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Funktionentheorie

Organised by
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Ed Saff (Vanderbilt)

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Introduction by the Organisers

The present conference was organized by Walter Bergweiler (Kiel), Stephan Ruscheweyh (Würzburg) and Ed Saff (Vanderbilt).

The 24 talks gave an overview of recent results and current trends in function theory.

Workshop on Funktionentheorie

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Abstracts

Curvature flow in conformal mapping

Kenneth Stephenson

(joint work with Charles Collins and Tobin Driscoll)

In joint work with Charles Collins (Tennessee) and Tobin Driscoll (Delaware), the author investigates the conformal mapping of a non-planar Riemann surface to a rectangle in the plane. The methods involve circle packing, and the discussion centres on a simple prototype problem: A Riemann surface $\mathcal{S}$ is created as a nonplanar cone space by pasting 10 equilateral triangles together in a specified pattern. Four vertices on the boundary are designated as "corners". It is well known classically that there is a conformal map $F : \mathcal{S} \longrightarrow \mathcal{R}$ mapping S to a plane rectangle $\mathcal{R}$ with corners going to corners, as suggested in Figure 1.

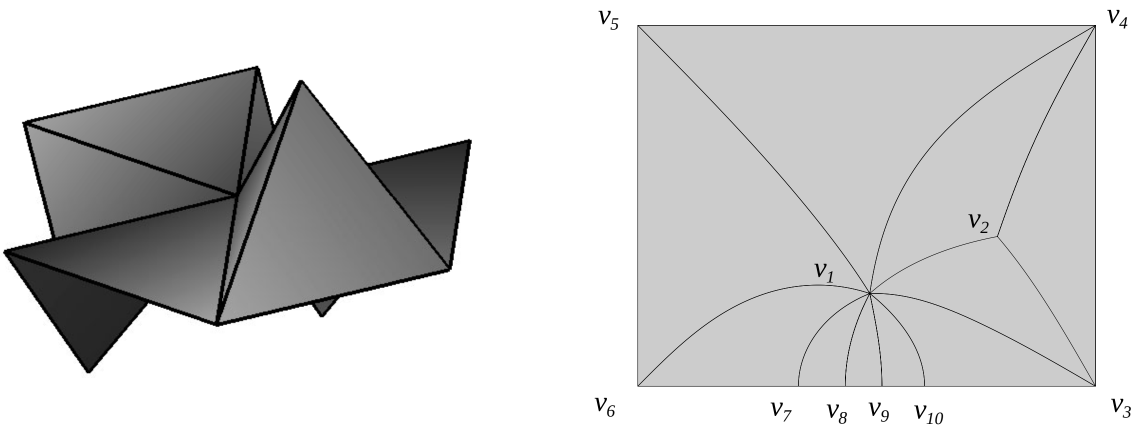

Figure 1. Conformally mapping an equilateral surface to a rectangle

Circle packing provides a means for numerically approximating F. A sequence of ever finer *insitu* circle packings Q_n are created in $\mathcal{S}$ based on its equilateral structure and a "repacking" computation then lays out circle packings P_n in the plane having the same combinatorics but with carriers that form rectangles $\mathcal{R}_n$. For each n the associated map $f_n : Q_n \longrightarrow P_n$ is defined as a "discrete conformal map". It has been established by Phil Bowers and the author that as n grows, appropriately normalized rectangles $\mathcal{R}_n$ converge to $\mathcal{R}$ and the discrete conformal maps f_n converge uniformly on compact subsets of $\mathcal{S}$ to F. (See [1] and for background, [3, 4].) The circle packing on the left in Figure 2 is P_6; the images of the 10 faces of $\mathcal{S}$ here are very close to their correct conformal shapes.

In studying this mapping, the authors parametrized the flattening process, both classical and discrete, in a natural way to obtain a continuous family of surfaces stretching from $\mathcal{S}$ to $\mathcal{R}$. One can observe experimentally the "flow" of radius adjustments as the circle packings are computed from one discrete surface to the next; that flow reflects the movement of "curvature" at the circle centres during

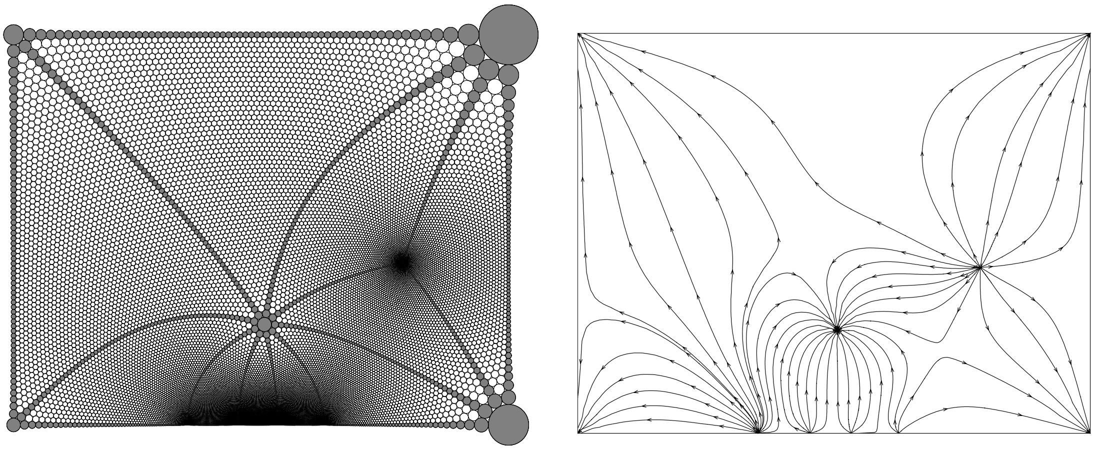

FIGURE 2. The packing P_6 and the associated "flow" field

the adjustment process. The surprise came in our observation that this flow was essentially independent of the parametrization stage. In other words, from the beginning to the end of the parametrization the circles seemed to move in accordance with an unchanging prescription about how to coordinate their size adjustments. On the right in Figure 2 is one of these simulated flow fields.

This field ultimately describes the flow of cone angle (curvature) among the ten cone points of $\mathcal{S}$ during the flattening process. The authors looked for a classical parallel and obtained it via a modification of the Schwarz-Christoffel (SC) method [2]. That modification introduces interior cone points and cuts to allow mapping to a non-planar surface. The experimental flows are nearly exact copies of the gradient field $\nabla \log |\Phi'(z)|$, where Φ' is the derivative of the mapping function generated by our modified SC method (and then lifted to $\mathcal{R}$). This raises a number of questions about the classical interpretation and the possible uses for this "curvature" flow.

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Julia polynomials and the Szegő kernel method

Igor Pritsker

Let G be a Jordan domain bounded by a rectifiable curve L of length l. The Smirnov space of analytic functions $E_2(G)$ is defined by the product $\langle f, g\rangle = \frac{1}{l}\int_L f(z)\overline{g(z)}|dz|$ (see [2], [3] and [10]). Consider the associated contour orthonormal polynomials $\{p_n(z)\}_{n=0}^{\infty}$. If G is a Smirnov domain, then polynomials are dense in $E_2(G)$ [2]. In this case, the Szegő kernel is given by $K(z,\zeta) = \sum_{k=0}^{\infty} \overline{p_k(\zeta)} p_k(z) = \frac{l}{2\pi}\sqrt{\varphi'(z)\overline{\varphi'(\zeta)}},\ z,\zeta \in G$, where φ is the conformal map of G onto the unit disk, normalized by $\varphi(\zeta) = 0$, $\varphi'(\zeta) > 0$ [11]. Julia polynomials approximate φ, with a construction resembling Bieberbach polynomials in the Bergman kernel method,

$$J_{2n+1}(z) = \frac{2\pi}{l}\int_{\zeta}^{z}\left(\sum_{k=0}^{n}\overline{p_k(\zeta)}p_k(t)\right)^2 dt \;/ \sum_{k=0}^{n}|p_k(\zeta)|^2, \quad n \in \mathbb{N}.$$

The uniform convergence of Bieberbach polynomials has been extensively studied, but methods based on the Szegő kernel did not receive a comprehensive attention. It is not difficult to see that J_{2n+1} converge to φ locally uniformly in G. We show in [9] that J_{2n+1} converge to φ uniformly on the closure of any Smirnov domain. This class contains all Ahlfors-regular domains [8], allowing arbitrary (even zero) angles at the boundary. For the piecewise analytic domains, we also give the estimate

$$\|\varphi - J_{2n+1}\|_{L_\infty(\overline{G})} \le C(G)\, n^{-\frac{\lambda}{4-2\lambda}}, \quad n \in \mathbb{N}, \tag{1}$$

where $\lambda\pi,\ 0 < \lambda < 2$, is the smallest exterior angle at the boundary of G. The rate of convergence for J_{2n+1} on compact subsets of G is essentially squared comparing to (1). These results have standard applications to the rate of decay for the contour orthogonal polynomials inside the domain, and to the rate of locally uniform convergence of Fourier series.

The approximating polynomials of this kind were first introduced via an extremal problem by Keldysh and Lavrentiev (cf. [5], [6] and [7]), who developed the ideas of Julia [4]. Set $\|f\|_p = \left(\int_L |f(z)|^p|dz|\right)^{1/p}$ for $f \in E_p(G),\ 0 < p < \infty$, where $E_p(G)$ is the Smirnov space [2]. Let $Q_{n,p}$ be a polynomial minimizing $\|P_n\|_p$ among all polynomials P_n such that $P_n(\zeta) = 1$. Julia [4] showed that the corresponding extremal problem in the class of all $E_p(G)$ functions is solved by $(\phi')^{1/p}$, where ϕ is the conformal map of G onto a disk $\{z : |z| < R\}$, normalized by $\phi(\zeta) = 0$ and $\phi'(\zeta) = 1$. Keldysh and Lavrentiev [7] proved that $Q_{n,p}$ converge to $(\phi')^{1/p}$ locally uniformly in G if and only if G is a Smirnov domain. Thus the polynomials $J_{n,p}(z) := \int_{\zeta}^{z} Q_{n,p}^p(t)dt$ provide an approximation to $\phi(z)$. If $p = 2$ then $J_{n,2}$ differ from J_{2n+1} just by a constant factor. This case was studied by Ahlfors [1], Warschawski [12] and Gaier [3]. Again, the locally uniform convergence of $J_{n,p}$ to ϕ in Smirnov domains is immediate for any $p \in (0,\infty)$. We prove the uniform convergence on $\overline{G}$ in arbitrary Smirnov domains, and give the convergence rates generalizing (1) for piecewise analytic domains.

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Conformal Pseudo–metrics and a free boundary value problem for analytic functions

Daniela Kraus

The starting point is the following free boundary value problem for analytic functions f which are defined on a domain $G \subset \mathbb{C}$ and map into the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Problem 1 *Let $z_1, \ldots, z_n$ be finitely many points in a bounded simply connected domain $G \subset \mathbb{C}$ and let $\phi : \partial G \to (0, \infty)$ be a continuous function. Show that there exists a holomorphic function $f : G \to \mathbb{D}$ with critical points z_j (counted with multiplicities) and no others such that*

$$\lim_{z \to \xi} \frac{|f'(z)|}{1 - |f(z)|^2} = \phi(\xi)$$

for all $\xi \in \partial G$.

If $G = \mathbb{D}$, $\phi \equiv 1$, Problem 1 was solved by Kühnau [5] in case of one critical point, which is sufficiently close to the origin, and for more than one critical point by Fournier and Ruscheweyh [2]. The method employed by Kühnau, Fournier and Ruscheweyh easily extends to more general domains G, say bounded by a Dini–smooth Jordan curve, but does not work for arbitrary bounded simply connected domains.

We present a completely new approach to Problem 1, which shows that this boundary value problem is not an isolated question in complex analysis, but is intimately connected to a number of basic (open) problems in conformal geometry and non–linear PDE. To solve Problem 1 for arbitrary bounded simply connected domains we divide it into the following two parts.

In a first step we construct a conformal metric in a bounded regular domain $G \subset \mathbb{C}$ with prescribed non–positive Gaussian curvature $\kappa(z)$ and prescribed singularities by solving the first boundary value problem for the Gaussian curvature equation $\Delta u = -\kappa(z)e^{2u}$ in G with prescribed singularities and continuous boundary data. More precisely, we have

Theorem 1 *Let $G \subset \mathbb{C}$ be a bounded and regular domain, let $z_1, z_2, \ldots, z_n \in G$ be finitely many distinct points and let $\alpha_1, \ldots, \alpha_n \in (0, \infty)$. Let $\phi : \partial G \to (0, \infty)$ be a continuous function and $\kappa : G \to (-\infty, 0]$ a bounded and locally Hölder continuous function with exponent α, $0 < \alpha \le 1$. Then there exists a unique pseudo–metric $\lambda : G \to [0, \infty)$ of curvature $\kappa(z)$ in $G\backslash\{z_1, z_2, \ldots, z_n\}$ with zeros of orders α_j at z_j and no others such that λ is continuous on $\overline{G}$ with $\lambda(z) = \phi(z)$ for $z \in \partial G$.*

Theorem 1 is related to the Berger–Nirenberg problem in Riemannian geometry, that is, the question which functions on a surface R can arise as the Gaussian curvature of a Riemannian metric on R. The special case, where $\kappa(z) \equiv -4$ and the domain G is bounded by finitely many analytic Jordan curves was treated by Heins [4].

In a second step we show every conformal pseudo–metric on a simply connected domain $G \subseteq \mathbb{C}$ with constant negative Gaussian curvature and isolated zeros of integer order is the pullback of the hyperbolic metric on $\mathbb{D}$ under an analytic map $f : G \to \mathbb{D}$:

Theorem 2 *Let $E = \{z_1, z_2, \ldots\}$ be a discrete set in a simply connected domain $G \subseteq \mathbb{C}$, let $\alpha_1, \alpha_2, \ldots$ be positive integers, and let $\lambda : G \to [0, \infty)$ be a pseudo–metric of constant curvature $\kappa = -4$ in $G\backslash E$ with zeros of orders α_j at z_j and no others. Then λ is the pullback of the hyperbolic metric under a holomorphic function $f : G \to \mathbb{D}$, i.e.*

$$\lambda(z) = \frac{|f'(z)|}{1 - |f(z)|^2}, \quad z \in G .$$

If $g : G \to \mathbb{D}$ is another holomorphic function such that

$$\lambda(z) = \frac{|g'(z)|}{1 - |g(z)|^2}, \quad z \in G ,$$

then $g = T \circ f$, where T is a conformal automorphism of the unit disk $\mathbb{D}$.

This extends a theorem of Liouville [6] which deals with the case that the pseudo–metric has no zeros at all.
Theorem 1 and Theorem 2 together allow in particular a quick and complete solution of Problem 1.

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Critical points of discrete potentials in space

J.K. Langley

(joint work with J. Rossi)

The following was conjectured in [1]: let $z_k \in \mathbb{C}, a_k > 0$,

$$z_k \to \infty, \quad \sum_{z_k \neq 0} \left| \frac{a_k}{z_k} \right| < \infty, \quad f(z) = \sum_{k=1}^{\infty} \frac{a_k}{z - z_k}. \tag{1}$$

Then f has infinitely many zeros.

The zeros of f correspond to equilibrium points of the electrostatic field generated by wires carrying charge density $a_k/2$, perpendicular to the plane at z_k. The conjecture is known to be true in two contrasting cases: (i) if the total charge $\sum_{k=1}^{\infty} a_k$ is finite (or, more generally, if $\sum_{|z_k| \leq r} a_k = o(\sqrt{r})$ as $r \to \infty$) [1]; (ii) if $\inf\{a_k\} > 0$ [2].

For point charges in space, the following was proved in [1]. Let $x_k \in \mathbb{R}^3$, with

$$x_k \to \infty, \quad \sum_{x_k \neq 0} \frac{a_k}{|x_k|} < \infty, \quad u(x) = \sum_{k=1}^{\infty} \frac{a_k}{|x - x_k|}. \tag{2}$$

If $\inf\{a_k\} > 0$ then u has infinitely many critical points in $\mathbb{R}^3$.

In this case the critical points of u are equilibrium points of the electrostatic field generated by charges a_k at x_k. Langley and Rossi [5] have recently shown that instead of the condition $\inf\{a_k\} > 0$ it suffices that the x_k have finite exponent of convergence, which follows at once from (2) if $\inf\{a_k\} > 0$. The Cartan lemma [3, p.366] is used to prove that there exist spheres $|x| = r_n \to \infty$ on which the maximum of $u(x)$ tends to 0, following which the method of [1] is applied.

The talk concludes with some results from [4] concerning zeros of $f(z)$ when the a_k are complex in (1). A number of methods are applied, including quasiconformal surgery.

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Efficient Discretization of Green Energy and Grunsky-Type Development of Functions Univalent in an Annulus

Marcus Stiemer

Let Γ be an analytic Jordan curve in the complex plane. In 1970, K. Menke introduced an extremal point system on Γ and applied it to approximate the logarithmic capacity of Γ and the conformal mapping Φ from the outer domain of the unit circle onto the outer domain of Γ with $\Phi(z) = dz + O(1)$, $z \to \infty$, $d > 0$ geometrically fast [2, 3, 4, 5, 6]. D. Gaier introduced the notation *Menke points* for systems of this type. In contrast to Fekete-points, which possess a worse distribution on analytic Jordan curves [10, 11], Menke-points consist of two sets of points that alternate on the curve Γ. An extension to the hyperbolic situation (see below) has been developed in [9].

Let now $F \subset \widehat{\mathbb{C}}$ be a set with connected complement Ω, such that the Green function $G(z, \zeta)$ in Ω with pole in $\zeta \in \Omega$ exists. Moreover, let Γ be an analytic Jordan curve in Ω with $E = \overline{\operatorname{Int}\Gamma}$.

The purpose of this work is to develop a Menke-type discretization for the measure of minimal Green energy on Γ with respect to Ω and to prove that this discretization provides a geometrically fast converging approximation to minimal Green energy.

Particularly for the hyperbolic situation, $F = \widehat{\mathbb{C}} \setminus \mathbb{D}$, $\Omega = \mathbb{D}$, we prove that Menke-points approximate the images of rotated roots of unity under the conformal mapping Φ from $\{1 < |z| < e^{1/C(E,F)}\}$ onto $\mathcal{R} = \mathbb{D} \setminus E$ with $\Phi(e^{1/C(E,F)}) = 1$ geometrically fast. Here, $C(E, F)$ denotes the capacity of the condenser (E, F). Thus, hyperbolic Menke points possess a better distribution on analytic Jordan curves than points of Fekete-type, which are called Tsuji-points in the hyperbolic situation [7, 8]. The latter has only been shown under additional assumptions so far.

The key to the presented proof is to utilize the connection between Green energy and the coefficients of the logarithmic development of functions univalent in an annulus. In particular, an extension of the Grunsky inequaltities to functions univalent in an annulus due to R. Kühnau is applied [1].

Finally, a pointwise geometrically fast approximation to the Green potential in $\mathcal{R} = \mathbb{D} \setminus E$ is derived and several numerical examples are presented.

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Boundary Interpolation in the Theory of Nonlinear Riemann-Hilbert Problems

Gunther Semmler

We study Riemann-Hilbert problems for a holomorphic function w in the unit disc $\mathbb{D}$ with the boundary condition

$$w(t) \in M_t \tag{1}$$

for all $t \in \mathbb{T}$. The restriction manifold

$$M := \bigcup_{t\in\mathbb{T}} \{t\} \times M_t$$

is supposed to be smooth so that the existence of solutions that are continuous on the closed unit disc is secured by well-known theorems. Given k points $z_1, \ldots, z_k$ in the unit disc, there is exactly one solution of the boundary value problem (1) satisfying the side conditions

$$w(z_j) = w_j,\ j = 1, \ldots, k \qquad w(t_0) = w_0 \in M_{t_0}$$

The ambition of our research is to replace these conditions solely by interpolation points on M, i.e. we require

$$w(t_j) = w_j, \qquad j = 0, \ldots, k \tag{2}$$

where $t_j \in \mathbb{T}$ and $w_j \in M_{t_j}$ are given. As a generalization of a result by Ruscheweyh and Jones for Blaschke products, we show that the interpolation problem (2) has a solution with winding number at most k about M. This raises the question to determine a solution of (2) with minimal winding number about M. For three interpolation points we define the notion of counterclockwise turning around M with respect to the holomorphic parametrization, which allows to finally solve this problem. For more than three interpolation points, the situation is more involved. It turns out that we can distinguish three classes of problems which will be called rigid, fragil, and flexible. Problems in these classes have different properties concerning uniqueness and stability of solutions.

It is remarkable that also for finite Blaschke products (which solve the most simple Riemann-Hilbert problem where $M_t = \mathbb{T}$), no solvability criterium for (2) is known. In order to find at least an algorithmic approach we transformed this problem to an interpolation problem for a rational funtion on the real line, the numerator and denumerator polynomial of which have the interlacing property.

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Restriction operators on Bergman space

Mihai Putinar

(joint work with B. Gustafsson and H.S. Shapiro)

Let Ω be a bounded planar domain and let $A^2(\Omega)$ be the associated Bergman space (of analytic square integrable functions). For a positive measure μ, compactly supported by Ω we consider the restriction operator:

$$R : A^2(\Omega) \longrightarrow L^2(\mu),\ Rf = f|_{\,\mathrm{supp}\mu}.$$

It is a trace class operator, whose modulus square R^*R has a complete system of eigenvectors $f_k \in A^2(\Omega)$, corresponding to a descending sequence of eigenvalues λ_k (after putting aside the null vectors). The typical eigenvalue problem for f_k can be written as an integral equation:

$$\lambda_k f_k(z) = \int K(z,w) f_k(w) d\mu(w).$$

This shows that each function f_k analytically extends across the boundary of Ω.

The system of functions f_k is doubly orthogonal with respect to the two inner products:

$$\lambda_k \langle f_k, f_m \rangle_{2,\Omega} = \delta_{km} \langle f_k, f_m \rangle_\mu.$$

Such doubly orthogonal systems have appeared a long time ago in function theory and approximation theory. Most of the references below illustrate such instances.

We are interested in qualitative properties of the eigenfunctions f_k. A central result in this direction is the following.

Theorem. *Let Ω be a bounded domain with smooth boundary, such that its Green function of the bi-Laplacian(associated to an arbitrary point of the boundary) is non-negative. Let $H(z, w)$ denote the reproducing kernel for all harmonic, square integrable functions in Ω, and assume that the positivity set: $P = \{z \in \Omega;\ H(z, w) > 0,\ w \in \partial\Omega\}$ is non-empty.*

Suppose that the positive measure μ is supported by a compact subset of P. Then each eigenfunction f_k does not vanish on the boundary of Ω and it possesses exactly k zeros in Ω.

For instance, if $\Omega = \mathbf{D}$ is the unit disk, then the conditions of the theorem are met for the set $P = \{z; |z| < \sqrt{2} - 1\}$. The analogous theorem for restrictions from the Hardy space was discovered by Fisher and Micchelli [7] and it played an important role in best approximation results and estimates on n-widths.

The proof of the theorem is based on potential theoretic techniques, starting from the observation that each eigenfunction f_k satisfies the balayage identity:

$$\lambda_k \int_\Omega |f_k(z)|^2 u(z) dArea(z) = \int |f_k(z)|^2 u(z) d\mu(z),$$

valid for an arbitrary harmonic function u, defined on a neighborhood of the closure of Ω.

This particular framework of doubly orthogonal systems can be used to estimate the growth of the contractive divisors in the Bergman space, best approximation in the $L^2(\mu)$ norm with control of the $L^2(\Omega, dArea)$ norm or exact identification of the inner measure μ from the matricial elements of the restriction operator.

This is a report on results published in:

B. Gustafsson, M. Putinar and H.S.Shapiro *Restriction operators, balayage and doubly orthogonal systems of analytic functions*, J. Funct. Analysis **199**(2003), 332-378.

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Measurable dynamics of transcendental entire functions on their Julia sets

Jan-Martin Hemke

One of the main ideas in complex dynamics is to divide the plane into the Fatou set of points, where the iterates behave stable, i.e. where they form a normal family, and its complement, the Julia set. By definition the dynamics in the Fatou set is easier and understood very well. We are interested in the dynamics of meromorphic functions on their Julia set and study it in terms of the Lebesgue measure. In [1] H. Bock proved, that for any non-constant meromorphic function, which is defined on the whole complex plane, one of the two following cases holds:

(1) The Julia set is the entire plane and almost every orbit is dense in the sphere $\hat{\mathbb{C}}$;

(2) almost every forward-orbit in the Julia set accumulates only in the post-singular set.

Here the post-singular set denotes the closure of the union of the forward-orbits of all singularities of the inverse function, which are the critical and asymptotic values. This result is a generalization of similar results for rational functions obtained by M. Lyubich [8] and C. McMullen [10].
It is natural to ask for a given function, which case holds. Since a non-empty Fatou set always implies (ii), one can restrict to the cases, in which the Julia set consists of the whole complex plane. If the Julia set is not the entire plane, and thus (ii) holds, it would still be interesting to know if the Julia set has positive measure, since otherwise the statement (ii) would be trivial.
In the paper mentioned H. Bock gives sufficient conditions for (i): If f is entire and the set of singularities of the inverse function is finite, all of these are pre-periodic but not periodic, then (i) is satisfied. Thus the function $f(z) = 2\pi i \exp(z)$ is an example for this first case, in which the post-singular set consists of the only asymptotic value zero and its image $2\pi i$. Other conditions concerning this case are given by L. Keen and J. Kotus [4]. Conversely it was already shown in 1984 independently by M. Rees [6] and M. Lyubich [7] that the function $f(z) = \exp(z)$ is an example for (ii). Here the post-singular set consists of the the closure of the forward-orbit of the only asymptotic value zero, which tends to infinity on the real axis. This result was generalized in [11] to functions $f_\lambda(z) = \lambda \exp(z)$, if $f_\lambda^n(0)$ tends to infinity sufficiently fast. M. Urbanski and A. Zdunik [3] even showed, that the Hausdorff-dimension of the remaining set is smaller than 2.
The difference between the dynamics of $\exp(z)$ and $2\pi i \exp(z)$ is caused by the different behavior of the asymptotic value zero under iteration. We consider functions of the type $f(z) = \int_0^z P(t)\exp(Q(t))dt + c$, with polynomials P and Q and $c \in \mathbb{C}$, such that Q is not constant and P not zero. Counting multiplicity these functions have exacty $\deg(Q)$ asymptotic values and $\deg(P)$ critical points and may even be characterized as those entire functions with this property. In the extremal case that all singularities of the inverse are pre-periodic but not periodic, the theorem of H. Bock implies (i). We consider the other extreme and may neglect the critical values but have to specify the speed of escape. We assume that every asymptotic values s escapes exponentially fast, i.e. that $|f^n(s)| \geq \exp(|f^{n-1}(s)|^\delta)$ for some $\delta > 0$ and almost all $n \in \mathbb{N}$. Then we can prove that the Julia set has positive measure and that (ii) is satisfied. If the degree of Q is at least three, using an argument introduced by H. Schubert in [13], we obtain that the measure of the Fatou set is even finite.

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On Periodic Rays of Certain Entire Functions

Lasse Rempe

A well-known theorem of Douady and Hubbard [M, Theorem 18.10] states that *periodic dynamic rays* of polynomials always have a periodic landing point. This result forms the basis of the combinatorial methods which have been an essential ingredient in the success story of polynomial dynamics since the early studies of the Mandelbrot set [DH].

In this talk, we will consider the analogous question for periodic rays of transcendental entire functions. For our purposes, a *periodic dynamic ray* of an entire function $f : \mathbb{C} \to \mathbb{C}$ is a maximal curve

$$\gamma : (t_0, \infty) \to I(f) := \{z \in \mathbb{C} : |f^n(z)| \to \infty\}$$

which satisfies $f^n(\gamma(t)) = \gamma(t+1)$ for some $n \geq 1$ and all $t > t_0$. (Here $t_0 \in [-\infty, \infty)$.) As usual, we say that γ *lands* at a point $z_0 \in \mathbb{C}h$ if $\lim_{t \to t_0} \gamma(t) = z_0$.

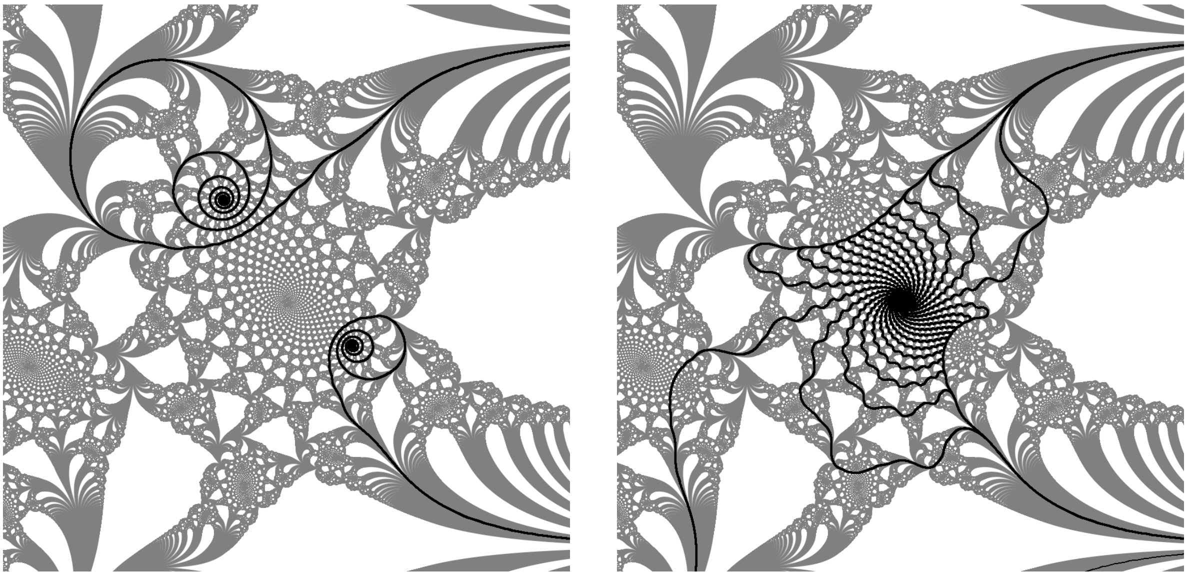
For the family of *exponential maps*[1]

$$E_\kappa k : z \mapsto \exp(z) + \kappa,$$

landing behavior of periodic rays has recently been used to great advantage by Schleicher (see e.g. [S2, RS]). However, it was previously not known whether periodic rays of exponential maps always land. We can now answer this question.

Theorem 1 (Periodic rays land [R1]**)** *Every periodic ray of every exponential map lands.*

[1]This family forms the simplest parameter space of transcendental entire functions, as exponential maps are the only such functions with only one singular value. Also, the exponential family can be considered to be the limit of the families of *unicritical polynomials*, $z \mapsto z^d + c$ [BDG], which are by far the best-understood polynomial families.



(a) (b)

FIGURE 3. Periodic rays for $z \mapsto \exp(z) + 1.0038 + 2.8999i$. (a) shows two rays forming a period 2-cycle; (b) shows a cycle of 25 rays landing at a common fixed point.

The proof of Douady and Hubbard's landing theorem for polynomials uses a hyperbolic contraction principle, and this argument can be carried over to several situations in which there is some form of expansion along the ray. However, it is conceivable that a periodic ray γ might accumulate on a singular value, whose orbit again accumulates everywhere on γ. In such a situation, a proof by hyperbolic contraction would be impossible. Thus, in order to apply this method to maps with large postsingular sets, it seems that one must *a priori* show that the given ray does not accumulate on singular values. The problem is that it can be very difficult to control the accumulation behavior of these rays; even for many tame exponential maps, there are many (nonperiodic) dynamic rays with complicated accumulation behavior [DJ, R2].

Our proof of Theorem 1 circumvents these difficulties by using a theorem of Schleicher [S1] on landing properties of *parameter* rays.[2] However, there is little hope for this method to generalise to higher-dimensional parameter spaces. For example, we currently know of no argument which would prove the analogue of Theorem for *cosine maps*,

$$z \mapsto a\exp(z) + b\exp(-z),$$

[2]Thus, we are reversing Douady's famous principle: we plough in the parameter plane to harvest in the dynamical plane.

where $a, b \in \mathbb{C}$. (Many results for the exponential family are known to generalise to this two-dimensional family; in particular, there is a complete classification of escaping points in terms of dynamic rays [Ro].)

On the other hand, we were able to show that the above problem is indeed the only obstruction for a large set of functions in the class

$$\mathcal{B} := \{f : \mathbb{C} \to \mathbb{C} \text{ entire}; \operatorname{sing}(f^{-1}) \text{ is bounded}\}.$$

Theorem 2 (Periodic rays with nonsingular accumulation sets [R3]**)** *Let f be either*

- *a cosine map $z \mapsto a\exp(z) + b\exp(-z)$ or*
- *a function $f \in \mathcal{B}$ all of whose singular values lie in the Julia set.*

If γ is a fixed dynamic ray of f which has no accumulation points in $\overline{\operatorname{sing}(f^{-1})}$, then γ lands.

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On the zeros of the solutions of a functional equation

Walter Hayman

We consider an entire function

$$f(z) = \sum_0^\infty a_n z^n$$

satisfying the equation

$$(a - qz)f(q^2 z) - (1 + a)f(qz) + f(z) = 0\,, \quad 0 < |q| < 1.$$

Let z_n be the nth zero of $f(z)$ in order of nondecreasing moduli. Then

$$z_n = -q^{(1-2n)}\left\{1 + \sum_{\nu=1}^{k} b_\nu q^{n\nu} + O(|q|^{(k+1)n})\right\},$$

where the b_ν are constants depending on a and q. This verifies a conjecture of Mourad Ismail [1], concerning the zeros of q–Bessel functions. The above result also contains as a special case an identity of Ramanujan [4].

The method builds on an earlier paper by Walter Bergweiler and the author [3] which applies to a wider class of functional equations but gives only the first term in the asymptotic series. In this case the zeros may approach a finite number of distinct geometric progressions. We compare the coefficients of $f(z)$ and so $f(z)$ itself with certain theta–functions.

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On the number of zeros of certain rational harmonic functions

Dmitry Khavinson

(joint work with Genevra Neumann)

A. Wilmshurst [Wil 98] showed that there is an upper bound on the number of zeros of a harmonic polynomial $f(z) = p(z) - \overline{q(z)}$, where p and q are analytic polynomials of different degree, answering the question of T. Sheil-Small [SS 92]. Let $n = \deg p > \deg q = m$. Wilmshurst showed that n^2 is a sharp upper bound when $m = n-1$ and conjectured that the upper bound is actually $m(m-1)+3n-2$. D. Khavinson and G. Świątek [KS 03] showed that Wilmshurst's conjecture holds for the case $n > 1, m = 1$ using methods from complex dynamics. When hearing of this result, P. Poggi-Corradini asked whether this approach can be extended to the case $f(z) = p(z)/q(z) - \overline{z}$, where p and q are analytic polynomials.

In this note, we apply the approach from [KS 03] to prove

Theorem *Let $r(z) = p(z)/q(z)$ be a rational function where p and q are relatively prime, analytic polynomials and such that $n = \deg\ r = \max(\deg\ p, \deg\ q) > 1$. Then*

$$\#\{z \in \mathbb{C} : \overline{r(z)} = z\} \leq 5n - 5$$

We note that the zeros of $\overline{r(z)} - z$ are isolated, because each zero is also a fixed point of $Q(z) = \overline{r(\overline{r(z)})}$, an analytic rational function of degree n^2. This also follows from a result of P. Davis [Da 74] (Chapter 14) concerning the Schwarz functions of analytic curves. (The Schwarz function is an analytic function $S(z)$ that gives the equation of a curve in the form $\overline{z} = S(z)$, *cf.* [Da 74].) A rational Schwarz function implies that the curve is a line or a circle, so the degree must be one.

We also note that $\overline{r(z)} - z$ will not have a zero at ∞.

L. Geyer [Ge 03] has recently shown that the $3n - 2$ bound on the number of zeros of $f(z) = p(z) - \overline{z}$ where $\deg\ p = n$ is sharp for all $n > 1$. D. Bshouty and A. Lyzzaik [BL 03] have recently given an elementary proof for $n = 4, 5, 6, 8$. Hence, a sharp bound on the number of zeros of $f(z) = \overline{r(z)} - z$ must be at least $3n - 2$.

Let us discuss applications of the result to gravitational microlensing. An n-point gravitational lens can be modeled as follows: Suppose that we have n point masses (such as stars). Construct a plane through the center of mass of these point masses, such that the line of sight from the observer to the center of mass is orthogonal to this plane. This plane is called the lens plane (or deflector plane). Suppose that the lens plane is between the observer and the light source. (We are assuming that the distance between the point masses is small compared to the distance between the observer and the lens plane, as well as the distance between the lens plane and the light source.) The plane containing our light source which is parallel to the lens plane is called the source plane. Due to deflection of light by masses multiple images of the light source are formed. This phenomenon is known as gravitational microlensing and is modeled by a lens equation. The lens equation defines a mapping from the lens plane to the source plane. Suppose that our light source is located at postion w in the source plane. In this model, if z satisfies the lens equation, then our gravitational lens will map z to w; hence z corresponds to the position of a lensed image. The number of lensed images is the number of solutions of the lens equation. See [Wa 98] for an introduction to gravitational lensing and [St 97] for an introduction to a complex formulation of lensing theory.

To set up a lens equation for our n-point gravitational lens, the point masses are projected onto positions in the lens plane. The projection of the j-th point mass has a scaled mass of m_j and is located at a scaled coordinate of z_j in the lens plane, where m_j is a positive constant and z_j is a complex constant. Suppose that we have a light source located at a scaled coordinate of w in the source plane.

Following [Wit 90], this lens equation will be given by

$$w = z + \gamma\overline{z} - sign(\sigma)\Sigma_{j=1}^{n} m_j/(\overline{z} - \overline{z_j}),$$

where the normalized shear γ and the optical depth (or normalized surface density) $\sigma \neq 0$ are real constants. See [Wit 90] and [Pa 86] for a derivation of the normalized lens equation for microlensing.

We can rewrite this lens equation in terms of the rational harmonic function $f(z) = \overline{r(z)} - z$ by letting $r(z) = \overline{w} - \gamma z + \mathrm{sign}(\sigma)\Sigma_{j=1}^{n} m_j/(z - z_j)$. We thus see that the zeros of $f(z)$ are solutions of the lens equation for a light source at position w. H. Witt [Wit 90] showed for $n > 1$ that the maximum number of observed images is at most $n^2 + 1$ when $\gamma = 0$ and $(n+1)^2$ when $\gamma \neq 0$. S. H. Rhie [Rh 01] conjectured that for $n > 1$ such a gravitational lens gives at most $5n - 5$ images for the case $\gamma = 0$ and $\sigma > 0$. In the $\gamma = 0$ case, deg $r = n$; hence, our theorem settles this conjecture. Further, for the case $\gamma \neq 0$, we see that deg $r = n + 1$, so our theorem gives an upper bound of $5(n+1) - 5 = 5n$ lensed images.

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An extension of the Schwarz–Carathéodory reflection principle

Oliver Roth

1. A reflection principle for conformal metrics

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk in the complex plane $\mathbb{C}$. An open subarc of the unit circle $\partial\mathbb{D} := \{z \in \mathbb{C} : |z| = 1\}$ is an open connected proper subset of $\partial\mathbb{D}$.

Theorem 1 *Let I be an open subarc of $\partial\mathbb{D}$ and let R be a Riemann surface, which carries a complete real analytic conformal Riemannian metric $\lambda(w)\,|dw|$. Then a non–constant analytic map $f : \mathbb{D} \to R$ can be continued analytically across I with $f(I) \subset R$ if and only if there exists a holomorphic function $h : I \to \mathbb{C}$ such that*

$$\lim_{z \to \xi} \frac{\lambda(f(z))\,|f'(z)|}{|h'(z)|} = 1, \qquad \xi \in I. \tag{1}$$

Remarks.

(a) Note that $\lambda(f(z))\,|f'(z)|$ in (1) is the pullback of the metric $\lambda(w)\,|dw|$ under the map f. Hence $\lambda(f(z))\,|f'(z)|$ is a well-defined function on $\mathbb{D}$.

(b) The phrase "$f : \mathbb{D} \to R$ can be continued analytically across I with $f(I) \subset R$" means there exists a domain $\Omega \supset \mathbb{D}$ with $I \subset \Omega$ and an analytic map $F : \Omega \to R$ such that $F = f$ in $\mathbb{D}$. This map F is the unique analytic continuation of f to Ω.

(c) A function $h : M \to \mathbb{C}$ is said to be holomorphic on a set $M \subseteq \mathbb{C}$, if it is defined and holomorphic in an open set $V \subseteq \mathbb{C}$ containing M.

(d) The special case $R = \mathbb{C}$ and $\lambda(w) = 1$ of Theorem 1 may be regarded as a version of the classical Schwarz–Carathéodory reflection principle [3, 7] for holomorphic functions $f : \mathbb{D} \to \mathbb{C}$. Just as with the Schwarz–Carathéodory reflection principle, Theorem 1 readily generalizes to non–constant analytic maps $f : D \to R$, where (i) D is a domain in $\mathbb{C}$ with an open free analytic boundary arc I or (ii) D is a bordered Riemann surface with border Γ and $I \subset \Gamma$.

(e) For the special case $R = \mathbb{D}$ and $\lambda(w) = 1/(1 - |w|^2)$ Theorem 1 reduces to the Fournier–Ruscheweyh reflection principle [4, 5].

(f) The restraint in Theorem 1 that $\lambda(w)\,|dw|$ is a *complete* and *real analytic* conformal Riemannian metric can slightly be relaxed. For the 'if' part it suffices to assume $\lambda(w)\,|dw|$ is a complete conformal Riemannian metric, which is real analytic in a neighborhood $U \subset R$ of $f(I)$. For the 'only if' part we need only $\lambda(w)\,|dw|$ is real analytic in a neighborhood of $f(I)$. These assumptions cannot further be weakened.

2. Analytic continuation of Beurling–Riemann maps

In 1953 Arne Beurling [2] proved the following extension of the Riemann mapping theorem[3].

Theorem A *Let $\Phi(w)$ be a positive, continuous and bounded function defined for $|w| < \infty$ and let w_0 be a given point in the w-plane. Then there exists an analytic and univalent function $f : \mathbb{D} \to \mathbb{C}$ normalized by*

$$f(0) = w_0, \qquad f'(0) > 0, \tag{2}$$

and satisfying the non–linear boundary condition

$$\lim_{|z|\to 1} \left(|f'(z)| - \Phi(f(z))\right) = 0. \tag{3}$$

Moreover, if $\log \Phi(w)$ *is superharmonic, then there is exactly one such function.*

We call any normalized, analytic and univalent function $f : \mathbb{D} \to \mathbb{C}$ satisfying (3) a *Beurling–Riemann mapping function* (for $\Phi(w)$). Note that every Beurling–Riemann mapping function $f(z)$ is a Lipschitz map from $(\mathbb{D}, |\cdot|)$ to $(\mathbb{C}, |\cdot|)$,

$$|f(z_1) - f(z_2)| \le M \cdot |z_1 - z_2|, \qquad z_1, z_2 \in \mathbb{D},$$

with $M := \sup_{w\in\mathbb{C}} \Phi(w) < \infty$. Hence $f(z)$ has a continuous extension to $\overline{\mathbb{D}}$, and $\partial f(\mathbb{D})$ is a closed curve, which admits the conformal parametrization

$$\partial f(\mathbb{D}) : \quad f(e^{it}), \qquad 0 \le t \le 2\pi.$$

Moreover, $|f'(z)|$ has a continuous extension to $\overline{\mathbb{D}}$ with $|f'(z)| \neq 0$.

If a Beurling–Riemann mapping function can be continued analytically across an open subarc I of the unit circle, then the corresponding function $\Phi(f(z))$ will be real analytic on I since $\Phi(f(z)) = |f'(z)| > 0$ there. A partial converse is given by the following theorem, which is essentially another special case of Theorem 1.

Theorem 2 *Let $\Phi(w)$ be a positive, continuous and bounded function defined for $|w| < \infty$, let w_0 be a given point in the w-plane, and let $f(z)$ be a Beurling–Riemann mapping function for $\Phi(w)$ normalized by (2). If $\Phi(w)$ is real analytic in a neighborhood of $f(I)$ for some open subarc I of the unit circle, then $f(z)$ has an analytic continuation across I.*

In particular, if $\Phi(w)$ is real analytic in a neighborhood of $\partial f(\mathbb{D})$, then every Beurling–Riemann mapping function for $\Phi(w)$ has an analytic extension to some

[3] See [1, 4, 6] for recent generalizations of and variations on Beurling's theorem.

disk $|z| < \rho$, $\rho > 1$. Hence, at least in this special case, the analytic properties of the function $\Phi(w)$ are reflected by the analytic properties of the corresponding mapping functions.

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Schwarzians of Hyperbolically Convex Functions

G. Brock Williams

(joint work with Roger W. Barnard, Leah Cole, and Kent Pearce)

The Schwarzian derivative S_f of an analytic function $f : \Omega \to \mathbb{C}$ is given by

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

The Schwarzian itself contains a great deal of geometric information about the function f, but much more is encoded in the Schwarz norm

$$||S_f||_\Omega = \sup_{z\in\Omega}\{\eta_\Omega^{-2}(z)\,|S_f(z)|\},$$

where η_Ω is the hyperbolic density of Ω.

The Schwarz norm of f is completely Möbius invariant and is 0 if and only if f is a Möbius transformation. Thus the Schwarzian derivative provides an effective means of describing how much an analytic map differs from a Möbius transformation. For functions f defined on the unit disc $\mathbb{D}$, this also serves to describe how the range of f differs from a disc. Olli Lehto has made this notion precise, defining a pseudo-metric on the space of all simply connected proper subsets of $\mathbb{C}$ modulo Möbius transformations [1].

As a general principle, regions which are close to discs in Lehto's pseudo-metric share some of the properties of discs. Thus it is natural to ask "how far from a disc can a convex set be?" [6] For convex sets in euclidean geometry, this question was answered by Zeev Nehari who showed that if f is convex, then $||S_f||_\mathbb{D} \le 2$, with equality if and only if $f(\mathbb{D})$ is a euclidean strip [7]. Similarly, Diego Mejía and Christian Pommerenke proved that the extremal spherically convex domains are also strips [3].

In this talk, we complete the classification in all three classical geometries of the convex domains which are furthest from being a disc, by establishing the sharp upper bound on the Schwarz norm of functions from the disc onto hyperbolically convex regions. In particular, we show that the bound is attained by a map onto a domain bounded by two hyperbolic geodesics, a sort of "hyperbolic strip." This result had earlier been conjectured in several papers of Diego Mejía and Christian Pommerenke [2, 4, 5].

Our major tools are the Julia variation as extended by Roger Barnard and John Lewis, estimates on elliptic integrals, and a critical new Step Down Lemma. We formulate two new variations which preserve hyperbolic convexity. The first variation allows us to show there is an extremal domain with at most four sides. Our Step Down Lemma and the second variation then reduces the number of sides to at most two. We then directly compute the Schwarz norm for the remaining possibilities using special functions techniques.

This talk represents joint work with Roger W. Barnard, Leah Cole, and Kent Pearce.

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Metric properties of Green's functions

Vilmos Totik

Extensions of the classical Markov inequality

$$\|P_n'\|_{[-1,1]} \leq n^2 \|P_n\|_{[-1,1]}$$

(where P_n is a polynomial of degree at most n) to more general sets are closely related to smoothness of Green's functions. If E is a compact set on the plane, then the n-th Markoff constant M_n for E is defined as the smallest M_n for which

$$\|P_n'\|_E \leq M_n \|P_n\|_E.$$

Let $g_{\mathbb{C}\setminus E}$ be the Green's function of the unbounded component of $\mathbb{C}\setminus E$ with pole at infinity (we assume that E is of positive logarithmic capacity). A standard way of estimating M_n is to use the Bernstein-Walsh lemma

$$|P_n(z)| \le e^{ng_{\mathbb{C}\setminus E}(z)}\|P_n\|_E, \qquad z\in\mathbb{C}$$

and then to use the Cauchy integral formula for the derivative of P_n. This approach gives e.g. that if $g_{\mathbb{C}\setminus E}$ is Hölder continuous: $g_{\mathbb{C}\setminus E}(z)\le C\ \mathrm{dist}(z,E)^\alpha$, then $M_n \le C'n^{1/\alpha}$. Thus, smoothness of Green's function implies a growth restriction on the Markov factors M_n. The converse is not clear, and in the talk first a situation is mentioned when the connection is completely known, and this is the case of Cantor type sets.

Let $\varepsilon_1,\varepsilon_2,\ldots$ be a sequence from the interval $(0,1)$, and starting from $\mathcal{C}_0=[0,1]$ do the Cantor construction with the modification that at level n we remove the middle ε_n part of all remaining intervals. If $\mathcal{C}_n$ denotes the set after making n such steps, then $\mathcal{C}_n$ consists of 2^n intervals of total length $(1-\varepsilon_1)\cdots(1-\varepsilon_n)$. Consider the Cantor set $\mathcal{C}=\cap_n\mathcal{C}_n$. It is of measure zero if and only if $\sum_n\varepsilon_n=\infty$, and it is of positive capacity if and only if $\sum_k|\log(1-\varepsilon_k)|/2^k<\infty$ (see e.g. [5, Section V.6]). Now for Cantor sets we have (see [6], [7], [8])

- **(a):** $M_n=e^{o(n)} \iff g_{\mathbb{C}\setminus E}$ continuous $\iff \sum_j 2^{-j}\log(1-\varepsilon_j)>-\infty$,
- **(b):** $M_n=O(n^k)$ for some $k \iff g_{\mathbb{C}\setminus E}\in$ Lip α for some $\alpha>0$ $\iff \sum_{j=1}^n\log(1-\varepsilon_j)\ge -cn$,
- **(c):** $M_n=O(n^2)\iff g_{\mathbb{C}\setminus E}\in$ Lip$1/2 \iff \sum_j\varepsilon_j^2<\infty$.

Note that $M_n\ge cn^2$ and $g_{\mathbb{C}\setminus E}(-r)\ge cr^{1/2}$ for all $E\subseteq[0,1]$, i.e. the growth rates in (c) are optimal.

In the special case $\varepsilon_j=1/(j+1)$ we get a compact set $E\subset[0,1]$ of linear measure 0 such that $g_{\mathbb{C}\setminus E}\in$ Lip$1/2$ and $M_n=O(n^2)$.

As we can see, there is a big difference between the conditions on ε_j in (b) and (c). An explanation was given by V. Andrievskii [1] who proved that for $E\subset[0,1]$ the condition $g_{\mathbb{C}\setminus E}(z)\le C|z|^{1/2}$ implies that the set E is locally of full capacity at 0, i.e.

$$\lim_{t\to 0}\frac{\mathrm{cap}([0,t]\cap E)}{\mathrm{cap}([0,t])}=1.$$

Recently a characterization of optimal Hölder smoothness of Green's function was given by L. Carleson ([3]): for $E\subset[0,1]$ we have $g_{\mathbb{C}\setminus E}(z)\le C|z|^{1/2}$ if and only if $\sum_k\theta_k<\infty$, where with some $0<\varepsilon<1/3$

$$\theta_k=2^k\Big(\ \mathrm{cap}([0,2^{-k}])-\ \mathrm{cap}\big((E\cap[0,2^{-k}])\cup[0,\varepsilon 2^{-k}]\cup[(1-\varepsilon)2^{-k},2^{-k}]\big)\Big).$$

Returning to measuring density of sets with linear Lebesgue measure, T. Erdélyi, A. Kroó and J. Szabados [4] used for $E\subset[0,1]$ the function $\Theta_E(t)=|[0,t]\setminus E|$ to measure density, and they proved some local Markov inequalities in terms of this Θ_E. In [7] we used the same measure Θ_E (if E is not on $[0,1]$ then take its circular projection onto $\mathcal{R}_+$ and use the Θ function for the projected set), and proved that

$$g_{\mathbb{C}\setminus E}(z) \le C\sqrt{|z|}\exp\left(C\int_{|z|}^{1}\frac{\Theta_E^2(u)}{u^3}du\right)\log\frac{2}{\mathrm{cap}(E)},$$

and this is sharp, for if $\Theta \nearrow$, $\Theta(t) \le t$, then there is an $E \subset [0,1]$ such that $\Theta_E(t) \le \Theta(t)$ and

$$g_{\mathbb{C}\setminus E}(-r) \ge c\sqrt{r}\exp\left(c\int_{r}\frac{\Theta^2(u)}{u^3}du\right).$$

This result was extended in [2] by V. Andrievskii.

Finally, we talk about characterization of Hölder continuity with some positive exponent in the spirit of Wiener's regularity test. Let E be a compact subset on the plane such that 0 is on the boundary of the unbounded component of $\mathbb{C}\setminus E$. With

$$E^n = \left\{z \in E \,\middle|\, 2^{-n} \le |z| \le 2^{-n+1}\right\}$$

the continuity of $g_{\mathbb{C}\setminus E}$ at 0 was characterized by Wiener (see e.g. [9, Theorem III.62]): $g_{\mathbb{C}\setminus E}$ is continuous at 0 if and only if

$$\sum_{n=1}^{\infty}\frac{n}{\log(1/\,\mathrm{cap}(E^n))} = \infty.$$

For $\varepsilon > 0$ set

$$\mathcal{N}_E(\varepsilon) = \{n \in \mathbb{N} \mid \ \mathrm{cap}(E^n) \ge \varepsilon 2^{-n}\},$$

and we say that a subsequence $\mathcal{N} = \{n_1 < n_2 < \dots\}$ of the natural numbers is of positive lower density if

$$\liminf_{N\to\infty}\frac{|\mathcal{N}\cap\{0,1,\dots,N\}|}{N+1} > 0,$$

which is clearly the same condition as $n_k = O(k)$. Now (see [3]) under the cone condition (i.e. there is a cone with vertex at 0 not intersecting E) Green's function $g_{\overline{\mathbb{C}}\setminus E}$ is Hölder continuous at 0 (i.e. $g_{\mathbb{C}\setminus E}(z) \le C|z|^\alpha$ for some $\alpha > 0$) if and only if $\mathcal{N}_E(\varepsilon)$ is of positive lower density for some $\varepsilon > 0$. Here the cone condition cannot be omitted, but the rings $\{2^{-n} \le |z| \le 2^{-n+1}\}$ in the definition of E^n can be replace by the disks $\{|z| \le 2^{-n}\}$.

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Random matrices in an external source and multiple orthogonal polynomials

Arno B.J. Kuijlaars
(joint work with Pavel Bleher)

We consider the random matrix ensemble

$$\frac{1}{Z_n} e^{-n\operatorname{Tr}(V(M)-AM)} dM \tag{1}$$

defined on $n \times n$ Hermitian matrices M, where A is a given Hermitian matrix, called the external source. The ensemble is unitary invariant if $A = 0$, and then the eigenvalue correlations can be described with orthogonal polynomials. The universal behavior of local eigenvalue statistics in the large n limit can then be obtained from precise asymptotic formulae for the orthogonal polynomials. This was done in [2, 8] with the steepest descent method for Riemann-Hilbert (RH) problems.

For a general external source A the ensemble (1) is not unitary invariant. Suppose A has p distinct eigenvalues $a_1, \dots, a_p$ of multiplicity $n_1, \dots, n_p$, respectively. Then the average characteristic polynomial $P_n(z) = \mathbb{E} \det[zI - M]$ satisfies

$$\int P_n(x) x^k e^{-n(V(x)-a_j x)} dx = 0, \qquad k = 0, \dots, n_j - 1, \quad j = 1, \dots, p,$$

and these relations characterize the polynomial P_n, see [3]. The polynomials are known as multiple orthogonal polynomials of type II and they are characterized by a $(p+1) \times (p+1)$-matrix RH problem [10]. For $p = 2$, the RH problem is to find an analytic $Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{3\times 3}$ such that

- for $x \in \mathbb{R}$, we have

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & e^{-n(V(x)-a_1 x)} & e^{-n(V(x)-a_2 x)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2}$$

- as $z \to \infty$, we have

$$Y(z) = \left(I + O\left(\frac{1}{z}\right)\right) \begin{pmatrix} z^n & 0 & 0 \\ 0 & z^{-n_1} & 0 \\ 0 & 0 & z^{-n_2} \end{pmatrix}. \tag{3}$$

This RH problem has a unique solution and $Y_{11}(z) = P_n(z)$.

The m-point correlation function for the eigenvalues of (1) has determinantal form [11]

$$R_m(\lambda_1, \dots, \lambda_m) = \det\left(K_n(\lambda_i, \lambda_j)\right)_{1 \le i,j \le n}$$

with a kernel K_n built out of multiple orthogonal polynomials of type I and II, see [3]. For the case $p = 2$ the kernel can be expressed in terms of the solution of the Riemann-Hilbert problem as follows

$$K_n(x,y) = \frac{e^{-\frac{1}{2}n(V(x)+V(y))}}{2\pi i(x-y)} \begin{pmatrix} 0 & e^{na_1 y} & e^{na_2 y} \end{pmatrix} Y^{-1}(y) Y(x) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{4}$$

The expression (4) is based on a Christoffel-Darboux formula for multiple orthogonal polynomials [3, 7].

The large n limit of the Gaussian case ($V(M) = \frac{1}{2}M^2$) with 2 eigenvalues $a_1 = a$, $a_2 = -a$ of equal multiplicity exhibits a phase transition for the value $a = 1$. For $a > 1$ the eigenvalues are asymptotically distributed on two disjoint intervals $[-z_1, -z_2] \cup [z_2, z_1]$, while for $a \leq 1$ the eigenvalues accumulate on a single interval $[-z_1, z_1]$. The limiting mean eigenvalue density is given by $\rho(x) = \frac{1}{\pi}\Im|\xi(x)|$, where $\xi(x)$ satisfies the third order equation (Pastur's equation [9])

$$\xi^3 - x\xi^2 - (a^2-1)\xi + xa^2 = 0. \tag{5}$$

For $a = 1$, the density has a $|x|^{1/3}$ behavior near $x = 0$.

We establish universality of local eigenvalue correlations in the large n limit. In [4] we apply the steepest descent method to the RH problem (2), (3) with $V(x) = \frac{1}{2}x^2$, $a_1 = a$, $a_2 = -a$, $n_1 = n_2$, and we assume $a > 1$. A main tool is the Riemann surface for the equation (5) and the functions defined on it. The results are that for x_0 in the bulk,

$$\lim_{n\to\infty} \frac{1}{n\rho(x_0)} \hat{K}_n\left(x_0 + \frac{x}{n\rho(x_0)}, x_0 + \frac{y}{n\rho(x_0)}\right) = \frac{\sin \pi(x-y)}{\pi(x-y)}. \tag{6}$$

At the edge point z_1, we have for a certain $c > 0$,

$$\lim_{n\to\infty} \frac{1}{(cn)^{2/3}} \hat{K}_n\left(z_1 + \frac{x}{(cn)^{2/3}}, z_1 + \frac{y}{(cn)^{2/3}}\right) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y} \tag{7}$$

where Ai is the Airy function. Similar expressions are valid at $-z_1$ and at $\pm z_2$. The kernel $\hat{K}_n$ in (6) and (7) is a modification of K_n

$$\hat{K}_n(x,y) = e^{n(h(x)-h(y))} K_n(x,y)$$

for a certain function h, which does not affect the eigenvalue correlation functions.

For $0 < a < 1$, the steepest descent analysis of the RH problem proceeds in a different way [1], but we again find the sine kernel in the bulk and the Airy kernel at the edges. For $a = 1$, the local eigenvalue correlations near $x = 0$ are given in terms of Pearcey integrals [5, 6].

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Behaviour of kernel functions under homotopies of planar domains

Eric Schippers

The main results are 1) a variational formula for Green's function of finitely connected planar domains, and 2) the demonstration of the monotonicity of various domain functions under set inclusion. The variational formula shows that up to first order, a general homotopy behaves like the normal variation of Hadamard [5]. The consideration of general homotopies is necessary in order to obtain monotonicity of the domain functions.

The variational formula is obtained by isolating the normal part of the variation. Let Γ_{t_0} and Γ_t be parametrize one of the boundary components of domains D_t and D_{t_0} (here t is the homotopy variable). For t is close to t_0, let $n_{t_0}(t,\tau)$ be the distance from $\Gamma_{t_0}(\tau)$ to the curve Γ_t along the normal to Γ_{t_0}. Let

$$\nu_{t_0}(\tau) = \left.\frac{d}{dt}\right|_{t_0} n_{t_0}(t,\tau);$$

we then have that

$$g_t(z,\zeta) - g_{t_0}(z,\zeta) = \frac{t-t_0}{2\pi}\int_{\partial D_{t_0}} \frac{\partial g_{t_0}}{\partial n_u}(u,z)\frac{\partial g_{t_0}}{\partial n_u}(u,\zeta)\nu_{t_0}(u)ds_u + O(|t-t_0|^2)$$

where ds is arc length and n is the outward unit normal. The remainder term is harmonic and bounded on compact sets. This idea was applied in special cases by Barnard and Lewis [1].

With the use of this formula, it is quite easy to prove the monotonicity of various expressions in the derivatives of Green's function simply by differentiating the expression in the homotopy variable. More precisely, one desires theorems of the form $D_1 \subset D_2 \implies \Phi(D_1) \geq \Phi(D_2)$, where Φ is some functional depending on the domain. If one can construct a homotopy D_t between D_1 and D_2, one can apply the variational formula above to show that $\Phi(D_t)$ is monotonic. For example, for Green's function g let

$$K(\zeta,\eta) = -\frac{2}{\pi}\frac{\partial^2 g}{\partial\zeta\partial\overline{\eta}} \quad \text{and} \quad L(\zeta,\eta) = -\frac{2}{\pi}\frac{\partial^2 g}{\partial\zeta\partial\eta}.$$

These are the familiar Bergman kernel and an analogue of the Garabedian kernel for the Bergman space. The following expression decreases as the domain increases:

$$\Re\Delta\left(\sum_{\mu,\nu}\alpha_\mu\alpha_\nu\frac{\partial^{2m}L}{\partial\zeta^m\partial\eta^n}(\zeta_\mu,\zeta_\nu)\right) - \Delta\left(\sum_{\mu,\nu}\alpha_\mu\overline{\alpha}_\nu\frac{\partial^{2m}K}{\partial\zeta^m\partial\overline{\eta}^m}(\zeta_\mu,\zeta_\nu)\right) \leq 0,$$

where ζ_μ are points in the domain and $\alpha_\mu \in \mathbb{C}$ for $\mu = 1,\ldots n$. For simply connected domains this result was obtained by the author in [4]. The case $m = 0$ is due to Nehari [3]. The theorem was obtained by Bergman and Schiffer [2] in the case that $m = 0$ and the outer domain is the plane.

The original motivation of the author for constructing monotonic quantities was in order to obtain distortion theorems for bounded univalent functions. In the simply connected case Green's function can be written in terms of the mapping function and vice versa. The monotonicity theorems for domain functions in some sense are intrinsic versions of inequalities for mapping functions; by choosing D_2 to be the unit disc, and D_1 to be the image of the unit disc under a mapping function, one recovers estimates for the mapping function.

Considering expressions in higher derivatives of Green's function is a natural way to generate inequalities for higher derivatives of the mapping function. Indeed the above inequality easily leads to sharp inequalities for odd derivatives of bounded univalent functions. Inequalities for even derivatives of the mapping function seem to be more difficult. The following quantity is monotonic for all λ, points ζ_μ and parameters α_μ, β_μ, and generates inequalities for even derivatives:

$$\sum_{\mu,\nu}\alpha_\mu\overline{\alpha}_\nu\frac{\partial^{2m}K}{\partial\zeta^m\partial\overline{\eta}^m}(\zeta_\mu,\zeta_\nu) + 2\lambda\Re\left(\sum_{\mu,\nu}\beta_\mu\alpha_\nu\frac{\partial^{2m+1}L}{\partial\zeta^{m+1}\partial\eta^m}(\zeta_\mu,\zeta_\nu)\right) + \lambda^2\sum_{\mu,\nu}\beta_\mu\overline{\beta}_\nu\frac{\partial^{2m+2}K}{\partial\zeta^{m+1}\partial\overline{\eta}^{m+1}}(\zeta_\mu,\zeta_\nu).$$

Although this expression appears complicated, it is the simplest monotonic quantity in which an odd derivative of L appears. Many more such monotonic quantities can be constructed.

Some questions arise naturally. 1) For this method, it is crucial that the boundaries of the domains must be homotopic, and hence two domains must be of the same topological type in order to compare them. For which expressions is this condition necessary for monotonicity to hold? 2) Can one detect the connectivity from these domain functions?

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Zero distribution and asymptotics of Bergman orthogonal polynomials

Nikos Stylianopoulos
(joint work with Erwin Mina Diaz, Eli Levin and Ed Saff)

Let G be a bounded simply-connected domain in the complex plane $\mathbb{C}$, whose boundary $L := \partial G$ is a Jordan curve and let $\{P_n\}_{n=0}^{\infty}$ denote the sequence of Bergman polynomials of G. This is defined as the sequence

$$P_n(z) = \gamma_n z^n + \cdots, \quad \gamma_n > 0, \quad n = 0, 1, 2, \ldots,$$

of polynomials that are orthonormal with respect to the inner product

$$(f, g) := \int_G f(z)\overline{g(z)}dm(z),$$

where dm stands for the 2-dimensional Lebesgue measure.

One purpose of the talk is to report on results, obtained jointly with Eli Levin and Ed Saff in [2], concerning the asymptotic behaviour of the zeros of the Bergman polynomials $\{P_n\}$. In order to state these results we need to consider the two conformal maps associated with L. That is, with $\mathbb{D} := \{w : |w| < 1\}$, let $\Omega := \overline{\mathbb{C}}\setminus\overline{G}$ and $\Delta := \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ denote, respectively, the exterior (in $\overline{\mathbb{C}}$) of $\overline{G}$ and $\overline{\mathbb{D}}$. Then, the *exterior* conformal map Φ associated with G is the conformal map $\Phi : \Omega \to \Delta$, normalised so that

$$\Phi(z) = cz + \mathcal{O}(1), \quad z \to \infty, \quad c > 0.$$

The constant

$$\operatorname{cap} L = 1/c,$$

is called the (logarithmic) capacity of L. With $\zeta \in G$, let φ_ζ be an *interior* conformal mapping of G onto the unit disk $\mathbb{D}$, such that $\varphi_\zeta(\zeta) = 0$. Our first result characterises the asymptotic behaviour of the zeros of P_n's in terms of the analytic properties of φ_ζ, by means of two measures. Namely, the *normalised counting measure of the zeros* of P_n, denoted by ν_{P_n}, and the *equilibrium measure* for L, denoted by μ_L. With the above notations, our result can be stated as follows (see [2, Thm 2.1]):

The following two statements are equivalent:

(i) φ_ζ *has a singularity on* L.
(ii) *There is a subsequence* $\mathcal{N} \subset \mathbb{N}$ *such that*

$$\nu_{P_n} \stackrel{*}{\longrightarrow} \mu_L, \quad \text{as } n \to \infty,\ n \in \mathcal{N}.$$

Note that the fact φ_ζ has a singularity on L is independent of the choice of φ_ζ, since any two conformal mappings of G onto $\mathbb{D}$ are related by a Möbius transformation. The complimentary case where φ_ζ has no singularities on L is more complicated, and different situations may arise. Here, we consider the special case where the boundary L of G consists of two circular arcs, L_α and L_β, that meet each other at right angles at the points i and $-$ i. In this case we have the following result (see [2, Thm 3.3]):

There exists a Jordan arc Γ *joining the two vertices of* G, *and a certain measure* μ *supported on* Γ, *such that*

$$\nu_{P_n} \stackrel{*}{\longrightarrow} \mu, \quad n \to \infty.$$

This "critical arc" Γ is characterised by the property that

$$\Gamma = \{z \in \overline{G} : |\Phi(z_\alpha)| = |\Phi(z_\beta)|\},$$

where for any point z on $\overline{G}$, z_α and z_β denote, respectively, the reflections of z with respect to L_α and L_β.

Another purpose of the talk is to report on, as yet unpublished, results obtained jointly with Erwin Mina Diaz and Ed Saff. These results concern the asymptotic behaviour of the zeros of the *weighted* Bergman polynomials $\{P_{n,w}\}_{n=0}^\infty$, of lens shaped-domains G of the type studied above. These are the polynomials orthonormal with respect to the weighted inner product

$$(f,g)_w := \int_G f(z)\overline{g(z)}|w(z)|^2 dm(z),$$

where w is an entire function with finitely many zeros in $\mathbb{C}$.

Finally, we present a conjecture concerning the asymptotic behaviour of the Bergman polynomials $\{P_n\}$. More precisely, consider the following two formulas:

$$\gamma_n = \sqrt{\frac{n+1}{\pi}}\frac{1}{\operatorname{cap} L^{n+1}}\,\{1+\alpha_n\},$$

$$P_n(z) = \sqrt{\frac{n+1}{\pi}}\Phi'(z)\Phi^n(z)\ \{1+\beta_n\},\ z \in \overline{\Omega}.$$

If the boundary L of G is an *analytic* Jordan curve, then a result due to T. Carleman gives,

$$\alpha_n = \mathcal{O}(\rho^{2n}) \text{ and } \beta = \mathcal{O}(\rho^n),\ n \to \infty,$$

for some $\rho < 1$; see e.g. [1, pp. 12–13]. In the case where L is *smooth*, typically $L \in C(p+1,s)$, where $p+1 \in \mathbb{N}$ and $p+s > \frac{1}{2}$, then a result of P.K. Suetin ([3, Thms 1.1 and 1.2]) gives,

$$\alpha_n = \mathcal{O}\left(\frac{1}{n^{2(p+s)}}\right) \text{ and } \beta_n = \mathcal{O}\left(\frac{\log n}{n^{p+s}}\right),\ n \to \infty.$$

Our *conjecture*, which is based on certain theoretical results and strong numerical evidence, is concerned with boundary curves that encountered very frequently in the applications and can be stated as follows:

If L is a piecewise analytic Jordan curve without cusps, then

$$\gamma_n = \sqrt{\frac{n+1}{\pi}} \frac{1}{\operatorname{cap} L^{n+1}} \{1 + \mathcal{O}\left(\frac{1}{n^2}\right)\}, \ n \to \infty,$$

$$P_n(z) = \sqrt{\frac{n+1}{\pi}} \Phi'(z)\Phi^n(z) \ \{1 + \mathcal{O}\left(\frac{1}{n}\right)\}, \ z \in \Omega, \ n \to \infty.$$

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Asymptotics of Hermite-Padé Polynomials to the Exponential Function

Herbert Stahl

1. Abstract of the Talk

Hermite-Padé polynomials and their associated approximants are in a very natural way generalizations of Taylor polynomials, Padé approximants, and continued fractions (cf. [2], [1]). Historically, they are, perhaps, most famous for their role in Hermite's proof of the transcendency of the number e (cf. [8], [11], [12]).

Within the last 15 years a considerable up-swing of interest and research in this topic could be observed in complex and constructive approximation theory, where the field is typically connected with questions like multiple orthogonality, higher order recurrence relations, and/or the approximation of functions with branch points (cf. surveys in [14], [3], [1], [7], [17]). Many of the basic questions about the convergence of the approximants and the asymptotics of the polynomials are still open.

The talk is based on recent research about quadratic Hermite-Padé polynomials associated with the exponential function. After a somewhat broader introduction to the subject, new results about the asymptotic behavior of the polynomials have been presented. The central element of the asymptotic relations is a concrete, compact Riemann surfaces with 3 sheets over $\overline{\mathbb{C}}$. Details of its definition can be found in [18], Subsection 2.2. Specific results will be summarized further below in the present abstract. First we repeat the definition of Hermite-Padé polynomials and the associated approximants.

2. Definition of Hermite-Padé Polynomials

Let $\mathfrak{f} = (f_0, \dots, f_m)$, $m \geq 1$, be a system of $m+1$ functions; all functions are assumed to be analytic in a neighborhood of the origin.

Definition 1 Hermite-Padé Polynomials of Type I (Latin polynomials in K. Mahler's terminology in [13]): For any multi-index $n = (n_0, \dots, n_m) \in \mathbb{N}^{m+1}$ there exists a vector of polynomials $(p_0, \dots, p_m) \in \mathcal{P}^*_{n_0-1} \times \mathcal{P}_{n_1-1} \times \dots \times \mathcal{P}_{n_m-1}$ such that

$$\sum_{j=0}^{m} p_j(z) f_j(z) \;=\; O(z^{|n|-1}) \quad \text{as} \quad z \to 0, \tag{1}$$

where $|n| := n_0 + \dots + n_m$ and $\mathcal{P}^*_k := \{\, p \in \mathcal{P}_k \,|\, p \text{ monic}, p \not\equiv 0 \,\}$. The vector $(p_0, \dots,\ p_m)$ is called Hermite-Padé form of type I, and its elements are the Hermite-Padé polynomials of type I.

Definition 2 Hermite-Padé Polynomials of Type II (German polynomials in K. Mahler's terminology in [13]): For any multi-index $n = (n_0, \dots, n_m) \in \mathbb{N}^{m+1}$ there exists a vector of polynomials $(\mathfrak{p}_0, \dots, \mathfrak{p}_m) \in \mathcal{P}^*_{N_0} \times \mathcal{P}_{N_1} \times \dots \times \mathcal{P}_{N_m}$ with $N_j := |n| - n_j$, $j = 0, \dots, m$, such that

$$\mathfrak{p}_i(z) f_j(z) - \mathfrak{p}_j(z) f_i(z) \;=\; O(z^{|n|+1}) \quad \text{as} \quad z \to 0, \tag{2}$$

for $i, j = 0, \dots, m$, $i \neq j$. The vector $(\mathfrak{p}_0, \dots, \mathfrak{p}_m)$ is called Hermite-Padé form of type II, and its elements are the Hermite-Padé polynomials of type II.

The assumption $p_0 \in \mathcal{P}^*_{n_0-1}$ and $\mathfrak{p}_0 \in \mathcal{P}^*_{N_0}$ implies a normalization of the whole form $(p_0, \dots, p_m)$ and $(\mathfrak{p}_0, \dots, \mathfrak{p}_m)$, respectively. There may exist situations in which a normalization by the first component is not possible, however, one of the $m+1$ components always is appropriate for normalization.

3. Definition of Hermite-Padé Approximants

With each of the two types of Hermite-Padé polynomials a specific type of Hermite-Padé approximants is associated; these are the algebraic approximants in case of type I polynomials and the simultaneous rational approximants in case of type II polynomials. We start with the simultaneous rational approximants.

If $f_0(0) \neq 0$, then one can assume without loss of generality in Definition 2 that $f_0 \equiv 1$, and under this assumption the relations (2) reduce to

$$\mathfrak{p}_0(z) f_j(z) - \mathfrak{p}_j(z) \;=\; O(z^{|n|+1}) \quad \text{as} \quad z \to 0 \quad \text{for} \quad j = 1, \dots, m. \tag{3}$$

Defintion 3 Hermite-Padé Simultaneous Rational Approximants: For a given multi-index $n \in \mathbb{N}^{m+1}$ let $\mathfrak{p}_0, \dots, \mathfrak{p}_m$ be the Hermite-Padé polynomials of type II defined by (2) respectively (3). Then the vector of rational functions

$$\left(\frac{\mathfrak{p}_1}{\mathfrak{p}_0}(z), \dots, \frac{\mathfrak{p}_m}{\mathfrak{p}_0}(z) \right) \tag{4}$$

with common denominator polynomial $\mathfrak{p}_0$ is called (Hermite-Padé) simultaneous rational approximant to the (reduced) system of functions $\mathfrak{f}_{red} = (f_1, \ldots, f_m)$.

One immediately sees that for $m = 1$ in Definition 3 we have the Padé approximant to f_1 with numerator and denominator degrees (n_1, n_0).

As counterpart to the simultaneous rational approximants we have the algebraic Hermite-Padé approximants, which are defined with the help of polynomials of type I, but in their case the system of functions has to be an algebraic one.

Let f be a function analytic at the origin. We define the algebraic system of functions $\mathfrak{f}$ as

$$\mathfrak{f} = (f_0, \ldots, f_m) := (1, f, \ldots, f^m). \tag{5}$$

Defintion 4 Algebraic Hermite-Padé Approximants: For a given multi-index $n \in \mathbb{N}^{m+1}$ let $p_0, \ldots, p_m \in \mathcal{P}^*_{n_0-1} \times \ldots \times \mathcal{P}_{n_m-1}$ be the Hermite-Padé polynomials of type I defined by (1) with the special choice of (5). Let the algebraic function $y = y(z)$ be defined by the relation

$$\sum_{j=0}^{m} p_j(z) y(z)^j \equiv 0. \tag{6}$$

From the m branches of y we select the branch $y = y_n$ that has the highest contact to f at the origin; this branch y_n is the algebraic Hermite-Padé approximant to f associated with the multi-index n.

Again, it is immediate that for $m = 1$ Definition 4 leads to an Padé approximant, but this time with numerator and denominator degrees $(n_0 - 1, n_1 - 1)$.

4. The Special Case of the Exponential Function

In the talk we have reported about new research on asymptotics of Hermite-Padé polynomials of both types associated with systems of exponential functions. The order of the system is $m = 2$ and the multi-indices are all of the form $(n, \ldots, n) \in \mathbb{N}^{m+1}$ with $n \in \mathbb{N}$ and $n \to \infty$. Thus, we are dealing with quadratic diagonal Hermite-Padé polynomials to the system $\mathfrak{f} = (1, \exp, \exp^2)$.

After the investigations in the classical period, from where we here only mention [8], [11], [12], our specific line of research in Hermite-Padé approximants had been taken up P. B. Borwein in [4], and more or less the same problem has been studied from a point of view of special functions in [6] and [5]. In these later investigations several questions about the asymptotic distribution of the zeros of the polynomials, and especially about the asymptotic behavior of the larger zeros remained open, and these open questions have triggered our new research.

The leading idea in this new research is a rescaling of the independent variable in such a way that the zeros of the polynomials, which almost all normally diverge to infinity, now have finite asymptotic distributions.

The rescaling method was introduced by G. Szegö in [20] for the investigation of Taylor polynomials to the exponential function, and has later been taken up very successfully for the investigation of poles and zeros of Padé approximants by E.B. Saff and R.S. Varga; the results interesting here can be found in [15]

With the rescaling method it has become possible to prove asymptotic relations for quadratic Hermite-Padé polynomials. In these relations an algebraic function of third degree and the associated Riemann surface play a central role.

The new results for polynomials of type I have just been appeared in [18], very precise results about the asymptotic distributions of zeros will appear soon in [19], and results about the asymptotic behavior of polynomials of type II are in preparation.

An alternative approach to the asymptotic analysis based on a matrix Riemann-Hilbert problem has been developed by A.B.J. Kuijlaars, W. Van Assche, and F. Wielonsky in [9]. A survey of these results is contained in [10].

A generalisation of P. B. Borwein's investigations in [4] to general $m > 2$ has been done by F. Wielonsky in [21] and [22], and it has led to best results for the measure of irrationality of the number e. Investigations of quadratic Hermite-Padé approximants from a numerical point of view can be found in [16].

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Entire functions with no unbounded Fatou components

Aimo Hinkkanen

Let f be a transcendental entire function of order less than 1/2. We introduce a condition on the regularity of growth of f and show that it implies that every component of the Fatou set of f is bounded.

The Fatou set $\mathcal{F}(f)$ of f is defined to be the set of those points z in the complex plane $\mathbb{C}$ that have a neighbourhood U such that the family $\{f^n|U : n \geq 1\}$ of the restrictions of the iterates f^n of f to U is a normal family. The Julia set $\mathcal{J}(f)$ of f is $\mathcal{J}(f) = \mathbb{C} \setminus \mathcal{F}(f)$.

I.N. Baker asked in 1981 whether every component of $\mathcal{F}(f)$ is bounded if the growth of f is sufficiently small. This would then imply, in particular, that $\mathcal{F}(f)$ has no Baker domains and no completely invariant components. The best possible growth condition in terms of order would be of order 1/2, minimal type at most, as shown by the functions $f(z) = \cos\sqrt{\varepsilon z + (3\pi/2)^2}$, for $0 < \varepsilon < 3\pi$, for which $\mathcal{F}(f)$ has unbounded components. Baker proved that under this growth condition, a component D of $\mathcal{F}(f)$ is bounded except possibly if it is a wandering domain (that is, all $f^n(D)$ are contained in distinct components of $\mathcal{F}(f)$) or if D or one of its forward images is in a Baker domain cycle of length at least 2. Stallard extended Baker's result to cover Baker domain cycles of any length.

The problem remains if D is a wandering domain; one may then assume that D is simply connected for otherwise all components of $\mathcal{F}(f)$ are bounded for other reasons as shown by Baker.

A number of authors have shown that if f is a transcendental entire function of order less than 1/2 satisfying an extra condition on the regularity of growth of the maximum modulus $M(r, f)$ then all wandering domains and hence all components of $\mathcal{F}(f)$ are bounded. We prove that this conclusion holds if f has the following additional property where $m(r, f)$ denotes the minimum modulus of f: suppose that there exist positive numbers R_0, L, δ, and C with $R_0 > e$, $M(R_0, f) > e$,

$L > 1$, and $0 < \delta \leq 1$ such that for every $r > R_0$ there exists $t \in (r, r^L]$ with

$$\frac{\log m(t,f)}{\log M(r,f)} \geq L\left(1 - \frac{C}{(\log r)^\delta}\right). \tag{1}$$

One can ask whether every transcendental entire function of order less than 1/2 satisfies (1). This is still an open question. If we are not close to or inside an annulus containing very few zeros of f, it would seem plausible that the condition (1) should be easy to satisfy, with a wide margin, by taking t to be a value arising from the $\cos \pi\rho-$theorem. This is because then $\log m(t,f)/\log M(t,f)$ is greater than a fixed constant while $\log M(t,f)/\log M(r,f)$ should be quite large. So there should be a potential problem at most if we are in an annulus where f behaves like a polynomial. But in that case we should be able to take t close to r^L, and then the three numbers $\log m(t,f)$, $\log M(t,f)$, and $L \log M(r,f)$, should be close together. There may be some error term required to estimate $\log m(t,f)/(L \log M(r,f))$ from below, but (1) allows for such a term.

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Parameter Space of the Exponential Family and Infinite-Dimensional Thurston Theory

Markus Förster
(joint work with Lasse Rempe and Dierk Schleicher)

The talk deals with the investigation of the parameter space of the exponential family

$$\{E_\kappa : z \mapsto e^z + \kappa;\quad \kappa \in \mathbb{C}/2\pi i\mathbb{Z}\}.$$

For each parameter κ we consider the dynamical system generated by iteration of the function E_κ. The exponential family can be considered as a model family for transcendental dynamics in the spirit of quadratic polynomials, for every E_κ has only one singular value, the asymptotic value κ. We are interested in the set I of parameters for which the singular value is escaping, i.e. for which κ is contained in the set

$$I(E_\kappa) := \{z \in \mathbb{C} :\ |E_\kappa^{\circ n}(z)| \to \infty \quad \text{as } n \to \infty\}$$

of *escaping points.* We call such parameters *escaping parameters.* For the quadratic family $\{p_c : z \mapsto z^2 + c\}$, the set of escaping parameters can be viewed as a collection of external rays (*parameter rays*) which do or do not land on the bifurcation locus, the boundary of the Mandelbrot set. The parameter rays are the main tool of understanding the topological and bifurcation structure of the Mandelbrot set. In the case of the exponential family the bifurcation locus

$$\mathcal{B} := \{\kappa : \nexists \text{ neighborhood } U \ni \kappa \text{ s.t. } \forall \kappa' \in U \ E_\kappa \text{ and } E_{\kappa'} \text{ are conjugated}\}$$

also turns out to be the boundary of I, and I is still a disjoint union of *parameter rays*, see the theorem below.

The main idea to construct these parameter rays is to carry over structure from the dynamic plane into the parameter plane. Dierk Schleicher and Johannes Zimmer [SZ] have precisely described for any κ the set $I(E_\kappa)$ of escaping points, which consists of uncountably many *dynamic rays* $g^\kappa_{\underline{s}}(t)$ going off to $+\infty$ together with some (but not all) end points of them. This gives rise to a combinatorial description of $I(E_\kappa)$: each escaping point can be assigned a unique pair $(\underline{s}, t)$ of an integer sequence $\underline{s} \in \mathbb{Z}^{\mathbb{N}}$ (which codes the ray $g^\kappa_{\underline{s}}$ the point belongs to) and a real number $t \geq t_{\underline{s}}$ (which determines the position on the ray), where $t_{\underline{s}} \geq 0$ is independent of κ. The sequence $\underline{s} = (s_1, s_2, \dots)$ is derived from itineraries, i.e. symbolic dynamics, and the potential t indicates the speed of escape. Most importantly, the combinatorial data $(\underline{s}, t)$ gives a precise prediction of the orbit of $z = g^\kappa_{\underline{s}}(t)$: for large n we have

$$E_\kappa^{\circ n}(z) = F^{\circ n}(t) + 2\pi i s_{n+1} + O\left((F^{\circ (n+1)}(t))^{-1}\right) , \tag{1}$$

where $F(t) := e^t - 1$. Moreover, the set $X \subset \mathbb{Z}^{\mathbb{N}} \times \mathbb{R}^+_0$ of possible combinatorial pairs, endowed with the discrete topology in the first coordinate and the usual one in the second coordinate, is mapped for all κ bijectively onto $I(E_\kappa)$ by the continuous map

$$\phi_\kappa(\underline{s}, t) : X \to I(E_\kappa) \ ; \quad (\underline{s}, t) \mapsto g^\kappa_{\underline{s}}(t)$$

except if κ is an escaping parameter. We extended this result to the parameter space in the following sense.

Theorem (M. F., L. Rempe, D. Schleicher '03) *Let I be the set of* escaping parameters*:*

$$I := \{\kappa : \ \kappa \in I(E_\kappa)\} ,$$

the parameters for which the singular orbit escapes under E_κ. There is a continuous bijection $\phi : X \to I$ satisfying

$$\phi(\underline{s}, t) = \kappa \quad \Longleftrightarrow \quad g^\kappa_{\underline{s}}(t) = \kappa \ .$$

The maps $G_{\underline{s}}(t) := \phi(\underline{s}, t)$ are differentiable rays, which precisely form the path-connected components of I.

This result has been obtained by carefully estimating derivatives and winding numbers of dynamic rays ([FS1], [FRS]). Since the proof is technical and impossible to modify for any other setting, we reprove the existence and uniqueness of every combinatorial pair $(\underline{s}, t) \in X$ using *spider theory* [FS2]. The spider algorithm provides a constructive method of realizing given combinatorics and can be implemented as a computer program. It provides a much more conceptual proof which unlike the previous proof uses nothing but the asymptotics (1) of the escaping singular orbits in the dynamic plane. Spider theory is inspired by Thurston's topological characterization of rational maps [DH]. It establishes a correspondence between parameters assuming $(\underline{s}, t) \in X$ and fixed points of a certain self-mapping on Teichmüller space, which is easily described in terms of pull-backs of *spiders*. Spiders are a substantially simplified model of Teichmüller space. They have been invented by John H. Hubbard and have been used by several people in several contexts. However, this is the first time that spiders are applied to a case of infinite degree and an infinite-dimensional Teichmüller space.

The spiders constructed for this purpose are objects consisting of infinitely many feet, which model the escaping singular orbit and represent the projection into moduli space, as well as a leg attached to each foot modulo homotopy, which models the dynamic ray associated to the respective orbit point. By the asymptotic behavior (1) we have very good control of how the actual singular orbit and the dynamic rays eventually have to behave if κ assumes the prescribed combinatorics. This allows us to only consider legs and feet with rather special properties. The iterated map on the space of spiders (*spider map*) is defined by pulling back the spider along the inverse branches of E_κ as given by the entries of $\underline{s}$, where κ is the first foot.

Showing that the spider map possesses exactly one fixed point for a given pair $(\underline{s}, t) \in X$ consists of finding an invariant compact subset of spiders in order to apply the Banach fixed point theorem for the existence and a contraction argument for the uniqueness. The definition of the infinitesimal Teichmüller metric on the spider space involves the discussion of L^1-integrable meromorphic quadratic differentials, which describe the cotangent space and give rise to the dual norm on the tangent space. The push-forward of quadratic differentials turns out to be adjoint to the spider map acting on the tangent space, so that the contraction of the spider map can be understood in terms of mass loss of quadratic differentials. In order to find a compact invariant subset we carefully construct a configuration ofszi feet with definite estimates on absolute values and mutual distances as well as estimates on winding numbers of the feet.

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Growth of harmonic functions in the unit disc and an application

Igor Chyzhykov

1. Analytic and harmonic functions in the unit disc. Let $D = \{z \in \mathbb{C} : |z| < 1\}$. We denote by $A(D)$ the class of analytic function in D. For $f \in A(D)$ let $M(r, f) = \max\{|f(z)| : |z| = r\}$, $0 < r < 1$, $T(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta$.

Usually, the orders of the growth of analytic functions in D are defined as

$$\rho_M[f] = \limsup_{r\uparrow 1} \frac{\log^+ \log^+ M(r, f)}{-\log(1-r)},\ \rho_T[f] = \limsup_{r\uparrow 1} \frac{\log^+ T(r, f)}{-\log(1-r)}.$$

It is well known that $\rho_T[f] \le \rho_M[f] \le \rho_T[f] + 1$, and all cases are possible.

In 1960th M. M. Djrbashian using the Riemann-Liouville fractional integral obtained a parametric representation of the class of analytic (meromorphic) functions f in D of finite order of the growth [Chap. IX, Dj].

Here we confine by the case when $f(z)$ has no zeros and of finite order of the growth, hence $\log |f(z)|$ is harmonic.

For $\psi\colon [0, 2\pi] \to \mathbb{R}$ we define the modulus of continuity $\omega(\delta; \psi) = \sup\{|\psi(x) - \psi(y)| : |x - y| \le \delta, x, y \in [0, 2\pi]\}$, $\delta > 0$.

Following [HL, Z] we say that $\psi \in \Lambda_\gamma$ if $\omega(\delta; f) = O(\delta^\gamma)$ $(\delta \downarrow 0)$.

The fractional integral of order $\alpha > 0$ for $h\colon (0, 1) \to \mathbb{R}$ is defined by the formulas [Dj, HL]

$$D^{-\alpha}h(r) = \frac{1}{\Gamma(\alpha)} \int_0^r (r - x)^{\alpha-1} h(x)\, dx, \quad D^0 h(r) \equiv h(r).$$

Let $H(D)$ be the class of harmonic functions in D.

We put $u_\alpha(re^{i\varphi}) = r^{-\alpha} D^{-\alpha} u(re^{i\varphi})$, where the fractional integral is taken on the variable r. Let $B(r, u) = \max\{u(z) : |z| \le r\}$.

Our starting point is the following theorem

Theorem B (M. Djrbashian). *Let $u \in H(D)$, $\alpha > -1$. Then*

$$u(re^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} P_\alpha(r, \varphi - \theta)\, d\psi(\theta), \tag{2}$$

where $\psi \in BV[0, 2\pi]$,

$$P_\alpha(r, t) = \Gamma(1 + \alpha)\Big(\Re \frac{2}{(1 - re^{it})^{\alpha+1}} - 1\Big),$$

if and only if

$$\sup_{0<r<1} \int_0^{2\pi} |u_\alpha(re^{i\varphi})|\, d\varphi < M_\alpha.$$

Remark 1. Actually, for $\alpha = 0$ it is the classical result of Nevanlinna on representation of $\log|F(z)|$ when $F \in N$.

Remark 2. Note that $P_0(r,t)$ is the Poisson kernel; $P_\alpha(r,t) = D^\alpha(r^\alpha P_0(r,t))$.

Applying methods from [Dj] and [HL] (see also [Chap.7, Z]), we prove the following theorem (cf. Theorem 40 [HL]).

Theorem 1. *Let $u(z) \in H(D)$, $\alpha \geq 0$, $0 < \gamma < 1$. Then $u(z)$ has form (2) where ψ is of bounded variation on $[0, 2\pi]$, and $\psi \in \Lambda_\gamma$, if and only if*

$$B(r,u) = O((1-r)^{\gamma-\alpha-1}), \quad r \uparrow 1$$

and

$$\sup_{0<r<1} \int_0^{2\pi} |u_\alpha(re^{i\varphi})|\, d\varphi < +\infty.$$

2. An application to growth of analytic functions. For $\psi \in BV[0,2\pi]$ we denote

$$\tau[\psi] = \liminf_{\delta \downarrow 0} \frac{\log^+ \frac{1}{\omega(\delta;\psi)}}{-\log \delta} \geq 0.$$

The quantity $\tau[\psi]$ compares $\omega(\delta;\psi)$ with δ^γ as $\delta \to 0$.

Theorem 2. *Let $F \in A(D)$, and*

$$\log|F(re^{i\varphi})| = \frac{1}{2\pi} \int_0^{2\pi} P_\alpha(r, \varphi - t)\, d\psi(t),$$

where $\psi \in BV[0,2\pi]$, $\tau[\psi] = \tau \in [0,1)$. Then $\rho_M[F] = \alpha + 1 - \tau$, $\rho_T[F] \leq \alpha$. If, in addition, ψ is not absolutely continuous, then $\rho_T[F] = \alpha$.

Corollary. *Suppose that the conditions of Theorem 2 hold, and $\tau = 0$. Then $\rho_M[F] = \rho_T[F] + 1 = \alpha + 1$.*

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On conformal invariants in problems of constructive function theory

V. V. Andrievskii

This is a survey of some recent results by the author and his collaborators in the constructive theory of functions of a real variable. The results are achieved by the application of methods and techniques of modern geometric function theory and potential theory in the complex plane.

Let $E \subset \mathbb{C}$ be a compact set of positive logarithmic capacity $\mathrm{cap}(E)$ with connected complement $\Omega := \overline{\mathbb{C}} \setminus E$ with respect to $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $g_\Omega(z) = g_\Omega(z, \infty)$ be the Green function of Ω with pole at infinity, and μ_E be the equilibrium measure for the set E. The properties of g_Ω and μ_E play an important role in many problems concerning polynomial approximation of continuous functions on E and the behavior of polynomials with a known uniform norm along E.

We discuss some of these problems for the case when E is a subset of the real line $\mathbb{R}$. The main idea of our approach is to use conformal invariants such as the extremal length and module of a family of curves. The basic conformal mapping can be described as follows.

Let $E \subset [0,1]$ be a regular set such that $0 \in E$, $1 \in E$. Then $[0,1] \setminus E = \sum_{j=1}^{N} (a_j, b_j)$, where N is finite or infinite.

Denote by $\mathbb{H} := \{z : \Im(z) > 0\}$ the upper half-plane and consider the function

$$F(z) = F_E(z) := \exp\left(\int_E \log(z - \zeta)\, d\mu_E(\zeta) - \log \mathrm{cap}(E) \right), \quad z \in \mathbb{H}.$$

Using the reflection principle we can extend F to a function analytic in $\overline{\mathbb{C}} \setminus [0,1]$ by the formula

$$F(z) := \overline{F(\overline{z})}, \quad z \in \mathbb{C} \setminus \overline{\mathbb{H}}.$$

F is univalent and maps $\overline{\mathbb{C}} \setminus [0,1]$ onto a (with respect to ∞) starlike domain $\overline{\mathbb{C}} \setminus K_E$ with the following properties: $\overline{\mathbb{C}} \setminus K_E$ is symmetric with respect to the real line $\mathbb{R}$ and coincides with the exterior of the unit disk with $2N$ slits.

Note that

$$g_\Omega(z) = \log |F(z)|, \quad z \in \mathbb{C} \setminus E.$$

There is a close connection between the capacities of the compact sets K_E and E, namely

$$4\mathrm{cap}(E)\mathrm{cap}(K_E) = 1.$$

The main idea of our results is the investigation of the local properties of the Green function g_Ω, i.e., local properties of conformal mapping F.

The lecture is organized as follows. In part 1 we describe the connection between uniformly perfect subsets in $\mathbb{R}$ and John domains. It allows us to extend well-known theorem about constructive description of functions with a given majorant of their best uniform polynomial approximations to the case of C-dense compact subset of $\mathbb{R}$.

In part 2 we give sharp uniform bounds for exponentials of logarithmic potentials if the logarithmic capacity of the subset, where they are at most 1, is known.

In part 3 we give a new interpretation (and a generalization) of recent remarkable result by Totik [7] concerning the smoothness properties of g_Ω and μ_E. We also demonstrate that if for $E \subset [0,1]$ the Green function satisfies the 1/2-Hölder condition locally at the origin, then the density of E at 0, in terms of logarithmic capacity, is the same as that of the whole interval $[0,1]$.

In part 4 the Nikol'skii-Timan-Dzjadyk theorem concerning polynomial approximation of functions on the interval $[-1,1]$ is generalized to the case of approximation of functions given on a compact set on the real line.

A new necessary condition and a new sufficient condition for the approximation of the reciprocal of an entire function by reciprocals of polynomials on $[0,\infty)$ with geometric speed of convergence are provided in part 5.

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Dynamics on fractal spheres

Mario Bonk

The point of this talk was to argue the dynamics of Kleinian groups on the 2-sphere and the dynamics of a rational function under iteration lead to closely related problems in the theory of analysis on metric spaces.

We first introduce a setting for *Kleinian groups* that can be considered as the "standard picture" in this respect. Let M be a closed hyperbolic 3-orbifold, and $\Gamma = \pi_1(M)$ the fundamental group of M. The universal covering space of M is hyperbolic 3-space $\mathbb{H}^3$, the group Γ acts on $\mathbb{H}^3$ by deck transformations, and the orbifold is given by the quotient $M = \mathbb{H}^3/\Gamma$.

The action $\Gamma \curvearrowright \mathbb{H}^3$ is isometric, discrete, and cocompact. Let us call a group *standard* if it admits an action on $\mathbb{H}^3$ with these properties. The basic problem is to characterize this standard situation from the point of view of geometric group theory.

There is a well-developed theory due to Gromov of groups that resemble fundamental groups of negatively curved manifolds [Gr]. These groups are called *hyperbolic (in the sense of Gromov)* [GhHa]. If Γ is a group as above, then Γ is hyperbolic and its boundary at infinity $\partial_\infty\Gamma$ is homeomorphic to the standard

2-sphere $\mathbb{S}^2$ (abbreviated $\partial_\infty \Gamma \approx \mathbb{S}^2$). According to a conjecture by Cannon [Ca] this should characterize standard groups.

Cannon's conjecture. *Suppose G is a Gromov hyperbolic group with $\partial_\infty G \approx \mathbb{S}^2$. Then G is standard.*

In this situation it is enough to show that $\partial_\infty G$ is homeomorphic to $\mathbb{S}^2$ by a quasisymmetric homeomorphism (abbreviated $\partial_\infty G \overset{qs}{\approx} \mathbb{S}^2$). Indeed, G acts in a natural way on $\partial_\infty G$ by uniformly quasi-Möbius homeomorphisms. If $\partial_\infty G \overset{qs}{\approx} \mathbb{S}^2$, then this action $G \curvearrowright \partial_\infty G$ conjugates to an action $G \curvearrowright \mathbb{S}^2$ of G on the standard 2-sphere by uniformly quasiconformal homeomorphisms. A well-known theorem due to Sullivan [Su] and to Tukia [Tu] then implies that this action is conjugate to an action of G on $\mathbb{S}^2$ by Möbius transformations. From this it easily follows that G is standard.

We are lead to the general problem when a fractal 2-sphere such as $\partial_\infty G$ in the above situation is quasisymmetrically equivalent to the standard 2-sphere. This question was studied by B. Kleiner and myself [BK1]. As an application of our results we obtained the following partial result for Cannon's conjecture.

Theorem 1. *Suppose G is a Gromov hyperbolic group with $\partial_\infty G \approx \mathbb{S}^2$. If there exists an Ahlfors 2-regular 2-sphere Z such that $\partial_\infty G \overset{qs}{\approx} Z$, then G is standard.*

Recall that a (compact) metric space Z is called *Ahlfors Q-regular* for $Q > 0$ if the Hausdorff Q-measure of small balls $B(a, R)$ in Z behaves like R^Q up to multiplicative constants independent of the balls.

A stronger result can be obtained by using the concept of the *Ahlfors regular conformal dimension* $\dim_{AR} X$ of a metric space X. By definition this is the infimum of all numbers $Q > 0$ for which there exists an Ahlfors Q-regular space Y with $X \overset{qs}{\approx} Y$. Whenever X is the boundary of a Gromov hyperbolic group G, the set of these numbers Q is nonempty. In particular, $\dim_{AR} \partial_\infty G$ is well defined, and it is not hard to show that $\dim_{AR} \partial_\infty G$ is at least as large as the topolgical dimension of $\partial_\infty G$.

Theorem 2 [BK2]**.** *Suppose G is a Gromov hyperbolic group with $\partial_\infty G \approx \mathbb{S}^2$. If there exists an Ahlfors Q-regular 2-sphere Z such that $\partial_\infty G \overset{qs}{\approx} Z$ and $Q = \dim_{AR} \partial_\infty G$, then G is standard.*

In other words, if the infimum by which $\dim_{AR} \partial_\infty G$ is defined is attained as a minimum, then G is standard. Note that Theorem 2 contains Theorem 1, because we have $\dim_{AR} \partial_\infty G \geq 2$.

In view of these results it seems worthwhile to study the general question when the Ahlfors regular conformal dimension of a fractal 2-sphere is attained as a minimum. Interesting examples are provided by post-critically finite rational maps R on the Riemann sphere $\overline{\mathbb{C}}$. The analog of the standard picture in the Kleinian group case is given by the following setting.

Let $R\, \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ a holomorphic map of $\overline{\mathbb{C}}$ into itself, i.e., a rational function. Let Ω_R denote the set of critical points of R, and $P_R = \bigcup_{n \in \mathbb{N}} R^n(\Omega_R)$ be the set of post-critical points of R (here R^n denotes the nth iterate of R). We make the following assumptions on R:

(i) R is post-critically finite, i.e., P_R is a finite set,
(ii) R has no periodic critical point; this implies that $J_R = \overline{\mathbb{C}}$ for the Julia set of R,
(iii) the orbifold $\mathcal{O}_R$ associated with R is hyperbolic (see [DoHu] for the definition of $\mathcal{O}_R$); this implies that the dynamics of R on $J_R = \overline{\mathbb{C}}$ is expanding.

A characterization of post-critically finite rational maps is due to Thurston. The right framework is the theory of topologically holomorphic self-maps $f\ \mathbb{S}^2 \to \mathbb{S}^2$ of the sphere. By definition these maps have the local form $z \mapsto z^n$ with $n \in \mathbb{N}$ in appropriate local coordinates, and one defines the critical set, the post-critical set, and the associated orbifold similarly as for rational maps. In our context, Thurston's theorem can be stated as follows [DoHu]:

Theorem (Thurston)**.** *Let $f\ \mathbb{S}^2 \to \mathbb{S}^2$ be a post-critically finite topologically holomorphic map with hyperbolic orbifold. Then f is equivalent to a rational map R if and only if f has no "Thurston obstructions".*

Equivalence has to be understood in an appropriate sense. If f and R are both expanding, this just means conjugacy of the maps.

The definition of a Thurston obstruction is as follows. A *multicurve* $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ is a system of Jordan curves in $\mathbb{S}^2 \setminus P_f$ with the following properties: the curves have pairwise empty intersection, are pairwise non-homotopic in $\mathbb{S}^2 \setminus P_f$, and non-peripheral (this means that each of the complementary components of a curve contains at least two points in P_f). A multicurve Γ is called *f-stable* if for all j every component of $f^{-1}(\gamma_j)$ is either peripheral or homotopic in $\mathbb{S}^2 \setminus P_f$ to one of the curves γ_i.

If Γ is an f-stable multicurve, fix i and j and label by α the components $\gamma_{i,j,\alpha}$ of $f^{-1}(\gamma_j)$ homotopic to γ_i in $\mathbb{S}^2 \setminus P_f$. Then f restricted to $\gamma_{i,j,\alpha}$ has a mapping degree $d_{i,j,\alpha} \in \mathbb{N}$. Let $m_{i,j} = \sum_\alpha \frac{1}{d_{i,j,\alpha}}$ and define the *Thurston matrix* $A(\Gamma)$ of the f-stable multicurve Γ by $A(\Gamma) = (m_{ij})$. This is a matrix with nonnegative coefficients; therefore, it has a largest eigenvalue $\lambda(f, \Gamma) \geq 0$. Then Γ is a *Thurston obstruction* if $\lambda(f, \Gamma) \geq 1$.

Post-critically finite rational maps are related to *subdivision rules* [CFP]. For example, if R is a real rational map (i.e, $R(\overline{\mathbb{R}}) \subseteq \overline{\mathbb{R}}$) satisfying the above conditions (i)–(iii), then $R^{-1}(\overline{\mathbb{R}})$ is a graph providing a subdivision of the upper and lower half-planes whose combinatorics determines R (up to conjugacy by a real Möbius transformation). The combinatorics of the graphs $R^{-n}(\overline{\mathbb{R}})$ is determined by iterating the subdivisions of the upper and lower half-planes by the complementary components of $R^{-1}(\overline{\mathbb{R}})$ n-times. One can ask whether every rational map satisfying (i)–(iii) (or at least a sufficiently high iterate) is associated with a (two-tile) subdivision rule. This reduces to the following problem.

Problem. *Let R be a rational function satisfying (i)–(iii). Does there exist a quasicircle $C \subseteq \overline{\mathbb{C}}$ such that $P_R \subseteq C$ and $C \subseteq R^{-1}(C)$?*

Conversely, one can start with a (two-tile) subdivision rule of the sphere $\mathbb{S}^2$. One can associate a natural (class of) metric(s) on $\mathbb{S}^2$ associated with a subdivision rule (of appropriate type). If we denote by X the sphere $\mathbb{S}^2$ equipped with this metric, then the subdivision rule produces a topologically holomorphic expanding map $fX \to X$ which is post-critically finite. It turns out that f is conjugate to a rational function R if and only if $X \overset{qs}{\approx} \mathbb{S}^2$. So we have a situation that is very similar to the Kleinian group setting. In view of this it would be very interesting to find $\dim_{AR} X$ for these fractal spheres. In discussions with L. Geyer and K. Pilgrim we were lead to a conjecture on the Ahlfors regular conformal dimension of these spaces X. To state this conjecture let $Q \geq 2$ and Γ be an f-stable multicurve, define the modified Thurston matrix $A(\Gamma, Q)$ as $A(\Gamma, Q) = (m^Q_{i,j})$, where $m^Q_{ij} = \sum_\alpha \frac{1}{d^{Q-1}_{i,j,\alpha}}$, and let $\lambda(f, \Gamma, Q)$ be the largest nonnegative eigenvalue of $A(\Gamma, Q)$.

Conjecture. *If X comes from a subdivision rule with associated expanding map f, then $\dim_{AR} X$ is the infimum of all $Q \geq 2$ such that $\lambda(f, \Gamma, Q) < 1$ for all f-stable multicurves Γ. Moreover, $\dim_{AR} X$ is never attained unless $X \overset{qs}{\approx} \mathbb{S}^2$.*

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Inverse Source Problem in a 3-D Ball from Meromorphic approximation on 2-D Slices

L. Baratchart
(joint work with J. Leblond and E.B. Saff)

1. Notations and Preliminaries

Let $\mathbb{T}$ be the unit circle, $\mathbb{D}$ the unit disk, $\mathcal{P}_K$ the set of probability measures on a compact set K, $\mathcal{P}_n$ the space of algebraic polynomials of degree $\leq n$, H^∞ the Hardy space of the disk, $H_n^\infty = \{h/q_n;\ h \in H^\infty,\ q_n \in \mathcal{P}_n\}$ the set of meromorphic functions with n poles in $\mathbb{D}$ that are bounded near the boundary, Ω the unit ball of $\mathbb{R}^3$ and $\mathcal{S}_2$ the unit sphere.

The Green capacity of K is the nonnegative number $C_{\mathbb{T},K}$ given by

$$\frac{1}{C_{\mathbb{T},K}} = \inf_{\mu \in \mathcal{P}_K} \int\int \log\left|\frac{1-\bar{t}z}{z-t}\right| d\mu(t)d\mu(x).$$

If $C_{\mathbb{T},K} > 0$, there is a unique measure $\omega_K \in \mathcal{P}_K$ to meet the infimum, called the Green equilibrium measure on K. The measure ω_K is difficult to compute in general, but charges the endpoints if K is a system of arcs. We need the notion of extremal domain, which is specialized below to the case of a disk

Theorem [8] *Let f be holomorphic in $\overline{\mathcal{C}_\varepsilon} = \{z;\ 1-\varepsilon < |z| < 1\}$ and continuous in $\overline{\overline{\mathcal{C}_\varepsilon}}$. Set*

$\mathcal{V}_f = \{V;\ V$ connected open in $\overline{\mathbb{D}}$ with $\overline{\mathcal{C}_\varepsilon} \subset V,\ f$ extends holomorphically to $V\}$.

There is a unique $V_m \in \mathcal{V}_f$ such that $C_{\mathbb{T},\mathbb{D}\setminus V_m} = \inf_{V\in\mathcal{V}_f} C_{\mathbb{T},\mathbb{D}\setminus V}$ which contains every other member of $\mathcal{V}_f$ with this property.

We shall be concerned here with the class :

$\mathcal{BLP} \triangleq \{f$ continuous in $\overline{\mathcal{C}_\varepsilon}$, holomorphic in $\overset{\circ}{\overline{\mathcal{C}_\varepsilon}}$,can be continued analytically in $\mathbb{D}$ except for finitely many poles,branchpoints, and log singularities$\}$

For such functions, more is known on the structure of extremal domains.

Theorem [9] *If $f \in \mathcal{BLP}$, then $\overline{\mathbb{D}} \setminus V_f$ consists of its poles, its branchpoints, its* log *singularities, and finitely many analytic cuts. A cut ends up either at a branchpoint, a* log *singularity, or at an end of another cut. The diagram thus formed has no loop.*

For more than two points, $\overline{\mathbb{D}} \setminus V_f$ is a trajectory of a rational quadratic differential, but there is no easy computation. The situation is similar to that in problems of Tchebotarev-Lavrentiev type, where one must find the continuum of minimal capacity that connects prescribed groups of points [6, 7]. The difference is that, here, the connectivity is not known *a priori*.

2. Meromorphic Approximation

By a best meromorphic approximant with at most n poles of f, we mean some $g_n \in H_n^\infty$ such that :

$$\|f - g_n\|_{L^\infty(\mathbb{T})} = \inf_{g \in H_n^\infty} \|f - g\|_{L^\infty(\mathbb{T})}.$$

Clearly this notion is conformally invariant.

By the Adamjan-Arov-Krein theory [1], a best meromorphic approximant with at most n poles uniquely exists provided that $f \in C(\mathbb{T})$. Moreover, it can be computed from the singular value decomposition of the Hankel operator with symbol f.

If g_n is the sequence of best meromorphic approximants to f, whose poles are numbered as ξ_j, n for $1 \leq j \leq d_n \leq n$, we form the sequence of counting probability measures $\mu_n = \sum_j \delta_{\xi_{j,n}}/d_n$.

Theorem [2] *If $f \in \mathcal{BLP}$ is not single-valued, the counting measure μ_n of the poles of its best meromorphic approximants converges weak* to the Green equilibrium distribution of $\overline{\mathbb{D}} \setminus V_f$. Moreover, each neighborhood of a pole of f contains at least one pole of the approximant as $n \to \infty$, and only finitely many poles can remain in a compact subset of V_f.*

3. An Inverse Source Problem in 3-D.

If we are given m_1 monopolar sources $S_1, \ldots, S_{m_1}$ and m_2 dipolar sources $C_1, \ldots, C_{m_2}$ in Ω, the potential u satisfies :

$$\begin{cases} -\Delta u = F \quad \text{in } \Omega \\ \\ \dfrac{\partial u}{\partial \nu}_{|\mathcal{S}_2} = \phi \quad \text{current flux} \\ \\ u_{|\mathcal{S}_2} = g \quad \text{electric potential} \end{cases}$$

$$F = \sum_{j=1}^{m_1} \lambda_j \, \delta_{S_j} + \sum_{k=1}^{m_2} p_k \, . \, \nabla \, \delta_{C_k}$$

The inverse problem is to locate the monopolar sources S_j with their intensities λ_j and the dipolar sources C_k with their momentums p_k from the knowledge of Φ and u on $\mathcal{S}_2$. Such problems arise in Electro-Encephalography, see for instance [3, 5].

The fundamental solution is $(4\pi\|X\|)^{-1}$ so the potential assumes the form :

$$u(X) = h(X) - \sum_{j=1}^{m_1} \frac{\lambda_j}{4\pi \, \|X - S_j\|} + 3 \sum_{k=1}^{m_2} \frac{< p_k \, , \, X - C_k >}{4\pi \, \|X - C_k\|^3}$$

where h is harmonic. Using the Green formula and the expansion into spherical harmonics, one can then recover $h_{|\mathcal{S}_2}$, although we do not explain this in details here. This is just to say we can assume $h = 0$ by subtraction. We shall assume that all sources lie in general position, in the sense that none of them lies on the vertical axis $\{x = y = 0\}$.

Put : $\xi_j = x_{S_j} + iy_{S_j}$ where $S_j = (x_{S_j}, y_{S_j}, z_{S_j})^T$, $\xi_k = x_{C_k} + iy_{C_k}$ where $C_k = (x_{C_k}, y_{C_k}, z_{C_k})^T$, and let the dipolar moments be expressed in coordinates as : $p_k = (p_{k,x}, p_{k,y}, p_{k,z})$.

When we slice the ball Ω along the horizontal plane $\{z = z_p\}$, the intersection with $\mathcal{S}_s$ is a circle $\mathcal{C}_p$ of radius r_p with $r_p^2 = 1 - z_p^2$. If we let $\xi = x + iy$ be the complex variable in the plane $\{z = z_p\}$, the restriction $g_{|\mathcal{C}_p}$ is the trace on $\mathcal{C}_p$ of the function $f(\xi)$ given by

$$\frac{i}{4\pi} \times \left[-\sum_{j=1}^{m_1} \frac{\Lambda_{j,p}}{(\xi - \xi_{j,p}^-)^{1/2}} + 3 \sum_{k=1}^{m_2} \frac{R_{k,p}(\xi)}{(\xi - \xi_{k,p}^-)^{3/2}} \right]$$

where

$$Q_{l,p}(\xi) = |\xi - \xi_l|^2 + (z_p - z_l)^2 = -\frac{1}{\xi} \xi_l \, (\xi - \xi_{l,p}^-) \, (\xi - \xi_{l,p}^+), \; l = \{j, k\},$$

with

$$|\xi_{l,p}^-| < r_p, \; |\xi_{l,p}^+| > r_p, \; \xi_{l,p}^-/\xi_l \in \mathbb{R}, \; \xi_{l,p}^+/\xi_l \in \mathbb{R}$$

and where

$$\Lambda_{j,p} = \frac{\lambda_j \sqrt{\xi}}{\sqrt{\xi_j (\xi - \xi_{j,p}^+)}},$$

$$R_{k,p}(\xi) = \frac{\sqrt{\xi} \left[\tilde{p}_k \xi^2 + 2(p_{k,z} h_{p,k} - \mathrm{Re}\, \{\tilde{p}_k \xi_k\})\xi + \overline{\tilde{p}_k}\, r_p^2 \right]}{2\sqrt{\xi_k}(\xi - \xi_{j,p}^+)^{3/2}}$$

with

$$\tilde{p}_k = p_{k,x} - ip_{k,y} \text{ and } h_{p,k} = z_p - z_k.$$

Although $f(\xi)$ may not lie in $\mathcal{BLP}$, its square does. We can in principle locate the branchpoints using the convergence of poles in meromorphic approximation from the previous section. To solve the inverse problem, it remains to connect $\xi_{l,p}^-$ with the original sources :

Proposition *For f as above, each branchpoint $\xi_{j,p}^-$ or $\xi_{k,p}^-$ has maximum modulus when $z_p = z_{S_j}$ in which case they coincide with the corresponding source.*

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