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Mathematical Theory of Water Waves

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Abstract. The water-wave problem is the study of the two- and three-dimensional irrotational flow of a perfect fluid bounded above by a free surface subject to the forces of gravity and surface tension. It is a paradigm for most modern methods in nonlinear functional analysis and nonlinear dispersive wave theory. Its mathematical study calls upon many different approaches, as iteration methods, bifurcation theory, dynamical systems theory, complex variable methods, PDE methods, the calculus of variations, positive operator theory, topological degree theory, KAM theory, and symplectic geometry.

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Introduction by the Organisers

The 1990s saw vigorous activity in the mathematical theory of water waves by several independent international research groups, and in response we organised a mini-workshop in Oberwolfach in 2001 entitled *Recent Developments in the Mathematical Theory of Water Waves.* The 2001 meeting, which was devoted to the exact equations for water waves as written down by Euler, was a great success. A collection of papers originating at the meeting were published in a special issue of the *Philosophical Transactions of the Royal Society* and lead to significant progress on certain famous and outstanding problems in water waves, particularly local existence and uniqueness, stability, three-dimensional waves, justification of model equations, convexity results for Stokes waves, and effective and accurate numerical schemes.

In view of the interest in water waves generated by the 2001 meeting and our subsequent endeavours, it appeared timely to bring the various research groups together in another Oberwolfach workshop to intensify research in this subject. The following topics were chosen as priority research areas for the workshop since

(i) they represent issues which have been almost fully settled for model equations, but remain almost fully open for the exact water-wave problem; and (ii) pose mathematical challenges whose resolution is likely to prove beneficial in a wide range of situations beyond the water-wave problem:

- Comparison of the diverse mathematical formulations of the hydrodynamic equations which have recently been found by different researchers;
- Numerical and analytical construction of fully localized three-dimensional solitary waves;
- Numerical and analytical investigations of stability and instability mechanisms for periodic wave trains;
- New scenarios for the generation of freak waves in long wave form;
- Construction of doubly-periodic three-dimensional water waves;
- New results for Dirichlet-Neumann operators.

Significant new results in these areas were reported at the conference and are summarised in the extended abstracts below. The workshop was attended by twenty-three researchers from eight countries; there was a good mix of researchers who had attended the 2001 meeting and those who had not, and a number of younger researchers new to this field attended. Twenty 45-minute talks were held in a friendly and informal atmosphere and, in true Oberwolfach spirit, many collaborative discussions took place.

Walter L. Craig
Mark D. Groves
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Workshop: Mathematical Theory of Water Waves

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Abstracts

Analytical Aspects of Boundary Integral Methods for Three-Dimensional Water Waves

J. Thomas Beale

Much of my work of the last several years has been concerned with developing simple, direct methods for computing singular or nearly singular integrals, such as a layer potential on a curve or surface. Several aspects are described here, beginning with the connection to water waves. The papers referred to and other recent papers can be found at my web site.

Numerical methods based on a boundary integral formulation of the water wave equations have long been used to simulate fully nonlinear, exact, time-dependent water waves. For surface waves in three dimensions it appears that choices of discretization can strongly affect the numerical stability as well as accuracy. In [1] we were concerned with the formulation, accuracy, and stability in the case of a doubly periodic surface with Lagrangian markers. One version of the method, similar to ones in use, was proved to converge to the actual solution as long as it is smooth. The normal velocity at the surface is determined from the potential by solving an integral equation. This integral equation is a realization of the Dirichlet-Neuman operator. The singular integrals are computed at points on the moving surface using a simple discretization with a regularization and then adding a correction. The wave-like character of the motion depends on the positivity of a certain operator related to the Dirichlet-Neumann operator, as well as the sign of the normal pressure gradient. As a consequence, positivity for the discrete single layer potential seems to be important to maintain numerical stability for the discrete approximations. The estimates use a class of discrete versions of pseudodifferential operators. The error analysis leading to convergence depends on preserving a structure in the linearized discrete equations which appears in the original system.

The work on water waves led to a more general approach [2, 3] for computing a singular or nearly singular integral, such as a harmonic function given by a layer potential on a curve in two dimensions or a surface in three dimensions. The singularity is regularized and a standard quadrature is used for the regularized integral. Correction terms added for the errors due to regularization and discretization. These corrections are found by local analysis near the singularity. This technique could be useful, for instance, in viscous fluid calculations with moving interfaces, since a pressure term due to a boundary force can be written as a layer potential. The accurate evaluation of a layer potential at a point near the curve or surface on which it is defined is not routine, since the integral is nearly singular. In work with M.-C. Lai [2], we solved boundary value problems in two dimensions by computing the integral at grid points near the curve as described and using these values to find those at all points. A similar approach works in three dimensions, with the surface integrals computed in overlapping coordinate grids on the surface [3]. To solve a boundary value problem, we first need to solve an integral equation for the strength of a dipole layer on the surface. We proved in [3] that the solution of the discrete integral equation converges to the exact solution. Special care is needed where the grids overlap.

In work with G. Baker [4] we used a special choice of regularization in a boundary integral calculation of the motion of two layers of inviscid fluid. We considered the general Rayleigh-Taylor flow with a heavy fluid over a lighter fluid of positive density. The problem is ill-posed; the calculation was resolved with fixed regularization, but the limit of zero regularization is not understood.

Current work with John Strain is intended to develop a numerical method for moving interfaces in viscous fluid. The interface exerts a force on the fluid; the force could be surface tension or the response of an elastic material. The immersed boundary method of C. Peskin was designed for this model and has been widely applied to biological problems. We have begun with the case of two-dimensional Stokes flow. In our approach, the jumps in pressure and velocity gradient at the interface lead to a representation of the velocity by layer potentials. With periodic boundary conditions, we use Ewald summation to write the boundary integral in a local part, approximated analytically, and a smooth part, computed as a Fourier series. The decomposition is essentially the same as described above with regularization. The velocity can be found accurately on or off the interface. The motion of the interface is found using an adaptive method previously developed by Strain.

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Hydrodynamical Instabilities and Energy Estimates for the Free Boundary Problems of the Euler Equation

Chongchun Zeng

(joint work with Jalal Shatah)

We consider the evolution of free surfaces of incompressible and inviscid fluids. Neglecting the gravity, we are interested in the cases of 1.) the motion of a droplet in the vacuum with or without surface tension and 2.) the motion of the interface between two fluids with surface tension. The evolution of these fluid boundaries and the velocity fields is determined by the free boundary problem of the Euler's equation.

In the first problem, i.e. one fluid case, let $\Omega_t \subset \mathbb{R}^n$, $n \geq 2$, be the bounded and smooth moving fluid domain. The velocity field $v(t,\cdot) : \Omega_t \to \mathbb{R}^n$ satisfies the Euler's equation

$$\text{(1F)} \qquad \begin{cases} v_t + \nabla_v v = -\nabla p, & x \in \Omega_t \subset \mathbb{R}^n \\ \nabla \cdot v = 0, & x \in \Omega_t \end{cases}$$

where $p(t,\cdot)$ is the pressure. The boundary of the domain Ω_t moves with the fluid velocity and the pressure at the boundary may or may not contain the surface tension, that is

$$\text{(BC1)} \qquad \begin{cases} \mathbf{D}_t = \partial_t + v \cdot \nabla \quad \text{is tangent to} \quad \bigcup_t \Omega_t \subset \mathbb{R}^{n+1} \\ p(t,x) = \epsilon^2 \kappa(t,x), \quad x \in \partial\Omega_t, \quad 0 \leq \epsilon \leq 1 \end{cases}$$

where $\kappa(t,x)$ is the mean curvature of the boundary $\partial\Omega_t$ at $x \in \partial\Omega_t$, and $\mathbf{D}_t$ is the material derivative. This is equivalent to saying the velocity of $\partial\Omega_t$ is given by $v \cdot N$ where N is the unit normal to $\partial\Omega_t$. The case $\epsilon = 0$ corresponds to the zero surface tension problem.

In the second problem, i.e. the interface motion between two fluids, Ω_t^+ and Ω_t^- in $\mathbb{R}^n$, $n \geq 2$, are the bounded and smooth moving fluid domains. We assume that $\mathbb{R}^n = \Omega_t^+ \cup \Omega_t^- \cup S_t$ where $S_t = \partial\Omega_t^\pm$, and let $p_\pm : \Omega_t^\pm \to \mathbb{R}$, $v_\pm : \Omega_t^\pm \to \mathbb{R}^n$, and the constant $\rho_\pm > 0$ denote the pressure, the velocity vector field, and the density respectively. On the interface S_t, we let $N_\pm(t,x)$, $x \in S_t$ denote the unit outward normal of $\Omega_t^\pm$ (thus $N_+ + N_- = 0$), $H(t,x) \in (T_x S_t)^\perp$ denote the mean curvature vector, and $\kappa_\pm = H \cdot N_\pm$. We also assume that there is surface tension on the interface given by the mean curvature. Thus the free boundary problem for the Euler equation that we consider here is given by

$$\text{(2F)} \qquad \begin{cases} \rho(v_t + \nabla_v v) = -\nabla p, & x \in \mathbb{R}^n \setminus S_t \\ \nabla \cdot v = 0, & x \in \mathbb{R}^n \setminus S_t. \end{cases}$$

The boundary conditions for the interface evolution and the pressure are

$$\text{(BC2)} \qquad \begin{cases} \partial_t + v_\pm \cdot \nabla \quad \text{is tangent to} \quad \bigcup_t S_t \subset \mathbb{R}^{n+1}, \\ p_+(t,x) - p_-(t,x) = \kappa_+(t,x), \quad x \in S_t, \end{cases}$$

where we introduced the notation $v = v_+ \mathbb{1}_{\Omega_+} + v_- \mathbb{1}_{\Omega_-} : \mathbb{R}^n \setminus S_t \to \mathbb{R}^n$, etc.

These free boundary value problems have been studied intensively by many authors. Due to the limitation on the length of this article, we skip the background description. Our goal in studying these problems is to obtain a priori energy estimates and our approach is based on the well-known fact that the free boundary problems can be formulated as Lagrangian systems on infinite dimension manifolds of volume preserving diffeomorphisms.

Problem (1F, BC1) has a Lagrangian formulation given by

$$I(u) = \iint_{\Omega_0} \frac{|u_t|^2}{2} dy dt - \epsilon^2 \int S(u) dt,$$

where $u(t,\cdot) \in \Gamma = \{\Phi : \Omega_0 \to \mathbb{R}^n, \text{volume preserving homeomorphisms}\}$, and $S(u)$ is the surface area of $u(\partial\Omega_t)$. Using this variational derivation, one can write the Euler-Langrangian equation of $I(u)$ as

(E-L) $$\bar{\mathscr{D}}_t u_t + \epsilon^2 S'(u) = 0,$$

where $\bar{\mathscr{D}}$ is the Riemannian connection on Γ induced by the L^2 metric on $T\Gamma$. The above form of the equation makes it relatively easy to identify the correct linearized problem

$$\bar{\mathscr{D}}_t^2 \bar{w} + \bar{\mathscr{R}}(u_t, \bar{w})u_t + \epsilon^2 \bar{\mathscr{D}}^2 S(u)(\bar{w}) = 0, \quad \bar{w}(t,\cdot) \in T_{u(t,\cdot)}\Gamma,$$

where $\bar{\mathscr{R}}$ is the curvature tensor of Γ. Keeping the highest order terms in the above equation we obtain

(LN) $$\bar{\mathscr{D}}_t^2 \bar{w} + \bar{\mathscr{R}}_0(v)\bar{w} + \epsilon^2 \bar{\mathscr{A}} \bar{w} = \text{ lower order terms},$$

where $\bar{\mathscr{R}}_0(v)$ is a first order differential operator and $\bar{\mathscr{A}}$ is a third order differential operator. In Eulerian coordinates these terms are given by

$$\mathscr{R}_0(v)(w,w) = \int_{\partial u(\Omega_0)} -\nabla_N p_{v,v} |w{\cdot}N|^2 \, dS, \quad \mathscr{A}(u)(w,w) = \int_{\partial u(\Omega_0)} |\nabla^\top w{\cdot}N|^2 \, dS$$

where N is the unit normal and $\nabla^\top$ is the tangential gradient on the boundary of $u(\Omega_0)$. Here once again we are led in a natural way to distinguish the two problems in the following manner.

1) For $\epsilon > 0$ two time derivatives are associated with $\bar{\mathscr{A}}$, which is a positive semi-definite operator similar to three spatial differentiation, thus roughly speaking, $\partial_t \sim (\partial_x)^{\frac{3}{2}}$. Therefore one may be led to believe that the regularity of the Lagrangian coordinates given by $\partial_t u = v$ is $\frac{3}{2}$ order better than v, which reflects the regularizing effect of the surface tension. However this is not true for the Lagrangian coordinates since $\mathscr{A}$ is degenerate, and the regularity improvement of the $\partial\Omega_t$ is geometric and is not reflected in the Lagrangian coordinates system.

2) For $\epsilon = 0$ the leading term involves $\mathscr{R}_0(v)$ and thus the Rayleigh-Taylor instability may occur unless we impose the condition

(RT) $$-\nabla_N p_{v,v}(t,x) > a > 0 \quad x \in \partial\Omega_t.$$

In this case two time derivatives are associated with $\bar{\mathscr{R}}$ which is a positive semi-definite operator similar to one spatial differentiation. Thus, $\partial_t \sim (\partial_x)^{\frac{1}{2}}$ and comments similar to above hold on the regularity of $\partial\Omega_t$.

3) For $\epsilon > 0$ one can directly obtain nonlinear estimates by multiplying (E-L) by $(\mathscr{D}^2 S)^k S'$.

4) The control that any power $\mathscr{R}_0(v)$, with (RT) condition, can give over vector fields is limited by the smoothness of the boundary $\partial\Omega_t$. This fact makes the velocity field v inappropriate vector field to estimate because it is smoother than what these operators allow. In fact, a divergence free vector field suitable for the energy estimates turns out to be $J = \nabla\kappa_{\mathcal{H}}$ which is less smooth than v and one may verify that it satisfies

$$\bar{\mathscr{D}}_t^2 \bar{J} + \bar{\mathscr{R}}_0(v)\bar{J} + \epsilon^2 \bar{\mathscr{A}} \bar{J} = \text{ lower order terms.}$$

5) Since $\mathscr{A}$ and $\mathscr{R}_0(v)$ are degenerate for vector fields which are tangential to the boundary $\partial\Omega_t$ one needs to add the vorticity ω which controls the rotational part of the velocity which is tangential to the boundary.

Following from these facts, we define the energy $\mathcal{E} = E + E_{RT}$ where

$$\begin{aligned} E &= \int_\Omega \tfrac{1}{2}|\mathscr{A}^{k-1}\mathscr{D}_t J|^2 + \tfrac{\epsilon^2}{2}|\mathscr{A}^{k-\frac{1}{2}} J|^2\, dx + |\omega|^2_{H^{3k-1}(\Omega)}, \\ E_{RT} &= \int_\Omega \tfrac{1}{2}\mathscr{R}_0(v)\mathscr{A}^{k-1}J \cdot \mathscr{A}^{k-1}J dx. \end{aligned}$$

The following definition of neighborhood of domains will be needed in the statement of energy estimates

Definition 1. *Suppose $\Omega_* \subset \mathbb{R}^n$ is a bounded domain so that $\partial\Omega_*$ locally is given by the graphs of $H^{\frac{3}{k}-\frac{1}{2}}$ functions. Let $\Lambda = \Lambda(\Omega_*, 3k-\frac{1}{2}, \delta)$ be the collection of all domains Ω satisfying that there exists a diffeomorphism $F : \partial\Omega_* \to \partial\Omega \subset \mathbb{R}^n$, so that $|F - id_{\partial\Omega_*}|_{H^{3k-\frac{1}{2}}(\partial\Omega_*)} < \delta$.*

Assuming $k > \frac{1}{3}(\frac{n}{2}+1)$ and $\delta > 0$ is sufficiently small, we have the following two theorems which closes the a priori estimates local in time.

Theorem 2. *For any $\Omega \in \Lambda_0$, we have*

$$\epsilon^2|\kappa|^2_{H^{3k-1}(\partial\Omega)} \le 3E + C_0\epsilon^2, \quad |v|^2_{H^{3k}(\Omega)} \le C_0(E + E_0)$$

and, if we also assume (RT),

$$|\kappa|^2_{H^{3k-2}(\partial\Omega)} \le C_* E_{RT} + C_0,$$

for constants $C_* > 0$ depending only on a in assumption (RT) and $C_0 > 0$ only on the set Λ.

Theorem 3. *There exists $t^* > 0$, depending only on $|v(0,\cdot)|_{H^{3k}(\Omega_t)}$ and the set Λ, such that, for all $t \in [0, t^*]$, any solution of (1F, BC1) with $\Omega_0 \in \Lambda(\Omega_*, 3k-\frac{1}{2}, \frac{\delta}{2})$ satisfies*

$|\frac{d}{dt}\mathcal{E}| \le Q$, *where* $Q =$ *a polynomial of* $(|v|_{H^{3k}(\Omega_t)}, |\kappa|_{H^{3k-2}(\partial\Omega_t)}, \epsilon|\kappa|_{H^{3k-1}(\partial\Omega_t)})$.

For the second problem, i.e. the interface problem between two fluids, one may verify that the Lagrangian coordinates maps satisfy:
1) $\Phi_\pm : \bar{\Omega}^\pm \to \Phi_\pm(\bar{\Omega}^\pm)$ a volume preserving homeomorphism.
2) $S \triangleq \partial\Phi_\pm(\Omega^\pm) = \Phi(\partial\Omega^\pm)$.
Define

$$\Gamma = \{\Phi = \Phi_+ \mathbb{1}_{\Omega^+} + \Phi_- \mathbb{1}_{\Omega^-}; \quad \Phi_\pm \text{ satisfy 1 and 2 above}\}.$$

The system (2F, BC2) is actually the Euler-Lagrangian equation with the Lagrangian action

$$I(u) = \iint_{\mathbb{R}^n \setminus S_0} \frac{\rho|u_t|^2}{2} dydt - \int S(u)dt, \quad u(t,\cdot) \in \Gamma.$$

The similar program can be carried out as in the one fluid case. It turns out that, on the one hand, the principle part of $\bar{\mathscr{R}}(v,\cdot)v$ is a second order semi negative definite operator which implies the Kelvin-Helmholtz instability. On the other hand, the principle part of $\bar{\mathcal{D}}^2 S$ is a third order semi-positive definite operator and thus it regularizes the evolution of the interface and the a priori estimates follows.

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Mathematical Analysis of Vortex Sheets

Sijue Wu

We consider the motion of the interface separating two domains of the same fluid in R^2 that moves with different velocity along the tangential direction of the interface. We assume that the fluids occupying the two domains separated by the interface are of constant densities that are equal, inviscid, incompressible and irrotational. We also assume that the surface tension is zero, and there are no external forces. The interface in the aforementioned fluid motion is a so-called vortex sheet.

In general, there are two approaches in the study of vortex sheet evolution. One is to solve the initial value problem of the incompressible Euler equation in R^2:

$$\begin{cases} v_t + v \cdot \nabla v + \nabla p = 0 \\ \operatorname{div} v = 0, \\ v(x,y,0) = v_0(x,y) \end{cases} \qquad (x,y) \in R^2,\ t \geq 0 \tag{1}$$

where the initial incompressible velocity $v_0 \in L^2_{\text{loc}}(R^2)$, in which the vorticity $\omega_0 = \operatorname{curl} v_0$ is a finite Radon measure. Here v is the fluid velocity, p is the pressure, and the density of the fluid is assumed to be one. Notice that a vortex sheet gives a measure valued vorticity supported on the interface. This approach was posed by DiPerna and Majda in 1987 [1]. In 1991, J.M. Delort [2] proved the existence of weak solutions global in time of Equation (1) for measure-valued initial vorticity in $H^{-1}_{\text{loc}}(R^2)$ that has a distinguished sign. However non-uniqueness of the weak solutions in $v \in L^2(R^2 \times (-T,T))$ was demonstrated by examples of V. Scheffer [3] and A. Shnirelman [4], and weak solutions give little information about the nature of the evolution.

The second approach assumes further that the free interface between the two fluids remains a curve in R^2 at a later time, therefore reduces Equation (1) to the following evolutionary differential-integral equation:

$$\partial_t \overline{z}(\alpha,t) = \frac{1}{2\pi i}\, p.v. \int \frac{1}{z(\alpha,t)-z(\beta,t)}\, d\beta. \tag{2}$$

Here $z(\alpha,t)$ is the parametric equation of the vortex sheet curve $\Gamma(t)$ in complex variables at time t, α is the circulation variable, $1/|z_\alpha|$ is the vortex strength, and $\overline{z}$ denotes the complex conjugate of z. (2) is the so-called Birkhoff-Rott equation. A rigorous justification of the equivalence between Equations (1) and (2) for smooth vortex sheet curves and smooth vortex strength can be found in [5]. A steady solution of (2) is the flat sheet $z(\alpha,t) = \alpha$.

Equation (2) has been actively investigated over the last four decades. A well-known property of (2) is that perturbations of the flat sheet grow due to the Kelvin-Helmholtz instability, following from a linearization of Equation (2) about the flat sheet. Sulem, Sulem, Bardos and Frisch [6] proved that (2) is well-posed locally in time in analytic class. Duchon and Robert [7] obtained the global existence of solutions of Equation (2) for a special class of initial data that is close to the flat sheet. However, numerous results show that a vortex sheet can develop a curvature singularity in finite time from analytic data. D.W. Moore [8, 9] was the first to provide analytical evidence that predicts the occurrence and time of singularity formation, which was verified numerically by Meiron, Baker and Orszag [10] and by Krasny [11]. For each $\nu > 0$, Duchon and Robert [7] and Caflisch and Orellana [12] constructed specific examples of solutions $z = z_\nu(\alpha, t)$ of Equation (2), with $z_\nu(\cdot, 0)$ real analytic, but $z_\nu(\cdot, t_\nu) \notin C^{1+\nu}(R)$ for some small time $t_\nu > 0$. These examples show that the initial value problem of the Birkhoff-Rott equation (2) is ill-posed in $C^{1+\nu}(R)$, $\nu > 0$, and in Sobolev spaces $H^s(R)$, $s > 3/2$ in the Hadamard sense. However the existence of solutions in spaces less regular than $C^{1+\nu}$ or H^s, the nature of vortex sheet at and beyond singularity formation remained unknown.

This suggests that we look for solutions in the largest possible spaces where Equation (2) makes sense. In a recent work [13, 14], we considered the Birkhoff-Rott equation (2) in the chord-arc class. The reasons for looking for solutions in this class are the following: firstly, experimental results and numerical computations (see [15]) suggest that vortex sheets tend to roll up into infinite spirals after the singularity formation time. The chord-arc class contains infinitely rolled-up spiral curves, therefore can be a possible space in which to study the behavior of solutions after the singularity formation time and prove existence. Secondly, the chord-arc class is nearly the largest class in which the Birkhoff-Rott equation (2) makes sense in L^2 [13, 14]. The equivalence of (2) with (1) in the chord-arc class is recently verified in [16].

Let Γ be a rectifiable Jordan curve in R^2 given by $\xi = \xi(s)$ in arc-length s. We say Γ is chord-arc, if there is a constant $M \geq 1$, such that

$$|s_1 - s_2| \leq M|\xi(s_1) - \xi(s_2)|, \quad \text{for all } s_1,\ s_2.$$

The infimum of all such constants M is called the chord-arc constant. Examples of chord-arc curves include Lipschitz curves and logarithmic spirals.

The chord-arc class has a natural correspondence with the BMO class. For a chord-arc curve $\xi = \xi(s)$, s the arc-length, it is proved in [17] that $\xi'(s)$ exists almost everywhere, and there is a choice of the argument function $b \in \text{BMO}$, such that $\xi'(s) = e^{ib(s)}$. In particular, if the chord-arc constant M is close to 1, then $\|b\|_{mathrmBMO}$ is close to 0. On the other hand, if $b \in \text{BMO}$, and $\|b\|_{\text{BMO}} < 1$, then $\xi(s) = \xi_0 + \int_0^s e^{i\,b(s')}\,ds'$ defines a chord-arc curve, with chord-arc constant $\leq 1/\{1 - \|b\|_{\text{BMO}}\}$.

Define

$$\|f\|_{\mathrm{BMO}(a,b),\delta_0} = \sup_{\text{all } I\subset(a,b),|I|\le\delta_0} \tfrac{1}{|I|}\int_I |f(\alpha)-f_I|\,d\alpha < \infty,$$

here $f_I = \frac{1}{|I|}\int_I f(\alpha)\,d\alpha$, I is an interval. We proved the following three results in [14].

Theorem 1. *Assume that* $z \in H^1([0,T], L^2_{\mathrm{loc}}(R)) \cap L^2([0,T], H^1_{\mathrm{loc}}(R))$ *and that* z *is a solution of the Birkhoff-Rott equation (2) for* $0 \le t \le T$*, satisfying the following (cf.* [13, 14]*):*

(i) There are constants $m > 0$, $M > 0$, *independent of* t, *such that*

(3) $$m|\alpha-\beta| \le |z(\alpha,t)-z(\beta,t)| \le M|\alpha-\beta| \quad \text{for all } \alpha,\ \beta\in R,\ 0\le t\le T.$$

Then there is a constant $c(m,M) > 0$ *as follows: if also*

(ii) on some fixed interval (a,b), *there exist a determination of* $\ln z_\alpha$ *and a constant* $\delta_0 > 0$, *independent of* t, *satisfying*

(4) $$\sup_{[0,T]} \|\ln z_\alpha(\cdot,t)\|_{\mathrm{BMO}(a,b),\delta_0} \le c(m,M),$$

then $z_\alpha \in C((a,b)\times(0,T))$, *and for each* $t_0 \in (0,T)$, $z_\alpha(\cdot,t_0)$ *is analytic on* (a,b).

Notice that assumption (ii) is satisfied if $z_\alpha \in C([a,b]\times[0,T])$. Roughly Theorem 1 states that if during some positive time period $[0,T]$, a certain section of the vortex sheet curve $z = z(\cdot,t)$ is chord-arc and doesn't roll up too fast and if the vortex strength $1/|z_\alpha|$ is bounded away from zero and infinity, then for all fixed $t \in (0,T)$, this section of the vortex sheet curve $z = z(\cdot,t)$ is smooth, in fact real analytic.

A consequence of Theorem 1 is that after the singularity formation time, the part of the vortex sheet near the singularity points cannot be a chord-arc curve that doesn't roll-up too fast, and meanwhile has a vortex strength that is bounded away from zero and infinity.

G. Lebeau [18] proved a version of Theorem 1 under some stronger assumptions that the solution of the Birkhoff-Rott equation (2): $z = z(\alpha,t) \in C^{1+\nu}$, for some $\nu > 0$, the vortex strength is bounded away from zero and infinity, and the vortex sheets $\Gamma(t)$ are closed chord-arc curves.

Let Rez and Imz be the real and imaginary parts of the complex number z respectively. Regarding the existence of solutions, we proved the following

Theorem 2. *For any real valued function* $w_0 \in H^{\frac{3}{2}}(R)$, *there exists*

$$T = T(\|w_0\|_{H^{\frac{3}{2}}}) > 0,$$

such that the Birkhoff-Rott equation (2) has a solution $z = z(\alpha,t)$ *for* $0 \le t \le T$, *satisfying* $\ln z_\alpha \in C([0,T], H^{\frac{3}{2}}(R)) \cap C^1([0,T], H^{\frac{1}{2}}(R))$ *and* $\mathrm{Im}\{(1+i)\ln z_\alpha(\alpha,0)\} = w_0(\alpha)$, *with the property that there exist constants* $m,\ M > 0$ *independent of* t *such that*

$$m|\alpha-\beta| \le |z(\alpha,t)-z(\beta,t)| \le M|\alpha-\beta|, \quad \text{for all } \alpha,\beta\in R,\ 0\le t\le T.$$

Moreover, for each fixed $t_0 \in (0,T)$, $z(\cdot,t_0)$ *is real analytic (cf.* [13, 14]*).*

Theorem 2 states that if only half of the data $z_\alpha(\alpha, 0)$ is given, there is a solution of the Birkhoff-Rott equation (2) for a finite time period. Theorem 2 is a generalization of the existence result of Duchon and Robert [7] to general data.

The following result implies that Theorem 2 is optimal, in the sense that in general there is no solution of equation (2) satisfying properties (i) and (ii) as stated in Theorem 2 beyond the initial time $t = 0$ for arbitrarily given initial data.

Theorem 3. *Assume that* $z \in H^1([0,T], L^2_{\text{loc}}(R)) \cap L^2([0,T], H^1_{\text{loc}}(R))$ *is a solution of the Birkhoff-Rott equation (2) for* $0 \le t \le T$, $T > 0$, *satisfying properties (i) and (ii) on some interval* (a,b) *as stated in Theorem 1. Assume further that* $w_0 = \text{Im}\{(1+i)\ln z_\alpha(\cdot, 0)\}$ *is real analytic on* (a,b). *Then* $z_\alpha \in C((a,b) \times [0,T))$ *and* $\text{Re}\{(1+i)\ln z_\alpha(\cdot, 0)\}$ *is also real analytic on* (a,b) *(cf.* [13, 14]*).*

We remark that by assigning the full data $z(\alpha, 0)$ for the Birkhoff-Rott equation (2) it is equivalent to assigning the initially incompressible velocity (with the initial vorticity supported on a curve) for the incompressible Euler equation (1).

Theorems 2–4 were proved after we derived the elliptic nature of the Birkhoff-Rott equation (2) in chord-arc class. Notice that equation (2) is fully nonlinear. Roughly speaking, we derived the ellipticity of (2) by taking a derivative with respect to α to (2), and by using results and techniques from harmonic analysis. However, since the solution class we considered (i.e. chord-arc) has minimum regularities, taking derivatives to the equation doesn't make sense. A strict proof (see [14]) for the ellipticity of (2) in chord-arc class was through taking difference quotient with respect to α. Theorems 2–4 follow from exploiting the elliptic nature of equation (2) and developing new analysis tools (see [14]).

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Dynamics of Solitary Gravity-Capillary Waves

Paul A. Milewski

We now know that both the Euler equations and various models allow for localized solitary gravity-capillary waves in two- and three-dimensions in water of arbitrary depth [2],[3], whereas, with gravity alone, solitary waves exist only in shallow water and only in two-dimensions. With the exception of extremely shallow water (Bond number less than 1/3) these capillary-gravity waves are of the "wavepacket" type. They can be seen as bifurcating from the minimum of the phase-speed curve $c(k)$ where

$$c'(k) = \frac{1}{k}\left(c_g(k) - c(k)\right) = 0. \tag{1}$$

Thus, in a wave-packet expansion, the carrier wave and the envelope have the same leading order speed and the surface displacement is an oscillatory solitary pulse. Although the existence (numerical, asymptotic and rigorous) of these waves has been studied by various authors, little is known about their dynamics. The most obvious equation describing their behavior, the (focusing) Nonlinear Schrödinger (NLS) equation, cannot predict even the *existence* of the two families of waves that are observed. These two solutions are denoted waves of depression and elevation in the literature and correspond to whether the crest of the envelope is located at the crest or trough of the carrier. The NLS equation would predict solutions with an arbitrary phase between the carrier and the envelope. We approach this problem by writing a more general small amplitude equation that describes the waves:

$$R_t + i\mathcal{H}R = \mathcal{N}(R), \tag{2}$$

where $i\mathcal{H}$ has the Fourier symbol for right-travelling capillary-gravity waves in deep water $sign(k)(|k|(1+k^2))^{1/2}$ and $\mathcal{N}(R)$ is the formal small amplitude expansion for the nonlinear terms. This equation has capillary-gravity solitary waves (see Figures below) of both types, and does not appear to have any asymmetric waves.

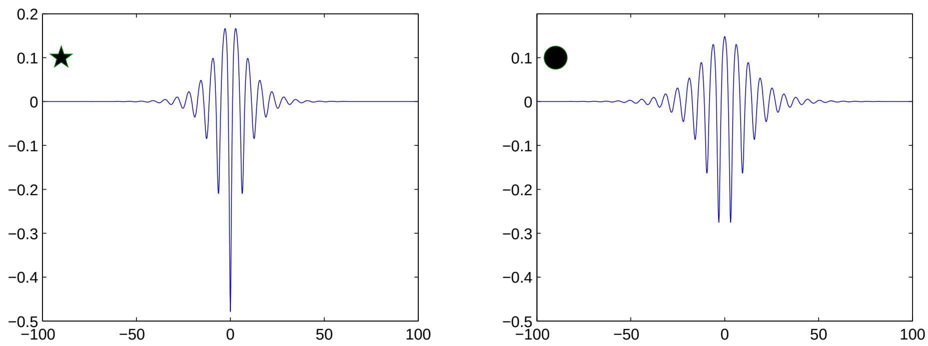

FIGURE 1. Wavepacket solitary waves of elevation and depression.

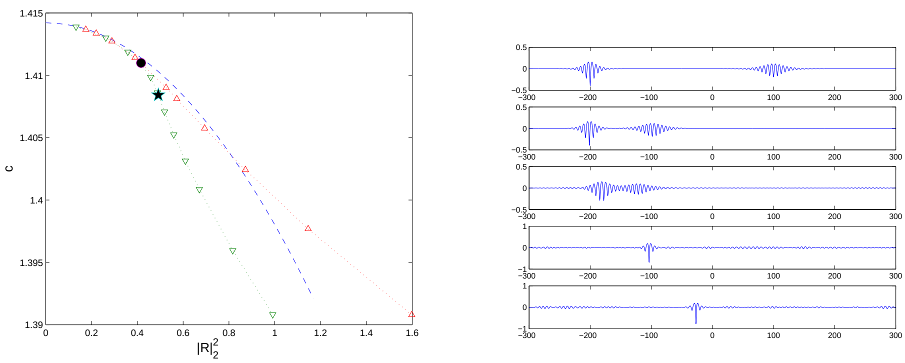

FIGURE 2. [Left] Depression (triangles) and elevation (circles) subcritical solitary wave-packet branches bifurcating from the minimum of $c(k)$. The NLS curve (dashed) is shown. [Right] Inelastic collisions of depression waves

It also contains the defocusing NLS in the appropriate asymptotic limit and thus is an appropriate setting to study the fluid problem *and* the limitations of NLS. In time dependent calculations, the solitary waves of depression are stable and those of elevation unstable as has been observed in another model (a fifth order KdV [1] for shallow water). Collisions between waves exhibit a variety of phenomena with waves sometimes losing their "travelling" status (and becoming wave-packets) to inelastic collisions (see Figure above). We are also pursuing similar questions for three-dimensional wave-packet solitary waves (as shown in Figure 3).

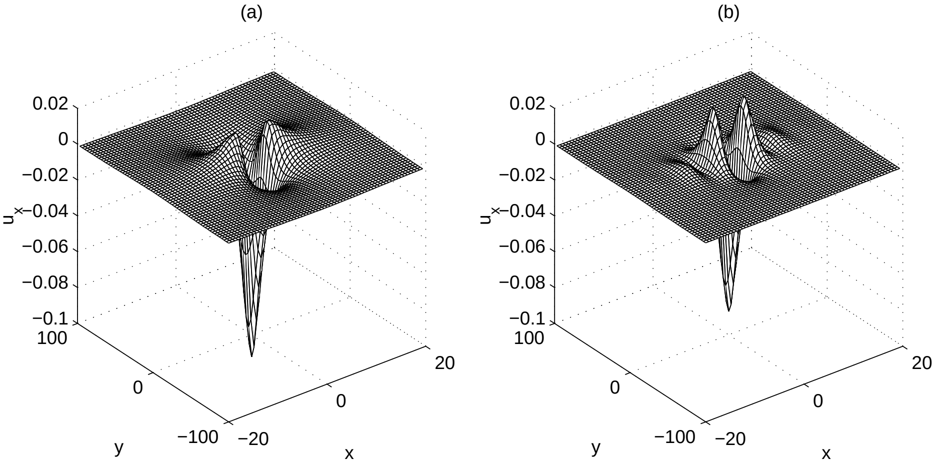

Figure 3. Shallow and deep water capillary-gravity wavepacket solitary waves from [2]

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Stability of Periodic Waves for the Nonlinear Schrödinger Equation

Mariana Haragus

(joint work with Thierry Gallay)

We consider the one-dimensional defocusing nonlinear Schrödinger equation (NLS)

$$\mathrm{i}U_t(x,t) + U_{xx}(x,t) - |U(x,t)|^2 U(x,t) \;=\; 0\;, \tag{1}$$

where $x \in \mathbb{R}$, $t \in \mathbb{R}$, $U(x,t) \in \mathbb{C}$. This equation possesses a six-parameter family of *quasi-periodic* solutions of the general form

$$U(x,t) \;=\; \mathrm{e}^{\mathrm{i}(px-\omega t)}\, V(x-ct)\;, \quad x \in \mathbb{R}\;, \quad t \in \mathbb{R}\;, \tag{2}$$

where p, ω, c are real parameters and the wave profile V is a *complex-valued periodic* function of its argument. We investigate the stability properties of these particular solutions [1, 2].

A crucial role in our stability analysis is played by the following four continuous symmetries of the NLS equation:

- *phase invariance:* $U(x,t) \mapsto U(x,t)\,\mathrm{e}^{\mathrm{i}\phi}$, $\phi \in \mathbb{R}$;
- *translation invariance:* $U(x,t) \mapsto U(x+\xi,t)$, $\xi \in \mathbb{R}$;
- *Galilean invariance:* $U(x,t) \mapsto \mathrm{e}^{-\mathrm{i}\left(\frac{v}{2}x+\frac{v^2}{4}t\right)}U(x+vt,t)$, $v \in \mathbb{R}$;
- *dilation invariance:* $U(x,t) \mapsto \lambda U(\lambda x, \lambda^2 t)$, $\lambda > 0$.

These symmetries are also useful to understand the structure of the set of all quasi-periodic solutions of (1). Assume that $U(x,t)$ is a solution of (1) of the form (2), where $V : \mathbb{R} \to \mathbb{C}$ is a bounded function. In view of the Galilean and the dilation invariance, we can assume without loss of generality that $c = 0$ and $\omega \in \{-1; 0; 1\}$. Setting $U(x,t) = \mathrm{e}^{-\mathrm{i}\omega t} W(x)$, we see that $W(x) = \mathrm{e}^{\mathrm{i}px} V(x)$ is a bounded solution of the ordinary differential equation

$$(3) \qquad W_{xx}(x) + \omega W(x) - |W(x)|^2 W(x) \;=\; 0\;, \quad x \in \mathbb{R}\;.$$

If $\omega = 0$ or $\omega = -1$, it is straightforward to verify that $W \equiv 0$ is the only bounded solution of (3), thus we assume from now on that $\omega = 1$. Equation (3) is actually the stationary Ginzburg-Landau equation and the set of its bounded solutions is well-known. There are two kinds of solutions of (3) which lead to quasi-periodic solutions of the NLS equation of the form (2):

- A family of *periodic solutions* $W(x) = (1-p^2)^{1/2}\,\mathrm{e}^{\mathrm{i}(px+\phi)}$, where $p \in [-1,1]$ and $\phi \in [0, 2\pi]$. The corresponding solutions of (1) are called *plane waves.* The general form of these waves is

$$U(x,t) \;=\; \mathrm{e}^{\mathrm{i}(px-\omega t)}\, V\;,$$

  where $p \in \mathbb{R}$, $\omega \in \mathbb{R}$, and $V \in \mathbb{C}$ satisfy $\omega = p^2 + |V|^2$.
- A family of *quasi-periodic solutions* of the form $W(x) = r(x)\,\mathrm{e}^{\mathrm{i}\phi(x)}$, where the modulus $r(x)$ and the derivative of the phase $\phi(x)$ are periodic with the same period. Any such solution can be written in the equivalent form $W(x) = \mathrm{e}^{\mathrm{i}px}\, Q(2kx)$, where $p \in \mathbb{R}$, $k > 0$, and $Q : \mathbb{R} \to \mathbb{C}$ is 2π-periodic. In particular,

$$(4) \qquad U(x,t) = \mathrm{e}^{-\mathrm{i}t} W(x) = \mathrm{e}^{\mathrm{i}(px-t)} Q(2kx)$$

  is a quasi-periodic solution of (1) of the form (2) (with $c = 0$ and $\omega = 1$). We refer to such a solution as a *periodic wave*, because its profile $|U(x,t)|$ is a (non-trivial) periodic function of the space variable x. Important quantities related to the periodic wave (4) are the period of the modulus $T = \pi/k$, and the Floquet multiplier $\mathrm{e}^{\mathrm{i}pT}$. For small amplitude solutions ($|Q| \ll 1$) the minimal period T is close to π, hence $k \approx 1$, and the Floquet multiplier is close to -1, so that we can choose $p \approx 1$.

While the plane waves form a three-parameter family, the periodic waves form a six-parameter family of solutions of (1).

The stability question is well-understood for *plane waves* [5], but it turns out to be much more difficult for *periodic waves.* In contrast to dissipative systems for which nonlinear stability of periodic patterns has been established for rather general classes of perturbations, including localized ones (see e.g. [4]), no such result is available so far for dispersive equations. While this problem remains open, we treat here two particular questions: orbital stability with respect to periodic perturbations, and spectral stability with respect to bounded or localized perturbations.

Our first result shows that the periodic waves of (1) are *orbitally stable* within the class of solutions which have the same period and the same Floquet multiplier as the original wave:

Theorem 1 (Orbital stability [2])
Let $X = H^1_{\rm per}([0,2\pi],\mathbb{C})$ and assume that $U_{\rm per}(x,t) = {\rm e}^{{\rm i}(px-t)}Q_{\rm per}(2kx)$ is a solution of (1) with $p \in \mathbb{R}$, $k > 0$, and $Q_{\rm per} \in X$. Then there exist positive constants C_0 and ϵ_0 such that, for all $R \in X$ with $\|R\|_X \le \epsilon_0$, the solution $U(x,t) = {\rm e}^{{\rm i}(px-t)}Q(2kx,t)$ of (1) with initial data $U(x,0) = {\rm e}^{{\rm i}px}(Q_{\rm per}(2kx) + R(2kx))$ satisfies, for all $t \in \mathbb{R}$,

$$\inf_{\phi,\xi\in[0,2\pi]} \|Q(\cdot,t) - {\rm e}^{{\rm i}\phi}Q_{\rm per}(\cdot-\xi)\|_X \le C_0\|R\|_X \ . \tag{5}$$

The proof of Theorem 1 relies upon the general approach to orbital stability developed by Grillakis, Shatah, and Strauss [3]. The main difficulty is to verify the assumptions of the stability theorem in [3]. We first check that the second variation of the energy functional at the periodic wave has exactly one negative eigenvalue. This result is first established for small waves (by perturbation arguments), and then a continuity argument allows to extend it to waves of arbitrary size. We next consider the *structure function* (which is called "$d(\omega)$" in [3]) and show, by a direct calculation, that its Hessian matrix has a negative determinant. Both properties together imply orbital stability. We point out that this stability result holds uniformly for all quasi-periodic solutions of (1) with small amplitude [1].

Next, we investigate the *spectral stability* of the *small* periodic waves with respect to bounded, or localized, perturbations. Although spectral stability is weaker than nonlinear stability, it provides valuable information about the linearization of the system at the periodic wave. Our second result is:

Theorem 2 (Spectral stability [1])
Let $Y = L^2(\mathbb{R},\mathbb{C})$ or $Y = C_b(\mathbb{R},\mathbb{C})$. There exists $\delta_1 > 0$ such that the following holds. Assume that $U_{\rm per}(x,t) = {\rm e}^{{\rm i}(px-t)}\,Q_{\rm per}(2kx)$ is a solution of (1) with $Q_{\rm per} \in X$, $\|Q_{\rm per}\|_X \le \delta_1$, and $p,k \approx 1$. Then the spectrum of the linearization of (1) about the periodic wave $U_{\rm per}$ in the space Y entirely lies on the imaginary axis. Consequently, this wave is spectrally stable in Y.

The proof of Theorem 2 is based on the so-called Bloch-wave decomposition, which reduces the spectral study of the linearized operator in the space Y to the study of the spectra of a family of linear operators in a space of periodic functions. The advantage of such a decomposition is that the resulting operators have compact resolvent, and therefore only point spectra. The main step in the analysis consists in locating these point spectra. For our problem, we rely on perturbation arguments for linear operators in which we regard the operators resulting from the Bloch-wave decomposition as small perturbations of operators with constant coefficients. The latter ones are actually obtained from the linearization of (1) about zero, and Fourier analysis allows to compute their spectra explicitly. The restriction to small amplitudes is essential in this perturbation argument, and we do not know whether spectral stability holds for large waves.

Finally, we point out that these results can be extended to the focusing NLS equation

$$\mathrm{i}U_t(x,t) + U_{xx}(x,t) + |U(x,t)|^2 U(x,t) \;=\; 0\;.$$

In contrast to the defocusing case, the focusing NLS equation possesses two different families of quasi-periodic solutions of the form (2), one for $\omega > 0$ and the other for $\omega < 0$. While for periodic perturbations the same orbital stability result holds [2], it turns out that the small periodic waves are spectrally *unstable* in this case [1]. The instability is of side-band type, and therefore cannot be detected in the periodic set-up used for the analysis of orbital stability.

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Leapfrogging in Coupled KdV Equations

Douglas Wright

The Korteweg-de Vries (KdV) equation, $u_t + (u_{xx} + u^2)_x = 0$, is a well-known model for many processes involving the evolution of long waves. Korteweg & de Vries and Boussinesq initially derived this equation to model the behavior of surface water waves in a flat-bottomed canal. (Here $x \in \mathbf{R}$ is the spatial dimension, t is time and u is roughly proportional to the surface elevation of the water.) The KdV equation famously possesses solitary wave solutions of the form $u(x,t) = \frac{3c}{2}\mathrm{sech}^2\left(\frac{\sqrt{c}}{2}(x - ct - x_0)\right)$ where the wave-speed c is any positive number.

Gear & Grimshaw in [3] derived a system of coupled KdV equations to model interactions of long waves, for example in a stratified fluid. Their model is of the form

$$\begin{aligned} u_t + \left(u_{xx} + u^2 + \epsilon_1 v_{xx} + \epsilon_2 \partial_u H(u,v)\right)_x &= 0 \\ v_t + \left(v_{xx} + v^2 + \epsilon_1 u_{xx} + \epsilon_2 \partial_v H(u,v)\right)_x &= 0. \end{aligned} \tag{1}$$

Here $\epsilon_1,\ \epsilon_2$ are constants, and H is a smooth real valued function (precisely, it is a polynomial). The functions u and v can be thought of as the displacement from horizontal of the fluid interfaces. Equations similar to (1) also arise in the study

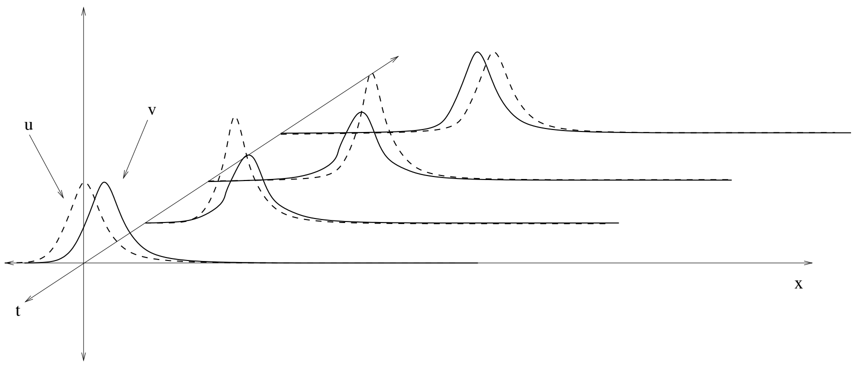

FIGURE 1. Sketch of one half of a "period" of leapfrogging. After this point, the process repeats but with the roles of u and v reversed.

of two-dimensional atomic lattices and head-on collisions of solitary water waves. Notice that if $\epsilon_1 = \epsilon_2 = 0$, then (1) simplifies to two completely uncoupled KdV equations.

Numerically computed solutions of (1) sometimes display a phenomenon called "leapfrogging". This is a solution which looks like a solitary wave in each component and where each wave oscillates about a shared center of mass. Figure 1 provides a sketch of this behavior. There have been a number of numerical and formal investigations into this phenomenon (see [6]), but little rigorous analysis.

I have, with A. Scheel in our paper [8], developed a novel explanation of leapfrogging when the coupling is weak. We do this as follows. First we study the existence solitary wave solutions to (1) when $|\epsilon_1| + |\epsilon_2|$ is close to zero. Our approach is perturbative; an enormous amount of information is known about the existence and stability of solitary waves in single KdV equations and, as a consequence, we have a more or less complete understanding of solitary wave solutions in the uncoupled problem. We are able to use a Liapunov-Schmidt analysis to determine the existence of a variety of solitary wave solutions for weak coupling. Specifically we prove the existence of four different types of solitary waves. The first type is $O(1)$ in the u component and $O(|\epsilon_1| + |\epsilon_2|)$ in the v component. The second is the same as the first but with the roles of u and v switched[1]. The third (and more interesting) type of solution is $O(1)$ in both components simultaneously. The u and v components are even functions on their own and share a common center of mass. We call this solution a *piggybacking* solitary wave as it appears that the wave in one component is riding on the back of the other. We have also determined a criterion for the existence of a fourth type of solitary wave which is $O(1)$ in both components but the components do not share a common center—these are particularly interesting because there are few examples of solitary waves in dispersive equations which are asymmetric. See Figure 2 for a sketch of these different types of solitary waves.

[1]These solutions have been discovered previously using variational means in [1] and [2].

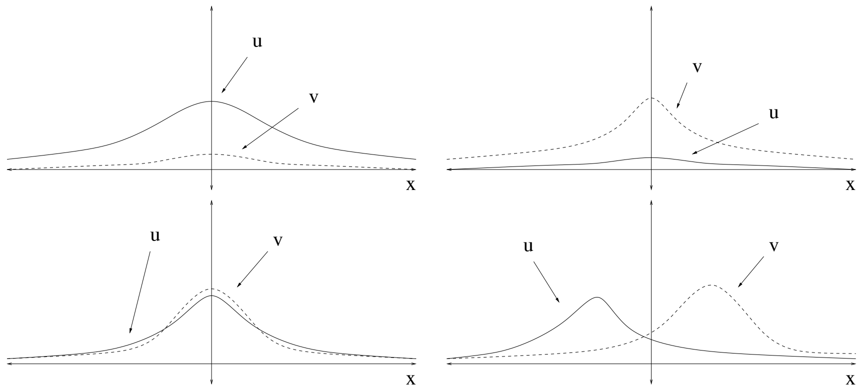

FIGURE 2. The four different types of solitary waves. The wave in the bottom left corner is the piggybacking solitary wave.

After we have established the existence of these different types of solitary waves, we then examine their stability. The first two types of solitary waves are orbitally stable, and this follows from an application of the abstract theory of Grillakis, Shatah & Strauss (see [4] and [5]). This theory does not apply to the piggybacking solitary wave. Instead, using reduction methods and perturbation techniques, we compute the spectrum of the linearization of (1) about the piggybacking solution[2]. We find for ϵ_1 and ϵ_2 non-zero that the spectrum of the linearization consists of: **(i)** a double zero eigenvalue **(ii)** the essential spectrum (which consists of the entire imaginary axis)[3] and either **(iii-a)** two real-valued eigenvalues (one positive and one negative) each of size $O(\sqrt{|\epsilon_1| + |\epsilon_2|})$ or **(iii-b)** a complex conjugate pair whose real parts are positive and of $O(|\epsilon_1| + |\epsilon_2|)$ and whose imaginary parts are $O(\sqrt{|\epsilon_1| + |\epsilon_2|})$.

Note that in either case **(iii-a)** or **(iii-b)** these waves are linearly unstable. Moreover, it is this last scenario **(iii-b)** which explains leapfrogging. Since the imaginary parts of these eigenvalues are non-zero, the instability is oscillatory, just as leapfrogging is. Moreover, this result establishes that leapfrogging is a transient phenomenon. In addition to our analytic results, in [8] we carry out a number of numerical simulations of leapfrogging over long time intervals which display this transience. What we observe is that the amplitude of the leapfrogging oscillation grows in magnitude while simultaneously "radiation" is emitted behind the waves. Eventually the solitary wave splits in two and the emission of radiation ceases. That is, it breaks into the superposition of two solitary waves (one each of the first two types described above) which have different speeds. The radiation emitted during the leapfrogging lags behind both waves and slowly disperses.

[2]In fact, we compute this spectrum for the asymmetric solitary wave as well. The results are not different than in the piggybacking case.

[3]This complicates the perturbation analysis and makes necessary the use of an exponentially weighted function space, see [7].

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Hamiltonian Expansions for Water Waves over a Random Bottom

Walter Craig & Catherine Sulem

This pair of talks describes the asymptotic behavior of wave motions in the free surface of a perfect fluid (water, in applications) which lies over a fluid region with variable bathymetry. The hydrodynamical significance of the work is to studies of nonlinear wave propagation in coastal regions, as well as to global scale propagation of tsunamis; cases in which the bottom topography is approximately but not perfectly known. It is a well known fact that in a channel of fixed constant depth, the Euler equations for incompressible irrotational flows have solitary wave solutions which propagate energy and momentum rapidly over long distances with little attenuation. The question is to what extent this capability is retained when the fluid is not of constant depth, in which case coherent wave motions will encounter environments varying in space and modulation and scattering effects may result in significant attenuation. Our work is a reappraisal of the paper of Rosales & Papanicolaou [6], who consider the scaling regime of the Korteweg - deVries equations, and who treat the problem as one of homogenization, that is, of finding effective coefficients for the long wave models of free surface water waves in a channel with a variable depth bed. The case in which the bottom topography is periodic is shown to homogenize completely in [6], a result which is recovered and extended in Craig, Guyenne, Nicholls & Sulem [4] using method of Hamiltonian perturbation theory for PDE. The fact that it fully homogenizes can be explained by a scale separation lemma for periodic coefficients in the latter reference.

In the present work we take up the case of a random bottom, namely we consider situations in which the bottom of the fluid region is taken to be a realization of a stationary ergodic process which exhibits a sufficiently strong property of mixing. We show that this problem does not homogenize fully, and that there are realization dependent phenomena that are as important as the dispersion and the nonlinear effects in the long wave limit. We furthermore show that in this limit, the random effects are governed by a canonical limit process which is equivalent to a white noise through Donsker's invariance principle, with one free parameter the variance σ. The coherent wave motions of the KdV limit are shown to be preserved, while at the same time the random effects on solutions of the KdV are described, and the degree of scattering due to the variable bottom is quantified. In this result we extend the random topography discussed in [6] to the case $\sigma > 0$.

1. Water waves Hamiltonian

The problem of free surface water waves concerns the time evolution of a fluid region $S(\eta, b) := \{x \in \mathbb{R}^{d-1}, b(x) < y < \eta(x,t)\}$, in which we solve for a potential function $\varphi(x,y,t)$ and a free surface $\eta(x,t)$ satisfying the ideal fluid equations. The acceleration of gravity is g. It suffices to specify the boundary values for the potential on the free surface, $\varphi(x, \eta(x,t), t) = \xi(x,t)$ because the resulting elliptic boundary value problem at each instant of time is well posed.

The Hamiltonian for the problem of water waves is given in Zakharov [7], and elaborated in Craig & Sulem [5] in terms of the Dirichlet – Neumann operator for the fluid region;

$$H(\eta,\xi) = \int \tfrac{1}{2}\xi(x)G(\eta,\beta)\xi(x) + \tfrac{g}{2}\eta^2(x)\,dx \ , \tag{1}$$

where the bottom of the channel is described by $y = b(x) := -h + \beta(x)$, and where $G(\eta,\beta)$ is the Dirichlet – Neumann operator for the fluid domain. The operator $G(\eta,\beta)$ is analytic in both arguments (η,β), and its Taylor expansion in the case of a variable bottom is described in [4].

The long wave regime is probed through a series of scaling transformations, a theory of which is given in Craig, Guyenne & Kalisch [3]. For the Boussinesq scaling regime one sets

$$X = \varepsilon x \ , \quad \varepsilon\beta'(X/\varepsilon) = \beta(x) \ , \quad \varepsilon\xi'(X) = \xi(x) \ , \quad \varepsilon^2\eta'(X) = \eta(x) \ , \tag{2}$$

(and subsequently drops the primes, for notational convenience). The principal terms that emerge from the resulting expansion in powers of ε are

$$\begin{aligned} H(\eta,\beta;\varepsilon) \quad = \quad & \tfrac{\varepsilon^3}{2}\int (h|D_X\xi|^2 + g\eta^2)dX - \tfrac{\varepsilon^4}{2}\int \beta(x)|D_X\xi|^2 dX \\ + \quad & \tfrac{\varepsilon^5}{2}\int (\xi D\eta D_X\xi - \tfrac{h^3}{3}\xi D_X^4\xi)dX \\ & -\tfrac{\varepsilon^5}{2}\int \Big(\beta(x)D\tanh(hD)\beta(x)\Big)|D_X\xi|^2 dX + O(\varepsilon^6) \ , \end{aligned} \tag{3}$$

which retains some of the influence of the short scale variations of the bottom topography $\beta(x)$. Further transformations of this expression give rise to the Hamiltonian for the KdV equation and for the scattering that results from reflections from the variable bottom topography [4], these are discussed in Section 3 below.

2. Stationary ergodic processes

The problem is concerned with a statistical ensemble of fluid regions, expressed through the choice of the bottom boundary from a set Ω of realizations ω, which we indicate by writing $\beta(x,\omega)$. The set of realizations is a probability space $(\Omega, \mathcal{M}, \mathsf{P})$ with an ergodic one parameter group of measure preserving translations $\{\tau_y\}_{y\in\mathbb{R}}$ (and a filtration of the measurable sets, $\mathcal{M}_y$, $y \in \mathbb{R}$ adapted to $\{\tau_y\}_{y\in\mathbb{R}}$). For our purpose we take $\Omega \subseteq \mathcal{S}'$ the space of tempered distributions, and we will require a mixing condition to hold on the probability measure P;

$$|\mathsf{P}(A \cap \tau_y(B)) - \mathsf{P}(A)\mathsf{P}(B)| < \varphi(y)\sqrt{\mathsf{P}(A)\mathsf{P}(B)}\ , \tag{4}$$

for sets $A \in \mathcal{M}_{\{y\le 0\}}$ and $B \in \mathcal{M}_{\{y\ge 0\}}$, with a mixing rate that satisfies

$$\textstyle\int_0^\infty \varphi^{1/2}(y)\,dy < +\infty\ . \tag{5}$$

The variance of the process $\beta(x,\omega)$ is defined to be

$$\sigma_\beta^2 := 2\textstyle\int_0^\infty \mathsf{E}(\beta(0,\omega)\beta(0,\tau_y omega))\,dy\ , \tag{6}$$

which is finite by (5). We note that if $\beta(x,\omega) = \partial_x\gamma(x,\omega)$ for some stationary process $\gamma \in C^1$, then $\sigma_\beta = 0$. The principal terms of the Hamiltonian (3) that we must understand in the limit of small ε are

$$-\tfrac{1}{2}\textstyle\int \beta(X/\varepsilon)|u(X)|^2\,dX\ , \tag{7}$$

$$-\tfrac{1}{2}\textstyle\int \big(\beta(X/\varepsilon)D\tanh(hD)\beta(X/\varepsilon)\big)|u(X)|^2\,dX\ , \tag{8}$$

where $u(X) := \partial_X\xi(X)$. These integrals are in the form of a stationary ergodic process of the short scale variables $x = X/\varepsilon$ being tested by a function of the long scale variables X. This observation leads to a study of the asymptotic behavior in the small parameter ε of integrals of the form

$$\textstyle\int_{-\infty}^{+\infty} \gamma(X/\varepsilon)f(X)\,dX\ , \tag{9}$$

for $f(X) \in \mathcal{S}$ for example. Our main tool for averaging theory of the terms (7)(8) in the Hamiltonian is the following distributional version of Donsker's invariance principle [1].

Theorem 2.1. *Suppose that $\gamma(x;\omega)$ is a stationary ergodic process which is mixing, with a rate $\varphi(y)$ which satisfies the condition* (5). *Then the integral* (9) *behaves as follows as $\varepsilon \to 0$*

$$\textstyle\int \gamma(X/\epsilon,\omega)f(X)\,dX = \int \big(\mathsf{E}(\gamma) + \sqrt{\varepsilon}\sigma_\gamma\partial_X B_\omega(X)\big)f(X)\,dX + o(\sqrt{\varepsilon})\ , \tag{10}$$

where $B_\omega(X)$ is normal Brownian motion.

3. The Korteweg deVries equation with random coefficients

Following the analysis of the previous section, one derives the KdV approximation for surface water waves in the long-wave scaling regime. Using $r(X,t)$ to represent the principal component of the solution, which propagates to the right, and $s(X,t)$ the component of the solution that is reflected by interactions with the bottom topography, the equations of motion are

$$\partial_t r = -\partial_X\big(c_0(X,\varepsilon,\omega)r + \varepsilon^2(c_1\partial_X^2 r + c_2 r^2)\big) \tag{11}$$

$$\partial_t s = \partial_X\big(c_0(X,\varepsilon,\omega)s\big) - \frac{1}{4}\sqrt{g/h}\big(\sigma_\beta\partial_X\Gamma_\omega(X)r\big)\ . \tag{12}$$

Here, $\Gamma_\omega(X) = \partial_X B_\omega(X)$ is given by a normalized white noise process, the overall wavespeed $c_0(X, \varepsilon, \omega)$ is a realization dependent function, given by the expresson

$$c_0^2(X, \varepsilon, \omega) = g\big(h - \varepsilon^{3/2}\sigma_\beta \Gamma_\omega(X) - \varepsilon^2 a\big) \ , \tag{13}$$

and the coefficients c_1, c_2 and a are averaged quantities, mean values stemming from integrals such as in (7)(8). Solutions to (11)(12) are to be understood in a distributional sense, and as limits of a regularisation procedure. This is one of the topics discussed in [2], in which in particular it is shown that the property of propagation of coherent waveforms is preserved with a random bottom, affected only by some random distortion due to the bathymetry, and by a random element of scattering as per (12).

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Well-Posedness of the Generalized Proudman-Johnson Equation

Hisashi Okamoto

The singularity/regularity problem of Euler's equations for incompressible inviscid fluid motion is a notoriously difficult problem. Related problems for water-waves are equally difficult. I therefore wish to consider models which are easier to solve but interesting enough to show singular or near-singular points of solutions.

The model which I consider is a generalized version of an equation by Proudman and Johnson discovered in 1961. With t, time variable, and x, $1D$ spatial variable, the model equation reads:

$$f_{txx} + f f_{xxx} - a f_x f_{xx} = \nu f_{xxxx}. \tag{1}$$

I consider this equation in $0 < x < 1$ with the periodic boundary condition. Here, a is a real parameter. For more details, see [4]. Well-posedness of (1) which is local in time is proved in [4] with appropriate function spaces of L^2-type.

The author and X. Chen, in [1], proved that (1) is well-posed globally in time if $-3 \le a \le 1$. Numerical experiments suggest that blow-up of solutions occur when $a < -3$ or $1 < a$; see [4].

In a recent paper [3] the author considers the case of $\nu = 0$, where the global well-posedness is proved when $-2 < a < 1$, and blow-ups are mathematically proved to exist if $-\infty < a \leq 2$. No mathematical conclusion can be drawn if $1 < a$, although blow-up is expected to occur.

Our result supports the result in [2] that the convection term suppresses the blow-up of solutions and the well-known proposition that viscosity helps the global existence.

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Deriving Modulation Equations via Lagrangian and Hamiltonian Reduction

Alexander Mielke

Modulation equations can be seen as effective macroscopic equations describing the evolution of a microscopically period pattern. We discuss general strategies how to pass from the microscopic systems to a macroscopic one by using the Hamiltonian or the Lagrangian structure.

The derivation of macroscopic equations for discrete models (or continuous models with microstructure) can be seen as a kind of reduction of the infinite dimensional system to a simpler subclass. If we choose well-prepared initial conditions, we hope that the solution will stay in this form and evolve according to a slow evolution with macroscopic effects only. We may interpret this as a kind of (approximate) invariant manifold, and the macroscopic equation describes the evolution on this manifold. We refer to [Mie91] for exact reductions of Hamiltonian systems and to [DHM06, GHM06, Mie06, GHM07] for the full details concerning this note.

As the easiest example we consider the one-dimensional Klein-Gordon equation

$$u_{tt} = u_{xx} - au - bu^3, \quad (t, x) \in [0, \infty) \times \mathbb{R}.$$

The sum of the kinetic and potential energy gives the Hamiltonian

$$H(u, u_t) = \int_{\mathbb{R}} \left(\tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 + \tfrac{a}{2}u^2 + \tfrac{b}{4}u^4 \right).$$

As we are interested in modulated waves we embed this system $\mathbb{R}$ into the cylinder $\Xi = \mathbb{R} \times \mathbb{S}^1$, where $\mathbb{S}^1$ contains the additional microscopic phase variable. The continuous Hamiltonian system is

$$\text{(1)} \qquad \begin{aligned} &\partial_t^2 u = \Delta_{(1,0)} u - au + bu^3 \quad \text{with } a>0, \quad u \in \mathrm{L}^2(\Xi), \\ &\text{and } \Delta_{(1,0)} u(x,\phi) := u_{xx}(x,\phi). \end{aligned}$$

Introducing $p = \partial_\tau u$, this is a canonical Hamiltonian system with

$$H^{\mathrm{cont}}(u,p) = \int_\Xi \tfrac{1}{2}p^2 + \tfrac{1}{2}\big(\nabla_{(1,0)}u\big)^2 + \tfrac{a}{2}u^2 + \tfrac{b}{4}u^4 \,\mathrm{d}x\,\mathrm{d}\phi.$$

Like the original KG equation the enlarged problem (1) is translationally invariant in the x direction. Moreover, it is invariant under translations in the phase direction ϕ. This leads to the two first integrals $I^{\mathrm{sp}}(u,p) = \int_\Xi p\,\partial_x u \,\mathrm{d}x\,\mathrm{d}\phi$ and $I^{\mathrm{ph}}(u,p) = \int_\Xi p\,\partial_\phi u \,\mathrm{d}x\,\mathrm{d}\phi$. Using the symmetry transformation

$$(\widetilde{u},\widetilde{p}) = T^{\mathrm{sp}}_{ct} T^{\mathrm{ph}}_{(\omega - c\theta)t}(u,p), \qquad \widetilde{\mathcal{H}} = \mathcal{H} - cI^{\mathrm{sp}} - (\omega - c\theta)I^{\mathrm{ph}}$$

the associated canonical Hamiltonian system $\mathbf{\Omega}^{\mathrm{can}}(\widetilde{u}_t,\widetilde{p}_t) = \mathrm{D}\widetilde{\mathcal{H}}(\widetilde{u},\widetilde{p})$ on $\mathrm{L}(\Xi)^2$ is still fully equivalent to a family of uncoupled KG chains.

Introducing a suitable scaling, which anticipates the desired microscopic and macroscopic behavior, will expose the desired limit. For this we let

$$(\widetilde{u}(x,\phi), \widetilde{p}(x,\phi)) = (\varepsilon U(\varepsilon x, \phi - \theta x), \varepsilon P(\varepsilon x, \phi - \theta x)),$$

which keeps the canonical structure (after moving a factor ε arising from $\mathrm{d}y = \varepsilon\,\mathrm{d}x$ into the time parameterization $\tau = \varepsilon^2 t$). We obtain the new Hamiltonian

$$\begin{aligned} \mathcal{H}_\varepsilon(U,P) = \int_\Xi \tfrac{1}{2\varepsilon^2}\Big(\big[P - \omega U_\phi - \varepsilon c U_y\big]^2 + \big(\nabla_{(\varepsilon,\theta)}U\big)^2 \\ + aU^2 - \big[\omega P U_\phi + \varepsilon c P U_y\big]^2 \Big) + \tfrac{b}{4}U^4 \,\mathrm{d}y\,\mathrm{d}\phi, \end{aligned}$$

where $\nabla_{(\varepsilon,\theta)} = \varepsilon U_y + \theta U_\phi$. The modulation ansatz now reads

$$(U(y,\phi), P(y,\phi)) = R_\varepsilon(A)(y,\phi) = (\operatorname{Re} A(y)\mathrm{e}^{\mathrm{i}\phi}, \omega \operatorname{Re} A(y)\mathrm{e}^{\mathrm{i}\phi}) + O(\varepsilon),$$

and leads to $\mathcal{H}_\varepsilon(R_\varepsilon(A)) = \mathbb{H}_{\mathrm{nlS}}(A) + O(\varepsilon)$ and $\mathrm{D}R_\varepsilon(A)^*\mathbf{\Omega}^{\mathrm{can}}\mathrm{D}R_\varepsilon(A) = \mathbf{\Omega}^{\mathrm{red}} + O(\varepsilon)$ with

$$\mathbb{H}_{\mathrm{nlS}}(A) = \int_{\mathbb{R}} \omega\omega''|A_y|^2 + \tfrac{3b}{8}|A|^4 \,\mathrm{d}y \qquad \text{and} \qquad \mathbf{\Omega}^{\mathrm{red}} = 2\mathrm{i}\omega.$$

Thus, the macroscopic limit is the one-dimensional nonlinear Schrödinger equation

$$2\mathrm{i}\omega A_\tau = -2\omega\omega'' A_{yy} + \tfrac{3}{2}b|A|^2 A.$$

Of course, a mathematically rigorous justification of the nonlinear Schrödinger equation as a modulation equation was known long before (see [KSM92, Sch98, GM04, GM06]). However, the emphasis here is to see how the Hamiltonian and Lagrangian structures need to be transformed to converge to the desired limits.

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Criticality of Water Waves, and the Generation of Dark Solitary Waves in Shallow Water

Thomas J. Bridges

Criticality, uniform flows and bulk quantities such as mass flux, total head and the momentum flux are at the heart of the subject of open-channel hydraulics. However, attempts to generalize criticality to non-trivial flows and unsteady flows have been largely unsuccessful. The main questions are: what is the appropriate mathematical model for criticality? And, what are the implications of criticality?

Recent results [1, 2, 3] show that criticality is equivalent to degeneracy of a relative equilibrium (RE) characterization of the flow. RE are solutions which are aligned with a group orbit. The water wave problem has natural symmetries so that a surprising variety of well-known flows are in fact RE. An example is the classical Stokes traveling wave interacting with a mean flow.

In [2] the concept of "secondary criticality" of water waves is introduced, to indicate when a non-uniform state, such as a Stokes traveling wave, goes through criticality. The theory shows that secondary criticality of water waves signals a bifurcation to a new class of *steady dark solitary waves* which are biasymptotic to a Stokes wave with a phase jump in between, and synchronized with the Stokes wave. The bifurcation to these new solitary waves – from Stokes *gravity* waves in shallow water – is pervasive, even at low amplitude. The theory works because the hydraulic quantities can be related to the governing equations in a precise way using the Hamiltonian formulation of water waves.

Surprisingly, this approach to the study of criticality has led to a new theory for the nonlinear behavior near a degenerate RE. The case of degenerate periodic orbits, interpreted as degenerate RE, was reported in [1], and the general case of a symmetric Hamiltonian system with an n-dimensional symmetry group is treated in [5]. Using symplectic Jordan chain theory, one shows that the degeneracy of an RE leads to an additional pair of eigenvalues, in the linearization about an RE. Applying normal form theory shows that the saddle-center bifurcation of eigenvalues leads to a homoclinic bifurcation in the reduced system transverse to the group orbit. A new observation is that the curvature of the pullback of the momentum map to the Lie algebra determines the leading order behavior of the nonlinear normal form for the homoclinic bifurcation. There is also an induced geometric phase in the homoclinic bifurcation. The backbone of the analysis is the use of singularity theory for smooth mappings between manifolds applied to the pullback of the momentum map. In the Thom-Boardman classification of singularities, degeneracy of a relative equilibrium corresponds to a singularity of type $\Sigma^1(\mathbf{P})$ where $\mathbf{P}$ is the pullback of the momentum map. Higher order singularities, $\Sigma^{11}(\mathbf{P})$, et cetera, are also possible. But the leading order singularity $\Sigma^1(\mathbf{P})$ is the most interesting, and most likely to occur in applications.

The theory of criticality can also be extended to unsteady flows [2]. For example, the theory leads to a new formulation of the Benjamin-Feir instability for Stokes waves in finite depth coupled to a mean flow, which takes the criticality matrix as an organizing center. Unsteady criticality appears to generate *unsteady* dark solitary waves in the nonlinear problem.

There is also an interesting connection between unsteady criticality and wave breaking [4]. It is well known that the Stokes traveling wave is unstable to *superharmonic* (SH) perturbations. (Superharmonic perturbations have the same wavelength as the basic Stokes wave.) The Stokes traveling wave is a relative equilibrium, and the point where SH instability occurs is precisely where the momentum of the Stokes wave has a maximum considered as a function of the wavespeed. In other words, SH instability is associated precisely with degeneracy of a relative equilibrium. Applying the nonlinear theory shows that a homoclinic bifurcation – in time – will occur. This theory explains the results found in numerical simulations of the unstable Stokes wave [4]. However, this theory is formal as the time dependent water wave equations have an infinite number of purely imaginary eigenvalues in addition to the saddle center bifurcation. It is an open problem to determine the dynamics of an infinite-dimensional Hamiltonian system near a saddle-center bifurcation, and the nature of the induced homoclinic bifurcation.

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Existence and Stability of Fully Localised Three-Dimensional Solitary Gravity-Capillary Water Waves

Mark D. Groves

The classical *water-wave problem* concerns the irrotational flow of a perfect fluid of unit density subject to the forces of gravity and surface tension. The fluid motion is described by the Euler equations in a domain bounded below by a rigid horizontal bottom $\{y = -h\}$ and above by a free surface which is described as a graph $\{y = \eta(x, z, t)\}$, where the function η depends upon the two horizontal spatial directions x, z and time t. In terms of an Eulerian velocity potential $\phi(x, y, z, t)$ the mathematical problem is to solve the equations

$$\begin{aligned} &\phi_{xx} + \phi_{yy} + \phi_{zz} = 0, && -h < y < \eta, \\ &\phi_y = 0, && y = -h, \\ &\phi_y = \eta_t + \eta_x\phi_x + \eta_z\phi_z, && y = \eta \end{aligned}$$

and

$$\begin{aligned} &\phi_t + \tfrac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) + g\eta \\ &\qquad - T\left[\frac{\eta_x}{\sqrt{1+\eta_x^2+\eta_z^2}}\right]_x - T\left[\frac{\eta_z}{\sqrt{1+\eta_x^2+\eta_z^2}}\right]_z = 0, \quad y = \eta, \end{aligned}$$

where g and T are respectively the acceleration due to gravity and the coefficient of surface tension. A *fully localised solitary wave* is a solution to the above problem of the form $\eta(x, z, t) = \eta(x - ct, z)$, $\phi(x, y, z, t) = \phi(x - ct, y, z)$, where c is a positive constant and $\eta(x - ct, z) \to 0$ as $|(x - ct, z)| \to \infty$. Figure 1 shows a numerical computation of a fully localised solitary wave in certain parameter regime (Parau, Vanden-Broeck & Cooker [6]).

Solutions of the hydrodynamic problem for solitary waves are characterised as critical points of the *energy*

$$H(\eta, \xi) = \int_{\mathbb{R}^2} \left\{ \tfrac{1}{2}\xi G(\eta)\xi + \tfrac{1}{2}g\eta^2 + T(\sqrt{1 + \eta_x^2 + \eta_z^2} - 1) \right\} \mathrm{d}x\,\mathrm{d}z$$

subject to fixed values of the *momentum*

$$I(\eta, \xi) = \int_{\mathbb{R}^2} \eta\xi_x \,\mathrm{d}x\,\mathrm{d}z,$$

both of which are conserved quantities of the hydrodynamic problem. The role of the Lagrange multiplier is played by the wave speed c and $G(\eta)$ is the *Dirichlet-Neumann operator* defined by $G(\eta)\xi := \nabla\phi.(-\eta_x, -\eta_z, 1)|_{y=\eta}$, where the potential function ϕ is the harmonic extension of ξ into the fluid domain with Neumann

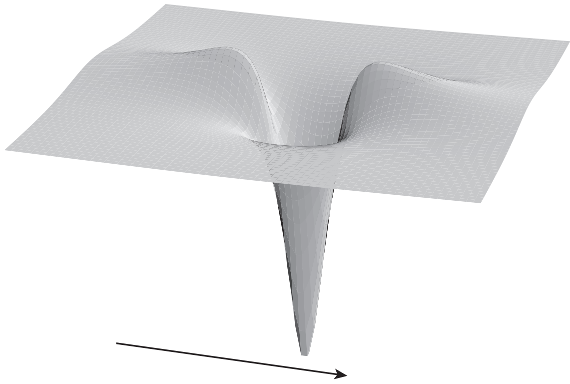

FIGURE 1. A fully localised solitary wave; the arrow shows the x direction.

data at $y = -h$ (e.g. see Craig & Nicholls [2]). *Minimisers* of H subject to the constraint $I = 2\mu$ are of particular interest due to their stability properties (see below).

Proceeding formally, let us first fix η and minimise $H(\eta, \cdot)$ subject to the constraint $I(\eta, \cdot) = 2\mu$. This problem admits a unique minimiser ξ_η, which satisfies $G(\eta)\xi_\eta = \lambda_\eta \eta_x$ for some nonzero Lagrange multiplier λ_η. The next step is to minimise $J(\eta) := H(\eta, \xi_\eta)$; an explicit calculation shows that $J(\eta) = \mathcal{K}(\eta) + \mu^2/\mathcal{L}(u)$, where

$$\mathcal{K}(\eta) = \int_{\mathbb{R}^2} \left\{ \tfrac{1}{2} g\eta^2 + T(\sqrt{1 + \eta_x^2 + \eta_z^2} - 1) \right\} \mathrm{d}x \, \mathrm{d}z,$$
$$\mathcal{L}(\eta) = \int_{\mathbb{R}^2} \eta K(\eta)\eta \, \mathrm{d}x \, \mathrm{d}z, \quad K(\eta) = -(G(\eta)^{-1}\eta_x)_x.$$

The following lemma attaches a rigorous meaning to the operator K and shows that K, L are analytic functionals on a neighbourhood of the origin in $H^{s+3/2}(\mathbb{R}^2)$ for $s > 1$; it is proved by generalising the methods introduced by Nicholls & Reitich [5].

Lemma 1. *Choose $s > 1$. The operator $K(\eta) \in \mathcal{B}(H^{s+1}(\mathbb{R}^2), H^s(\mathbb{R}^2))$ is an analytic function of $\eta \in B_R(0) \subset H^{s+3/2}(\mathbb{R}^2)$ for some $R > 0$.*

The key ingredients in the proof of the main theorem are (i) the minimisation technique for quasilinear variational problems due to Buffoni [1]; and (ii) the estimates on nonlocal operators in the concentration-compactness method due to Groves & Sun [3].

Theorem 2. *Suppose that $T/gh^2 > 1/3$ and take μ_0 sufficiently small. There exists a set $M \in (0, \mu_0)$ of positive measure with the property that the following statements hold for each $\mu \in M$.*

(i) The functional J has a nonzero minimiser in $B_R(0) \subset H^3(\mathbb{R}^2)$.

(ii) Let $\{\eta_n\} \in B_R(0) \subset H^3(\mathbb{R}^2)$ be a minimising sequence for J. There exists a sequence $\{(x_n, z_n)\} \subset \mathbb{R}^2$ such that $\{\eta_n(\cdot + x_n, \cdot + z_n)\}$ converges to a nonzero minimiser of J in $B_R(0) \subset H^3(\mathbb{R}^2)$.

Corollary 3. *Let* $H_*^{1/2}(\mathbb{R}^2) = \{\xi \in H_{\mathrm{loc}}^{1/2}(\mathbb{R}^2) : \nabla\xi \in H^{-1/2}(\mathbb{R}^2)\}/\mathbb{R}$ *with* $\|\xi\|_{H_*^{1/2}(\mathbb{R}^2)} = \|\nabla\xi\|_{H^{-1/2}(\mathbb{R}^2)}$, *take* β, μ *as specified in Theorem 2 and consider the problem of minimising* $H(\eta,\xi)$ *on the manifold* $\{I(\eta,\xi) = 2\mu\}$.

(i) This problem admits a minimiser in a neighbourhood of the origin in the energy space $\mathcal{E} = \{(\eta,\xi) \in H^3(\mathbb{R}^2) \times H_*^{1/2}(\mathbb{R}^2)\}$.

(ii) Let $\{(\eta_n,\xi_n)\} \in B_R(0) \subset \mathcal{E}$ *be a minimising sequence. There exists a sequence* $\{(x_n,z_n)\} \subset \mathbb{R}^2$ *with the property that* $\{(\eta_n(\cdot + x_n, \cdot + z_n), \xi_n(\cdot + x_n, \cdot + z_n))\}$ *converges to a nonzero minimiser in* $B_R(0) \subset \mathcal{E}$.

It is a general principle that a solution of a nonlinear evolution equation which can be characterised as a minimiser of one conserved quantity on a level set of another is orbitally stable; this principle has the technical requirements that (i) the equation is globally well posed in its energy space (the space in which minimisation is acccomplished); (ii) the minimiser is unique (up to spatial translations); and (iii) all minimising sequences converge (up to adjustments by spatial translations). It is an open question whether conditions (i) and (ii) are satisfied for the water-wave problem, and we therefore weaken the definition of stability here. The *set* $\mathcal{S}$ of minimisers identified in Corollary 3 is said to be *conditionally energetically stable* (Mielke [4]): any solution $(\eta,\xi) \in C^1([0,T],\mathcal{X})$ of the hydrodynamic problem, where $\mathcal{X}$ is continuously embedded in $\mathcal{E}$, whose initial datum is near an element of $\mathcal{S}$ (in the metric of $\mathcal{E}$) remains close to an element of $\mathcal{S}$ (in the metric of $\mathcal{E}$) over its lifetime $[0,T]$.

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Traveling Doubly-Periodic Gravity Water Waves in Resonant Cases

Gérard Iooss
(joint work with Pavel Plotnikov)

We consider traveling water waves which form a bi-periodic horizontal pattern on the free surface, in absence of surface tension, for a potential flow in an infinitely deep fluid layer. Such waves may be considered as the result of the nonlinear superposition of two plane waves making an angle 2θ between them. There are two parameters in the problem: θ and the dimensionless bifurcation parameter $\mu = gL/c^2$ (L is the wave length along the direction of the traveling wave and c is the velocity of the wave), bifurcation occurs for $\mu = \cos\theta$. This talk is concerned with the building of an asymptotic expansion of these waves, including the *resonant cases.* This case means that the *dispersion relation* for the wave vectors $K = (n, \tau m)$, $(n, m) \in \mathbb{Z}^2$ which takes the form

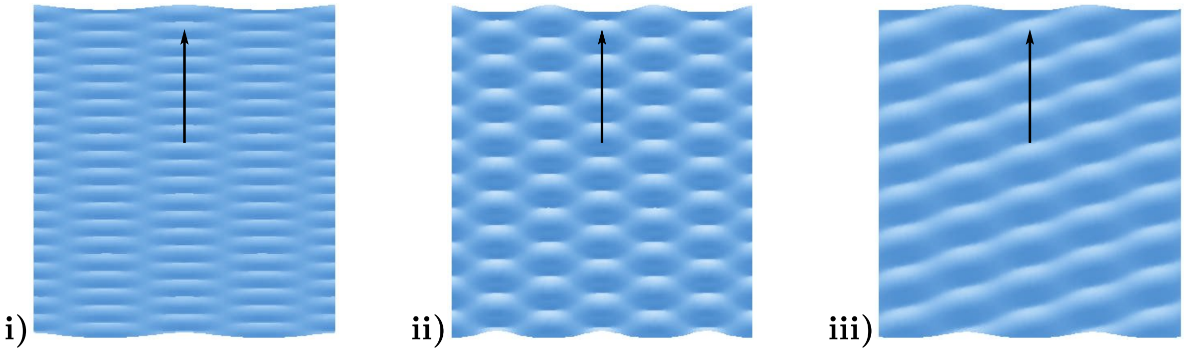
(1) $$n^2 + \tau^2 m^2 = n^4(1 + \tau^2), \quad \tau = \tan\theta$$

has not only the basic solution $(1, 1)$ in $\mathbb{N}^2$.

1. We show how to build an asymptotic expansion of bi-periodic *monomodal traveling waves in all cases*, in terms of the amplitudes ε_1 and ε_2 of the two incident plane waves (monomodal means that at main order $(\varepsilon_1, \varepsilon_2)$ only the 4 wave vectors $(\pm 1, \pm\tau)$ occur. We show in case of *symmetric waves* that the expansion may be written in terms of powers of $\varepsilon = \varepsilon_1 = \varepsilon_2$. For the *non-symmetric waves* $(\varepsilon_1 \neq \varepsilon_2)$, the expansion is in terms of powers of ε_1 and ε_2 until the degree $p - 2$, where p is the minimal value of the integer $m \geq 2$ solution of the dispersion relation (1). Then, higher orders of the expansion are in powers of ε with coefficient functions of $\alpha \in (0, 1)$ where $\varepsilon_1 = \alpha\varepsilon$, $\varepsilon_2 = (1 - \alpha)\varepsilon$.

2. For multimodal waves, i.e. waves such that at the main order at least 8 wave vectors $(\pm n_0, \pm\tau m_0)$, $(\pm n_1, \pm\tau m_1)$ occur ((n_0, m_0) and (n_1, m_1) being solutions of (1)), we give general results for a large family of resonant angles θ, about the construction of formal expansions of such solutions in terms of two parameters (which reduce to one parameter in the symmetric case).

3. The proof of existence of these waves in all cases is valid only for symmetric monomodal waves, as shown in Figures i) and ii) below, the proof is detailed in the talk by Pavel Plotnikov, where for the treatment of the small divisor problem we use the Nash-Moser theorem (see [1]), here complicated in the resonant cases by the infinite dimensional kernel, as in the situation treated in [2].



Symmetric 3-dim traveling wave, i) $\theta = 5.7^o$, ii) $\theta = 16.7^o$, iii) non symmetric waves $\theta = 16.7^o, \varepsilon_2/\varepsilon_1 = 0.3$. The arrow is the direction of propagation of the waves. Crests are white and troughs are dark.

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Small Divisor Problem in the Theory of Three-Dimensional Water Gravity Waves

Pavel Plotnikov

(joint work with Gérard Iooss)

We consider periodic waves at the surface of an infinitely deep perfect fluid, only subjected to gravity g and resulting from the nonlinear interaction of two symmetric traveling waves making an angle 2θ between them. This talk deals with the non-resonant case when the dispersion equation has the only basic solution in the lattice of periods, and the linearized problem has the only symmetric doubly-periodic solution. The main goal is the proof of existence of solutions to the nonlinear problem bifurcating from the trivial solution. The essential difficulty here is that we assume the absence of surface tension, which leads to a so-called *small divisor problem*. We show that the linearized operator at a non-trivial point can be reduced, by the change of independent variables, to a canonical pseudodifferential operator with constant coefficients in the principal part. The peculiarity of our problem is that the small divisors form clusters in the lattice of periods. We employ a modification of the Weyl theory on uniform distributions of irrational numbers modulo 1 to deduce the effective estimates of small divisors and to prove the invertibility of the principle part of the canonical operator. The most substantial ingredient of our approach is the descent method which allows to reduce the canonical pseudodifferential equation on the 2-dimensional torus to a Fredholm type equation (see [1],[2]). Finally we exploit the Nash-Moser implicit function theorem and prove the existence of bifurcating doubly-periodic symmetric waves for any value of the parameter in a Cantor set dense at 0.

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Particle Trajectories in Stokes Waves

Adrian Constantin

Ocean waves are classified as either sea or swell. Irregular patterns made up of various waves with different speeds, wavelengths and heights are called sea. When these waves move past the area of influence of the generating winds, they sort themselves into groups with similar speed and wavelengths. This process produces swell that is characteristically a regular pattern of undulation of the ocean surface. Swell often moves thousands of miles away from a storm to a shore somewhere (for example, swell originating from Antarctic storms has been recorded close to the Alaskan coast after more than 5000 miles). These progressing groups of swell with the same origin and wavelength are called wave trains: two-dimensional waves (that is, the motion is identical in any direction parallel to the crest line), periodic, travelling at constant speed. Wave trains propagating at the water's free surface with an irrotational flow (i.e. of zero vorticity) within the fluid are called *Stokes waves*. Watching the sea or a lake it is often possible to trace such a wave as it propagates on the water's surface. Contrary to a possible first impression, what one observes traveling across the sea is not the water but a wave pattern, as enunciated intuitively in the fifteenth century by Leonardo da Vinci in the following form: "... the wave flees the place of its creation, while the water does not; like the waves made in a field of grain by the wind, where we see the waves running across the field while the grain remains in its place" [12]. In other words, the displacements of the water particles induce a much more rapid motion of the free surface wave, a fact supported by field evidence [13]. It is widely believed (see for example any classical textbook on water waves) that particles in water above a flat bed execute a circular motion as such a wave passes over. Support for this conclusion is given by the only known explicit solution with a non-flat free surface for the governing equations in water of infinite depth [8], solution for which all particle paths are circles of diameters decreasing with the distance from the free surface. More convincingly perhaps, the conclusion seems to be supported by experimental evidence: photographs [7, 17, 18] or movie films [2] of small buoyant particles in laboratory wave tanks (see also the comments in [10]). Due to the mathematical intractability of the governing equations for water waves, the classical approach [11, 16, 18] towards explaining this aspect of water waves consists in analyzing the particle motion after linearizing the governing equations. At least within the linear water wave theory, it appears that all particle paths are closed (cf. [6, 7, 9, 11, 13, 17, 18]). However, an analysis of the average energy flow within linear water wave theory (see [6, 9]) indicates that during the passage of a periodic surface wave the water particles in the fluid experience on average a net displacement in the direction of wave propagation, termed *Stokes drift* – see [19] for recent calculations. The accommodation of this observation with the classical theory on the particle motion below the surface of a wave train is perhaps best summarized by Longuet-Higgin's [14]: "In progressive gravity waves of very small amplitude it is well known that the orbits of the particles are either elliptical or circular. In steep waves, however, the orbits become quite distorted, as shown by the existence of a mean horizontal drift or mass-transport in irrotational waves."

Actually, for a periodic traveling water wave propagating over a flat bed in an irrotational flow, no particle trajectory is closed, unless the free surface is flat. Over a period, each trajectory consists of a backward/forward movement of the particle, and the path is an elliptical arc (which degenerates on the flat bed) but with a forward drift. Interestingly, we reach this conclusion both working with the governing equations for water waves (nonlinear theory) as well as within the framework of linear water theory. The methods differ considerably: while for linear water waves we rely on phase plane analysis [5], the nonlinear theory consists in the analysis of a free boundary value problem for harmonic functions [4].

Finally, we would like to point out that our conclusion is not in contradiction to the photographs referred to above. For the purpose of these photographs, the water was kept in a narrow glass container with parallel walls with small metallic particles mixed in the water and a long exposure (as long as half the period of the surface wave) made it possible to distinguish path curves as elliptical arcs (cf. [17]). Thus the photograph in [15] on page 686 from an experiment performed to trace a particle trajectory at the surface is indicative of the path of every particle above the flat bed. With respect to Gerstner's explicit solution [8], notice that the Gerstner flow has non-constant vorticity [3]. We show that within linear as well as within nonlinear irrotational water wave theory, the particle paths are almost closed and the more we approach the free surface, the more pronounced the deviation from a closed orbit becomes.

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Three-Dimensional Gravity Capillary Solitary Waves and Related Problems

Jean-Marc Vanden-Broeck

Three dimensional gravity capillary water waves are considered. Accurate numerical computations are presented for the full Euler equations. The codes are based on boundary integral equation formulations. Solitary waves with decaying oscillations in the direction of propagation and monotonic decay in the direction perpendicular to the direction of propagation are found in water of infinite depth [1]. Further results in water of finite depth and for interfacial waves are also presented ([2], [3] and [4]). Our findings are consistent with computations on model equations by Milewski [5], asymptotic calculations by Kim and Akylas [6] and analytical results by Groves and Sun [7]. In addition related problems involving gravity capillary free surface flows generated by moving references are studied. It is shown that accurate solutions can be computed by including a small Rayleigh viscosity in the dynamic boundary condition. The properties of the three dimensional waves are similar to those found before for two-dimensional waves (see Hunter and Vanden-Broeck [8], Vanden-Broeck and Dias [9] for numerical work and Dias and Iooss [10] for a review of analytical work).

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Stable, High-Order Computation of Traveling Water Waves and Their Linear Dynamic Stability

David P. Nicholls

The motion of the surface of an ideal fluid under the influence of gravity and capillarity, the "Water Wave Problem", is one of the oldest and most fundamental in fluid mechanics. Successful modeling techniques for this problem are crucial in many applications such as the generation and propagation of tsunamis. In a pair of recent papers [11, 12] the author, in collaboration with F. Reitich, developed a novel, stable and high-order Boundary Perturbation scheme for the reliable numerical simulation of *traveling* solutions of the water wave equations. In this talk we discuss this Boundary Perturbation technique (the method of Transformed Field Expansions) and its extension to address the equally important topic of dynamic *stability* of these traveling wave forms. More specifically we describe, and provide the theoretical justification for, a new numerical algorithm to compute the spectrum of the linearized water wave problem as a function of the amplitude of the traveling wave [10].

The question of existence and stability of traveling water waves has received a great deal of attention, and a full account of the results known to date is well beyond the scope of this abstract (however, please see the survey article of Dias & Kharif [4]). While much is known about two-dimensional waves (water with one depth dimension and one lateral dimension), there are few results about existence and stability of genuinely three-dimensional patterns.

Regarding existence of three-dimensional traveling water waves, the most general results to date are those of Craig & Nicholls [2] who, in the presence of resonance, established existence of traveling capillary-gravity water waves with arbitrary fundamental period. Other existence results in three dimensions include those of Sun [14] who viewed the traveling wave as generated by a surface pressure, and Groves & Mielke [6] and Groves [5] who have studied traveling waves using a "spatial dynamics" approach. The results most closely related to those we present are those of Reeder & Shinbrot [13] who demonstrate the existence and parametric analyticity of "short-crested" capillary-gravity waves of sufficiently small amplitude. Akin to the method we adopt, Reeder & Shinbrot also use a "domain flattening" change of variables. Our results expand on those of [13] in two important directions: First, our derivations demonstrate that the free boundary and velocity potential are *jointly* analytic in space and bifurcation parameter (a fact that does not follow from separate analytic dependence); and second, our developments allow for the interaction of wavetrains of *arbitrary* amplitude ratio, i.e. not necessarily short-crested waves. To attain the latter, our approach entails the use of *multi-dimensional* perturbation parameters.

Regarding dynamic stability, results can be classified not only by the type of basic traveling wave they consider, but also by the class of permitted perturbations which may grow or decay from this equilibrium state. For example, the work of Benjamin & Feir [1] and Zakharov [15] focused upon two-dimensional Stokes waves and the evolution of two-dimensional perturbations. By contrast, the calculations of Chen & Saffman [3] and MacKay & Saffman [9] focus upon *three*-dimensional perturbations of two-dimensional traveling waves, while Ioualalen *et al*, e.g. [7, 8], consider three-dimensional perturbations of *three*-dimensional patterns (short-crested waves). Another means of classification is the periodicity requirements of the traveling wave and/or perturbation. Finally, we must specify a notion of stability: Nonlinear, linear, or spectral? We will consider traveling waves which can be generated by the algorithm of Nicholls & Reitich [11, 12] (e.g. periodic Stokes waves in two dimensions and periodic short-crested waves in three dimensions) and quite general *quasi*-periodic perturbations which need not have the same periodicity as the traveling wave. Additionally, the method considers the spectrum of the water wave problem linearized about these solutions and thus constitutes a study of spectral stability.

Zakharov's realization of the water wave problem as a Hamiltonian system [15] implies that the best stability that one can expect is *weak* stability (that small disturbances will remain small) and not *strong* stability (where small disturbances decay exponentially fast). In the context of a spectral stability analysis, this notion of weak stability is characterized by the spectrum of the linearization of the dynamical water wave problem about a traveling solution being pure imaginary; for trivial waves ("flat water") this can be verified explicitly.

For non-trivial traveling profiles a straightforward approach to determining stability is to linearize the water wave problem about an approximate traveling wave and simulate the spectrum via a numerical eigensolver. This linearization/eigensolve approach has been investigated by Ioualalen *et al* [7, 8] and is an active line of research being pursued by the speaker in collaboration with W. Craig using a *stabilized* numerical scheme. However, this method does ignore some of the information available to us regarding the traveling waves. We recall that [2, 13, 11] demonstrated that traveling waves come in branches (two dimensions) or surfaces (three dimensions) which can be specified by a parameter ε which is meant to represent wave height or wave slope. In the linearization/eigensolve procedure outlined above this information is used solely to compute the basic traveling wave and information regarding the dependence of the spectrum upon ε is lost. Our point of view is that thinking of the spectrum as an (analytic) function of ε gives valuable insights into the nature of the onset of instability in this problem.

One can imagine the spectrum “moving” smoothly as a function of the branch parameter ε; this is guaranteed by our new theorem [10] for generic choices of perturbation quasiperiod. The question now arises: Can spectrum on the imaginary axis (e.g. at $\varepsilon = 0$) move into the right-half of the complex plane resulting in instability of the base traveling wave? MacKay & Saffman [9], using the Hamiltonian structure of the water wave problem, showed that a necessary (though not sufficient) condition for eigenvalues to move off the imaginary axis is that they collide. This observation is important for us as we propose to measure the strength of an instability by finding, for each configuration (i.e. choice of perturbation quasiperiod), the value of ε of the *first* eigenvalue collision.

It is well-known that collisions can occur in the *linear* problem, i.e. there may be eigenvalues of multiplicity higher than one for $\varepsilon = 0$. These instances of resonance represent an easily identified source of potential instability (though not all such resonances give rise to instability, see [9] § 4) for traveling waves. Unfortunately, due to the nature of our scheme as it is currently formulated, we are unable to address configurations which feature these resonances (i.e. higher multiplicity of the eigenvalues). While this may appear to be a shortcoming of our approach, it is a reflection of the somewhat imprecise nature of our (and [9, 7, 8]) instability criterion: Collision of eigenvalues. If we choose this as our test, then in a resonant configuration instability already exists at $\varepsilon = 0$ and more finely-tuned techniques must be used. However, if we are in a (generic) non-resonant configuration then our new method can be used to give reliable and highly accurate estimates of the onset and strength of instabilities in traveling water waves.

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Initial Value Problems for Free-Surface Flows in 3D Fluids

David M. Ambrose
(joint work with Nader Masmoudi)

We study the well-posedness of free-surface problems in irrotational 3D fluids. The method is quite similar to the method used previously by the speaker and by the speaker and Masmoudi for two-dimensional fluids. The primary difference in the formulation of the problems is that in this higher-dimensional case, there is no direct analogue of arclength.

In the case of a two-dimensional free surface surrounded by three-dimensional fluids, we are able to choose two non-physical tangential velocities to enforce a suitable parameterization of the surface. This choice of non-physical tangential velocity does not change the dynamics; instead, it only reparameterizes the free surface. Thus, we are able to place two conditions on the parameterization. We now describe the surface as $\mathbf{X}(\alpha,\beta,t) = (x(\alpha,\beta,t), y(\alpha,\beta,t), z(\alpha,\beta,t))$, with α and β the spatial parameters. In [4, 2, 5], we make the choice that

$$\mathbf{X}_\alpha \cdot \mathbf{X}_\alpha = \mathbf{X}_\beta \cdot \mathbf{X}_\beta, \qquad \mathbf{X}_\alpha \cdot \mathbf{X}_\beta = 0.$$

This choice of parameterizations has a beneficial feature; if $\mathbf{X}$ has $s+1$ derivatives, then we might expect $E = \mathbf{X}_\alpha \cdot \mathbf{X}_\alpha = \mathbf{X}_\beta \cdot \mathbf{X}_\beta$ to have only s derivatives. This is not the case. It turns out that E has the same regularity as $\mathbf{X}$.

There are difficulties in the three-dimensional problems which were not present in the two-dimensional problems, but the general lines of the previous method can be followed. The main difficulty in the three-dimensional problems is in working with singular integrals (including the Birkhoff-Rott integral) on a two-dimensional domain. This is much more delicate than the one-dimensional singular integrals encountered in the earlier cases. Nevertheless, we are able to work with these integrals, and we are able to write systems of evolution equations for each of the vortex sheet with surface tension, the irrotational water wave, and Darcy flows. Well-posedness of each problem can then be proved.

1. The vortex sheet with surface tension

The vortex sheet with surface tension is the interface between two fluids shearing past each other. The dependent variables we use in describing the motion are κ, the mean curvature of the free surface, and μ, the jump in velocity potential across the free surface. We will denote by Λ the operator $\sqrt{-\Delta}$. We find in [4] the following evolution equations:

$$\kappa_t = \frac{1}{4\sqrt{E}}\left[\partial_\alpha \frac{1}{\sqrt{E}}\Lambda\frac{1}{\sqrt{E}}\mu_\alpha + \partial_\beta\frac{1}{\sqrt{E}}\Lambda\frac{1}{\sqrt{E}}\mu_\beta\right] + \mathbf{T}\cdot\nabla\kappa + f_1, \tag{1}$$

$$\Lambda\mu_t = \tau\Lambda\kappa + \frac{1}{\sqrt{E}}\Big((\mu_\alpha H_1 + \mu_\beta H_2)^2(\kappa)\Big) + \mathbf{T}\cdot\nabla\Lambda\mu + f_2. \tag{2}$$

In these equations, H_1 and H_2 are Riesz transforms, and $\mathbf{T}$ is the vector of transport speeds. The positive, constant coefficient of surface tension is τ, and f_1 and f_2 are collections of lower-order terms.

As was the case in the free-surface problems in two-dimensional fluids studied by the speaker, the terms f_i arise in two ways. One is as commutators, such as the commutator of the Riesz transform and multiplication by a smooth function. The other way we generate smooth remainders is by approximating the Birkhoff-Rott integral with Riesz transforms and other singular integral operators. In the current setting, some care must be taken in making this approximation of the Birkhoff-Rott integral. In [4], we prove that both of these kinds of remainders are smooth enough to allow energy estimates on this system to be performed.

For the system (1), (2), we are able to perform energy estimates. These estimates lead to a proof of well-posedness of the vortex sheet with surface tension in 3D. Since (1), (2) form a quasilinear system in which one time derivative is like 3/2 of a space derivative, before performing estimates, we simply need to symmetrize the system. To prove well-posedness, an iterative scheme is used. The approximate problem is a linear equation, and it can be proved to have solutions. The approximate system is set up in such a way that the energy estimates are still possible to be carried out. Then, we are able to pass to the limit, getting existence of solutions to (1), (2).

2. Darcy Flow in 3D Fluids

We now consider the case of two fluids subject to Darcy's Law, separated by a sharp interface, without the effect of surface tension. The fluids have (possibly different) constant viscosity and density. This problem is treated in [2].

As before, we use an isothermal parameterization, and we use mean curvature, κ, as the dependent variable. As in the case of Hele-Shaw flow without surface tension [1], we derive a quasi-linear parabolic evolution equation. We thus find short-time well-posedness as long as a condition is satisfied, and this condition can be interpreted in the equal-density case as requiring that the more viscous fluid be displacing the less viscous fluid.

The evolution equation for curvature can be written as

$$\kappa_t = -k\Lambda(\kappa) + \mathbf{T} \cdot \nabla\kappa + f.$$

We must have k uniformly positive at the initial time to guarantee short-time well-posedness. The quantity k is

$$k(\alpha, \beta, t) = \frac{Rh + 2A_\mu \mathbf{W} \cdot \hat{\mathbf{n}}}{2\sqrt{E}}.$$

Here, R is essentially the difference in density between the two fluids (multiplied by the constant acceleration due to gravity) and A_μ is the difference in viscosity. The normal component of the Birkhoff-Rott integral, $\mathbf{W} \cdot \hat{\mathbf{n}}$, is the normal velocity of the free surface. Also, $h = (0, 0, 1) \cdot \hat{\mathbf{n}}$. Given the assumption that k is positive, energy estimates are possible, and then well-posedness follows by standard methods.

We remark that a similar approach has recently been used by Cordoba and Gancedo for this problem, in the viscosity-matched case [6].

3. The Water Wave in 3D

For the irrotational water wave, we again use an isothermal parameterization, and we use κ as a dependent variable. Unlike for the vortex sheet with surface tension, μ is not a good enough variable to use in the case of the water wave (i.e., it would be quite difficult to find estimates uniform in surface tension using μ). Instead, we introduce a variable B, which is related to the difference between our nonphysical tangential velocities, and the Lagrangian tangential velocities of fluid particles on the free surface. This is analogous to the approach of [3]. We are able to find a system of evolution equations for κ and B :

$$\kappa_t = \frac{1}{2\sqrt{E}} B + \mathbf{T} \cdot \nabla\kappa + f_1,$$

$$B_t = -\tau \left[\mathcal{L}\mathcal{L}^* \kappa + \frac{1}{\sqrt{E}} \kappa \Lambda(H_1 \kappa_\alpha + H_2 \kappa_\beta) \right] - c\Lambda\kappa + \mathbf{T} \cdot \nabla B + f_2.$$

The operators $\mathcal{L}$ and $\mathcal{L}^*$ are adjoints, and each acts like 3/2 of a derivative. The quantity c is important; it is

$$c = -\nabla p \cdot \hat{\mathbf{n}} > 0.$$

That c is uniformly positive at the initial time is now known as the Generalized Taylor Condition; Wu has proved that this condition is satisfied as long as the interface is non-self-intersecting [7].

After formulating this system of quasilinear evolution equations, we are able to perform energy estimates which are uniform in surface tension. Passing to the limit as surface tension goes to zero, we have a new proof of existence of irrotational water waves in 3D without surface tension.

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Well-Posedness of the KDV Equation in a Quarter Plane or a Finite Domain

Shu-Ming Sun
(joint work with J.L. Bona and B.-Y. Zhang)

The research considered here concerns small amplitude long waves on the surface of an ideal fluid of finite depth over a flat horizontal bottom under the force of gravity. Interest is focused upon waves which propagate essentially in the positive horizontal x-direction and without significant variation in the transverse y-direction of a standard xyz-Cartesian frame in which gravity acts in the negative z-direction. For such waves, the full three-dimensional Euler equations can be reduced to approximate models with only one independent spatial variable.

Assume that h_0 is the undisturbed depth and the free surface is represented by $z = u(x,t) = h(x,t) - h_0$ where t is proportional to elapsed time and $h(x,t)$ is the depth of the water column over the spatial point x at time t. Under the classical small-amplitude, long wave-length assumptions which feature a balance between nonlinear and dispersive effects, the evolution equation

$$u_t + u_x + uu_x + u_{xxx} = 0 \tag{1}$$

is a formal reduction of two-dimensional Euler equations usually called the Korteweg-de Vries (KdV) equation. The auxiliary data attached to the evolution equation are the initial and boundary conditions

$$u(x,0) = u_0(x)\,, \quad x > 0\,; \qquad u(0,t) = g(t)\,, \quad t > 0 \tag{2}$$

where the domain is $x > 0$ and $t > 0$, which is called a quarter-plane problem.

The initial-boundary-value problem (IBVP) (1)–(2) arises when modeling the effect in a channel of a wave maker mounted at one end [1], or in modeling near-shore zone motions generated by waves propagating from deep water. Indeed, the IBVP is a natural model whenever waves determined at an entry point propagate into a quiescent patch of a medium for which disturbances are governed approximately by the KdV equation. It can be imagined that water at rest in a channel is set in motion by a wave maker mounted at the left end of the channel. If the frequency and amplitude of the wave-maker oscillations are appropriately restricted, this will generate small-amplitude long waves that propagate down the channel, and thus will be brought into being wave motion that corresponds more or less exactly to the KdV regime.

When comparison between experimentally produced waves is made with model prediction, one usually has to depend upon numerical approximation of its solution. Whenever a numerical scheme is set up for calculating the solutions of the equation, the spatial domain is necessarily of limited extent and two-point boundary value problems cannot be avoided. For this, a bounded domain $[0, L]$ is used and two extra boundary conditions

$$u(L,t) = g_1(t), \quad u_x(L,t) = g_2(t) \tag{3}$$

are imposed. In practice, it is often the case that $u_0(x) \equiv 0$ with $g_1(t) = g_2(t) = 0$, corresponding to an initially undisturbed medium with homogeneous boundary conditions at the right end.

Here, our research mainly focuses on the proof of the well-posedness of these problems ((1)–(2) or (1)–(3)) by making use of modern methods for the study of nonlinear dispersive wave equation. Roughly speaking, local and global well-posedness are obtained for initial data $u_0(x)$ in the class $H^s(R^+)$ and boundary data $g(t)$ in $H_{loc}^{\frac{1+s}{3}}(R^+)$ for the possible smallest s. The precise theorems can be stated as follows. The first theorem is the local and global well-posedness of (1)–(2) in $x > 0, t > 0$ [2].

Theorem 1. *For given $-3/4 < s \leq 1$, there exists a constant $b \in (0, \frac{1}{2})$ such that for $u_0(x) \in H^s(R^+), g(t) \in H^{\frac{3b+s-1/2}{3}}(R^+)$ with certain compatibility condition, there exists a T dependent on the corresponding norms of u_0, g so that the IBVP (1)–(2) is locally well-posed in $H^s(R^+)$ and the solution $u \in C([0,T]; H^s(R^+))$ satisfies*

$$\|u(\cdot,t)\|_{H^s(R^+)} \leq C\Big(\|u_0\|_{H^s(R^+)} + \|g\|_{H^{\frac{3b+s-1/2}{3}}(R^+)}\Big).$$

If $g \in H^{\frac{s+1}{3}}(R^+)$, then

$$\|u(\cdot,t)\|_{H^s(R^+)} \leq C\Big(\|u_0\|_{H^s(R^+)} + \|g\|_{H^{\frac{s+1}{3}}(R^+)}\Big).$$

Moreover,, the IBVP (1)–(2) is globally well-posed (i.e., T is independent of the initial and boundary data) in $H^s(R^+) \times H_{loc}^{\frac{s+1+\epsilon}{3}}(R^+)$ for $\epsilon > 0$ and $0 \leq s < 3$ or in $H^s(R^+) \times H_{loc}^{\frac{s+1}{3}}(R^+)$ for $s \geq 3$.

The next one is the local and global well-posedness of two-point boundary value problem (1)–(3) in $x \in (0, L), t > 0$ [3].

Theorem 2. *For given $s > -1$ and any $r > 0$, there exists a $T^* = T(r) > 0$ such that if $(u_0(x), g(t), g_1(t), g_2(t)) \in H^s(0,L) \times H^{\frac{(s+1)}{3}}(0,T^*) \times H^{\frac{(s+1)}{3}}(0,T^*) \times H^{\frac{s}{3}}(0,T^*)$ satisfies some compatibility conditions and $\|(u_0(x), g(t), g_1(t), g_2(t))\| < r$, then there exists a unique solution $u \in C([0,T^*); H^s(0,L))$ for the IBVP (1)–(3) and the problem is locally well-posed. Moreover, for given $-1 < s < 0$ and any $T > 0$ (global well-posedness) with some $\epsilon > 0$, if $g_1(t) = g_2(t) = 0, (u_0(x), g(t)) \in H^s(0,L) \times H^{\frac{\epsilon+1}{3}}(0,T)$, then there exists a unique solution $u \in C([0,T); H^s(0,L)) \cap C((0,T); L^2(0,L))$. If $g(t)$ is smooth in $H^m(0,T)$, then $u \in C((0,T); H^m(0,L))$ for any $m > 0$. For $s \geq 0$ and $T > 0$, if $(u_0(x), g(t)) \in H^s(0,L) \times H^{\frac{s+1}{3}+\delta(s)}(0,T)$ with some compatibility conditions and $\delta(s) = \epsilon$ if $0 \leq s < 3$ and 0 if $s \geq 3$, then there is a unique solution $u \in C((0,T); H^s(0,L))$.*

We also obtained that if the initial and boundary conditions are chosen from function classes that include suitable decay of $u_0(x)$ as $x \to +\infty$, then for fixed $T > 0$, the solution of (1)–(3) will converge to the solution of (1)–(2) as L goes to infinity [4].

Theorem 3. *Let $u_\infty(x,t)$ be the solution of the IBVP (1) and (2) with $u_0(x) \in H^s(R^+)$ and $g(t) \in H^{\frac{s+1+\epsilon}{3}}(0,T)$ for some s in $[0,3]$, where $\epsilon > 0$ is any positive constant. Assume u_0 is supported on $[0,N]$, say. Let $u_L(x,t)$ be the solution of the two-point boundary-value problem (1)–(3) for $0 \leq x \leq L$ and $t \geq 0$ with the same initial condition and the boundary condition indicated in (2) and $g_1(t) = g_2(t) = 0$ in (3), where $L > N$. Assume that the compatibility condition $u_0(0) = g(0)$ is satisfied if $1/2 < s \leq 3$. Then, $u_\infty(x,t)$ and $u_L(x,t)$ exist for $t \in [0,T]$ and the inequality*

$$\sup_{t\in[0,T]} \|u_\infty(\cdot,t) - u_L(\cdot,t)\|_{H^s(0,L)} \leq Ce^{-bL} \,,$$

holds, where C only depends on the corresponding norms of $u_0(x)$ and $g(t)$. In case $u_0 \in H^s(R^+)$ and $g \in H^{\frac{s+1}{3}}(0,T)$, the same result holds at least on some time interval $[0,T^*]$ for some $T^* \in (0,T]$.

It is worth remark that the constant C can be shown to be of the form $e^{\gamma T}$ under reasonable assumptions on the boundary data g with $u_0 \equiv 0$. The constant γ is, for physically relevant data g, of order one, as is the constant b. In consequence, we see that if solutions u_L on a time interval $[0,T]$ are in question and the data is physically relevant, then L must be chosen to be of the form

$$L \geq O(T) + |\log \delta|$$

to have an approximation to the solution of the quarter-plane problem of error at most δ, uniformly on $[0,T]$. Notice that once $L \geq O(T)$ the error decays exponentially with larger values of L, a very satisfactory result from a practical perspective.

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Recent Results in the Theory of Ocean Waves

Jerry L. Bona

(joint work with Jean-Claude Saut)

This lecture focused upon one of the possible generative mechanisms for what are often termed Rogue Waves or sometimes Freak Waves. These are very big waves that appear occasionally in the deep ocean. Such waves have been reported by mariners for centuries, but have only been taken seriously fairly recently. They are unlike tsunamis in a number of ways. First, they persist in relatively small portions of space-time, unlike tsunamis which can propagate coherently for thousands of miles. Secondly, they are of truly large amplitude even in the deep ocean, whereas tsunamis are of small amplitude in deep water, often unnoticeable there in fact. Their existence only becomes clear when they enter shallower water. And while we have a pretty clear idea of the ways tsunamis can be generated, it is otherwise with Rogue waves.

One of the suggested mechanisms for the generation of Rogue waves is what we will call *concurrence*. Roughly speaking, this simply amounts to the possibility that small waves spread out in the ocean might, on occasion, get together *en masse* and add up to something really significant. It is our purpose to investigate the plausibility of this mechanism within the mathematical framework of classical water wave models. What is reported is joint work with Jean-Claude Saut (see [1], [2], [3]).

We begin with the Korteweg-de Vries model

$$u_t + u_x + uu_x + u_{xxx} = 0.$$

It is elementary to see that for the linearized version of this equation, a kind of dispersive focusing can occur by placing shorter and shorter wavelength components out near $x = +\infty$. It is an interesting bit of analysis to see that the initial value problem for the *nonlinear* KdV equation has the same property, which we term *dispersive blow-up*.

An immediate objection to this analysis as far as its application to Rogue waves is concerned is that the model is uni-directional and we are making use of waves traveling in the wrong direction. This can be remedied by consideration of a Boussinesq system of equations, which does allow for two-way propagation of waves. It turns out this system also exhibits the dispersive blow-up phenomenon. Moreover, this latter result can be generalized to a fully three dimensional Boussinesq system. Thus, an initial wave and velocity configuration that is too small to even bed seen with the naked eye can, in this approximation, concentrate wave components and, in finite time, lead to a wave with a infinite amplitude.

A further objection can be raised, which is that the foregoing theory makes use of the unbounded group and phase velocities that obtain within certain of the Boussinesq (and the KdV) approximations. As the full Euler equations do not possess this property, it is still not clear whether or not the theory might pertain to the development of real Rogue waves by concurrence.

A final result was mentioned, that grows out of the preceding. Another way of looking at what was established for the KdV equation and for Boussinesq systems is that these equations are not well posed in L^∞-type spaces. That is, no matter how small the initial data is restricted as far as its maximum values are concerned, the resulting solution can take on values as large as we like in finite time. Looked at this way, the issue is clarified. The authors have been able to show that at least the linearized, two-dimensional Euler equations are not well posed in L_∞, thereby coming closer to being able to say that concurrence is a possible mechanism for the formation of Rogue waves.

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