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Stochastic Processes under Constraints (hybrid meeting)

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Abstract. The analysis of random processes under various constraints and conditions has been a central theme in the theory of stochastic processes, which links together several mathematical subdisciplines. The connection between potential theory and a certain type of conditioning of Markov processes via Doob's h-transform can be seen as a classical highlight. The last decades have seen further exciting and highly interesting developments which are related to the title of the workshop such as the analysis of persistence exponents for various classes of processes and various types of penalization problems. Many of these problems are rooted in questions from statistical mechanics. The workshop aims to investigate the topic stochastic processes under constraints from all these different perspectives.

Mathematics Subject Classification (2010): 60F10, 60F17, 60G10, 60G15, 60G18, 60G22, 60G50, 60G51.

Introduction by the Organizers

Topic of the workshop

Conditional distributions, which appear from constraining paths of stochastic processes, have always been one of the central topics in the probability theory and compose a very active field of research with many exciting recent developments.

The aim of the workshop consists in bringing together leading researchers from different research communities, in which constrained stochastic systems play an important role. This in particular includes topics such as persistence probabilities, quasistationary distributions and penalizations as well as parts of mathematical statistical mechanics.

Course of the workshop

18 talks have been held during this workshop which took place in a hybrid format (with both hosted and virtual participants). During coffee breaks and other free times, a virtual place allowed all participants to meet and discuss.

Persistence. Half of the 18 talks that have been held during this workshop were in relation with persistence probabilities or survival probabilities, enlighting the wide variety of behaviours and of types of results depending on the models, as well as the wide variety of methods and related questions. Exact estimates have been provided for:

- the probability of extinction at infinity of multitype continuous state branching processes (using Lamperti's representation)
- the escape probability of transient reflecting processes in a quadrant (using Laplace transform and functional equations)
- the survival probability for a d-dimensional run-tumble particle (analogous to the Sparre Andersen theorem)
- exit time of an interval for integrated self-similar Markov processes (by identifying the Mellin transform of some stopped process)

Asymptotic estimates have been obtained for other models, let us mention the study of:

- the tail distribution of the extinction time of branching processes in correlated Gaussian environment (using asymptotics of the persistence probability of fractional Brownian motion)
- the persistence rate of Gaussian stationary processes (depending on the behaviour of the spectral measure at 0)
- the persistence probability for the autoregressive and moving averages processes, furthermore the persistence rate depends on the parameter in an analytic way
- an equivalent is established for the persistence probability for weighted sums of Gaussian stationary processes

Penalization (processes under constraints).

- asymptotic behaviour of the probability that a random walk in cone is at 0 at time n (the asymptotic expansion expressed in terms of a sum of polyharmonic functions)
- Limit theorems (central, local, large deviations) for random walk (in the domain of attraction of a stable distribution) and conditioned to stay in a half plane (uses asymptotic of the Green function)
- study of random walk avoiding the convex hull of its last positions and of its initial position, almost sure convergence of its speed

- Reinforced random walks (loop erasure or by geodesic reinforcement) applied to finding geodesics in a graph in the case of series parallel graphs and of the losange graph
- study of the conditional probability of absorbed processes before the absorbing time and its convergence, at exponential rate, to the quasi-stationary distribution (based on a Lyapunov criteria); consequences to the study of measure valued Pólya processes and their almost sure convergence to the quasi-stationary distribution
- Random forests are constructed by branching random walks, applications of random forests to multiresolution analysis

Other models.

- large deviations principle for some lacunary sums (sums of uncorrelated, not independent, identically distributed random variables, obtained by trigonometric functions)
- study of finiteness of perpetual integral of Lévy process and more precisely caracterization of this property by integrals on transient sets
- study of a continuous time Derrida-Retaux model corresponding to a painting scheme on a Yule tree (given by a McKean-Vlasov type differential equation), in particular study of existence, uniqueness, asymptotic behaviour of the expectation

Workshop (hybrid meeting): Stochastic Processes under Constraints

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Abstracts

Extinction times of multitype, continuous-state branching processes

Loïc Chaumont
(joint work with Marine Marolleau)

1. Multitype continuous state branching process (MCSBP)

A MCSBP is an $\mathbb{R}_+^d$-valued Markov process satisfying,

$$E_{\mathrm{r}}(e^{-\langle\lambda,\mathrm{Z}_t\rangle}) = e^{-\langle \mathrm{r},\mathrm{u}_t(\lambda)\rangle}\,,\quad \mathrm{r},\lambda\in\mathbb{R}_+^d,$$

where $\mathrm{u}_t(\lambda)=(u_t^{(j)}(\lambda),\, j=1,\dots,d)$ satisfies,

$$\frac{\partial}{\partial t}u_t^{(j)}(\lambda)+\varphi_j(\mathrm{u}_t(\lambda))=0\,,\quad \lambda\in[0,\infty)^d\,,$$

and the branching mechanism φ_j is given by

$$\varphi_j(\lambda) = -\sum_{i=1}^d a_{i,j}\lambda_i+\frac{1}{2}q_j\lambda_j^2+\int_{(0,\infty)^d}(e^{-\langle\lambda,x\rangle}-1+\langle\lambda,x\rangle 1_{\{|\mathrm{x}|<1\}})\,\pi_j(dx)$$

$a_{i,j}\geq 0$, for $i\neq j$, $q_j\geq 0$ and π_j is a measure on $(0,\infty)^d$ such that

$$\int_{(0,\infty)^d}[(1\wedge|x|^2)+\sum_{i\neq j}(1\wedge x_i)]\pi_j(dx)<\infty\,.$$

For $j=1,\dots,d$, φ_j is the Laplace exponent of a d-dimensional Lévy process $X^{(j)}={}^t(X^{1,j},\dots,X^{d,j})$, such that

$X^{i,j}$, $i\neq j$ are subordinators and
$X^{j,j}$ is a spectrally positive Lévy process.

Theorem 1. *Any multitype branching process* Z*, issued from* $\mathrm{r}=(r_1,\dots,r_d)$ *admits the following representation,*

$$(Z_t^{(1)},\dots,Z_t^{(d)})=\mathrm{r}+\left(\sum_{j=1}^d X^{1,j}_{\int_0^t Z_s^{(j)}\,ds},\dots,\sum_{j=1}^d X^{d,j}_{\int_0^t Z_s^{(j)}\,ds}\right),$$

where the processes,

$$X^{(j)}={}^t(X^{1,j},\dots,X^{d,j})\,,\quad j=1,\dots,d,$$

are independent d-dimensional Lévy processes, such that

$X^{i,j}$, $i\neq j$ are subordinators and
$X^{j,j}$ is a spectrally positive Lévy process.

The leading process of Lamperti equation is the stochastic field:

$$\mathrm{t} = (t_1, \dots, t_d) \mapsto \mathbb{X}_\mathrm{t} = \begin{pmatrix} X^{1,1}_{t_1} & \cdots & X^{1,d}_{t_d} \\ \vdots & \vdots & \vdots \\ X^{d,1}_{t_1} & \cdots & X^{d,d}_{t_d} \end{pmatrix}$$

or equivalently the additive Lévy process:

$$\mathbf{X}_\mathrm{t} \quad = \quad \mathbb{X}_\mathrm{t} \cdot \mathbf{1}\,, \quad \mathbf{1} = (1, 1, \dots, 1) = \sum_{j=1}^d X^{i,j}_{t_j}, \quad i = 1, \dots, d$$

Then $E(e^{-\langle \lambda, \mathbf{X}_\mathrm{t} \rangle}) = e^{-\langle \mathrm{t}, \varphi(\lambda) \rangle}$, where $\varphi := (\varphi_1, \dots, \varphi_d)$. $(\mathbf{X}_\mathrm{t},\, \mathrm{t} \in [0,\infty)^d)$: spectrally positive additive Lévy field (spaLf). Then the multivariate Lamperti representation can be written as

$$\begin{aligned} (Z^{(1)}_t, \dots, Z^{(d)}_t) \quad &= \quad \mathrm{r} + \left(\sum_{j=1}^d X^{1,j}_{\int_0^t Z^{(j)}_s\, ds}, \dots, \sum_{j=1}^d X^{d,j}_{\int_0^t Z^{(j)}_s\, ds} \right)\,, \\ \mathrm{Z}_t \quad &= \quad \mathrm{r} + \mathbf{X}_{\int_0^t \mathrm{Z}_s\, ds}\,, \end{aligned}$$

where we denote $\int_0^t \mathrm{Z}_s\, ds = (\int_0^t Z^{(j)}_s\, ds, j = 1, \dots, d)$.

Theorem 2. *For each* $\mathrm{r} \in [0,\infty)^d$ *there is a multivariate random time* $\mathbf{T}_\mathrm{r} = (T^{(1)}_\mathrm{r}, \dots, T^{(d)}_\mathrm{r})$ *such that*

$$\mathbf{X}_{\mathbf{T}_\mathrm{r}} = \left(\sum_{j=1}^d X^{i,j}_{T^{(j)}_\mathrm{r}}, i = 1, \dots, d \right) = -\mathrm{r}, \quad a.s.\ on\ \{\mathbf{T}_\mathrm{r} < \infty\}$$

and $\mathbf{T}_\mathrm{r} \le \mathrm{t}$, *for all* t *such that* $\mathbf{X}_\mathrm{t} = -\mathrm{r}$.

Moreover, for each $\mathrm{r}, \mathrm{r}' \in [0,\infty)^d$,

$$\mathbf{T}_{\mathrm{r}+\mathrm{r}'} \overset{(law)}{=} \mathbf{T}_\mathrm{r} + \tilde{\mathbf{T}}_{\mathrm{r}'}\,,$$

where $\tilde{\mathbf{T}}_{\mathrm{r}'}$ *is an independent copy of* $\mathbf{T}_{\mathrm{r}'}$.

If $\mathbb{P}(\mathbf{T}_\mathrm{r} < \infty) > 0$, then there is $\phi : [0,\infty)^d \to (0,\infty)^d$ such that

$$\mathbb{E}(e^{-\langle \lambda, \mathbf{T}_\mathrm{r} \rangle}) = e^{-\langle \phi(\lambda), \mathrm{r} \rangle}, \quad \lambda \in \mathbb{R}^d_+\,.$$

Theorem 3. *The first hitting time process* $\mathbf{T}_\mathrm{r} = \inf\{\mathrm{t} : \mathbf{X}_\mathrm{t} = -\mathrm{r}\}$ *satisfies:*

1. $\mathbf{T}_\mathrm{r} < \infty$ *holds with positive probability for all* $\mathrm{r} \in [0,\infty)^d$ *if and only if*

$$(H) \qquad D := \{\lambda \in \mathbb{R}^d_+ : \varphi_j(\lambda) > 0,\, j \in [d]\} \ne \emptyset.$$

2. *Suppose that* (H) *holds, then the mapping* $\phi : (0,\infty)^d \to D$ *is a diffeomorphism whose inverse corresponds to the mapping* $\varphi = (\varphi_1, \dots, \varphi_d) : D \to (0,\infty)^d$, *that is*

$$\varphi(\phi(\lambda)) = \lambda\,, \quad \lambda \in (0,\infty)^d.$$

From $\mathbb{E}(e^{-\langle\lambda,\mathbf{T}_{\rm r}\rangle}) = e^{-\langle\phi(\lambda),{\rm r}\rangle}, \quad \lambda\in\mathbb{R}_+^d$, we derive that

$$\mathbb{P}(\mathbf{T}_{\rm r}<\infty) = e^{-\langle\phi(0),{\rm r}\rangle},$$

so that $(\mathbf{X}_{\rm t},\, {\rm t}\in[0,\infty)^d)$ hits almost surely all levels $-{\rm r}\in[0,\infty)^d$ if and only if

$$\phi(0)=0\,.$$

Kyprianou and Palau 2018 : If Z is critical or subcritical (that is $\phi(0)=0$) then $\mathbb{P}_{\rm r}\left(\lim_{t\to\infty}{\rm Z}_t=\mathbf{0}\right)=1$. More generally:

Proposition 4. *Let* Z *be a MCSBP, then for all* ${\rm r}\in\mathbb{R}_+^d$,

$$\mathbb{P}_{\rm r}\left(\lim_{t\to\infty}{\rm Z}_t=\mathbf{0}\right)\le e^{-\langle{\rm r},\phi(\mathbf{0})\rangle} \quad \textit{and}$$
$$\mathbb{P}_{\rm r}\left(\lim_{t\to\infty}{\rm Z}_t=\mathbf{0}\right)=1 \quad \textit{if and only if} \quad \phi(\mathbf{0})=\mathbf{0}.$$

If moreover $\lim_{t\to\infty}{\rm Z}_t$ *exists in* $[0,\infty]^d$ *a.s. then for all* ${\rm r}\in\mathbb{R}_+^d$,

$$\mathbb{P}_{\rm r}\left(\lim_{t\to\infty}{\rm Z}_t=\mathbf{0}\right)=e^{-\langle{\rm r},\phi(\mathbf{0})\rangle}.$$

Our goal is to distinguish between the two following types of extinction:

$$\begin{aligned}
&1) \quad \left\{\lim_{t\to\infty}{\rm Z}_t=\mathbf{0} \text{ and } {\rm Z}_t>\mathbf{0},\text{ for all } t>0\right\}\\
&\mapsto \quad \text{extinction occurs at infinity.}\\
&2) \quad \{{\rm Z}_t=\mathbf{0},\text{ for some } t>0\}\\
&\mapsto \quad \text{extinction occurs at a finite time.}
\end{aligned}$$

Call φ_{ii} the Laplace exponent of the diagonal Lévy processes $X^{i,i}$, that is

$$E(e^{-sX_t^{i,i}})=e^{-t\varphi_{ii}(s)}.$$

Then define the conditions

$$(G_i) \quad \int^\infty \frac{ds}{\varphi_{ii}(s)}<\infty.$$

Theorem 5. *Let* Z *be a MCSBP such that none of the processes* $X^{i,j}$ *for* $i,j\in[d]$, $i\neq j$ *is a compound Poisson process.*

1. *Assume that* (G_i) *is satisfied for all* $i\in[d]$. *Then* Z *can only become extinct at a finite time. Moreover,*

   $$\mathbb{P}_{\rm r}({\rm Z}_t=\mathbf{0},\textit{ for some } t>0)=e^{-\langle{\rm r},\phi(\mathbf{0})\rangle},$$

   for all ${\rm r}\in\mathbb{R}_+^d$.
2. *Assume that* (G_i) *is not satisfied for some* $i\in[d]$. *Then,* Z *cannot become extinct at a finite time. Moreover,*

   (i) *if* $\phi(\mathbf{0})=\mathbf{0}$, *then* Z *becomes extinct at infinity, almost surely, that is,*

   $$\mathbb{P}_{\rm r}\left(\lim_{t\to\infty}{\rm Z}_t=\mathbf{0}\textit{ and } {\rm Z}_t>\mathbf{0},\textit{ for all } t>0\right)=1,$$

   for all ${\rm r}\in\mathbb{R}_+^d\setminus\{\mathbf{0}\}$, (*ii*) *if* $\lim_{t\to\infty} \mathrm{Z}_t$ *exists in* $[0,\infty]^d$ *a.s., then*

$$\mathbb{P}_{\mathrm{r}}\left(\lim_{t\to\infty} \mathrm{Z}_t = \mathbf{0} \ \textit{ and } \ \mathrm{Z}_t > \mathbf{0}, \textit{ for all } t > 0\right) = e^{-\langle \mathrm{r}, \phi(\mathbf{0})\rangle},$$

for all $\mathrm{r} \in \mathbb{R}^d_+ \setminus \{\mathbf{0}\}$.

Escape probability for transient reflected processes in a quadrant

Sandro Franceschi
(joint work with Vladimir Fomichov, Jevgenijs Ivanovs)

Obliquely reflected Brownian motion in a quadrant is a famous process in probability and queueing theory as presented in the classical survey [3]. Results about escape probabilities of transient processes presented below have been recently studied in [2]. These results rely on an analytic approach initially developped for random walks in the quadrant and presented in the reference book [1].

1. Obliquely reflected Brownian motion

We consider $(Z(t))_{t\in\mathbb{R}}$ an obliquely reflected Brownian motion in the quadrant with drift $\mu = (\mu_1, \mu_2)$, starting from (u, v) and with reflection parameters $r_1 > 0$ and $r_2 > 0$, see Figure 1. Such a process behaves as a standard planar Brownian motion with drift inside the quarter plane, it reflects instantaneously on the boundary, the direction of reflection follows the vector $(r_2, 1)$ along the horizontal axis and $(1, r_1)$ along the vertical axis, the amount of time spent in the corner is zero. More precisely, the process is defined as the unique solution of the following Skorokhod problem. For some standard planar Brownian motion $W(t)$ called the free process, $Z(t)$ is defined for $t \in \mathbb{R}$ by

$$\begin{cases} Z_1(t) = u + W_1(t) + \mu_1 t + l_1(t) + r_2 l_2(t), \\ Z_2(t) = v + W_2(t) + \mu_2 t + r_1 l_1(t) + l_2(t), \end{cases}$$

where $l_1(t)$ (resp. $l_2(t)$) is a non-decreasing process which increases only when $Z_1(t) = 0$ (resp. $Z_2(t) = 0$). In fact l_1 (resp. l_2) is the local time of the process on the vertical (resp. horizontal) axis.

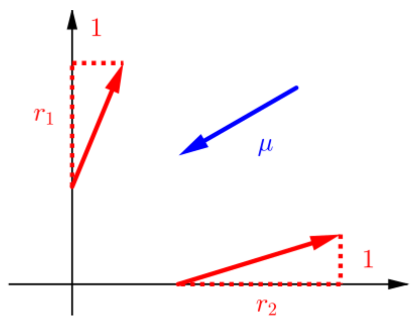

Figure 1. Drift and reflection vectors.

Proposition 1 (Recurrence and transience). *The process Z is positive recurrent if and only if a stationary distribution exists, if and only if*

$$\mu_1 - r_2\mu_2 < 0 \quad \textit{and} \quad \mu_2 - r_1\mu_1 < 0.$$

The process Z is transient if and only if $Z(t) \to \infty$ when $t \to \infty$, if and only if

$$\mu_1 - r_2\mu_2 > 0 \quad \textit{or} \quad \mu_2 - r_1\mu_1 > 0.$$

2. Escape probability

We now assume that

$$\mu_1 < 0, \quad \mu_2 < 0,$$

and that

$$\mu_1 - r_2\mu_2 > 0 \quad \underline{\text{and}} \quad \mu_2 - r_1\mu_1 > 0.$$

We are in the transient case and then the process escapes to infinity. In fact the process escapes to infinity along one of the axes, see Figure 2. We define the probability of escape along the horizontal axis

$$p_1(u,v) := \mathbb{P}_{(u,v)}(Z_1(t) \to \infty)$$

and the probability of escape along the vertical axis

$$p_2(u,v) := \mathbb{P}_{(u,v)}(Z_2(t) \to \infty).$$

These probabilities depend of the starting point (u,v) and satisfy the following properties.

Proposition 2 (Escape probabilities). *For all starting point (u,v) we have*

$$p_1(u,v) + p_2(u,v) = 1$$

and

$$\mathbb{P}(Z_1(t) \to \infty \textit{ and } Z_2(t) \to \infty) = 0.$$

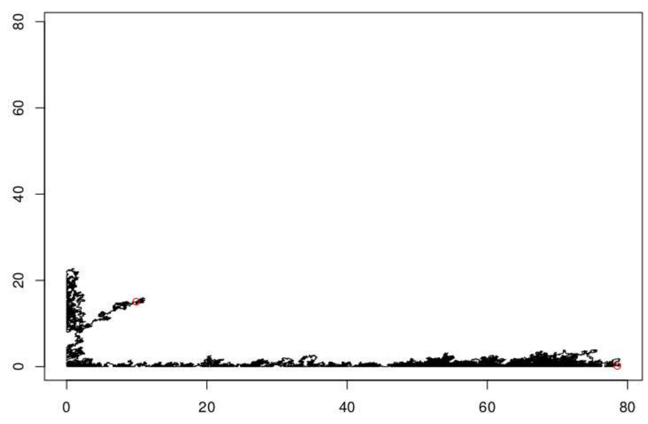

Figure 2. Sample path of reflected Brownian motion escaping along the horizontal axis.

Main Goal. The main goal is to study p_1 the escape probability along the horizontal axis.

Proposition 3 (PDE and Neumann conditions)**.** *The escape probability along the horizontal axis p_1 is characterized by the partial differential equation*

$$\Delta p_1 + \mu \cdot \nabla p_1 = 0$$

inside the quadrant with Neumann boundary condition

$$(\partial_u p_1 + r_1 \partial_v p_1)(0, v) = 0 \quad \text{and} \quad (r_2 \partial_u p_1 + \partial_v p_1)(u, 0) = 0$$

and the following limits

$$\lim_{u\to\infty} p_1(u, v) = 1 \quad \text{and} \quad \lim_{v\to\infty} p_1(u, v) = 0.$$

Sketch of proof. This partial differential equation is obtained thanks to Itô's formula. □

3. Functional equations

We now introduce the Laplace transform of the escape probability p_1

$$L(x, y) := \int_0^\infty \int_0^\infty e^{-xu-yv} p_1(u, v)\mathrm{d}u\mathrm{d}v$$

and the Laplace transforms on the boundaries

$$L_1(x) := \int_0^\infty e^{-xu} p_1(u, 0)\mathrm{d}u \quad \text{and} \quad L_2(y) := \int_0^\infty e^{-yv} p_1(0, v)\mathrm{d}v.$$

Remark 4. *Assuming that the starting point (u, v) is distributed as the product of two independent exponential laws of parameters x and y then $xyL(x, y)$ is the probability of escape along the horizontal axis.*

Let us define the kernel

$$K(x, y) := \frac{1}{2}(x^2 + y^2) + \mu_1 x + \mu_2 y,$$

which is the Laplace exponent of the free process W and the functions and the constant

$$k_1(x, y) := \mu_1 + \frac{1}{2}(y - r_2 x), \quad k_2(x, y) := \mu_2 + \frac{1}{2}(x - r_1 y), \quad c := \frac{1}{2}(r_1 + r_2).$$

Proposition 5 (Functional equation)**.** *For $x > 0$ and $y > 0$, the following functional equation holds*

$$K(x, y)L(x, y) = k_1(x, y)L_1(x) + k_2(x, y)L_2(y) + c p_1(0, 0).$$

Sketch of proof. This functional equation can be obtained performing integrations by parts of $\iint_{\mathbb{R}^2_+} (\triangle p_1 + \mu \cdot \nabla p_1)$ which is equal to zero thanks to the partial differential equation previously obtained. This equation can also be obtained by an approximation of the Poissonian case presented below. □

Proposition 6 (Escape probability starting from the corner)**.** *We have*

$$p_1(0,0) = \frac{r_1(\mu_1 - r_2\mu_2)(\mu_2 + r_2\mu_1)}{\mu_1\mu_2(1 - r_1r_2)(r_1 + r_2)}.$$

Sketch of proof. The value of $p_1(0,0)$ can be computed evaluating the functional equation in some well chosen points which cancel the kernel K and k_1 or k_2. □

Remark 7. *It is easy to remark that*

$$K(x,y)\frac{1}{xy} = k_1(x,y)\frac{1}{x} + k_2(x,y)\frac{1}{y} + c.$$

Then, defining

$$F(x,y) := L(x,y) - \frac{p_1(0,0)}{xy}, \quad F_1(x) := L_1(x) - \frac{p_1(0,0)}{x}, \quad F_2(y) := L_2(y) - \frac{p_1(0,0)}{y},$$

we obtain a simpler functional equation without constant term

$$K(x,y)F(x,y) = k_1(x,y)F_1(x) + k_2(x,y)F_2(y).$$

4. Boundary value problem

Our goal in this section is to find an explicit formula for L_1 the Laplace transform of the escape probability. To that purpose we are going to establish a boundary value problem satisfied by F_1 and solve it.

Let us introduce for algebraic functions $X^\pm$ and $Y^\pm$ which cancel the kernel, that is such that $K(X^\pm(y), y) = K(x, Y^\pm(x)) = 0$. We define

$$X^\pm(y) := -\mu_1 \pm \sqrt{\mu_1^2 - y^2 - 2\mu_2 y} \quad \text{and} \quad Y^\pm(x) := -\mu_2 \pm \sqrt{\mu_2^2 - x^2 - 2\mu_1 x}.$$

The functions $X^\pm$ and $Y^\pm$ respectively admit

$$y^\pm := -\mu_2 \pm \sqrt{\mu_2^2 + \mu_1^2} \quad \text{and} \quad x^\pm := -\mu_1 \pm \sqrt{\mu_1^2 + \mu_2^2}$$

as branch points and are respectively analytic on the cut planes $\mathbb{C} \setminus ((-\infty, y^-] \cup [y^+, \infty))$ and $\mathbb{C} \setminus ((-\infty, x^-] \cup [x^+, \infty))$.

Proposition 8 (Carleman boundary value problem)**.** *The function F_1 is analytic on the half plane $\{x \in \mathbb{C} : \Re x > -\mu_1\}$ and for all $x \in -\mu_1 + i\mathbb{R}$ the following boundary condition is satisfied*

$$F_1(\overline{x}) = G(x)F_1(x)$$

where $G(x) := \frac{k_1}{k_2}(x, Y^+(x))\frac{k_2}{k_1}(\overline{x}, Y^+(x))$.

Sketch of proof. Evaluating the functional equation of Remark 7 at the points $(X^\pm(y), y)$ we obtain two equations

$$0 = k_1(X^+(y), y)F_1(X^+(y)) + k_2(X^+(y), y)F_2(y),$$
$$0 = k_1(X^-(y), y)F_1(X^-(y)) + k_2(X^-(y), y)F_2(y).$$

Eliminating $F_2(y)$ we obtain

$$F_1(X^+(y)) = \frac{k_1}{k_2}(X^-(y), y)\frac{k_2}{k_1}(X^+(y), y)F_1(X^-(y)).$$

To conclude it is enough to remark that for $y \in (y^+, \infty)$ we have $X^\pm(y) \in -\mu_1 + i\mathbb{R}$, $X^+(y) = \overline{X^-(y)}$ and $y = Y^+(X^\pm(y))$. □

Remark 9 (Conformal gluing function)**.** *The function $w(x) := -(x + \mu_1)^2$ is analytic and bijective from the half plane $\{x \in \mathbb{C} : \Re x > -\mu_1\}$ to the cut plane $\mathbb{C} \setminus [0, \infty)$. Furthermore, for all x in the boundary of the half plane, i.e. for $x \in -\mu_1 + i\mathbb{R}$, we have*

$$w(x) = w(\overline{x}).$$

This function is said to be a conformal gluing function and is needed to solve the above boundary value problem.

Theorem 10 (Integral expression for the Laplace transform of the escape probability)**.** *For some constant C which can be explicitly computed we have*

$$L_1(x) = \frac{p_1(0,0)}{x} + C \exp\left(\frac{1}{2i\pi}\int_{-\mu_1+i\mathbb{R}} G(t)\frac{w'(t)}{w(t) - w(x)}\mathrm{d}t\right).$$

Sketch of proof. This formula is obtained solving the above boundary value problem in a classical way using Sokhotski-Plemelj formula. □

Remark 11 (Correlated Brownian motion and complex hyperbola)**.** *These results generalize to Brownian motion with non-identity covariance matrix* [2]*. In this case, the complex integral is not made on a half line but on a hyperbola. Then the conformal gluing function w is more complicated and can be expressed in terms of generalized Tchebychev polynomials thanks to hypergeometric functions.*

5. Compound Poisson model

Escape probability for obliquely reflected compound Poisson process with negative jumps has also been studied in [2] using the same analytic method. Let N_1, N_2 be independent Poisson processes with rates $\lambda_1, \lambda_2 > 0$ respectively, and let $J_k^{(1)}, J_k^{(2)}$, $k \geq 1$, be independent standard exponential random variables that are also independent of N_1, N_2. Consider that the free process is now a drifted compound Poisson process $X = (X_1, X_2)$ given by

$$X_i(t) = c_i t - \frac{1}{q_i}\sum_{k=1}^{N_i(t)} J_k^{(i)}, \quad i = 1, 2,$$

where $c_i, q_i > 0$ are fixed parameters. Note that q_1, q_2 are the rate parameters of the individual exponential jumps. The case of common jumps (shocks) can also be studied. The effective drift is now $(\mu_1, \mu_2) = (c_1 - \lambda_1/q_1, c_2 - \lambda_2/q_2)$. We assume the same transience conditions stated in section 2 and consider L, L_1 and

L_2 the Laplace transforms of the escape probability. The kernel is now equal to the Laplace exponent of X and is given by

$$K(x,y) := xc_1 + yc_2 - (\lambda_1 + \lambda_2) + \frac{\lambda_1}{1 + x/q_1} + \frac{\lambda_2}{1 + y/q_2},$$

and we define

$$k_1(x,y) := c_2 - \frac{\lambda_2 q_2}{(q_2 + y)(q_2 - r_2 x)}, \quad k_2(x,y) := c_1 - \frac{\lambda_1 q_1}{(q_1 + x)(q_1 - r_1 y)},$$
$$L_0 := c_2 L_1 \left(q_2/r_2\right) + c_1 L_2 \left(q_1/r_1\right).$$

Thanks to Markov property we obtain the following functional equation.

Proposition 12 (Poissonian functional equation)**.** *We have*

$$K(x,y)L(x,y) = k_1(x,y)\left[L_1(x) - L_1\left(q_2/r_2\right)\right] + k_2(x,y)\left[L_2(y) - L_2\left(q_1/r_1\right)\right] + L_0.$$

Remark 13 (An important circle)**.** *This equation leads to a boundary value problem on a circle. Its resolution gives a complex integral formula on a circle for* L_1.

Remark 14 (Random walks in the quadrant)**.** *The same escape phenomenon can also be observed for random walks in the quadrant and the analytic approach could also be applied.*

References

[1] G. Fayolle, R. Iasnogorodski, V. Malyshev, *Random walks in the quarter-plane: Algebraic methods, boundary value problems, applications to queueing systems and analytic combinatorics*, Springer Publishing Company, Incorporated, 2nd edition, (2017).

[2] V. Fomichov, S. Franceschi, J. Ivanovs, *Probability of total domination for transient reflecting processes in a quadrant*, (2020), arXiv:2006.11826.

[3] R. J. Williams, *Semimartingale reflecting Brownian motions in the orthant*, Stochastic networks, **71** (1995), 125–137.

Universal survival probability for a d-dimensional run-and-tumble particle

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(joint work with F. Mori, P. Le Doussal)

We consider an active run-and-tumble particle (RTP) in d dimensions. The RTP process is defined as follows. A particle starts at the origin of a d-dimensional space. At $t = 0$, it chooses a random direction (uniformly), a random speed v from an aribitrary distribution $W(v)$ with positive support, and a random time of flight τ from an exponential distribution $p(\tau) = \gamma\, e^{-\gamma\, \tau}$ where γ is called the tumbling rate. It then moves ballistically in the chosen direction with the chosen speed v during the random run-time τ. At the end of this period, it instantaneously 'tumbles', i.e., chooses a new direction, a new spped from the same distribution $W(v)$ and a new run-time τ from the same exponential run-time distribution $p(\tau)$. The process continues up to a total time t.

We compute exactly the persistence probability $S(t)$ that the x-component of the position of the RTP does not change sign up to time t. We show that $S(t)$ is independent of the spatial dimension d and the speed distribution $W(v)$ for any finite time t (and not just for large t), as a consequence of the celebrated Sparre Andersen theorem for discrete-time random walks in one dimension. We further demonstrate, as a consequence, the universality of the distribution of $t_{\max}$ denoting the time of the occurrence of the maximum of the x-component, as well as the distribution of the number of lower records up to time t for this RTP process. We then generalise our method to two other variants of the RTP model where a finite waiting time during 'tumbling' is considered. The results vary from model to model, but in each model, the results are shown to be universal, i.e., independent of d and $W(v)$.

References

[1] F. Mori, P. Le Doussal, S. N. Majumdar, and G. Schehr, *Universal Survival Probability for a d-Dimensional Run-and-Tumble Particle*, Phys. Rev. Lett. **124** 090603 (2020)

[2] F. Mori, P. Le Doussal, S. N. Majumdar, and G. Schehr, *Universal Properties of a Run-and-Tumble Particle in Arbitrary Dimension*, arxiv: 2006.06989 (to appear in Phys. Rev. E (2020))

Random walks and branching processes in correlated Gaussian environment: persistence and applications

Alexis Devulder

(joint work with Frank Aurzada, Nadine Guillotin-Plantard and Françoise Pène)

We use persistence results for discrete fractional Brownian motion, and some more general gaussian processes, to estimate the probability that these processes cross some (large) $-x$ before y satisfying some technical conditions. We then apply this to obtain the annealed persistence for Random Walks in Random Environments in such potentials. Finally, applying this last result and some other technics, we get asymptotics of the annealed tail distribution of the extinction time, the total population size, and the maximum population of some critical Branching Processes in (correlated) Random Environments. That is, we apply persistence results to prove other persistence properties, but also to prove some other (non persistence) results.

More precisely, we consider a nearest-neighbor *random walk in random environment* (RWRE) $S = (S_n)_{n\geq 0}$ in $\mathbb{Z}$, defined as follows. Let $\omega := (\omega_i)_{i\in\mathbb{Z}}$ be a stationary sequence of random variables with values in $(0,1)$ defined on a probability space $(\Omega, \mathcal{F}, P)$. A realization of ω is called an *environment*. Conditionally on ω, $S := (S_n)_{n\geq 0}$ is a Markov chain such that $P_\omega[S_0 = 0] = 1$ and for every $n \in \mathbb{N}$, $k \in \mathbb{Z}$ and $i \in \mathbb{Z}$,

$$P_\omega[S_{n+1} = k | S_n = i] = \begin{cases} \omega_i & \text{if } k = i+1, \\ 1-\omega_i & \text{if } k = i-1, \\ 0 & \text{otherwise.} \end{cases}$$

This defines the *quenched law* P_ω. We also define the *annealed law* by

$$\mathbb{P}[\cdot] := \int P_\omega[\cdot] \, \mathrm{d}P(\omega).$$

It is worth noting that $(S_n)_{n\in\mathbb{N}}$ is not Markovian under $\mathbb{P}$. This model has many applications in physics and in biology. The case when $(\omega_i)_i$ is a sequence of independent identically distributed random variables has been widely studied since the seminal works by Solomon [10], and by Sinai [8].

In the present paper, we consider instead a correlated context that has been introduced in statistical physics (see Oshanin, Rosso and Schehr [7]), for which very few results are known. Before defining our setup more precisely, we introduce the *potential* $V = (V(k),\ k \in \mathbb{Z})$. It is defined as follows:

$$X_i := \log((1-\omega_i)/\omega_i), \qquad V(0) := 0, \qquad V(k+1) := V(k) + X_{k+1}$$

for every $i \in \mathbb{Z}$ and $k \in \mathbb{Z}$.

In order to remain concise, we consider here only the particular case in which the potential V is a discrete two-sided fractional Brownian motion (FBM), i.e. $(V(k),\ k \in \mathbb{Z}) = (B_H(k),\ k \in \mathbb{Z})$, with Hurst parameter $H \in [\frac{1}{2}, 1)$, where B_H is a centered real Gaussian process such that $B_H(0) = 0$ with covariance function given by

$$E\big[B_H(t)B_H(s)\big] = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right), \qquad s \in \mathbb{R},\ t \in \mathbb{R}.$$

We stress that all the results presented here have, in [3], more general (although sometimes slightly less precise) versions for more general correlated Gaussian environments.

Our first main result is the following.

Theorem 1 (particular case of [3], Theorem 1)**.** *Let $H \in [\frac{1}{2}, 1)$. If $V = B_H$ on $\mathbb{Z}$, then there exist a $c > 0$ and an $N_0 \in \mathbb{N}$ such that, for every $N \geq N_0$,*

$$(\log N)^{-\left(\frac{1-H}{H}\right)} e^{-c\sqrt{\log\log N}} \leq \mathbb{P}\Big[\min_{k=1,\dots,N} S_k > -1\Big] \leq (\log N)^{-\left(\frac{1-H}{H}\right)} (\log\log N)^c.$$

Remark. In many cases with physical relevance, the persistence probability behaves asymptotically as $N^{-\theta+o(1)}$ when $N \to +\infty$, with a $\theta > 0$ called the *persistence exponent* or the *survival exponent*. However, in the previous theorem and in other persistence results in random environment [1], [5] and [12], the persistence probability is a power of $\log N$ instead of a power of N. In particular, the persistence exponent of our RWRE $(S_n)_n$ is $\frac{1-H}{H}$.

We now consider Branching Processes in Random Environment (BPRE). They are an important generalization of the Galton-Watson process where the reproduction law depends on a random environment indexed by time. This model was first introduced by Smith and Wilkinson [9]. In most studies, the reproduction laws are supposed to be independent and identically distributed, and they are often assumed to be geometrical laws.

It is however also natural to consider BPRE for which the reproduction laws of the different generations are correlated. Such BPRE with correlations have been widely studied in biology (see e.g. Haccou and Vatutin [6]), but there are very few results about them in mathematics.

Our goal is to study some important quantities of such branching processes, such as the annealed tail distribution of their total population size, of their maximum population, and of their extinction time. To this aim, we use a correspondence between recurrent random walks in random environment and critical branching processes in random environment with geometric distribution of offspring sizes (see e.g. Afanasyev [1], and [3] Section 1.2).

Given ω, let $(O_{n,k})_{n\geq 0,k\geq 1}$ be a family of independent random variables and $(Z_n)_{n\in\mathbb{N}}$ be such that $Z_0 = 1$

$$(1) \quad Z_{n+1} := \sum_{k=1}^{Z_n} O_{n,k}, \quad P_\omega(O_{n,k} = N) = (1-\omega_n)\omega_n^N, \quad (k,n,N) \in \mathbb{N}^* \times \mathbb{N}^2.$$

The process $Z := (Z_n)_{n\in\mathbb{N}}$ is a BPRE, and the number of offsprings $O_{n,k}$ of the k-th particle of generation n (of the BPRE Z) is, under P_ω, a geometric random variable on $\mathbb{N}$ with mean e^{-X_n}. So the BPRE Z is critical, and in particular there is almost surely extinction of this BPRE (see e.g. Tanny [11], eq. (2)).

Thanks to the previously cited correspondance between some RWRE and BPRE, Theorem 1 leads to the following result.

Corollary 2 (Total population size; particular case of [3], Corollary 1.1). *Let $H \in [\frac{1}{2}, 1)$. If $V = B_H$ on $\mathbb{Z}$, then there exist a $c > 0$ and an $N_0 \in \mathbb{N}$ such that, for every $N \geq N_0$,*

$$(\log N)^{-\left(\frac{1-H}{H}\right)} e^{-c\sqrt{\log\log N}} \leq \mathbb{P}\left[\sum_{j=0}^{\infty} Z_j > N\right] \leq (\log N)^{-\left(\frac{1-H}{H}\right)} (\log\log N)^c.$$

Let $\mathcal{T} := \inf\{n \geq 1;\, Z_n = 0\}$ be the extinction time of the BPRE Z. Our second main result deals with the (annealed) survival probability $\mathbb{P}[\mathcal{T} > N]$ of the BPRE Z.

Theorem 3 (Extinction time, particular case of [3], Theorem 2). *Under the assumptions of Theorem 1, there exist $c > 0$, $C > 0$ and $N_0 \in \mathbb{N}$ such that, for every $N \geq N_0$,*

$$(2) \qquad N^{-(1-H)} (\log N)^{-c} \leq \mathbb{P}[\mathcal{T} > N] \leq C N^{-(1-H)}.$$

An easy consequence of the previous results is the following estimate on the maximum population size $\sup_{j\geq 0} Z_j$ of the BPRE Z before its extinction.

Corollary 4 (Maximum population size of BPRE, particular case of [3], Corollary 1.2). *If $H \in [\frac{1}{2}, 1)$ and $V = B_H$ on $\mathbb{Z}$, then there exist a $c > 0$ and an*

$N_0 \in \mathbb{N}$ *such that, for every* $N \geq N_0$,

$$(\log N)^{-\left(\frac{1-H}{H}\right)} e^{-c\sqrt{\log\log N}} \leq \mathbb{P}\left[\sup_{j\geq 0} Z_j > N\right] \leq (\log N)^{-\left(\frac{1-H}{H}\right)} (\log\log N)^c.$$

Finally, the following Theorem 5 may be of independent interest, since it controls the probability that a (discrete) FBM first hits a large negative level before hitting a (small) positive level. We set

$$\widetilde{T}(x) := \begin{cases} \inf\{k \in \mathbb{N};\ B_H(k) \geq x\} & \text{if} \quad x > 0, \\ \inf\{k \in \mathbb{N};\ B_H(k) \leq x\} & \text{if} \quad x < 0. \end{cases}$$

In the following theorem, which is useful to prove our Theorem 1, we estimate the probability that the discrete FBM $(B_H(k))_{k\in\mathbb{N}}$ hits $-x$ before y, for $y > 0$ and large positive x satisfying some technical conditions. It is a consequence of persistence results for FBM [4].

Theorem 5. *Recall that* $H \in [\frac{1}{2}, 1)$. *Let* $\alpha > 1$. *There exist a* $c = c(\alpha) > 0$ *and an* $x_\alpha > 0$ *such that for any* $y > e$ *and any* $x > \max(y, x_\alpha)$ *such that* $\log x \leq [\log(x/y)]^\alpha$, *we have*

$$(x/y)^{-(1-H)/H} \big[\log(x/y)\big]^{-c} \leq P\left[\widetilde{T}(-x) < \widetilde{T}(y)\right] \leq c(x/y)^{-(1-H)/H} \big[\log x\big]^c.$$

It is well known that more precise results can be obtained with martingale techniques when $H = 1/2$, however these methods fail when $H \neq 1/2$.

References

[1] Afanasyev, V.I. On the maximum of a critical branching process in a random environment. *Discrete Math. Appl.* **9** (3) (1999), 267–284.

[2] Athreya, K. B. and Karlin, S. On branching processes with random environments: I: Extinction probabilities. *Ann. Math. Stat.* **42** (1971), 1499-1520.

[3] Aurzada, F., Devulder A., Guillotin-Plantard, N. and Pène F. Random walks and branching processes in correlated Gaussian environment. *Journal of Statistical Physics* **166**, no 1 (2017), 1–23.

[4] Aurzada, F., Guillotin-Plantard, N. and Pène, F. Persistence probabilities for stationary increments processes. arXiv:1606.00236 (2016).

[5] Devulder, A. Persistence of some additive functionals of Sinaï's walk. *Ann. Inst. Henri Poincaré Probab. Stat.* 52 (2016), 1076–1105.

[6] Haccou, P. and Vatutin, V. Establishment success and extinction risk in autocorrelated environments. *Theoretical Population Biology* **64** (2003), 303–314.

[7] Oshanin, G.; Rosso, A.; and Schehr, G. Anomalous Fluctuations of Currents in Sinai-Type Random Chains with Strongly Correlated Disorder. *Phys. Rev. Lett.* **110** (2013), 100602.

[8] Sinai, Ya. G. The limiting behavior of a one-dimensional random walk in a random medium. *Th. Probab. Appl.* **27** (1982), 256–268.

[9] Smith, W. L. and Wilkinson, W. E. On branching processes in random environments. *Ann. Math. Stat.* **40** (1969), 814–827.

[10] Solomon, F. Random walks in a random environment. *Ann. Probab.* **3** (1975), 1–31.

[11] Tanny, D. A necessary and sufficient condition for a branching process in a random environment to grow like the product of its means. *Stochastic Process. Appl.* **28** (1988), 123–139.

[12] Vatutin, V. A. Total population size in critical branching processes in a random environment. *Math. Notes* **91** (2012), 12–21.

Persistence and exit times for some integrated self-similar Markov processes

Christophe Profeta

For $b \in \mathbb{R}$ and X a real-valued process starting from $x < b$, the persistence problem consists in studying the asymptotics of

$$\mathbb{P}_x\left(\sup_{s\leq t} X_s < b\right) \qquad \text{as } \; t \to +\infty,$$

or equivalently, defining $T_b = \inf\{t > 0,\ X_t \geq b\}$,

$$\mathbb{P}_x(T_b > t) \qquad \text{as } \; t \to +\infty.$$

This is a classic problem which has been studied for many families of processes, and we shall give here several results when X is the integral of either a stable Lévy process, or a skew Bessel process. As a consequence of our computations, we shall also solve the exit time problem from an interval in the skew Bessel process case, and compute the area under a normalized excursion in the (spectrally positive) stable case.

1. A general approach

We start by giving a heuristic method for computing the asymptotics of the persistence probability when X is given by the integral of an H-self-similar Markov process Y taking values in $\mathbb{R}$. The idea is the following : as Y is self-similar and Markovian, one may expect to observe a kind of asymptotic independence given by

$$(1) \qquad \mathbb{P}_x(T_b > t) \underset{t\to+\infty}{=} t^{-\theta+o(1)} \qquad \Longleftrightarrow \qquad \mathbb{P}_x(Y_{T_b} > z) \underset{z\to+\infty}{=} z^{-\theta/H+o(1)}$$

where θ is a positive constant, which does not depend on x and b. The key is then to observe that the Mellin transform of Y_{T_b} only involves the distribution of X_1, and not of the pair (X_1, Y_1). Therefore, when the law of X_1 is known, one may explicitly compute the asymptotics of the survival function Y_{T_b}. This is for instance the case for stable Lévy processes, and one can then show that the equivalence (1) holds true, see [6] and [2].

2. The case of skew Bessel processes

We next apply the previous methodology to the case

$$Y_t = |R_t|^\gamma \left(\mathbf{1}_{\{R_t\geq 0\}} - c\mathbf{1}_{\{R_t<0\}}\right), \qquad\qquad t \geq 0,$$

where $\gamma, c > 0$ and R is a skew Bessel process of dimension $\delta \in (1, 2)$ and skewness parameter $\eta \in (-1, 1)$. Unfortunately, the law of X_1 is unknown in this situation, even in the Brownian case. To circumvent this problem, the idea is to look at a slightly larger stopping time, i.e.

$$\zeta_b = \inf\{t > 0,\ X_t \geq b \text{ and } Y_t = 0\}.$$

For this stopping time, we expect that $\mathbb{P}(\zeta_b > t) \simeq \mathbb{P}(T_b > t)$ as the time it takes for a Bessel process to hit zero should be negligible with respect to the time it has taken for its integral to reach the level b. In other words, instead of computing the law of Y_{T_b} we shall rather look for that of X_{ζ_b}. It turns out that this distribution has a simple expression, which allows to obtain the following asymptotics.

Theorem 1. *There exists a constant $\kappa > 0$ independent from $(X_0, Y_0) = (x, y)$ such that*

$$\mathbb{P}_{(x,y)}\left(T_b > t\right) \sim \kappa\, h(b-x, y)\, t^{-\theta}, \qquad t \to +\infty,$$

where

$$\theta = \frac{2+\gamma}{2\pi} Arctan\left(\frac{\sin(\nu\pi)}{c^{\nu}\frac{1-\eta}{1+\eta} + \cos(\nu\pi)}\right) \qquad \text{with} \qquad \nu = \frac{2-\delta}{2+\gamma},$$

and where h admits the representation, with $\sigma_0 = \inf\{t > 0,\ Y_t = 0\}$,

$$h(x,y) = \begin{cases} \mathbb{E}_{(0,y)}\left[(x - X_{\sigma_0})_+^{\frac{2\theta}{2+\gamma}}\right] & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Since Y is a continuous process, the previous computations may be adapted to solve the exit time problem from an interval $[a, b]$ with $a < x < b$.

Theorem 2. *The probability that X reaches the level b before the level a is given by*

$$\mathbb{P}_x\left(T_b < T_a\right) = \frac{\Gamma(\nu)}{\Gamma(1+\alpha)\Gamma(\beta)}\left(\frac{x-a}{b-a}\right)^{\alpha} {}_2F_1\left[\begin{matrix}\alpha & 1-\beta \\ & 1+\alpha\end{matrix}; \frac{x-a}{b-a}\right].$$

where α and β are explicit constants related to θ.

3. Some spectrally positive areas

We have so far computed several distributions related to X and Y taken at some stopping times. Another related problem is the study of the area under a normalized excursion of Y. When Y is a Brownian motion, this distribution is nowadays referred as Airy distribution and has been the subject of numerous studies, generally relying on the Feynman-Kac formula, see for instance Janson [3] or Majumdar & Comtet [5]. The case of Bessel processes has also been investigated in the physics literature recently, using similar methods.

We show here how our previous computations allow to compute the area $\mathcal{A}_{ex}$ under the normalized excursion of a spectrally positive α-stable process L. The idea is to observe that this distribution is closely related to the distribution of the pair (σ_0, X_{σ_0}) where as before $\sigma_0 = \inf\{t > 0,\ L_t = 0\}$ and $X = \int L$. By computing explicitly the Laplace transform of this pair, we deduce the following result :

Theorem 3. *The double Laplace transform of $\mathcal{A}_{ex}$ is given by :*

$$\int_0^{+\infty} (1 - e^{-\lambda t})\mathbb{E}\left[e^{-t^{1+\frac{1}{\alpha}}\mathcal{A}_{ex}}\right] t^{-(1+\frac{1}{\alpha})} dt = \alpha\Gamma\left(1 - \frac{1}{\alpha}\right)\left(\frac{\Phi'_\alpha(0)}{\Phi_\alpha(0)} - \frac{\Phi'_\alpha(\lambda)}{\Phi_\alpha(\lambda)}\right).$$

where the M-Wright function Φ_α is defined by

$$\Phi_\alpha(x) = \frac{1}{\pi}\int_0^{+\infty} e^{-\sin(\frac{\pi\alpha}{2})\frac{z^{1+\alpha}}{1+\alpha}} \cos\left(\cos\left(\frac{\pi\alpha}{2}\right)\frac{z^{1+\alpha}}{1+\alpha} - zx\right) dz.$$

Furthermore :

$$\left(\mathbb{E}[\mathcal{A}_{ex}^n]\right)^{\frac{1}{n}} \underset{n\to+\infty}{\asymp} n^{1-\frac{1}{\alpha}} \qquad \textit{and} \qquad \ln \mathbb{P}(\mathcal{A}_{ex} > x) \underset{x\to+\infty}{\asymp} -x^{\frac{\alpha}{\alpha-1}}.$$

Other areas may be computed similarly, such as the area under the meander of L, or the area under L conditioned to stay positive for instance.

Open questions. These results yield several related problems :

(1) Can we study the persistence problem for the integral of a general self-similar Markov process ?
$\Longrightarrow$ Maybe using the Lamperti representation, see [1]
(2) Can we compute the asymptotics of the exit time, i.e. $\mathbb{P}_x(T_a \wedge T_b > t)$ as $t \to +\infty$?
$\Longrightarrow$ In the Brownian case, the existence of a limit has been proven in [4].
(3) Can we solve the same exit time problem for the integral of a stable Lévy process ?
(4) Can we compute the area under a general stable excursion ?

References

[1] L. Chaumont, H. Pantí & V. Rivero. The Lamperti representation of real-valued self-similar Markov processes. *Bernoulli*, **19** (5B), 2494–2523, 2013.

[2] J.-F. Jabir and C. Profeta. A stable Langevin model with diffusive-reflective boundary conditions. *Stochastic Process. Appl.*, Vol. **129** (11), 4269–4293, 2019.

[3] S. Janson. Brownian excursion area, Wright's constants in graph enumeration, and other Brownian areas. *Probab. Surv.* **4**, 80–145, 2007.

[4] A. Lachal and T. Simon. Chung's law for homogeneous Brownian functionals. *Rocky Mountain J. Math.*, **40** (2), 561–579, 2010.

[5] S. N. Majumdar and A. Comtet. Airy distribution function: from the area under a Brownian excursion to the maximal height of fluctuating interfaces. *J. Stat. Phys.* **119**, no. 3-4, 777–826, 2005.

[6] C. Profeta and T. Simon. Persistence of integrated stable processes. *Probab. Theory Relat. Fields.* **162** (3), 463–485, 2015.

[7] C. Profeta. Persistence and exit times for some additive functionals of skew Bessel processes. *J. Theor. Probab.* https ://doi.org/10.1007/s10959-019-00966-1, 2019.

[8] C. Profeta. The area under a spectrally positive stable excursion and other related processes. *Preprint.* https://hal.archives-ouvertes.fr/hal-02928052, 2020.

Persistence of Gaussian Stationary Processes

Naomi Feldheim
(joint work with Ohad Feldheim, Sumit Mukherjee, Shahaf Nitzan)

The persistence of a stochastic process f above a certain level ℓ, that is, the probability that $f(t) > \ell$ for all t in some large interval, is a classical topic of study (see the recent surveys in mathematics [2] and physics [3]). Here we investigate the persistence probability for the class of Gaussian stationary processes (GSP's) above a fixed level (usually, above the mean). This quantity has been extensively studied since the 1950's, by Slepian [13], Newell-Rosenblatt [11] and many others, with old and new applications in mathematical physics, engineering and other areas of probability [3, 4, 5, 12]. Nonetheless, until recently, good estimates of the persistence decay were known only for particular cases (e.g. [1, 13]), and for families of processes with either summable or non-negative correlations. The state of the art in the latter case was recently achieved by Dembo-Mukherjee [5], who were able to determine the log persistence of non-negatively correlated GSP's up to a constant factor.

A few years ago, by introducing a spectral point of view, the speaker together with O. Feldheim were able to provide general conditions under which the log persistence is bounded between two linear functions [6]. This extended a result by Antezana-Buckley-Marzo-Olsen for the *sinc-kernel* process [1], and provided the first general result on persistence of GSP's which does not require summability or non-negativity of correlations. However, these tools alone were insufficient to provide answers to two long-standing questions formulated by Slepian in his well known 1962 paper [13]:

- What are the possible asymptotic behaviors of the persistence probability of a GSP on large intervals?
- What features of the covariance function determine this behavior?

Spectral methods were recently used by Krishna-Krishnapur [10] in order to give a lower bound of e^{-cN^2} on the persistence of any GSP over $\mathbb{Z}$, provided that the spectral measure has a non-trivial absolutely continuous part. This gave rise to other interesting questions:

- Is there a GSP that achieves a persistence of the order of e^{-cN^2}?
- Is it possible for a GSP over $\mathbb{R}$ to have an even lower persistence?

Recently, in [8], we were able to provide nearly complete answers to all of these questions, in the case where the spectral measure has a non–trivial absolutely continuous component. Our main results are as follows.

Let $f : \mathbb{R} \to \mathbb{R}$ be a GSP with continuous covariance kernel $r(t) = \mathbb{E}[f(0)f(t)]$. Let ρ be its spectral measure, that is, the finite, symmetric non-negative measure on $\mathbb{R}$ such that $r(t) = \widehat{\rho}(t) = \int_{\mathbb{R}} e^{-i\lambda t} d\rho(\lambda)$. We always assume that there exists some $\delta > 0$ for which $\int |\lambda|^{\delta} w(\lambda) d\lambda < \infty$.

Denote the *persistence probability* by

$$P_f(N) := \mathbb{P}\Big(f(t) > 0,\ \forall t \in (0, N] \cap T\Big).$$

Theorem 1. Let f be a GSP over $\mathbb{R}$ or $\mathbb{Z}$. Suppose that its spectral measure is absolutely continuous with density $w(\lambda|)$ satisfying $c_1|\lambda|^\alpha \leq w(\lambda|) \leq c_2|\lambda||^\alpha$ for all λ in a neighborhood of 0 (and some $\alpha > -1$, $c_1, c_2 > 0$). Then , for large enough N:

$$\log P_f(N) \begin{cases} \asymp -N^{1+\alpha} \log N, & \alpha < 0 \\ \asymp -N, & \alpha = 0 \\ \lesssim -N \log N, & \alpha > 0. \end{cases}$$

Moreover, if $w(\lambda)$ vanishes on an interval containing 0, then $\log P_f(N) \lesssim -N^2$.

Note that when the spectrum vanishes at the origin ($\alpha > 0$), we have only given upper bounds on the persistence. This is because the persistence in these cases may be indeed much smaller then this upper bound, depending on the tail of the spectrum. An instructivce example is given in the following result from [8].

Theorem 2. Let f be GSP over $\mathbb{R}$ which has spectral density $w(\lambda)$ such that $w(\lambda) = 0$ for $|\lambda| < 1$ and $w(\lambda|) \geq \lambda|^{-\eta}$ for some $\eta > 0$ and all $|\lambda| > 1$, then $\log P_f(N) \leq -e^{CN}$.

The results so far estimate the order of magnitude of $\log P_f(N)$, up to a constant. Can we give conditions under which there is an exact asymptotic constant, that is, under which the limit

$$\theta_f := -\lim_{N\to\infty} \frac{1}{N} \log P_f(N)$$

exists? In a new paper [9], currently in final stages of preparation, we treat this question in detail. We define the *persistence exponent* of a spectral measure ρ and a level ℓ to be

$$\theta_\rho^\ell = -\lim_{N\to\infty} \frac{1}{N} \log \mathbb{P}(\inf_{[0,T]} f_\rho > \ell),$$

whenever the limit exists. We prove:

Theorem 3. Let f be a GSP over $\mathbb{R}$, whose spectral measure ρ has contniuous positive density in $[-a, a]$ (for some $a > 0$). Then θ_ρ^ℓ exists for all ℓ.

Moreover, we prove continuity properties of θ_ρ^ℓ:

Theorem 4. Let ρ be a spectral measure with continuous density near 0, and let $\ell \in \mathbb{R}$.

(1) θ_ρ^ℓ is continuous in ℓ.
(2) θ_ρ^ℓ is continuous in ρ with respect to the metric $d(\rho_1, \rho_2) = \int d|\rho_1 - \rho_2| + |\rho_1'(0) - \rho_2'(0)|$.
(3) If ν is a purely singular measure with $\nu([-a, a]) = 0$, then $\theta_\rho^\ell = \theta_{\rho+\nu}^\ell$.
(4) Suppose ρ has density which is bounded and compactly supported. If ρ_ϵ is the spectral measure which corresponds to the sequnce $\{f(n\epsilon)\}_{n\in\mathbb{Z}}$, then $\theta_{\rho_\epsilon}^\ell \to \theta_\rho^\ell$ as $\epsilon \to 0$.

There is much hope to use similar methods in order to answer questions about other stochastic processes under constraints, such as:

- Find the order of log-persistence probability for GSPs in higher dimensions
- What does the exponent θ_ρ^ℓ depend on? Can it be computed for certain examples?
- What is the "shape of persistence", that is, what is the typical behavior of a GSP conditioned to persist above ℓ on $[0, T]$?
- Can one generalize these results to some models which are non-Gaussian or non-stationary?

References

[1] J. Antezana, J. Buckley, J. Marzo and J. Olsen, *Gap probabilities for the cardinal sine*, Journ. Math. Anal. Appl. **396 (2)**, (2012), 466–472.

[2] F. Aurzada and T. Simon, *Persistence probabilities and exponents*, In: Lévy Matters V. Functionals of Lévy processes. Lect. Notes Math., **2149**, (2015), 183–224

[3] A.J. Bray, S.N. Majumdar and G. Schehr, *Persistence and First-Passage Properties in Non-equilibrium Systems*, Advances in physics **62 (3)**, (2013), 225–361.

[4] A. Dembo and S. Mukherjee, *No zero-crossings for random polynomials and the heat equation*, Ann. Probab. (2015), **43 (1)**, 85–118.

[5] A. Dembo and S. Mukherjee, *Persistence of Gaussian processes: non-summable correlations*, Probability Theory and Related Fields **169** (2017), 1007–1039.

[6] N.D. Feldheim and O.N. Feldheim, *Long gaps between sign-changes of Gaussian Stationary Processes*, Inter. Math. Res. Notices **11** (2015), 3012–3034.

[7] N. Feldheim, O. Feldheim, B. Jaye, F. Nazarov and S. Nitzan, *On the probability that a Stationary Gaussian Process with spectral gap remains non-negative on a long interval*, Inter. Math. Res. Notices (2018), rny248.

[8] N. Feldheim, O. Feldheim and S. Nitzan, *Persistence of Gaussian Stationary Processes: a Spectral Perspective*, to appear in Annals of Probability. preprint arXiv:1709.00204.

[9] N. Feldheim, O. Feldheim and S. Mukherjee, *Persistence and Ball Exponents for Gaussian stationary processes*, in preparation.

[10] M. Krishna and M. Krishnapur, *Persistence probabilities in centered, stationary, Gaussian processes in discrete time*, Indian Journal of Pure and Applied Mathematics **47 (2)** (2016), 183–194.

[11] G.F. Newell and M. Rosenblatt, *Zero Crossing Probabilities for Gaussian Stationary Processes*, Annals Mathematical Statistics **33 (4)** (1962), 1306–1313.

[12] G. Schehr and S.N. Majumdar, *Real roots of random polynomials and zero crossing properties of diffusion equation*, Journal of Statistical Physics **132** (2008), 235–273.

[13] D. Slepian, *The one-sided barrier problem for Gaussian noise*, Bell Systems Technical Journal **41** (1962), 463–501.

Persistence exponents via perturbation theory: AR(1)- and MA(1)-processes

Marvin Kettner

We study persistence probabilities in the context of Markov chains. For a real-valued stochastic process $(X_n)_{n\in\mathbb{N}}$ and $N \in \mathbb{N}$ the persistence probability is given by $p_N := \mathbb{P}(X_0 \geq 0, \ldots, X_N \geq 0)$. A first goal in persistence is to determine the asymptotic behaviour of p_N if $N \to \infty$. We look at processes such that p_N converges to zero at exponential speed, i.e. $p_N = \lambda^{N+o(N)}$ with $\lambda \in (0, 1)$. The value λ is called the persistence exponent.

It is well-known that persistence probabilities of Markov chains are related to eigenvalue problems of the canonical integral operator, which is defined by the corresponding transition kernel. We refer to the recent work [2] and the references given there. However, quantitative statements about the persistence exponent are known only in a few particular examples. We want to deal with this problem by using methods of perturbation theory [3].

For an autoregressive process of order one with standard normally distributed innovations, i.e. $X_n = \rho X_{n-1} + \xi_n$, $n \in \mathbb{N}$, with $(\xi_i)_{i\geq 1}$ i.i.d., $\xi_1 \sim \mathcal{N}(0,1)$ and $\rho \in \mathbb{R}$, we prove a perturbation result for the persistence exponent. By considering a proper modification of the canonical integral operator and applying results from perturbation theory, we show that the persistence exponent λ_ρ admits a power series representation in ρ at 0, i.e. $\lambda_\rho = \sum_{n=0}^{\infty} \rho^n K_n$ with $K_n \in \mathbb{R}$. Moreover, we show that the radius of convergence of this power series is at least $\frac{1}{3}$. Additionally, we derive an iterative formula for the coefficients K_n and we compute explicitly the first ones. It would be desirable to get a closed-form expression for the n-th coefficient but we have not been able to do this. In addition, it remains an interesting open problem to determine the radius of convergence. This part of the talk is based on the article [1].

For a moving average process of order one with standard normally distributed innovations, i.e. $X_n = \rho\xi_{n-1} + \xi_n$, $n \in \mathbb{N}$, with $(\xi_i)_{i\geq -1}$ i.i.d., $\xi_0 \sim \mathcal{N}(0,1)$ and $\rho \in \mathbb{R}$, we prove a similar result. By relating the persistence problem to an eigenvalue problem of an operator on a Hilbert space of analytic functions, we show that the persistence exponent λ_ρ can be expressed as a power series in ρ at 0. Here again, we have an iterative formula for the coefficients of this power series and we compute the first coefficients. As for the autoregressive case, a closed-form expression for the coefficients would be desirable. Moreover, we conjecture that the radius of convergence is one but we have not been able to prove this. This part of my talk is joint work with Frank Aurzada (Darmstadt) and Christophe Profeta (Évry).

We believe that our techniques and results have a greater generality, e.g. for other innovation distributions and also for autoregressive and moving average processes of a higher order.

The talk is based on my research work during my doctoral studies at the Technical University of Darmstadt and the results will be handed in as a dissertation shortly.

References

[1] Frank Aurzada and Marvin Kettner. Persistence exponents via perturbation theory: AR(1)-processes. *Journal of Statistical Physics*, 177(4):651–665, 2019.

[2] Frank Aurzada, Sumit Mukherjee, and Ofer Zeitouni. Persistence exponents in Markov chains. *arXiv preprint arXiv:1703.06447*, 2017.

[3] Tosio Kato. *Perturbation theory for linear operators*, volume 132. Springer Science & Business Media, 1966.

Persistence of weighted sums of GSPs

Sumit Mukherjee
(joint work with Frank Aurzada)

Suppose $\{\xi_i\}_{i\geq 0}$ is a centered Gaussian Stationary Process (GSP) with non negative correlation function $\rho(i) := \mathbb{E}[\xi_0\xi_i]$. Let $\{\sigma(i)\}_{i\geq 1}$ be a sequence of positive reals. Define the weighted partial sum $S_k := \sum_{i=1}^n \sigma(i)\xi_i$. In this talk, we study the asymptotics of the persistence probability of the process $\{S_k\}_{k\geq 1}$ on the negative half line, i.e.

$$p_n := \mathbb{P}(\max_{1\leq k\leq n} S_k < 0).$$

If $\rho(.) = 1_{.=0}$, the process $\{\xi_i\}_{i\geq 1}$ is i.i.d. In this case, the asymptotics of p_n was studied recently in [2], where the authors show that if

$$s(n)^2 := \sum_{i=1}^n \sigma(i)^2 \to \infty, \tag{1}$$

then under the uniform asymptotic condition

$$\max_{1\leq k\leq n} \sigma(k)^2 = o\Big(s(n)^2\Big)$$

we have

$$p_n \sim \frac{L(s_n^2)}{s_n}.$$

Here $L(.)$ is a slowly varying function which can be characterized explicitly. In particular this gives the log asymptotics $\log p_n \sim -\log s(n)$. In fact, in this case the same conclusion holds as soon as $\{\xi_i\}_{i\geq 1}$ are i.i.d. with first two moments finite. The justification for the universal asymptotics of p_n is via convergence of the $\{S_k\}_{k\geq 1}$ process to Brownian motion.

A natural question is to what happens when $\{\xi_i\}_{i\geq 1}$ is a correlated GSP. In this case, we show that the answer depends on whether $\sum_{i=1}^\infty \rho(i) < \infty$ or not, i.e. whether $\rho(.)$ is summable or not. If $\rho(.)$ is summable, then under the assumption that

$$\lim_{n\to\infty} \frac{\sigma(n+\ell)}{\sigma(n)} = 1 \text{ for all } \ell \geq 1, \tag{2}$$

we show that the $\{S_k\}_{k\geq 1}$ process again converges to Brownian motion. Developing on this, we show that the same asymptotics

$$\log p_n \sim -\log s(n)$$

holds in the summable case, provided (1) and (2) hold, plus the extra assumption that the sequence $\frac{\sigma(n)}{s(n)}$ is eventually decreasing up to a constant, i.e. there exists constants C, N such that for all $m \geq n \geq N$ we have

$$\frac{\sigma(m)}{s(m)} \leq C\frac{\sigma(n)}{s(n)}.$$

We give examples to demonstrate that several natural choices of weight function satisfies these conditions.

For the case of non summable correlations, the picture is not universal and significantly richer. In this case we focus specifically on the case of polynomially growing weight function $\sigma(i) \sim i^p$, and polynomially decaying correlation function $\rho(i) \sim \kappa i^{2H-2}$, for some $H \in (1/2, 1)$. As it turns out, the limiting process is no longer driven by Brownian motion, but instead is given by a stochastic integral $\int_0^t x^p dB_H(x)$ with respect to Fractional Brownian Motion with index H. In this case we have

$$\log p_n \sim -\theta(p, H) \log n,$$

where $\theta(p, H)$ is the exponent of this limiting process. Compare this with $\rho(.)$ summable and $\sigma(i) \sim i^p$, in which case the general result concludes that

$$\log p_n \sim -\Big(p + \frac{1}{2}\Big) \log n.$$

Even though the exponent $\theta(p, H)$ is implicit, we can analyze the exponent to get the following conclusions:

$$\begin{aligned} \lim_{H \downarrow 1/2} \theta(p, H) =& p + \frac{1}{2}, \\ \lim_{H \uparrow 1} \frac{\theta(p, H)}{1 - H} =& 1, \\ \lim_{p \downarrow -H} \frac{\theta(p, H)}{p + H} =& 1, \\ p^{2H-2} \lesssim \theta(p, H) \lesssim& p^{2H-2} \log p \text{ as } p \to \infty. \end{aligned}$$

In particular this confirms that $\theta(p, H)$ is neither $p + 1/2$ (which is the exponent for summable ρ), nor $1 - H$ (which is the exponent for FBM). The exact rate of growth of $\theta(p, H)$ as $p \to \infty$ remains open. We believe understanding this growth rate will lead to a better understanding of the persistence of GSPs with non integrable correlation function.

During the course of our proofs, we develop a stronger version of the continuity result for persistence exponents from [1], which is the main tool used for proving the results outlined above. We also show that the persistence exponent of a GSP (discrete or continuous time) is strictly positive if and only if the correlation function is integrable. Even though similar results are known in the literature, an exact characterization such as this was missing.

References

[1] Dembo, Amir; Mukherjee, Sumit. No zero-crossings for random polynomials and the heat equation. Ann. Probab. 43 (2015), no. 1, 85–118.

[2] Denisov, Denis; Sakhanenko, Alexander; Wachtel, Vitali. First-passage times for random walks with nonidentically distributed increments. Ann. Probab. 46 (2018), no. 6, 3313–3350.

Large Deviation Principles for Lacunary Sums

Nina Gantert
(joint work with C. Aistleitner, Z. Kabluchko, J. Prochno, K. Ramanan)

The study of lacunary series is a classical and still flourishing topic in harmonic analysis that has attracted considerable attention. First, let us define a lacunary (trigonometric) sum. Let $(a_k)_{k\in\mathbb{N}}$ be an increasing sequence of positive integers satisfying the Hadamard gap condition $a_{k+1}/a_k > q > 1$ for all $k \in \mathbb{N}$, and let

$$S_n(\omega) = \sum_{k=1}^{n} \cos(2\pi a_k \omega), \qquad n \in \mathbb{N},\ \omega \in [0,1].$$

Then S_n is called a lacunary (trigonometric) sum, and can be viewed as a random variable defined on the probability space $\Omega = [0,1]$ endowed with the standard Lebesgue measure. Lacunary sums are known to exhibit several properties that are typical for sums of independent random variables. For example, a central limit theorem for S_n as $n \to \infty$ has been obtained by Salem and Zygmund, while a law of the iterated logarithm is due to Erdős and Gál. In this paper we initiate the investigation of large deviation principles for lacunary sums. Our results can be summarized as follows. Under the large gap condition $a_{k+1}/a_k \to \infty$, we prove that the sequence $(S_n/n)_{n\in\mathbb{N}}$ does indeed satisfy a large deviation principle with the same rate function $\widetilde{I}$ as a sum of independent random variables with the arcsine distribution. On the other hand, we show that the large deviation principle may fail to hold when we only assume the Hadamard gap condition. However, we show that in the special case when $a_k = q^k$ for some $q \in \{2,3,\ldots\}$, S_n/n satisfies a large deviation principle with a rate function I_q that is different from $\widetilde{I}$, and describe an algorithm to compute an arbitrary number of terms in the Taylor expansion of $I_q(x)$. In addition, we also prove that I_q converges pointwise to $\widetilde{I}$ as $q \to \infty$. Finally, we construct a random perturbation $(a_k)_{k\in\mathbb{N}}$ of the sequence $(2^k)_{k\in\mathbb{N}}$ for which $a_{k+1}/a_k \to 2$ as $k \to \infty$, but for which at the same time $(S_n/n)_{n\in\mathbb{N}}$ satisfies a large deviation principle with the same rate function $\widetilde{I}$ as in the independent case, which is surprisingly different from the rate function I_2 one might naïvely expect. All these results together show that large deviation principles for lacunary sums are very sensitive to the arithmetic properties of the sequence $(a_k)_{k\in\mathbb{N}}$. This is particularly interesting since no such arithmetic effects are visible in the central limit theorem or in the law of the iterated logarithm for lacunary trigonometric sums.

Polyharmonic functions and random walks in cones

Kilian Raschel
(joint work with François Chapon, Éric Fusy)

In the continuous setting, polyharmonic functions are functions which cancel some power of the usual Laplacian. More precisely, a function v on some domain K of $\mathbb{R}^d$ satisfying

$$\Delta^p v = 0$$

for some $p \geq 1$, where Δ is the usual Laplacian in $\mathbb{R}^d$, is said to be *polyharmonic* of order p, or *polyharmonic* for short. So polyharmonic functions of order 1 are just harmonic functions. Obviously, a polyharmonic function v_p of order p satisfies $\Delta v_p = v_{p-1}$, where v_{p-1} is polyharmonic of order $p-1$. For example, polynomials are polyharmonic. Harmonic functions have been tremendously investigated and pioneer works on polyharmonic functions go back to the work of Almansi [1]. One can consult for instance the monograph [2] for an introduction to this topic.

In particular, Almansi [1] proved that if the domain K is star-like with respect to the origin, then every polyharmonic function of order p admits a unique decomposition

$$f(x) = \sum_{k=0}^{p-1} |x|^{2k} h_k(x), \tag{1}$$

where each h_k is harmonic on K and $|x|$ is the Euclidean length of x, hence completely characterising continuous polyharmonic functions on such domains.

In comparison with the continuous case, much less is known in the discrete setting, where the Laplacian has to be replaced by a discrete difference operator (see, e.g., (5)). Some progress in understanding discrete polyharmonic functions has been made in the last two decades. For instance, one may cite [10], where the authors investigated polyharmonic functions for the Laplacian on trees, and proved a similar result as Almansi's theorem (1) for homogeneous trees. Recent works of Woess and co-authors [14, 16] are generalising this previous work.

Our original motivation to study discrete polyharmonic functions comes from the following framework. Consider a walk in $\mathbb{Z}^d$ with step set $\mathcal{S}$ confined in some cone $K \subset \mathbb{Z}^d$. Denote by $q(x, y; n)$ the number of n-length excursions between x and y staying in the cone K. To simplify, we only consider the case where y is the origin, but all considerations below can be generalised to $y \neq 0$. In various cases [12], the asymptotics of $q(x, 0; n)$ as $n \to \infty$ is known to admit the form

$$q(x, 0; n) \sim v_0(x)\gamma^n n^{-\alpha_0}, \tag{2}$$

where $v_0(x) > 0$ is a function depending only on x, $\gamma \in (0, |\mathcal{S}|]$ is the exponential growth, and α_0 is the critical exponent. It is easy to see that the function $v_0(x)$ in (2) defines a discrete harmonic function. Indeed, plugging (2) into the obvious recursive relation

$$q(x, 0; n+1) = \sum_{s \in \mathcal{S}} q(x+s, 0; n)\mathbf{1}_{\{x+s \in K\}}, \tag{3}$$

dividing by $\gamma^{n+1}n^{-\alpha_0}$ and letting $n \to \infty$, we obtain

$$v_0(x) = \frac{1}{\gamma}\sum_{s\in\mathcal{S}} v_0(x+s)\mathbf{1}_{\{x+s\in K\}}, \tag{4}$$

which proves that, with the assumption that $v_0(x) = 0$ for $x \notin K$, $v_0(x)$ is discrete harmonic for the Laplacian operator

$$Lf(x) = \frac{1}{\gamma}\sum_{s\in\mathcal{S}} f(x+s) - f(x), \tag{5}$$

that is, $Lv_0 = 0$. Denisov and Wachtel [12] go further and show that

- the exponential growth γ is $\min_{\mathbb{R}_+^d} \sum_{(s_1,\ldots,s_d)\in\mathcal{S}} x_1^{s_1}\cdots x_d^{s_d}$, it does not depend on K;
- the critical exponent α_0 equals $1+\sqrt{\lambda_1+(d/2-1)^2}$, where d is the dimension and λ_1 is the principal Dirichlet eigenvalue on some spherical domain constructed from K.

As a leading example, consider the simple random walk in the quarter plane, with step set $\{\leftarrow,\uparrow,\rightarrow,\downarrow\}$. In this case, the number of excursions $q((i,j),0;n)$ is 0 if $m = \frac{n-i-j}{2}$ is not a non-negative integer, and otherwise takes the value

$$q((i,j),0;n) = \frac{(i+1)(j+1)n!(n+2)!}{m!(m+i+j+2)!(m+i+1)!(m+j+1)!}, \tag{6}$$

see [8]. The asymptotics (2) is then

$$q((i,j),0;n) \sim \frac{4}{\pi}4^n\frac{v_0(i,j)}{n^3}, \tag{7}$$

where $v_0(i,j) = (i+1)(j+1)$ is the well-known unique (up to multiplicative constants) harmonic function positive within the quarter plane with Dirichlet boundary conditions. Other examples of such asymptotics may be found for instance in [3, 9, 11].

Our aim in this discrete setting is to study more precise estimates than (2), by considering complete asymptotic expansions of the following form, as $n \to \infty$,

$$q(x,0;n) \sim \gamma^n \sum_{p\geq 0}\frac{v_p(x)}{n^{\alpha_p}}. \tag{8}$$

From such an asymptotic expansion and using similar ideas as in (3), (4) and (5), it is rather easy to prove that the terms v_p are polyharmonic functions, in the sense that a power $L^k v_p$ of the Laplacian operator vanishes. We will provide examples of such asymptotic expansions (at least for the first terms) and of the set of exponents $\{\alpha_p\}_{p\geq 0}$ appearing in (8).

On the other hand, the functional equation approach has proved to be fruitful when studying random walk problems. The reference book on this topic is the monograph [13] by Fayolle, Iasnogorodski and Malyshev. This method has been used in [15] to construct harmonic functions, both in the discrete and continuous settings. Basically, the method consists of drawing from the harmonicity condition a functional equation satisfied by the generating function (in the discrete setting) or by the Laplace transform (in the continuous setting) of a harmonic function. Solving some boundary value problem for these quantities leads, via Cauchy or Laplace inversion, to the sought harmonic function. We will provide an implementation of this method to construct bi-harmonic functions, which can be generalised to polyharmonic functions.

The main features of our results are as follows:

- We shine a light on a new link between discrete polyharmonic functions and complete asymptotic expansions in the enumeration of walks.
- Our approach provides tools to study complete asymptotics expansions as in (8), but does not allow to prove their existence. On the other hand, the powerful approach of Denisov and Wachtel [12] seems restricted to the first term in the asymptotics (2). Indeed, one of the main tools in [12] is a coupling result of random walks by Brownian motion, which only provides an approximation of polynomial order, see [12, Lem. 17].
- We introduce a new class of functional equations, for which the method of Tutte's invariants introduced in [17, 4, 5] proves to be useful.
- In the unweighted planar case, it has been shown [7] that knowing the rationality of the exponent α_0 in (8) was sufficient to decide the non-D-finiteness of the series of excursions. However, for walks with big steps in dimension two or walk models in dimension three, this information is not enough [6]. As a potential application of our results, we might use arithmetic information on the other exponents α_p to study the algebraic nature, for example the transcendance, of the associated combinatorial series.

References

[1] E. Almansi, Sull'integrazione dell'equazione differenziale $\Delta^{2n} = 0$, *Annali di Mat. (3)*, 2:1–51, 1899.

[2] N. Aronszajn, T. M. Creese, and L. J. Lipkin, *Polyharmonic functions*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1983.

[3] C. Banderier and P. Flajolet, Basic analytic combinatorics of directed lattice paths, *Theoretical Computer Science*, 281(1-2):37–80, 2002.

[4] O. Bernardi and M. Bousquet-Mélou, Counting colored planar maps: algebraicity results, *J. Combin. Theory Ser. B*, 101(5):315–377, 2011.

[5] O. Bernardi, M. Bousquet-Mélou, and K. Raschel, Counting quadrant walks via Tutte's invariant method, *Preprint arXiv:1708.08215*, 2017.

[6] A. Bostan, M. Bousquet-Mélou, and S. Melczer, Counting walks with large steps in an orthant, *Preprint arXiv:1806.00968*, 2018.

[7] A. Bostan, K. Raschel, and B. Salvy, Non-D-finite excursions in the quarter plane, *J. Combin. Theory Ser. A*, 121:45–63, 2014.

[8] M. Bousquet-Mélou, Counting walks in the quarter plane, In *Mathematics and computer science, II (Versailles, 2002)*, Trends Math., pages 49–67. Birkhäuser, Basel, 2002.

[9] M. Bousquet-Mélou and M. Mishna, Walks with small steps in the quarter plane, In *Algorithmic probability and combinatorics*, volume 520 of *Contemp. Math.*, pages 1–39. Amer. Math. Soc., Providence, RI, 2010.

[10] J. M. Cohen, F. Colonna, K. Gowrisankaran, and D. Singman, Polyharmonic functions on trees, *Amer. J. Math.*, 124(5):999–1043, 2002.

[11] J. Courtiel, S. Melczer, M. Mishna, and K. Raschel, Weighted lattice walks and universality classes, *J. Combin. Theory Ser. A*, 152:255–302, 2017.
[12] D. Denisov and V. Wachtel, Random walks in cones, *Ann. Probab.*, 43(3):992–1044, 2015.
[13] G. Fayolle, R. Iasnogorodski, and V. Malyshev, *Random walks in the quarter plane*, volume 40 of *Probability Theory and Stochastic Modelling*, Springer, Cham, second edition, 2017.
[14] T. Hirschler and W. Woess, Polyharmonic functions for finite graphs and Markov chains, *Preprint arXiv:1901.08376*, 2019. Frontiers in Analysis and Probability: in the Spirit of the Strasbourg-Zürich Meetings, Springer (to appear).
[15] K. Raschel, Random walks in the quarter plane, discrete harmonic functions and conformal mappings, *Stochastic Process. Appl.*, 124(10):3147–3178, 2014. With an appendix by S. Franceschi.
[16] E. Sava-Huss and W. Woess, Boundary behaviour of λ-polyharmonic functions on regular trees, *Preprint arXiv:1904.10290*, 2019.
[17] W. T. Tutte, Chromatic sums revisited, *Aequationes Math.*, 50(1-2):95–134, 1995.

A Characterization of the Finiteness of Perpetual Integrals of Lévy Processes

Mladen Savov
(joint work with M. Kolb)

Let $\xi = (\xi_s)_{s\geq 0}$ be a one dimensional Lévy process such that $\lim_{s\to\infty} \xi_s = \infty$ almost surely. In this talk we present an analytical criterion for the finiteness of perpetual integrals of ξ, that is quantities of the type

$$I^x = \int_0^\infty f(x+\xi_s)ds, \tag{1}$$

where f is a non-negative, locally bounded measurable function. The availability of a tool which distinguishes when I^x is almost surely finite or not is of importance in a number of settings. To describe informally one such scenario let us consider a motion whose random dynamics is described by ξ. However, the evolution of the system is affected by some potential (say f). Then the large time behaviour of the system depends on the finiteness of I^x and the availability of an analytic criterion which captures this finiteness can inform us a priori about this behaviour.

Analytical conditions for the finiteness of I^x have been known in the literature either for special classes of Lévy processes or for special classes of functions f. However, their derivation depends on more general and seemingly side results such as the strong law of large numbers when f is non-increasing or the hitting of points when ξ is assumed of finite mean and in possession of local time and it is not immediately clear how they can be extended to the general case, see the considerations in [2]. The work of Batty [1] for Brownian motion provides a first insight as to what a general criterion for finiteness of I^x for Lévy processes might be. Unfortunately, the methodology in [1] relies on the continuity of the Brownian path, a property which is not directly available for Lévy processes. However, on close inspection of the problem it turns out that the crucial stopping times which appear in [1] are in fact announceable (previsible) for Lévy processes which implies the fact that the processes are almost surely continuous at those random moments. This opens the door for the derivation of a general analytic criterion which distinguishes when I^x is almost surely finite or not.

Fix $x \in \mathbf{R}$. Let $E \subseteq \mathbf{R}$ be such that almost surely $x+\xi$ spends a finite amount of time in $\mathbf{R} \setminus E$. $\mathbf{R} \setminus E$ is called transient set for $x+\xi$. Then we have that

(2)

$$I^x = \infty \iff \int f 1_E(x+y)U(dy) = \infty, for\ all\ \mathbf{R} \setminus E\ transient\ sets\ for\ x+\xi,$$

where U is the potential measure of ξ. This criterion is very natural as it disregards the transient sets for $x+\xi$ and involves the potential measure which evaluates the expected time the process spends in each Borel set. The criterion (2) and the arguments that lead up to its proof can be used to recover the special cases in [2, 3]. The drawback of (2) consists in the fact that at present it is not clear how the transient sets can be classified for general Lévy processes. However, as we have highlighted in [4] one may bypass this problem by considering the set-theoretical properties of a particular set which plays an important role in the course of the proof.

The results we present in this talk are published in Bernoulli, see [4].

References

[1] Batty, C. J. K., Asymptotic stability of Schrödinger semigroups: path integral methods, *Math. Ann.* **292**, 457–492, (1992)

[2] Döring, L., A. E. Kyprianou, Perpetual Integrals for Lévy Processes, *J. Theor. Probab.* **29**, 1192–1198, (2015)

[3] Erickson, K. and Maller, R. Generalised Ornstein-Uhlenbeck processes and the convergence of Lévy integrals. Séminaire de Probabilités XXXVIII, 70–94, Lecture Notes in Mathematics, 1857, Springer, Berlin, 2005

[4] Kolb, M. and Savov, M. A Characterization of the Finiteness of Perpetual Integrals of Lévy Processes. *Bernoulli*, **26** No.2, 1453–1472, (2020)

Lyapunov criteria for the convergence of conditional distributions of absorbed Markov processes

Nicolas Champagnat

(joint work with Denis Villemonais)

This presentation is based on the results of [3].

We consider a general Markov process with absorbtion $(X_t, t \geq 0)$ with state space $E \cup \partial$ where E and ∂ are disjoint measurable spaces. Here, time can be either continuous: $t \in [0,\infty)$, or discrete: $t \in \mathbb{Z}_+ := \{0,1,\dots\}$. We assume that ∂ is absorbing, i.e. that, $\mathbb{P}_x$-a.s. for all $x \in E \cup \partial$, $X_t \in \partial$ for all $t \geq \tau_\partial$, where

$$\tau_\partial = \inf\{t \geq 0,\ X_t \in \partial\}.$$

Typical applications are population dynamics, where absorption corresponds to extinction of one or several populations. For example, one-dimensional Galton-Watson processes (in discrete-time) or birth and death processes in $\mathbb{Z}_+$ absorbed at 0 (in continuous time) correspond to the state spaces $E = \mathbb{N} := \{1, 2, \ldots\}$ and $\partial = \{0\}$. Markov processes of continuous population densities, such as diffusion processes, may also be considered, corresponding typically to the state spaces $E = (0, \infty)$ and $\partial = \{0\}$.

We assume a.s. absorption, i.e. $\tau_\partial < \infty$ $\mathbb{P}_x$-a.s. for all $x \in E$, and a positive probability of non-absorption up to any time, i.e. $\mathbb{P}_x(t < \tau_\partial) > 0$ for all $x \in E$ and $t \geq 0$. This means that the stationary behavior of the process—absorption—is trivial. However observed populations often exhibit seemingly stationary behavior before extinction. Other phenomena of stabilization of stochastic processes before absorption include *mortality plateau* [7], metastable dynamics [1] and numerical methods for molecular simulations [5]. An impotant tool to study of such phenomena is provided by so-called quasi-stationary distributions (QSD), first proposed in [4]. We denote by $\mathcal{M}_1(E)$ the set of probability measures on E.

Definition 1. (i)**:** *A probability measure $\alpha \in \mathcal{M}_1(E)$ is called a QSD iff, for all $t \geq 0$, $\mathbb{P}_\alpha(X_t \in \cdot \mid t < \tau_\partial) = \alpha$.*
(ii)**:** *A probability measure $\alpha \in \mathcal{M}_1(E)$ is called a quasi-limiting distribution (QLD) iff there exists $\mu \in \mathcal{M}_1(E)$ such that $\mathbb{P}_\mu(X_t \in \cdot \mid t < \tau_\partial)$ converges to α for the total variation distance.*

The proofs of the following basic properties can be found e.g. in [7].

Proposition 2. (i)**:** *α is a QSD $\Longleftrightarrow$ is a QLD.*
(ii)**:** *If α is a QSD, then there exists $\lambda_0 > 0$ such that $\mathbb{P}_\alpha(t < \tau_\partial) = e^{-\lambda_0 t}$.*
(iii)**:** *If α is a QSD, then $\alpha P_t = e^{-\lambda_0 t}\alpha$ and $\alpha L = -\lambda_0 \alpha$, where P_t is the semigroup of the killed process $\mu P_t f = \mathbb{E}_\mu[f(X_t)\mathbb{1}_{t<\tau_\partial}]$ acting on the set of bounded measurable functions f, and L its infinitesimal generator.*

For one-dimensional birth and death processes absorbed at 0, it is knwon that there exists a unique QSD iff ∞ is an entrance boundary [10]. For example, for the linear birth and death process with birth rate $b_n = bn$ and death rate $d_n = dn$ from state $n \in \mathbb{Z}_+$, absorption is almost sure for $b \leq d$, there exist no QSD for $b = d$ and there exist infinitely many QSD for $b < d$. On the contrary, if $b_n = bn^{1+\varepsilon}$ and $d_n = dn^{1+\varepsilon}$ for $b < d$ and $\varepsilon > 0$, there exists a unique QSD. For one-dimensional diffusions in $[0, \infty)$ absorbed at 0, similar results are expected. For example, it is known that Ornstein-Uhlenbeck process absorbed at 0 have infinitely many QSD [6]. However, as far as we know, the conjecture that there exists a unique QSD iff ∞ is an entrance boundary is still not proved in full generality [2].

Inspired by classical criteria for exponential ergodicity [8], we consider the following set of assumptions.

Assumption (E). There exist $n_1 \in \mathbb{N}$, $\theta_1, \theta_2, c_1, c_2, c_3 > 0$, $\varphi_1, \varphi_2 : E \to \mathbb{R}_+$, a measureable set $K \subset E$ and $\nu \in \mathcal{M}_1(K)$ such that

(E1) *(Local Dobrushin coefficient)* $\forall x \in K$, $\mathbb{P}_x(X_{n_1} \in \cdot) \geq c_1 \nu(\cdot \cap K)$.

(E2) *(Global Lyapunov criterion)* $\theta_1 < \theta_2$, $\inf_{x\in E}\varphi_1(x) \geq 1$, $\sup_{x\in K}\varphi_1(x) < \infty$, $\inf_{x\in K}\varphi_2(x) > 0$, $\sup_{x\in E}\varphi_2(x) \leq 1$ and

$$P_1\varphi_1(x) \leq \theta_1\varphi_1(x) + c_2\mathbb{1}_K(x), \ \forall x \in E$$
$$P_1\varphi_2(x) \geq \theta_2\varphi_2(x), \ \forall x \in E.$$

(E3) *(Local Harnack inequality)* $\sup_{n\in\mathbb{Z}_+}\dfrac{\sup_{y\in K}\mathbb{P}_y(n<\tau_\partial)}{\inf_{y\in K}\mathbb{P}_y(n<\tau_\partial)} \leq c_3$.

(E4) *(Aperiodicity)* For all $x \in K$, there exists $n_4(x)$ such that, for all $n \geq n_4(x)$, $\mathbb{P}_x(X_n \in K) > 0$.

In Assumption (E2), the function φ_2 allows to control the probability of survival of the population and the inequality $\theta_1 < \theta_2$ means that entrance in K occurs faster than extinction. Assumption (E1) is a classical coupling condition within K. Assumption (E3) means that there's no initial point in K such that survival is much better than for other initial conditions in K. This technical condition is not involved in classical ergodicity criteria for irreducible processes, but is needed in cases with absorption.

Under Assumption (E), we obtain the following results.

Theorem 3. *There exist $\nu_{QSD} \in \mathcal{M}_1(E)$, $C > 0$ and $\alpha \in (0,1)$ such that*

$$\|\mathbb{P}_\mu(X_n \in \cdot \mid n < \tau_\partial) - \nu_{QSD}\|_{TV} \leq C\,\alpha^n\frac{\mu(\varphi_1)}{\mu(\varphi_2)},$$

for all $\mu \in \mathcal{M}_1(E)$ such that $\mu(\varphi_1) < \infty$ and $\mu(\varphi_2) > 0$.

This result implies that ν_{QSD} is the unique QSD such that $\nu_{\mathrm{QSD}}(\varphi_1) < \infty$ and $\nu_{\mathrm{QSD}}(\varphi_2) > 0$ and that its domain of attraction contains all probability measures μ such that $\mu(\varphi_1) < \infty$ and $\mu(\varphi_2) > 0$.

Theorem 4. *There exists a function $\eta : E \to \mathbb{R}_+$ such that*

$$\eta(x) = \lim_{n\to+\infty} e^{-\lambda_0 t}\mathbb{P}_x(n < \tau_\partial), \quad \forall x \in E, \tag{1}$$

where the convergence is geometric in $L^\infty(\varphi_1)$. In addition, $\nu_{QSD}(\eta) = 1$, $e^{-\lambda_0} \geq \theta_2$, $P_t\eta = e^{-\lambda_0 t}\eta$ and $L\eta = -\lambda_0\eta$.

As a first application, we consider the simple case of sub-critical Galton-Watson processes. Let m denote the expected number of offspring of an individual ($m < 1$). Since the absorption probability is uniformly lower bounded, it is sufficient to take $\varphi_2 \equiv 1$. If the reproduction law has some finite exponential moment, let $b \in (m, 1)$ and $\varphi_1(x) = e^{ax}$. For $a > 0$ small enough,

$$P_1\varphi_1(x) = \mathbb{E}_x e^{aX_1} = \left(\mathbb{E}_1 e^{aX_1}\right)^x \leq e^{abx} = \theta_1(x)\varphi_1(x)$$

with $\theta_1(x) = e^{-a(1-b)x} \to 0$ when $x \to +\infty$. This entails (E2). (E1) and (E3) follow from irreducibility with $K = \{1\}$, and (E4) is straightforward. In this case, it is known that there exist infinitely many QSD [9], and Theorem 3 entails that the domain of attraction of ν_{QSD} contains all initial distributions having a finite exponential moment.

Our criterion is flexible and extends easily to more general processes, such as processes dominated by state-dependent branching processes: assume that

$$X_{n+1} \le \sum_{i=1}^{X_n} \xi_{i,n}^{(X_n)},$$

where $(\xi_{i,n}^{(x)})_{i,n,x\ge 1}$ are independent r.v., i.i.d. for fixed x, such that $\mathbb{E}\xi_{1,1}^{(x)} \le 1-\varepsilon$ for x large enough for some $\varepsilon > 0$, and $(\xi_{1,1}^{(x)})_{x\ge 1}$ have a uniformly bounded exponential moment of the same parameter. Then (E) is satisfied. This is still true under similar conditions for multi-dimensional state-dependent branching processes and if the r.v. $\xi^{(x)}$ have bounded polynomial moments only.

To conclude, our criterion can be checked for general perturbed dynamical systems: assume that $D \subset \mathbb{R}^d$ has positive Lebesgue measure and let $\partial \notin D$. Then (E) is satisfied by the process

$$X_{n+1} = \begin{cases} f(X_n) + \xi_n & \text{if } X_n \neq \partial \text{ and } f(X_n) + \xi_n \in D, \\ \partial & \text{otherwise,} \end{cases}$$

provided that $f : \mathbb{R}^d \to \mathbb{R}^d$ is a locally bounded measurable function such that $|x| - |f(x)| \to +\infty$ when $|x| \to +\infty$ and $(\xi_n)_{n\ge 1}$ is an i.i.d. non-degenerate Gaussian sequence in $\mathbb{R}^d$.

References

[1] A. Bovier and F. den Hollander. *Metastability. A potential-theoretic approach* Grundlehren der Mathematischen Wissenschaften **351**. Springer (2015).

[2] N. Champagnat and D. Villemonais. *Uniform convergence of conditional distributions for absorbed one-dimensional diffusions.* Adv. in Appl. Probab. **50(1)** (2018),178–203.

[3] N. Champagnat and D. Villemonais. *General criteria for the study of quasi-stationarity*, Arxiv e-prints (2017).

[4] J. N. Darroch and E. Seneta. *On quasi-stationary distributions in absorbing discrete-time finite Markov chains.* J. Appl. Probab. **2** (1965), 88–100.

[5] C. Le Bris, T. Lelièvre, M. Luskin and D. Perez. *A mathematical formalization of the parallel replica dynamics*, Monte Carlo Methods Appl. **18(2)** (2012), 119–146.

[6] M. Lladser and J. San Martin. *Domain of attraction of the quasi-stationary distributions for the Ornstein-Uhlenbeck process.* J. Appl. Probab. **37(2)** (2000), 511–520.

[7] S. Méléard and D. Villemonais. *Quasi-stationary distributions and population processes.* Probab. Surv. **9** (2012), 340–410.

[8] S. Meyn and R.L. Tweedie. *Markov chains and stochastic stability.* Cambridge University Press, Cambridge, 2nd edition (2009).

[9] E. Seneta and D. Vere-Jones. *On quasi-stationary distributions in discrete-time Markov chains with a denumerable infinity of states.* J. Appl. Probab. **3** (1966), 403–434.

[10] E.A. van Doorn. *Quasi-stationary distributions and convergence to quasi-stationarity of birth-death processes.* Adv. Appl. Probab. **23(4)** (1991), 683–700.

Almost sure convergence of Measure Valued Pólya Processes

Denis Villemonais
(joint work with Cécile Mailler)

In [4] and [6], the authors provide a new approach to Pólya urn processes with infinitely many colors, where the content of an urn is represented as a measure over a set of colors. The construction of these so-called measure-valued Pólya urn processes is done as follows. Let E be a Polish space endowed with its Borel σ-field, a positive measure finite measure m_0 over E (the original content of the urn), a sequence of independent identically distributed random kernels $(R^{(n)})_{n\geq 1}$ over E and a weight kernel P over E. The measure valued Pólya process is a Markov chain $(m_n)_{n\geq 0}$ constructed iteratively via the relation

$$m_{n+1} = m_n + R^{(n+1)}_{Y_{n+1}},$$

where $Y_{n+1} \sim m_n P/m_n P(E)$ is chosen independently from the past. The authors of [4] and [6] were interested in the long time behaviour (in probability) of m_n when the urn is *balanced*, which means that $R^{(n)}$ is deterministic and $R^{(n)}_x(E) = 1$ for all $x \in E$. In the recent paper [7], we study the almost sure convergence of $m_n P/m_n P(E)$, allowing $R^{(n)}$ to be random and unbalanced (meaning that $R^{(n)}_x(E)$ can take different values than 1 and that its expectation may depend on x). In order to do so, we use stochastic approximation methods, one of the main difficulty being that the state space of our process is allowed to be non-compact.

Before going further in the presentation of our main result, note that one recovers the usual Pólya urn with finitely many colours, say $d \geq 2$, and with random replacement matrix $M^{(n)}$ with no weights ($P = I$), where $(M^{(n)})_n$ is a sequence of i.i.d. random matrices with non-negative entries and mean M. Indeed, setting $S = \max_{x=1}^d \sum_{i=1}^d M_{x,i}$ and $m_n = \frac{1}{S}\sum_{i=1}^d U_i(n)\delta_i$, where $U_i(n)$ is the number of balls of color i in the urn at time n, one can easily check that $(m_n)_{n\geq 0}$ is a measure valued Pólya process with replacement kernel $R^{(n)}_x = \frac{1}{S}\sum_{i=1}^d M^{(n)}_{x,i}\delta_i$, for all $n \geq 0$ and $1 \leq x \leq d$. Doing so, we recover the setting of the classical references [1] and [2], where the authors study Pólya urn processes with irreducible replacement rules.

In order to state our main result, we introduce the following assumptions, where $R = \mathbb{E}(R^{(n)})$ and $Q = RP$ (a more general and more technical assumption is presented in [7], which in particular allows the removal of balls from the urn). We assume that

(i) for all $x \in E$, $Q_x(E) \leq 1$ and there exists a probability measure μ on $[0, +\infty)$ with positive mean and such that, for all $x \in E$, the law of $R^{(i)}_x P(E)$ stochastically dominates μ. In particular, setting $c_1 = \int_0^\infty x \,\mathrm{d}\mu(x)$,

$$0 < c_1 \leq \inf_{x\in E} Q_x(E) \leq \sup_{x\in E} Q_x(E) \leq 1;$$

(ii) there exists a lower semi-continuous and locally bounded function $V : E \to [1, +\infty)$ such that
  (a) for all $N \geq 1$, the set $\{x \in E \colon V(x) \leq N\}$ is relatively compact;
  (b) there exist two constants $\theta \in (0, c_1)$ and $K \geq 0$ such that

$$Q_x \cdot V \leq \theta V(x) + K \quad (\forall x \in E),$$

  (c) there exist three constants $1 < r < 2 \leq p$ and $A > 0$ such that

$$\mathbb{E}\left[R_x^{(1)}(E)^r\right] \vee \mathbb{E}\left[Q_x^{(1)}(E)^p\right] \leq AV(x) \quad (\forall x \in E).$$

Under the above assumption, $Q - I$ is the infinitesimal generator of a sub-Markov semi-group $(P_t)_{t\in[0,+\infty)}$ over the set of bounded measurable functions on E, which allows us to define the following assumption (which plays the role of the irreducibility in [1, 2]):

(iii) the continuous time sub-Markov process X with infinitesimal generator $Q - I$ has a quasi-stationary distribution ν over E and the convergence of $\mathbb{P}(X_t \in \cdot \mid X_t \neq \partial)$ (where ∂ is the cemetery point for X) holds true uniformly in total variation norm over the set of initial distributions $\{\alpha \mid \alpha \cdot V^{1/q} \leq C\}$, for all constants $C > 0$.

Finally, we make the following technical assumption:

(iv) for all bounded continuous function $f : E \to \mathbb{R}$, $x \in E \mapsto R_x f$ and $x \in E \mapsto Q_x f$ are continuous.

Our main result is the following one.

Theorem. *Under the above assumptions, if $m_0 \cdot V < \infty$ and $m_0 P \cdot V < \infty$, then the sequence of random measures $(m_n/n)_{n\geq 0}$ converges almost surely to νR with respect to the topology of weak convergence. Moreover,* $\sup_n\{m_n P \cdot V^{1/q}/n\} < +\infty$ *almost surely, where $q = p/(p-1)$.*

Furthermore, if $\nu R(E) > 0$, then $(\tilde{m}_n)_{n\in\mathbb{N}}$ converges almost surely to $\nu R/\nu R(E)$ with respect to the topology of weak convergence.

During the talk, an application of this result, fully developed in [7], has been presented. In this example, we focused on the long time behaviour of a self-reinforced process interacting with its past. More precisely, consider the process $(Y_t)_{t\geq 0}$ in $\mathbb{R}^d$ evolving as follows:

- Y evolves following the SDE

$$dY_t = dB_t + b(Y_t)\, dt,\ Y_0 \in \mathbb{R}^d,$$

  where b is locally Hölder continuous,
- and, with rate $\kappa(Y_t) \geq 1$, the process jumps with respect to its empirical occupation measure $\frac{1}{t}\int_0^t \delta_{Y_s}\, \mathrm{d}s$.

If one assumes in addition that $\limsup_{x\to+\infty} \frac{\langle b(x),x\rangle}{|x|} < -\frac{3}{2}\|\kappa\|_\infty^{1/2}$, then

- the solution to the SDE $dX_t = dB_t + b(X_t)\,dt$ admits a unique quasi-stationary distribution ν with an exponential moment of order $\|\kappa\|_\infty^{1/2}$
- almost surely,

$$\frac{1}{t}\int_0^t \delta_{Y_s}\,\mathrm{d}s \xrightarrow[t\to+\infty]{} \nu$$

in the topology of weak convergence.

The aim of this example is two-fold: first it makes a non obvious link between the theory of diffusion processes interacting with their past and the theory of Pólya urns; second it illustrates how the results proved in [5] interact nicely with the set of assumptions (i-iv) above.

One desirable extension of our result would be to find a reasonable setting under which the conclusion holds without assuming that $Q_x(E)$ is uniformly lower bounded. A step in this direction has been done in [3], where a diffusion process interacting with its past when it hits a boundary is considered. Another natural problem, currently under investigation, is to prove a central limit theorem for these processes.

References

[1] D. Aldous, B. Flannery, and J.L. Palacios, *Two applications of urn processes the fringe analysis of search trees and the simulation of quasi-stationary distributions of markov chains. Probability in the engineering and informational sciences*, **2**, No 3 (1988), 293– 307.

[2] K.B. Athreya and S. Karlin, *Embedding of urn schemes into continuous time Markov branching processes and related limit theorems*, Annals of Mathematical Statistics, **39** (1968), 1801–1817.

[3] M. Benaïm, N. Champagnat and D. Villemonais, *Stochastic approximation of quasi-stationary distributions for diffusion processes in a bounded domain*, Arxiv:1904.08620, Jul. 2020.

[4] A. Bandyopadhyay and D. Thacker, *Pólya urn schemes with infinitely many colors*, Bernoulli **23**, No. 4B (2017), 3243–3267.

[5] N. Champagnat and D. Villemonais, *General criteria for the study of quasi- stationarity*, Arxiv:1712.08092, Dec. 2017.

[6] C. Mailler and J.-F. Marckert, *Measure-valued Pólya urn processes*, Electron. J. Probab. **22** No. 26 (2017).

[7] C. Mailler and D. Villemonais, *Stochastic approximation on noncompact measure spaces and application to measure-valued Pólya processes*, Ann. Appl. Probab. **30** No. 5 (2020), 2393–2438.

On a continuous-time Derrida-Retaux model

Yueyun Hu

(joint work with Bastien Mallein, Michel Pain)

To study the depinning transition in the limit of strong disorder, Derrida and Retaux (J. Stat. Phys. (2014)) introduced a discrete-time max-type recursive model. It is believed that for a large class of recursive models, including Derrida and Retaux' model, there is a highly non-trivial phase transition. We present here a continuous-time version of Derrida and Retaux model, built on a Yule tree, which yields an exactly solvable model belonging to this universality class.

More specifically, the *continuous-time Derrida–Retaux process* $(X_t)_{t\geq 0}$ is defined as the (unique) solution of the McKean–Vlasov type stochastic differential equation

$$\begin{cases} X_t &= X_0 - \int_0^t 1_{\{X_s>0\}}ds + \int_0^t \int_{[0,1]} F_s^{-1}(u)N(ds,du) \\ F_t(x) &:= \mathbb{P}(X_t \leq x), \text{ for all } x \geq 0, \, t \geq 0, \\ X_t &\geq 0, \quad a.s. \end{cases}$$

where N is a Poisson point process on $\mathbb{R}_+ \times [0,1]$ with intensity $dsdu$ and $F_s^{-1}(u) := \inf\{x \geq 0 : F_s(x) > u\}$.

An exactly solvable case is when μ_0, the law of X_0, is a mixture of an exponential distribution and a Dirac mass at 0:

$$\mu_0 = p\delta_0 + (1-p)\lambda e^{-\lambda x} 1_{\{x>0\}}dx, \qquad p \in [0,1], \lambda > 0.$$

Denote by $F_\infty(p,\lambda)$ the associated *free energy*:

$$F_\infty(p,\lambda) := \lim_{t\to\infty} e^{-t}\mathbb{E}(X_t),$$

where it is easy to see the existence of the limit.

We study in details the phase transition near criticality and confirm the infinite order phase transition predicted by physicists.

Theorem 1. *Fix some $\lambda \in (0,e)$ and let $p \in (0,1)$ vary.*

(i) If $\lambda \in (1,e)$, setting $p_c = \lambda - \lambda \log \lambda$, then

$$F_\infty(p,\lambda) \sim C \exp\left(-\pi\sqrt{2\lambda}(p_c - p)^{-1/2}\right) \quad \text{as } p \uparrow p_c.$$

(ii) If $\lambda = 1$, we have $F_\infty(1,1) = 0$ and

$$F_\infty(p,\lambda) \sim C(1-p)^{2/3} \exp\left(-\frac{\pi}{\sqrt{2}}(1-p)^{-1/2}\right) \quad \text{as } p \uparrow 1.$$

(iii) If $\lambda \in (0,1)$, we have $F_\infty(1,\lambda) = 0$ and

$$F_\infty(p,\lambda) \sim C(1-p)^{1/(1-\lambda)} \quad \text{as } p \uparrow 1.$$

We present the scaling limit of this model at criticality, which we believe to be universal.

Many questions remain open when μ_0 is a general distribution on $\mathbb{R}_+$, for instance we may ask:

Question 2. *What are necessary and sufficient conditions on μ_0 so that $F_\infty(\mu_0) = 0$?*

We also make the following

Conjecture 3. *If $F_\infty(\mu_0) = 0$, then $X_t \to 0$ in probability as $t \to \infty$.*

The ants walk: Finding geodesics on graphs using reinforcement learning

Cécile Mailler
(joint work with Daniel Kious, Bruno Schapira)

The aim of this talk is to define and analyse a probabilistic model for ants finding shortest paths between their nest and a source of food. The fact that ants can find shortest paths by successive random explorations and without any means of communication other than the pheromones they leave behind them has been observed in the biology literature (see e.g. [1]). In this talk, as in the corresponding pre-print [2], we introduce the first (to our knowledge) probabilistic model for this fascinating phenomenon and show that, in this model, the ants do "find the shortest path(s)".

1. Definition of the model

We consider a sequence of random walkers on a finite graph $\mathcal{G} = (V, E)$ with two distinguished nodes N and F (for "nest" and "food" when the walkers are interpreted as ants). At the beginning of time, all edges of $\mathcal{G}$ are given weight 1. The idea is that the walkers explore the graph from N to F one after each other, and the weights of the edges are updated after each walker reaches F. More precisely, for all $n \geq 1$, the n-th walker starts a random walk from N and walks randomly on the graph until it reaches F. At every step, the walker chooses one of the neighbouring edges with probability proportional to their weights and crosses the chosen edge to the next vertex. Once the n-th walker has reached F, we update the weights of the edges by adding 1 to a subset of the trace of this walker. We look at two possible rules for the choice of this subset of edges to reinforce:

- In the *loop-erased* version of the model, we reinforce the loop-erased time-reversed trace of walker n. This corresponds to how a hiker without a map would go back from F to N by walking backwards on their own trace, but avoiding unnecesary loops: when facing a choice between several edges they crossed on their way to F, they choose the edge that they crossed the earliest on their way forward.
- In the *uniform-geodesic* version of the model, we reinforce the shortest path from N to F inside the trace of the walker (i.e. we only look at the subgraph of all edges that were crossed by this specific walker). If there are several shortest path between N and F, we choose one uniformly at random.

We call this stochastic process the loop-erased or uniform-geodesic ant process.

The interpretation of the model in terms of ants is as follows: (1) the ants only lay pheromones behind them on their way back from the food to the nest, (2) each ant goes back to the nest either following the loop-erasure of their forward trajectory reversed in time (for the loop-erased ant process), or following the shortest path in the subgraph that they have explored on the way forward (for the geodesic ant process), and (3) each ant can sense from the amount of pheromones how many of its predecessors have crossed an edge on their way back to the nest, and crosses each neighboring edge with probability proportional to this number. We conjecture that, following this simple unsupervised reinforcement-learning algorithm, the colony of ants *eventually finds the shortest path(s) between the nest and the food*, more precisely, asymptotically when time goes to infinity, a proportion 1 of all ants go from the nest to the food following a geodesic.

2. Results

We are able to prove that the conjecture is true for all series-parallel graph, as well as for the losange graph of Figure 1(e). For all $n \geq 0$ and $e \in E$, we let $W_n(e)$ denote the weight of edge e at time n (i.e. after the n-th update). By definition, $W_0(e) = 1$ for all $e \in E$.

The family of series-parallel graph is defined recursively as follows: a series-parallel graph is either (i) the graph with two nodes N and F linked by one edge, (ii) two series-parallel graphs merged in parallel, or (iii) two series-parallel graphs merged in series (see Figure 1(a-d))

Theorem 1. *If $\mathcal{G}$ is a series-parallel graph, in the loop-erased version of the model, there exists $(\chi_e)_{e\in E}$ a sequence of random variables such that*

$$\frac{W_e(n)}{n} \to \chi_e, \text{ almost surely when } n \to +\infty,$$

and $\chi_e \neq 0$ almost surely if and only if e belongs to a shortest path between N and F.

Remark 2. *If there is only one shortest path between N and F, then $\chi_e = 1$ almost surely for all edges e that belong to this shortest path.*

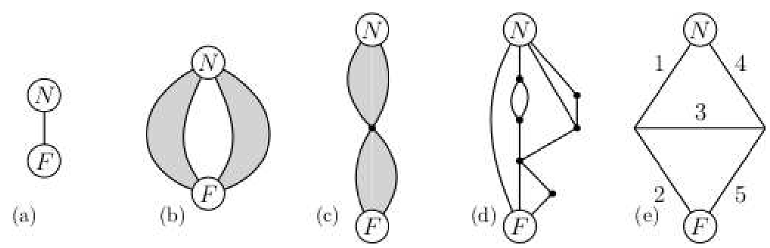

Figure 1. (a-c) The recursive definition of series-parallel graphs, (d) A series-parallel graph, (e) The losange graph

In the losange case, we let $W_i(n)$ be the weight of edge number i (as in Figure 1(e)) at time n.

Theorem 3. *If $\mathcal{G}$ is the losange graph of Figure 1(e), in the uniform-geodesic version of the model, there exists $(\chi_i)_{1\leq i\leq 5}$ a sequence of random variables such that*

$$\frac{W_i(n)}{n} \to \chi_i, \text{ almost surely when } n \to +\infty.$$

Moreover, $\chi_3 = 0$ almost surely, and $\chi_i \in (0,1)$ almost surely for $i \in \{1,2,4,5\}$.

The proofs rely on three main methods: (1) the electric conductances method for random walks on a weighted graph (see, e.g. [3]), (2) couplings with generalised Pólya urn methods, and (3) stochastic approximation methods.

3. Open problems

This work is the first study of a probabilistic model for what is known as "ant colony optimisation", and as such, it raises numerous open questions:

- Can we extend Theorems 1 and 3 to a larger class of graphs? to all graphs?
- Although we believe that Theorem 1 also holds for the uniform-geodesic version of the model (with the caveat that the "if and only if" needs to be replaced by an "if", we also expect the proof to be much more intricate. Similarly, we believe that Theorem 3 holds for the loop-erased version of the model, but expect the proof to be more intricate. These two questions remain open.
- Reinforcement rules other than the loop-erased and the uniform-geodesic ones could be defined: would they also lead to the ants *finding shortest paths*? Preliminary calculation seem to suggest that, if we reinforced the whole trace of each ant, instead of only reinforcing a subset, the ants do not always find the shortest path.

References

[1] *Ant colony optimization.* M. Dorigo and T. Stützle. MIT Press, 2004.
[2] *Finding geodesics on graphs using reinforcement learning.* D. Kious, C. Mailler and B. Schapira. ArXiV:2010.04820
[3] *Random walk: a modern introduction.* G.F. Lawler and V. Limic. Cambridge University Press, 2010.

Some applications of random forests

Fabienne Castell
(joint work with Luca Avena, Alexandre Gaudillière, Clothilde Mélot)

Let $G = \{\mathcal{X}, \mathcal{E}, w\}$ be a weighted connected graph on a finite set of vertices $\mathcal{X}$, where $w : \mathcal{X} \times \mathcal{X} \mapsto \mathbb{R}^+$ is a weight function, and $\mathcal{E} = \{(x,y) : x \neq y \text{ s.t. } w(x,y) > 0\}$ is the set of edges. We associate two basic objects with G.

The first one is the graph Laplacian, i.e. is the operator acting on functions $f : \mathcal{X} \mapsto \mathbb{R}$ as

$$\mathcal{L}f(x) = \sum_{y\in\mathcal{X}} w(x,y)(f(y) - f(x)) .$$

We assume that $\mathcal{L}$ is reversible w.r.t. some (probability) measure μ:

$$\mu(x)w(x,y) = \mu(y)w(y,x)\,.$$

$-\mathcal{L}$ is therefore a symmetric operator on $\ell_2(\mathcal{X},\mu)$, and we denote by $\lambda_0 = 0 < \lambda_1 \leq \cdots \leq \lambda_{n-1}$ its eigenvalues (n is the cardinality of $\mathcal{X}$.).

The second one is a random spanning oriented forest Φ_q. A spanning oriented forest ϕ is a collection of trees oriented from their leaves to their root, whose set of vertices is the whole of $\mathcal{X}$. It will be viewed as a set of edges. The law of Φ_q depends on a parameter $q > 0$ and is defined by

$$\forall \phi \text{ spanning oriented forest,, } \mathbb{P}(\Phi_q = \phi) = \frac{1}{Z(q)} w(\phi) q^{|\rho(\phi)|}\,,$$

where $\rho(\phi)$ is the set of roots of the trees of ϕ, $|\rho(\phi)|$ is the cardinality of $\rho(\phi)$, $w(\phi) = \prod_{(x,y)\in\phi} w(x,y)$, and $Z(q)$ is a normalizing constant.

The random forest Φ_q has been investigated in [1]. Two important features of this forest are the following ones:

(1) $\rho(\Phi_q)$ is "well spread" in the set of vertices $\mathcal{X}$ of G. For instance, the mean time for the Markov process with generator $\mathcal{L}$ to reach $\rho(\Phi_q)$ starting from a vertex $x \in \mathcal{X}$ does not depend on x, when the mean is taken w.r.t. the random forest and the Markov process.
(2) there exists a coupling of the forests Φ_q for all the values of $q > q_0$.

We exploit these features in two different problems which are the estimation of the distribution of the Laplacian spectrum on one part, and the construction of a multiresolution analysis of functions defined on $\mathcal{X}$ on the other part.

Estimate of the spectrum of $\mathcal{L}$. The coupling of the forests Φ_q allows one to obtain Monte-Carlo estimates of the fonction $q \mapsto m_1(q) = \frac{1}{n}\mathbb{E}(|\rho(\Phi_q)|) = \frac{1}{n}\sum_{i=0}^{n-1} \frac{q}{q+\lambda_i}$ (and more generally of the function $m_k(q) = \frac{1}{n}\sum_{i=0}^{n-1}(\frac{q}{q+\lambda_i})^k$) in a time which is lenear in n. In a work in progress, we use these estimates to obtain estimates of the cumulative distribution function of the spectrum of $\mathcal{L}$: $F_{\mathcal{L}}(t) = \frac{1}{n}\sum_{i=0}^{n-1} \mathbb{1}_{\lambda_i \leq t}$.

Multiresolution analysis on G.. A multiresolution analysis on G is an iterative algorithm allowing a sparse coding of a "smooth" function $f : \mathcal{X} \mapsto \mathbb{R}$. It is therefore widely used for compressing or classifying discrete signals defined on a regular grid, setting where this algorithm has been developed.The first step of this algorithm is based on the construction of a basis of $\ell_2(\mathcal{X},\mu)$ made of two parts: the "scale functions" $(\phi_{\bar{x}}, \bar{x} \in \bar{\mathcal{X}})$, indexed by a subset $\bar{\mathcal{X}}$ of $\mathcal{X}$, and the "wavelets functions" $(\psi_{\breve{x}}, \breve{x} \in \breve{\mathcal{X}})$ (with $\breve{\mathcal{X}}$ the complement of $\bar{\mathcal{X}}$) such that

- $\phi_{\bar{x}}$ is localized around $\bar{x}$;
- $\psi_{\breve{x}}$ is localized around $\breve{x}$;
- the wavelets coefficient of f , $\breve{f}(\breve{x}) = \langle f, \psi_{\breve{x}}\rangle$ is small if f is "smooth" in the neighborhood of $\breve{x}$ (high frequency localization).
- the reconstruction of f using its "approximation" coefficients $\bar{f}(\bar{x}) = \langle f, \phi_{\bar{x}}\rangle$ and its wavelets coefficient $\breve{f}(\breve{x})$ is numerically stable.

The space localization refers here to the graph structure of $G = (\mathcal{X}, \mathcal{E}, w)$, while the frequency localization can be defined w.r.t. the eigenfunctions of $\mathcal{L}$: a "smooth function" is a linear combination of the eigenfunctions corresponding to low eigenvalues $\lambda_0, \lambda_1, \cdots, \lambda_k$ ($k << n$)) of $\mathcal{L}$. The subsequent step of the algorithm is analogous the the first one, with $\bar{f}$ as an input. Thus, it requires to define a weight function $\bar{w}$ on $\bar{\mathcal{X}}$. Due to the efficiency of this algorithm for audio signals or images processing, many generalizations to other graphs than the regular grid have been proposed. Such a generalization raises the following questions:

- How to choose $\bar{\mathcal{X}}$ and $\bar{w}$?
- How to choose the scale functions and the wavelet functions satisfying the previous requirements?

Based on the fact that $\rho(\Phi_q)$ is well spread in $\mathcal{X}$, we propose in [3, 2, 4] an answer to these questions, where we choose $\bar{\mathcal{X}}$ as $\rho(\Phi_q)$. The choice of the scale functions and of the weight $\bar{w}$ is made through the construction of a solution $(\Lambda, \bar{\mathcal{L}})$ to the Markov intertwining equation:

$$\Lambda \mathcal{L} = \bar{\mathcal{L}} \Lambda \,. \tag{1}$$

In (1), $\bar{\mathcal{L}}$ is a Markov generator on $\bar{\mathcal{X}} = \rho(\Phi_q)$, providing a natural candidate for $\bar{w}$, while Λ is a rectangular matrix indexed by $\bar{\mathcal{X}} \times \mathcal{X}$, whose rows $\nu_{\bar{x}} = \Lambda(\bar{x}, \cdot)$ are positive measures on $\mathcal{X}$. When (1) is satisfied, $\nu_{\bar{x}}$ is an element of an eigenspace of $\mathcal{L}$ whose dimension is $|\bar{\mathcal{X}}|$ (frequency localization), and it is quite natural to define $\bar{f}(\bar{x})$ as $\nu_{\bar{x}}(f)$. The problem is that solving (1) requires the spectral decomposition of $\mathcal{L}$, which is too costly, and often results in $\nu_{\bar{x}}$ not being space-localized. Henceforth, we study in [4] approximate space-localized solutions to (1), which are used in [2] to construct the multiresolution analysis. This construction has led to the free software [5].

References

[1] L. Avena, A. Gaudillière, *Two applications of random spanning forests.* J. Theoret. Probab. **31** (2018), no. 4, 1975–2004.

[2] L. Avena, F. Castell, F., A. Gaudillière, C. Mélot, *Intertwining wavelets or Multiresolution analysis on graphs through random forests.* Appl. Comput. Harmon. Anal. **48** (2020), no. 3, 949–992.

[3] L. Avena, F. Castell, F., A. Gaudillière, C. Mélot, *Random forests and Network analysis.* J. Stat. Phys. **173** (2018), no. 3-4, 985–1027.

[4] L. Avena, F. Castell, F., A. Gaudillière, C. Mélot, *Approximate and exact solutions of intertwining equations through random spanning forests.* To appear in "In and Out of Equilibrium 3. Celebrating Vladas Sidoravicius", Progress in Probability.

[5] free Software *IntertwiningWavelet* toolbox: https://archimede.pages.math.cnrs.fr/intertwiningwavelet/

Random walks avoiding their convex hull with a finite memory

Andrew R. Wade
(joint work with Francis Comets, Mikhail V. Menshikov)

Introduction. We describe recent work [2] on a multidimensional random walk that interacts with its previous history via an excluded volume effect. Self-interaction may be *local*, as in reinforced or excited random walks, where the walker is biased by its occupation measure in the immediate vicinity, or *global*, where the interaction is mediated by e.g. the centre of mass or convex hull of the whole trajectory. Such non-Markovian processes arise naturally in systems with learning, resource depletion, or physical interaction. For example, an animal may tend to avoid previously visited regions as it forages for fresh resources [3].

Fix $d \geq 2$ (ambient dimension) and $k \geq d-1$ an integer (the 'memory'). We study the process $X = (X_0, X_1, X_2, \dots)$ in $\mathbb{R}^d$, where, roughly speaking, given $X_0, \dots, X_n$, the next position X_{n+1} is uniform on the unit ball centred at X_n but conditioned so that the line segment from X_n to X_{n+1} does not intersect the convex hull of $\{0, X_{n-k}, X_{n-k+1}, \dots, X_n\}$ at any point other than X_n. See Figure 1.

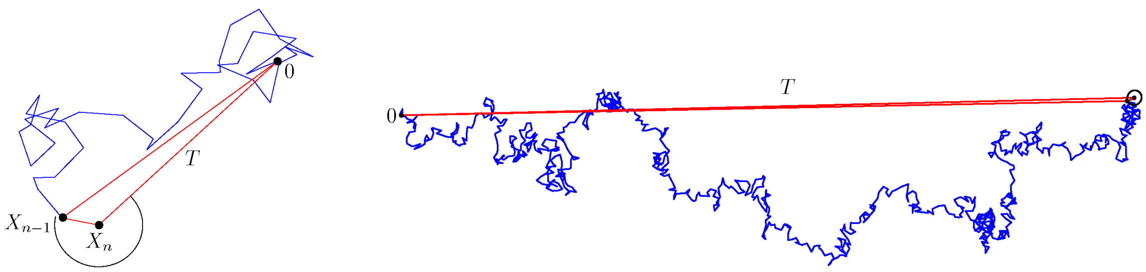

Figure 1. The model with $d = 2$, $k = 1$ for $n = 40$ (*left*) and 1000 steps (*right*). The excluded convex hull is the triangle T with vertices X_n, X_{n-1}, and 0, and X_{n+1} is uniform on the unit-radius disc centred at X_n, excluding the sector generated by the angle at X_n of T.

The inspiration for this model comes from the case '$k = \infty$', which is a variation on a model of Angel *et al.* [1] that avoids the convex hull of its entire previous trajectory. The $k = \infty$ ('infinite memory') model is conjectured to be *ballistic*, i.e., to exhibit a positive limiting speed and a limiting direction; a 'lim inf' speed result is known [4]. Our first main result is to establish ballisticity for the *finite* k model described above.

Theorem 1. *Let $k \geq d-1$. There exist a positive constant $v_{d,k}$ and a uniform random unit vector ℓ such that $\lim_{n\to\infty} n^{-1} X_n = v_{d,k}\ell$, a.s.*

The constants $v_{d,k}$ seem hard to compute in general, but:

Theorem 2. *If $d = 2$ and $k = 1$, then $v_{2,1} = \frac{8}{9\pi^2} \approx 0.09006327$.*

Ballisticity I: Renewal structure. Fix $\delta \in (0, 1/8)$ from now on. For $x \in \mathbb{R}^d$ define $\Pi(x) := \prod_{i=1}^{k} B(x + \frac{i}{2}\hat{x}; \delta) \subseteq (\mathbb{R}^d)^k$. Say X has *good geometry* at time n, and that G_n occurs, if $(X_{n-k}, \dots, X_n)$ is such that the support of $(X_{n+1}, \dots, X_{n+k})$ contains $\Pi(X_n)$. See Figure 2.

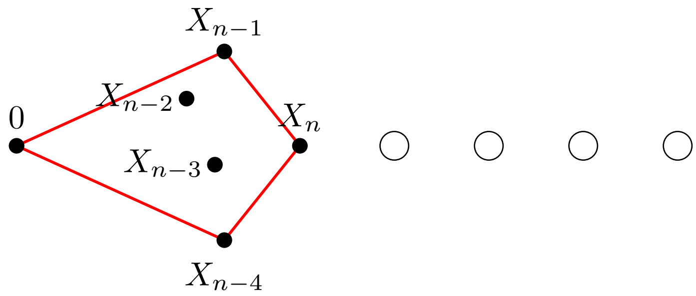

FIGURE 2. A configuration with good geometry for $d = 2$ and $k = 4$.

We show that, if the configuration has good geometry, then we can extract from the law of $(X_{n+1}, \dots, X_{n+k})$ a component that is uniform on $\Pi(X_n)$. Thus we can identify 'renewals', i.e., times at which $(X_{n+1}, \dots, X_{n+k})$ follows the uniform distribution on $\Pi(X_n)$. These renewals occur rather frequently; even if the geometry is not good, it is fairly likely to be good after a few more steps. Thus the inter-renewal times have a uniform exponential tail bound. The renewal structure entails that (i) the process between renewals has strictly positive *radial drift*; and (ii) the transverse fluctuations are symmetric.

This essentially yields the limiting direction, but does not give a *speed*. The segments of the process between renewal times are *not homogeneous*, due to the special role played by the origin in the construction of the process, and so the radial drift is *not constant*.

Ballisticity II: Coupling. Let $\tau_1, \tau_2, \dots$ denote the renewal times. Our limiting speed will follow from:

Proposition 3. *There are positive constants λ and u such that, for all $\gamma \in (0, 1)$,*

$$\mathbb{E}[\tau_{n+1} - \tau_n \mid \mathcal{F}_{\tau_n}] = \lambda + o(n^{-\gamma}); \text{ and } \mathbb{E}[X_{\tau_{n+1}} - X_{\tau_n} \mid \mathcal{F}_{\tau_n}] = u\hat{X}_{\tau_n} + o(n^{-\gamma}).$$

We get this result from a *spatially homogeneous* version of the process for which the above equalities hold exactly (with no $o(n^{-\gamma})$ term) and a coupling over the inter-renewal interval. This homogeneous process is like the normal process, but with the origin 'sent to infinity' in the direction $-\ell$ for a fixed unit vector ℓ. See Figure 3.

If $\ell = \hat{X}_n$ (as in Figure 3) then the next-step transition law of the ℓ-process is the same as the original process. The coupling idea is that at time τ_n, we set $\ell = \hat{X}_{\tau_n}$ and run both processes until the next renewal on the same probability space, with increments coupled in the maximal way. We show that $\hat{X}$ will not deviate much from ℓ over the entire time, and hence the coupling has a good chance of success. These are the main steps in the proof of Theorem 1.

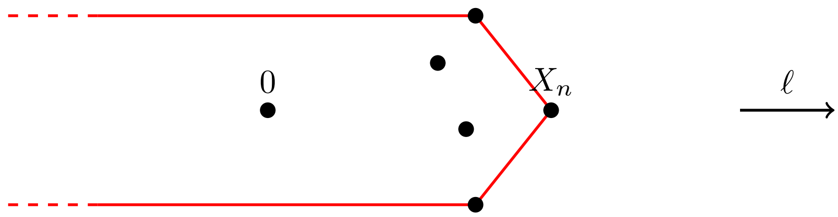

FIGURE 3. An example of the ℓ-process with $d = 2$ and $k = 4$.

The unit-memory case in the plane. The magnitude $\theta_n \in [0, \pi]$ of the interior angle of the convex hull of $\{0, X_{n-1}, X_n\}$ at X_n (see Figure 4) determines the *local drift* and ultimately the *global speed*, via

$$v_{2,1} = \lim_{n\to\infty} \mathbb{E}\left[\frac{2\sin\theta_n}{6\pi - 3\theta_n}\right]. \tag{1}$$

The process θ_n is not Markov, but not far from being so (cf. Figure 4):

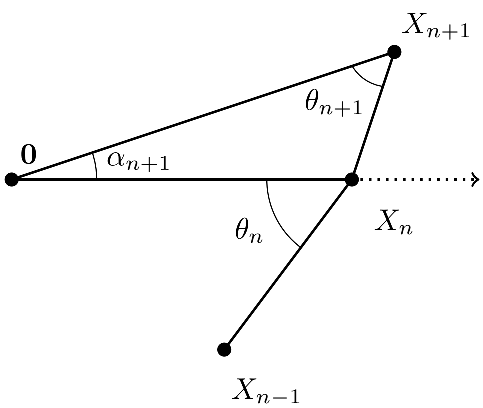

FIGURE 4. $d = 2$, $k = 1$.

Lemma 4. *We have $\theta_{n+1} = |(2\pi - \theta_n)U_{n+1} - \pi| - \alpha_{n+1}$, where $U_1, U_2, \dots$ are i.i.d. $U[0,1]$ and $\alpha_n \to 0$ a.s. Moreover, as $n \to \infty$, $\theta_n \to \theta$ in distribution, where*

$$\theta \overset{d}{=} |(2\pi - \theta)U - \pi|, \ \theta \in [0, \pi], \tag{2}$$

with $U \sim U[0,1]$ and independent of θ.

The distribution of θ determined by (2) has probability density $f(t) = \frac{2}{3\pi^2}(2\pi - t)$, $t \in [0, \pi]$. Then we can compute from (1) that $v_{2,1} = \frac{8}{9\pi^2}$, as stated in Theorem 2.

Final remarks. The renewal structure depends crucially on the finite memory, but one might seek to approach the $k = \infty$ model by taking a limit. Simulations suggest that $v_{d,k} \le v_{d,k+1}$, which one might hope to establish via coupling the process with memory k and the process with memory $k + 1$, but this does not seem straightforward. There is likely some 'discontinuity' as $k \to \infty$, because for the finite-k model one anticipates a central limit theorem for $n^{-1/2}(X_n - v_{d,k} n \hat{X}_n)$, while for the $k = \infty$ model, fluctuations are conjectured to be of larger order [1]. Nevertheless, one might imagine:

Conjecture 5. *We have* $\lim_{k\to\infty} v_{d,k} = v_{d,\infty}$*, the (conjectural) speed of the* $k = \infty$ *model.*

Finally, we remark that including the origin in the convex hull to exclude is crucial; otherwise the process is driftless on large scales one would expect it to be diffusive.

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References

[1] O. Angel, I. Benjamini & B. Viràg, Random walks that avoid their past convex hull. *Electron. Commun. Probab.* **8** (2003) 6–16.
[2] F. Comets, M.V. Menshikov & A.R. Wade, Random walks avoiding their convex hull with a finite memory. *Indagationes Math.* **31** (2020) 117–146.
[3] P.E. Smouse, S. Focardi, P.R. Moorcroft, J.G. Kie, J.D. Forester & J.M. Morales, Stochastic modelling of animal movement. *Phil. Trans. Roy. Soc. Ser. B Biol. Sci.* **365** (2010) 2201–2211.
[4] M. Zerner, On the speed of a planar random walk avoiding its past convex hull. *Ann. Inst. H. Poincaré Probab. Statist.* **41** (2005) 887–900.

Asymptotics for the Green function of an asymptoticaly stable random walk in a half space

Denis Denisov
(joint work with Vitali Wachtel)

We consider a random walk $\{S_n, n \ge 1\}$ on $\mathbb{R}^d$, $d \ge 1$, where

$$S_n = X_1 + \cdots + X_n$$

and $\{X_n, n \ge 1\}$ is a family of independent copies of a random vector

$$X = (X^{(1)}, \ldots, X^{(d)}) = (X^{(1)}, X^{(2,d)})$$

Let

$$\mathbb{R}^d_+ = \{(x^{(1)}, \ldots, x^{(d)}) : x^{(1)} > 0\}.$$

For $x = (x^{(1)}, \ldots, x^{(d)}) : x^{(1)} \ge 0$ let

$$\tau_x^- := \min\{\, n \ge 1 : x + S_n \notin \mathbb{R}^d_+ \,\}$$

be the first time the random walk exits the (positive) half space.

Let $\Delta = [0,1)^d$. For $x, y \in \mathbb{R}^d_+$ we study the asymptotic behaviour of local probabilities

$$G_n(x,y) = \mathbb{P}(x + S_n = y, \tau_x^- > n)$$

when X is a lattice random vector and

$$G_n(x,y) = \mathbb{P}(x + S_n \in y + \Delta, \tau_x^- > n),$$

when X is a non-lattice random vector, as n, x, y increases. We also study the behaviour of the Green function

$$G(x, y) = \delta(x, y) + \sum_{n=1}^{\infty} G_n(x, y)$$

as x and/or y increases.

The main assumption is that

$$\frac{S_n}{c_n} \overset{d}{\to} \zeta_\alpha, \tag{1}$$

where ζ_α is a multivariate stable law of index $\alpha \in (0, 2]$ and c_n is a scaling sequence, which is regularly varying of index $1/\alpha$. Furthermore, we assume that

$$\mathbb{P}(\zeta_\alpha \in \mathbb{R}^d_+) = \mathbb{P}(\zeta_\alpha^{(1)} > 0) = \rho \in (0, 1)$$

ensuring that the Spitzer-Doney condition holds

$$\mathbb{P}(S_n^{(1)} > 0) \to \rho, \quad n \to \infty.$$

In case $d = 1$ these questions have been considered in [18],[11] and [1]. For general cones Gaussian estimates for $G_n(x, y)$ have been obtained by Varopoulos [15, 16] for random walks with bounded increments. For Lipschitz domains under the assumption of existence of sufficiently many moments Gaussian estimates for $G_n(x, y)$ have been obtained by Varopoulos [17]. Asymptotics for $G(x, y)$ for integer-valued random walks in half-space have been obtained in [14] under the assumptions close to $\mathbb{E}|X|^d < \infty$. In [7] the asymptotics for $G(x, y)$ have been obtained in the Gaussian case for general cones when x and y are far from the boundary or fixed. Next in [13] the asymptotics for the Green function have been studied for convex cones. Exact formula for the Green function of a rotationally invariant Lévy process has been obtained in [19]. For Lamperti Markov chains asymptotics for $G(x, y)$ when x is fixed have been found in [3].

In the multidimensional case one can study random walks with finite variance by approximating them with the Brownian motion. In this situation an important role is played by harmonic function for the killed random walks, which can also be approximated by the harmonic function of the Brownian motion. More generally, this approach can be applied to Markov chains by approximating Markov chains with the corresponding limiting diffusion process. We used this approach for ordered random walks [6], random walks on cones [7, 10], integrated random walks [8, 2], random walks over moving boundaries [4, 5, 9]. One of the aims of this work is to start consideration of multidimensional random walks with infinite variance in unbounded domains. Here, the half space is the first step which should enable us to consider at least convex domains.

We are now ready to formulate some of the results. We first analyse $G_n(0, y)$ and then obtaining asymptotics for $G(0, y)$ by summing up $G_n(0, y)$ and afterwards asymptotics for $G(x, y)$. There are three main regimes to consider $G_n(0, y)$: small deviations, normal deviations and large deviations. For small and normal deviations we use the approach of [18] to small and normal deviations extended to the multidimensional situation. This approach is based on analysis of a recursive equality, which is a corollary to the Baxter-Spitzer identity. For large deviations we cannot use approach [12]. However, it seems that recursive approach of [18] can be generalised to this situation as well.

We will present several of our results in the non-lattice case now. In the normal deviations regime the following results holds.

Theorem 1. *Suppose that* (1) *holds and the distribution of* X *is non-lattice. Then, there exist a random vector* M *with the density* p_M *such that*

$$c_n^d(S_n \in y + \Delta \mid \tau_0^- > n) - p_M(y/c_n) \to 0 \text{ as } n \to \infty$$

uniformly in $\mathbb{R}_+^d$.

Our next theorem refines Theorem 1 in the mentioned domain of small deviations, i.e. when $x^{(1)}/c_n \to 0$. Let $\chi^+ := S_{\tau^+}^{(1)}$ be the first (strict) ladder height and let $(\chi_n)_{n=1}^\infty$ be a sequence of i.i.i. copies of χ^+. Let

$$H(u) := \mathrm{I}\{u > 0\} + \sum_{k=1}^{\infty} \mathbf{P}(\chi_1^+ + \ldots + \chi_k^+ < u) \tag{2}$$

be the renewal function of the increasing ladder height process. Clearly, H is a left-continuous function.

Theorem 2 *Suppose that* (1) *holds and the distribution of* X *is non-lattice. Then*

$$c_n^d \mathbb{P}(S_n \in y + \Delta \mid \tau_0^- > n) \sim g_\alpha\left(0, \frac{y^{(2,d)}}{c_n}\right) \frac{\int_{y^{(1)}}^{y^{(1)}+\Delta} H(u)du}{n\mathbb{P}\left(\tau^- > n\right)}, \quad \text{as }, n \to \infty$$

uniformly in $y^{(1)} \in (0, \delta_n c_n]$, *where* $\delta_n \to 0$ *as* $n \to \infty$ *and* g_α *is the density of the limiting law* ζ_α.

Further results include estimates and asymptotics for $G_n(0, y)$ in the large deviations regime, asymptotics for $G_n(x, y)$and $G(x, y)$.

References

[1] Caravenna, F. and Chaumont, L. An invariance principle for random walk bridges conditioned to stay positive. *Electron. J. Probab.* **18**: paper 60, 32 pp., 2013.

[2] Denisov, D., Kolb, M. and Wachtel, V. Local asymptotics for the area of random walk excursions *J. London Math. Soc.* **91** (2): 495-513

[3] Denisov, D., Korshunov, D. and Wachtel, V. Renewal theory for transient Markov chains with asymptotically zero drift. *Trans. Amer. Math. Soc.* **373**(10): 7253–7286, 2020.

[4] Denisov, D., Sakhanenko, A. and Wachtel, V. First-passage times for random walks with non-identically distributed increments *Ann. Probab.*, **46**(6): 3313-3350, 2018.

[5] Denisov, D., Sakhanenko, A. and Wachtel, V. First-passage times over moving boundaries for asymptotically stable walks arXiv:1801.04136.

[6] Denisov, D. and V. Wachtel, V. Conditional limit theorems for ordered random walks. *Elec. J. Probab.*, **15**: 292-322, 2010.

[7] Denisov, D. and Wachtel, V. Random walks in cones. *Ann. Probab.*, **43**: 992-1044, 2015.

[8] Denisov, D. and Wachtel, V. Exit times for integrated random walks. *Ann. Inst. H. Poincare Probab. Statist.*, **51**: 167-193, 2015.

[9] Denisov, D. and Wachtel, V. Exact asymptotics for the moment of crossing a curved boundary by an asymptotically stable random walk *Theory of Prob. Appl.* **60**(3): 481-500, 2016

[10] Denisov, D. and Wachtel, V. Alternative constructions of a harmonic function for a random walk in a cone. *Elec. J. Probab.*, **24**: paper no. 92, 26pp, 2019.

[11] Doney, R. A. Local behaviour of first passage probabilities. *Probab. Theory Relat. Fields* **152**: 559–88, 2012.

[12] Doney, R. and Jones, E. Large deviation results for random walks conditioned to stay positive. *Electron. Commun. Probab.* **17**: paper no. 38, 1-11, 2012.

[13] Duraj J., Raschel K., Tarrago, P. and Wachtel V. Martin boundary of random walks in convex cones *arXiv*:2003.03647, 2020

[14] Uchiyama, K. Green's functions of random walks on the upper half plane. *Tohoku Math. J.* **66**(2): 289–307, 2014

[15] Varopoulos, N.Th. Potential theory in conical domains. *Math. Proc. Camb. Phil. Soc.*, **125**: 335-384, 1999.

[16] Varopoulos, N. Th. Potential theory in conical domains. II. *Math. Proc. Camb. Phil. Soc.*, **129**: 301-320, 2000.

[17] Varopoulos, N. Th. Potential Theory in Lipschitz Domains. *Canad. J. of Math.*, **59**: 1057-1120, 2001.

[18] Vatutin, V. A. and Wachtel, V. Local probabilities for random walks conditioned to stay positive. *Probab. Theory Relat. Fields* **143**: 177–217, 2009.

[19] Tamura, Y. and Tanaka, H. On a formula on the potential operators of absorbing Lévy processes in the half space. *Stoch. Proc. Appl.* **118**: 199–212, 2008

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