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І. І. Голіченко[1], *к. ф.-м. н., доц.*
М. П. Моклячук[2], *д. ф.-м. н., проф.*

Задача фільтрації для періодично корельованих випадкових послідовностей із пропущенними значеннями

[1] Національний технічний університет України "Київський політехнічний інститут Ігоря Сікорського", м. Київ 03056, Україна
e-mail: idubovetska@gmail.com
[2] Київський національний університет імені Тараса Шевченка, 01601, м. Київ, вул. Володимирська 64
e-mail: moklyachuk@gmail.com

I. I. Golichenko[1], *Ph.D., Assoc. Prof.*
M. P. Moklyachuk[2], *Dr.Sci., Prof.*

Filtering problem for periodically correlated stochastic sequences with missing observations

[1] National Technical University of Ukraine "Igor Sikorsky Kyiv Politechnic Institute", Kyiv 03056, Ukraine
e-mail: idubovetska@gmail.com
[2] Taras Shevchenko National University of Kyiv, 64 Volodymyrska Str., Kyiv 01601, Ukraine
e-mail: moklyachuk@gmail.com

Досліджується задача оптимального оцінювання лінійних функціоналів від невідомих значень періодично корельованої стохастичної послідовності за спостереженнями послідовності із пропущеними значеннями. Знайдено формули для обчислення значень середньоквартичних похибок та спектральних характеристик оптимальних оцінок функціоналів у випадку, коли спектральні щільності послідовностей точно відомі. Отримано формули для визначення найменш сприятливих спектральних щільностей та мінімаксних спектральних характеристик оптимальних лінійних оцінок функціоналів у випадку спектральної невизначеності, коли спектральні щільності послідовностей точно не відомі, але задано множини допустимих спектральних щільностей.
Ключові слова: періодично корельована стохастична послідовність, мінімаксна (робастна) оцінка, найменш сприятлива спектральна щільність, мінімаксні спектральні характеристики.

The problem of the mean-square optimal estimation of the linear functionals which depend on the unknown values of a periodically correlated stochastic sequence from observations of the sequence with missings is considered. Formulas for calculation the mean-square error and the spectral characteristic of the optimal estimate of the functionals are proposed in the case where spectral densities of the sequences are exactly known. Formulas that determine the least favorable spectral densities and the minimax spectral characteristics are proposed in the case of spectral uncertainty, when spectral densities of sequences are not exactly known but the class of admissible spectral densities is given.
Key Words: periodically correlated stochastic sequence, minimax (robust) estimate, least favorable spectral density, minimax spectral characteristics.

Introduction

W. R. Bennett in 1958 [1] started to explore cyclostationarity as a phenomenon and property of the process, which describes signals in channels of communication. Studying the statistical characteristics of information transmission, he calls the group of telegraph signals the cyclostationary process, that is the process whose group of statistics changes periodically with time. W. A. Gardner and L. E. Franks [3] highlights the greatest similarity of cyclostationary processes, which are a subclass of nonstationary processes, with stationary processes. W. A. Gardner [4], W. A. Gardner, A. Napolitano and L. Paura [5] presented bibliography of works in which properties and applications of cyclostationary processes were studied. Recent developments and applications of cyclostationary signal analysis are reviewed in the papers by A. Napolitano [28], [29]. In other sources cyclostationary processes are called periodically stationary, periodically nonstationary, periodically correlated. We will use the term periodically correlated processes.

E. G. Gladyshev in 1961 [6] was the first who published the analysis of spectral properties and representation of periodically correlated sequences based on its connection with vector stationary sequences. He formulated the necessary and sufficient conditions for determining of periodically correlated sequence in terms of the correlation function. A. Makagon carried in his works [17], [18] detailed spectral analysis of periodically correlated sequences. Main ideas of the research of periodically correlated sequences are outlined in the book by H. L. Hurd and A. Miamee [12].

The problem of estimation of unknown values of random processes is one of the very important and topical subsections of the theory of stochastic processes. Processes that are observed can be completely defined by its characteristics (correlation function, spectral density, canonical decomposition) or their characteristics can be defined only by the set of admissible values of characteristics. The linear extrapolation and interpolation problems for stationary stochastic processes under the condition that spectral densities are known exactly were first introduced by A. N. Kolmogorov [15]. Solutions of the extrapolation and filtering problems for stationary processes and sequences with rational spectral densities were offered by N. Wiener [34] and A. M. Yaglom [35]. Prediction problems for vector-valued stationary processes were investigated by Yu. A. Rozanov [32] and E. J. Hannan [11].

Since processes often accompanied by undesirable noise it is naturally to assume that the exact value of spectral density is unknown and the model of process is given by a set of restrictions on spectral density. K. S. Vastola and H. V. Poor [33] showed for certain classes of spectral densities that the Wiener filter is very sensitive to minor changes of spectral model unlike the robust Wiener filter. That is the filter is the least sensitive to the worst case of uncertainty. Thus, it is reasonable to use the minimax (robust) estimation method, which allows to define the optimal estimate for all densities from a certain class of the admissible spectral densities simultaneously. Ulf Grenander [10] was the first who proposed the minimax approach to the extrapolation problem for stationary processes. A survey of results in minimax-robust methods of data processing can be found in the paper by S. A. Kassam and H. V. Poor [14]. Formulation and investigation of the problems of extrapolation and interpolation of linear functionals which depend on the unknown values of stationary sequences from observations with and without noise are presented by M. P. Moklyachuk in the paper [23]. Similar problems of optimal estimation of vector-valued stationary sequences and processes were examined by M. P. Moklyachuk and O. Yu. Masyutka [24], by O. Yu. Masyutka, I. I. Golichenko and M. P. Moklyachuk [22].

In their book M. M. Luz and M. P. Moklyachuk [16] investigated the minimax estimation problems for linear functionals which depends on unobserved values of stochastic sequences with stationary increments. In their book I. I. Golichenko and M. P. Moklyachuk [27] presented results of investigation of the interpolation, extrapolation and filtering problems for linear functionals from periodically correlated stochastic sequences and processes.

The interpolation and extrapolation problems of linear functionals from periodically correlated stochastic sequences with missing observations were investigated by I. I. Golichenko and M. P. Moklyachuk in [7], [9], by I. I. Golichenko, O. Yu. Masyutka and M. P. Moklyachuk in [8]. The results of the study of the extrapolation, interpolation and filtering problems for linear functionals constructed from unobserved values of multidimensional stochastic sequences and processes are presented in the papers by O. Yu. Masyutka, M. P. Moklyachuk and M. I. Sidei [19]–[21], [26]. We also refer to the book by M. P. Moklyachuk, O. Yu. Masyutka and I. I. Golichenko [25] where results of the investigation of the problem of mean square optimal estimation (forecasting, interpolation, and filtering) of linear functionals constructed from unobserved values of periodically correlated isotropic random fields are described.

In this paper we study the problem of optimal linear estimation of the functional $A\zeta = \sum\limits_{j\in Z_s} a(j)\zeta(-j)$, which depends on the unknown values of a periodically correlated stochastic sequence $\zeta(-j)$, $j \in Z_s = \{T+1, T+2, \dots\} \setminus \bigcup\limits_{i=1}^{s} \{M_i \cdot T+1, \dots, (M_i+N_i)\cdot T\}$. Estimation is based on observations of the sequence $\zeta(j)+\theta(j)$ at points $j \in \{\dots, -(T+2), -(T+1)\} \setminus S$, $S = \bigcup\limits_{i=1}^{s} \{-(M_i+N_i)\cdot T, \dots, -M_i \cdot T-1\}$. $\theta(j)$ is an uncorrelated with $\zeta(j)$ periodically correlated stochastic sequence. Formulas for calculation the mean square error and the spectral characteristic of the optimal estimate of the functional $A\zeta$ are proposed in the case where spectral densities are exactly known. Formulas that determine the least favorable spectral densities and the minimax spectral characteristic are proposed for the given class of admissible spectral densities.

1 Periodically correlated and multidimensional stationary sequences

The term *periodically correlated* process was introduced by E. G. Gladyshev [6] while W. R. Bennett [1] called random and periodic processes *cyclostationary* process.

Periodically correlated sequences are stochastic sequences that have periodic structure (see, for example, the book by H. L. Hurd and A. Miamee [12]).

Definition 1.1. A complex valued stochastic sequence $\zeta(n), n \in \mathbb{Z}$ with zero mean, $\mathsf{E}\zeta(n) = 0$, and finite variance, $\mathsf{E}|\zeta(n)|^2 < +\infty$, is called cyclostationary or periodically correlated (PC) with period T (T-PC) if for every $n, m \in \mathbb{Z}$

$$\mathsf{E}\zeta(n+T)\overline{\zeta(m+T)} = R(n+T, m+T) = R(n,m) \tag{1}$$

and there are no smaller values of $T > 0$ for which (1) holds true.

Definition 1.2. A complex valued T-variate stochastic sequence $\vec{\xi}(n) = \{\xi_\nu(n)\}_{\nu=1}^{T}, n \in \mathbb{Z}$ with zero mean, $\mathsf{E}\xi_\nu(n) = 0, \nu = 1, \dots, T$, and $\mathsf{E}||\vec{\xi}(n)||^2 < \infty$ is called stationary if for all $n, m \in \mathbb{Z}$ and $\nu, \mu \in \{1, \dots, T\}$

$$\mathsf{E}\xi_\nu(n)\overline{\xi_\mu(m)} = R_{\nu\mu}(n,m) = R_{\nu\mu}(n-m).$$

If this is the case, we denote $R(n) = \{R_{\nu\mu}(n)\}_{\nu,\mu=1}^{T}$ and call it the *covariance matrix* of T-variate stochastic sequence $\vec{\xi}(n)$.

Proposition 1.1. (E. G. Gladyshev [6]). A stochastic sequence $\zeta(n)$ is PC with period T if and only if there exists a T-variate stationary sequence $\vec{\xi}(n) = \{\xi_\nu(n)\}_{\nu=1}^{T}$ such that $\zeta(n)$ has the representation

$$\zeta(n) = \sum_{\nu=1}^{T} e^{2\pi i n\nu/T}\xi_\nu(n), \ n \in \mathbb{Z}. \tag{2}$$

The sequence $\vec{\xi}(n)$ is called *generating sequence* of the sequence $\zeta(n)$.

Proposition 1.2. (E. G. Gladyshev [6]). A complex valued stochastic sequence $\zeta(n), n \in \mathbb{Z}$ with zero mean and finite variance is PC with period T if and only if the T-variate blocked sequence $\vec{\zeta}(n)$ of the form

$$[\vec{\zeta}(n)]_p = \zeta(nT+p), \ n \in \mathbb{Z}, p = 1, \dots, T \tag{3}$$

is stationary.

We will denote by $f^{\vec{\zeta}}(\lambda) = \left\{f^{\vec{\zeta}}_{\nu\mu}(\lambda)\right\}_{\nu,\mu=1}^{T}$ the matrix valued spectral density function of the T-variate stationary sequence $\vec{\zeta}(n) = (\zeta_1(n), \dots, \zeta_T(n))^\top$ arising from the T-blocking (3) of a univariate T-PC sequence $\zeta(n)$.

2 The classical projection method of filtering

Let $\zeta(j)$ and $\theta(j)$ be uncorrelated T-PC stochastic sequences. Consider the problem of optimal linear estimation of the functional

$$A\zeta = \sum_{j \in Z_s} a(j)\zeta(-j),$$

that depends on the unknown values of T-PC stochastic sequence $\zeta(-j)$, $j \in Z_s = \{T+1, T+2, \dots\} \setminus \bigcup_{i=1}^{s} \{M_i \cdot T + 1, \dots, (M_i + N_i) \cdot T\}, (M_i \geqslant 1, M_i > M_{i-1} + N_{i-1}, i = 1, ...s)$. Estimation is based on observations of the sequence $\zeta(j) + \theta(j)$ at points $j \in \{\dots, -(T+2), -(T+1)\} \setminus S, S = \bigcup_{i=1}^{s} \{-(M_i + N_i) \cdot T, \dots, -M_i \cdot T - 1\}$. Note that in every interval of known and unknown observations of sequence $\zeta(j) + \theta(j)$ the amount of observations is a multiple of the period T.

Let assume that the coefficients $a(j), j \in Z_s$ which determine the functional $A\zeta$ satisfy condition

$$\sum_{j \in Z_s} |a(j)| < \infty \tag{4}$$

and are of the form

$$a(j) = a\left(\left(j - \left[\frac{j}{T}\right]T\right) + \left[\frac{j}{T}\right]T\right) = a(\nu + \tilde{j}T) = a(\tilde{j})e^{2\pi i \tilde{j}\nu/T}, \tag{5}$$

where $\nu = 1, \dots, T, \tilde{j} = \left[\frac{j}{T}\right] \geqslant 1$; $\nu = T$ and $\tilde{j} = \lambda - 1$, if $j = T \cdot \lambda$, $\lambda \in \mathbb{Z}$, or

$$a(j) = a(T \cdot \lambda) = a(T + (\lambda - 1)T) = a(\lambda - 1)e^{2\pi i(\lambda-1)T/T}.$$

Under the condition (4) the functional $A\zeta$ has the finite second moment.

Using Proposition 1.2, the linear functional $A\zeta$ can be written as follows

$$A\zeta = \sum_{j\in Z_s} a(j)\zeta(-j) =$$

$$\sum_{\widetilde{j}\in \widetilde{Z}_s} a(\widetilde{j}) \sum_{\nu=1}^{T} e^{2\pi i\widetilde{j}\nu/T}\zeta(-(\nu+\widetilde{j}T)) =$$

$$\sum_{\widetilde{j}\in \widetilde{Z}_s} \sum_{\nu=1}^{T} a(\widetilde{j})e^{2\pi i\widetilde{j}\nu/T}\zeta_\nu(-\widetilde{j}) =$$

$$\sum_{\widetilde{j}\in \widetilde{Z}_s} \vec{a}^\top(\widetilde{j})\vec{\zeta}(-\widetilde{j}) = A\vec{\zeta},$$

where

$$\widetilde{Z}_s = \{1,2,3,\dots,\}\setminus\bigcup_{i=1}^{s}\{M_i,\dots,M_i+N_i-1\},$$

$$\vec{a}^\top(\widetilde{j}) = \left(a_1(\widetilde{j}),\dots,a_T(\widetilde{j})\right),$$

$$a_\nu(\widetilde{j}) = a(\widetilde{j})e^{2\pi i\widetilde{j}\nu/T},\ \nu=1,\dots,T, \qquad (6)$$

$\vec{\zeta}(\widetilde{j}) = \left\{\zeta_\nu(\widetilde{j})\right\}_{\nu=1}^{T}$ is T-variate stationary sequence, obtained by the T-blocking (3) of univariate T-PC sequence $\zeta(-j)$, $j\in Z_s$.

Let $\vec{\zeta}(j)$ and $\vec{\theta}(j)$ be uncorrelated T-variate stationary stochastic sequences with the spectral density matrices $f^{\vec{\zeta}}(\lambda) = \left\{f^{\vec{\zeta}}_{\nu\mu}(\lambda)\right\}_{\nu,\mu=1}^{T}$ and $f^{\vec{\theta}}(\lambda) = \left\{f^{\vec{\theta}}_{\nu\mu}(\lambda)\right\}_{\nu,\mu=1}^{T}$, respectively. Consider the problem of optimal linear estimation of the functional

$$A\vec{\zeta} = \sum_{\widetilde{j}\in \widetilde{Z}_s} \vec{a}^\top(\widetilde{j})\vec{\zeta}(-\widetilde{j}),$$

that depends on the unknown values of sequence $\vec{\zeta}(-\widetilde{j}), \widetilde{j}\in\widetilde{Z}_s$, based on observations of the sequence $\vec{\zeta}(\widetilde{j})+\vec{\theta}(\widetilde{j})$ at points $\widetilde{j}\in\{...,-2,-1\}\setminus\widetilde{S}$, $\widetilde{S}=\bigcup_{i=1}^{s}\{-(M_i+N_i)+1,\dots,-M_i\}$.

Let the spectral densities $f^{\vec{\zeta}}(\lambda)$ and $f^{\vec{\theta}}(\lambda)$ satisfy the minimality condition

$$\int_{-\pi}^{\pi} Tr\Big[(f^{\vec{\zeta}}(\lambda)+f^{\vec{\theta}}(\lambda))^{-1}\Big]d\lambda < +\infty. \qquad (7)$$

Condition (7) is necessary and sufficient in order that the error-free filtering of unknown values of the sequence $\vec{\zeta}(j)+\vec{\theta}(j)$ is impossible [32].

Define as $H = L_2(\Omega,\mathfrak{F},P)$ the Hilbert space generated by random variables ζ with zero mathematical expectation, $E\zeta=0$, finite variation, $E|\zeta|^2<\infty$, and inner product $(\zeta,\theta)=E\zeta\overline{\theta}$. Consider values $\zeta_\nu(j), \nu=1,...,T; j\in\mathbb{Z}$ and $\theta_\nu(j), \nu=1,...,T; j\in\mathbb{Z}$ as elements of H. Denote by $H^s[\vec{\zeta}+\vec{\theta}]$ the closed linear subspace in the Hilbert space H generated by elements $\{\zeta_\nu(\widetilde{j})+\theta_\nu(\widetilde{j}), \widetilde{j}\in\{...,-2,-1\}\setminus\widetilde{S}, \nu=1,...,T\}$.

Denote by $L_2(f)$ the Hilbert space of vector valued functions $\vec{b}(\lambda)=\{b_\nu(\lambda)\}_{\nu=1}^{T}$ that are integrable with respect to a measure with the density $f(\lambda)=\{f_{\nu\mu}(\lambda)\}_{\nu,\mu=1}^{T}$:

$$\int_{-\pi}^{\pi}\vec{b}^\top(\lambda)f(\lambda)\overline{\vec{b}(\lambda)}d\lambda =$$

$$\int_{-\pi}^{\pi}\sum_{\nu,\mu=1}^{T} b_\nu(\lambda)f_{\nu\mu}(\lambda)\overline{b_\mu(\lambda)}d\lambda < +\infty.$$

Denote by $L_2^s(f)$ the subspace in $L_2(f)$ generated by functions

$$e^{i\widetilde{j}\lambda}\delta_\nu, \delta_\nu=\{\delta_{\nu\mu}\}_{\mu=1}^{T},$$

$$\nu=1,\dots,T,\ \widetilde{j}\in\{...,-2,-1\}\setminus\widetilde{S},$$

where $\delta_{\nu\nu}=1, \delta_{\nu\mu}=0$ for $\nu\neq\mu$.

Every linear estimate $\widehat{A\vec{\zeta}}$ of the functional $A\vec{\zeta}$ from observations of the sequence $\vec{\zeta}(\widetilde{j})+\vec{\theta}(\widetilde{j})$ at points $\widetilde{j}\in\{...,-2,-1\}\setminus\widetilde{S}$ has the form

$$\widehat{A\vec{\zeta}} = \int_{-\pi}^{\pi}\vec{h}^\top(e^{i\lambda})(Z^{\vec{\zeta}}(d\lambda)+Z^{\vec{\theta}}(d\lambda)) =$$

$$\int_{-\pi}^{\pi}\sum_{\nu=1}^{T} h_\nu(e^{i\lambda})(Z^{\vec{\zeta}}_\nu(d\lambda)+Z^{\vec{\theta}}_\nu(d\lambda)), \quad (8)$$

where $Z^{\vec{\zeta}}(\Delta)=\left\{Z^{\vec{\zeta}}_\nu(\Delta)\right\}_{\nu=1}^{T}$ and $Z^{\vec{\theta}}(\Delta)=\left\{Z^{\vec{\theta}}_\nu(\Delta)\right\}_{\nu=1}^{T}$ are orthogonal random measures of the sequences $\vec{\zeta}(\widetilde{j})$ and $\vec{\theta}(\widetilde{j})$, and $\vec{h}(e^{i\lambda})=\left\{h_\nu(e^{i\lambda})\right\}_{\nu=1}^{T}$ is the spectral characteristic of the estimate $\widehat{A\vec{\zeta}}$. The function $\vec{h}(e^{i\lambda})\in L_2^s(f^{\vec{\zeta}}+f^{\vec{\theta}})$.

The mean square error $\Delta(\vec{h};f^{\vec{\zeta}},f^{\vec{\theta}})$ of the estimate $\widehat{A\vec{\zeta}}$ is calculated by the formula

$$\Delta(\vec{h};f^{\vec{\zeta}},f^{\vec{\theta}}) = E|A\vec{\zeta}-\widehat{A\vec{\zeta}}|^2 =$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[A(e^{i\lambda})-\vec{h}(e^{i\lambda})\right]^\top f^{\vec{\zeta}}(\lambda)\overline{\left[A(e^{i\lambda})-\vec{h}(e^{i\lambda})\right]}d\lambda+ \qquad (9)$$

$$+\frac{1}{2\pi}\int_{-\pi}^{\pi}\vec{h}^\top(e^{i\lambda})f^{\vec{\theta}}(\lambda)\overline{\vec{h}(e^{i\lambda})}d\lambda,$$

$$A(e^{i\lambda})=\sum_{\widetilde{j}\in\widetilde{Z}_s}\vec{a}(\widetilde{j})e^{-i\widetilde{j}\lambda}.$$

The spectral characteristic $\vec{h}(f^{\vec{\zeta}},f^{\vec{\theta}})$ of the optimal linear estimate of $A\vec{\zeta}$ minimizes the mean square error

$$\Delta(f^{\vec{\zeta}},f^{\vec{\theta}})=\Delta(\vec{h}(f^{\vec{\zeta}},f^{\vec{\theta}});f^{\vec{\zeta}},f^{\vec{\theta}})=$$
$$\min_{\vec{h}\in L_2^s(f^{\vec{\zeta}}+f^{\vec{\theta}})}\Delta(\vec{h};f^{\vec{\zeta}},f^{\vec{\theta}})=\min_{\widehat{A\vec{\zeta}}}E\left|A\vec{\zeta}-\widehat{A\vec{\zeta}}\right|^2. \quad (10)$$

With the help of the Hilbert space projection method proposed by A. N. Kolmogorov [15] we can find a solution of the optimization problem (10). The optimal linear estimate $\widehat{A\vec{\zeta}}$ is a projection of the functional $A\vec{\zeta}$ on the subspace $H^s[\vec{\zeta}+\vec{\theta}]$. The projection is characterized by following conditions

1) $\widehat{A\vec{\zeta}}\in H^s[\vec{\zeta}+\vec{\theta}]$,
2) $A\vec{\zeta}-\widehat{A\vec{\zeta}}\perp H^s[\vec{\zeta}+\vec{\theta}]$.

The condition 2) gives us the possibility to derive the formula for spectral characteristic of the estimate

$$\vec{h}^{\top}(f^{\vec{\zeta}},f^{\vec{\theta}})=$$
$$\left(A^{\top}(e^{i\lambda})f^{\vec{\zeta}}(\lambda)-C^{\top}(e^{i\lambda})\right)\left[f^{\vec{\zeta}}(\lambda)+f^{\vec{\theta}}(\lambda)\right]^{-1}=$$
$$A^{\top}(e^{i\lambda})-\left(A^{\top}(e^{i\lambda})f^{\vec{\theta}}(\lambda)+C^{\top}(e^{i\lambda})\right)\times$$
$$\times\left[f^{\vec{\zeta}}(\lambda)+f^{\vec{\theta}}(\lambda)\right]^{-1}, \quad (11)$$

where

$$C(e^{i\lambda})=\sum_{j\in\widetilde{S}}\vec{c}(n)e^{ij\lambda}+\sum_{j=0}^{\infty}\vec{c}(n)e^{ij\lambda},$$

where $\vec{c}(j), j\in\widetilde{S}\cup\{0,1,2,...\}$, are unknown vectors of coefficients.

Denote by $U=\widetilde{S}\cup\{0,1,2,...\}$.

Condition 1) is satisfied if the system of equalities

$$\int_{-\pi}^{\pi}\vec{h}(f^{\vec{\zeta}},f^{\vec{\theta}})e^{-im\lambda}d\lambda=0, m\in U \quad (12)$$

holds true.

The last equalities (12) provide the following relations

$$\sum_{\widetilde{j}\in\widetilde{Z}_s}\vec{a}^{\top}(\widetilde{j})\frac{1}{2\pi}\int_{-\pi}^{\pi}f^{\vec{\zeta}}(\lambda)(f^{\vec{\zeta}}(\lambda)+f^{\vec{\theta}}(\lambda))^{-1}e^{-i\lambda(\widetilde{j}+m)}d\lambda-$$
$$\sum_{\widetilde{j}\in\widetilde{S}}\vec{c}^{\top}(\widetilde{j})\frac{1}{2\pi}\int_{-\pi}^{\pi}(f^{\vec{\zeta}}(\lambda)+f^{\vec{\theta}}(\lambda))^{-1}e^{-i\lambda(m-\widetilde{j})}d\lambda-$$
$$\sum_{\widetilde{j}=0}^{\infty}\vec{c}^{\top}(\widetilde{j})\frac{1}{2\pi}\int_{-\pi}^{\pi}(f^{\vec{\zeta}}(\lambda)+f^{\vec{\theta}}(\lambda))^{-1}e^{-i\lambda(m-\widetilde{j})}d\lambda=\vec{0},$$
$$m\in U. \quad (13)$$

Denote the Fourier coefficients of the matrix functions $(f^{\vec{\zeta}}(\lambda)+f^{\vec{\theta}}(\lambda))^{-1}$ and $f^{\vec{\zeta}}(\lambda)(f^{\vec{\zeta}}(\lambda)+f^{\vec{\theta}}(\lambda))^{-1}$ as

$$B(m,\widetilde{j})=\frac{1}{2\pi}\int_{-\pi}^{\pi}(f^{\vec{\zeta}}(\lambda)+f^{\vec{\theta}}(\lambda))^{-1}e^{-i\lambda(m-\widetilde{j})}d\lambda,$$
$$m\in U,\widetilde{j}\in U,$$
$$R(m,\widetilde{j})=\frac{1}{2\pi}\int_{-\pi}^{\pi}f^{\vec{\zeta}}(\lambda)(f^{\vec{\zeta}}(\lambda)+f^{\vec{\theta}}(\lambda))^{-1}e^{-i\lambda(m+\widetilde{j})}d\lambda,$$
$$m\in U,\ \widetilde{j}\in\widetilde{Z}_s.$$

Using the notations above we can rewrite relation (13) in the form of the system of equations

$$\sum_{\widetilde{j}\in\widetilde{Z}_s}R(m,\widetilde{j})\vec{a}(\widetilde{j})=\sum_{\widetilde{j}\in\widetilde{S}}B(m,\widetilde{j})\vec{c}(\widetilde{j})+\sum_{\widetilde{j}=0}^{\infty}B(m,\widetilde{j})\vec{c}(\widetilde{j}),$$
$$m\in U. \quad (14)$$

Denote by

$$\mathbf{\vec{a}}^{\top}=\Bigg(\underbrace{\vec{0}^{\top},...,\vec{0}^{\top}}_{\sum_{i=1}^{s}N_i+1},\vec{a}^{\top}(1),...,\vec{a}^{\top}(M_1-1),$$
$$\underbrace{\vec{0}^{\top},....,\vec{0}^{\top}}_{N_1},...,\vec{a}^{\top}(M_s+N_s),\vec{a}^{\top}(M_s+N_s+1),...\Bigg)$$

a vector that has zero vectors $\vec{0}^{\top}=(\underbrace{0,...,0}_{T})$, vectors $\vec{a}(1),...,\vec{a}(M_1-1),...,\vec{a}(M_s+N_s),...$, are constructed from coefficients of the functional $A\zeta$ by formula (6).

Denote by $\mathbf{\vec{c}}^{\top}=(\vec{c}^{\top}(m))_{m\in U}$ a vector of the unknown coefficients.

The last system of equations (14) can be rewritten in the matrix form

$$\mathbf{R\vec{a}}=\mathbf{B\vec{c}}.$$

The linear operator $\mathbf{B}$ is defined by the matrix

$$\mathbf{B}=\begin{pmatrix}B_{s,s}&B_{s,s-1}&\dots&B_{s,1}&B_{s,n}\\B_{s-1,s}&B_{s-1,s-1}&\dots&B_{s-1,1}&B_{s-1,n}\\\dots&\dots&\dots&\dots&\\B_{1,s}&B_{1,s-1}&\dots&B_{1,1}&B_{1,n}\\B_{n,s}&B_{n,s-1}&\dots&B_{n,1}&B_{n,n}\end{pmatrix},$$

constructed with the help of the block-matrices

$$B_{l,k}=\left\{B_{l,k}(m,\widetilde{j})\right\}_{m=-(M_l+N_l)+1}^{-M_l}{}_{\widetilde{j}=-(M_k+N_k)+1}^{-M_k},$$

$$l,k=1,...,s,$$

$$B_{l,n}=\left\{B_{l,n}(m,\widetilde{j})\right\}_{m=-(M_l+N_l)+1}^{-M_l}{}_{\widetilde{j}=0}^{\infty},\ l=1,...,s,$$

$$B_{n,l}=\left\{B_{n,l}(m,\widetilde{j})\right\}_{m=0}^{\infty}{}_{\widetilde{j}=-(M_l+N_l)+1}^{-M_l},\ l=1,...,s,$$

$$B_{n,n}=\left\{B_{n,n}(m,\widetilde{j})\right\}_{m=0}^{\infty}{}_{\widetilde{j}=0}^{\infty},$$

$$B_{l,k}(m,\widetilde{j})=B(m,\widetilde{j}),\forall l,k=\in\{1,...,s\}\cup\{n\}.$$

The linear operator $\mathbf{R}$ is defined by the corresponding matrix, which is constructed in the same manner as matrix $\mathbf{B}$.

The unknown coefficients $\vec{c}(m), m\in U$ are determined from the equation

$$\vec{\mathbf{c}}=\mathbf{B}^{-1}\mathbf{R}\vec{\mathbf{a}},\tag{15}$$

where the m-th component of the vector $\vec{\mathbf{c}}$ is the m-th component of vector $\mathbf{B}^{-1}\mathbf{R}\vec{\mathbf{a}}$:

$$\vec{c}(m)=(\mathbf{B}^{-1}\mathbf{R}\vec{\mathbf{a}})(m),\ m\in U.\tag{16}$$

We will suppose that the operator $\mathbf{B}$ has the inverse matrix.

The mean-square error of the optimal estimate $\widehat{A\vec{\zeta}}$ is calculated by the formula (9) and is of the form

$$\Delta(\vec{h},f^{\vec{\zeta}},f^{\vec{\theta}})=E|A\vec{\zeta}-\widehat{A\vec{\zeta}}|^2=$$
$$\sum_{\widetilde{j}\in U}\sum_{m\in U}\vec{a}^{\top}(\widetilde{j})\frac{1}{2\pi}\int_{-\pi}^{\pi}f^{\vec{\zeta}}(\lambda)(f^{\vec{\zeta}}(\lambda)+f^{\vec{\theta}}(\lambda))^{-1}\times$$
$$\times f^{\vec{\theta}}(\lambda)e^{-i\lambda(\widetilde{j}-m)}d\lambda\cdot\overline{\vec{a}(m)}+$$
$$\sum_{\widetilde{j}\in U}\sum_{m\in U}\vec{c}^{\top}(\widetilde{j})\frac{1}{2\pi}\int_{-\pi}^{\pi}(f^{\vec{\zeta}}(\lambda)+f^{\vec{\theta}}(\lambda))^{-1}\times$$
$$\times e^{-i\lambda(m-\widetilde{j})}d\lambda\cdot\overline{\vec{c}(m)}=$$
$$\langle\mathbf{D}\vec{\mathbf{a}},\vec{\mathbf{a}}\rangle+\langle\mathbf{B}\vec{\mathbf{c}},\vec{\mathbf{c}}\rangle,\tag{17}$$

where $\langle a,b\rangle$ denotes the scalar product, $\mathbf{D}$ is defined by the corresponding matrix, which is constructed in the same manner as matrix $\mathbf{B}$, with elements

$$D(m,\widetilde{j})=\frac{1}{2\pi}\int_{-\pi}^{\pi}f^{\vec{\zeta}}(\lambda)(f^{\vec{\zeta}}(\lambda)+f^{\vec{\theta}}(\lambda))^{-1}\times$$
$$\times f^{\vec{\theta}}(\lambda)e^{-i(\widetilde{j}-m)\lambda}d\lambda,\ m,\widetilde{j}\in U.$$

The following statement holds true.

Theorem 2.1. *Let $\zeta(j)$ and $\theta(j)$ be uncorrelated T-PC stochastic sequences with the spectral density matrices $f^{\vec{\zeta}}(\lambda)$ and $f^{\vec{\theta}}(\lambda)$ of T-variate stationary sequences $\vec{\zeta}(\widetilde{j})$ and $\vec{\theta}(\widetilde{j})$, respectively. Assume that $f^{\vec{\zeta}}(\lambda)$ and $f^{\vec{\theta}}(\lambda)$ satisfy the minimality condition (7). Assume that condition (4) is satisfied and operator $\boldsymbol{B}$ is invertible. The spectral characteristic $\vec{h}(f^{\vec{\zeta}},f^{\vec{\theta}})$ and the mean square error $\Delta(f^{\vec{\zeta}},f^{\vec{\theta}})$ of the optimal linear estimate of the functional $A\vec{\zeta}$ based on observations of the sequence $\vec{\zeta}(\widetilde{j})+\vec{\theta}(\widetilde{j})$ at points $\widetilde{j}\in\{...,-1\}\setminus\widetilde{S}$, are calculated by formulas (11) and (17).*

Consider the mean-square estimation problem of functional $A\zeta=\sum_{j\in Z_s}a(j)\zeta(-j)$, $Z_s=\{T+1,T+2,...\}\setminus\{MT+1,...,(M+N)T\}$ based on observations of the sequence $\zeta(j)+\theta(j)$ at points $j\in\{...,-(T+2),-(T+1)\}\setminus\{-(M+N)T,...,-MT-1\}$. Using Proposition 1.2, the linear functional $A\zeta$ can be written as follows

$$A\zeta=\sum_{j\in Z_s}a(j)\zeta(-j)=\sum_{\widetilde{j}\in\widetilde{Z}_s}\vec{a}^{\top}(\widetilde{j})\vec{\zeta}(-\widetilde{j})=A\vec{\zeta},$$

where

$$\widetilde{Z}_s=\{1,2,3,\dots,\}\backslash\{M,\dots,M+N-1\}.$$

The estimate $\widehat{A\vec{\zeta}}$ of functional $A\vec{\zeta}$ from observations of sequence $\vec{\zeta}(\widetilde{j})+\vec{\theta}(\widetilde{j})$ at points $\widetilde{j}\in\{...,-1\}\setminus\{-(M+N)+1,...,-M\}$ is defined by spectral characteristic $\vec{h}(f^{\vec{\zeta}},f^{\vec{\theta}})$ (see formula (8)). The spectral characteristic is calculated by the following formula

$$\vec{h}^{\top}(f^{\vec{\zeta}},f^{\vec{\theta}})=\left(A^{\top}(e^{i\lambda})f^{\vec{\zeta}}(\lambda)-C^{\top}(e^{i\lambda})\right)\times$$
$$\times\left[f^{\vec{\zeta}}(\lambda)+f^{\vec{\theta}}(\lambda)\right]^{-1},\tag{18}$$

where $C(e^{i\lambda})=\sum_{j=-(M+N)+1}^{-M}\vec{c}(j)e^{ij\lambda}+\sum_{j=0}^{\infty}\vec{c}(j)e^{ij\lambda}$, unknown coefficients $\vec{c}(j), j\in\{-(M+N)+1,...,-M\}\cup\{0,1,...\}$ are calculated by formula

$$\vec{\mathbf{c}}=\mathbf{B}^{-1}\mathbf{R}\vec{\mathbf{a}}.$$

Linear operators $\mathbf{B},\mathbf{R}$ are defined by compound matrices, for example

$$\mathbf{B}=\begin{pmatrix}B_{1,1}&B_{s,n}\\B_{n,s}&B_{n,n}\end{pmatrix}$$

is constructed with the help of the block-matrices $B_{1,1}, B_{s,n}, B_{n,s}, B_{n,n}$ with the elements

$$B_{1,1} = \left\{B_{1,1}(m,\widetilde{j})\right\}_{m=-(M+N)-1}^{-M}{}_{\widetilde{j}=-(M+N)-1}^{-M},$$

$$B_{s,n} = \left\{B_{s,n}(m,\widetilde{j})\right\}_{m=-(M+N)-1}^{-M}{}_{\widetilde{j}=0}^{\infty},$$

$$B_{n,s} = \left\{B_{n,s}(m,\widetilde{j})\right\}_{m=0}^{\infty}{}_{\widetilde{j}=-(M+N)-1}^{-M},$$

$$B_{n,n} = \left\{B_{n,n}(m,\widetilde{j})\right\}_{m=0}^{\infty}{}_{\widetilde{j}=0}^{\infty}.$$

The vector

$$\vec{\mathbf{a}}^{\top} = \Bigg(\underbrace{\vec{0}^{\top},...,\vec{0}^{\top}}_{N+1}, \vec{a}^{\top}(1),...,\vec{a}^{\top}(M-1),$$

$$\underbrace{\vec{0}^{\top},....,\vec{0}^{\top}}_{N}, \vec{a}^{\top}(M+N), \vec{a}^{\top}(M+N+1),...\Bigg)$$

is a vector with vectors $\vec{a}(1),...,\vec{a}(M-1),\vec{a}(M+N),...,$ constructed from coefficients of the functional $A\zeta$ by formula (6).

The mean square error $\Delta(f^{\vec{\zeta}}, f^{\vec{\theta}})$ is calculated by the formula

$$\Delta(f^{\vec{\zeta}}, f^{\vec{\theta}}) = \langle \mathbf{D}\vec{\mathbf{a}}, \vec{\mathbf{a}}\rangle + \langle \mathbf{B}\vec{\mathbf{c}}, \vec{\mathbf{c}}\rangle, \tag{19}$$

where linear operator $\mathbf{D}$ is defined by the corresponding matrix, which is constructed in the same manner as matrix $\mathbf{B}$ above.

The following corollary from the theorem 2.1 holds true.

Corollary 2.1. Let $\zeta(j)$ and $\theta(j)$ be uncorrelated T-PC stochastic sequences with the spectral density matrices $f^{\vec{\zeta}}(\lambda)$ and $f^{\vec{\theta}}(\lambda)$ of T-variate stationary sequences $\vec{\zeta}(\widetilde{j})$ and $\vec{\theta}(\widetilde{j})$, respectively. Assume that $f^{\vec{\zeta}}(\lambda)$ and $f^{\vec{\theta}}(\lambda)$ satisfy the minimality condition (7). Assume that condition (4) is satisfied and operator $\mathbf{B}$ is invertible. The spectral characteristic $\vec{h}(f^{\vec{\zeta}}, f^{\vec{\theta}})$ and the mean square error $\Delta(f^{\vec{\zeta}}, f^{\vec{\theta}})$ of the optimal linear estimate of the functional $A\vec{\zeta}$ based on observations of the sequence $\vec{\zeta}(\widetilde{j}) + \vec{\theta}(\widetilde{j})$ at points $\widetilde{j} \in \{..., -1\} \setminus \{-(M+N)+1, ..., -M\}$, are calculated by formulas (18) and (19).

Consider the mean-square estimation problem of functional $A\zeta = \sum\limits_{j\in Z_s} a(j)\zeta(-j)$, $Z_s = \{T+1, T+2, ...\} \setminus \{MT+1, ..., MT+T\}$ based on observations of the sequence $\zeta(j)+\theta(j)$ at points $j \in \{..., -(T+2), -(T+1)\} \setminus \{-(MT+T), ..., -(MT+1)\}$. Using Proposition 1.2, the linear functional $A\zeta$ can be written as follows

$$A\zeta = \sum_{j\in Z_s} a(j)\zeta(-j) = \sum_{\widetilde{j}\in \widetilde{Z}_s} \vec{a}^{\top}(\widetilde{j})\vec{\zeta}(-\widetilde{j}) = A\vec{\zeta},$$

where

$$\widetilde{Z}_s = \{1,2,3,\ldots,\}\backslash\{M\}.$$

The estimate $\widehat{A\vec{\zeta}}$ of functional $A\vec{\zeta}$ from observations of sequence $\vec{\zeta}(\widetilde{j}) + \vec{\theta}(\widetilde{j})$ at points $\widetilde{j} \in \{..., -1\} \setminus \{-M\}$ is defined by spectral characteristic $\vec{h}(f^{\vec{\zeta}}, f^{\vec{\theta}})$. The spectral characteristic is calculated by the following formula

$$\vec{h}^{\top}(f^{\vec{\zeta}}, f^{\vec{\theta}}) = \left(A^{\top}(e^{i\lambda})f^{\vec{\zeta}}(\lambda) - C^{\top}(e^{i\lambda})\right) \times \times \left[f^{\vec{\zeta}}(\lambda) + f^{\vec{\theta}}(\lambda)\right]^{-1}, \tag{20}$$

where $C(e^{i\lambda}) = \vec{c}(-M)e^{-iM\lambda} + \sum\limits_{j=0}^{\infty} \vec{c}(j)e^{ij\lambda}$, unknown coefficients $\vec{c}(j), j \in \{-M\} \cup \{0,1,...\}$ are calculated by formula

$$\vec{\mathbf{c}} = \mathbf{B}^{-1}\mathbf{R}\vec{\mathbf{a}}.$$

Linear operators $\mathbf{B}, \mathbf{R}$ are defined by compound matrices, for example

$$\mathbf{B} = \begin{pmatrix} B_{-M,-M} & B_{-M,n} \\ B_{n,-M} & B_{n,n} \end{pmatrix}$$

is constructed with the help of the block-matrices $B_{-M,-M}, B_{-M,n}, B_{n,-M}, B_{n,n}$ with the elements

$$B_{-M,-M} = B(-M,-M),$$

$$B_{-M,n} = \left\{B_{-M,n}(-M,\widetilde{j})\right\}, \widetilde{j} = 0,1,...,$$

$$B_{n,-M} = \{B_{n,-M}(m,-M)\}, m = 0,1,...,$$

$$B_{n,n} = \left\{B_{n,n}(m,\widetilde{j})\right\}_{m=0}^{\infty}{}_{\widetilde{j}=0}^{\infty}.$$

The vector

$$\vec{\mathbf{a}}^{\top} = \Big(\vec{0}^{\top}, \vec{0}^{\top}, \vec{a}^{\top}(1), ..., \vec{a}^{\top}(M-1), \vec{0}^{\top},$$

$$\vec{a}^{\top}(M+1), \vec{a}^{\top}(M+2), ...\Big)$$

is a vector with vectors $\vec{a}(1),...,\vec{a}(M-1),\vec{a}(M+1),...,$ constructed from coefficients of the functional $A\zeta$ by formula (6).

The mean square error $\Delta(f^{\vec{\zeta}}, f^{\vec{\theta}})$ is calculated by the formula

$$\Delta(f^{\vec{\zeta}}, f^{\vec{\theta}}) = \langle \mathbf{D}\vec{\mathbf{a}}, \vec{\mathbf{a}}\rangle + \langle \mathbf{B}\vec{\mathbf{c}}, \vec{\mathbf{c}}\rangle, \qquad (21)$$

where linear operator $\mathbf{D}$ is defined by the corresponding matrix, which is constructed in the same manner as matrix $\mathbf{B}$ above.

The next corollary from the theorem 2.1 holds true.

Corollary 2.2. Let $\zeta(j)$ and $\theta(j)$ be uncorrelated T-PC stochastic sequences with the spectral density matrices $f^{\vec{\zeta}}(\lambda)$ and $f^{\vec{\theta}}(\lambda)$ of T-variate stationary sequences $\vec{\zeta}(\widetilde{j})$ and $\vec{\theta}(\widetilde{j})$, respectively. Assume that $f^{\vec{\zeta}}(\lambda)$ and $f^{\vec{\theta}}(\lambda)$ satisfy the minimality condition (7). Assume that condition (4) is satisfied and operator $\mathbf{B}$ is invertible. The spectral characteristic $\vec{h}(f^{\vec{\zeta}}, f^{\vec{\theta}})$ and the mean square error $\Delta(f^{\vec{\zeta}}, f^{\vec{\theta}})$ of the optimal linear estimate of the functional $A\vec{\zeta}$ based on observations of the sequence $\vec{\zeta}(\widetilde{j})+\vec{\theta}(\widetilde{j})$ at points $\widetilde{j} \in \{..., -1\}\backslash\{-M\}$, are calculated by formulas (20) and (21).

The filtering problem of linear functional for the case with factorization of density matrices $f^{\vec{\zeta}}(\lambda)$ and $f^{\vec{\theta}}(\lambda)$ of T-variate stationary sequences $\vec{\zeta}(\widetilde{j})$ and $\vec{\theta}(\widetilde{j})$ is considered in the article [2].

3 Minimax (robust) method of filtering problem

Let $f(\lambda)$ and $g(\lambda)$ be the spectral density matrices of T-variate stationary sequences $\vec{\zeta}(j)$ and $\vec{\theta}(j)$, obtained by T-blocking (3) of T-PC sequences $\zeta(j)$ and $\theta(j)$, respectively.

Formulas (11) and (17), (18) and (19), (20) and (21) may be applied for finding the spectral characteristic and the mean square error of the optimal linear estimate of the functional $A\vec{\zeta}$ only under the condition that the spectral density matrices $f(\lambda)$ and $g(\lambda)$ are exactly known. If the density matrices are not known exactly while a set $D = D_f \times D_g$ of possible spectral densities is given, the minimax (robust) approach to estimation of functionals from unknown values of stationary sequences is reasonable. In this case we find the estimate which minimizes the mean square error for all spectral densities from the given set simultaneously.

Definition 3.1. For a given class of pairs of spectral densities $D = D_f \times D_g$ the spectral density matrices $f^0(\lambda) \in D_f$, $g^0(\lambda) \in D_g$ are called the least favorable in D for the optimal linear estimation of the functional $A\vec{\zeta}$ if

$$\Delta(f^0, g^0) = \Delta(\vec{h}(f^0, g^0); f^0, g^0) = \max_{(f,g)\in D} \Delta(\vec{h}(f, g); f, g).$$

Definition 3.2. For a given class of pairs of spectral densities $D = D_f \times D_g$ the spectral characteristic $\vec{h}^0(\lambda)$ of the optimal linear estimate of the functional $A\vec{\zeta}$ is called minimax (robust) if

$$\vec{h}^0(\lambda) \in H_D = \bigcap_{(f,g)\in D} L_2^s(f + g),$$

$$\min_{\vec{h}\in H_D} \max_{(f,g)\in D} \Delta(\vec{h}; f, g) = \max_{(f,g)\in D} \Delta(\vec{h}^0; f, g).$$

Taking into consideration these definitions and the obtained relations we can verify that the following lemma holds true.

Lemma 3.1. The spectral density matrices $f^0(\lambda) \in D_f$, $g^0(\lambda) \in D_g$, that satisfy the minimality condition (7), are the least favorable in the class D for the optimal linear estimation of $A\vec{\zeta}$, if the Fourier coefficients of the matrix functions

$$(f^0(\lambda) + g^0(\lambda))^{-1}, \quad f^0(\lambda)(f^0(\lambda) + g^0(\lambda))^{-1},$$
$$f^0(\lambda)(f^0(\lambda) + g^0(\lambda))^{-1}g^0(\lambda)$$

define matrices $\mathbf{B}^0, \mathbf{R}^0, \mathbf{D}^0$, that determine a solution of the constrained optimization problem

$$\max_{(f,g)\in D} (\langle \mathbf{R}\vec{\mathbf{a}}, \mathbf{B}^{-1}\mathbf{R}\vec{\mathbf{a}}\rangle) + \langle \mathbf{D}\vec{\mathbf{a}}, \vec{\mathbf{a}}\rangle) = \langle \mathbf{R}^0\vec{\mathbf{a}}, (\mathbf{B}^0)^{-1}\mathbf{R}^0\vec{\mathbf{a}}\rangle) + \langle \mathbf{D}^0\vec{\mathbf{a}}, \vec{\mathbf{a}}\rangle. \qquad (22)$$

The minimax spectral characteristic $\vec{h}^0 = \vec{h}(f^0, g^0)$ is given by (11), if $\vec{h}(f^0, g^0) \in H_D$.

The least favorable spectral densities $f^0(\lambda) \in D_f$, $g^0(\lambda) \in D_g$ and the minimax spectral characteristic $\vec{h}^0 = \vec{h}(f^0, g^0)$ form a saddle point of the function $\Delta(\vec{h}; f, g)$ on the set $H_D \times D$. The saddle point inequalities

$$\Delta(\vec{h}^0; f, g) \leqslant \Delta(\vec{h}^0; f^0, g^0) \leqslant \Delta(\vec{h}; f^0, g^0),$$
$$\forall\vec{h} \in H_D, \forall f \in D_f, \forall g \in D_g$$

hold true when $\vec{h}^0 = \vec{h}(f^0, g^0)$, $\vec{h}(f^0, g^0) \in H_D$ and (f^0, g^0) is a solution of the constrained optimization problem

$$\Delta\left(\vec{h}(f^0, g^0); f, g\right) \to \sup, \ (f, g) \in D_f \times D_g. \qquad (23)$$

The linear functional $\Delta(\vec{h}(f^0,g^0);f,g)$ is calculated by the formula

$$\Delta(\vec{h}(f^0,g^0);f,g)=$$
$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(A^\top(e^{i\lambda})g^0(\lambda)+(C^0(e^{i\lambda}))^\top\right)\times$$
$$\times(f^0(\lambda)+g^0(\lambda))^{-1}f(\lambda)(f^0(\lambda)+g^0(\lambda))^{-1}\times$$
$$\times\left(A^\top(e^{i\lambda})g^0(\lambda)+(C^0(e^{i\lambda}))^\top\right)^*d\lambda+$$
$$+\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(A^\top(e^{i\lambda})f^0(\lambda)-(C^0(e^{i\lambda}))^\top\right)\times$$
$$\times(f^0(\lambda)+g^0(\lambda))^{-1}g(\lambda)(f^0(\lambda)+g^0(\lambda))^{-1}\times$$
$$\times\left(A^\top(e^{i\lambda})f^0(\lambda)-(C^0(e^{i\lambda}))^\top\right)^*d\lambda,$$

where $C^0(e^{i\lambda})=\sum_{j\in U}\vec{c}^{\,0}(j)e^{ij\lambda}$, according to the formula (16) column vectors $\vec{c}^{\,0}(j)=((\mathbf{B}^0)^{-1}\mathbf{R}^0\vec{\mathbf{a}})(j)$.

The constrained optimization problem (23) is equivalent to the unconstrained optimization problem, [30]:

$$\Delta_D(f,g)=-\Delta(\vec{h}(f^0,g^0);f,g)+\delta((f,g)\,|D_f\times D_g)\to\inf,\quad(24)$$

where $\delta((f,g)|D_f\times D_g)$ is the indicator function of the set $D=D_f\times D_g$. A solution of the problem (24) is characterized by the condition $0\in\partial\Delta_D(f^0,g^0)$, where $\partial\Delta_D(f^0,g^0)$ is the subdifferential of the convex functional $\Delta_D(f,g)$ at point (f^0,g^0), [31].

The form of the functional $\Delta(\vec{h}(f^0,g^0);f,g)$ admits finding the derivatives and differentials of the functional in the space $L_1\times L_1$. Therefore the complexity of the optimization problem (24) is determined by the complexity of calculating of subdifferentials of the indicator functions $\delta((f,g)|D_f\times D_g)$ of the sets $D_f\times D_g$, [13].

The form of the functional $\Delta(\vec{h}(f^0,g^0);f,g)$ is convenient for application the Lagrange method of indefinite multipliers for finding solution to the problem (24). Using the Lagrange method of indefinite multipliers and the form of subdifferentials of the indicator functions we derived relations that determine the least favourable spectral densities in some classes of spectral densities (see books [27], [24]).

4 The least favorable spectral densities in the class $D=D_0\times D_{1\delta}$

Let $f(\lambda)$ and $g(\lambda)$ be the spectral density matrices of T-variate stationary sequences $\vec{\zeta}(j)$ and $\vec{\theta}(j)$, obtained by T-blocking (3) of T-PC sequences $\zeta(j)$ and $\theta(j)$, respectively.

Consider the problem of minimax estimation of the functional $A\vec{\zeta}$ based on observations of the sequence $\vec{\zeta}(\widetilde{j})+\vec{\theta}(\widetilde{j})$ at points $\widetilde{j}\in\{...,-2,-1\}\setminus\widetilde{S}$, $\widetilde{S}=\bigcup_{i=1}^{s}\{-(M_i+N_i)+1,\dots,-M_i\}$, under the condition that the spectral density matrices $f(\lambda)$ and $g(\lambda)$ belong to the class $D=D_0\times D_{1\delta}$, where

$$D_0^1=\left\{f(\lambda)\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{Tr}\;f(\lambda)d\lambda=p\right.\right\},$$

$$D_{1\delta}^1=\left\{g(\lambda)\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\mathrm{Tr}(g(\lambda)-g^1(\lambda))\right|d\lambda\leqslant\delta\right.\right\},$$

$$D_0^2=\left\{f(\lambda)\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}f_{kk}(\lambda)d\lambda=p_k,k=1,...,T\right.\right\},$$

$$D_{1\delta}^2=\left\{g(\lambda)\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\mathrm{Tr}(g_{kk}(\lambda)-g_{kk}^1(\lambda))\right|d\lambda\leqslant\delta_k,\right.\right.$$
$$\left.k=1,....,T\right\},$$

where $g^1(\lambda)=\{g_{kl}^1(\lambda)\}_{k,l=1}^T$ is known positive definite Hermitian matrix, $\delta,\delta_k,p,p_{kk},k=1,...,T$, are known and fixed numbers.

The classes $D_0^i,i=1,2$, describe densities with the moment restrictions. The classes $D_{1\delta}^i,i=1,2$, describe the "δ–neighborhood" models in the space L_1 of a fixed bounded spectral density $g^1(\lambda)$.

With the help of the method of Lagrange multipliers we can find that solution $(f^0(\lambda),g^0(\lambda))$ of the constrained optimization problem (23) satisfy the following relations for these sets of admissible spectral densities.

For the pair $D_0^1\times D_{1\delta}^1$ we have relations

$$(g^0(\lambda)\overline{A(e^{i\lambda})}+\overline{C^0(e^{i\lambda})})(g^0(\lambda))^\top(A(e^{i\lambda})+C^0(e^{i\lambda}))^\top=\alpha^2(f^0(\lambda)+g^0(\lambda))^2,\quad(25)$$

$$(f^0(\lambda)\overline{A(e^{i\lambda})}-\overline{C^0(e^{i\lambda})})(f^0(\lambda))^\top(A(e^{i\lambda})-C^0(e^{i\lambda}))^\top=\beta^2\psi(\lambda)(f^0(\lambda)+g^0(\lambda))^2,\quad(26)$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\mathrm{Tr}(g^0(\lambda)-g^1(\lambda))\right|d\lambda=\delta,\qquad(27)$$

where α,β are Lagrange multipliers, $|\psi(\lambda)|\leqslant1$ and $\psi(\lambda)=\mathrm{sign}\;(\mathrm{Tr}(g^0(\lambda)-g^1(\lambda)))$ if $\mathrm{Tr}(g^0(\lambda)-g^1(\lambda))\neq0$.

For the pair $D_0^2 \times D_{1\delta}^2$ we have relations

$$(g^0(\lambda)\overline{A(e^{i\lambda})} + \overline{C^0(e^{i\lambda})})\times \\ \times (g^0(\lambda))^\top (A(e^{i\lambda}) + C^0(e^{i\lambda}))^\top = \\ (f^0(\lambda) + g^0(\lambda)) \left\{\alpha_k^2\delta_{kl}\right\}_{k,l=1}^T (f^0(\lambda) + g^0(\lambda)), \tag{28}$$

$$(f^0(\lambda)\overline{A(e^{i\lambda})} - \overline{C^0(e^{i\lambda})})\times \\ (f^0(\lambda))^\top (A(e^{i\lambda}) - C^0(e^{i\lambda}))^\top = \\ (f^0(\lambda)+g^0(\lambda)) \left\{\beta_k^2\psi_k(\lambda)\delta_{kl}\right\}_{k,l=1}^T (f^0(\lambda)+g^0(\lambda)), \tag{29}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\mathrm{Tr}(g_{kk}^0(\lambda) - g_{kk}^1(\lambda))\right| d\lambda = \delta_k, k = 1, ..., T, \tag{30}$$

where α_k^2, β_k^2 are Lagrange multipliers, δ_{kl} are Kronecker symbols, $|\psi_k(\lambda)| \leqslant 1$ and $\psi_k(\lambda) = \mathrm{sign}\ (\mathrm{Tr}(g_{kk}^0(\lambda) - g_{kk}^1(\lambda)))$ if $\mathrm{Tr}(g_{kk}^0(\lambda) - g_{kk}^1(\lambda)) \neq 0, k = 1, ..., T$.

Hence the following theorem and corollaries hold true.

Theorem 4.1. *Let the spectral densities $f^0(\lambda)$ and $g^0(\lambda)$ satisfy the minimality condition (7). The least favorable spectral densities $f^0(\lambda)$, $g^0(\lambda)$ in the class $D_0^1 \times D_{1\delta}^1$ for the optimal linear filtering of the functional $A\vec{\zeta}$ are determined by relations (25)–(27). The least favorable spectral densities $f^0(\lambda)$, $g^0(\lambda)$ in the class $D_0^2 \times D_{1\delta}^2$ for the optimal linear filtering of the functional $A\vec{\zeta}$ are determined by relations (28)–(30). The minimax spectral characteristic of the optimal estimate of the functional $A\vec{\zeta}$ is determined by the formula (11).*

Corollary 4.1. Assume that the spectral density $g(\lambda)$ is known. Let the spectral density $f^0(\lambda)+g(\lambda)$ satisfies the minimality condition (7). The least favorable spectral density $f^0(\lambda)$ in the class D_0^1 or D_0^2 for the optimal linear filtering of the functional $A\vec{\zeta}$ based on observations of $\vec{\zeta}(\widetilde{j})$ at points $\widetilde{j} \in \{..., -2, -1\} \setminus \widetilde{S}$ is determined by relations (25), (28), respectively, and by the constrained optimization problem (22).

Corollary 4.2. Assume that the spectral density $f(\lambda)$ is known. Let the spectral density $f(\lambda) + g^0(\lambda)$ satisfies the minimality condition (7). The least favorable spectral density $g^0(\lambda)$ in the class $D_{1\delta}^1$ or $D_{1\delta}^2$ for the optimal linear filtering of the functional $A\vec{\zeta}$ based on observations of $\vec{\zeta}(\widetilde{j})$ at points $\widetilde{j} \in \{..., -2, -1\} \setminus \widetilde{S}$ is determined by relations (26)–(27), (29)–(30), respectively, and by the constrained optimization problem (22).

5 The least favorable spectral densities in the class $D = D_0 \times D_V^W$

Let $f(\lambda)$ and $g(\lambda)$ be the spectral density matrices of T-variate stationary sequences $\vec{\zeta}(j)$ and $\vec{\theta}(j)$, obtained by T-blocking (3) of T-PC sequences $\zeta(j)$ and $\theta(j)$, respectively.

Consider the problem of minimax estimation of the functional $A\vec{\zeta}$ based on observations of the sequence $\vec{\zeta}(\widetilde{j}) + \vec{\theta}(\widetilde{j})$ at points $\widetilde{j} \in \{..., -2, -1\} \setminus \widetilde{S}$, under the condition that the spectral density matrices $f(\lambda)$ and $g(\lambda)$ belong to the class $D = D_0 \times D_V^W$, where

$$D_0^3 = \left\{ f(\lambda) \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\lambda)d\lambda = P \right.\right\},$$

$$D_U^W = \left\{ g(\lambda) \left| U(\lambda) \leqslant g(\lambda) \leqslant W(\lambda), \right.\right. \\ \left.\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\lambda)d\lambda = Q \right\},$$

where P, Q are known positive definite Hermitian matrices, $U(\lambda), W(\lambda)$ are fixed spectral densities.

The class D_0^3 describes densities with the moment restrictions. The class D_U^W describes the "strip" models of spectral densities.

With the help of the method of Lagrange multipliers we can find that solution $(f^0(\lambda), g^0(\lambda))$ of the constrained optimization problem (23) satisfy the following relations for this set of admissible spectral densities.

For the class $D_0^3 \times D_U^W$ we have relations

$$(g^0(\lambda)\overline{A(e^{i\lambda})} + \overline{C^0(e^{i\lambda})})\times \\ (g^0(\lambda))^\top (A(e^{i\lambda}) + C^0(e^{i\lambda}))^\top = \\ (f^0(\lambda) + g^0(\lambda))\overline{\vec{\alpha}}\vec{\alpha}^\top (f^0(\lambda) + g^0(\lambda)), \tag{31}$$

$$(f^0(\lambda)\overline{A(e^{i\lambda})} - \overline{C^0(e^{i\lambda})})\times \\ (f^0(\lambda))^\top (A(e^{i\lambda}) - C^0(e^{i\lambda}))^\top = \\ (f^0(\lambda)+g^0(\lambda))(\overline{\vec{\beta}}\vec{\beta}^\top + \Gamma_1(\lambda) + \Gamma_2(\lambda))(f^0(\lambda)+g^0(\lambda)), \tag{32}$$

where $\vec{\alpha}, \vec{\beta}$ are Lagrange multipliers, $\Gamma_1(\lambda) \leqslant 0$ and $\Gamma_1(\lambda) = 0$ if $g^0(\lambda) > V(\lambda)$, $\Gamma_2(\lambda) \geqslant 0$ and $\Gamma_2(\lambda) = 0$ if $g^0(\lambda) < W(\lambda)$.

Hence the following theorem and corollaries hold true.

Theorem 5.1. *Let the spectral densities $f^0(\lambda)$ and $g^0(\lambda)$ satisfy the minimality condition (7). The least favorable spectral densities $f^0(\lambda)$, $g^0(\lambda)$ in the class $D_0^3 \times D_U^W$ for the optimal linear filtering of the functional $A\vec{\zeta}$ are determined by relations (31), (32). The minimax spectral characteristic of the optimal estimate of the functional $A\vec{\zeta}$ is determined by the formula (11).*

Corollary 5.1. Assume that the spectral density $g(\lambda)$ is known. Let the spectral density $f^0(\lambda)+g(\lambda)$ satisfies the minimality condition (7). The least favorable spectral density $f^0(\lambda)$ in the class D_0^3 for the optimal linear filtering of the functional $A\vec{\zeta}$ based on observations of $\vec{\zeta}(\widetilde{j})$ at points $\widetilde{j} \in \{..., -2, -1\} \setminus \widetilde{S}$ is determined by relation (31) and by the constrained optimization problem (22).

Corollary 5.2. Assume that the spectral density $f(\lambda)$ is known. Let the spectral density $f(\lambda) + g^0(\lambda)$ satisfies the minimality condition (7). The least favorable spectral density $g^0(\lambda)$ in the class D_U^W for the optimal linear filtering of the functional $A\vec{\zeta}$ based on observations of $\vec{\zeta}(\widetilde{j})$ at points $\widetilde{j} \in \{..., -2, -1\} \setminus \widetilde{S}$ is determined by relation (32) and by the constrained optimization problem (22).

6 Conclusions

In this article we study the filtering problem of the functional $A\zeta$ which depends on the unobserved values of a periodically correlated stochastic sequence $\zeta(j)$. Estimates are based on observations of a periodically correlated stochastic sequence $\zeta(j) + \theta(j)$ with missing observations, that means that observations of $\zeta(j) + \theta(j)$ are known at points $j \in \{\dots, -(T+2), -(T+1), \}\setminus S$. The sequence $\theta(j)$ is an uncorrelated with $\zeta(j)$ additive noise.

The filtering problem is considered under conditions of spectral certainty and spectral uncertainty. In the first case of spectral certainty the spectral density matrices $f(\lambda)$ and $g(\lambda)$ of the T-variate stationary sequences $\vec{\zeta}(n)$ and $\vec{\theta}(n)$, obtained by T-blocking of T-PC sequences $\zeta(j)$ and $\theta(j)$, respectively, are suppose to be known exactly. With the help of Hilbert space projection method formulas for calculating the spectral characteristic and the mean-square error of the optimal estimate of the functionals are proposed. In the second case of spectral uncertainty the spectral density matrices are not exactly known while a class $D = D_f \times D_g$ of admissible spectral densities is given. Using the minimax (robust) estimation method we derived relations that determine the least favorable spectral densities and the minimax spectral characteristic of the optimal estimate of the functional $A\zeta$. The problem is investigated in details for two special classes of admissible spectral densities.

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