and C. Mantegazza, *Line energies for gradient vector fields in the plane*, Calc. Var. Partial Differential Equations **9** (1999), 327–355.
[2] P. Aviles and Y. Giga, *A mathematical problem related to the physical theory of liquid crystal configurations*, Proc. Centre Math. Anal. Austral. Nat. Univ. (1987), 1–16.
[3] C. De Lellis and F. Otto, *Structure of entropy solutions to the eikonal equation*, JEMS **5** (2003), 107–145.
[4] C. De Lellis, F. Otto, and M. Westdickenberg, *Structure of entropy solutions for multi-dimensional scalar conservation laws*, Arch. Ration. Mech. Anal. **170** (2003), 137–184.
[5] A. De Simone, R. V. Kohn, S. Müller, and F. Otto, *A compactness result in the gradient theory of phase transitions* Proc. of the Royal Society of Edinburgh **131A** (2001), 833–844.
[6] M. Lecumberry. *Geometric structure of micromagnetic walls and shock waves in scalar conservation laws*, PhD thesis, Université de Nantes (2004).
[7] X. Lamy and F. Otto, *On the regularity of weak solutions to Burgers' equation with finite entropy production*, Calc. Var. Partial Differential Equations **57** (2018), Art. 94.
[8] E. Marconi, *On the structure of weak solutions to scalar conservation laws with finite entropy production*, arXiv:1909.07257 (2019).
[9] E. Marconi *The rectifiability of the entropy defect measure for Burgers equation*, arXiv:2004.09932 (2020).
[10] E. Marconi *Characterization of minimizers of Aviles-Giga functionals in special domains*, in preparation (2021).

## Well balancedness and error balance: observations and ideas related to the approximation of (hyperbolic) balance laws

Mario Ricchiuto

Consider the solution of nonlinear hyperbolic balance laws written in 1D as

$$\partial_t u + \partial_x f(u) + S(u; d(x)) = 0 \tag{1}$$

with some initial and appropriate boundary conditions, and with $S$ a very general source depending on some data $d$, and not only of $u$ but also on its derivatives: $S = S(u, \partial_t u, \partial_x u, \partial_{xx} u, \text{etc}; d)$. When approximating discretely solutions of (1) the usual notion of consistency wrt constant $u$ does not apply, as this is not a relevant solution of the problem. The concept of *well balanced* schemes is nothing else that an answer to this question. Relevant notions can be obtained by focusing on particular steady (and unsteady) states. For example, defining a global flux

$$\mathsf{g}(u; x) = f(u) + \int_{x_0}^{x} S(u(s); d(s)) ds \tag{2}$$

we can use the notion of consistency wrt constant $\mathsf{g}$, which is appropriate for (1). This notion does not necessary tells us something on the structure of $u$, but for special cases if can be expressed in terms of some set of steady state physical invariants. Very well known examples exist e.g. for the shallow water equations (cf [2, 3] and references therein). Another issue is that the mapping $\mathsf{g} \mapsto u$ (and reverse) may be quite complex. Nevertheless, the explicit and consistent use of the global flux for the definition of the numerical fluxes and of the approximation has been shown to be very effective [2]. Note that we can go even further and set

$$\mathcal{G}(u; x, t) = f(u) + \int_{x_0}^{x} (S(u(s); d(s)) + \partial_t u(s)) ds \tag{3}$$

This will give a genuine notion of spatial consistency which encompasses all terms in the PDE, and has potential for including propagating solutions in the picture.

This talk elaborates on several ideas to exploit and extend the above ideas. The starting point is a well known result concerning finite volume schemes dating back to [4]: for piecewise linear data (solution) and piecewise constant sources, assuming that the slopes are associated to cell equilibira, and integrating the balance law along the charactersitics provides schemes of the form

$$\Delta x \frac{du_i}{dt} + \phi_i^{i+1/2} + \phi_i^{i-1/2} = 0 \;\text{ with }\; \phi_i^{i+1/2} + \phi_{i+1}^{i+1/2} = \phi^{i+1/2} := \int_{x_i}^{x_{i+1}} (\partial_x f + S) \tag{4}$$

The scheme of [4] corresponds to the upwind splitting $\phi_{i+1}^{i+1/2} = \{\max(f', 0)/|f'|\}\phi^{i+1/2}$.

The scheme above is naturally consistent with a global flux formulation as trivially

$$\phi_i^{i+1/2} + \phi_{i+1}^{i+1/2} = \phi^{i+1/2} := \int_{x_i}^{x_{i+1}} (\partial_x f + S) = \int_{x_i}^{x_{i+1}} \partial_x \mathrm{g} = \mathrm{g}_{i+1} - \mathrm{g}_i \tag{5}$$

Defining pointwise global fluxes $\mathrm{g}_{i+1} = \mathrm{g}_i + \int_{x_i}^{x_{i+1}} S$, we can recast (4) in finite volume form

$$\Delta x \frac{du_i}{dt} + \hat{\mathrm{g}}_{i+1/2} - \hat{\mathrm{g}}_{i-1/2} = 0\,, \quad \hat{\mathrm{g}}_{i+1/2} = \mathrm{g}_i + \phi_i^{i+1/2} = \mathrm{g}_{i+1} - \phi_{i+1}^{i+1/2} \tag{6}$$

with the second equality in the definition of the *global numerical flux* $\hat{\mathrm{g}}_{i+1/2}$ a direct consequence of the consistency condition (5).

The talk further elaborates on this analogy in two directions. The first is related to the high order extensions of scheme (6) in space and time. Instead of introducing a polynomial reconstruction of the global flux as in [2], we investigate the use of a generalized notion of consistency based on (3). The definition of a numerical flux that includes the time derivative leads to a more general prototype

$$\hat{\mathcal{G}}_{i+1/2} - \hat{\mathcal{G}}_{i-1/2} = \Delta x \widehat{\frac{du_i}{dt}} + \hat{\mathrm{g}}_{i+1/2} - \hat{\mathrm{g}}_{i-1/2} = 0 \tag{7}$$

which expresses a global balance of all the terms involved in the PDE, and in which a *consistent numerical time derivative* appears which can be in general written as

$$\Delta x \widehat{\frac{du_i}{dt}} = \Delta x \frac{du_i}{dt} + \sum_j \Gamma_j \widehat{\frac{du_j}{dt}}\,, \quad \sum_j \Gamma_j = 0 \tag{8}$$

and where the $\Gamma_j$ coefficient (or matrices depending on the numerical flux) are error balancing terms, which improve the consistency of the scheme wrt a give numerical flux. Starting from the upwind flux, a second and third order examples obtained in this way are shown. The issue of marching in time (8) efficiently is resolved either by resorting to implicit time stepping, or by a predictor corrector iteration (see [1] and references therein)

(9)

$$\begin{aligned}\Delta x(u_i^{n+1} - u_i^n) + \int_{t^n}^{t^{n+1}} (\hat{\mathrm{g}}_{i+1/2} - \hat{\mathrm{g}}_{i-1/2}) = \Delta x(u_i^* - u_i^n) + \int_{t^n}^{t^{n+1}} (\hat{\mathrm{g}}_{i+1/2} - \hat{\mathrm{g}}_{i-1/2}) \\ - \int_{t^n}^{t^{n+1}} (\hat{\mathcal{G}}_{i+1/2} - \hat{\mathcal{G}}_{i-1/2})\end{aligned}$$

with $u^*$ a first order predictor obtained using a standard first order explicit Euler time discretization of (4), and where the time integral of the spatial flux is approximated to second order as $\Delta t(\hat{\mathrm{g}}^* + \hat{\mathrm{g}}^n)/2$. Despite its apparent complexity, when combining all the elements the corrector can be coded as

$$u_i^{n+1} = \frac{1}{2}u_i^* + \frac{1}{2}u_i^n - \frac{\Delta t}{2\Delta x}(\hat{\mathrm{g}}_{i+1/2} - \hat{\mathrm{g}}_{i-1/2})^* - \Delta x \widehat{(u_i^* - u_i^n)} \tag{10}$$

which is not more expensive than a standard high order RK2 explicit method. High order defect correction techniques can be used to go beyond second order. Several results involving complex source terms show the capability of retaining well balancing for very general steady as well as time dependnent problems, and no a-priori knowledge of any invariants.

The second issue issue explored in the talk is the multidimensional case. Scheme (4) can be written in this context as a multidimensional residual distribution, using an appropriate fully consistent treatment of the time derivative, and explicit predictor corrector time stepping as in (10). This has bee done in several references. For example in [3] we have shown how such method allows a well balanced treatment of moving water equilibria in shallow water flows with and without friction. The question which is addressed here is the meaning of global flux consistency in multiple dimensions. For a conservation law

$$\partial_t u + \nabla \cdot \vec{f}(u) + S(u; d(\vec{x})) = 0 \tag{11}$$

The natural generalization of the global flux formulation is

$$\begin{aligned} \partial_t u + \nabla \cdot (\vec{f}(u) + \vec{\sigma}) &= 0 \\ \nabla \cdot \vec{\sigma} &= S \end{aligned} \tag{12}$$

which now is a conservation law with a solenoidal involution ! The presentation explores this aspect further for residual distribution schemes. On Cartesian meshes, we will show an embedded compatibility with the solenoidal involution (12), based on the use of a rewriting of nodal residual distribution schemes in a staggered formalism with nodal unknowns, edge fluxes and cell sources. Other avenues to embed this type of constraints are also highlighted.

## References


[1] R. Abgrall, M. Ricchiuto, *High order methods for CFD*, Stein, de Borst, Hughes, Eds, Encyclopedia of Computational Mechanics. John Wiley & Sons, Ltd. (2017)

[2] Y. Cheng, A. Chertock, M. Herty, A. Kurganov, and T. Wu, *A new approach for designing moving-water equilibria preserving schemes for the shallow water equations*, J.Sci.Comp. **80** (2019), 538–554.

[3] M. Ricchiuto, *An explicit residual based approach for shallow water flows*, J.Comput.Phys. **280** (2015), 306–344.

[4] P.L. Roe, *Upwind differencing schemes for hyperbolic conservation laws with source terms*, In: Carasso, Serre and Raviart Ed.s, Nonlinear hyperbolic problems, Springer Berlin Heidelberg (1987), 41–51.

## Convex limiting and entropy fixes for finite element discretizations of hyperbolic conservation laws

Dmitri Kuzmin
(joint work with Manuel Quezada de Luna and Hennes Hajduk)

Algebraic flux correction (AFC) is a general framework for enforcing inequality constraints in numerical methods for conservation laws [6]. In this report, we review recent advances in the field of AFC for hyperbolic problems of the form

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{f}(u) = 0 \qquad \text{in } \Omega \times \mathbb{R}_+. \tag{1}$$

Using continuous ($\mathbb{P}_1$ or $\mathbb{Q}_1$) finite elements for discretization in space, we obtain

$$\sum_{j \in \mathcal{N}_i} m_{ij} \frac{\mathrm{d}u_j}{\mathrm{d}t} + \sum_{j \in \mathcal{N}_i \backslash \{i\}} (\mathbf{f}(u_j) - \mathbf{f}(u_i)) \cdot \mathbf{c}_{ij} = 0, \tag{2}$$

where $m_{ij}$ are entries of the consistent mass matrix, $\mathbf{c}_{ij}$ are entries of the discrete gradient/divergence operator, and $\mathcal{N}_i$ is the stencil of node $i$. Note that many other space discretizations can be written in this generic form.

Since the Galerkin space discretization (2) may violate maximum principles and entropy conditions, we replace it by the modified semi-discrete scheme [5, 6]

$$m_i \frac{\mathrm{d}u_i}{\mathrm{d}t} = \sum_{j \in \mathcal{N}_i \backslash \{i\}} [d_{ij}(u_j - u_i) - (\mathbf{f}(u_j) - \mathbf{f}(u_i)) \cdot \mathbf{c}_{ij} + f^*_{ij}], \tag{3}$$

where $m_i = \sum_{j \in \mathcal{N}_i} m_{ij} > 0$ and $f^*_{ij}$ is a suitably constrained approximation to

$$f_{ij} = m_{ij}(\dot{u}_i - \dot{u}_j) + d_{ij}(u_i - u_j).$$

The coefficients $d_{ij} = \max\{\lambda^{\max}_{ij}|\mathbf{c}_{ij}|, \lambda^{\max}_{ji}|\mathbf{c}_{ji}|\}$ are defined using the maximum wave speed $\lambda^{\max}_{ij}$ of the Riemann problem with the initial states $u_i$ and $u_j$ [1]. The nodal time derivatives $\dot{u}_j = \frac{\mathrm{d}u_j}{\mathrm{d}t}$ are defined by the solution of system (2).

By construction, our AFC scheme (3) reduces to (2) for $f^*_{ij} = f_{ij}$. The choice $f^*_{ij} = 0$ corresponds to an algebraic Lax-Friedrichs (ALF) method. Guermond and Popov [1] proved that this "first-order" approximation to (2) is invariant domain preserving (IDP) in the sense that the nodal values $u_i$ stay in a convex set $\mathcal{G}$ if all initial values belong to this set. Moreover, the validity of a fully discrete entropy inequality can be shown for any convex entropy pair. The crux of the proofs presented in [1] is representation of the ALF scheme in terms of the *bar states*

$$\bar{u}_{ij} = \frac{u_j + u_i}{2} - \frac{(\mathbf{f}(u_j) - \mathbf{f}(u_i)) \cdot \mathbf{c}_{ij}}{2d_{ij}} \tag{4}$$

such that $\bar{u}_{ij} \in \mathcal{G}$ if $\mathcal{G}$ is an invariant set of (1) and $u_i, u_j \in \mathcal{G}$. A "second-order" IDP scheme for general hyperbolic problems was designed in [2] using *convex limiting* based on a localized flux-corrected transport (FCT) algorithm.

The *monolithic* convex limiting strategy that we favor in the present report differs from FCT-like predictor-corrector approaches in that the fluxes $f^*_{ij}$ are used

to correct the right-hand side of (3) rather than a low-order solution obtained with the ALF method. The semi-discrete IDP limiting criterion is given by [5]

$$\bar{u}_{ij} \in \mathcal{G}_i \quad \Rightarrow \quad \bar{u}_{ij}^* := \bar{u}_{ij} + \frac{f_{ij}^*}{2d_{ij}} \in \mathcal{G}_i, \tag{5}$$

where $\mathcal{G}_i \subseteq \mathcal{G}$ is a convex set of admissible values and $\bar{u}_{ij}^*$ is a flux-corrected counterpart of the ALF bar state $\bar{u}_{ij}$ defined by (4). Additionally, we constrain the fluxes $f_{ij}^*$ to satisfy an entropy stability condition which implies the validity of a semi-discrete entropy inequality; see [7, 8] for details. Integration in time is performed using a strong stability preserving (SSP) Runge–Kutta method.

Figure 1 presents the results of numerical studies for the KPP problem [4]. Extensions of monolithic convex limiting (MCL) to hyperbolic systems, high-order finite elements, and discontinuous Galerkin methods can be found in [3, 5, 8].

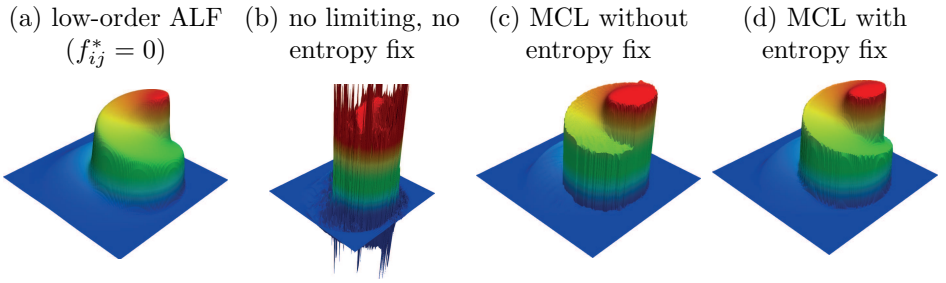


FIGURE 1. Numerical solutions of the KPP problem [4] at $t = 1$ obtained using a uniform mesh of $2 \times 128^2$ linear elements [8].

## REFERENCES

[1] J.-L. Guermond and B. Popov, *Invariant domains and first-order continuous finite element approximation for hyperbolic systems*, SIAM J. Numer. Anal. **54** (2016), 2466-2489.

[2] J.-L. Guermond, M. Nazarov, B. Popov and I. Tomas, *Second-order invariant domain preserving approximation of the Euler equations using convex limiting*, SIAM J. Sci. Computing **40** (2018), A3211-A3239.

[3] H. Hajduk, *Monolithic convex limiting in discontinuous Galerkin discretizations of hyperbolic conservation laws*, Computers & Mathematics with Applications **87** (2021), 120-138.

[4] A. Kurganov, G. Petrova and B. Popov, *Adaptive semidiscrete central-upwind schemes for nonconvex hyperbolic conservation laws*, SIAM J. Sci. Comput. **29** (2007), 2381-2401.

[5] D. Kuzmin, *Monolithic convex limiting for continuous finite element discretizations of hyperbolic conservation laws*, Computer Methods Appl. Mech. Engrg. **361** (2020), 112804.

[6] D. Kuzmin, R. Löhner and S. Turek, *Flux-Corrected Transport. Principles, Algorithms, and Applications*, Springer, 1st edition: 2005, 2nd edition: 2012, pp. 145-238.

[7] D. Kuzmin and M. Quezada de Luna, *Algebraic entropy fixes and convex limiting for continuous finite element discretizations of scalar hyperbolic conservation laws*, Computer Methods Appl. Mech. Engrg. **372** (2020), 113370.

[8] D. Kuzmin and M. Quezada de Luna, *Entropy conservation property and entropy stabilization of high-order continuous Galerkin approximations to scalar conservation laws*, Computers & Fluids **213** (2020), 104742.

## Consistency and Convergence of Finite Volume Methods for Hyperbolic Balance Laws

Jiequan Li
(joint work with Matania Ben-Artzi)

This talk reports my recent works [3, 4] jointly with Matania Ben-Artzi from Hebrew University from Jerusalem.

I want to address fundamental issues of consistency and convergence on (particularly high order) finite volume methods for hyperbolic balance laws. The talk focuses on one-dimensional conservation laws

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0, \tag{1}$$

where $\mathbf{u}$ is the conservative vector, and $\mathbf{f}(\mathbf{u})$ is the associated flux density function. The conclusions also apply to multi-dimensional cases [4]. As it is well-known that discontinuities may be present in solutions, (1) should be understood in the weak (distributional) sense.

**Definition 1** (weak solution)**.** Let $\mathbf{u}(x,t) \in L^1(\Re) \cap L^\infty(\Re)$. For every rectangle $\Omega = [x_1, x_2] \times [t_1, t_2] \subseteq \Re \times \overline{\Re_+}$, if there holds

$$\int_{t_1}^{t_2} \int_{x_1}^{x_2} [\mathbf{u}(x,t)\phi_t + \mathbf{f}(\mathbf{u})(x,t)\phi_x] dxdt = 0 \tag{2}$$

for every test function $\phi(x,t) \in C_0^\infty(\Omega)$, then $\mathbf{u}(x,t)$ is a weak solution to (1).

Formally we use the Gauss-Green formula for (1) to obtain the *integral balance law*,

$$\int_{x_1}^{x_2} \mathbf{u}(x,t_2)dx - \int_{x_1}^{x_2} \mathbf{u}(x,t_2)dx = \int_{t_1}^{t_2} \mathbf{f}(\mathbf{u}(x_1,t))dt - \int_{t_1}^{t_2} \mathbf{f}(\mathbf{u}(x_2,t))dt. \tag{3}$$

But it is not obvious [6, Section 1.3]. Our first result is about the regularity of fluxes and the validation of integral balance law (3).

**Theorem 1.** *Let $\mathbf{u}(x,t) \in L^1(\Re) \cap L^\infty(\Re)$ be a solution of (1) in the sense of Definition 1. Also assume that $\mathbf{u}(x,t)$ satisfies the following properties.*

(i) *$\mathbf{u}(x,t)$ is locally bounded in $\Re \times \overline{\Re_+}$.*
(ii) *For every fixed interval $[x_1, x_2] \subset \Re$ the mass*

$$\mathbf{m}(x,t) = \int_{x_1}^{x_2} \mathbf{u}(x,t)dx \tag{4}$$

*is a well-defined and continuous function of $t \in \overline{\Re_+}$.*

*Then we have:*

(i) *For every fixed time interval $[t_1, t_2] \subset \overline{\Re_+}$, the flux*

$$\mathbf{g}(x) = \int_{t_1}^{t_2} \mathbf{f}(\mathbf{u}(x,t))dt \tag{5}$$

*is locally Lipschitz continuous in* $x \in \Re$.

(ii) $\mathbf{u}(x,t)$ *satisfies the integral balance law* (3) *in every rectangle* $\Omega$.

This theorem actually lays the foundation of finite volume methods that are schematically stated in the following three steps.

**Finite Volume Methods.**

Step 1. With the data $\mathbf{Q}^n(x)$ in a finite subspace of a persistence space $\xi \in \mathcal{V}$ (e.g. piecewise polynomials by WENO), approximate the flux over time level $[t_n, t_{n+1}]$ by a high order flow solver, e.g., using Generalized Riemann Problem (GRP) solvers,

$$\mathbf{f}_{j+\frac{1}{2}}^{n,n+1} \approx \frac{1}{k_n} \int_{t_n}^{t_{n+1}} \mathbf{f}(\mathbf{u}(x_{j+\frac{1}{2}}, t))dt, \tag{6}$$

where $\mathbf{u}(x,t)$ be the exact solution of (1), $k_n = t_{n+1} - t_n$ is the time increment, $x = x_{j+\frac{1}{2}}$ is a cell boundary of $I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$, and mesh size $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$.

Step 2. Use a full discrete finite volume formula to advance the solution from a time level $t = t_n$ to the next time level $t = t_{n+1}$,

$$\bar{\mathbf{u}}_j^{n+1} = \bar{\mathbf{u}}_j^n - \frac{k_n}{h_j}\left(\mathbf{f}_{j+\frac{1}{2}}^{n,n+1} - \mathbf{f}_{j-\frac{1}{2}}^{n,n+1}\right), \quad \bar{\mathbf{u}}_j^n = \frac{1}{h_j}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \mathbf{u}(x,t_n)dx. \tag{7}$$

Step 3. Project the solution $\mathbf{u}(x,t_{n+1})$ of (1) to obtain a "new" data $\mathbf{Q}^{n+1}(x)$ in $\mathcal{V}$ at the next time level $t = t_{n+1}$.

Symbolically, we can write the finite volume formula in an operator form

$$\mathbf{Q}^{n+1}(x) = \mathcal{P} \circ \mathcal{E} \circ \mathcal{A}[\mathbf{Q}^n(x)], \tag{8}$$

where $\mathbf{Q}^{n+1}(x)$ is the approximation of $\mathbf{u}(x,t_{n+1})$, $\mathcal{A}$ is the operator of the flux approximation in Step 1, $\mathcal{E}$ is the evolution operator in Step 2, and $\mathcal{P}$ is the projection operator in Step 3. Note that the evolution operator is exact. So the errors of finite volume approximation come from the flux approximation $\mathcal{A}$ and the data projection $\mathcal{P}$. Thanks to the conservation property, the data projection of high order a does not affect the cell average

$$\frac{1}{h_j}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \mathbf{Q}^{n+1}(x)dx = \frac{1}{h_j}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \mathbf{u}(x,t_{n+1})dx. \tag{9}$$

Thus the flux consistency becomes crucial and is defined below.

**Flux consistency.**

The finite volume approximation (8) is consistent of order $q$, $q > 0$, with the integral balance law (3) if for every admissible set of initial data in $\mathcal{V}$, there holds,

(10)
$$\left| k_n \left[ \mathbf{f}_{j+\frac{1}{2}}^{n,n+1} - \mathbf{f}_{j-\frac{1}{2}}^{n,n+1} \right] - \int_{t_n}^{t_{n+1}} \left[ \mathbf{f}(\mathbf{u}(x_{j+\frac{1}{2}}, t)) - \mathbf{f}(\mathbf{u}(x_{j-\frac{1}{2}}, t)) \right] dt \right| \leq C(k_n)^{2+q},$$

where $C > 0$ depends only on the persistence space $\mathcal{V}$.

This concept of flux consistency is different from the classical Lax consistency of conservative schemes in [8] that is adequate just for first order finite difference methods [7] but not, in particular, for high order finite volume approximation. Besides, it is important to measure the error in terms of the time increment $k_n$ (equivalently mesh size $h_j$), rather than local solution oscillations $\Delta\mathbf{u}$ because there is no control of the oscillation of solutions to hyperbolic systems (e.g. compressible Euler equations). Most approximations uses the (exact or approximate) Godunov flux (e.g. in semi-discrete finite volume methods), which produces errors in terms of $\Delta\mathbf{u}$.

This NEW concept brings us the following observations, which are verified through numerical experiments [9].

(I) **Godunov flux.**
- As initial data is piecewise constant, the order is INFINITY $q = \infty$;
- As initial data consists of piecewise polynomials, the order is ZERO $q = 0$ if the solution is discontinuous;
- As initial data consists of piecewise polynomials, the order is ONE $q = 1$ if the solution is smooth.

(II) **GRP flux.**
- As initial data consists of piecewise polynomials, the order is ONE $q = 1$ if the solution is discontinuous;
- As initial data consists of piecewise polynomials, the order is TWO $q = 2$ if the solution is smooth.

These observations stress the important of flux approximations. The order $q > 0$ is indeed the necessary condition for the convergence of approximate solutions. See below.

**Lax-Wendroff type convergence.**

With the aid of this consistency we can prove the following Lax-Wendroff type convergence theorem.

**Theorem 2.** *Assume that the finite volume approximation is consistent of order $q > 0$ and $\{k_n \downarrow 0\}$ be a decreasing sequence of time steps. Let $\{\Upsilon^{k_n}(x,t)\}_n^\infty$ be the corresponding solutions subject to the initial data in $\mathcal{V}$. Suppose that*

(i) *The sequence $\{\Upsilon^{k_n}(x,t)\}_n^\infty$ is uniformly bounded in $L^\infty([0,T], L^1(\Re))$.*
(ii) *The sequence $\{\Upsilon^{k_n}(x,t)\}_n^\infty$ converges in $C([0,T], L^1_{loc}(\Re))$ to a function $\mathbf{v}(x,t)$ (in particular it is uniformly bounded in this space).*

*Then $\mathbf{v}(x,t)$ is a solution to the balance law (3) in $\Re \times [0,T]$.*

A key estimate in the proof is about total error is $\mathcal{O}(k_n)$, which benefits from the consistency requirement (10). Otherwise the estimate would collapse for hyperbolic systems (although it is still true for one-dimensional scalar equations thanks to the BV estimate).

**Godunov compatibility.**

The physical admissibility (entropy stability) of the weak solution $\mathbf{v}(x,t)$ is always a crucial issue. For the Godunov scheme, the cell entropy inequality is always true, thanks to the Jensen inequality and the consistency of infinite order,

$$\begin{aligned} \eta(\bar{\mathbf{u}}_j^{n+1}) &\le \frac{1}{h_j}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \eta(\mathbf{u}(x,t_{n+1}))dx \\ &\le \eta(\bar{\mathbf{u}}_j^{n}) - \frac{k_n}{h_j}\left[\mathbf{F}^n_{j+\frac{1}{2}} - \mathbf{F}^n_{j-\frac{1}{2}}\right], \end{aligned} \tag{11}$$

where $\eta$ is the convex entropy function and $\mathbf{F}$ is the associated entropy flux. For high order finite volume methods, if we solve the corresponding generalized Riemann problem precisely, then we have the *integral entropy inequality*

$$\begin{aligned} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \eta(\mathbf{u}(x,t_{n+1}))dx \le & \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \eta(\mathbf{Q}^n(x))dx \\ & - \int_{t_n}^{t_{n+1}} \left[\mathbf{F}(\mathbf{u}(x_{j+\frac{1}{2}},t)) - \mathbf{F}(\mathbf{u}(x_{j-\frac{1}{2}},t))\right] dt. \end{aligned} \tag{12}$$

Once the GRP solver [1, 2] is applied with tolerant error in (10), the entropy production from the flux evaluation is $\mathcal{O}(k_n^3)$ that is sufficient for the estimate. The difficulty lies the data projection, which does NOT satisfy

$$\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \eta(\mathbf{Q}^{n+1}(x)) \le \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \eta(\mathbf{u}(x,t_{n+1}))dx, \tag{13}$$

in general. Hence we have to revisit the Godunov scheme and propose the *Godunov compatibility* so that the Godunov scheme is taken as the reference. Indeed, it is well-accepted as the foundation of modern CFD.

**Assumption.** Let $\mathfrak{B}_K$ be the ball of radius $K>0$ and let $\mathbf{u}_0 \in \mathfrak{B}_K$. The Godunov scheme $\Phi^{k,G}$ applied to $\boldsymbol{\theta}^{0,G} = \mathbf{u}_0^{k,av}$, the cell average of $\mathbf{u}_0$, converges to a unique solution of the balance law. Furthermore, if $\mathbf{v}_0 \in \mathfrak{B}_K$ is another initial function and $\boldsymbol{\psi}^{0,G} = \mathbf{v}_0^{k,av}$ , then

$$\|\Phi^{k,G}\boldsymbol{\theta}^{0,G} - \Phi^{k,G}\boldsymbol{\psi}^{0,G}\|_1 \le (1+Ck)\|\boldsymbol{\theta}^{0,G} - \boldsymbol{\psi}^{0,G}\|_1, \tag{14}$$

where $C>0$ depends only on $K$.

**Definition 2.** [**Godunov Compatibility**] The finite volume approximation $\Phi^k$ (consistent of order $q>0$) is *compatible with the Godunov scheme* if the following conditions hold.

(i) The finite volume approximation $\Phi^k$ coincides with the Godunov scheme on piecewise constant functions; if $\boldsymbol{\xi} \in \mathcal{V}$ is piecewise constant then

$$\Phi^k \xi = \Phi^{k,G} \boldsymbol{\xi}. \tag{15}$$

(ii) Let $H$ be an admissible set. Then

$$\int_{\Re} |\Phi^k \boldsymbol{\xi}(x) - \Phi^{k,G} \boldsymbol{\xi}^{k,av}(x)| dx = o(k), \quad \boldsymbol{\xi} \in H \tag{16}$$

where $o(k)$ is uniform for all $\boldsymbol{\xi} \in H$.

Then we have the following conclusion.

**Theorem 3.** *Assume the validity of the Assumption above and that the finite volume approximation $\Phi^k$ is consistent of order $q > 0$ and compatible with the Godunov scheme. Let $\{\widetilde{\theta^n}\}_{n=0}^N$ ($N = k^{-1}T$) be the discrete set of finite volume solutions. Then the limit function in Theorem 2 is unique, namely, under the hypotheses of Theorem 2 is a unique limit function for all converging subsequences.*

Hence an open problem remains for the Godunov scheme: *How to prove the entropy convergence of the Godunov approximate solutions to the system of hyperbolic balance laws?* In other words, all theoretical issues on finite volume methods boil down to the analysis of the Godunov method [5] and the validation of the Godunov compatibility that we address here.

## References


[1] M. Ben-Artzi and J. Falcovitz, *Generalized Riemann problems in computational fluid dynamics*. Cambridge Monographs on Applied and Computational Mathematics, 11. Cambridge University Press, Cambridge, 2003.

[2] M. Ben-Artzi and J. Li, *Hyperbolic balance laws: Riemann invariants and the generalized Riemann problem.* Numer. Math. **106** (2007), 369–425.

[3] M. Ben-Artzi and J. Li, *Consistency of Finite Volume Approximation to Nonlinear Hyperbolic Balance Laws*, Math. Comp., **90** (2021), 141–169.

[4] M. Ben-Artzi and J. Li, *Regularity of fluxes in hyperbolic balance laws*, arXiv:2006.09253, 2020.

[5] A. Bressan, M. T. Chiri and W. Shen, *A posteriori error estimates for numerical solutions to hyperbolic conservation laws*, (2021), arXiv:2010.00428.

[6] C. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics*, 4th ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. **325**, Springer-Verlag, Berlin, 2016

[7] R. Eymard, T. Galloüet and R. Herbin, *Finite volume methods, Handbook of numerical analysis,* Vol. VII, Handb. Numer. Anal., VII, North-Holland, Amsterdam, 2000, pp. 713–1020.

[8] P. Lax and B. Wendroff, *Systems of conservation laws,* Comm. Pure Appl. Math. **13** (1960), 217–237.

[9] J. Li and Y. Wang, *Thermodynamical effects and high resolution methods for compressible fluid flows*, J. Comput. Phys. **343** (2017), 340–354.

## Existence and uniqueness in viscoelasticity of Kelvin-Voigt type with nonconvex stored energy

Athanasios E. Tzavaras
(joint work with Konstantinos Koumatos, Corrado Lattanzio, Stefano Spirito)

We consider the Cauchy problem for viscoelastic materials of strain-rate type in Lagrangean coordinates

$$\partial_{tt} y - \operatorname{div}(S(\nabla y)) - \Delta \partial_t y = 0\,, \tag{1}$$

where $y : (0,T) \times \mathbb{T}^d \to \mathbb{R}^d$, $\mathbb{T}^d$ is the $d$-dimensional torus, $d = 2, 3$, $T > 0$ is arbitrary but finite, and with initial data

$$y|_{t=0} = y_0, \qquad \partial_t y|_{t=0} = v_0\,.$$

For this model the Piola-Kirchhoff stress tensor has the form

$$T_R = S(F) + \partial_t F\,, \quad S(F) = \frac{\partial W}{\partial F}(F)\,. \tag{2}$$

The elastic part of the stress is given as the gradient of a strain-energy function, $S = DW$, while the viscous part of the stress is linear, leading to (2). Such constitutive relations fit under the general framework of viscoelasticity of strain-rate type, and specifically into the class of Kelvin-Voigt type materials.

The system (1) is expressed as a hyperbolic-parabolic system,

$$\begin{aligned} \partial_t v - \operatorname{div}(S(F)) - \Delta\, v &= 0 \\ \partial_t F - \nabla\, v &= 0 \\ \operatorname{curl} F &= 0\,, \end{aligned} \tag{3}$$

with $F = \nabla y$ for the deformation gradient, and is supplemented with periodic boundary conditions and initial data The constraint curl $F = 0$ is propagated from the initial data $F_0 = \nabla y_0$ by the kinematic compatibility equation $F_t = \nabla v$.

The strain energy function $W(F)$ is in general allowed to be non-convex, and it is assumed coercive with growth of polynomial type

$$c(|F|^p - 1) \le W(F) \le C(1 + |F|^p) \tag{4}$$

and growth conditions for its derivatives up to second order. Instead of convexity we adopt the Andrews-Ball condition imposing monotonicity at infinity; namely, $W(F)$ satisfies for $R > 0$

$$(S(F_1) - S(F_2), F_1 - F_2) \ge 0, \qquad \forall\, |F_1|, |F_2| \ge R\,, \tag{5}$$

where $(F, G) = \operatorname{tr} FG^T$ denotes the inner product. On occasion a strengthened version of (5) is employed, requesting that for $C > 0$, $R > 0$

$$(S(F_1) - S(F_2), F_1 - F_2) \ge C(|F_1|^{p-2} + |F_2|^{p-2})|F_1 - F_2|^2,\ \forall\, |F_1|, |F_2| \ge R\,. \tag{6}$$

The main results covered in this talk are the following:

**Theorem 1.** *Let $W$ satisfy* (4)-(5) *for $p \geq 2$. For data $(v_0, F_0) \in L^2(\mathbb{T}^d) \times L^p(\mathbb{T}^d)$ with $F_0 = \nabla y_0$ a.e. in $\mathbb{T}^d$ there exists a weak solution*

$$(v, F) \in L^\infty(0, T; L^2(\mathbb{T}^d)) \cap L^2(0, T; H^1(\mathbb{T}^d)) \times L^\infty(0, T; L^p(\mathbb{T}^d))$$

*satisfying the weak form of the equations* (3) *and the energy inequality*

$$\int \frac{|v|^2}{2} + W(F)\, dx + \int_0^t \int |\nabla v|^2\, dxds \leq \int \frac{|v_0|^2}{2} + W(F_0)\, dx. \tag{7}$$

*Moreover:*
*(i) If $F_0 \in H^1(\mathbb{T}^d)$ then $F \in L^\infty(0, T; H^1(\mathbb{T}^d))$.*
*(ii) If $F_0 \in H^1(\mathbb{T}^d)$ and Hypothesis* (6) *holds, then*

$$F \in L^\infty(0, T; H^1(\mathbb{T}^d)) \quad \nabla |F|^{\frac{p}{2}} \in L^2(0, T; L^2(\mathbb{T}^d)). \tag{8}$$

*(iii) For weak solutions of class $F \in L^\infty(0, T; H^1(\mathbb{T}^d))$, if the dimension $d = 2$ and $p \geq 2$ or $d = 3$ and $2 \leq p \leq 4$, then $(v, F)$ verifies the energy identity*

$$\int \frac{|v|^2}{2} + W(F)\, dx + \int_0^t \int |\nabla v|^2\, dxds = \int \frac{|v_0|^2}{2} + W(F_0)\, dx. \tag{9}$$

The proof is based on an a-priori estimate showing transfer of dissipation for initial data $F_0 \in H^1(\mathbb{T}^d)$ and for stored energies satisfying the hypotheses (5) or (6) in conjunction with an idea developed in [1] that shows propagation of compactness for the system (3), see [2].

By contrast, for (3) in one-space dimension with non-monotone stress, a sequence of exact solutions is constructed emanating from oscillating initial data and such that the oscillations propagate and produce sustained oscillations.

On the issue of uniqueness in two-space dimensions we prove:

**Theorem 2.** *Let $p \geq 2$ and $d = 2$ and assume $W$ satisfies* (4), (6) *and the growth condition $|D^2 W(F)| \leq C(1 + |F|^s)$ for some $p - 2 \leq s < p$. Then, for $(v_0, F_0) \in L^2(\mathbb{T}^2) \times H^1(\mathbb{T}^2)$ the weak solution in Theorem 1 is unique.*

## References

[1] G. Friesecke, G. Dolzmann, *Implicit time discretization and global existence for a quasi-linear evolution equation with nonconvex energy*, SIAM J. Math. Anal. **28** (1997), 363–380.
[2] K. Koumatos, C. Lattanzio, S. Spirito and A.E. Tzavaras, *Existence and uniqueness for a viscoelastic Kelvin-Voigt model with nonconvex stored energy*, arXiv:2012.10344, (2020).

## On the sticky particle solutions to the multi-dimensional pressureless Euler equations

Stefano Bianchini
(joint work with Sara Daneri)

We consider the pressureless Euler system in $[0,T]\times\mathbb{R}^d$

$$\begin{cases}\partial_t\rho+\operatorname{div}(\rho v)=0\\ \partial_t(\rho v)+\operatorname{div}(\rho v\otimes v)=0,\end{cases}\tag{1}$$

where $\rho$ is the distribution of particles and $v$ is their velocity.

Such a model has been proposed by Zeldovich [16] as a simplified model for the early stages of the formation of galaxies, when a dust of particles moving without pressure should start to collide and aggregate into bigger and bigger clusters.

Since then, several authors devoted attention to the search of sticky particle solutions, namely solutions to (1) which satisfy the following adhesion principle: if two particles of fluid do not interact, then they move freely keeping constant velocity, otherwise they join with velocity given by the balance of momentum.

The great majority of the results in the literature (see e.g. [2, 3, 5, 7, 8, 9, 10, 11]) are concerned with the one-dimensional pressureless dynamics. In this case, exploiting the density of finite particle solutions, one can obtain from quite general initial data a global measure solution of (1) satisfying a suitable entropy condition (see [7] and independently [8]).

In general dimension, much less is known. For initial data given by a finite number of particle pointing each in a given direction, it is easy to show that a global sticky particle solution always exists and is unique. However, in dimension $d\geq 2$, one sees immediately already from a finite number of particles that the sticky particle solutions do not depend continuously on the initial data.

In [4] it is shown that, in general, both existence and uniqueness might fail: it is indeed possible to build initial data of non-existence or non-uniqueness for the sticky particle solutions, in contrast to what had been erroneously stated in [14].

In [6] measure valued solutions to (1) on a compactification of the state space have been constructed for general initial data as limits of variational in time discretizations.

Thus the natural question of whether one can still find particle solutions for a large class of data (hence excluding the counterexamples in [4]) remained unanswered. In this paper we give a positive answer to this question.

We consider the problem of existence and uniqueness in a larger class of solutions which we call *dissipative* since in particular their kinetic energy is decreasing but their trajectories might cross without joining at later times. By *free flow* we mean a flow in which trajectories are disjoint straight lines which never intersect.

Our main result is the following:

**Theorem 1.** *There is a set $D_0\subset\mathcal{P}_{2,1}(\mathbb{R}^d\times\mathbb{R}^d)$ such that, for any $\nu_0\in D_0$ there exists a unique dissipative solution $\eta$ with initial data $\nu_0$ and it is given by a free*

*flow. Such a set is a dense $G_\delta$ set (i.e. of second category) in the weak topology on $\mathcal{P}_{2,1}(\mathbb{R}^d \times \mathbb{R}^d)$.*

Since our notion of dissipative solution includes the classical sticky particle solutions, the above theorem implies that, even though the sticky particle solutions are not well-posed for every measure-type initial data, there exists a comeager set of initial data in the weak topology giving rise to a unique sticky particle solution. Moreover, for any of these initial data the sticky particle solution is unique also in the larger class of dissipative solutions (where trajectories are allowed to cross) and is given by a trivial free flow concentrated on trajectories which do not intersect. In particular for such initial data there is only one dissipative solution and its dissipation is equal to zero. Thus, for a comeager set of initial data the problem of finding sticky particle solutions is well-posed, but the dynamics that one sees is trivial.

## References


[1] L. Boudin. A solution with bounded expansion rate to the model of viscous pressureless gases. *SIAM J. Math. Anal.* **32** (2002), 172-193.

[2] Y. Brenier, W. Gangbo, G. Savaré and M. Westdickenberg. Sticky particle dynamics with interactions. *J. Math. Pures Appl.* **99** (2013), 577-617.

[3] Y. Brenier and E. Grenier. Sticky particles and scalar conservation laws. *SIAM Jour. Num. Anal.* **35** (1998), 2317-2328.

[4] A. Bressan and T. Nguyen. Non-existence and non-uniqueness for multidimensional sticky particle systems. *Kinetic and related models* **7** (2014), 205-218.

[5] F. Cavalletti, M. Sedjro and M. Westdickenberg. A simple proof of global existence for the 1D pressureless gas dynamics equations. *SIAM J. Math. Anal.* **47** (2015), 66-79.

[6] F. Cavalletti, M. Sedjro and M. Westdickenberg. A variational time discretization for the compressible Euler equations. *Trans. Amer. Math. Soc.* **371** (2019), 5083-5155.

[7] W. E, Y.G. Rykov and Y.G. Sinai. Generalized variational principles, global weak solutions and behaviour with random initial data for systems of conservation laws arising in adhesion particle dynamics. *Comm. Math. Phys.* **177** (1996), 349-380.

[8] E. Grenier. Existence globale pour le systeme des gaz sans pression. *C. R. Acad. Sci. Paris Sér. I Math.* **321** (1995), 171-174.

[9] F. Huang and Z. Wang. Well-posedness for pressureless flows. *Comm. Math. Phys.* **222** (2001), 117-146.

[10] L. Natile and G. Savaré. A Wasserstein approach to the one-dimensional sticky particle system. *SIAM J. Math. Anal.* **41** (2009), 1340-1365.

[11] T. Nguyen and A. Tudorascu. Pressureless Euler/Euler-Poisson system via adhesion dynamics and scalar conservation laws. *SIAM J. Math. Anal.* **40** (2008), 754-775.

[12] T. Nguyen and A. Tudorascu. One-dimensional pressureless gas systems with/without viscosity. *Comm. Part. Diff. Eq.* **40** (2015), 1619-1665.

[13] F. Poupaud and M. Rascle Measure solutions to the linear transport equations with non-smooth coefficients *Comm. Partial Differential Equations* **22** (1997), 337-358.

[14] M. Sever. An existence theorem in the large for zero-pressure gas dynamics. *Diff. Int. Eq.* **14** (2001), 1077-1092.

[15] A. Sobolevskii. The small viscosity method for a one-dimensional system of equations of gas dynamic type without pressure. *Doklady Mathematics* **56** (1997), 707-709.

[16] Ya. B. Zeldovich. Gravitational instability: an approximate theory for large density perturbations. *Astron. & Astrophys.* **5** (1970), 84-89.

## The Active Flux Method for Hyperbolic Problems: A review of the method and recent results of our group

Christiane Helzel
(joint work with Erik Chudzik, David Kerkmann)

Recently, Eymann and Roe [4] proposed a new numerical method for hyperbolic conservation laws, the so-called *Active Flux* method. While the Active Flux method is a finite volume method, it is based on the use of cell average values as well as point values of the conserved quantities. The point values are located at the grid cell boundaries and are used both for the flux computation as well as for the reconstruction. In it's original form the method is third order accurate. This is achieved by using a continuous, piecewise quadratic reconstruction and the use of Simpson's rule for the flux computation. The flux computation requires point values of the conserved quantities at all the nodes of Simpson's rule. The computation of these point values plays a crucial role. Eymann and Roe suggested to use exact evolution formulas which are available for advection and acoustics. Other approximative approaches for the computation of point values, which can also be used for nonlinear problems, have been discussed in [1] and [5].

An important property of the Active Flux method is it's local stencil in space and time. While a DG method for example also uses a local stencil in space, the Runge-Kutta time discretisation increases the stencil. According to Roe [7], this increased stencil might be responsable for the loss of stability typically observed in higher order DG methods. The Active Flux method does not suffer from such a severe loss of stability. We investigated the linear stability of the two-dimensional Active Flux method, for details see [3]. For the advection equation we found that stability for time steps corresponding to CFL $\leq 1$ can only be obtained if the numerical fluxes are computed exactly. If Simpson's rule is used instead, then the method requires slightly smaller time steps. For the acoustic equations we observed stability for CFL $\leq 1/2$. This is the optimal stability limit since the method was constructed in such a way that acoustic waves are allowed to propagate at most through half a grid cell.

Currently we investigate the use of the Active Flux methods on Cartesian grids with cut cells and on adaptively refined Cartesian grids. In both situations we benefit from the local stencil of the Active Flux method.

### 1. Large Time Step Stability and Construction of Cartesian Grid Embedded Boundary Methods

Cartesian grid cut cell methods allow the approximation of hyperbolic problems in complicated geometries. This approach cuts solid bodies out of a background Cartesian mesh. Away from the boundary a regular Cartesian grid is used but near the boundary cut cells might be orders of magnitude smaller than regular cells. Existing cut cell approaches require some form of stabilisation in order to allow time steps which are governed by a stability condition for the regular part of the domain. For a one-dimensional test situation the continuous reconstruction of

the Active Flux method and the use of an exact evolution formula for the update of the point values lead to a third order accurate approximation which is stable with regard to time steps determined by the regular part of the grid. If Simpson's rule is used instead then the accuracy near the cut cell is reduced to second order while stability is maintained. The same stability is observed in two-dimensional computations. However, for arbitrary cut cell constellations we observe a reduction of the accuracy along the boundary to second order. Preliminary results can be found in [6].

### 2. Adaptive Mesh Refinement in Cartesian Grid Active Flux Methods

Together with Donna Calhoun from Boise State University, the main developer of the ForestClaw software [2], we currently investigate the use of the Active Flux method on a hierarchy of Cartesian grids with different refinement. The local stencil of the Active Flux method simplifies the exchange of data between neighbouring grid patches. Furthermore, the reconstruction used by the Active Flux method can directly be used to carry out the transition from coarse to fine grids. Our preliminary results confirm that the Active Flux method retains its third order accuracy on such adaptively refined grids.

### References

[1] W. Barsukow, *The Active Flux Scheme for Nonlinear Problems*, Journal of Scientific Computing **86** (2021).
[2] D. Calhoun and C. Burstedde, *ForestClaw: A parallel algorithm for patch-based adaptive mesh refinement on a forest of quadtrees*, arXiv: 1703.03116 (2017).
[3] E. Chudzik, C. Helzel and D. Kerkmann, *The Cartesian Grid Active Flux Method: Linear stability and bound preserving limiting*, Applied Mathematics and Computation **393** (2021), 125501.
[4] T.A. Eymann and P.L. Roe, *Multidimensional active flux schemes*, In: 21st AIAA computational fluid dynamics conference (2013), p. 2940.
[5] C. Helzel, D. Kerkmann and L. Scandurra, *A New ADER Method Inspired by the Active Flux Method*, Journal of Scientific Computing **80** (2019), 1463-1497.
[6] C. Helzel and D. Kerkmann, *An Active Flux Method for Cut Cell Grids*, in Klöfkorn et al. (eds) Proceedings of FVCA 2020, Springer Proceedings in Mathematics & Statistics, vol 323.
[7] P. Roe *Designing CFD methods for bandwidth – A physical approach*, Computers & Fluids, **214** (2021), 104774.

## Hyperbolic Transport across Fluidic Interfaces

Christian Rohde

The dynamics of compressible two-phase flow can be described by diffuse-interface models as alternatives to classical sharp-interface approaches. Typically, these systems are extensions of the Euler/Navier-Stokes equations but exhibit structural changes that render standard numerical methods for these hyperbolic(-parabolic) systems to be inadequate. In this short note we consider as particular instance

the Navier-Stokes-Korteweg (NSK) equations for isothermal flow, and report on approximations that can be accessed by hyperbolic techniques again. We focus on the verification of a discrete energy inequality for a finite-volume (FV) method.

Let a fluid in $d$-dimensional domain $D$ be given which can appear in a liquid and a vapour phase. For fixed temperature and density $\rho > 0$ let $p = p(\rho)$ be the non-monotone (Van-der-Waals-like) pressure which is related to the Helmholtz free energy $W = W(\rho)$ by $p(\rho) = -W(\rho) + \rho W'(\rho)$. The phases are identified with the monotone-increasing branches of $p$. As approximations of the classical third-order Navier-Stokes-Korteweg system (see e.g. [2]) relaxed versions have been suggested, for a survey see [4]. Let a kernel $K : \mathbb{R}^d \to [0, \infty)$ be given and define the scaled version

$$K^\alpha(\mathbf{x}) = \alpha^{-\frac{\mathbf{d}}{\mathbf{2}}} \mathbf{K}\big(\sqrt{\alpha}\mathbf{x}\big) \qquad (\alpha > \mathbf{0}).$$

With the unknowns density $\rho^\alpha = \rho^\alpha(\mathbf{x}, \mathbf{t}) > \mathbf{0}$, and velocity field $\mathbf{v}^\alpha = \mathbf{v}^\alpha(\mathbf{x}, \mathbf{t}) \in \mathbb{R}^{\mathbf{d}}$ the relaxed approximations write as

$$(1) \qquad \begin{aligned} \rho^\alpha_t \; + \qquad & \nabla \cdot (\rho^\alpha \mathbf{v}^\alpha) \qquad && = 0, \\ (\rho^\alpha \mathbf{v}^\alpha)_t \; + \; & \nabla \cdot (\rho^\alpha \mathbf{v}^\alpha \otimes \mathbf{v}^\alpha + \mathbf{p}_\alpha(\rho^\alpha)\mathcal{I}) \; && = \nabla \cdot \mathcal{T}[\mathbf{v}^\alpha] + \alpha \rho^\alpha \nabla (\mathbf{K}^\alpha * \rho^\alpha) \end{aligned}$$

in $(0, T) \times D$ for $T > 0$. In (1), $\mathcal{T}$ denotes the viscous part of the stress tensor and $p_\alpha(\rho) = p(\rho) + \rho^2/2$.

The system (1) is thermodynamically consistent in the sense that smooth solutions satisfy for e.g. $\mathbf{v}(\mathbf{t}, \cdot) \equiv \mathbf{0}$ on $\partial D$ the energy inequality

$$(2) \qquad \begin{aligned} \frac{d}{dt} \int_D \Big( W(\rho^\alpha(t, \mathbf{x})) + \alpha \int_{\mathbf{D}} \mathbf{K}^\alpha(\mathbf{x} - \mathbf{y}) \big(\rho^\alpha(\mathbf{t}, \mathbf{x}) - \rho^\alpha(\mathbf{t}, \mathbf{y})\big)^{\mathbf{2}} \mathbf{dy} \\ + \frac{1}{2} \rho^\alpha(t, \mathbf{x}) |\mathbf{v}^\alpha(\mathbf{t}, \mathbf{x})|^{\mathbf{2}} \Big) \, \mathbf{dx} \leq \mathbf{0}. \end{aligned}$$

For appropriate kernel functions $K$, it can expected that one recovers the classical NSK system in the limit $\alpha \to \infty$.

Numerical discretisations of (1) target at a discrete counterpart of (2). For hyperbolic systems that are equipped with a strictly-convex entropy, Tadmor has suggested entropy-conservative schemes in [6] that base on numerical fluxes in terms of the entropy variables. In fact, the lefthand-side operator in (1) corresponds to a hyperbolic system with convex entropy if $\alpha > \max\{-W''(r) \mid r > 0\}$ holds. We restrict ourselves to the case $d = 1$, $D = \mathbb{R}$, and set $\mathcal{T} \equiv 0$. Let $\mathbf{g}^* = (g_1^*, g_2^*)^T \in \mathbb{R}^2$ denote some Tadmor (numerical) flux for the flux on the lefthand-side in (1). For uniform mesh width $\Delta x > 0$ and $t > 0$ let $\mathbf{u}_{\Delta x}(t) : \mathbb{R} \to \mathbb{R}^2$ denote the FV approximation that is assembled from the cell averages $\{\mathbf{u}_j(t)\}_{j \in \mathbb{Z}}$. With $\mathbf{g}^*_{j+\frac{1}{2}}(t) = \mathbf{g}^*(\mathbf{w}_j(t), \mathbf{w}_{j+1}(t))$ ($\mathbf{w} = (w_1, w_2)^T \in \mathbb{R}^2$ being the tranformed entropy variable of the conservative variable $\mathbf{u} = (\rho, \rho v)^T$), the FV method is then given

by

$$\mathbf{u}_j'(t) + \frac{1}{\Delta x}\left(\mathbf{g}^*_{j+\frac{1}{2}}(t) - \mathbf{g}^*_{j-\frac{1}{2}}(t)\right) = \alpha \begin{pmatrix} 0 \\ h^*_{j+\frac{1}{2}}(t)\dfrac{(K^\alpha * \rho)_{\Delta x}(t, \mathbf{x_{j+1}}) - (\mathbf{K}^\alpha * \rho)_{\mathbf{\Delta x}}(\mathbf{t}, \mathbf{x_j})}{\Delta x} \end{pmatrix}. \tag{3}$$

In (3), the operator $(\cdot)_{\Delta x}$ denotes a discrete convolution and $h^*(t) : \mathbb{R}^2 \to \mathbb{R}$ is supposed to be a consistent approximation of the density. We have set $h^*_{j+\frac{1}{2}}(t) = h^*(\mathbf{w}_j(t), \mathbf{w}_{j+1}(t))$. For regular kernels $K$, the discrete convolution and $h^*$ can be chosen such that the FV method (3) is conservative and satisfies a discrete entropy relation. Precisely, we have (see [4, 5] for special cases)

**Theorem 1.** *There is a Tadmor flux* $\mathbf{g}^*$ *with the property*

$$h^*(\mathbf{w}, \tilde{\mathbf{w}})w_2 = g_1^*(\mathbf{w}, \tilde{\mathbf{w}}) \quad \forall\, \mathbf{w}, \tilde{\mathbf{w}} \in \mathbb{R}^2,$$

*which implies that the semi-discrete FV method* (3) *for* (1) *is entropy conservative, i.e.,*

$$\frac{d}{dt}\sum_{j\in\mathbb{Z}} \Delta x \left( W\big(\rho_j(t)\big) + \sum_{k\in\mathbb{Z}} \Delta x \Big( K^\alpha(\mathbf{x_j} - \mathbf{x_k})\big(\rho_\mathbf{j}(\mathbf{t}) - \rho_\mathbf{k}(\mathbf{t})\big)^\mathbf{2} \Big) + \frac{\rho_\mathbf{j}(\mathbf{t})\mathbf{v}_\mathbf{j}^\mathbf{2}(\mathbf{t})}{\mathbf{2}} \right) = 0.$$

Using straightforward discretisations of $\mathcal{T}$ one obtains the discrete counterpart of (2). The result can be generalised to arbitrary space dimensions and standard meshes using FV discretisations. The generalisation to higher-order methods like the ones used in e.g. [1, 3, 5] remains an open problem. However, numerical experiments indicate that the use of the Tadmor fluxes leads to monotone decaying discrete energies in relevant two-phase flow regimes.

## References


[1] A. Chertock, P. Degond, and J. Neusser, An asymptotic-preserving method for a relaxation of the Navier-Stokes-Korteweg equations. J. Comput. Phys. **335** (2017), 387-403.

[2] J.E. Dunn and J. Serrin, On the thermomechanics of interstitial working. Arch. Rational Mech. Anal. **88** (1985), no. 2, 95-133.

[3] T. Hitz, J. Keim, C.-D. Munz, C. Rohde, A parabolic relaxation model for the Navier-Stokes-Korteweg equations. J. Comput. Phys. **421** (2020), 109714, 24 pp.

[4] C. Rohde, Fully resolved compressible two-phase flow: modelling, analytical and numerical issues. New trends and results in mathematical description of fluid flows, 115-181, Nečas Center Ser., Birkhäuser/Springer, Cham, 2018.

[5] J. Neusser, C. Rohde, and V. Schleper, Relaxation of the Navier-Stokes-Korteweg equations for compressible two-phase flow with phase transition. Internat. J. Numer. Methods Fluids **79** (2015), no. 12, 615-639.

[6] E. Tadmor, Entropy stability theory for difference approximations of nonlinear conservation laws and related time-dependent problems. Acta Numer. **12** (2003), 451-512.


*Reporters: Philipp Öffner and Yuhuan Yuan*

## Participants

**Prof. Dr. Remi Abgrall**
Institut für Mathematik
Universität Zürich
Winterthurerstrasse 190
8057 Zürich
SWITZERLAND

**Prof. Dr. Fabio Ancona**
Dipartimento di Matematica "Tullio Levi-Civita"
Università di Padova
Via Trieste, 63
35121 Padova
ITALY

**Dr. Roberta Bianchini**
Consiglio Nazionale delle Ricerche
IAC
via dei Taurini 19
00185 Roma 00185
ITALY

**Prof. Dr. Stefano Bianchini**
SISSA - ISAS
Via Bonomea 265
34136 Trieste
ITALY

**Prof. Dr. Alina Chertock**
Department of Mathematics
North Carolina State University
SAS Hall 4152
Raleigh, NC 27695-8205
UNITED STATES

**Dr. Maria Teresa Chiri**
Department of Mathematics, Penn State University
McAllister Building, Pollock Rd
16802 State College 16801
UNITED STATES

**Prof. Dr. Rinaldo M. Colombo**
INdAM Unit - c/o DII
Università degli Studi di Brescia
Via Branze, 38
25123 Brescia
ITALY

**Dr. Sara Daneri**
Gran Sasso Science Institute (GSSI)
Viale Francesco Crispi, 7
67100 L'Aquila
ITALY

**Prof. Dr. Pierre Degond**
Department of Mathematics
Imperial College London
Huxley Building
London SW7 2AZ
UNITED KINGDOM

**Prof. Dr. Bruno Despres**
Université Pierre et Marie Curie
Laboratoire Jacques-Louis Lions
Boite courrier 187
4 place Jussieu
75252 Paris Cedex 05
FRANCE

**Prof. Dr. Michael Dumbser**
DICAM
Universita degli Studi di Trento
Via Mesiano, 77
38123 Trento
ITALY

**Prof. Dr. Eduard Feireisl**
Institute of Mathematics of the AS CR
Charles University
Žitná 25
115 67 Praha 1
CZECH REPUBLIC

**Prof. Dr. Heinrich Freistühler**
Fachbereich Mathematik und Statistik
Universität Konstanz
Universitätsstrasse 10
78457 Konstanz
GERMANY

**Prof. Dr. Mauro Garavello**
Dipartimento di Matematica e Applicazioni
Università di Milano-Bicocca
Edificio U5
Via Roberto Cozzi 55
20125 Milano
ITALY

**Prof. Dr. Gregor Gassner**
Department Mathematik/Informatik
Universität zu Köln
Weyertal 86 - 90
50931 Köln
GERMANY

**Prof. Dr. Jan Giesselmann**
Fachbereich Mathematik
Technische Universität Darmstadt
Dolivostraße 15
64293 Darmstadt
GERMANY

**Dr. Paola Goatin**
INRIA - Team ACUMES
BP 93
2004, route des Lucioles
06902 Sophia-Antipolis Cedex
FRANCE

**Prof. Dr. Graziano Guerra**
Dipartimento di Matematicà e Applicazioni
Università di Milano-Bicocca
Edificio U5
Via Roberto Cozzi 55
20125 Milano
ITALY

**Prof. Dr. Christiane Helzel**
Mathematisches Institut
Heinrich-Heine-Universität Düsseldorf
Universitätsstraße 1
40225 Düsseldorf
GERMANY

**Prof. Dr. Michael Herty**
LuF Continuous Optimization
IGPM - 114620
RWTH Aachen University
Templergraben 55
52062 Aachen
GERMANY

**Prof. Dr. Helge Holden**
Department of Mathematical Sciences
Norwegian University of Science and Technology
A. Getz vei 1
7491 Trondheim
NORWAY

**Prof. Dr. Christian Klingenberg**
Institut für Mathematik
Universität Würzburg
Emil-Fischer-Strasse 40
97074 Würzburg
GERMANY

**Prof. Dr. Dietmar Kröner**
Abteilung für Angewandte Mathematik
Universität Freiburg
Hermann-Herder-Strasse 10
79104 Freiburg i. Br.
GERMANY

**Prof. Dr. Alexander Kurganov**
Department of Mathematics
Southern University of Science and Technology (SUSTech)
1088 Xueyuan Ave. Nanshan
Shenzhen, Guangdong Province 518 055
CHINA

**Prof. Dr. Dmitri Kuzmin**
Fakultät für Mathematik, LS III
Technische Universität Dortmund
Vogelpothsweg 87
44227 Dortmund
GERMANY

**Elise Le Meledo**
Institut für Mathematik
Universität Zürich
Winterthurerstr. 190
8057 Zürich
SWITZERLAND

**Prof. Dr. Fengyan Li**
Dept. of Mathematical Sciences
Rensselaer Polytechnic Institute
110 8th Street
Troy NY 12180-3590
UNITED STATES

**Prof. Dr. Jiequan Li**
Institute of Applied Physics
and Computational Mathematics,
Beijing
Beijing 100088
CHINA

**Dr. Raphael Loubère**
Institut de Mathématiques de Bordeaux
(IMB)
Université de Bordeaux
351, cours de la Liberation
33405 Talence Cedex
FRANCE

**Prof. Dr. Mária Lukáčová-Medvid'ová**
Institut für Mathematik
Fachbereich
Mathematik/Physik/Informatik
Johannes-Gutenberg-Universität Mainz
Staudingerweg 9
55128 Mainz
GERMANY

**Dr. Elio Marconi**
Departement Mathematik und
Informatik der Universität Basel
Fachbereich Mathematik
Spiegelgasse 1
4051 Basel
SWITZERLAND

**JProf. Dr. Sandra May**
Fakultät für Mathematik
Technische Universität Dortmund
Vogelpothsweg 87
44227 Dortmund
GERMANY

**Prof. Dr. Tien Khai Nguyen**
Department of Mathematics
North Carolina State University
Campus Box 8205
Raleigh, NC 27695-8205
UNITED STATES

**Prof. Dr. Sebastian Noelle**
Institut für Geometrie und
Praktische Mathematik
RWTH Aachen
Templergraben 55
52061 Aachen
GERMANY

**Dr. Philipp Öffner**
Institut für Mathematik
Johannes-Gutenberg-Universität Mainz
Staudingerweg 9
55128 Mainz
GERMANY

**Prof. Dr. Felix Otto**
Max-Planck-Institut für Mathematik
in den Naturwissenschaften
Inselstrasse 22 - 26
04103 Leipzig
GERMANY

**Mr Marco Petrella**
Seminar for Applied Mathematics
ETH Zürich
Rämistr. 101
P.O. Box HG G 58.3
8092 Zürich
SWITZERLAND

**Prof. Dr. Gabriella A. Puppo**
Dipartimento di Matematica
Universita di Roma "La Sapienza"
Istituto "Guido Castelnuovo"
Piazzale Aldo Moro, 5
00185 Roma
ITALY

**Dr. Mario Ricchiuto**
INRIA Bordeaux Sud-Ouest
200, Ave. de la Vieille Tour
33405 Talence Cedex
FRANCE

**Prof. Dr. Nils Henrik Risebro**
Department of Mathematics
University of Oslo
Blindern
P.O. Box 1053
0316 Oslo
NORWAY

**Prof. Dr. Christian Rohde**
Institut für Angewandte Analysis und
Numerische Simulation
Universität Stuttgart
Pfaffenwaldring 57
70569 Stuttgart
GERMANY

**Dr. Elena Rossi**
Università degli Studi di Modena e
Reggio Emilia
Dipartimento di Scienze e Metodi
dell'Ingegneria - DISMI
Via Amendola, 2
42122 Reggio Emilia
ITALY

**Simon Schneider**
Institut für Mathematik
Johannes-Gutenberg-Universität Mainz
Staudingerweg 9
55128 Mainz
GERMANY

**Prof. Dr. László Székelyhidi Jr.**
Mathematisches Institut
Universität Leipzig
Postfach 10 09 20
04009 Leipzig
GERMANY

**Prof. Dr. Eitan Tadmor**
Center for Scientific Computation
and Mathematical Modeling (CSCAMM)
University of Maryland
CSIC Building # 406
Paint Branch Drive
College Park MD 20742-3289
UNITED STATES

**Dr. Andrea Thomann**
Institut für Mathematik
Fachbereich
Mathematik/Physik/Informatik
Johannes-Gutenberg-Universität Mainz
Staudingerweg 9
55128 Mainz
GERMANY

**Tessa Thorsen**
Department of Mathematics
University of Maryland
College Park, MD 20742
UNITED STATES

**Dr. Davide Torlo**
INRIA Bordeaux / Sud-Ouest
Team CARDAMOM
200, Avenue de la Vieille Tour
33405 Talence Cedex
FRANCE

**Prof. Dr. Konstantina Trivisa**
Department of Mathematics
University of Maryland
College Park, MD 20742-4015
UNITED STATES

**Prof. Athanasios E. Tzavaras**
Applied Mathematics and
Computational Sciences
4700 King Abdullah University of
Science
and Technology (KAUST)
Thuwal 23955-6900, Jeddah
SAUDI ARABIA

**Prof. Dr. Gerald Warnecke**
Fakultät für Mathematik
Institut für Analysis und Numerik
Otto-von-Guericke-Universität
Magdeburg
Universitätsplatz 2
39106 Magdeburg
GERMANY

**Prof. Dr. Franziska Weber**
Department of Mathematical Sciences
Carnegie Mellon University
5000 Forbes Avenue
Pittsburgh, PA 15213-3890
UNITED STATES

**Bettina Wiebe**
Institut für Mathematik
Johannes-Gutenberg-Universität Mainz
Staudingerweg 9
55128 Mainz
GERMANY

**Dr. Yuhuan Yuan**
Institut für Mathematik
Johannes-Gutenberg-Universität Mainz
Staudingerweg 9
55128 Mainz
GERMANY