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On Leibniz Cohomology

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ON LEIBNIZ COHOMOLOGY

JÖRG FELDVOSS AND FRIEDRICH WAGEMANN

ABSTRACT. In this paper we prove the Leibniz analogue of Whitehead's vanishing theorem for the Chevalley-Eilenberg cohomology of Lie algebras. As a consequence, we obtain the second Whitehead lemma for Leibniz algebras. Moreover, we compute the cohomology of several Leibniz algebras with adjoint or irreducible coefficients. Our main tool is a Leibniz analogue of the Hochschild-Serre spectral sequence, which is an extension of the dual of a spectral sequence of Pirashvili for Leibniz homology from symmetric bimodules to arbitrary bimodules.

INTRODUCTION

In [1], the authors study the cohomology of semi-simple Leibniz algebras, i.e., the cohomology of finite-dimensional Leibniz algebras $\mathfrak{L}$ with an ideal of squares $\mathrm{Leib}(\mathfrak{L})$ such that the corresponding canonical Lie algebra $\mathfrak{L}_{\mathrm{Lie}} := \mathfrak{L}/\mathrm{Leib}(\mathfrak{L})$ is semi-simple, and conjecture that $\mathrm{HL}^2(\mathfrak{L}, \mathfrak{L}_{\mathrm{ad}}) = 0$. In [16], the authors determine the deviation of the second Leibniz cohomology of a complex Lie algebra with adjoint or trivial coefficients from the corresponding Chevalley-Eilenberg cohomology. With these motivations in mind, we systematically transpose Pirashvili's results and tools from homology (see [31]) to cohomology, generalize the dual of one of Pirashvili's spectral sequences from symmetric bimodules to arbitrary bimodules, and prove the conjecture mentioned above.

Obtaining this kind of vanishing results would be easy with a strong analogue of the Hochschild-Serre spectral sequence for Leibniz cohomology. Recall that the Hochschild-Serre spectral sequence for a Lie algebra extension $0 \to \mathfrak{k} \to \mathfrak{g} \to \mathfrak{q} \to 0$ arises from a filtration of the standard cochain complex of $\mathfrak{g}$ by cochains which vanish in case one inserts for a certain fixed number q elements of the ideal $\mathfrak{k}$ in q arguments of the cochain (see [19, Sections 2 and 3]). When trying to generalize this filtration from Lie algebras to Leibniz algebras, one needs to choose whether to filter from the left or from the right. Another difficulty is that the arising spectral sequence does not converge to the cohomology of the Leibniz algebra, but rather to the cohomology of some quotient complex. Furthermore, one must impose that the ideal acts trivially from the left (right) on the left (right) Leibniz algebra. This last issue excludes the application of the spectral sequence to many interesting ideals in the Leibniz algebra. Pirashvili [31, Theorem C] has constructed an analogue of the Hochschild-Serre spectral sequence using the filtration from the right for right

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Leibniz algebras and indicated how to use it together with a long exact sequence in order to extract cohomology. We use Pirashvili's framework and extend the dual of his spectral sequence from symmetric bimodules to arbitrary bimodules (see Theorem 3.4). The two main changes of perspective with respect to [31] are the systematic use of arbitrary bimodules and computations in which we consider ground fields of all characteristics. We hope that this might be useful for further applications in the future.

The main application of Theorem 3.4 is Theorem 4.3 in which we compute the cohomology of a finite-dimensional semi-simple Leibniz algebra over a field of characteristic zero with coefficients in an arbitrary finite-dimensional bimodule. The case $n = 2$ of Theorem 4.3 is the second Whitehead lemma for Leibniz algebras. But note that contrary to Chevalley-Eilenberg cohomology, Leibniz cohomology vanishes in any degree $n \geq 2$. This is one of several instances that we found by our computations in this paper which indicates that Leibniz cohomology behaves more uniformly than Chevalley-Eilenberg cohomology. We also show by examples that the theorem fails in prime characteristic or for infinite-dimensional modules (see Examples E and F, respectively).

As an immediate consequence of Theorem 4.3 we obtain the rigidity of finite-dimensional semi-simple Leibniz algebras in characteristic zero (see Corollary 4.7). More generally, we obtain a complete description of the cohomology of a finite-dimensional semi-simple left Leibniz algebra with coefficients in the adjoint bimodule and its (anti-)symmetric counterparts (see Theorem 4.5). In particular, we deduce that a finite-dimensional semi-simple non-Lie Leibniz algebra in characteristic zero always possesses outer derivations (see Corollary 4.6) which might be somewhat surprising as this shows that derivations of non-Lie Leibniz algebras are more complicated than derivations of Lie algebras.

In addition to the results just mentioned, we dualize another spectral sequence obtained by Pirashvili for Leibniz homology (see [31, Theorem A]) that relates the Leibniz cohomology of a Lie algebra to its Chevalley-Eilenberg cohomology (see Theorems 2.5 and 2.6). As an application we generalize some known results on rigidity to complete Lie algebras (see Corollary 2.9 and Corollary 2.10) and to parabolic subalgebras of finite-dimensional semi-simple Lie algebras (see Proposition 2.11). Moreover, we compute the Leibniz cohomology for the non-abelian two-dimensional Lie algebra (see Example A) and the three-dimensional Heisenberg algebra (see Example B) with coefficients in irreducible Leibniz bimodules. The authors believe that Leibniz cohomology is an important invariant of a Lie algebra that behaves more uniformly than Chevalley-Eilenberg cohomology. The motivation for including so many details in Section 2 was to provide the reader with a solid foundation for computing this invariant in arbitrary characteristics.

The subject of Leibniz algebras, and especially its (co)homology theory, owes a great deal to Jean-Louis Loday and Teimuraz Pirashvili (see [25], [24], [26], [31], and [27]). Many fundamental definitions and tools are due to them. While Loday and Pirashvili work with right Leibniz algebras, we work with left Leibniz algebras. Obviously, results for left Leibniz algebras are equivalent to the corresponding results for right Leibniz algebras. For the convenience of the reader we shall indicate where to find the corresponding formulae for left Leibniz algebras, even when they have been invented in the framework of right Leibniz algebras and are due to Loday and Pirashvili.

Teimuraz Pirashvili spotted an error in a first version of this article (see [32]) which we have subsequently corrected. The error is related to another Hochschild-Serre type spectral sequence (see Remark (c) after Corollary 3.5) which we have removed from the present version because its E_2-term is more involved than we originally thought.

In this paper we will follow the notation used in [14]. All tensor products are over the relevant ground field and will be denoted by $\otimes$. For a subset X of a vector space V over a field $\mathbb{F}$ we let $\langle X\rangle_{\mathbb{F}}$ be the subspace of V spanned by X. We will denote the space of linear transformations from an $\mathbb{F}$-vector space V to an $\mathbb{F}$-vector space W by $\mathrm{Hom}_{\mathbb{F}}(V,W)$. In particular, $V^* := \mathrm{Hom}_{\mathbb{F}}(V,\mathbb{F})$ will be the space of linear forms on a vector space V over a field $\mathbb{F}$. Moreover, $S^2(V)$ will denote the symmetric square of a vector space V. Finally, the identity function on a set X will be denoted by id_X, and the set $\{0,1,2,\dots\}$ of non-negative integers will be denoted by $\mathbb{N}_0$.

1. Preliminaries

In this section we recall some definitions and collect several results that will be useful in the remainder of the paper.

A *left Leibniz algebra* is an algebra $\mathfrak{L}$ such that every left multiplication operator $L_x : \mathfrak{L} \to \mathfrak{L}$, $y \mapsto xy$ is a derivation. This is equivalent to the identity

$$x(yz) = (xy)z + y(xz) \tag{1.1}$$

for all $x, y, z \in \mathfrak{L}$, which in turn is equivalent to the identity

$$(xy)z = x(yz) - y(xz) \tag{1.2}$$

for all $x, y, z \in \mathfrak{L}$. We will call both identities the *left Leibniz identity*. There is a similar definition of a *right Leibniz algebra* but in this paper we will only consider left Leibniz algebras.

Every left Leibniz algebra has an important ideal, its Leibniz kernel, that measures how much the Leibniz algebra deviates from being a Lie algebra. Namely, let $\mathfrak{L}$ be a left Leibniz algebra over a field $\mathbb{F}$. Then

$$\mathrm{Leib}(\mathfrak{L}) := \langle x^2 \mid x \in \mathfrak{L}\rangle_{\mathbb{F}}$$

is called the *Leibniz kernel* of $\mathfrak{L}$. The Leibniz kernel $\mathrm{Leib}(\mathfrak{L})$ is an abelian ideal of $\mathfrak{L}$, and $\mathrm{Leib}(\mathfrak{L}) \neq \mathfrak{L}$ when $\mathfrak{L} \neq 0$ (see [14, Proposition 2.20]). Moreover, $\mathfrak{L}$ is a Lie algebra if, and only if, $\mathrm{Leib}(\mathfrak{L}) = 0$. It follows from the left Leibniz identity (1.2) that $\mathrm{Leib}(\mathfrak{L}) \subseteq C_\ell(\mathfrak{L})$, where $C_\ell(\mathfrak{L}) := \{c \in \mathfrak{L} \mid \forall\, x \in \mathfrak{L} : cx = 0\}$ denotes the *left center* of $\mathfrak{L}$.

By definition of the Leibniz kernel, $\mathfrak{L}_{\mathrm{Lie}} := \mathfrak{L}/\mathrm{Leib}(\mathfrak{L})$ is a Lie algebra which we call the *canonical Lie algebra* associated to $\mathfrak{L}$. In fact, the Leibniz kernel is the smallest ideal such that the corresponding factor algebra is a Lie algebra (see [14, Proposition 2.22]).

Next, we will briefly discuss left modules and bimodules of left Leibniz algebras. Let $\mathfrak{L}$ be a left Leibniz algebra over a field $\mathbb{F}$. A *left $\mathfrak{L}$-module* is a vector space M over $\mathbb{F}$ with an $\mathbb{F}$-bilinear left $\mathfrak{L}$-action $\mathfrak{L} \times M \to M$, $(x,m) \mapsto x \cdot m$ such that

$$(xy) \cdot m = x \cdot (y \cdot m) - y \cdot (x \cdot m) \tag{1.3}$$

is satisfied for every $m \in M$ and all $x, y \in \mathfrak{L}$.

By virtue of [14, Lemma 3.3], every left $\mathfrak{L}$-module is an $\mathfrak{L}_{\mathrm{Lie}}$-module, and vice versa. Therefore left Leibniz modules are sometimes called Lie modules. Consequently, many properties of left Leibniz modules follow from the corresponding properties of modules for the canonical Lie algebra.

The correct concept of a module for a left Leibniz algebra $\mathfrak{L}$ is the notion of a Leibniz bimodule. An $\mathfrak{L}$-*bimodule* is a left $\mathfrak{L}$-module M with an $\mathbb{F}$-bilinear right $\mathfrak{L}$-action $M \times \mathfrak{L} \to M$, $(m, x) \mapsto m \cdot x$ such that

$$(x \cdot m) \cdot y = x \cdot (m \cdot y) - m \cdot (xy) \tag{1.4}$$

and

$$(m \cdot x) \cdot y = m \cdot (xy) - x \cdot (m \cdot y) \tag{1.5}$$

are satisfied for every $m \in M$ and all $x, y \in \mathfrak{L}$. In fact, all three identities (1.3), (1.4), and (1.5) are instances of the left Leibniz identity, written down for the left Leibniz algebra $\mathfrak{L} \oplus M$ which is considered as an abelian extension in the theory of non-associative algebras, where the element m coccurs on the right, in the middle, or on the left, respectively (for the details see [14, Section 3]).

The usual definitions of the notions of *sub-(bi)module*, *irreducibility*, *complete reducibility*, *composition series*, *homomorphism*, *isomorphism*, etc., hold for left Leibniz modules and Leibniz bimodules.

Let $\mathfrak{L}$ be a left Leibniz algebra over a field $\mathbb{F}$, and let M be an $\mathfrak{L}$-bimodule. Then M is said to be *symmetric* if $m \cdot x = -x \cdot m$ for every $x \in \mathfrak{L}$ and every $m \in M$, and M is said to be *anti-symmetric* if $m \cdot x = 0$ for every $x \in \mathfrak{L}$ and every $m \in M$. We call

$$M_0 := \langle x \cdot m + m \cdot x \mid x \in \mathfrak{L}, m \in M \rangle_{\mathbb{F}}$$

the *anti-symmetric kernel* of M. It is known that M_0 is an anti-symmetric $\mathfrak{L}$-sub-bimodule of M (see [14, Proposition 3.12]) such that $M_{\mathrm{sym}} := M/M_0$ is symmetric (see [14, Proposition 3.13]).

Moreover, for any $\mathfrak{L}$-bimodule M we will need its *space of right* $\mathfrak{L}$*-invariants*

$$M^{\mathfrak{L}} := \{m \in M \mid \forall\, x \in \mathfrak{L} : m \cdot x = 0\}$$

and the *annihilator*

$$\mathrm{Ann}^{\mathrm{bi}}_{\mathfrak{L}}(M) := \{x \in \mathfrak{L} \mid \forall\, m \in M : x \cdot m = 0 = m \cdot x\}\,.$$

Our first result will be useful in the proof of Theorem 4.2.

Lemma 1.1. *Let $\mathfrak{L}$ be a left Leibniz algebra, and let M be an $\mathfrak{L}$-bimodule such that $M^{\mathfrak{L}} = 0$. Then M is symmetric. In particular,* $\mathrm{Leib}(\mathfrak{L}) \subseteq \mathrm{Ann}^{\mathrm{bi}}_{\mathfrak{L}}(M)$.

Proof. Since M_0 is anti-symmetric, it follows from the hypothesis that

$$M_0 = M_0^{\mathfrak{L}} \subseteq M^{\mathfrak{L}} = 0\,.$$

Hence we obtain from the definition of M_0 that M is symmetric. The second part is then an immediate consequence of [14, Lemma 3.10]. □

It is clear from the definition of $M^{\mathfrak{L}}$ that an $\mathfrak{L}$-bimodule M is anti-symmetric if, and only if, $M^{\mathfrak{L}} = M$. We will use Lemma 1.1 to show that the symmetry of non-trivial irreducible Leibniz bimodules can also be characterized by the behavior of their spaces of right invariants. As a preparation for this, we need to know that the latter space is a sub-bimodule.

Lemma 1.2. *Let $\mathfrak{L}$ be a left Leibniz algebra, and let M be an $\mathfrak{L}$-bimodule. Then $M^{\mathfrak{L}}$ is a sub-bimodule of M.*

Proof. It follows from (1.4) that $M^{\mathfrak{L}}$ is invariant under the left action on M, and it follows from (1.5) that $M^{\mathfrak{L}}$ is invariant under the right action on M. ☐

Remark. More generally, the proof of Lemma 1.2 shows that $M^{\mathfrak{I}}$ is an $\mathfrak{L}$-sub-bimodule of M for every left ideal $\mathfrak{I}$ of $\mathfrak{L}$.

Now we can characterize the symmetry of a non-trivial irreducible Leibniz bimodule by the vanishing of its space of right invariants. In particular, for non-trivial irreducible Leibniz bimodules we obtain the converse of Lemma 1.1. (Recall that an irreducible bimodule M is a bimodule that has exactly two sub-bimodules, namely, 0 and M. In particular, an irreducible bimodule is by definition a non-zero vector space.)

Corollary 1.3. *Let $\mathfrak{L}$ be a left Leibniz algebra, and let M be an irreducible $\mathfrak{L}$-bimodule. Then M is symmetric with non-trivial $\mathfrak{L}$-action if, and only if, $M^{\mathfrak{L}} = 0$.*

Proof. Since M is irreducible, we obtain from Lemma 1.2 that $M^{\mathfrak{L}} = 0$ or $M^{\mathfrak{L}} = M$. Suppose first that M is symmetric with non-trivial $\mathfrak{L}$-action. Then we have that $M^{\mathfrak{L}} = 0$. On the other hand, the converse immediately follows from Lemma 1.1. ☐

Recall that every left $\mathfrak{L}$-module M of a left Leibniz algebra $\mathfrak{L}$ determines a unique symmetric $\mathfrak{L}$-bimodule structure on M by defining $m\cdot x := -x\cdot m$ for every element $m \in M$ and every element $x \in \mathfrak{L}$ (see [14, Proposition 3.15 (b)]). We will denote this symmetric $\mathfrak{L}$-bimodule by M_s. Similarly, every left $\mathfrak{L}$-module M with trivial right action is an anti-symmetric $\mathfrak{L}$-bimodule (see [14, Proposition 3.15 (a)]). We will denote this module by M_a. Note that for any irreducible left $\mathfrak{L}$-module M the $\mathfrak{L}$-bimodules M_s and M_a are irreducible, and every irreducible $\mathfrak{L}$-bimodule arises in this way from an irreducible left $\mathfrak{L}$-module (see [27, p. 415]).

Similar to the boundary map in [25] for the homology of a right Leibniz algebra with coefficients in a right module one can also introduce a coboundary map $\widetilde{\mathrm{d}}^\bullet$ for the cohomology of a left Leibniz algebra with coefficients in a left module as follows.

Let $\mathfrak{L}$ be a left Leibniz algebra over a field $\mathbb{F}$, and let M be a left $\mathfrak{L}$-module. For any non-negative integer n set $\mathrm{CL}^n(\mathfrak{L}, M) := \mathrm{Hom}_{\mathbb{F}}(\mathfrak{L}^{\otimes n}, M)$ and consider the linear transformation $\widetilde{\mathrm{d}}^n : \mathrm{CL}^n(\mathfrak{L}, M) \to \mathrm{CL}^{n+1}(\mathfrak{L}, M)$ defined by

$$\begin{aligned}(\widetilde{\mathrm{d}}^n f)(x_1, \dots, x_{n+1}) \quad &:= \quad \sum_{i=1}^{n+1} (-1)^{i+1} x_i \cdot f(x_1, \dots, \hat{x}_i, \dots, x_{n+1}) \\ &+ \sum_{1 \le i < j \le n+1} (-1)^i f(x_1, \dots, \hat{x}_i, \dots, x_i x_j, \dots, x_{n+1})\end{aligned}$$

for any $f \in \mathrm{CL}^n(\mathfrak{L}, M)$ and all elements $x_1, \dots, x_{n+1} \in \mathfrak{L}$. Note that in the second sum of the coboundary map $\widetilde{\mathrm{d}}^n$ the term $x_i x_j$ appears in the j-th position. (Note that the convention in Loday's book (see [25, (10.6.2)]) is for the homology of right Leibniz algebras, and it is different.) Moreover, here and in the remainder of the paper we identify the tensor power $\mathfrak{L}^{\otimes n}$ with the corresponding Cartesian power.

Now let M be an $\mathfrak{L}$-bimodule and for any non-negative integer n consider the linear transformation $\mathrm{d}^n : \mathrm{CL}^n(\mathfrak{L}, M) \to \mathrm{CL}^{n+1}(\mathfrak{L}, M)$ defined by

$$\begin{aligned}(\mathrm{d}^n f)(x_1, \dots, x_{n+1}) &:= \sum_{i=1}^{n} (-1)^{i+1} x_i \cdot f(x_1, \dots, \hat{x}_i, \dots, x_{n+1}) \\ &+ (-1)^{n+1} f(x_1, \dots, x_n) \cdot x_{n+1} \\ &+ \sum_{1 \le i < j \le n+1} (-1)^i f(x_1, \dots, \hat{x}_i, \dots, x_i x_j, \dots, x_{n+1})\end{aligned}$$

for any $f \in \mathrm{CL}^n(\mathfrak{L}, M)$ and all elements $x_1, \dots, x_{n+1} \in \mathfrak{L}$, where as before the term $x_i x_j$ appears in the j-th position.

It is proved in [10, Lemma 1.3.1] that $\mathrm{CL}^\bullet(\mathfrak{L}, M) := (\mathrm{CL}^n(\mathfrak{L}, M), \mathrm{d}^n)_{n \in \mathbb{N}_0}$is a cochain complex, i.e., $\mathrm{d}^{n+1} \circ \mathrm{d}^n = 0$ for every non-negative integer n. Of course, the original idea of defining Leibniz cohomology as the cohomology of such a cochain complex for right Leibniz algebras is due to Loday and Pirashvili [26, Section 1.8]. Hence one can define the *cohomology of* $\mathfrak{L}$ *with coefficients in an* $\mathfrak{L}$*-bimodule* M by

$$\mathrm{HL}^n(\mathfrak{L}, M) := \mathrm{H}^n(\mathrm{CL}^\bullet(\mathfrak{L}, M)) := \mathrm{Ker}(\mathrm{d}^n)/\mathrm{Im}(\mathrm{d}^{n-1})$$

for every non-negative integer n. (In this definition we use that $\mathrm{d}^{-1} := 0$.)

If M is a symmetric $\mathfrak{L}$-bimodule, then we have the identity $\widetilde{\mathrm{d}}^n = \mathrm{d}^n$ for any non-negative integer n. Namely,

$$\begin{aligned}(\widetilde{\mathrm{d}}^n f)(x_1, \dots, x_{n+1}) &= \sum_{i=1}^{n} (-1)^{i+1} x_i \cdot f(x_1, \dots, \hat{x}_i, \dots, x_{n+1}) \\ &+ (-1)^{n+2} x_{n+1} \cdot f(x_1, \dots, x_n) \\ &+ \sum_{1 \le i < j \le n+1} (-1)^i f(x_1, \dots, \hat{x}_i, \dots, x_i x_j, \dots, x_{n+1}) \\ &= (\mathrm{d}^n f)(x_1, \dots, x_{n+1})\end{aligned}$$

for any $f \in \mathrm{CL}^n(\mathfrak{L}, M)$ and all elements $x_1, \dots, x_{n+1} \in \mathfrak{L}$. In particular, as mentioned earlier, any left $\mathfrak{L}$-module M can be turned into a symmetric $\mathfrak{L}$-module M_s, and the fact that $\mathrm{d}^\bullet$ is a coboundary map for $\mathrm{CL}^\bullet(\mathfrak{L}, M_s)$ shows that $\widetilde{\mathrm{CL}}^\bullet(\mathfrak{L}, M) := (\mathrm{CL}^n(\mathfrak{L}, M), \widetilde{\mathrm{d}}^n)_{n \in \mathbb{N}_0}$ is a cochain complex, i.e., $\widetilde{\mathrm{d}}^{n+1} \circ \widetilde{\mathrm{d}}^n = 0$ for every non-negative integer n. Hence one can define the *cohomology of* $\mathfrak{L}$ *with coefficients in a left* $\mathfrak{L}$*-module* M by

$$\widetilde{\mathrm{HL}}^n(\mathfrak{L}, M) := \mathrm{H}^n(\widetilde{\mathrm{CL}}^\bullet(\mathfrak{L}, M)) := \mathrm{Ker}(\widetilde{\mathrm{d}}^n)/\mathrm{Im}(\widetilde{\mathrm{d}}^{n-1})$$

for every non-negative integer n. (As in the previous definition we set $\widetilde{\mathrm{d}}^{-1} := 0$.)

Now we are ready to state the next result (see [31, Lemma 2.2] for the analogous result in Leibniz homology) whose second part generalizes [14, Corollary 4.4 (b)] to arbitrary degrees and which will be crucial in Section 4. (Note that the second part has already been obtained in [10, Proposition 1.3.16]). For the convenience of the reader we include a detailed proof.

Lemma 1.4. *Let* $\mathfrak{L}$ *be a left Leibniz algebra over a field* $\mathbb{F}$*, and let* M *be a left* $\mathfrak{L}$*-module. Then the following statements hold:*

(a) *If* M *is considered as a symmetric* $\mathfrak{L}$*-bimodule* M_s*, then*

$$\mathrm{HL}^n(\mathfrak{L}, M_s) = \widetilde{\mathrm{HL}}^n(\mathfrak{L}, M)$$

for every integer $n \geq 0$.

(b) *If* M *is considered as an anti-symmetric* $\mathfrak{L}$*-bimodule* M_a*, then*

$$\mathrm{HL}^0(\mathfrak{L}, M_a) = M$$

and

$$\mathrm{HL}^n(\mathfrak{L}, M_a) \cong \widetilde{\mathrm{HL}}^{n-1}(\mathfrak{L}, \mathrm{Hom}_{\mathbb{F}}(\mathfrak{L}, M)) = \mathrm{HL}^{n-1}(\mathfrak{L}, \mathrm{Hom}_{\mathbb{F}}(\mathfrak{L}, M)_s)$$

for every integer $n \geq 1$*, where* $\mathrm{Hom}_{\mathbb{F}}(\mathfrak{L}, M)$ *is a left* $\mathfrak{L}$*-module via*

$$(x \cdot f)(y) := x \cdot f(y) - f(xy)$$

for every $f \in \mathrm{Hom}_{\mathbb{F}}(\mathfrak{L}, M)$ *and any elements* $x, y \in \mathfrak{L}$.

Proof. By virtue of the computation before Lemma 1.4, we only need to prove part (b). Note that the first part of (b) is just [14, Corollary 4.2 (b)].

First, we show that $\mathrm{Hom}_{\mathbb{F}}(\mathfrak{L}, M)$ is a left $\mathfrak{L}$-module via the given action. Let $f \in \mathrm{Hom}_{\mathbb{F}}(\mathfrak{L}, M)$ and $x, y, z \in \mathfrak{L}$ be arbitrary. Then we obtain from the defining identity of a left Leibniz module (1.3) and the left Leibniz identity (1.2) that

$$\begin{aligned} ((xy) \cdot f)(z) &= (xy) \cdot f(z) - f((xy)z) \\ &= x \cdot (y \cdot f(z)) - y \cdot (x \cdot f(z)) - f(x(yz)) + f(y(xz)), \end{aligned}$$

and

$$\begin{aligned} (x \cdot (y \cdot f))(z) &= x \cdot (y \cdot f)(z) - (y \cdot f)(xz) \\ &= x \cdot (y \cdot f(z)) - x \cdot f(yz) - y \cdot f(xz) + f(y(xz)), \end{aligned}$$

as well as

$$\begin{aligned} (y \cdot (x \cdot f))(z) &= y \cdot (x \cdot f)(z) - (x \cdot f)(yz) \\ &= y \cdot (x \cdot f(z)) - y \cdot f(xz) - x \cdot f(yz) + f(x(yz)). \end{aligned}$$

Hence $((xy) \cdot f)(z) = (x \cdot (y \cdot f)(z) - (y \cdot (x \cdot f)(z)$ for every $z \in \mathfrak{L}$, or equivalently, $(xy) \cdot f = x \cdot (y \cdot f) - y \cdot (x \cdot f)$.

Now we will prove the second part of (b). Let n be any positive integer. Consider the linear transformations $\varphi^n : \mathrm{CL}^n(\mathfrak{L}, M) \to \mathrm{CL}^{n-1}(\mathfrak{L}, \mathrm{Hom}_{\mathbb{F}}(\mathfrak{L}, M))$ defined by $\varphi^n(f)(x_1, \dots, x_{n-1})(x) := f(x_1, \dots, x_{n-1}, x)$ for any elements $x_1, \dots, x_{n-1}, x \in \mathfrak{L}$ and $\psi^n : \mathrm{CL}^{n-1}(\mathfrak{L}, \mathrm{Hom}_{\mathbb{F}}(\mathfrak{L}, M)) \to \mathrm{CL}^n(\mathfrak{L}, M)$ defined by $\psi^n(g)(x_1, \dots, x_{n-1}, x_n)$ $:= g(x_1, \dots, x_{n-1})(x_n)$ for any elements $x_1, \dots, x_{n-1}, x_n \in \mathfrak{L}$. Then φ^n and ψ^n are inverses of each other.

Next, we will show that $\widetilde{\mathrm{d}}^{n-1} \circ \varphi^n = \varphi^{n+1} \circ \mathrm{d}^n$. Compute

$$\begin{aligned}
(\widetilde{\mathrm{d}}^{n-1} \circ \varphi^n)(f)(x_1, \dots, x_n)(x) &= \widetilde{\mathrm{d}}^{n-1}(\varphi^n(f))(x_1, \dots, x_n)(x) \\
&= \sum_{i=1}^{n} (-1)^{i+1} (x_i \cdot \varphi^n(f))(x_1, \dots, \hat{x}_i, \dots, x_n)(x) \\
&+ \sum_{1 \le i < j \le n} (-1)^i \varphi^n(f)(x_1, \dots, \hat{x}_i, \dots, x_i x_j, \dots, x_n)(x) \\
&= \sum_{i=1}^{n} (-1)^{i+1} x_i \cdot \varphi^n(f)(x_1, \dots, \hat{x}_i, \dots, x_n)(x) \\
&- \sum_{i=1}^{n} (-1)^{i+1} \varphi^n(f)(x_1, \dots, \hat{x}_i, \dots, x_n)(x_i x) \\
&+ \sum_{i \le i < j \le n} (-1)^i f(x_1, \dots, \hat{x}_i, \dots, x_i x_j, \dots, x_n, x) \\
&= \sum_{i=1}^{n} (-1)^{i+1} x_i \cdot f(x_1, \dots, \hat{x}_i, \dots, x_n, x) \\
&+ \sum_{i=1}^{n} (-1)^i f(x_1, \dots, \hat{x}_i, \dots, x_n, x_i x) \\
&+ \sum_{1 \le i < j \le n} (-1)^i f(x_1, \dots, \hat{x}_i, \dots, x_i x_j, \dots, x_n, x)
\end{aligned}$$

and

$$\begin{aligned}
(\varphi^{n+1} \circ \mathrm{d}^n)(f)(x_1, \dots, x_n)(x) &= \varphi^{n+1}(\mathrm{d}^n(f))(x_1, \dots, x_n)(x) \\
&= \mathrm{d}^n(f)(x_1, \dots, x_n, x) \\
&= \sum_{i=1}^{n} (-1)^{i+1} x_i \cdot f(x_1, \dots, \hat{x}_i, \dots, x_n, x) \\
&+ (-1)^{n+1} f(x_1, \dots, x_n) \cdot x \\
&+ \sum_{1 \le i < j \le n} (-1)^i f(x_1, \dots, \hat{x}_i, \dots, x_i x_j, \dots, x_n, x) \\
&+ \sum_{i=1}^{n} (-1)^i f(x_1, \dots, \hat{x}_i, \dots, x_n, x_i x)
\end{aligned}$$

for any elements $x_1, \dots, x_n, x \in \mathfrak{L}$. Since M is anti-symmetric, the second of the last four summands vanishes, and thus the two compositions are equal. From the identity $\widetilde{\mathrm{d}}^{n-1} \circ \varphi^n = \varphi^{n+1} \circ \mathrm{d}^n$ for every integer $n \geq 1$ we obtain that

$$\varphi^n(\mathrm{Ker}(\mathrm{d}^n)) \subseteq \mathrm{Ker}(\widetilde{\mathrm{d}}^{n-1})$$

and

$$\varphi^n(\mathrm{Im}(\mathrm{d}^{n-1})) \subseteq \mathrm{Im}(\widetilde{\mathrm{d}}^{n-2})$$

for every integer $n \geq 1$. Hence φ^n induces an isomorphism of vector spaces between $\mathrm{HL}^n(\mathfrak{L}, M)$ and $\widetilde{\mathrm{HL}}^{n-1}(\mathfrak{L}, \mathrm{Hom}_{\mathbb{F}}(\mathfrak{L}, M))$ for every integer $n \geq 1$. In order to see the remainder of the assertion, apply part (a). $\square$

In the special case of the trivial one-dimensional Leibniz bimodule we obtain from Lemma 1.4 the following result which will be needed in Section 4 (see [25, Exercise E.10.6.1] for the analogous result in Leibniz homology).

Corollary 1.5. *Let $\mathfrak{L}$ be a left Leibniz algebra over a field $\mathbb{F}$. Then for every integer $n \geq 1$ there are isomorphisms*

$$\mathrm{HL}^n(\mathfrak{L}, \mathbb{F}) \cong \widetilde{\mathrm{HL}}^{n-1}(\mathfrak{L}, \mathfrak{L}^*) = \mathrm{HL}^{n-1}(\mathfrak{L}, (\mathfrak{L}^*)_s)$$

of vector spaces, where $\mathfrak{L}^*$ is a left $\mathfrak{L}$-module via $(x \cdot f)(y) := -f(xy)$ for every linear form $f \in \mathfrak{L}^*$ and any elements $x, y \in \mathfrak{L}$.

Remark. Note that [20, Theorem 3.5] is an immediate consequence of the case $n = 2$ of Corollary 1.5 and [14, Corollary 4.4 (a)].

2. A relation between Chevalley-Eilenberg cohomology and Leibniz cohomology for Lie algebras

Let $\mathfrak{g}$ be a Lie algebra, and let M be a left $\mathfrak{g}$-module that is also viewed as a symmetric Leibniz $\mathfrak{g}$-bimodule M_s. In this section, we will investigate how the Chevalley-Eilenberg cohomology $\mathrm{H}^\bullet(\mathfrak{g}, M)$ and the Leibniz cohomology $\mathrm{HL}^\bullet(\mathfrak{g}, M_s)$ are related. The tools set forth in this section have been developed by Pirashvili, and we follow the analogous treatment for homology given in [31] very closely.

The Chevalley-Eilenberg cohomology of a Lie algebra $\mathfrak{g}$ with trivial coefficients is not isomorphic (up to a degree shift) to the Chevalley-Eilenberg cohomology of $\mathfrak{g}$ with coadjoint coefficients as it is the case for Leibniz cohomology (see Corollary 1.5). Instead these cohomologies are only related by a long exact sequence (see Proposition 2.1). The cohomology measuring the deviation from such an isomorphism will appear in a spectral sequence (see Theorem 2.5) which can be used to relate the Leibniz cohomology of a Lie algebra to its Chevalley-Eilenberg cohomology (see Proposition 2.2).

The exterior product map $m_n : \Lambda^n\mathfrak{g} \otimes \mathfrak{g} \to \Lambda^{n+1}\mathfrak{g}$ given on homogeneous tensors by $x_1 \wedge \ldots \wedge x_n \otimes x \mapsto x_1 \wedge \ldots \wedge x_n \wedge x$ induces a monomorphism

$$m^\bullet : \overline{\mathrm{C}}^\bullet(\mathfrak{g}, \mathbb{F})[-1] \hookrightarrow \mathrm{C}^\bullet(\mathfrak{g}, \mathfrak{g}^*) \,,$$

where $\overline{\mathrm{C}}^\bullet(\mathfrak{g}, \mathbb{F})$ is the truncated cochain complex

$$\overline{\mathrm{C}}^0(\mathfrak{g}, \mathbb{F}) := 0 \text{ and } \overline{\mathrm{C}}^n(\mathfrak{g}, \mathbb{F}) := \mathrm{C}^n(\mathfrak{g}, \mathbb{F}) \quad \text{for every integer } n > 0 \,.$$

The cochain complex $\mathrm{CR}^\bullet(\mathfrak{g})$ is defined by $\mathrm{CR}^\bullet(\mathfrak{g}) := \mathrm{Coker}(m^\bullet)[-1]$, and the corresponding chain complex is defined by $\mathrm{CR}_\bullet(\mathfrak{g}) := \mathrm{Ker}(m_\bullet)[1]$ (see [35, (1.2.8), p. 9] for the definition of the degree shift and see [31, Section 1] for the definition of $\mathrm{CR}_\bullet(\mathfrak{g})$). We will mainly use the cochain complex in our paper, but the chain complex appears in Theorem 2.5 and its proof. Observe that classes in $\mathrm{CR}^n(\mathfrak{g})$ are represented by cochains of degree $n+1$ with values in $\mathfrak{g}^*$, i.e., they have $n+2$ arguments. From the short exact sequence

$$0 \to \overline{\mathrm{C}}^\bullet(\mathfrak{g}, \mathbb{F})[-1] \to \mathrm{C}^\bullet(\mathfrak{g}, \mathfrak{g}^*) \to \mathrm{CR}^\bullet(\mathfrak{g})[1] \to 0$$

of cochain complexes we obtain the following long exact sequence:

Proposition 2.1. *For every Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$ there is a long exact sequence*

$$\begin{aligned} 0 &\to \mathrm{H}^2(\mathfrak{g},\mathbb{F}) \to \mathrm{H}^1(\mathfrak{g},\mathfrak{g}^*) \to \mathrm{HR}^0(\mathfrak{g}) \\ &\to \mathrm{H}^3(\mathfrak{g},\mathbb{F}) \to \mathrm{H}^2(\mathfrak{g},\mathfrak{g}^*) \to \mathrm{HR}^1(\mathfrak{g}) \to \cdots \end{aligned}$$

and an isomorphism $\mathrm{H}^1(\mathfrak{g},\mathbb{F}) \cong \mathrm{H}^0(\mathfrak{g},\mathfrak{g}^*)$.

Remark. If we assume that the characteristic of the ground field $\mathbb{F}$ is not 2, then $\mathrm{HR}^0(\mathfrak{g}) \cong [S^2(\mathfrak{g})^*]^{\mathfrak{g}}$ is the space of invariant symmetric bilinear forms on $\mathfrak{g}$ (see [31, p. 403]). As a consequence, we obtain from Proposition 2.1 in the case that $\mathrm{char}(\mathbb{F}) \neq 2$ the five-term exact sequence

$$0 \to \mathrm{H}^2(\mathfrak{g},\mathbb{F}) \to \mathrm{H}^1(\mathfrak{g},\mathfrak{g}^*) \to [S^2(\mathfrak{g})^*]^{\mathfrak{g}} \to \mathrm{H}^3(\mathfrak{g},\mathbb{F}) \to \mathrm{H}^2(\mathfrak{g},\mathfrak{g}^*)\,,$$

which generalizes [13, Proposition 1.3 (1) & (3)]. Note that the map $[S^2(\mathfrak{g})^*]^{\mathfrak{g}} \to \mathrm{H}^3(\mathfrak{g},\mathbb{F})$ is the classical Cartan-Koszul map defined by $\omega \mapsto \overline{\omega} + \mathrm{B}^3(\mathfrak{g},\mathbb{F})$, where $\overline{\omega}(x \wedge y \wedge z) := \omega(xy, z)$ for any elements $x, y, z \in \mathfrak{g}$ (see [31, p. 403]).

For a Lie algebra $\mathfrak{g}$ and a left $\mathfrak{g}$-module M viewed as a symmetric Leibniz $\mathfrak{g}$-bimodule M_s, we have a natural monomorphism

$$\mathrm{C}^\bullet(\mathfrak{g}, M) \hookrightarrow \mathrm{CL}^\bullet(\mathfrak{g}, M_s)\,.$$

The cokernel of this morphism is by definition (up to a shift in the degree) the cochain complex $\mathrm{C}^\bullet_{\mathrm{rel}}(\mathfrak{g}, M)$:

$$\mathrm{C}^\bullet_{\mathrm{rel}}(\mathfrak{g}, M) := \mathrm{Coker}(\mathrm{C}^\bullet(\mathfrak{g}, M) \to \mathrm{CL}^\bullet(\mathfrak{g}, M_s))[-2]\,.$$

We therefore have another long exact sequence. (For the isomorphisms in degrees 0 and 1 see [14, Corollary 4.2 (a)] and [14, Corollary 4.4 (a)], respectively.)

Proposition 2.2. *Let $\mathfrak{g}$ be a Lie algebra, and let M be a left $\mathfrak{g}$-module considered as a symmetric Leibniz $\mathfrak{g}$-bimodule M_s. Then there is a long exact sequence*

$$\begin{aligned} 0 &\to \mathrm{H}^2(\mathfrak{g},M) \to \mathrm{HL}^2(\mathfrak{g},M_s) \to \mathrm{H}^0_{\mathrm{rel}}(\mathfrak{g},M) \\ &\to \mathrm{H}^3(\mathfrak{g},M) \to \mathrm{HL}^3(\mathfrak{g},M_s) \to \mathrm{H}^1_{\mathrm{rel}}(\mathfrak{g},M) \to \cdots \end{aligned}$$

and isomorphisms

$$\mathrm{HL}^0(\mathfrak{g}, M_s) \cong \mathrm{H}^0(\mathfrak{g}, M),\;\; \mathrm{HL}^1(\mathfrak{g}, M_s) \cong \mathrm{H}^1(\mathfrak{g}, M)\,.$$

Remark. If we again assume that the characteristic of the ground field $\mathbb{F}$ is not 2, it follows from Theorem 2.5 below in conjunction with the remark after Proposition 2.1 that $\mathrm{H}^0_{\mathrm{rel}}(\mathfrak{g},\mathbb{F}) \cong \mathrm{HR}^0(\mathfrak{g}) \cong [S^2(\mathfrak{g})^*]^{\mathfrak{g}}$ is the space of invariant symmetric bilinear forms on $\mathfrak{g}$. So when $\mathrm{char}(\mathbb{F}) \neq 2$, we obtain the five-term exact sequence

$$0 \to \mathrm{H}^2(\mathfrak{g},\mathbb{F}) \to \mathrm{HL}^2(\mathfrak{g},\mathbb{F}) \to [S^2(\mathfrak{g})^*]^{\mathfrak{g}} \to \mathrm{H}^3(\mathfrak{g},\mathbb{F}) \to \mathrm{HL}^3(\mathfrak{g},\mathbb{F})$$

as a special case of Proposition 2.2 (cf. [20, Proposition 3.2] for fields of characteristic zero). Note that Corollary 1.5 implies that the second terms of the five-term exact sequences in Proposition 2.1 and in Proposition 2.2 for $M := \mathbb{F}$ are isomorphic, but the fifth terms are not necessarily isomorphic (see the remark after Example A below).

Observe that as for $\mathrm{CR}^n(\mathfrak{g})$, representatives of classes in $\mathrm{C}^n_{\mathrm{rel}}(\mathfrak{g}, M)$ have $n + 2$ arguments.

On the quotient cochain complex $\mathrm{C}^{\bullet}_{\mathrm{rel}}(\mathfrak{g}, M)$ there is the following filtration

$$\mathcal{F}^p\mathrm{C}^n_{\mathrm{rel}}(\mathfrak{g}, M) = \{[c] \in \mathrm{C}^n_{\mathrm{rel}}(\mathfrak{g}, M) \mid c(x_1, \ldots, x_{n+2}) = 0 \text{ if } \exists\, j \leq p+1 : x_{j-1} = x_j\}\,.$$

Note that the condition is independent of the representative c of the class $[c]$. This defines a finite decreasing filtration

$$\mathcal{F}^0\mathrm{C}^n_{\mathrm{rel}}(\mathfrak{g}, M) = \mathrm{C}^n_{\mathrm{rel}}(\mathfrak{g}, M) \supset \mathcal{F}^1\mathrm{C}^n_{\mathrm{rel}}(\mathfrak{g}, M) \supset \cdots \supset \mathcal{F}^{n+1}\mathrm{C}^n_{\mathrm{rel}}(\mathfrak{g}, M) = \{0\}\,. \tag{2.1}$$

Then we have the following result:

Lemma 2.3. *This filtration is compatible with the Leibniz coboundary map* $\mathrm{d}^{\bullet}$.

Proof. The Leibniz coboundary map, acting on a cochain c, is an alternating sum of operators $\mathrm{d}_{ij}(c)$, $\delta_i(c)$ and $\partial(c)$, where $\mathrm{d}_{ij}(c)$ is the term involving the product of the i-th and the j-th element, $\delta_i(c)$ is the term involving the left action of the i-th element, and $\partial(c)$ is the term involving the right action of the $(n+1)$-th element. As the bimodule is symmetric, the term involving the right action can be counted among the terms involving the left actions.

We have to show that $\mathrm{d}^{\bullet}(\mathcal{F}^p\mathrm{C}^n_{\mathrm{rel}}(\mathfrak{g}, M)) \subseteq \mathcal{F}^p\mathrm{C}^{n+1}_{\mathrm{rel}}(\mathfrak{g}, M)$. We thus consider the different terms of $\mathrm{d}^{\bullet}(c)$ with two equal elements as arguments in the first $p+1$ positions and have to show that all terms are zero. For $\mathrm{d}_{ij}(c)$ with $i, j \leq p+1$, the assertion is clear because either the two equal elements do not occur in the product, and then it is correct, or at least one of them occurs, and then from the product terms of the sum of d_{ij} and d_{ij+1} (or d_{ij} and d_{ij-1}) we obtain an element $x_i x \otimes x + x \otimes x_i x$, which is a sum of symmetric elements thanks to

$$x_i x \otimes x + x \otimes x_i x = (x_i x + x) \otimes (x_i x + x) - x_i x \otimes x_i x - x \otimes x\,.$$

Even more elementary, the assertion holds for $\mathrm{d}_{ij}(c)$ with $i, j \geq p+1$. For $\mathrm{d}_{ij}(c)$ with $i \leq p+1$ and $j \geq p+2$, the assertion is clear in case x_i is not one of the equal elements. In case it is, the two terms corresponding to the product action of the two equal elements cancel as they are equal and have different sign.

For the action terms $\delta_i(c)$ the reasoning is similar. In case $i \leq p+1$, either the two equal elements do not occur and the assertion holds, or both occur and cancel each other because of the alternating sign. For $\delta_i(c)$ with $i \geq p+2$, the assertion is clear in any case. $\square$

The lemma implies that there is a spectral sequence of a filtered cochain complex associated to this filtration which thanks to (2.1) converges in the strong (i.e., finite) sense to $\mathrm{H}^n_{\mathrm{rel}}(\mathfrak{g}, M)$.

The next step is then to compute the 0-th term of this spectral sequence, i.e., the associated graded vector space of the filtration

$$E_0^{p,q} := \mathcal{F}^p\mathrm{C}^{p+q}_{\mathrm{rel}}(\mathfrak{g}, M) \,/\, \mathcal{F}^{p+1}\mathrm{C}^{p+q}_{\mathrm{rel}}(\mathfrak{g}, M)\,.$$

Observe that

$$\mathcal{F}^p\mathrm{C}^{p+q}_{\mathrm{rel}}(\mathfrak{g}, M) =$$
$$\{c \in \mathrm{CL}^{p+q+2}(\mathfrak{g}, M_s) \mid c(x_1, \ldots, x_{p+q+2}) = 0 \text{ if } \exists\, j \leq p{+}1 : x_{j-1} = x_j\} \,/\, \mathrm{C}^{p+q+2}(\mathfrak{g}, M).$$

In the quotient space $E_0^{p,q}$, the term $\mathrm{C}^{p+q+2}(\mathfrak{g}, M)$, by which both filtration spaces are divided, disappears.

Observe that the filtration can be expressed as

$$\mathcal{F}^p\mathrm{C}^{p+q}_{\mathrm{rel}}(\mathfrak{g}, M) = \{[c] \in \mathrm{C}^{p+q}_{\mathrm{rel}}(\mathfrak{g}, M) \mid c|_{\mathrm{Ker}(\otimes^{p+1}\mathfrak{g} \to \Lambda^{p+1}\mathfrak{g}) \otimes (\otimes^{q+1}\mathfrak{g})} = 0\}\,.$$

This is useful, because by elementary linear algebra, we have

$$F^{\perp} / G^{\perp} = \mathrm{Hom}_{\mathbb{F}}(G/F, M)\,,$$

where $F^{\perp} := \{f : E \to M \mid f_{|F} = 0\}$ and $G^{\perp} := \{f : E \to M \mid f_{|G} = 0\}$ for $F \subseteq G \subseteq E$.

In order to be able to find $E_0^{p,q}$, we therefore have to compute

$$\mathrm{Ker}(\otimes^{p+2}\mathfrak{g} \to \Lambda^{p+2}\mathfrak{g}) \otimes (\otimes^{q}\mathfrak{g}) \,/\, \mathrm{Ker}(\otimes^{p+1}\mathfrak{g} \to \Lambda^{p+1}\mathfrak{g}) \otimes (\otimes^{q+1}\mathfrak{g})\,.$$

By using the isomorphism (see the proof of Theorem A in [31])

$$\mathrm{Ker}(\otimes^{p+2}\mathfrak{g} \to \Lambda^{p+2}\mathfrak{g}) \,/\, \mathrm{Ker}(\otimes^{p+2}\mathfrak{g} \to \Lambda^{p+1}\mathfrak{g} \otimes \mathfrak{g}) \cong \mathrm{Ker}(\Lambda^{p+1}\mathfrak{g} \otimes \mathfrak{g} \to \Lambda^{p+2}\mathfrak{g})$$

and by applying that $\mathrm{Hom}_{\mathbb{F}}$ and $\otimes$ are adjoint functors, we obtain that

$$\begin{aligned}
E_0^{p,q} &= \mathrm{Hom}_{\mathbb{F}}(\mathrm{Ker}(\Lambda^{p+1}\mathfrak{g} \otimes \mathfrak{g} \to \Lambda^{p+2}\mathfrak{g}) \otimes \mathrm{CL}_q(\mathfrak{g}), M)\\
&= \mathrm{Hom}_{\mathbb{F}}(\mathrm{Ker}(\Lambda^{p+1}\mathfrak{g} \otimes \mathfrak{g} \to \Lambda^{p+2}\mathfrak{g}), \mathrm{Hom}_{\mathbb{F}}(\mathrm{CL}_q(\mathfrak{g}), M))\\
&= \mathrm{Hom}_{\mathbb{F}}(\mathrm{CR}_p(\mathfrak{g}), \mathrm{CL}^q(\mathfrak{g}, M))\,,
\end{aligned}$$

by definition of the chain complex $\mathrm{CR}_\bullet(\mathfrak{g})$ as the kernel of $m_\bullet$ (up to a degree shift).

In the case of a finite-dimensional Lie algebra $\mathfrak{g}$, we can use the isomorphism $\mathrm{Hom}_{\mathbb{F}}(U \otimes V, W) \cong U^* \otimes \mathrm{Hom}_{\mathbb{F}}(V, W)$ to write the E_0-term as

$$E_0^{p,q} = [\mathrm{Ker}(\Lambda^{p+1}\mathfrak{g} \otimes \mathfrak{g} \to \Lambda^{p+2}\mathfrak{g})]^* \otimes \mathrm{CL}^q(\mathfrak{g}, M)\,.$$

In this particular case, one may observe that the first tensor factor is the kernel of the exterior multiplication map $m_\bullet$, and thus

$$\mathrm{Ker}(\Lambda^{p+1}\mathfrak{g} \otimes \mathfrak{g} \to \Lambda^{p+2}\mathfrak{g})^* = \mathrm{Ker}(m_\bullet)[1]^* = \mathrm{Coker}(m^\bullet)[-1] = \mathrm{CR}^p(\mathfrak{g})\,.$$

Therefore the term $E_0^{p,q}$ takes in this particular case the form

$$E_0^{p,q} = \mathrm{CR}^p(\mathfrak{g}) \otimes \mathrm{CL}^q(\mathfrak{g}, M)\,.$$

Next, we will determine the differential on $E_0^{p,q}$:

Lemma 2.4. *The differential* d_0 *on* $E_0^{p,q} \cong \mathrm{Hom}_{\mathbb{F}}(\mathrm{CR}_p(\mathfrak{g}), \mathrm{CL}^q(\mathfrak{g}, M))$ *is induced by the coboundary operator* $\mathrm{d}^\bullet$ *on* $\mathrm{C}^\bullet_{\mathrm{rel}}(\mathfrak{g}, M)$. *More precisely, we have that*

$$d_0^{p,q}(f) := \mathrm{d}^q_{\mathrm{CL}^q(\mathfrak{g},M)} \circ f$$

for every linear transformation $f \in \mathrm{Hom}_{\mathbb{F}}(\mathrm{CR}_p(\mathfrak{g}), \mathrm{CL}^q(\mathfrak{g}, M))$.

Proof. By definition, the differential d_0 of the spectral sequence is the differential which is induced by the Leibniz coboundary map $\mathrm{d}^\bullet$ on the associated graded quotients

$$\mathrm{d}_0 : \mathcal{F}^p\mathrm{C}^{p+q}_{\mathrm{rel}}(\mathfrak{g}, M) \,/\, \mathcal{F}^{p+1}\mathrm{C}^{p+q}_{\mathrm{rel}}(\mathfrak{g}, M) \to \mathcal{F}^p\mathrm{C}^{p+q+1}_{\mathrm{rel}}(\mathfrak{g}, M) \,/\, \mathcal{F}^{p+1}\mathrm{C}^{p+q+1}_{\mathrm{rel}}(\mathfrak{g}, M)\,.$$

In order to examine which terms $\mathrm{d}_{ij}(c)$, $\delta_i(c)$ and $\partial(c)$ are zero for a cochain $c \in \mathcal{F}^p\mathrm{C}^{p+q}_{\mathrm{rel}}(\mathfrak{g}, M)$, we have to insert two consecutive equal elements in the arguments of c within the first $p + 2$ arguments.

Now, by the same reasoning as in the proof of Lemma 2.3, the terms $\mathrm{d}_{ij}(c)$ vanish in case $i, j \le p+2$, because in case the equal elements are not involved, the formula for $\mathrm{d}_{ij}(c)$ diminishes the number of arguments by one, and as c is of degree p in the filtration, this then gives zero. In case the elements occur, they create once again a symmetric element of the form $x_i x \otimes x + x \otimes x_i x$. Also for $\mathrm{d}_{ij}(c)$ with $i \le p + 2$ and $j \ge p + 3$, the terms are zero when the equal elements are not involved, and are zero in addition with $\mathrm{d}_{ij+1}(c)$ (or $\mathrm{d}_{ij-1}(c)$), in case of multiplying with one of the equal elements. The terms $\delta_i(c)$ for $i \leq p+1$ vanish as the corresponding formula diminishes the number of arguments by one in case the equal elements do not occur, and annihilate each other in case they occur.

Thus, there remain the terms $\mathrm{d}_{ij}(c)$ with $i, j \geq p{+}3$, $\delta_i(c)$ with $i \geq p{+}3$, and $\partial(c)$, which form together the coboundary map of the cochain complex $\mathrm{CL}^\bullet(\mathfrak{g}, M)$. □

Consequently, in the general case we obtain for the first term of the spectral sequence:

$$E_1^{p,q} = \mathrm{Hom}_{\mathbb{F}}(\mathrm{CR}_p(\mathfrak{g}), \mathrm{HL}^q(\mathfrak{g}, M_s))\,,$$

and in the case of a finite-dimensional Lie algebra $\mathfrak{g}$ we have that

$$E_1^{p,q} = \mathrm{CR}^p(\mathfrak{g}) \otimes \mathrm{HL}^q(\mathfrak{g}, M_s))\,.$$

Now we proceed to identify the differential d_1 on $E_1^{p,q}$. The differential d_1 is still induced by the Leibniz coboundary map on the filtered cochain complex. As the classes in $\mathrm{HL}^q(\mathfrak{g}, M_s)$ are represented by cocycles, the part of the Leibniz coboundary operator $\mathrm{d}^q_{\mathrm{CL}^q(\mathfrak{g},M_s)}$ constituting the differential must be zero. The action of one of the remaining terms on $\mathrm{HL}^q(\mathfrak{g}, M_s)$ must also be zero since the Cartan relations for Leibniz cohomology (due to Loday and Pirashvili [26, Proposition 3.1]) imply that a Leibniz algebra acts trivially on its cohomology. (For the reader interested in left Leibniz algebras, a proof of these formulas adapted to this case can be found in [10, Proposition 1.3.2].) Note that the Cartan relations only hold for $q \geq 1$. But as the Leibniz $\mathfrak{g}$-bimodule M is symmetric, the action of $\mathfrak{g}$ on $\mathrm{HL}^0(\mathfrak{g}, M_s)$ is also trivial. Therefore, the remaining terms of the differential constitute the coboundary operator $\mathrm{d}^p_{\mathrm{CL}^p(\mathfrak{g})}$ on $\mathrm{CR}^p(\mathfrak{g})$.

Consequently, in the general case we obtain for the second term of the spectral sequence:

$$E_2^{p,q} = \mathrm{HR}^p(\mathfrak{g}, \mathrm{HL}^q(\mathfrak{g}, M_s))\,,$$

where the right-hand side denotes the HR-cohomology with values in the trivial $\mathfrak{g}$-module $\mathrm{HL}^\bullet(\mathfrak{g}, M_s)$. It is the cohomology of the cochain complex arising from applying the exact functor $\mathrm{Hom}_{\mathbb{F}}(-, \mathrm{HL}^\bullet(\mathfrak{g}, M_s))$ to the chain complex $\mathrm{CR}_\bullet(\mathfrak{g})$. It follows from the exactness of this functor and the Universal Coefficient Theorem (for example, see [35, Theorem 3.6.5]) that we can express the E_2-term as

$$E_2^{p,q} = \mathrm{Hom}_{\mathbb{F}}(\mathrm{HR}_p(\mathfrak{g}), \mathrm{HL}^q(\mathfrak{g}, M_s))\,,$$

and in the special case of a finite-dimensional Lie algebra $\mathfrak{g}$, we have that

$$E_2^{p,q} = \mathrm{HR}^p(\mathfrak{g}) \otimes \mathrm{HL}^q(\mathfrak{g}, M_s)\,.$$

This discussion proves the following result which (up to dualization) is Theorem A in [31]. In the case of trivial coefficients (and possibly topological Fréchet Lie algebras) the second part of Theorem 2.5 has also been obtained by Lodder (see [28, Theorem 2.10]).

Theorem 2.5. *Let $\mathfrak{g}$ be a Lie algebra, and let M be a left $\mathfrak{g}$-module considered as a symmetric Leibniz $\mathfrak{g}$-bimodule M_s. Then there is a spectral sequence converging to $\mathrm{H}^\bullet_{\mathrm{rel}}(\mathfrak{g}, M)$ with second term*

$$E_2^{p,q} = \mathrm{Hom}_{\mathbb{F}}(\mathrm{HR}_p(\mathfrak{g}), \mathrm{HL}^q(\mathfrak{g}, M_s))\,.$$

Moreover, if $\mathfrak{g}$ is finite dimensional, then the E_2-term of this spectral sequence can be written as

$$E_2^{p,q} = \mathrm{HR}^p(\mathfrak{g}) \otimes \mathrm{HL}^q(\mathfrak{g}, M_s))\,.$$

Remark. As the spectral sequence of Theorem 2.5 is the spectral sequence of a filtered cochain complex, the higher differentials in this spectral sequence are again induced by the Leibniz coboundary operator $\mathrm{d}^\bullet$. We will see in Example B below an instance of a concrete computation of the differential d_2.

Our main application of the spectral sequence will be the next theorem which is a refinement of the cohomological analogue of [31, Corollary 1.3]:

Theorem 2.6. *Let $\mathfrak{g}$ be a Lie algebra, let M be a left $\mathfrak{g}$-module considered as a symmetric Leibniz $\mathfrak{g}$-bimodule M_s, and let n be a non-negative integer. If $\mathrm{H}^k(\mathfrak{g}, M) = 0$ for every integer k with $0 \le k \le n$, then $\mathrm{HL}^k(\mathfrak{g}, M_s) = 0$ for every integer k with $0 \le k \le n$ and $\mathrm{HL}^{n+1}(\mathfrak{g}, M_s) \cong \mathrm{H}^{n+1}(\mathfrak{g}, M)$ as well as $\mathrm{HL}^{n+2}(\mathfrak{g}, M_s) \cong \mathrm{H}^{n+2}(\mathfrak{g}, M)$. In particular, $\mathrm{H}^\bullet(\mathfrak{g}, M) = 0$ implies that $\mathrm{HL}^\bullet(\mathfrak{g}, M_s) = 0$.*

Proof. The proof follows the proof of Corollary 1.3 in [31] very closely.

According to Proposition 2.2, it suffices to prove that $\mathrm{H}^k(\mathfrak{g}, M) = 0$ for every integer k with $0 \le k \le n$ implies that $\mathrm{H}^n_{\mathrm{rel}}(\mathfrak{g}, M) = 0$ for every integer k with $0 \le k \le n$. We proceed by induction on n. In the case $n = 0$, the hypothesis yields that $E_2^{0,0} = 0$ for the second term of the spectral sequence of Theorem 2.5, and therefore we obtain from the convergence of the spectral sequence that $\mathrm{H}^0_{\mathrm{rel}}(\mathfrak{g}, M) = 0$ which initializes the induction.

So suppose now that $n \ge 1$ and $\mathrm{H}^k(\mathfrak{g}, M_s) = 0$ for every integer k with $0 \le k \le n + 1$. By induction hypothesis, we obtain that $\mathrm{H}^n_{\mathrm{rel}}(\mathfrak{g}, M) = 0$ for every integer k with $0 \le k \le n$. Hence it follows from Proposition 2.2 that $\mathrm{HL}^k(\mathfrak{g}, M_s) = 0$ for every integer k with $0 \le k \le n$ and $\mathrm{HL}^{n+1}(\mathfrak{g}, M_s) \cong \mathrm{H}^{n+1}(\mathfrak{g}, M) = 0$. Consequently, the second term $E_2^{p,q}$ of the spectral sequence in Theorem 2.5 is zero for $p + q \le n + 1$, and therefore $\mathrm{H}^{n+1}_{\mathrm{rel}}(\mathfrak{g}, M) = 0$.

Finally, the isomorphisms in degree $n{+}1$ and $n{+}2$, respectively, are an immediate consequence of Proposition 2.2. □

Remark. Note that the converse of Theorem 2.6 is also true, namely, $\mathrm{H}^k(\mathfrak{g}, M) = 0$ for every integer k with $0 \le k \le n$ if, and only if, $\mathrm{HL}^k(\mathfrak{g}, M_s) = 0$ for every integer k with $0 \le k \le n$. In particular, $\mathrm{H}^\bullet(\mathfrak{g}, M) = 0$ if, and only if, $\mathrm{HL}^\bullet(\mathfrak{g}, M_s) = 0$.

Next, we illustrate the use of the spectral sequence of Theorem 2.5 and the associated long exact sequences (see Propositions 2.1 and 2.2) by two examples. We begin by computing the Leibniz cohomology of the smallest non-nilpotent Lie algebra with coefficients in an arbitrary irreducible Leibniz bimodule (see also [31, Example 1.4 i)] for trivial coeffcients in characteristic $\neq 2$). Note that for a ground field of characteristic 2 the Leibniz cohomology of this Lie algebra is far more complicated than for a field of characteristic $\neq 2$.

In the case that the irreducible Leibniz bimodule is of finite dimension $\neq 1$ we can prove more generally for an arbitrary supersolvable Lie algebra the following vanishing result.

Proposition 2.7. *Let $\mathfrak{g}$ be a finite-dimensional supersolvable Lie algebra over a field $\mathbb{F}$, and let M be a finite-dimensional irreducible Leibniz $\mathfrak{g}$-bimodule such that $\dim_{\mathbb{F}} M \neq 1$. Then $\mathrm{HL}^n(\mathfrak{g}, M) = 0$ for every positive integer n. Moreover, if M is symmetric, then $\mathrm{HL}^n(\mathfrak{g}, M) = 0$ for every non-negative integer n.*

Proof. If M is symmetric, then the assertion is an immediate consequence of Theorem 2.6 in conjunction with [3, Theorem 3].

Now suppose that M is not symmetric. Then it follows from [14, Theorem 3.14] that M is anti-symmetric. We obtain from Lemma 1.4 (b) that

$$\mathrm{HL}^n(\mathfrak{g}, M) \cong \mathrm{HL}^{n-1}(\mathfrak{g}, \mathrm{Hom}_{\mathbb{F}}(\mathfrak{g}, M)_s) \cong \mathrm{HL}^{n-1}(\mathfrak{g}, (\mathfrak{g}^* \otimes M)_s)$$

for every positive integer n. By definition of supersolvability, the adjoint $\mathfrak{g}$-module has a composition series

$$\mathfrak{g}_{\mathrm{ad},\ell} = \mathfrak{g}_n \supset \mathfrak{g}_{n-1} \supset \cdots \supset \mathfrak{g}_1 \supset \mathfrak{g}_0 = 0$$

such that $\dim_{\mathbb{F}} \mathfrak{g}_j/\mathfrak{g}_{j-1} = 1$ for every integer $1 \leq j \leq n$. From the short exact sequences $0 \to \mathfrak{g}_{j-1} \to \mathfrak{g}_j \to \mathfrak{g}_j/\mathfrak{g}_{j-1} \to 0$, we obtain by dualizing, tensoring each term with M, and symmetrizing the short exact sequences:

$$0 \to [(\mathfrak{g}_j/\mathfrak{g}_{j-1})^* \otimes M]_s \to (\mathfrak{g}_j^* \otimes M)_s \to (\mathfrak{g}_{j-1}^* \otimes M)_s \to 0$$

for every integer $1 \leq j \leq n$. Since M is irreducible and $\dim_{\mathbb{F}} \mathfrak{g}_j/\mathfrak{g}_{j-1} = 1$, we conclude that $[(\mathfrak{g}_j/\mathfrak{g}_{j-1})^* \otimes M]_s$ is an irreducible symmetric Leibniz $\mathfrak{g}$-bimodule. Moreover, we have that $\dim_{\mathbb{F}}[(\mathfrak{g}_j/\mathfrak{g}_{j-1})^* \otimes M]_s \neq 1$ as $\dim_{\mathbb{F}} M \neq 1$. Hence we obtain inductively from the long exact cohomology sequence that $\mathrm{HL}^n(\mathfrak{g}, M) \cong \mathrm{HL}^{n-1}(\mathfrak{g}, (\mathfrak{g}^* \otimes M)_s) = 0$ for every positive integer n. □

Remark. It follows from Lie's theorem that every finite-dimensional irreducible Leibniz bimodule of a finite-dimensional solvable Lie algebra over an algebraically closed field of characteristic zero is one-dimensional (see [22, Corollary 4.1 A]). Consequently, in this case the hypothesis of Proposition 2.7 is never satisfied, and thus this result is only applicable over non-algebraically closed fields of characteristic zero or over fields of prime characteristic.

By virtue of Proposition 2.7, in the next example it is enough to consider one-dimensional Leibniz bimodules.

Example A. Let $\mathbb{F}$ denote an arbitrary field, and let $\mathfrak{a} := \mathbb{F}h \oplus \mathbb{F}e$ be the non-abelian two-dimensional Lie algebra over $\mathbb{F}$ with multiplication determined by $he = e = -eh$. For any scalar $\lambda \in \mathbb{F}$ one can define a one-dimensional left $\mathfrak{a}$-module $F_\lambda := \mathbb{F}1_\lambda$ with $\mathfrak{a}$-action defined by $h \cdot 1_\lambda := \lambda 1_\lambda$ and $e \cdot 1_\lambda := 0$. Then the Chevalley-Eilenberg cohomology of $\mathfrak{a}$ with coefficients in F_λ is as follows:

$$\mathrm{H}^n(\mathfrak{a}, F_\lambda) \cong \begin{cases} \mathbb{F} & \text{if } \lambda = 0 \text{ and } n = 0, 1 \text{ or } \lambda = 1 \text{ and } n = 1, 2 \\ 0 & \text{otherwise}\,. \end{cases}$$

In particular, if $\lambda \neq 0, 1$, then $\mathrm{H}^\bullet(\mathfrak{a}, F_\lambda) = 0$.

First, let us consider F_λ as a symmetric Leibniz $\mathfrak{a}$-bimodule $(F_\lambda)_s$. Then it follows from Theorem 2.6 that $\mathrm{HL}^\bullet(\mathfrak{a}, (F_\lambda)_s) = 0$ for $\lambda \neq 0, 1$.

In order to be able to compute the Leibniz cohomology for $\lambda = 0, 1$, and for the anti-symmetric Leibniz $\mathfrak{a}$-bimodules $(F_\lambda)_a$, let M be an arbitrary left $\mathfrak{a}$-module considered as a symmetric Leibniz $\mathfrak{a}$-bimodule M_s. Since $\mathrm{H}^n(\mathfrak{a}, M) = 0$ for every integer $n \geq 3$, we obtain from Proposition 2.2 the short exact sequence

$$0 \to \mathrm{H}^2(\mathfrak{a}, M) \to \mathrm{HL}^2(\mathfrak{a}, M_s) \to \mathrm{H}^0_{\mathrm{rel}}(\mathfrak{a}, M) \to 0 \tag{2.2}$$

and the isomorphisms

$$\mathrm{HL}^n(\mathfrak{a}, M_s) \cong \mathrm{H}^{n-2}_{\mathrm{rel}}(\mathfrak{a}, M) \text{ for every integer } n \geq 3\,. \tag{2.3}$$

Moreover, we have that $\mathrm{HL}^0(\mathfrak{a}, M_s) \cong M^{\mathfrak{a}}$ and $\mathrm{HL}^1(\mathfrak{a}, M_s) \cong \mathrm{H}^1(\mathfrak{a}, M)$.

For the computation of the relative cohomology spaces $\mathrm{H}^n_{\mathrm{rel}}(\mathfrak{a}, M)$ we need the coadjoint Chevalley-Eilenberg cohomology of $\mathfrak{a}$. It is easy to verify that

$$\dim_{\mathbb{F}} \mathrm{H}^0(\mathfrak{a}, \mathfrak{a}^*) = 1\,,$$

$$\dim_{\mathbb{F}} \mathrm{H}^1(\mathfrak{a}, \mathfrak{a}^*) = \begin{cases} 2 & \text{if } \mathrm{char}(\mathbb{F}) = 2 \\ 1 & \text{if } \mathrm{char}(\mathbb{F}) \neq 2\,, \end{cases}$$

and

$$\dim_{\mathbb{F}} \mathrm{H}^2(\mathfrak{a}, \mathfrak{a}^*) = \begin{cases} 1 & \text{if } \mathrm{char}(\mathbb{F}) = 2 \\ 0 & \text{if } \mathrm{char}(\mathbb{F}) \neq 2\,. \end{cases}$$

Consequently, we have to consider the cases $\mathrm{char}(\mathbb{F}) = 2$ and $\mathrm{char}(\mathbb{F}) \neq 2$ differently.

Let us first assume that $\mathrm{char}(\mathbb{F}) \neq 2$. Then it follows from Proposition 2.1 that $\mathrm{HR}^0(\mathfrak{a}) \cong \mathrm{H}^1(\mathfrak{a}, \mathfrak{a}^*) \cong \mathbb{F}$ and $\mathrm{HR}^n(\mathfrak{a}) = 0$ for every integer $n \geq 1$. Hence we derive from the spectral sequence of Theorem 2.5 that $\mathrm{H}^n_{\mathrm{rel}}(\mathfrak{a}, M) \cong \mathrm{HL}^n(\mathfrak{a}, M_s)$ for every non-negative integer n. In conclusion, we obtain from (2.2) and (2.3) that

$$\mathrm{HL}^2(\mathfrak{a}, M_s) \cong M^{\mathfrak{a}} \oplus \mathrm{H}^2(\mathfrak{a}, M) \tag{2.4}$$

and

$$\mathrm{HL}^n(\mathfrak{a}, M_s) \cong \mathrm{HL}^{n-2}(\mathfrak{a}, M_s) \text{ for every integer } n \geq 3\,. \tag{2.5}$$

As an immediate consequence, in the case of $\mathrm{char}(\mathbb{F}) \neq 2$ we deduce that $\dim_{\mathbb{F}} \mathrm{HL}^n(\mathfrak{a}, \mathbb{F}) = 1$ for every non-negative integer n and

$$\dim_{\mathbb{F}} \mathrm{HL}^n(\mathfrak{a}, (F_1)_s) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n > 0\,. \end{cases}$$

In summary, we have for the Leibniz cohomology of $\mathfrak{a}$ over a field $\mathbb{F}$ of characteristic $\neq 2$ with coefficients in a one-dimensional symmetric Leibniz bimodule that

$$\dim_{\mathbb{F}} \mathrm{HL}^n(\mathfrak{a}, (F_\lambda)_s) = \begin{cases} 1 & \text{if } \lambda = 0 \text{ and } n \text{ is arbitrary or if } \lambda = 1 \text{ and } n > 0 \\ 0 & \text{otherwise}\,. \end{cases}$$

Next, let us assume that $\mathrm{char}(\mathbb{F}) = 2$. Then it follows from Proposition 2.1 that $\mathrm{HR}^0(\mathfrak{a}) \cong \mathrm{H}^1(\mathfrak{a}, \mathfrak{a}^*) \cong \mathbb{F}^2$, $\mathrm{HR}^1(\mathfrak{a}) \cong \mathrm{H}^2(\mathfrak{a}, \mathfrak{a}^*) \cong \mathbb{F}$, and $\mathrm{HR}^n(\mathfrak{a}) = 0$ for every integer $n \geq 2$. Hence in the spectral sequence of Theorem 2.5, we have only two non-zero columns, namely the $p = 0$ and the $p = 1$ column. In the $p = 0$ column, we have spaces $\mathbb{F}^2 \otimes \mathrm{HL}^q(\mathfrak{a}, M_s) \cong \mathrm{HL}^q(\mathfrak{a}, M_s) \oplus \mathrm{HL}^q(\mathfrak{a}, M_s)$, while in the $p = 1$ column, we have just $\mathrm{HL}^q(\mathfrak{a}, M_s)$ for every integer $q \geq 0$. Therefore, the spectral sequence degenerates at the term E_2, and for every integer $n \geq 1$ we obtain that

$$\mathrm{H}^n_{\mathrm{rel}}(\mathfrak{a}, M) \cong \mathrm{HL}^n(\mathfrak{a}, M_s) \oplus \mathrm{HL}^n(\mathfrak{a}, M_s) \oplus \mathrm{HL}^{n-1}(\mathfrak{a}, M_s)\,,$$

and

$$\mathrm{H}^0_{\mathrm{rel}}(\mathfrak{a}, M) \cong E_2^{0,0} \cong \mathrm{HL}^0(\mathfrak{a}, M_s) \oplus \mathrm{HL}^0(\mathfrak{a}, M_s) \cong M^{\mathfrak{a}} \oplus M^{\mathfrak{a}}\,.$$

This, together with (2.2), (2.3), and induction yields the recursive relation

$$\mathrm{HL}^n(\mathfrak{a}, M_s) \cong \mathrm{HL}^{n-1}(\mathfrak{a}, M_s) \oplus \mathrm{HL}^{n-2}(\mathfrak{a}, M_s) \text{ for every integer } n \geq 2\,. \tag{2.6}$$

As a consequence, we obtain for the Leibniz cohomology of $\mathfrak{a}$ over a field $\mathbb{F}$ of characteristic 2 with coefficients in a one-dimensional symmetric Leibniz bimodule that

$$\dim_{\mathbb{F}} \mathrm{HL}^n(\mathfrak{a},(F_\lambda)_s)=\begin{cases} f_{n+1} & \text{if } \lambda=0\\ f_n & \text{if } \lambda=1\\ 0 & \text{otherwise}\end{cases}$$

for every non-negative integer n, where f_n denotes the n-th term of the standard Fibonacci sequence given by $f_0 := 0$, $f_1 := 1$, and $f_n := f_{n-1} + f_{n-2}$ for every integer $n \geq 2$. In particular, we have that

$$\mathrm{HL}^n(\mathfrak{a},(F_1)_s)\cong \mathrm{HL}^{n-1}(\mathfrak{a},\mathbb{F}) \text{ for every integer } n\geq 1\,.$$

Next, let us consider F_λ as an anti-symmetric Leibniz $\mathfrak{a}$-bimodule $(F_\lambda)_a$ with the same left $\mathfrak{a}$-action as above and with the trivial right $\mathfrak{a}$-action (see Section 1). Then we conclude from Lemma 1.4 (b) that

$$\dim_{\mathbb{F}} \mathrm{HL}^0(\mathfrak{a},(F_\lambda)_a)=1 \text{ for every } \lambda\in\mathbb{F}\,.$$

Let us now compute $\mathrm{HL}^n(\mathfrak{a},(F_\lambda)_a)$ for any integer $n \geq 1$. It follows from Lemma 1.4 (b) that

$$\mathrm{HL}^n(\mathfrak{a},(F_\lambda)_a)\cong \mathrm{HL}^{n-1}(\mathfrak{a},\mathrm{Hom}_{\mathbb{F}}(\mathfrak{a},F_\lambda)_s)\cong \mathrm{HL}^{n-1}(\mathfrak{a},(\mathfrak{a}^*\otimes F_\lambda)_s)\,. \tag{2.7}$$

A straightforward computation shows that

$$0\to F_\lambda\to \mathfrak{a}^*\otimes F_\lambda\to F_{\lambda-1}\to 0$$

is a short exact sequence of left $\mathfrak{a}$-modules. Then we obtain from the long exact cohomology sequence and another straightforward computation in the case $\lambda = 1$:

$$\dim_{\mathbb{F}}(\mathfrak{a}^*\otimes F_\lambda)^{\mathfrak{a}}=\begin{cases} 1 & \text{if } \lambda=0\\ 0 & \text{otherwise}\,,\end{cases}$$

$$\dim_{\mathbb{F}} \mathrm{H}^1(\mathfrak{a},\mathfrak{a}^*\otimes F_\lambda)=\begin{cases} 2 & \text{if } \lambda=0 \text{ and } \mathrm{char}(\mathbb{F})=2\\ 1 & \text{if } \lambda=0,2 \text{ and } \mathrm{char}(\mathbb{F})\neq 2\,,\\ 0 & \quad\text{otherwise}\end{cases}$$

and

$$\dim_{\mathbb{F}} \mathrm{H}^2(\mathfrak{a},\mathfrak{a}^*\otimes F_\lambda)=\begin{cases} 1 & \text{if } \lambda=0 \text{ and } \mathrm{char}(\mathbb{F})=2 \text{ or } \lambda=2 \text{ and } \mathrm{char}(\mathbb{F})\neq 2\\ 0 & \quad\text{otherwise}\,.\end{cases}$$

If $\mathrm{char}(\mathbb{F}) \neq 2$, we conclude by applying (2.7) and (2.4) to the symmetric Leibniz $\mathfrak{a}$-bimodule $M_s := (\mathfrak{a}^* \otimes F_\lambda)_s$ that

$$\dim_{\mathbb{F}} \mathrm{HL}^1(\mathfrak{a},(F_\lambda)_a)=\begin{cases} 1 & \text{if } \lambda=0\\ 0 & \text{otherwise}\,,\end{cases}$$

and

$$\dim_{\mathbb{F}} \mathrm{HL}^3(\mathfrak{a},(F_\lambda)_a)=\dim_{\mathbb{F}} \mathrm{HL}^2(\mathfrak{a},(F_\lambda)_a)=\begin{cases} 1 & \text{if } \lambda=0,2\\ 0 & \text{otherwise}\,,\end{cases}$$

Finally, we use (2.5) to deduce for every integer $n \geq 2$:

$$\dim_{\mathbb{F}} \mathrm{HL}^n(\mathfrak{a},(F_\lambda)_a)=\begin{cases} 1 & \text{if } \lambda=0,2\\ 0 & \text{otherwise}\,.\end{cases}$$

In summary, we have for the Leibniz cohomology of $\mathfrak{a}$ over a field $\mathbb{F}$ of characteristic $\neq 2$ with coefficients in a one-dimensional anti-symmetric Leibniz bimodule that

$$\dim_{\mathbb{F}} \mathrm{HL}^n(\mathfrak{a},(F_\lambda)_a) = \begin{cases} 1 & \text{if } \lambda = 0 \text{ and } n \text{ is arbitrary or } \lambda = 2 \text{ and } n \geq 2 \\ & \text{or } n = 0 \text{ and } \lambda \text{ is arbitrary} \\ 0 & \text{otherwise}. \end{cases}$$

If $\mathrm{char}(\mathbb{F}) = 2$, we obtain by applying (2.7) and (2.6):

$$\dim_{\mathbb{F}} \mathrm{HL}^n(\mathfrak{a},(F_\lambda)_a) = \begin{cases} 1 & \text{if } n = 0 \text{ and } \lambda \text{ is arbitrary} \\ f_{n+1} & \text{if } \lambda = 0 \text{ and } n \text{ is arbitrary} \\ 0 & \text{otherwise}. \end{cases}$$

Remark. Since every invariant symmetric bilinear form on $\mathfrak{a}$ is a multiple of the Killing form, we have that $[S^2(\mathfrak{a})^*]^{\mathfrak{a}} \cong \mathbb{F}$. On the other hand, from the computations in Example A we obtain when $\mathrm{char}(\mathbb{F}) = 2$ that

$$\mathrm{H}^0_{\mathrm{rel}}(\mathfrak{a},\mathbb{F}) \cong \mathrm{HR}^0(\mathfrak{a}) \cong \mathrm{H}^1(\mathfrak{a},\mathfrak{a}^*) \cong \mathbb{F}^2 .$$

This shows that, in general, $\mathrm{HR}^0(\mathfrak{a}) \ncong [S^2(\mathfrak{a})^*]^{\mathfrak{a}}$ and $\mathrm{H}^0_{\mathrm{rel}}(\mathfrak{a},\mathbb{F}) \ncong [S^2(\mathfrak{a})^*]^{\mathfrak{a}}$.

Moreover, the computations in Example A show that $\dim_{\mathbb{F}} \mathrm{HL}^3(\mathfrak{a},\mathbb{F}) = 3$, but $\dim_{\mathbb{F}} \mathrm{H}^2(\mathfrak{a},\mathfrak{a}^*) \leq 1$. Hence the fifth terms of the five-term exact sequences in Proposition 2.1 and in Proposition 2.2 for $M := \mathbb{F}$ are not necessarily isomorphic.

Similar to Proposition 2.7, we have the following general vanishing result:

Proposition 2.8. *Let $\mathfrak{g}$ be a finite-dimensional nilpotent Lie algebra, and let M be a finite-dimensional non-trivial irreducible Leibniz $\mathfrak{g}$-bimodule. Then $\mathrm{HL}^n(\mathfrak{g},M) = 0$ for every positive integer n. Moreover, if M is symmetric, then $\mathrm{HL}^n(\mathfrak{g},M) = 0$ for every non-negative integer n.*

Proof. If M is symmetric, then the assertion is an immediate consequence of Theorem 2.6 and [3, Lemma 3].

According to [14, Theorem 3.14], we can suppose that M is anti-symmetric. We obtain from Lemma 1.4 (b) that

$$\mathrm{HL}^n(\mathfrak{g},M) \cong \mathrm{HL}^{n-1}(\mathfrak{g},\mathrm{Hom}_{\mathbb{F}}(\mathfrak{g},M)_s) \cong \mathrm{HL}^{n-1}(\mathfrak{g},(\mathfrak{g}^* \otimes M)_s)$$

for every positive integer n. By refining the descending central series of $\mathfrak{g}$, one can construct a composition series

$$\mathfrak{g}_{\mathrm{ad},\ell} = \mathfrak{g}_n \supset \mathfrak{g}_{n-1} \supset \cdots \supset \mathfrak{g}_1 \supset \mathfrak{g}_0 = 0$$

of the adjoint $\mathfrak{g}$-module such that $\mathfrak{g}_j/\mathfrak{g}_{j-1}$ is the trivial one-dimensional $\mathfrak{g}$-module $\mathbb{F}$ for every integer $1 \leq j \leq n$. From the short exact sequences

$$0 \to \mathfrak{g}_{j-1} \to \mathfrak{g}_j \to \mathbb{F} \to 0 ,$$

we obtain by dualizing, tensoring each term with M, and symmetrizing the short exact sequences:

$$0 \to M_s \to (\mathfrak{g}_j^* \otimes M)_s \to (\mathfrak{g}_{j-1}^* \otimes M)_s \to 0$$

for every integer $1 \leq j \leq n$. Since M is a non-trivial irreducible left $\mathfrak{g}$-module, we conclude that M_s is a non-trivial irreducible symmetric Leibniz $\mathfrak{g}$-bimodule. Hence we obtain inductively from the long exact cohomology sequence that $\mathrm{HL}^n(\mathfrak{g},M) \cong \mathrm{HL}^{n-1}(\mathfrak{g},(\mathfrak{g}^* \otimes M)_s) = 0$ for every positive integer n. $\square$

Since the Leibniz cohomology of an abelian Lie algebra with trivial coefficients is known, in Example B we compute this cohomology for the smallest non-abelian nilpotent Lie algebra. Note that in [31, Example 1.4. iv)] the corresponding Leibniz homology has been computed. In fact, homology and cohomology of a finite-dimensional Leibniz algebra $\mathfrak{L}$ with *trivial* coefficients are isomorphic, as we have the duality isomorphism $\mathrm{CL}_\bullet(\mathfrak{L},\mathbb{F})^* \cong \mathrm{CL}^\bullet(\mathfrak{L},\mathbb{F})$ already on the level of cochain complexes. Therefore our results coincide with those of Pirashvili. We furthermore compute in Example C the Leibniz cohomology of the smallest nilpotent non-Lie Leibniz algebra with coefficients in the trivial Leibniz bimodule.

Example B. Let $\mathbb{F}$ denote an arbitrary field of characteristic $\neq 2$, and let $\mathfrak{h} := \mathbb{F}x\oplus\mathbb{F}y\oplus\mathbb{F}z$ be the three-dimensional Heisenberg algebra over $\mathbb{F}$ with multiplication determined by $xy = z = -yx$. Then the Chevalley-Eilenberg cohomology of $\mathfrak{h}$ with coefficients in the trivial module $\mathbb{F}$ is well-known:

$$\dim_{\mathbb{F}} \mathrm{H}^n(\mathfrak{h},\mathbb{F}) = \begin{cases} 1 & \text{if } n = 0,3 \\ 2 & \text{if } n = 1,2\,. \\ 0 & \text{if } n \geq 4 \end{cases}$$

Consequently, we have that $\dim_{\mathbb{F}} \mathrm{HL}^0(\mathfrak{h},\mathbb{F}) = 1$ and $\dim_{\mathbb{F}} \mathrm{HL}^1(\mathfrak{h},\mathbb{F}) = 2$.

As $\mathrm{H}^n(\mathfrak{h},\mathbb{F}) = 0$ for every integer $n \geq 4$, we obtain from Proposition 2.2 the following six-term exact sequence:

$$0 \to \mathrm{H}^2(\mathfrak{h},\mathbb{F}) \to \mathrm{HL}^2(\mathfrak{h},\mathbb{F}) \to \mathrm{H}^0_{\mathrm{rel}}(\mathfrak{h},\mathbb{F}) \to \mathrm{H}^3(\mathfrak{h},\mathbb{F}) \to \mathrm{HL}^3(\mathfrak{h},\mathbb{F}) \to \mathrm{H}^1_{\mathrm{rel}}(\mathfrak{h},\mathbb{F}) \to 0$$

and

$$\mathrm{HL}^n(\mathfrak{h},\mathbb{F}) \cong \mathrm{H}^{n-2}_{\mathrm{rel}}(\mathfrak{h},\mathbb{F}) \text{ for every integer } n \geq 4\,.$$

Since we assume that $\mathrm{char}(\mathbb{F}) \neq 2$, it follows from the remark after Proposition 2.2 that we can identify $\mathrm{H}^0_{\mathrm{rel}}(\mathfrak{h},\mathbb{F})$ with the space of invariant symmetric bilinear forms on $\mathfrak{h}$ and the map $\mathrm{H}^0_{\mathrm{rel}}(\mathfrak{h},\mathbb{F}) \to \mathrm{H}^3(\mathfrak{h},\mathbb{F})$ with the classical Cartan-Koszul map. It is easy to see that the latter map is zero for the Heisenberg algebra, which yields the surjectivity of the map $\mathrm{HL}^2(\mathfrak{h},\mathbb{F}) \to \mathrm{H}^0_{\mathrm{rel}}(\mathfrak{h},\mathbb{F})$ and the injectivity of the map $\mathrm{H}^3(\mathfrak{h},\mathbb{F}) \to \mathrm{HL}^3(\mathfrak{h},\mathbb{F})$. As a consequence, we obtain the following two short exact sequences:

$$0 \to \mathrm{H}^2(\mathfrak{h},\mathbb{F}) \to \mathrm{HL}^2(\mathfrak{h},\mathbb{F}) \to \mathrm{H}^0_{\mathrm{rel}}(\mathfrak{h},\mathbb{F}) \to 0\,,$$

$$0 \to \mathrm{H}^3(\mathfrak{h},\mathbb{F}) \to \mathrm{HL}^3(\mathfrak{h},\mathbb{F}) \to \mathrm{H}^1_{\mathrm{rel}}(\mathfrak{h},\mathbb{F}) \to 0\,.$$

In order to compute $\mathrm{H}^0_{\mathrm{rel}}(\mathfrak{h},\mathbb{F})$ and $\mathrm{H}^1_{\mathrm{rel}}(\mathfrak{h},\mathbb{F})$, we need the coadjoint Chevalley-Eilenberg cohomology of $\mathfrak{h}$. We have that $\dim_{\mathbb{F}} \mathrm{H}^0(\mathfrak{h},\mathfrak{h}^*) = 2$, $\dim_{\mathbb{F}} \mathrm{H}^1(\mathfrak{h},\mathfrak{h}^*) = 5$, $\dim_{\mathbb{F}} \mathrm{H}^2(\mathfrak{h},\mathfrak{h}^*) = 4$, and $\mathrm{H}^3(\mathfrak{h},\mathfrak{h}^*) = 1$. These dimensions can be computed directly, but for the complex numbers as a ground field it also follows from the main result of [29] in conjunction with Poincaré duality (for the latter see [36, Theorem 3.4]).

Similar to the discussion of the consequences of Proposition 2.2 above, we obtain from Proposition 2.1 the two short exact sequences

$$0 \to \mathrm{H}^2(\mathfrak{h},\mathbb{F}) \to \mathrm{H}^1(\mathfrak{h},\mathfrak{h}^*) \to \mathrm{HR}^0(\mathfrak{h}) \to 0\,,$$

$$0 \to \mathrm{H}^3(\mathfrak{h},\mathbb{F}) \to \mathrm{H}^2(\mathfrak{h},\mathfrak{h}^*) \to \mathrm{HR}^1(\mathfrak{h}) \to 0\,,$$

the isomorphism $\mathrm{HR}^2(\mathfrak{h}) \cong \mathrm{H}^3(\mathfrak{h},\mathfrak{h}^*)$, and $\mathrm{HR}^n(\mathfrak{h}) = 0$ for every integer $n \geq 3$.

From these two short exact sequences and the isomorphism we derive that

$$\dim_{\mathbb{F}} \mathrm{HR}^0(\mathfrak{h}) = \dim_{\mathbb{F}} \mathrm{H}^1(\mathfrak{h},\mathfrak{h}^*) - \dim_{\mathbb{F}} \mathrm{H}^2(\mathfrak{h},\mathbb{F}) = 5 - 2 = 3\,,$$

$$\dim_{\mathbb{F}} \mathrm{HR}^1(\mathfrak{h}) = \dim_{\mathbb{F}} \mathrm{H}^2(\mathfrak{h},\mathfrak{h}^*) - \dim_{\mathbb{F}} \mathrm{H}^3(\mathfrak{h},\mathbb{F}) = 4 - 1 = 3\,,$$

and

$$\dim_{\mathbb{F}} \mathrm{HR}^2(\mathfrak{h}) = \dim_{\mathbb{F}} \mathrm{H}^3(\mathfrak{h}, \mathfrak{h}^*) = 1\,,$$

respectively. Therefore we obtain from $\mathrm{H}^0_{\mathrm{rel}}(\mathfrak{h}, \mathbb{F}) \cong \mathrm{HR}^0(\mathfrak{h})$ that

$$\dim_{\mathbb{F}} \mathrm{HL}^2(\mathfrak{h}, \mathbb{F}) = \dim_{\mathbb{F}} \mathrm{H}^2(\mathfrak{h}, \mathbb{F}) + \dim_{\mathbb{F}} \mathrm{H}^0_{\mathrm{rel}}(\mathfrak{h}, \mathbb{F}) = 2 + 3 = 5\,.$$

Now we want to apply the spectral sequence of Theorem 2.5. For this let us compute the differential

$$\mathrm{d}_2^{0,1} : E_2^{0,1} = \mathrm{HR}^0(\mathfrak{h}) \otimes \mathrm{HL}^1(\mathfrak{h}, \mathbb{F}) \to E_2^{2,0} = \mathrm{HR}^2(\mathfrak{h}) \otimes \mathrm{HL}^0(\mathfrak{h}, \mathbb{F})\,.$$

In characteristic $\neq 2$, an element of $\mathrm{HR}^0(\mathfrak{h})$ is an invariant symmetric bilinear form ω. It is considered as a 1-cochain with values in $\mathfrak{h}^*$ and, as it is a representative of an element of a quotient cochain complex, it is zero in case it is skew-symmetric in all entries. Take furthermore a cocycle $c \in \mathrm{CL}^1(\mathfrak{h}, \mathbb{F})$ and compute for three elements $r, s, t \in \mathfrak{h}$:

$$\begin{aligned} \mathrm{d}^1(\omega \otimes c)(r,s,t) &= \omega(rs,-)c(t) + \omega(s,-)c(rt) - \omega(r,-)c(st) + \\ &+ \omega(s,r-)c(t) - \omega(r,s-)c(t) + \omega(r,t-)c(s) \end{aligned}$$

Now as c is a cocycle with trivial coefficients, c vanishes on products, thus the second and third terms are zero. Furthermore, the first and fourth term cancel by the invariance of the form and skew-symmetry of the Lie product. We are left with the two last terms $-\omega(r,s-)c(t) + \omega(r,t-)c(s)$, which are skew-symmetric in the three entries of the element in $\mathrm{HR}^2(\mathfrak{h})$ and vanish therefore as well. In conclusion, the differential $\mathrm{d}_2^{0,1}$ is zero, and we have that

$$\mathrm{H}^1_{\mathrm{rel}}(\mathfrak{h}) = \mathrm{HR}^0(\mathfrak{h}) \otimes \mathrm{HL}^1(\mathfrak{h}, \mathbb{F}) \oplus \mathrm{HR}^1(\mathfrak{h}) \otimes \mathrm{HL}^0(\mathfrak{h}, \mathbb{F})\,.$$

This implies in turn

$$\dim_{\mathbb{F}} \mathrm{HL}^3(\mathfrak{h}, \mathbb{F}) = \dim_{\mathbb{F}} \mathrm{H}^3(\mathfrak{h}, \mathbb{F}) + \dim_{\mathbb{F}} \mathrm{H}^1_{\mathrm{rel}}(\mathfrak{h}, \mathbb{F}) = 1 + 9 = 10\,.$$

It seems that all differentials d_2 are zero and thus that this scheme persists to yield the dimensions of the higher $\mathrm{H}^n_{\mathrm{rel}}(\mathfrak{h}, \mathbb{F})$ and thus of $\mathrm{HL}^n(\mathfrak{h}, \mathbb{F})$ (see the dimension formula in [31, Example 1.4 iv)]).

Remark. As a by-product of the above computations we obtain that the space $[S^2(\mathfrak{h})^*]^{\mathfrak{h}}$ of invariant symmetric bilinear forms on $\mathfrak{h}$ is three-dimensional when $\mathrm{char}(\mathbb{F}) \neq 2$.

We proceed by proving an extension of a result by Fialowski, Magnin, and Mandal (see Corollary 2 in [16]), namely, the fact that the vanishing of the center $C(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ implies $\mathrm{HL}^2(\mathfrak{g}, \mathfrak{g}_{\mathrm{ad}}) = \mathrm{H}^2(\mathfrak{g}, \mathfrak{g})$, where $\mathfrak{g}_{\mathrm{ad}}$ denotes the adjoint Leibniz $\mathfrak{g}$-bimodule induced by the left and right multiplication operator. Note that for Lie algebras, this bimodule is indeed symmetric.

Moreover, we observe that $\mathrm{H}^0(\mathfrak{g}, \mathfrak{g}) = C(\mathfrak{g})$. Therefore, it is an immediate consequence of the case $n = 0$ of Theorem 2.6 that the vanishing of the center implies $\mathrm{HL}^2(\mathfrak{g}, \mathfrak{g}_{\mathrm{ad}}) = \mathrm{H}^2(\mathfrak{g}, \mathfrak{g})$. By the same token for $n = 1$, we can extend this result to complete Lie algebras, i.e., to those Lie algebras $\mathfrak{g}$ for which $\mathrm{H}^0(\mathfrak{g}, \mathfrak{g}) = \mathrm{H}^1(\mathfrak{g}, \mathfrak{g}) = 0$:

Corollary 2.9. *Let $\mathfrak{g}$ be a complete Lie algebra. Then*

$$\mathrm{HL}^2(\mathfrak{g}, \mathfrak{g}_{\mathrm{ad}}) \cong \mathrm{H}^2(\mathfrak{g}, \mathfrak{g}) \text{ and } \mathrm{HL}^3(\mathfrak{g}, \mathfrak{g}_{\mathrm{ad}}) \cong \mathrm{H}^3(\mathfrak{g}, \mathfrak{g})\,.$$

A class of examples of complete Lie algebras over an algebraically closed field $\mathbb{F}$ of characteristic zero consists of those finite-dimensional Lie algebras $\mathfrak{g}$ for which $\mathfrak{g}$ has the same dimension as its Lie algebra of derivations and $\dim_{\mathbb{F}} \mathfrak{g}/\mathfrak{g}^2 > 1$ (see [8, Proposition 3.1]). Another example is the two-sided Witt algebra over a field of characteristic zero. Indeed, this infinite-dimensional simple Lie algebra is complete (see [12, Theorem A.1.1]). Hence we obtain from [33, Theorem 3.1] and [12, Theorem 4.1] in conjunction with the case $n = 3$ of Theorem 2.6 the following result:

Corollary 2.10. *Let $\mathcal{W} := \mathrm{Der}(\mathbb{F}[t, t^{-1}])$ be the two-sided Witt algebra over a field $\mathbb{F}$ of characteristic zero. Then $\mathrm{HL}^2(\mathcal{W}, \mathcal{W}_{\mathrm{ad}}) = 0$ and $\mathrm{HL}^3(\mathcal{W}, \mathcal{W}_{\mathrm{ad}}) = 0$. Moreover,*

$$\mathrm{HL}^4(\mathcal{W}, \mathcal{W}_{\mathrm{ad}}) \cong \mathrm{H}^4(\mathcal{W}, \mathcal{W}) \text{ and } \mathrm{HL}^5(\mathcal{W}, \mathcal{W}_{\mathrm{ad}}) \cong \mathrm{H}^5(\mathcal{W}, \mathcal{W}) .$$

Remark. Very recently, Camacho, Omirov, and Kurbanbaev also proved that the second adjoint Leibniz cohomology of $\mathcal{W}$ vanishes (see [7, Theorem 4]) by explicitly showing that every adjoint Leibniz 2-cocycle (resp. Leibniz 2-coboundary) is an adjoint Chevalley-Eilenberg 2-cocycle (resp. Chevalley-Eilenberg 2-coboundary) for $\mathcal{W}$.

We conclude this section with another application of Theorem 2.6. Let $\mathbb{F}$ be an algebraically closed field of characteristic zero, let n be a non-negative integer, and let $L_n(\mathbb{F}) \subseteq \mathbb{F}^{n^3}$ denote the affine variety of structure constants of the n-dimensional left Leibniz algebras over $\mathbb{F}$ with respect to a fixed basis of $\mathbb{F}^n$. Then the general linear group $\mathrm{GL}_n(\mathbb{F})$ acts on $L_n(\mathbb{F})$, and a point (= Leibniz multiplication law) $\phi \in L_n(\mathbb{F})$ is called *rigid* if the orbit $\mathrm{GL}_n(\mathbb{F}) \cdot \phi$ is open in $L_n(\mathbb{F})$. It follows from Corollary 2.10 in conjunction with [2, Théorème 3] that the infinite-dimensional two-sided Witt algebra over an algebraically closed field of characteristic zero is rigid as a Leibniz algebra.

It is well known that the Chevalley-Eilenberg cohomology of the non-abelian two-dimensional Lie algebra with coefficients in the adjoint module vanishes. According to Theorem 2.6, this implies that the corresponding Leibniz cohomology vanishes as well. Note that the non-abelian two-dimensional Lie algebra is the standard Borel subalgebra of the split three-dimensional simple Lie algebra $\mathfrak{sl}_2$.

Similarly, by applying Theorem 2.6 in conjunction with [34, Theorem 1] (see also [23, Section 1]) one obtains the following more general result in characteristic zero (cf. also [31, Proposition 2.3] for the rigidity of parabolic subalgebras). Recall that a subalgebra of a semi-simple Lie algebra $\mathfrak{g}$ is called *parabolic* if it contains a maximal solvable (= Borel) subalgebra of $\mathfrak{g}$.

Proposition 2.11. *Let $\mathfrak{p}$ be a parabolic subalgebra of a finite-dimensional semi-simple Lie algebra over a field of characteristic zero. Then $\mathrm{HL}^n(\mathfrak{p}, \mathfrak{p}_{\mathrm{ad}}) = 0$ for every non-negative integer n. In particular, parabolic subalgebras of a finite-dimensional semi-simple Lie algebra over an algebraically closed field of characteristic zero are rigid as Leibniz algebras.*

Remark. It would be interesting to know whether Proposition 2.11 remains valid in prime characteristic.

3. A Hochschild-Serre type spectral sequence for Leibniz cohomology

In this section we consider a Leibniz analogue of the Hochschild-Serre spectral sequence for the Chevalley-Eilenberg cohomology of Lie algebras that converges to some relative cohomology. It will play a predominant role in Section 4. The homology version of this spectral sequence with values in symmetric bimodules is due to Pirashvili (see [31, Theorem C]). Our arguments follow Pirashvili very closely, but we include all the details as it turns out that the spectral sequence holds more generally for arbitrary bimodules.

Let $\pi : \mathfrak{L} \to \mathfrak{Q}$ be an epimorphism of left Leibniz algebras, and let M be a $\mathfrak{Q}$-bimodule. Then M is also an $\mathfrak{L}$-bimodule via π. Moreover, the epimorphisms $\pi^{\otimes n} : \mathfrak{L}^{\otimes n} \to \mathfrak{Q}^{\otimes n}$ induce a monomorphism $\mathrm{CL}^\bullet(\mathfrak{Q}, M) \to \mathrm{CL}^\bullet(\mathfrak{L}, M)$ of cochain complexes. Now set

$$\mathrm{CL}^\bullet(\mathfrak{L}|\mathfrak{Q}, M) := \mathrm{Coker}(\mathrm{CL}^\bullet(\mathfrak{Q}, M) \to \mathrm{CL}^\bullet(\mathfrak{L}, M))[-1]$$

and

$$\mathrm{HL}^\bullet(\mathfrak{L}|\mathfrak{Q}, M) := \mathrm{H}^\bullet(\mathrm{CL}^\bullet(\mathfrak{L}|\mathfrak{Q}, M)) .$$

Then by applying the long exact cohomology sequence to the short exact sequence

$$0 \to \mathrm{CL}^\bullet(\mathfrak{Q}, M) \to \mathrm{CL}^\bullet(\mathfrak{L}, M) \to \mathrm{CL}^\bullet(\mathfrak{L}|\mathfrak{Q}, M)[1] \to 0$$

of cochain complexes one obtains the following result (see also [31, Proposition 4.1] for the corresponding result for Leibniz homology).

Proposition 3.1. *For every epimorphism $\pi : \mathfrak{L} \to \mathfrak{Q}$ of left Leibniz algebras and every $\mathfrak{Q}$-bimodule M there exists a long exact sequence*

$$\begin{aligned} 0 \quad \to \quad & \mathrm{HL}^1(\mathfrak{Q}, M) \to \mathrm{HL}^1(\mathfrak{L}, M) \to \mathrm{HL}^0(\mathfrak{L}|\mathfrak{Q}, M) \\ \to \quad & \mathrm{HL}^2(\mathfrak{Q}, M) \to \mathrm{HL}^2(\mathfrak{L}, M) \to \mathrm{HL}^1(\mathfrak{L}|\mathfrak{Q}, M) \to \cdots \end{aligned}$$

Let us now derive Pirashvili's analogue of the Hochschild-Serre spectral sequence for Leibniz cohomology (see [31, Theorem C] for the homology version). While Pirashvili considers only symmetric bimodules, we extend the dual of his spectral sequence to arbitrary bimodules. The construction of this spectral sequence is very similar to the construction of the spectral sequence in Theorem 2.5.

We consider the following filtration on the cochain complex $\mathrm{CL}^\bullet(\mathfrak{L}, M)[-1]$.

$$\mathcal{F}^p\mathrm{CL}^n(\mathfrak{L}, M)[-1] := \{c \in \mathrm{CL}^{n+1}(\mathfrak{L}, M) \mid c(x_1, \dots, x_{n+1}) = 0 \text{ if } \exists\, i \le p : \, x_i \in \mathfrak{I}\} .$$

This defines a finite decreasing filtration

$$\begin{aligned} (3.1) \quad \mathcal{F}^0\mathrm{CL}^n(\mathfrak{L}, M)[-1] = \mathrm{CL}^n(\mathfrak{L}, M)[-1] \supset \mathcal{F}^1\mathrm{CL}^n(\mathfrak{L}, M)[-1] \supset \cdots \\ \cdots \supset \mathcal{F}^{n+1}\mathrm{CL}^n(\mathfrak{L}, M)[-1] = \mathrm{CL}^n(\mathfrak{Q}, M)[-1] , \end{aligned}$$

Then we have the following result:

Lemma 3.2. *This filtration is compatible with the Leibniz coboundary map* $\mathrm{d}^\bullet$.

Proof. We have to prove that $\mathrm{d}^\bullet(\mathcal{F}^p\mathrm{CL}^n(\mathfrak{L}, M)[-1]) \subseteq \mathcal{F}^p\mathrm{CL}^{n+1}(\mathfrak{L}, M)[-1]$. For this, we consider the different terms $\mathrm{d}_{ij}(c)$, $\delta_i(c)$, and $\partial(c)$, which constitute $\mathrm{d}_0(c)$, where we have inserted an element of $\mathfrak{I}$ within the first p arguments. The vanishing is clear for the terms $\mathrm{d}_{ij}(c)$ with $i, j \le p$, because even if the element of $\mathfrak{I}$ occurs in the product, the product will again be in the ideal $\mathfrak{I}$. The vanishing is also clear for the terms $\mathrm{d}_{ij}(c)$ with $i, j \geq p+1$. Concerning the terms $\mathrm{d}_{ij}(c)$ with $i \leq p$ and $j \geq p+1$, we use the condition $\mathfrak{I} \subseteq C_\ell(\mathfrak{L})$ to conclude that these are zero.

The action terms follow a similar pattern. The terms $\delta_i(c)$ with $i \leq p$ vanish, because either the element of $\mathfrak{I}$ occurs in the arguments, or it acts on M, which is zero by assumption. The terms $\delta_i(c)$ with $i \geq p+1$ are zero for elementary reasons, as is the term $\partial(c)$. □

Thanks to (3.1), the associated spectral sequence converges in the strong (i.e., finite) sense to the cohomology $\mathrm{HL}^n(\mathfrak{L}|\mathfrak{Q}, M)$ of the quotient cochain complex $\mathrm{CL}^n(\mathfrak{L}, M)[-1]\,/\,CL^n(\mathfrak{Q}, M)[-1]$, and we get for the 0-th term of the spectral sequence

$$E_0^{p,q} = \mathrm{Hom}_{\mathbb{F}}(\mathfrak{Q}^p \otimes \mathfrak{L}^{q+1}, M)\,/\,\mathrm{Hom}_{\mathbb{F}}(\mathfrak{Q}^{p+1} \otimes \mathfrak{L}^q, M) \;\cong\; \mathrm{Hom}_{\mathbb{F}}(\mathfrak{Q}^p \otimes \mathfrak{I} \otimes \mathfrak{L}^q, M)\,,$$

where the isomorphism is induced by the inclusion $\mathfrak{I} \hookrightarrow \mathfrak{L}$. Since $\mathrm{Hom}_{\mathbb{F}}$ and $\otimes$ are adjoint functors, we obtain that

$$E_0^{p,q} = \mathrm{Hom}_{\mathbb{F}}(\mathfrak{Q}^p \otimes \mathfrak{I}, \mathrm{Hom}_{\mathbb{F}}(\mathfrak{L}^q, M)) = \mathrm{Hom}_{\mathbb{F}}(\mathfrak{Q}^p \otimes \mathfrak{I}, \mathrm{CL}^q(\mathfrak{L}, M))\,.$$

Next, we will determine the differential on $E_0^{p,q}$:

Lemma 3.3. *The differential* d_0 *on* $E_0^{p,q} \cong \mathrm{Hom}_{\mathbb{F}}(\mathfrak{Q}^p \otimes \mathfrak{I}, \mathrm{CL}^q(\mathfrak{L}, M))$ *is induced by the coboundary operator* $\mathrm{d}^\bullet$ *on* $\mathrm{CL}^q(\mathfrak{L}, M)$*. More precisely, we have that*

$$d_0^{p,q}(f) := \mathrm{d}^q_{\mathrm{CL}^q(\mathfrak{L},M)} \circ f$$

for every linear transformation $f \in \mathrm{Hom}_{\mathbb{F}}(\mathfrak{Q}^p \otimes \mathfrak{I}, \mathrm{CL}^q(\mathfrak{L}, M)))$.

Proof. The differential

$$\begin{aligned}\mathrm{d}_0 : \mathcal{F}^p\mathrm{CL}^n(\mathfrak{L}, M)[-1]\,/\,\mathcal{F}^{p+1}\mathrm{CL}^n(\mathfrak{L}, M)[-1] \to \\ \to \mathcal{F}^p\mathrm{CL}^{n+1}(\mathfrak{L}, M)[-1]\,/\,\mathcal{F}^{p+1}\mathrm{CL}^{n+1}(\mathfrak{L}, M)[-1]\end{aligned}$$

is induced by $\mathrm{d}^\bullet$. Thus, we have to examine which terms $\mathrm{d}_{ij}(c)$, $\delta_i(c)$, and $\partial(c)$ composing the value $\mathrm{d}_0(c)$ are non-zero when we put an element of $\mathfrak{I}$ within the first $p+1$ entries.

It is clear that $\mathrm{d}_{ij}(c) = 0$ for $i, j \leq p+1$ since this is true when the element of $\mathfrak{I}$ is not involved in the product, as the number of elements is diminished by one, and it is also true when the element of $\mathfrak{I}$ is in the product because $\mathfrak{I}$ is an ideal. We then have $\mathrm{d}_{ij}(c) = 0$ for $i \leq p+1$ and $j \geq p+2$, since in case the element of $\mathfrak{I}$ acts in the product, it acts trivially because of $\mathfrak{I} \subseteq C_\ell(\mathfrak{L})$. Furthermore, the terms $\delta_i(c)$ for $i \leq p+1$ are zero since elements of $\mathfrak{I}$ act trivially on M.

Therefore, we are left with the terms composing the differential $\mathrm{d}_0|_{\mathrm{CL}^q(\mathfrak{L},M)}$. □

Consequently, the first term of the spectral sequence is

$$E_1^{p,q} = \mathrm{Hom}_{\mathbb{F}}(\mathfrak{Q}^p \otimes \mathfrak{I}, \mathrm{HL}^q(\mathfrak{L}, M))\,.$$

Now we proceed to determine the differential d_1 on $E_1^{p,q}$. It is again induced by the Leibniz coboundary operator. As before, classes in $\mathrm{HL}^q(\mathfrak{L}, M)$ are represented by cocycles and thus the part of the Leibniz differential constituting the Leibniz coboundary operator $\mathrm{d}^q_{\mathrm{CL}^q(\mathfrak{L},M)}$ is zero. The remaining terms constitute the differential on $\mathfrak{Q}^p \otimes \mathfrak{I}$, again since by the Cartan relations for Leibniz cohomology (see [26, Proposition 3.1] for the case of right Leibniz algebras and [10, Proposition 1.3.2] for the case of left Leibniz algebras) a Leibniz algebra acts trivially on its cohomology. But one needs to be careful since the Cartan relations do only hold for $q \geq 1$. Therefore, for an arbitrary bimodule M, $\mathfrak{Q}$ will act non-trivially on $\mathrm{HL}^0(\mathfrak{L}, M)$. On the other hand, if the bimodule M is symmetric, however, the action is indeed trivial on $\mathrm{HL}^0(\mathfrak{L}, M)$.

Note that in the proof of the preceding lemma, in all action terms on $\mathfrak{I}$ or on $\mathrm{Hom}_{\mathbb{F}}(\mathfrak{I}, \mathrm{HL}^0(\mathfrak{L}, M))$ the action is from the left, thus, in order to interpret the remaining terms as the Leibniz boundary operator with values in $\mathfrak{I}$, we have to switch around the last action term. This is the reason for viewing $\mathfrak{I}$ and $\mathrm{Hom}_{\mathbb{F}}(\mathfrak{I}, \mathrm{HL}^0(\mathfrak{L}, M)))$ here as a symmetric $\mathfrak{Q}$-bimodule.

Finally, we obtain from the Universal Coefficient Theorem (for example, see [35, Theorem 3.6.5]) that for $q > 0$ the second term of the spectral sequence is

$$E_2^{p,q} = \mathrm{Hom}_{\mathbb{F}}(\mathrm{HL}_p(\mathfrak{Q}, \mathfrak{I}_s), \mathrm{HL}^q(\mathfrak{L}, M))\,.$$

Furthermore, for a finite-dimensional Leibniz algebra $\mathfrak{L}$ and a symmetric $\mathfrak{Q}$-bimodule M, for all $p, q \geq 0$, the E_1-term simplifies to

$$E_1^{p,q} = \mathrm{CL}^p(\mathfrak{Q}, \mathfrak{I}^*) \otimes \mathrm{HL}^q(\mathfrak{L}, M)\,,$$

and in this special case, for all $p, q \geq 0$, the E_2-term is

$$E_2^{p,q} = \mathrm{HL}^p(\mathfrak{Q}, (\mathfrak{I})_s^*) \otimes \mathrm{HL}^q(\mathfrak{L}, M)\,.$$

This discussion proves the following results:

Theorem 3.4. *Let $0 \to \mathfrak{I} \to \mathfrak{L} \xrightarrow{\pi} \mathfrak{Q} \to 0$ be a short exact sequence of left Leibniz algebras such that $\mathfrak{I} \subseteq C_\ell(\mathfrak{L})$. Then $\mathfrak{I}$ is in a natural way an anti-symmetric $\mathfrak{Q}$-bimodule via $a \cdot y := \pi^{-1}(a)y$ and $y \cdot a := y\pi^{-1}(a)$ for every element $a \in \mathfrak{Q}$ and every element $y \in \mathfrak{I}$. Moreover, there is a spectral sequence converging to $\mathrm{HL}^\bullet(\mathfrak{L}|\mathfrak{Q}, M)$ with second term*

$$E_2^{p,q} = \begin{cases} \mathrm{HL}^p(\mathfrak{Q}, \mathrm{Hom}_{\mathbb{F}}(\mathfrak{I}, \mathrm{HL}^0(\mathfrak{L}, M))_s) & \text{if } p \geq 0, q = 0 \\ \mathrm{Hom}_{\mathbb{F}}(\mathrm{HL}_p(\mathfrak{Q}, \mathfrak{I}_s), \mathrm{HL}^q(\mathfrak{L}, M)) & \text{if } p \geq 0, q \geq 1 \end{cases}$$

for every $\mathfrak{Q}$-bimodule M.

Corollary 3.5. *If in Theorem 3.4 the Leibniz algebra $\mathfrak{L}$ is finite dimensional and the $\mathfrak{Q}$-bimodule M is symmetric, then, for any integers $p, q \geq 0$, the E_2-term of the spectral sequence simply reads*

$$E_2^{p,q} = \mathrm{HL}^p(\mathfrak{Q}, (\mathfrak{I}^*)_s) \otimes \mathrm{HL}^q(\mathfrak{L}, M)\,,$$

where the linear dual $\mathfrak{I}^*$ of $\mathfrak{I}$ is a left $\mathfrak{L}$-module via $(x \cdot f)(y) := -f(xy)$ for every linear form $f \in \mathfrak{I}^*$ and any elements $x \in \mathfrak{L}, y \in \mathfrak{I}$.

Remarks.

(a) According to [14, Proposition 2.13], Theorem 3.4 applies to $\mathfrak{I} := \mathrm{Leib}(\mathfrak{L})$ and $\mathfrak{Q} := \mathfrak{L}_{\mathrm{Lie}}$ (see [31, Remark 4.2] for the analogous statement for Leibniz homology). Note that in the cohomology space $\mathrm{HL}^p(\mathfrak{Q}, (\mathfrak{I}^*)_s)$, the left $\mathfrak{Q}$-module $\mathfrak{I}^*$ is viewed as a symmetric bimodule while naturally it is an anti-symmetric $\mathfrak{Q}$-bimodule.

(b) The higher differentials in the spectral sequence are again induced by the Leibniz coboundary operator $\mathrm{d}^\bullet$. Observe that the spectral sequence of Corollary 3.5 is isomorphic to the spectral sequence of the cochain bicomplex $\mathrm{CL}^\bullet(\mathfrak{Q}, (\mathfrak{I}^*)_s) \otimes \mathrm{CL}^\bullet(\mathfrak{L}, M)$. Therefore the description of the higher differentials can be adapted from [21] (see, in particular, Remark 3.2 therein). For example, it is clear, if one of the two differentials in the bicomplex is zero, then all higher differentials vanish. We will see an instance of this case in Example C below.

(c) One might wonder what one gets when one uses the filtration by the last p arguments instead of the first p arguments. It turns out that this spectral sequence has an E_2-term that is more difficult to describe (and which we stated erroneously in a first version of this article), because one takes in the E_2-term the cohomology of a complex which appears as coefficients in the Leibniz cohomology that constitutes the E_1-term.

As in the previous section, we illustrate the use of the spectral sequence of Theorem 3.4 and the associated long exact sequence (see Proposition 3.1) by two examples.

In the first example we compute the Leibniz cohomology of the smallest nilpotent non-Lie left Leibniz algebra with trivial coefficients.

Example C. Let $\mathbb{F}$ denote an arbitrary field, and let $\mathfrak{N} := \mathbb{F}e \oplus \mathbb{F}f$ be the two-dimensional nilpotent left (and right) Leibniz algebra over $\mathbb{F}$ with multiplication determined by $ff = e$. Then $\mathrm{Leib}(\mathfrak{N}) = \mathbb{F}e$, and thus $\mathfrak{N}_{\mathrm{Lie}}$ is a one-dimensional abelian Lie algebra. Hence $\mathrm{HL}^n(\mathfrak{N}_{\mathrm{Lie}}, \mathbb{F}) \cong \mathbb{F}$ for every non-negative integer n. Moreover, we have that $\dim_{\mathbb{F}} \mathrm{HL}^0(\mathfrak{N}, \mathbb{F}) = 1$ and $\dim_{\mathbb{F}} \mathrm{HL}^1(\mathfrak{N}, \mathbb{F}) = 1$.

Next, we compute the higher cohomology with the help of the spectral sequence of Corollary 3.5. As observed in Remark (b) after Corollary 3.5, all higher differentials are zero in our case since the Leibniz coboundary operator of the abelian Lie algebra with values in the trivial module vanishes. With the input data $\dim_{\mathbb{F}} \mathrm{HL}^0(\mathfrak{N}, \mathbb{F}) = 1$ and $\dim_{\mathbb{F}} \mathrm{HL}^1(\mathfrak{N}, \mathbb{F}) = 1$, we therefore get from the spectral sequence

$$\dim_{\mathbb{F}} \mathrm{HL}^0(\mathfrak{N}|\mathfrak{N}_{\mathrm{Lie}}, \mathbb{F}) = 1 \text{ and } \dim_{\mathbb{F}} \mathrm{HL}^1(\mathfrak{N}|\mathfrak{N}_{\mathrm{Lie}}, \mathbb{F}) = 2\,.$$

In order to be able to apply now the long exact sequence from Proposition 3.1 and deduce the dimensions of the cohomology spaces, we want to argue that this sequence is split. In fact, it is split because the connecting homomorphism is surjective. This comes from the fact that the cochain complex $\mathrm{CL}^\bullet(\mathfrak{N}_{\mathrm{Lie}}, \mathbb{F})$ is one-dimensional in each degree and a generator can be hit via the connecting homomorphism which is easy to see directly (take a cochain in $\mathrm{CL}^n(\mathfrak{N}|\mathfrak{N}_{\mathrm{Lie}}, \mathbb{F})$ represented by an element in $\mathrm{CL}^{n+1}(\mathfrak{N}, \mathbb{F})$ with exactly one slot in e^* at the first place: the Leibniz product in this slot gives the only non-zero contribution). Consequently, the long exact sequence from Proposition 3.1 splits into short exact sequences

$$0 \to \mathrm{HL}^n(\mathfrak{N}, \mathbb{F}) \to \mathrm{HL}^{n-1}(\mathfrak{N}|\mathfrak{N}_{\mathrm{Lie}}, \mathbb{F}) \to \mathrm{HL}^{n+1}(\mathfrak{N}_{\mathrm{Lie}}, \mathbb{F}) \to 0\,,$$

starting from $n = 2$, where the right-hand term is one-dimensional. These short exact sequences, together with the spectral sequence where every differential is zero, permit us to determine all relative and absolute cohomology spaces. For example, we have $\dim_{\mathbb{F}} \mathrm{HL}^2(\mathfrak{N}, \mathbb{F}) = 1$, and then $\dim_{\mathbb{F}} \mathrm{HL}^2(\mathfrak{N}|\mathfrak{N}_{\mathrm{Lie}}, \mathbb{F}) = 3$, $\dim_{\mathbb{F}} \mathrm{HL}^3(\mathfrak{N}, \mathbb{F}) = 2$, and then $\dim_{\mathbb{F}} \mathrm{HL}^3(\mathfrak{N}|\mathfrak{N}_{\mathrm{Lie}}, \mathbb{F}) = 5$, and so on. In general, we obtain by induction that $\dim_{\mathbb{F}} \mathrm{HL}^n(\mathfrak{N}, \mathbb{F}) = 2^{n-2}$ for every integer $n \geq 2$ and $\dim_{\mathbb{F}} \mathrm{HL}^n(\mathfrak{N}|\mathfrak{N}_{\mathrm{Lie}}, \mathbb{F}) = 2^{n-1} + 1$ for every integer $n \geq 1$.

In the second example we compute the Leibniz cohomology of the smallest non-nilpotent non-Lie left Leibniz algebra with coefficients in one-dimensional bimodules. Note that contrary to the semidirect product of two one-dimensional Lie algebras in Example A the Leibniz algebra in Example D is the hemi-semidirect product of two one-dimensional Lie algebras. It turns out that this somewhat simplifies matters.

Example D. Let $\mathbb{F}$ denote an arbitrary field, and let $\mathfrak{A} := \mathbb{F}h \oplus \mathbb{F}e$ be the two-dimensional supersolvable left Leibniz algebra over $\mathbb{F}$ with multiplication determined by $he = e$. For any scalar $\lambda \in \mathbb{F}$ one can define a one-dimensional left $\mathfrak{A}$-module $F_\lambda := \mathbb{F}1_\lambda$ with $\mathfrak{A}$-action defined by $h \cdot 1_\lambda := \lambda 1_\lambda$ and $e \cdot 1_\lambda := 0$. Note that $\mathrm{Leib}(\mathfrak{A}) = \mathbb{F}e$, and thus $\mathfrak{A}_{\mathrm{Lie}}$ is a one-dimensional abelian Lie algebra. Then we obtain from [3, Lemma 1] and Theorem 2.6 that

$$\dim_{\mathbb{F}} \mathrm{HL}^n(\mathfrak{A}_{\mathrm{Lie}}, (F_\lambda)_s) = \begin{cases} 1 & \text{if } \lambda = 0 \text{ and } n \text{ is arbitrary} \\ 0 & \text{otherwise}\,. \end{cases}$$

Moreover, we deduce from Lemma 1.4 (b) that

$$\begin{aligned} \mathrm{HL}^n(\mathfrak{A}_{\mathrm{Lie}}, (F_\lambda)_a) &\cong \mathrm{HL}^{n-1}(\mathfrak{A}_{\mathrm{Lie}}, \mathrm{Hom}_{\mathbb{F}}(\mathfrak{A}_{\mathrm{Lie}}, F_\lambda)_s) \\ &\cong \mathrm{HL}^{n-1}(\mathfrak{A}_{\mathrm{Lie}}, (F_\lambda)_s) \end{aligned}$$

for every integer $n \geq 1$, and therefore

$$\dim_{\mathbb{F}} \mathrm{HL}^n(\mathfrak{A}_{\mathrm{Lie}}, (F_\lambda)_a) = \begin{cases} 1 & \text{if } \lambda = 0 \text{ and } n \text{ is arbitrary or if } \lambda \neq 0 \text{ and } n = 0 \\ 0 & \text{otherwise}\,. \end{cases}$$

In order to be able to apply the spectral sequence of Theorem 3.4, we first compute $\mathrm{HL}^\bullet(\mathfrak{A}_{\mathrm{Lie}}, [\mathrm{Leib}(\mathfrak{A})^*]_s)$. Observe that the module $\mathrm{Leib}(\mathfrak{A})^* = \mathbb{F}e^* \cong F_{-1}$ is non-trivial irreducible, and furthermore it is viewed as a symmetric $\mathfrak{A}_{\mathrm{Lie}}$-bimodule. Hence from the above it follows that $\mathrm{HL}^n(\mathfrak{A}_{\mathrm{Lie}}, [\mathrm{Leib}(\mathfrak{A})^*]_s) = 0$ for every non-negative integer n. This implies in turn that the spectral sequence of Theorem 3.4 collapses at the E_2-term and that

$$\begin{aligned} \mathrm{HL}^n(\mathfrak{A}|\mathfrak{A}_{\mathrm{Lie}}, (F_\lambda)_a) &= \mathrm{HL}^n(\mathfrak{A}_{\mathrm{Lie}}, \mathrm{Hom}_{\mathbb{F}}(\mathrm{Leib}(\mathfrak{A}), \mathrm{HL}^0(\mathfrak{A}, (F_\lambda)_a))_s) \\ &= \mathrm{HL}^n(\mathfrak{A}_{\mathrm{Lie}}, \mathrm{Hom}_{\mathbb{F}}(\mathrm{Leib}(\mathfrak{A}), F_\lambda)_s) \end{aligned}$$

for all non-negative integers n, while $\mathrm{HL}^n(\mathfrak{A}|\mathfrak{A}_{\mathrm{Lie}}, (F_\lambda)_s) = 0$ for all $n \geq 0$ by Corollary 3.5. Notice that as an $\mathfrak{A}$-bimodule $\mathrm{Hom}_{\mathbb{F}}(\mathrm{Leib}(\mathfrak{A}), F_\lambda)_s \cong [F_{\lambda-1}]_s$. We have already observed in Example C that the long exact sequence of Proposition 3.1 splits, and therefore we conclude from Proposition 3.1 that

$$\mathrm{HL}^n(\mathfrak{A}, (F_\lambda)_s) \cong \mathrm{HL}^n(\mathfrak{A}_{\mathrm{Lie}}, (F_\lambda)_s)$$

and

$$\mathrm{HL}^n(\mathfrak{A}, (F_\lambda)_a) \cong \mathrm{HL}^n(\mathfrak{A}_{\mathrm{Lie}}, (F_\lambda)_a) \oplus \mathrm{HL}^n(\mathfrak{A}_{\mathrm{Lie}}, (F_{\lambda-1})_s)$$

for all λ and all non-negative integers n. Consequently, we obtain that

$$\dim_{\mathbb{F}} \mathrm{HL}^n(\mathfrak{A}, (F_\lambda)_s) = \begin{cases} 1 & \text{if } \lambda = 0 \text{ and } n \text{ is arbitrary} \\ 0 & \text{otherwise}\,, \end{cases}$$

and

$$\dim_{\mathbb{F}} \mathrm{HL}^n(\mathfrak{A}, (F_\lambda)_a) = \begin{cases} 1 & \text{if } \lambda = 0, 1 \text{ and } n \text{ is arbitrary or if } \lambda \neq 0, 1 \text{ and } n = 0 \\ 0 & \text{otherwise}\,. \end{cases}$$

Remark. In particular, we have that $\dim_{\mathbb{F}} \mathrm{HL}^n(\mathfrak{A}, \mathbb{F}) = 1$ for every non-negative integer n. Note that this follows as well from the scheme of proof of Proposition 4.3 in [31] by using the isomorphism between Leibniz homology and cohomology with trivial coefficients. Indeed, the characteristic element $\mathrm{ch}(\mathfrak{A}) \in \mathrm{HL}^2(\mathfrak{A}_{\mathrm{Lie}}, \mathrm{Leib}(\mathfrak{A}))$ of $\mathfrak{A}$ is zero as $\mathrm{Leib}(\mathfrak{A}) = \mathbb{F}e \cong (F_1)_a$ and $\mathrm{HL}^2(\mathfrak{A}_{\mathrm{Lie}}, (F_1)_a) = 0$. Since also $\mathrm{HL}^\bullet(\mathfrak{A}_{\mathrm{Lie}}, [\mathrm{Leib}(\mathfrak{A})^*]_s)$ is zero, we can reason in the same way as Pirashvili does.

4. Cohomology of semi-simple Leibniz algebras

Recall that a left Leibniz algebra $\mathfrak{L}$ is called *semi-simple* if $\mathrm{Leib}(\mathfrak{L})$ contains every solvable ideal of $\mathfrak{L}$ (see [14, Section 7]). In particular, a finite-dimensional left Leibniz algebra $\mathfrak{L}$ is semi-simple if, and only if, $\mathrm{Leib}(\mathfrak{L}) = \mathrm{Rad}(\mathfrak{L})$, where $\mathrm{Rad}(\mathfrak{L})$ denotes the largest solvable ideal of $\mathfrak{L}$ (see [14, Proposition 7.4]). Moreover, a left Leibniz algebra $\mathfrak{L}$ is semi-simple if, and only if, the canonical Lie algebra $\mathfrak{L}_{\mathrm{Lie}}$ associated to $\mathfrak{L}$ is semi-simple (see [14, Proposition 7.8]).

Lemma 4.1. *Let $\mathfrak{L}$ be a finite-dimensional semi-simple left Leibniz algebra over a field of characteristic zero. Then $[\mathrm{Leib}(\mathfrak{L})^*]_s^{\mathfrak{L}_{\mathrm{Lie}}} = 0$, where $\mathrm{Leib}(\mathfrak{L})^*$ is a left $\mathfrak{L}$-module, and thus a left $\mathfrak{L}_{\mathrm{Lie}}$-module, via $(x \cdot f)(y) := -f(xy)$ for every linear form $f \in \mathrm{Leib}(\mathfrak{L})^*$ and any elements $x, y \in \mathfrak{L}$.*

Proof. It follows from Levi's theorem for Leibniz algebras (see [31, Proposition 2.4] or [4, Theorem 1]) that there exists a semi-simple Lie subalgebra $\mathfrak{s}$ of $\mathfrak{L}$ such that $\mathfrak{L} = \mathfrak{s} \oplus \mathrm{Leib}(\mathfrak{L})$ (see also [15, Corollary 2.14]). Note that then $\mathfrak{L}_{\mathrm{Lie}} \cong \mathfrak{s}$. Since $\mathfrak{s}$ is a Lie algebra and $\mathrm{Leib}(\mathfrak{L}) \subseteq C_\ell(\mathfrak{L})$, we obtain that $(s+x)(s+x) = sx$ for any elements $s \in \mathfrak{s}$ and $x \in \mathrm{Leib}(\mathfrak{L})$. This shows that $\mathrm{Leib}(\mathfrak{L}) = \mathfrak{s}\mathrm{Leib}(\mathfrak{L})$. Now let $\varphi \in [\mathrm{Leib}(\mathfrak{L})^*]_s^{\mathfrak{s}}$ be arbitrary. Since $(s \cdot \varphi)(x) = -\varphi(sx)$ for any $\varphi \in \mathrm{Leib}(\mathfrak{L})^*$, $s \in \mathfrak{s}$, and $x \in \mathrm{Leib}(\mathfrak{L})$, we conclude that $\varphi[\mathrm{Leib}(\mathfrak{L})] = \varphi[\mathfrak{s}\mathrm{Leib}(\mathfrak{L})] = 0$, which proves the assertion. □

The first main result in this section is the Leibniz analogue of Whitehead's vanishing theorem for the Chevalley-Eilenberg cohomology of finite-dimensional semi-simple Lie algebras over a field of characteristic zero (see [9, Theorem 24.1] or [19, Theorem 10]). Note that in the special case of a Lie algebra, Theorem 4.2 is an immediate consequence of Whitehead's classical vanishing theorem and Theorem 2.6.

Theorem 4.2. *Let $\mathfrak{L}$ be a finite-dimensional semi-simple left Leibniz algebra over a field of characteristic zero. If M is a finite-dimensional $\mathfrak{L}$-bimodule such that $M^{\mathfrak{L}} = 0$, then $\mathrm{HL}^n(\mathfrak{L}, M) = 0$ for every non-negative integer n.*

Proof. According to Lemma 1.1, the hypothesis $M^{\mathfrak{L}} = 0$ implies that M is symmetric. We can therefore use the spectral sequence of Corollary 3.5 with $\mathfrak{I} := \mathrm{Leib}(\mathfrak{L})$ and $\mathfrak{Q} := \mathfrak{L}_{\mathrm{Lie}}$. The E_2-term reads

$$E_2^{p,q} = \mathrm{HL}^p(\mathfrak{Q}, (\mathfrak{I}^*)_s) \otimes \mathrm{HL}^q(\mathfrak{L}, M)\,.$$

It follows from [14, Proposition 7.8] and the Ntolo-Pirashvili vanishing theorem for the Leibniz cohomology of a finite-dimensional semi-simple Lie algebra over a field of characteristic zero (see [30, Théorème 2.6] and the sentence after the proof of Lemma 2.2 in [31]) that $\mathrm{HL}^p(\mathfrak{Q}, (\mathfrak{I}^*)_s) = 0$ for every positive integer p. Hence the spectral sequence collapses, and we deduce

$$\mathrm{HL}^n(\mathfrak{L}|\mathfrak{Q}, M) = (\mathfrak{I}^*)_s^{\mathfrak{Q}} \otimes \mathrm{HL}^n(\mathfrak{L}, M)\,.$$

By virtue of Lemma 4.1, the relative cohomology $\mathrm{HL}^n(\mathfrak{L}|\mathfrak{Q}, M)$ vanishes for every non-negative integer n, and thus we obtain from Proposition 3.1 in conjunction with

[14, Proposition 4.1] and the Ntolo-Pirashvili vanishing theorem that $\mathrm{HL}^n(\mathfrak{L}, M) \cong \mathrm{HL}^n(\mathfrak{Q}, M) = 0$ for every non-negative integer n. $\square$

Remark. It is possible to prove Theorem 4.2 without using the Ntolo-Pirashvili vanishing theorem. Namely, the first time the Ntolo-Pirashvili vanishing theorem is used in the above proof, one can instead use Lemma 4.1, Whitehead's classical vanishing theorem, and Theorem 2.6, and the second time, by hypothesis, it is enough to apply just the last two results. As a consequence, the proof of Theorem 4.3 gives also a new proof of the Ntolo-Pirashvili vanishing theorem.

Next, we generalize the Ntolo-Pirashvili vanishing theorem from Lie algebras to arbitrary Leibniz algebras. The main tools in the proof are Corollary 1.3, Theorem 4.2, Corollary 1.5, and Lemma 1.4, where the second result and its use in this proof seems to be new.

Theorem 4.3. *Let $\mathfrak{L}$ be a finite-dimensional semi-simple left Leibniz algebra over a field of characteristic zero, and let M be a finite-dimensional $\mathfrak{L}$-bimodule. Then $\mathrm{HL}^n(\mathfrak{L}, M) = 0$ for every integer $n \geq 2$, and there is a five-term exact sequence*

$$0 \to M_0 \to \mathrm{HL}^0(\mathfrak{L}, M) \to M_{\mathrm{sym}}^{\mathfrak{L}_{\mathrm{Lie}}} \to \mathrm{Hom}_{\mathfrak{L}}(\mathfrak{L}_{\mathrm{ad},\ell}, M_0) \to \mathrm{HL}^1(\mathfrak{L}, M) \to 0\,.$$

Moreover, if M is symmetric, then $\mathrm{HL}^n(\mathfrak{L}, M) = 0$ for every integer $n \geq 1$.

Proof. The proof is divided into three steps. First, we will prove the assertion for symmetric $\mathfrak{L}$-bimodules. So suppose that M is symmetric. Since M is finite-dimensional, it has a composition series. It is clear that sub-bimodules and homomorphic images of a symmetric bimodule are again symmetric. By using the long exact cohomology sequence, it is therefore enough to prove the second part of the theorem for finite-dimensional irreducible symmetric $\mathfrak{L}$-bimodules. So suppose now in addition that M is irreducible and non-trivial. Then we obtain from Corollary 1.3 that $M^{\mathfrak{L}} = 0$, and thus Theorem 4.2 yields that $\mathrm{HL}^n(\mathfrak{L}, M) = 0$ for every non-negative integer n. Finally, suppose that $M = \mathbb{F}$ is the trivial irreducible $\mathfrak{L}$-bimodule. In this case it follows from Corollary 1.5 that $\mathrm{HL}^n(\mathfrak{L}, \mathbb{F}) \cong \mathrm{HL}^{n-1}(\mathfrak{L}, (\mathfrak{L}^*)_s)$ for every integer $n \geq 1$. Since $\mathfrak{L}_{\mathrm{Lie}}$ is perfect, we obtain from Corollary 1.5 that

$$(\mathfrak{L}^*)_s^{\mathfrak{L}} \cong \mathrm{HL}^0(\mathfrak{L}, (\mathfrak{L}^*)_s) \cong \mathrm{HL}^1(\mathfrak{L}, \mathbb{F}) \cong \mathrm{H}^1(\mathfrak{L}_{\mathrm{Lie}}, \mathbb{F}) = 0\,.$$

Therefore another application of Theorem 4.2 yields that

$$\mathrm{HL}^n(\mathfrak{L}, \mathbb{F}) \cong \mathrm{HL}^{n-1}(\mathfrak{L}, (\mathfrak{L}^*)_s) = 0$$

for every integer $n \geq 1$. This finishes the proof for symmetric $\mathfrak{L}$-bimodules.

If M is anti-symmetric, then we obtain the assertion from Lemma 1.4 (b) and the statement for symmetric bimodules. Finally, if M is arbitrary, then in the short exact sequence $0 \to M_0 \to M \to M_{\mathrm{sym}} \to 0$ the first term is anti-symmetric and the third term is symmetric. Hence another application of the long exact cohomology sequence in conjunction with the statement for the anti-symmetric and the symmetric case yields $\mathrm{HL}^n(\mathfrak{L}, M) = 0$ for every integer $n \geq 2$. Now we deduce the five-term exact sequence from the long exact cohomology sequence together with [14, Corollary 4.2], [14, Corollary 4.4 (b)], and the symmetric case. $\square$

Note that Theorem 4.3 contains [14, Theorem 7.15] as the special case $n = 1$ and the second Whitehead lemma for Leibniz algebras as the special case $n = 2$.

But contrary to Chevalley-Eilenberg cohomology, Leibniz cohomology vanishes in any degree $n \geq 2$.

The following example shows that the Ntolo-Pirashvili vanishing theorem (and therefore also Theorem 4.3) does not hold over fields of prime characteristic.

Example E. Let $\mathfrak{g} := \mathfrak{sl}_2(\mathbb{F})$ be the three-dimensional simple Lie algebra of traceless 2×2 matrices over a field $\mathbb{F}$ of characteristic $p > 2$. Moreover, let $\mathbb{F}_p$ denote the field with p elements, and let $L(n)$ $(n \in \mathbb{F}_p)$ denote the irreducible restricted $\mathfrak{g}$-module of heighest weight n. (If the ground field $\mathbb{F}$ is algebraically closed, these modules represent all isomorphism classes of restricted irreducible $\mathfrak{g}$-modules.) It is well known (see [11, Theorem 4]) that $\mathrm{H}^1(\mathfrak{g}, L(p-2)) \cong \mathbb{F}^2 \cong \mathrm{H}^2(\mathfrak{g}, L(p-2))$. (Note that by virtue of [11, Theorem 2], $\mathrm{H}^\bullet(\mathfrak{g}, M) = 0$ for every non-restricted irreducible $\mathfrak{g}$-module. In fact, $L(p-2)$ is the only irreducible $\mathfrak{g}$-module M such that $\mathrm{H}^1(\mathfrak{g}, M) \neq 0$ or $\mathrm{H}^2(\mathfrak{g}, M) \neq 0$.)

We obtain from Proposition 2.2 that

$$\mathrm{HL}^1(\mathfrak{g}, L(p-2)_s) \cong \mathrm{H}^1(\mathfrak{g}, L(p-2)) \cong \mathbb{F}^2 \neq 0$$

and

$$0 \neq \mathbb{F}^2 \cong \mathrm{H}^2(\mathfrak{g}, L(p-2)) \hookrightarrow \mathrm{HL}^2(\mathfrak{g}, L(p-2)_s)\,.$$

In particular, this shows that the Ntolo-Pirashvili vanishing theorem (and therefore also Theorem 4.3) is not true over fields of prime characteristic.

Remark. By using more sophisticated tools one can also say something about the Leibniz cohomology of anti-symmetric irreducible $\mathfrak{g}$-bimodules, where again $\mathfrak{g} := \mathfrak{sl}_2(\mathbb{F})$. We obtain from Lemma 1.4 (b) that

$$\mathrm{HL}^1(\mathfrak{g}, L(n)_a) \cong \mathrm{HL}^0(\mathfrak{g}, \mathrm{Hom}_{\mathbb{F}}(\mathfrak{g}, L(n))_s) \cong \mathrm{Hom}_{\mathbb{F}}(\mathfrak{g}, L(n))^{\mathfrak{g}}$$

and

$$\mathrm{HL}^2(\mathfrak{g}, L(n)_a) \cong \mathrm{HL}^1(\mathfrak{g}, \mathrm{Hom}_{\mathbb{F}}(\mathfrak{g}, L(n))_s) \cong \mathrm{H}^1(\mathfrak{g}, \mathrm{Hom}_{\mathbb{F}}(\mathfrak{g}, L(n)))\,.$$

Since $\mathfrak{g} \cong L(2)$ is a self-dual $\mathfrak{g}$-module, we have the following isomorphisms of $\mathfrak{g}$-modules:

$$\mathrm{Hom}_{\mathbb{F}}(\mathfrak{g}, L(n)) \cong L(2) \otimes L(n)\,.$$

Let us first consider the case $p > 3$. Then we obtain from the modular Clebsch-Gordan rule (see [5, Theorem 1.11 (a)] or Satz a) in Chapter 5 of [17]) that

$$L(2) \otimes L(2) \cong L(4) \oplus L(2) \oplus L(0)$$

and

$$L(2) \otimes L(p-4) \cong \begin{cases} L(3) \oplus L(1) & \text{if } p = 5 \\ L(p-2) \oplus L(p-4) \oplus L(p-6) & \text{if } p \geq 7\,. \end{cases}$$

Hence we conclude for $p > 3$ that

$$\mathrm{HL}^1(\mathfrak{g}, L(2)_a) \cong (L(2) \otimes L(2))^{\mathfrak{g}} \cong L(0)^{\mathfrak{g}} \cong \mathbb{F} \neq 0$$

and

$$\mathrm{HL}^2(\mathfrak{g}, L(p-4)_a) \cong \mathrm{H}^1(\mathfrak{g}, L(2) \otimes L(p-4)) \cong \mathrm{H}^1(\mathfrak{g}, L(p-2)) \cong \mathbb{F}^2 \neq 0\,.$$

Let us now consider $p = 3$. Note that in this case $L(2)$ is the *Steinberg module*, i.e., $L(2)$ is the unique projective irreducible restricted $\mathfrak{g}$-module. This implies that $L(2) \otimes L(n)$ is also projective for every highest weight $n \in \mathbb{F}_3$. Then we obtain from the modular Clebsch-Gordan rule (cf. [5, Theorem 1.11 (b) and (c)] or Satz b) and c) in Chapter 5 of [17]) for $p = 3$ that

$$L(2) \otimes L(n) \cong \begin{cases} L(2) & \text{if } n \equiv 0 \pmod 3 \\ P(1) & \text{if } n \equiv 1 \pmod 3\,, \\ P(0) \oplus L(2) & \text{if } n \equiv 2 \pmod 3) \end{cases}$$

where $P(n)$ denotes the projective cover (and at the same time also the injective hull) of $L(n)$. As a consequence, we have that

$$(L(2) \otimes L(n))^{\mathfrak{g}} \cong \begin{cases} \mathbb{F} & \text{if } n \equiv 2 \pmod 3 \\ 0 & \text{if } n \not\equiv 2 \pmod 3\,. \end{cases}$$

Therefore, we obtain that

$$\mathrm{HL}^1(\mathfrak{g}, L(2)_a) \cong (L(2) \otimes L(2))^{\mathfrak{g}} \cong \mathbb{F} \neq 0\,.$$

Moreover, by using the six-tem exact sequence relating Hochschild's cohomology of a restricted Lie algebra to its Chevalley-Eilenberg cohomology (see [18, p. 575]), we also conclude that

$$\mathrm{HL}^2(\mathfrak{g}, L(2)_a) \cong \mathrm{H}^1(\mathfrak{g}, L(2) \otimes L(2)) \cong \mathfrak{g}^* \cong \mathbb{F}^3 \neq 0\,.$$

The next example shows that the Ntolo-Pirashvili vanishing theorem (and therefore also Theorem 4.3) does not hold for infinite-dimensional modules.

Example F. Let $\mathfrak{g} := \mathfrak{sl}_2(\mathbb{C})$ be the three-dimensional simple complex Lie algebra of traceless 2×2 matrices, and let $V(\lambda)$ $(\lambda \in \mathbb{C})$ denote the *Verma module* of highest weight λ. (Here we identify every complex multiple of the unique fundamental weight with its coefficient.) Verma modules are infinite-dimensional indecomposable $\mathfrak{g}$-modules (see, for example, [22, Theorem 20.2 (e)]). Furthermore, it is well known (see [22, Exercise 7 (b) & (c) in Section 7.2]) that $V(\lambda)$ is irreducible if, and only if, λ is not a dominant integral weight (i.e., with our identification, λ is not a non-negative integer). Moreover, it follows from [36, Theorem 4.19] that

$$\mathrm{H}^n(\mathfrak{g}, V(\lambda)) \cong \begin{cases} \mathbb{C} & \text{if } \lambda = -2 \text{ and } n = 1, 2 \\ 0 & \text{otherwise}\,. \end{cases}$$

This in conjunction with Proposition 2.2 yields that

$$\mathrm{HL}^1(\mathfrak{g}, V(-2)_s) \cong \mathrm{H}^1(\mathfrak{g}, V(-2)) \cong \mathbb{C} \neq 0$$

and

$$0 \neq \mathbb{C} \cong \mathrm{H}^2(\mathfrak{g}, V(-2)) \hookrightarrow \mathrm{HL}^2(\mathfrak{g}, V(-2)_s)\,.$$

In particular, the Ntolo-Pirashvili vanishing theorem (and therefore also Theorem 4.3) is not true for infinite-dimensional modules.

We obtain as an immediate consequence of Theorem 4.3 the following generalization of [14, Corollary 7.9].

Corollary 4.4. *If $\mathfrak{L}$ is a finite-dimensional semi-simple left Leibniz algebra over a field of characteristic zero, then* $\mathrm{HL}^n(\mathfrak{L}, \mathbb{F}) = 0$ *for every integer* $n \geq 1$.

Remark. It is well known that the analogue of Corollary 4.4 does not hold for the Chevalley-Eilenberg cohomology of Lie algebras as $\mathrm{H}^3(\mathfrak{g}, \mathbb{F}) \neq 0$ for any finite-dimensional semi-simple Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$ of characteristic zero (see [9, Theorem 21.1]).

Next, we apply Theorem 4.3 to compute the cohomology of a finite-dimensional semi-simple left Leibniz algebra over a field of characteristic zero with coefficients in its adjoint bimodule and in its (anti-)symmetric counterparts.

Theorem 4.5. *For every finite-dimensional semi-simple left Leibniz algebra $\mathfrak{L}$ over a field of characteristic zero the following statements hold:*

(a)

$$\mathrm{HL}^n(\mathfrak{L},\mathfrak{L}_s)=\begin{cases}\mathrm{Leib}(\mathfrak{L}) & \text{if } n=0\\ 0 & \text{if } n\geq 1\,.\end{cases}$$

(b)

$$\mathrm{HL}^n(\mathfrak{L},\mathfrak{L}_a)=\begin{cases}\mathfrak{L} & \text{if } n=0\\ \mathrm{End}_{\mathfrak{L}}(\mathfrak{L}_{\mathrm{ad},\ell}) & \text{if } n=1\,,\\ 0 & \text{if } n\geq 2\end{cases}$$

where $\mathrm{End}_{\mathfrak{L}}(\mathfrak{L}_{\mathrm{ad},\ell})$ *denotes the vector space of endomorphisms of the left adjoint $\mathfrak{L}$-module $\mathfrak{L}_{\mathrm{ad},\ell}$.*

(c)

$$\mathrm{HL}^n(\mathfrak{L},\mathfrak{L}_{\mathrm{ad}})=\begin{cases}\mathrm{Leib}(\mathfrak{L}) & \text{if } n=0\\ \mathrm{Hom}_{\mathfrak{L}}(\mathfrak{L}_{\mathrm{ad},\ell},\mathrm{Leib}(\mathfrak{L})) & \text{if } n=1\,,\\ 0 & \text{if } n\geq 2\end{cases}$$

where $\mathrm{Hom}_{\mathfrak{L}}(\mathfrak{L}_{\mathrm{ad},\ell},\mathrm{Leib}(\mathfrak{L}))$ *denotes the vector space of homomorphisms from the left adjoint $\mathfrak{L}$-module $\mathfrak{L}_{\mathrm{ad},\ell}$ to the Leibniz kernel* $\mathrm{Leib}(\mathfrak{L})$ *considered as a left $\mathfrak{L}$-module.*

Proof. (a): According to [14, Proposition 4.1] and [14, Proposition 7.5] we have that that $\mathrm{HL}^0(\mathfrak{L},\mathfrak{L}_s)=(\mathfrak{L}_s)^{\mathfrak{L}}=C_\ell(\mathfrak{L})=\mathrm{Leib}(\mathfrak{L})$. Moreover, we obtain the statement for degree $n\geq 1$ from the second part of Theorem 4.3.

(b): It follows from [14, Corollary 4.2 (b)] that $\mathrm{HL}^0(\mathfrak{L},\mathfrak{L}_a)=\mathfrak{L}$, and it follows from [14, Corollary 4.4 (b)] that $\mathrm{HL}^1(\mathfrak{L},\mathfrak{L}_a)=\mathrm{End}_{\mathfrak{L}}(\mathfrak{L}_{\mathrm{ad},\ell})$. The remainder of the assertion is an immediate consequence of the first part of Theorem 4.3.

(c): As for the symmetric adjoint bimodule, we obtain from [14, Proposition 4.1] and [14, Proposition 7.5] that $\mathrm{HL}^0(\mathfrak{L},\mathfrak{L}_{\mathrm{ad}})=(\mathfrak{L}_{\mathrm{ad}})^{\mathfrak{L}}=C_\ell(\mathfrak{L})=\mathrm{Leib}(\mathfrak{L})$. Next, by applying the five-term exact sequence of Theorem 4.3 to the adjoint $\mathfrak{L}$-bimodule $M:=\mathfrak{L}_{\mathrm{ad}}$, we deduce that

$$\mathrm{HL}^1(\mathfrak{L},\mathfrak{L}_{\mathrm{ad}})\cong\mathrm{Hom}_{\mathfrak{L}}(\mathfrak{L}_{\mathrm{ad},\ell},\mathrm{Leib}(\mathfrak{L}))\,,$$

as the third term is $\mathfrak{L}_{\mathrm{Lie}}^{\mathfrak{L}_{\mathrm{Lie}}}=C(\mathfrak{L}_{\mathrm{Lie}})=0$. Finally, the assertion for degree $n\geq 2$ is again an immediate consequence of the first part of Theorem 4.3. □

Remark. Note that the vanishing part of Theorem 4.5 (c) confirms a generalization of the conjecture at the end of [1]. Moreover, parts (a) and (b) of Theorem 4.5 show that the statements in Theorem 4.3 are best possible.

In particular, one can derive from Theorem 4.5 (c) that finite-dimensional semi-simple non-Lie Leibniz algebras over a field of characteristic zero have outer derivations. In this respect non-Lie Leibniz algebras behave differently than Lie algebras (see, for example, [22, Theorem 5.3]).

Corollary 4.6. *Every finite-dimensional semi-simple non-Lie Leibniz algebra over a field of characteristic zero has derivations that are not inner.*

Proof. If one applies the contravariant functor $\mathrm{Hom}_{\mathbb{F}}(-,\mathrm{Leib}(\mathfrak{L}))$ to the short exact sequence

$$0\to\mathrm{Leib}(\mathfrak{L})\to\mathfrak{L}_{\mathrm{ad}}\to\mathfrak{L}_{\mathrm{Lie}}\to 0$$

considered as a short exact sequence of left $\mathfrak{L}$-modules, one obtains the short exact sequence

$$0\to\mathrm{Hom}_{\mathbb{F}}(\mathfrak{L}_{\mathrm{Lie}},\mathrm{Leib}(\mathfrak{L}))\to\mathrm{Hom}_{\mathbb{F}}(\mathfrak{L}_{\mathrm{ad},\ell},\mathrm{Leib}(\mathfrak{L}))\to\mathrm{Hom}_{\mathbb{F}}(\mathrm{Leib}(\mathfrak{L}),\mathrm{Leib}(\mathfrak{L}))\to 0$$

of left $\mathfrak{L}$-modules. Then the long exact cohomology sequence in conjunction with Lemma 1.4 (a) yields the long exact sequence

$$\begin{aligned}0\quad\to\quad&\mathrm{Hom}_{\mathfrak{L}}(\mathfrak{L}_{\mathrm{Lie}},\mathrm{Leib}(\mathfrak{L}))\to\mathrm{Hom}_{\mathfrak{L}}(\mathfrak{L}_{\mathrm{ad},\ell},\mathrm{Leib}(\mathfrak{L}))\to\mathrm{Hom}_{\mathfrak{L}}(\mathrm{Leib}(\mathfrak{L}),\mathrm{Leib}(\mathfrak{L}))\\ \to\quad&\widetilde{\mathrm{HL}}^1(\mathfrak{L},\mathrm{Hom}_{\mathbb{F}}(\mathfrak{L}_{\mathrm{Lie}},\mathrm{Leib}(\mathfrak{L})))=\mathrm{HL}^1(\mathfrak{L},\mathrm{Hom}_{\mathbb{F}}(\mathfrak{L}_{\mathrm{Lie}},\mathrm{Leib}(\mathfrak{L}))_s)\,.\end{aligned}$$

According to the second part of Theorem 4.3, the fourth term is zero. Since the third term contains the identity map, it is non-zero as by hypothesis $\mathfrak{L}$ is a not a Lie algebra. Hence in this case the second term is non-zero, and we obtain from Theorem 4.5 (c) that $\mathrm{HL}^1(\mathfrak{L},\mathfrak{L}_{\mathrm{ad}})\cong\mathrm{Hom}_{\mathfrak{L}}(\mathfrak{L}_{\mathrm{ad},\ell},\mathrm{Leib}(\mathfrak{L}))\neq 0$. □

Remark. After the submission of our paper we became aware of the preprint [6] in which the authors introduce a more general concept of inner derivations for Leibniz algebras than in our paper. Namely, a derivation D of a left Leibniz algebra $\mathfrak{L}$ is called *inner* if there exists an element $x\in\mathfrak{L}$ such that $\mathrm{Im}(D-L_x)\subseteq\mathrm{Leib}(\mathfrak{L})$. Then it is shown that every derivation of a finite-dimensional semi-simple Lie algebra over a field of characteristic zero is inner in this more general sense (see [6, Theorem 3.3]).

In the same way as at the end of Section 2 for the infinite-dimensional two-sided Witt algebra, by using [2, Théorème 3] in conjunction with Theorem 4.5 (c), one obtains the rigidity of any finite-dimensional semi-simple Lie algebra as a Leibniz algebra.

Corollary 4.7. *Every finite-dimensional semi-simple left Leibniz algebra over an algebraically closed field of characteristic zero is rigid as a Leibniz algebra.*

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