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HOMOLOGICAL TOPICS IN THE REPRESENTATION THEORY OF RESTRICTED LIE ALGEBRAS

Jörg Feldvoss

We present some recent developments in the application of homological methods to the represention theory of finite dimensional restricted Lie algebras.

§0. Introduction

In these notes, we give an account of some general features of restricted Lie algebra cohomology and discuss their application to some problems in representation theory. The concept of restrictedness that was introduced in the theory of modular Lie algebras by N. Jacobson in 1937 turned out to be fundamental. Restricted Lie algebras are much easier to deal with than modular Lie algebras in general, because they allow the definition of tori and a decomposition of elements into semisimple and nilpotent parts (Jordan-Chevalley-Seligman decomposition, cf. §1). These properties are similar to those of semisimple Lie algebras over algebraically closed fields of characteristic zero, where they are very useful in the classification and representation theory. With every finite dimensional restricted Lie algebra there is associated a family of finite dimensional (associative) Frobenius algebras, the so-called reduced universal enveloping algebras. These algebras play an important role in the representation theory of restricted Lie algebras since their module categories approximate to a certain degree (to be made precise below) the category of modules for the corresponding Lie algebra, and methods from the theory of associative algebras can be used in this context.

In the first two sections, we provide the necessary prerequisites from the theory of restricted Lie algebras and their representations. In particular, we introduce the toral and p-nilpotent radical of a finite dimensional restricted Lie algebra. In §2, we consider the category of all modules with a fixed

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character and indicate that it is equivalent to the module category of a finite dimensional Frobenius algebra. Since every simple module over a finite dimensional restricted Lie algebra is finite dimensional, in the case of an algebraically closed ground field, every simple module belongs to such a category. Moreover, we determine the simple modules for some three-dimensional restricted Lie algebras over algebraically closed fields. These examples will be used in the remaining sections to illustrate some of the results. Finally, we give some evidence that not every module over a reduced universal enveloping algebra is semisimple. Already at this early stage we use cohomological methods to prove the relevant results of N. Jacobson and G.P. Hochschild as efficiently as possible.

In 1954, G.P. Hochschild [Ho1] initiated the study of a cohomology theory for restricted Lie algebras. He established the usual elementary interpretations of low dimensional cohomology as extensions of modules or algebras and, more surprisingly, a connection between restricted and ordinary Lie algebra cohomology in form of a six-term exact sequence. Apart from a few papers in the late sixties [Chwe, May, Par], restricted cohomology has received considerable attention only quite recently in connection with the cohomology theory of algebraic groups [An, AJ, CPS, FP1, FP2, Hum3, Jan2, Jan3, Jan4, LN1, LN2, Na5, Sul1], associative algebras [Fa3, Fa4] and in its own right [Chiu2, Fa1, Fa5, FaS, Fe1, Fe2, Fe3, FeS1, FeS2, FP3, FP4, FP5, Jan1, Jan5, Na3, Na4, Sul2]. In the third section we present some vanishing and non-vanishing theorems for restricted Lie algebra cohomology, and in particular give a partial answer to Problem 7 in [Hum3]. Since the restricted universal enveloping algebra is a Frobenius algebra, it is possible to introduce complete restricted cohomology spaces which parallel those of the Tate cohomology of finite groups. Aside from a single occurence in [Par], this natural concept has apparently not been used in connection with Lie algebras. Our results especially yield structural characterizations of certain classes of finite dimensional solvable restricted Lie algebras, as well as some information about the block structure of their reduced universal enveloping algebras (cf. §4). This also enables us to describe the structure of the projective indecomposable modules over reduced universal enveloping algebras of a finite dimensional nilpotent restricted Lie algebra (cf. §5).

In §6, we define the complexity of a module over a reduced universal enveloping algebra and derive some of its elementary properties. More generally, based on the work of E.M. Friedlander and B.J. Parshall [FP3, FP4, FP5], we introduce the support variety of such a module and retrieve the complexity as its dimension. Due to the main result of [Jan1], support varieties often can be used as a substitute for the transfer mapping which plays a very important role in modular group cohomology but in general does not exist for restricted Lie algebra cohomology. We illustrate this by deriving two results on projective modules, namely a projectivity criterion using the vanishing of Ext-spaces of sufficiently high degree and the equivalence of projectivity and cohomological triviality for finite dimensional modules (for the latter see also §3). Another application of the geometric methods introduced in §6 is the classification of finite dimensional restricted Lie algebras with finitely many isomorphism classes of finite dimensional indecomposable restricted modules resp. with periodic cohomology. The solution of the first problem over algebraically closed fields was announced by W. Pfautsch and D. Voigt [PV] in the more general context of infinitesimal algebraic group schemes and the solution of the second problem (implying also a solution of the first one) over perfect fields was found by H. Strade and the author [FeS1]. This result is contained in the last section, where we also define the module type of an algebra and address the more general question of finding for certain classes of finite dimensional restricted Lie algebras the characters such that the corresponding reduced universal enveloping algebras are semisimple or of finite resp. tame module type (see also [Fa10]).

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§1. Restricted Lie Algebras

Let L always denote a Lie algebra over a field $\mathbb{F}$ of prime characteristic p. For our purposes we need an additional structure on L which was introduced by N. Jacobson in 1937. He observed an interesting connection between the *Frobenius mapping* $a \mapsto a^p$ of an associative $\mathbb{F}$-algebra A and the *commutator* $[a, b] := a \cdot b - b \cdot a$. In fact, he was interested in the $?^p$-closed Lie algebra of all $\mathbb{F}$-derivations, $\mathrm{Der}_{\mathbb{F}}(\mathbb{E})$, for a field extension $\mathbb{E} \supset \mathbb{F}$ in order to establish for purely inseparable field extensions of exponent one a *Galois correspondence* by using derivations instead of automorphisms.

In the abstract setting, a *restricted Lie algebra* (or a *Lie p-algebra*) L admits a mapping $?^{[p]} : L \to L$ such that

(RL1) $(x + y)^{[p]} = x^{[p]} + y^{[p]} + J(x, y) \qquad \forall\, x, y \in L,$

(RL2) $(\alpha x)^{[p]} = \alpha^p x^{[p]} \qquad \forall\, \alpha \in \mathbb{F}, x \in L,$

(RL3) $\mathrm{ad}_L\, x^{[p]} = (\mathrm{ad}_L\, x)^p \qquad \forall\, x \in L.$

Here $J(x, y) \in \langle x, y\rangle^p$, where $\langle x, y\rangle^p$ is the p-th term of the *descending central series* of the subalgebra of L generated by x and y. We refer to $?^{[p]}$ as the *restriction mapping* (or *p-mapping*) of L, and sometimes we denote the corresponding pair by $(L, ?^{[p]})$.

A subalgebra (resp. ideal) K of L is called a *p-subalgebra* (resp. *p-ideal*) if $K^{[p]} \subseteq K$. For any $S \subseteq L$ we denote by $\langle S\rangle$ (resp. $\langle S\rangle_p$) the *subalgebra* (resp. *p-subalgebra*) *generated by* S in L. If we define *restricted Lie algebra homomorphism, monomorphism, epimorphism*, and *isomorphism* as usual, then the well-known *Isomorphism Theorems* etc. hold for restricted Lie algebras.

Examples.

(i) Let L be an abelian Lie algebra and suppose that f is an arbitrary mapping on L. Then (L, f) is restricted if and only if f is p-semilinear. The p-ideals of L are just the f-invariant subspaces.

(ii) For any $\mathbb{F}$-vector space V, the commutator algebra

$$\mathfrak{gl}(V) := \mathrm{End}_{\mathbb{F}}(V)^- := (\mathrm{End}_{\mathbb{F}}(V), [?, ?], ?^p)$$

is a restricted Lie algebra. In particular, $\mathfrak{gl}_n(\mathbb{F}) := \mathrm{Mat}_n(\mathbb{F})^-$ is an n^2-dimensional restricted Lie algebra for any positive integer n. The so-called *Lie algebras of classical type* are p-subalgebras of $\mathfrak{gl}_n(\mathbb{F})$ for some positive integer n satisfying certain conditions, e.g. $\mathfrak{sl}_n(\mathbb{F}) := \{x \in \mathfrak{gl}_n(\mathbb{F}) \mid \mathrm{tr}(x) = 0\}$ is a restricted Lie algebra of type A_{n-1} for any integer $n \geq 2$, where $\mathrm{tr}(x)$ denotes the trace of the matrix x.

(iii) Let $P_n(\mathbb{F})$ denote the *n-th truncated polynomial algebra*, i.e., the commutative and associative $\mathbb{F}$-algebra with n generators $t_1, \dots, t_n$ subject to the relations $t_j^p = 0$ $(1 \leq j \leq n)$. According to the Leibniz Rule, the derivations $W_n(\mathbb{F}) := \mathrm{Der}_{\mathbb{F}}(P_n(\mathbb{F}))$ of $P_n(\mathbb{F})$ form a p-subalgebra of $\mathfrak{gl}(P_n(\mathbb{F}))$ and thus a finite dimensional restricted Lie algebra. The so-called *restricted Lie algebras of Cartan type* are p-subalgebras of $W_n(\mathbb{F})$ for some positive integer n satisfying certain conditions, e.g. the derived subalgebra $S_n(\mathbb{F})$ of $\{\sum_{j=1}^n f_j \frac{\partial}{\partial t_j} \mid \sum_{j=1}^n \frac{\partial f_j}{\partial t_j} = 0\}$ for any integer $n \geq 3$ is called *special Lie algebra*. It should be mentioned that the Lie algebras of classical type (including the exceptional types E, F, G resp. $\mathfrak{psl}_n(\mathbb{F})$ if $\mathrm{char}(\mathbb{F})$ divides n) and the restricted Lie algebras of Cartan type exhaust all finite dimensional restricted simple Lie algebras over algebraically closed fields of characteristic $p > 7$ [BW].

(iv) Let $A_n(\mathbb{F})$ denote the *n-th truncated Weyl algebra*, i.e., the associative $\mathbb{F}$-algebra with $2n$ generators $x_1, \dots, x_n, y_1, \dots, y_n$ subject to the relations $[x_i, x_j] = [y_i, y_j] = x_i^p = y_i^p = 0$ and $[x_i, y_j] = \delta_{ij} \cdot 1$ $(1 \leq i, j \leq n)$. Then $A_n(\mathbb{F})$ is a p^{2n}-dimensional central simple $\mathbb{F}$-algebra (cf. [Str4, p. 73]). The $(2n+1)$-dimensional nilpotent p-subalgebra

$$\mathcal{H}_n(\mathbb{F}) := \mathbb{F}x_1 \oplus \cdots \oplus \mathbb{F}x_n \oplus \mathbb{F}1 \oplus \mathbb{F}y_1 \oplus \cdots \oplus \mathbb{F}y_n$$

of $A_n(\mathbb{F})^-$ is the so-called *n-th Heisenberg algebra*.

(v) Let $L := \mathbb{F}t \oplus \mathbb{F}e$ be the non-abelian two-dimensional Lie algebra, where $[t, e] = e$. Since every derivation of L is inner and the center of L is zero (cf. (RL3)), L possesses a *unique* p-mapping defined via

$$(\theta t + \eta e)^{[p]} := \theta^p t + \theta^{p-1} \eta e,$$

i.e., $t^{[p]} = t$ and $e^{[p]} = 0$.

Remark. An arbitrary finite dimensional modular Lie algebra L can always be embedded into a *finite dimensional* restricted Lie algebra E, a so-called *p-envelope* of L (i.e., $\langle L\rangle_p = E$), such that $C(L)^{[p]} = 0 = C(E)^{[p]}$(see [Dzhu2, Str5, Str6, SF]).

An element $x \in L$ is called *semisimple* if $x \in \langle x^{[p]}\rangle_p$ and *p-nilpotent* if $x^{[p]^n} = 0$ for some positive integer n. Examples (i) and (v) suggest the so-called *Jordan-Chevalley-Seligman decomposition*:

Theorem 1.1. *Let L be a finite dimensional restricted Lie algebra over a perfect field $\mathbb{F}$. Every element $x \in L$ decomposes uniquely into $x = x_s + x_n$ such that x_s is semisimple, x_n is p-nilpotent and $[x_s, x_n] = 0$.* $\square$

A restricted Lie algebra L is called a *torus* if L is abelian and every element is semisimple. (Note that the first condition is superfluous if the ground field $\mathbb{F}$ is algebraically closed.) L is called *p-nilpotent* if every element is p-nilpotent. Maximal tori are useful to produce Cartan subalgebras. Indeed, the centralizer $C_L(T)$ of any maximal torus T of L is a Cartan subalgebra of L (and vice versa) (cf. [SF, Theorem II.4.1]).

Examples.

(vi) Consider the abelian restricted Lie algebra (L, f) over a perfect field. The Fitting-0-space (resp. Fitting-1-space) of L with respect to f is equal to the set of f-nilpotent (resp. f-semisimple) elements of L. In particular, the Fitting-1-space is a maximal torus of L.

(vii) A matrix $x \in \mathfrak{gl}_n(\mathbb{F})$ is semisimple if and only if x is diagonalizable after some field extension. A matrix $x \in \mathfrak{gl}_n(\mathbb{F})$ is p-nilpotent if and only if x is nilpotent. The subalgebra of all diagonal matrices is a maximal torus of $\mathfrak{gl}_n(\mathbb{F})$.

Note that an application of Example (vi) to $L := \langle x\rangle_p$ yields Theorem 1.1.

Let $C(L)$ denote the center of L. In fact, $C(L)$ is a p-ideal of L. Since every toral p-ideal is central, we obtain

Proposition 1.2. *Let L be a finite dimensional restricted Lie algebra. Then* $\operatorname{Tor}_p(L) := \{x \in C(L) \mid x \text{ is semisimple}\}$ *is the largest toral p-ideal of L and the following statements hold:*

(a) $\operatorname{Tor}_p(\operatorname{Tor}_p(L)) = \operatorname{Tor}_p(L)$.

(b) $\operatorname{Tor}_p(L/\operatorname{Tor}_p(L)) = 0$.

(c) *For any restricted Lie algebra homomorphism φ we have*

$$\varphi(\operatorname{Tor}_p(L)) \subseteq \operatorname{Tor}_p(\varphi(L)). \quad \square$$

Analogous to Proposition 1.2, we have (cf. [SF, pp. 67–69]):

Proposition 1.3. *Every finite dimensional restricted Lie algebra L possesses a largest p-nilpotent p-ideal $\operatorname{Rad}_p(L)$ and the following statements hold:*

(a) $\operatorname{Rad}_p(\operatorname{Rad}_p(L)) = \operatorname{Rad}_p(L)$.
(b) $\operatorname{Rad}_p(L/\operatorname{Rad}_p(L)) = 0$.
(c) *For any restricted Lie algebra homomorphism φ we have*

$$\varphi(\operatorname{Rad}_p(L)) \subseteq \operatorname{Rad}_p(\varphi(L)). \quad \square$$

(a), (b) and (c) in Proposition 1.2 resp. Proposition 1.3 are so-called *radical properties*. Note that $\operatorname{Tor}_p(L)$ and $\operatorname{Rad}_p(L)$ are "orthogonal" to each other, i.e., $\operatorname{Rad}_p(\operatorname{Tor}_p(L)) = 0 = \operatorname{Tor}_p(\operatorname{Rad}_p(L))$.

Examples.

(viii) $\operatorname{Rad}_p(\mathcal{H}_n(\mathbb{F})) = 0$ and the maximal torus of $\mathcal{H}_n(\mathbb{F})$ is just the one-dimensional center which therefore coincides with $\operatorname{Tor}_p(\mathcal{H}_n(\mathbb{F}))$.
(ix) Let $L := \mathbb{F}t \oplus \mathbb{F}e$ be the non-abelian two-dimensional Lie algebra. Then $\mathbb{F}t$ is a maximal torus of L, $\operatorname{Tor}_p(L) = 0$ and $\operatorname{Rad}_p(L) = \mathbb{F}e$.

§2. Representation Theoretic Background

Let L be a restricted Lie algebra over $\mathbb{F}$. One is particularly interested in the relationship between L and those Lie algebras that occur "naturally" in the sense of linear algebra. A (restricted) Lie algebra homomorphism $\rho : L \to \mathfrak{gl}(M)$, where M is an $\mathbb{F}$-vector space, is called a (*restricted*) *representation* of L in M, and M is then called a (*restricted*) *L-module*.

One of the early observations in the representation theory of modular Lie algebras is that Weyl's Theorem is never true in prime characteristic:

Theorem 2.1. [Jac1] *For every non-zero finite dimensional restricted Lie algebra there exists a finite dimensional module that is not semisimple.* $\square$

In preparation for our subsequent arguments we give a cohomological proof of Theorem 2.1. If every finite dimensional L-module is semisimple, then every short exact sequence of finite dimensional L-modules splits, i.e., in particular, the first ordinary cohomology space $H^1(L, M) := \operatorname{Ext}^1_{U(L)}(\mathbb{F}, M)$ of L with coefficients in an arbitrary finite dimensional L-module M vanishes. (Since the (ordinary) universal enveloping algebra $U(L)$ of L is a Hopf algebra, these conditions are in fact equivalent.) Consequently, it is enough to show the following cohomological non-vanishing result (see also [Dzhu4, Corollary 2 in §1] or [FaS, Corollary 2.2] for a more general result conjectured by G.B. Seligman):

Theorem 2.2. *For every non-zero finite dimensional restricted Lie algebra L there exists a finite dimensional module M such that $H^1(L, M) \neq 0$.* $\square$

Remark. By virtue of the long exact cohomology sequence and Theorem 3.1 (a), the module in Theorem 2.2 (but in general not in Theorem 2.1) can be chosen to be simple *and* restricted. Moreover, both results are also true for non-restricted modular Lie algebras. It is enough to give only a proof for a non-restricted version of Theorem 2.2. This can be done by using a result of R. Farnsteiner [Fa7, Corollary 2.4(2)], namely that for every simple restricted E-module S there is an isomorphism $H^1(E, S) \cong H^1(L, S) \oplus (E/L \otimes_{\mathbb{F}} S^L)$. If there is a non-trivial simple restricted E-module S with $H^1(E, S) \neq 0$, then $H^1(L, S) \neq 0$. Otherwise E is nilpotent (see Theorem 3.6 and its proof). Hence L is also nilpotent and $H^1(L, \mathbb{F}) \neq 0$.

For an abelian Lie algebra L we obtain by virtue of the standard resolution of the one-dimensional trivial L-module $\mathbb{F}$ that $H^1(L, \mathbb{F}) \cong L$. In the remaining case of a non-abelian (restricted) Lie algebra, Theorem 2.2 will follow immediately from Theorem 2.5 (or the first part of its proof) and Hochschild's five-term exact sequence (2) below.

Theorem 2.1 shows that it is not enough (even for the study of finite dimensional modules) to consider only simple modules. The next result shows that in contrast to the zero characteristic case there are no infinite dimensional simple modules over finite dimensional modular Lie algebras (cf. the Remark before Theorem 1.1). The proof has a strong ring theoretical flavour and shows that the "large" center $C(U(L))$ of the (ordinary) universal enveloping algebra $U(L)$ of L in the prime characteristic case is responsible for this difference. In fact, the dimensions of simple L-modules are bounded above, and there has been considerable interest in determining a least upper bound for some classes of Lie algebras, especially for simple Lie algebras of classical type [Ru1, VK, FP7][1] and Cartan type [Mil1, Mil2, Kry1] (cf. also [Pan, Kry2] and [Hum3, Problem 4]). Much more important for our purposes will be Corollary 2.4.

Theorem 2.3. [Cu1] *Every simple module over a finite dimensional restricted Lie algebra is finite dimensional.*

Proof. Consider the unitary subalgebra $O(L)$ of $U(L)$ generated by $\{x^p - x^{[p]} \mid x \in L\}$. (RL3) implies $O(L) \subseteq C(U(L))$. Let S be a (non-zero) simple

[1] Recently, A. Premet [Pre2] has made considerable progress by proving the Kac-Weisfeiler conjecture.

L-module. According to N. Jacobson's refinement of the Poincaré-Birkhoff-Witt-Theorem, $U(L)$ is a finitely generated $O(L)$-module and therefore the same holds for S. Then by the Generalized Nakayama Lemma (or some considerations on integral ring extensions) $\mathcal{M} := \mathrm{Ann}_{U(L)}(S) \cap O(L)$ is a maximal ideal of $O(L)$. Since $O(L)$ is a finitely generated $\mathbb{F}$-algebra, Hilbert's Nullstellensatz implies that $O(L)/\mathcal{M} \supseteq \mathbb{F}$ is a finite field extension, i.e., $\dim_{\mathbb{F}} S = \dim_{O(L)/\mathcal{M}} S \cdot \dim_{\mathbb{F}} O(L)/\mathcal{M}$ is finite. □

Then Schur's Lemma immediately yields

Corollary 2.4. [VK] *Let L be a finite dimensional restricted Lie algebra over an algebraically closed field. Then for every simple L-module S there exists a linear form $\chi \in L^*$ such that*

$$(x)_S^p - (x^{[p]})_S = \chi(x)^p \cdot \mathrm{id}_S \quad \forall\, x \in L. \tag{1}$$ □

Remark. Note that χ corresponds uniquely to the restriction of a central character of $U(L)$ to $O(L)$.

An arbitrary L-module M is said to have the *character* χ if and only if (1) holds for M instead of S. In the following we are going to investigate the category $\mathrm{Mod}(L, \chi)$ of all L-modules with a fixed character χ. By virtue of Corollary 2.4, every simple L-module belongs to such a category. But according to Theorem 2.1, not every finite dimensional L-module is semisimple. Moreover, (the isomorphism classes of) simple L-modules have been classified (as far as known to the author) only for

- nilpotent (restricted) Lie algebras (due to H. Zassenhaus [Zas1]),
- the p-dimensional Witt algebra W_1 (due to H.-J. Chang [Chang], see also [Str2] and [Dzhu2]),
- the three-dimensional simple Lie algebra $\mathfrak{sl}_2$ (due to N. Jacobson [Jac2] for the restricted case $\chi = 0$ and R.E. Block [Blo] for the non-restricted case $\chi \neq 0$),
- the $(p^2 - 2)$-dimensional Hamiltonian algebra H_2 (due to N.A. Koreshkov [Ko1], see also [Str1]),
- the $2p^2$-dimensional Jacobson-Witt algebra W_2 (due to N.A. Koreshkov [Ko2], see also [Wi]).

One reason to consider only $\mathrm{Mod}(L,\chi)$ is that this category is equivalent to the category of (unitary left) modules over a (finite dimensional associative) Frobenius algebra

$$u(L,\chi) := U(L)/U(L)\{x^p - x^{[p]} - \chi(x)^p \cdot 1 \mid x \in L\},$$

the so called *χ-reduced universal enveloping algebra* of L (cf. [SF, Corollary V.4.3] or [FP5, Proposition 1.2]). This allows a remarkable application of the representation theory of finite dimensional associative algebras to the study of $u(L,\chi)$-modules and indicates the similarity of many properties of $\mathrm{Mod}(L,\chi)$ to the modular representation theory of finite groups. In particular, we are interested in decomposing $u(L,\chi)$ into indecomposable two-sided ideals, the so-called *block ideals*, to reduce the problem of determining the structure of $u(L,\chi)$-modules further (cf. §4). Moreover, the group $\mathrm{Aut}_p(L)$ of restricted Lie algebra automorphisms of L acts on the coadjoint module L^* via

$$g \cdot \chi := \chi \circ g^{-1} \qquad \forall\ g \in \mathrm{Aut}_p(L), \chi \in L^*.$$

It is not difficult to see that if two characters χ and χ' of L belong to the same orbit under the action of $\mathrm{Aut}_p(L)$, then $u(L,\chi)$ and $u(L,\chi')$ are isomorphic (as unitary associative $\mathbb{F}$-algebras). Therefore it is enough to consider only one (suitable chosen) character for each $\mathrm{Aut}_p(L)$-orbit.

In order to have some explicit examples at hand, we determine the isomorphism classes of the simple modules for some three-dimensional restricted Lie algebras over an algebraically closed ground field.

Examples.

(i) Consider the three-dimensional Heisenberg algebra

$$L := \mathcal{H}_1(\mathbb{F}) = \mathbb{F}e_+ \oplus \mathbb{F}z \oplus \mathbb{F}e_-,$$

$$[e_+, e_-] = z,\ e_\pm^{[p]} = 0,\ z^{[p]} = z.$$

Put $I := \mathbb{F}z \oplus \mathbb{F}e_-$ and let $\omega \in I^*$. Then L^* has two different $\mathrm{Aut}_p(L)$-orbits, namely

$\underline{\chi(z) = 0}$: Since I is abelian, an application of [SF, Corollary V.7.6(2)] yields

$$\mathrm{Irr}(L,\chi) = \{F_\chi\} \cup \{\mathrm{Ind}_I^L(F_\omega,\chi) \mid 0 \neq \omega(z) \in \mathbb{F}_p, \omega(e_-) = \chi(e_-)\},$$

i.e., $|\,\mathrm{Irr}(L,\chi)| = p$.

$\underline{\chi(z) \neq 0}$: By the same argument as in the first case, we obtain that *every* simple module is (properly) induced, i.e.,

$$\operatorname{Irr}(L,\chi) = \{\operatorname{Ind}_I^L(F_\omega,\chi) \mid \omega(z)^p - \omega(z) = \chi(z)^p, \omega(e_-) = \chi(e_-)\}.$$

In particular, $|\operatorname{Irr}(L,\chi)| = p$ and by a dimension counting argument (see the proof of Theorem 5.2) we conclude that $u(L,\chi)$ is semisimple.

(ii) Consider the three-dimensional supersolvable restricted Lie algebra

$$L := \mathbb{F}t \oplus \mathbb{F}e \oplus \mathbb{F}z,$$

$$[t,e] = e + z,\ t^{[p]} = t,\ e^{[p]} = 0,\ z^{[p]} = z.$$

Then L has two different $\operatorname{Aut}_p(L)$-orbits so that the corresponding isomorphism classes of simple L-modules can be described by using the abelian p-ideal $I := \mathbb{F}e \oplus \mathbb{F}z$ as follows:

$\underline{\chi(e+z)^p = \chi(e)}$: Similarly to (i), [SF, Corollary V.7.6(2)] yields

$$\begin{aligned}\operatorname{Irr}(L,\chi) = &\{F_\Gamma \mid \Gamma \in L^*, \Gamma(t) \in \mathbb{F}_p, \Gamma(e) = \chi(e), \Gamma(e+z) = 0\}\\ &\cup \{\operatorname{Ind}_I^L(F_\gamma,\chi) \mid \gamma(e) = \chi(e), \gamma(z)^p - \gamma(z) = \chi(z)^p, \gamma(e+z) \neq 0\},\end{aligned}$$

i.e., $|\operatorname{Irr}(L,\chi)| = 2p-1$.

$\underline{\chi(e+z)^p \neq \chi(e)}$: Then every simple object in $\operatorname{Mod}(L,\chi)$ is (properly) induced by I, i.e.,

$$\operatorname{Irr}(L,\chi) = \{\operatorname{Ind}_I^L(F_\gamma,\chi) \mid \gamma(e) = \chi(e), \gamma(z)^p - \gamma(z) = \chi(z)^p\}.$$

Hence as in (i) we see that $u(L,\chi)$ is semisimple.

(iii) Consider the three-dimensional (restricted) simple Lie algebra

$$L := \mathfrak{sl}_2(\mathbb{F}) = \mathbb{F}e_+ \oplus \mathbb{F}h \oplus \mathbb{F}e_-,$$

$$[h, e_\pm] = \pm 2 \cdot e_\pm,\ [e_+, e_-] = h,\ h^{[p]} = h,\ e_\pm^{[p]} = 0.$$

Assume that $\operatorname{char}(\mathbb{F}) > 2$. (Note that otherwise $\mathfrak{sl}_2(\mathbb{F})$ is a three-dimensional Heisenberg algebra.) Let $B := \mathbb{F}h \oplus \mathbb{F}e_-$ denote a (standard) *Borel subalgebra* and $c := \frac{1}{2} \cdot h^2 + e_+e_- + e_-e_+$ the *Casimir element* of L. Most of the following is due to A.N. Rudakov and I.R. Shafarevic [RS] (see also [SF, pp. 208/209] and [FP5, §2]). There are three different coadjoint orbits of characters for L, namely the *restricted* orbit containing only the zero character and the so-called *regular nilpotent* resp. *regular semisimple* orbits for which a(ny) extension of the corresponding character of $O(L)$ to a central character of $U(L)$ annihilates resp. does not annihilate the Casimir element. The corresponding (isomorphism classes of) simple L-modules can be described as follows.

$\underline{\chi = 0}$: This case resembles in some respect the non-modular case. There are p isomorphism classes of simple objects in $\mathrm{Mod}(L, 0)$ distinguished by their *highest weights* $\lambda \in \mathbb{F}_p$. They can be represented as the unique simple quotient $S(\lambda)$ of the *restricted Verma modules* $V(\lambda, 0) := \mathrm{Ind}_B^L(F_\lambda, 0)$, where h (resp. e_-) act by multiplication with λ (resp. trivially) on the one-dimensional B-module F_λ.

If $\chi \neq 0$, conjugation by a suitable p-automorphism of L (see [SF, p. 208]) shows that we can always assume $\chi(e_-) = 0$. If $\hat{\chi}$ denotes a(ny) central character of $U(L)$ extending the character of $O(L)$ corresponding to χ, we obtain:

$\underline{\chi \neq 0 \text{ and } \hat{\chi}(c) = 0}$: There are $\frac{p+1}{2}$ isomorphism classes of simple objects in $\mathrm{Mod}(L, \chi)$ represented by χ-*reduced Verma modules* $V(\lambda, \chi) := \mathrm{Ind}_B^L(F_\lambda, \chi)$, where λ is a root of the separable polynomial $x^p - x - \chi(h)^p \cdot 1$. Note that $V(\lambda', \chi) \cong V(\lambda, \chi)$ if and only if $\lambda' + \lambda = p - 2$.

$\underline{\hat{\chi}(c) \neq 0}$: There are p isomorphism classes of simple objects in $\mathrm{Mod}(L, \chi)$ represented by reduced Verma modules as for regular nilpotent characters.

Most important for the structure of L are the *restricted* L-modules (i.e., L-modules with character 0), e.g. since the (co)adjoint module is restricted. Note that semisimple elements of L act semisimply on every $u(L, \chi)$-module (see Lemma 3.2), but p-nilpotent elements of L act *in general* only nilpotently on *restricted* L-modules (see Examples (i)–(iii) above).

In order to finish the proof of Theorem 2.2, we introduce the so-called *restricted cohomology*. Since the one-dimensional trivial L-module $\mathbb{F}$ is restricted, we can define the restricted cohomology spaces of L with coefficients in a restricted L-module M by means of

$$H_*^n(L, M) := \mathrm{Ext}_{u(L,0)}^n(\mathbb{F}, M) \qquad \forall\, n \geq 0.$$

G.P. Hochschild [Ho1] showed that there is a five-term exact sequence relating ordinary and restricted cohomology (see also Theorem 3.1(c)):

$$\begin{aligned}(2) \qquad 0 \longrightarrow H_*^1(L, M) &\longrightarrow H^1(L, M) \longrightarrow \mathrm{Hom}(L, M^L) \\ &\longrightarrow H_*^2(L, M) \longrightarrow H^2(L, M).\end{aligned}$$

The proof of the next result was inspired by ideas in [Fa1, Fe1, Fe2] (see also [Fa5]).

Theorem 2.5. [Ho2] *If every finite dimensional restricted L-module is semisimple, then L is a torus.*

Proof. As in the discussion after the proof of Theorem 2.1 for the ordinary cohomology, we see that $H^1_*(L,M)=0$ vanishes for every finite dimensional restricted L-module M. Let $0\to K\to P\to \mathbb{F}\to 0$ be a finite dimensional projective presentation of $\mathbb{F}$ over $u(L,0)$. If we apply the long exact cohomology sequence to the corresponding dual short exact sequence, we obtain the isomorphism $H^2_*(L,\mathbb{F})\cong H^1_*(L,K^*)$. The vanishing of the latter space follows from our assumption. Consider now (2) for $M:=\mathbb{F}$. We obtain the exactness of $0\to L/[L,L]\to \operatorname{Hom}(L,\mathbb{F})\to 0$, i.e., $[L,L]=0$ and L is abelian. Let x be an arbitrary element of L and put $X:=\langle x\rangle_p$. From Shapiro's Lemma for restricted cohomology, we deduce $H^1_*(X,\mathbb{F})\cong H^1_*(L,\operatorname{Hom}_X(u(L,0),\mathbb{F}))=0$, because $u(L,0)$ is finite dimensional. But $H^1_*(X,\mathbb{F})\cong X/\langle X^{[p]}\rangle$, i.e.,

$$x\in\langle X^{[p]}\rangle\subseteq\sum_{n\geq 1}\mathbb{F}x^{[p]^n}=\langle x^{[p]}\rangle_p,$$

and x is semisimple. □

Remark. In fact, the converse of Theorem 2.5 is also true (cf. [Ho2]), which will be shown in Lemma 3.2.

In §6 and §7 we will generalize Theorem 2.5 considerably. In order to do this, we will replace vanishing conditions by statements about the *growth* of the graded vector space $\operatorname{Ext}^{\bullet}_{u(L,\chi)}(M,N)$ (cf. §6). Our main goal in §7 will be to decide under which conditions on L and/or χ a classification of all (isomorphism classes of) finite dimensional indecomposable $u(L,\chi)$-modules is possible.

§3. Complete Restricted Cohomology

For every restricted Lie algebra L and every restricted L-module M there is a spectral sequence

$$\operatorname{Hom}_{\mathbb{F}}(\Lambda^i(L),H^j_*(L,M))\Longrightarrow H^n(L,M)$$

(cf. [FP5, §5] or [Fa5, Theorem 4.1]) which implies the second half of the next result. In the special case $n=1$, this is just Hochschild's five-term exact sequence (2) in §2.

Theorem 3.1. *Let L be a restricted Lie algebra and M be an arbitrary L-module. Then the following statements hold:*

(a) [Fa2] *If M is finite dimensional and does not contain a non-zero restricted submodule, then $H^n(L, M) = 0$ for every integer $n \geq 0$.*

(b) *If M is restricted, then we have for every integer $n \geq 0$:*

$$H^j(L, M) = 0 \quad \forall\, j \leq n \text{ if and only if } H^j_*(L, M) = 0 \quad \forall\, j \leq n.$$

(c) *If $H^j(L, M) = 0 \quad \forall\, j \leq n-2$, then $H^{n-1}(L, M) \cong H^{n-1}_*(L, M)$ and the following sequence is exact:*

$$0 \longrightarrow H^n_*(L, M) \longrightarrow H^n(L, M) \longrightarrow \operatorname{Hom}(L, H^{n-1}_*(L, M)) \longrightarrow H^{n+1}_*(L, M) \longrightarrow H^{n+1}(L, M).$$

Proof. In order to prove (a), we consider the (non-unitary) subalgebra C of $C(\mathfrak{gl}(M))$ generated by $\{(x)^p_M - (x)^{[p]}_M \mid x \in L\}$. Note that C is just the image of $O(L) - \{1\}$ (cf. the proof of Theorem 2.3) under the corresponding representation $(?)_M$ of M. Since M is finite dimensional, there is a Fitting decomposition $M = M_0(C) \oplus M_1(C)$ with respect to C. By assumption, the largest restricted L-submodule $M^C := \{m \in M \mid c \cdot m = 0 \quad \forall\, c \in C\}$ of M is zero and thus $M_0(C)^C = 0$. Then Engel's Theorem implies $M_0(C) = 0$, and we conclude $M = M_1(C)$. Hence $M = CM_1(C) = CM = C(U(L)^+)M$, where $U(L)^+$ denotes the augmentation ideal of the (ordinary) universal enveloping algebra $U(L)$ of L, and the assertion follows from a general cohomological vanishing result of R. Farnsteiner [Fa2, Corollary 5.2(1)]. Finally, (b) and (c) are special cases of the spectral sequence mentioned above. □

By virtue of Theorem 3.1, it is in many cases sufficient to consider *restricted* cohomology. Since the restricted universal enveloping algebra $u(L, 0)$ of a *finite dimensional* restricted Lie algebra L is a Frobenius algebra, it is possible (and indeed more natural) to introduce the so-called *complete restricted cohomology* (cf. [Par, Fe1, Fe2]). By the same token, one can introduce complete extension functors for reduced universal enveloping algebras, from which the complete restricted cohomology can be derived as usual (see §2).

Let X be an arbitrary $u(L, \chi)$-module. Since the χ-reduced universal enveloping algebra $u(L, \chi)$ is Frobenius, every member I^n of an injective resolution

$$0 \longrightarrow X \xrightarrow{\eta} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \longrightarrow \cdots$$

of X over $u(L,\chi)$ is projective. Let

$$\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} X \longrightarrow 0$$

be a projective resolution of X over $u(L,\chi)$ and put $P_{-n} := I^{n-1}$, $d_{-n} := d^{n-1}$ for every integer $n \geq 1$ and set $d_0 := \eta \circ \varepsilon$. Splicing the two resolutions together, one obtains the following acyclic chain complex of projective $u(L,\chi)$-modules:

$$\begin{array}{ccccccccccc} \cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} & P_0 & & \xrightarrow{d_0} & & P_{-1} & \xrightarrow{d_{-1}} P_{-2} \xrightarrow{d_{-2}} P_{-3} \to \cdots \\ & & \varepsilon \searrow & & \nearrow \eta & & \\ & & & X & & & \\ & & \nearrow & & \searrow & & \\ & 0 & & & & 0 & \end{array}$$

This diagram is referred to as a *complete projective resolution* of X over $u(L,\chi)$. Applying $\mathrm{Hom}_L(?,Y)$ for any $u(L,\chi)$-module Y and taking homology of the corresponding chain complex yields the *complete extension functors* $\hat{\mathrm{Ext}}^n_{u(L,\chi)}(X,Y)$ *with coefficients in* X *and* Y. Since the one-dimensional trivial L-module $\mathbb{F}$ has character 0, one can introduce the *complete restricted cohomology* with coefficients in a restricted L-module M by

$$\hat{H}^n_*(L,M) := \hat{\mathrm{Ext}}^n_{u(L,0)}(\mathbb{F},M) \qquad \forall\, n \in \mathbb{Z}.$$

This resembles the well-known Tate cohomology for finite groups and contains the more familiar Hochschild cohomology (cf. §2) for positive integers (cf. [Par, Fe1, Fe2]), but note that in contrast to the group algebra, $u(L,0)$ is in general *not* symmetric (see [Schue1, Hum2]):

$$\hat{H}^n_*(L,M) \cong \begin{cases} H^n_*(L,M) & \text{for} \quad n > 0 \\ M^L/\,\mathrm{Im}(s)_M & \text{for} \quad n = 0 \\ \mathrm{Ker}(s)_M/\nu_L^{-1}(\mathrm{Ker}(\varepsilon))M & \text{for} \quad n = -1 \\ \mathrm{Tor}^{u(L,0)}_{-n-1}(\mathbb{F}, {}_{\nu_L^{-1}}M) & \text{for} \quad n < -1 \end{cases}$$

Here ε denotes the *augmentation map* of $u(L,0)$, s the so-called *trace element* (defined by $\varepsilon = s \cdot \lambda$, where λ is the image of the identity element $1 \in u(L,0)$ under the *Frobenius isomorphism* $u(L,0) \cong u(L,0)^*$), ν_L the *Nakayama automorphism* of $u(L,0)$ induced by $x \mapsto x - \mathrm{tr}(\mathrm{ad}_L\, x)\cdot 1$ $(x \in L)$ and ${}_{\nu_L^{-1}}M$ the $u(L,0)$-module with twisted action $u \cdot_{\nu_L^{-1}} m := \nu_L^{-1}(u)\cdot m$ for $u \in u(L,0)$ and $m \in M$.

For later applications, we state two important features of complete restricted cohomology without proof (cf. [Fe2, Theorem 2.4 and Theorem 2.5]):

Reduction Theorem. *Let L be a finite dimensional restricted Lie algebra. Then for arbitrary $u(L,\chi)$-modules M, N we have natural isomorphisms*

$$\hat{\mathrm{Ext}}^n_{u(L,\chi)}(M,N) \cong \hat{H}^n_*(L, \mathrm{Hom}_{\mathbb{F}}(M,N)) \qquad \forall\, n \in \mathbb{Z}. \quad \square$$

Degree Shifting. *Let L be a finite dimensional restricted Lie algebra. Then for every integer r there exists a restricted L-module $M^{(r)}$ such that*

$$\hat{H}^n_*(L,M) \cong \hat{H}^{n-r}_*(L, M^{(r)}) \qquad \forall\, n \in \mathbb{Z}. \quad \square$$

As a first consequence we obtain the converse of Theorem 2.5:

Lemma 3.2. *Let L be a finite dimensional torus. Then $\hat{H}^n_*(L,M)$ vanishes for every restricted L-module M and every integer n. In particular, $u(L,\chi)$ is semisimple for any $\chi \in L^*$.*

Proof. By degree shifting, it is enough to show the assertion only for one integer, e.g. $n = 0$. Since $u(L,0)$ is finite dimensional, we can moreover assume (by virtue of the long exact sequence for complete restricted cohomology) that M is simple. If $M \not\cong \mathbb{F}$, we obtain $\hat{H}^0_*(L,M) = 0$ because $M^L = 0$. According to an old result of N. Jacobson (cf. [SF, Theorem II.3.6(1)]), L has a toral basis $\{t_1, \ldots, t_r\}$ (i.e., $t_i^{[p]} = t_i \;\forall\, 1 \le i \le r$) over the algebraic closure $\overline{\mathbb{F}}$ of $\mathbb{F}$. In this case we can easily compute the trace element $s = \prod_{i=1}^r (t_i^{p-1} - 1)$ and therefore conclude that $\varepsilon(s) = -1 \neq 0$. Hence

$$\hat{H}^0_*(L,\mathbb{F}) \otimes_{\mathbb{F}} \overline{\mathbb{F}} \cong \hat{H}^0_*(L \otimes_{\mathbb{F}} \overline{\mathbb{F}}, \overline{\mathbb{F}}) \cong \overline{\mathbb{F}}/(s \cdot \overline{\mathbb{F}}) = 0.$$

The second statement follows immediately from the first statement and the Reduction Theorem. $\square$

Remark. In fact, the above proof is a homological adaptation of Hochschild's original proof for the restricted case in [Ho2].

A restricted L-module M is called *cohomologically trivial* if the complete cohomology $\hat{H}^n_*(K,M)$ vanishes for every integer n and every p-subalgebra K of L.

Lemma 3.3. [Fe2] *Let L be a finite dimensional p-nilpotent restricted Lie algebra and M be an arbitrary restricted L-module. Then the following statements are equivalent:*

(a) *There is an integer n_0 such that $\hat{H}_*^{n_0}(L, M)$ vanishes.*
(b) *M is cohomologically trivial.*
(c) *M is projective.*
(d) *M is free.*

Proof. It is clear that the implications (d)$\Longrightarrow$(c) $\Longrightarrow$(b)$\Longrightarrow$(a) hold. Note that the augmentation ideal of the restricted universal enveloping algebra $u(L, 0)$ of a finite dimensional p-nilpotent restricted Lie algebra L is nilpotent, i.e., $u(L, 0)$ is a local algebra. By a standard argument (cf. Step 1 in the proof of [Fe2, Proposition 5.1]), it is therefore possible to show that $\mathrm{Tor}_1^{u(L,0)}(\mathbb{F}, M) = 0$ implies the freeness of M.[2]

Recall that $\hat{H}_*^{-2}(L, M)$ and $\mathrm{Tor}_1^{u(L,0)}(\mathbb{F}, M)$ can be identified in a natural way because $u(L, 0)$ is symmetric (i.e., $\nu_L = \mathrm{id}_{u(L,0)}$). In order to show the implication (a)$\Longrightarrow$(d), it is therefore enough to show that $\hat{H}_*^{n_0}(L, M) = 0$ implies $\hat{H}_*^{-2}(L, M) = 0$. By degree shifting there exists a restricted L-module M_0 such that

$$\hat{H}_*^{n}(L, M_0) \cong \hat{H}_*^{n+n_0+2}(L, M) \qquad \forall\, n \in \mathbb{Z}. \tag{3}$$

Hence our hypothesis yields the vanishing of $\hat{H}_*^{-2}(L, M_0)$. Then the above argument shows that M_0 is free. Consequently, we obtain $\hat{H}_*^{\bullet}(L, M_0) = 0$ and thus the vanishing of $\hat{H}_*^{-2}(L, M)$ follows directly from (3). □

Remark. Using Lemma 3.3 we will prove in Theorem 6.12 that statements (b) and (c) are equivalent for every *finite dimensional* restricted module over *any* finite dimensional restricted Lie algebra. Therefore it is possible to show that the equivalence of statements (c) and (d) (even for finite dimensional modules) characterizes finite dimensional p-nilpotent restricted Lie algebras (cf. [Fe2, Proposition 5.4]).

The main vanishing result for complete restricted cohomology of nilpotent restricted Lie algebras can be formulated as follows:

[2]Note that this is the finite dimensional version of a classical result of I. Kaplansky which says that every projective module over a local algebra is free.

Theorem 3.4. [Fe2] *Let L be a finite dimensional nilpotent restricted Lie algebra and M be an arbitrary restricted L-module. Suppose there is an integer n_0 such that $\hat{H}_*^{n_0}(L, M)$ vanishes, then the complete restricted cohomology $\hat{H}_*^n(L, M)$ vanishes for every integer n.*

Proof. Note that $T_0 := \mathrm{Tor}_p(L) = \{x \in L \mid x \text{ is semisimple}\}$ is the unique maximal torus of L. Then L/T_0 is p-nilpotent. By virtue of Lemma 3.2 and the Hochschild-Serre spectral sequence (cf. [Fe2, Proposition 3.9] for an elementary inductive argument), there is an isomorphism $\hat{H}_*^n(L, M) \cong \hat{H}_*^n(L/T_0, M^{T_0})$ for every integer n, and thus the assertion is an immediate consequence of Lemma 3.3. □

In particular, $\hat{H}_*^n(L, S)$ vanishes for every simple restricted L-module $S \not\cong \mathbb{F}$ and every integer n. Compare this with Lemma 3.2 which says that for a *torus* L the complete cohomology $\hat{H}_*^n(L, S)$ vanishes for *every* simple restricted L-module S and every integer n. Theorem 3.4 in conjunction with the long exact cohomology sequence and (the proof of) Theorem 2.5 yields the following non-vanishing result for nilpotent restricted Lie algebras:

Corollary 3.5. [Fa5, Fe2] *Let L be a finite dimensional nilpotent restricted Lie algebra and suppose L is not a torus. Then $\hat{H}_*^n(L, \mathbb{F}) \neq 0$ for every integer n.* □

Remark. Corollary 3.5 can be used to show that every finite dimensional non-toral nilpotent restricted Lie algebra has a *restricted* outer derivation.

Next, we give a cohomological characterization of nilpotent restricted Lie algebras that involves complete restricted cohomology with simple modules as coefficients.

Theorem 3.6. [Fe2] *Let L be a finite dimensional restricted Lie algebra. Then L is nilpotent if and only if $\hat{H}_*^1(L, S)$ vanishes for every simple restricted L-module $S \not\cong \mathbb{F}$.*

Proof. By virtue of Theorem 3.4, it is enough to prove that the condition is sufficient for the nilpotency of L. From $M^L = H_*^1(L, M) = 0$ for every non-trivial simple restricted L-module M one deduces on account of Hochschild's five-term exact sequence (see (2) in §2 or Theorem 3.1(c)) and Theorem 3.1(a) the vanishing of $H^1(L, M)$ for every non-trivial simple L-module M. Then an analogous cohomological characterization for *ordinary* modular nilpotent Lie algebras (cf. [Dzhu4, Theorem in §4]) concludes the proof of the theorem. □

Remark. In the non-modular case a classical cohomological vanishing theorem of J.H.C. Whitehead (cf. also [Fa3, Theorem 3.1]) implies that these conditions are equivalent to L being the direct product of a semisimple and a nilpotent Lie algebra. (Look at the proof of [Ba2, Theorem 3]!)

A Lie algebra is called *supersolvable* if there is a (descending) chain of ideals such that all factors are one-dimensional. One readily verifies that every subalgebra and every factor algebra of a supersolvable Lie algebra is supersolvable. A finite dimensional Lie algebra L over an algebraically closed field is supersolvable if and only if the derived subalgebra $[L, L]$ is nilpotent (cf. [Dzhu2, Theorem 3]). In fact, L is isomorphic to $T \oplus \mathrm{Nil}(L)$, where T is a torus in L and $\mathrm{Nil}(L)$ denotes the largest nilpotent ideal of L. The following theorem is a slight generalization of a restricted analogue of a result due to D.W. Barnes [Ba1]:

Theorem 3.7. *Let L be a finite dimensional supersolvable restricted Lie algebra and M be a finite dimensional[3] restricted L-module. If M does not contain a one-dimensional submodule, then the complete restricted cohomology $\hat{H}^n_*(L, M)$ vanishes for every integer n.*

Proof. Since the hypothesis and the conclusion are independent of the ground field $\mathbb{F}$, we can assume that $\mathbb{F}$ is algebraically closed. Set $N := \mathrm{Nil}(L)$. Because N is an ideal in L we have $\mathrm{Tor}_p(N) = \mathrm{Tor}_p(L) =: T_0$ which is the unique maximal torus of N. Hence we obtain from the decomposition $L = T \oplus N$ that L/T_0 is strongly solvable (i.e., a semidirect product of a torus and a p-nilpotent ideal).

By virtue of Lemma 3.2 and the Hochschild-Serre spectral sequence (or an elementary inductive argument as in [Fe2, Proposition 3.9]), there are isomorphisms

$$\hat{H}^n_*(L, M) \cong \hat{H}^n_*(L/T_0, M^{T_0}) \qquad \forall\, n \in \mathbb{Z}.$$

Since every simple restricted module of a strongly solvable restricted Lie algebra is one-dimensional, our hypothesis yields

$$\mathrm{Soc}_{L/T_0}(M^{T_0}) = 0, \text{ i.e., } M^{T_0} = 0,$$

and thus the assertion. $\square$

In particular, $\hat{H}^n_*(L, S)$ vanishes for every simple restricted L-module S that is not one-dimensional. In order to prove a non-vanishing result for

[3] It would be enough to assume that M is *artinian*.

supersolvable restricted Lie algebras which is analogous to Corollary 3.5, we introduce the following subgroup of (the additive group of) L^*:

$$G^L := \{\gamma \in L^* \mid \gamma([L,L]) = 0, \gamma(x^{[p]}) = \gamma(x)^p \ \forall\, x \in L\}$$

(cf. [SF, p. 242]). As a consequence of the Jordan-Chevalley-Seligman decomposition (cf. Theorem 1.1), we obtain that G^L is finite (cf. [SF, Proposition V.8.8(1)]). For every $\gamma \in G^L$ the one-dimensional vector space $F_\gamma := \mathbb{F}1_\gamma$ is a restricted L-module via $x \cdot 1_\gamma := \gamma(x)1_\gamma \ \forall\, x \in L$ and conversely, every one-dimensional restricted L-module occurs in this way. Hence we obtain from Theorem 3.7 and the long exact cohomology sequence in conjunction with degree shifting and (the proof of) Theorem 2.5:

Corollary 3.8. [Fa5, Fe2] *Let L be a finite dimensional supersolvable restricted Lie algebra and suppose L is not a torus. Then*

$$\hat{H}^n_*(L, \bigoplus_{\gamma \in G^L} F_\gamma) \neq 0 \quad \forall\, n \in \mathbb{Z}. \quad \square$$

Remark. By virtue of [Fe5, Lemma 2], G^L can be replaced by G^L_0 (see also the proof of Theorem 4.2(b)).

As an analogue to Theorem 3.4, we have the following cohomological characterization of supersolvable restricted Lie algebras:

Theorem 3.9. [Fe2, Fe3] *Let L be a finite dimensional restricted Lie algebra. Then L is supersolvable if and only if $\hat{H}^1_*(L,S)$ vanishes for every simple restricted L-module S that is not one-dimensional.*

Proof. According to Theorem 3.7, it is enough to prove that the condition is sufficient for the supersolvability of L. From $M^L = H^1_*(L,M) = 0$ we deduce by means of Hochschild's five-term exact sequence (see (2) in §2 or Theorem 3.1(c)) the vanishing of $H^1(L,M)$ for every simple restricted L-module M that is not one-dimensional. By virtue of Theorem 3.1(a) and [Ba2, Theorem 4], we obtain the assertion. $\square$

Remark. It is also possible to give cohomological characterizations of finite dimensional (strongly) solvable restricted Lie algebras in the spirit of [Sta, Theorem A] (cf. [Fe2, Proposition 5.8 and Proposition 5.12]). For a unifying approach towards a cohomological characterization of certain classes of finite dimensional solvable modular Lie algebras we refer the reader to [Fe3].

Since such satisfactory vanishing results as Theorem 3.4 are not true for more general classes of restricted Lie algebras (see e.g. Theorem 3.7) and the actual computation of restricted cohomology spaces is in most cases a very difficult (or even impossible) task, it is quite natural to introduce some cohomological invariants of L that are connected with both aspects mentioned. Define

$$\hat{v}_p(L) := \min\{\dim_{\mathbb{F}} M \mid 0 \neq M \text{ such that } \hat{H}^n_*(L, M) = 0 \ \forall\ n \in \mathbb{Z}\},$$

$$t_p(L) := \min\{\dim_{\mathbb{F}} M \mid 0 \neq M \text{ is cohomologically trivial}\}.$$

R. Farnsteiner introduced in [Fa4, Fa5] a similar invariant, namely

$$v_p(L) := \min\{\dim_{\mathbb{F}} M \mid 0 \neq M \text{ such that } H^n_*(L, M) = 0 \ \forall\ n > 0\}.$$

By virtue of Theorem 3.4, we have $\hat{v}_p(L) = v_p(L)$ for nilpotent restricted Lie algebras L.[4] Let $\operatorname{rk}(L)$ denote the maximal dimension of a torus of L. Then the following inequalities hold:

$$1 \leq v_p(L) \leq \hat{v}_p(L) \leq t_p(L) \leq p^{\dim_{\mathbb{F}} L - \operatorname{rk}(L)}. \tag{4}$$

The first three inequalities are obvious consequences of the definitions and the last one follows from Shapiro's Lemma for restricted cohomology in conjunction with Lemma 3.2. In particular, we obtain for any torus that $v_p(L) = \hat{v}_p(L) = t_p(L) = 1$ (see also Lemma 3.2). Moreover, Theorem 6.12 (cf. also the remark after Lemma 3.3) implies that

$$t_p(L) = \min\{\dim_{\mathbb{F}} M \mid 0 \neq M \text{ is a projective } u(L, 0)\text{-module}\}. \tag{5}$$

Since cohomology is additive, the problem of determining $t_p(L)$ is thus the same as that of determining the minimal dimensions of projective indecomposable $u(L, 0)$-modules (cf. [Hum3, Problem 5]). As an easy consequence of (5) we obtain that tori are characterized by the property $t_p(L) = 1$:

Proposition 3.10. *A finite dimensional restricted Lie algebra L is a torus if and only if $t_p(L) = 1$.*

Proof. Assume that $t_p(L) = 1$ and S is a one-dimensional projective $u(L, 0)$-module. Since the tensor product of a projective $u(L, 0)$-module with any $u(L, 0)$-module is always projective (cf. [Par, Lemma II.2.5]), we conclude that the one-dimensional trivial L-module $\mathbb{F} \cong S \otimes_{\mathbb{F}} S^*$ is projective. From the Reduction Theorem in conjunction with Theorem 2.5 we finally deduce that L is a torus. Since the other implication has already been observed, the proof is complete. □

[4] As a consequence of the results in [CNP] this is true in general.

Examples.

(i) For any *p-nilpotent* restricted Lie algebra we have

$$t_p(L) = \hat{v}_p(L) = v_p(L) = p^{\dim_{\mathbb{F}} L}$$

(cf. Lemma 3.3). But the equality $t_p(L) = \hat{v}_p(L)$ is not in general true as the non-abelian two-dimensional Lie algebra shows. In this case we have $\hat{v}_p(L) = 1 < p = t_p(L)$ for $p > 2$. (Of course, equality holds for $p = 2$ because of the lack of enough simple modules.)

(ii) If L is *strongly solvable*, then by virtue of (5), we can read off from the well-known description of the projective indecomposable $u(L,0)$-modules as induced modules (see the proof of Theorem 4.7):

$$t_p(L) = p^{\dim_{\mathbb{F}} L - \mathrm{rk}(L)}.$$

But in general not much is known about $\hat{v}_p(L)$.

(iii) Let L be a (restricted) *simple Lie algebra of classical type*. (5) in conjunction with Humphreys' dimension formula for projective indecomposable $u(L,0)$-modules [Hum1, Theorem 4.5] yields:

$$t_p(L) = p^{\frac{1}{2}\cdot[\dim_{\mathbb{F}} L - \mathrm{rk}(L)]}.$$

In fact, this can also be derived directly from an analogue of a result of L.E. Dickson for finite dimensional modular group algebras (cf. [Hum1, Proposition 4.3] or [Fe1, Satz II.3.1]) and the well-known dimension of the (projective) *Steinberg module*.

Consider $\mathfrak{sl}_2(\mathbb{F})$ for $p > 2$. If $p > 3$, then we obtain $\hat{v}_p(\mathfrak{sl}_2(\mathbb{F})) = 2$ since the two-dimensional natural $\mathfrak{sl}_2(\mathbb{F})$-module does not belong to the principal block of $u(\mathfrak{sl}_2(\mathbb{F}), 0)$ (see §4). If $p = 3$, then the adjoint module (= Steinberg module) is the only simple module with vanishing (complete) restricted cohomology and therefore we obtain in this case $\hat{v}_p(\mathfrak{sl}_2(\mathbb{F})) = 3 = t_p(\mathfrak{sl}_2(\mathbb{F}))$. In fact, one has explicit dimension formulas for the restricted cohomology of all indecomposable restricted $\mathfrak{sl}_2(\mathbb{F})$-modules (see [Fi, Chapter 3]). For $L \neq \mathfrak{sl}_2(\mathbb{F})$ there is only some information on 1-cohomology (see [Sul2, Pfe, Jan5]).

(iv) For a *restricted simple Lie algebra of Cartan type* over any *algebraically closed* field $\mathbb{F}$ of characteristic $p > 2$, one can read off from the work of R.R. Holmes and D.K. Nakano [Na1, HN2]:

$$t_p(W_1(\mathbb{F})) = p^{p-2},\ t_p(W_2(\mathbb{F})) = p^{2p^2-5},\ t_p(H_2(\mathbb{F})) = p^{p^2-6}.$$

About $\hat{v}_p$ nothing is known except for $W_1(\mathbb{F})$, $p = 5$ or $p = 7$ (see [Na4]) and $W_2(\mathbb{F})$, $p = 2$. E.g. in the latter case we have

$$\hat{v}_p(W_2(\mathbb{F})) = 8 = t_p(W_2(\mathbb{F}))$$

since the adjoint module is a projective $u(W_2(\mathbb{F}), 0)$-module which is very similar to $\mathfrak{sl}_2(\mathbb{F})$, $p = 3$ (cf. §4).[5]

Next, we prove an analogue of Proposition 3.10 for $\hat{v}_p$ in the nilpotent case. The equivalence of (a) and (b) is due to R. Farnsteiner [Fa5].

Theorem 3.11. *Let L be a non-zero finite dimensional nilpotent Lie algebra over an algebraically closed field. Then the following statements are equivalent:*

(a) $\hat{v}_p(L) = 1$.
(b) $\operatorname{Tor}_p(L) \not\subseteq \langle [L, L] \rangle_p$.
(c) $G^L \neq 0$.

Proof. (a)$\Longrightarrow$(b): Assume that $\hat{v}_p(L) = 1$ and F_γ is a one-dimensional restricted L-module such that $\hat{H}^n_*(L, F_\gamma) = 0$ for every $n \in \mathbb{Z}$. If $\gamma = 0$, then Corollary 3.5 implies that L is a torus, i.e., $\operatorname{Tor}_p(L) = L \not\subseteq 0 = \langle [L, L] \rangle_p$. If $\gamma \neq 0$, then we have $\gamma_{|\operatorname{Tor}_p(L)} \neq 0$. Otherwise F_γ is a trivial $L/\operatorname{Tor}_p(L)$-module (because $L/\operatorname{Tor}_p(L)$ is p-nilpotent) and as in the proof of Theorem 3.4 we obtain by applying again Corollary 3.5:

$$0 \neq \hat{H}^n_*(L/\operatorname{Tor}_p(L), F_\gamma) \cong \hat{H}^n_*(L, F_\gamma) = 0,$$

a contradiction. Since F_γ is restricted, we have $\gamma(\langle [L, L] \rangle_p) = 0$ and hence the assertion.

(b)$\Longrightarrow$(c): Set $A := L/\langle [L, L] \rangle_p$. Since $\langle L, L \rangle_p \subseteq \operatorname{Ker}(\gamma) = \operatorname{Ann}_L(F_\gamma)$, F_γ is a restricted A-module for any $\gamma \in G^L$. This establishes a one-to-one correspondence between one-dimensional restricted L-modules and one-dimensional restricted A-modules, i.e., $G^L \cong G^A$. By hypothesis, A is *not* p-nilpotent and therefore $G^A \neq 0$ is an immediate consequence of an old result of Jacobson establishing the existence of a toral basis (cf. [SF, Theorem II.3.6(1)]).

[5] For the p-dimensional Witt algebra the 1- and 2-cohomology is known [Dzhu1, Dzhu3] and for the other restricted simple Lie algebras of Cartan type there is some information on 1-cohomology [CS, Chiu1] which can be improved by using [Jan5] (cf. also [Fe1] especially for small characteristics).

Finally, the remaining implication (c)$\Longrightarrow$(a) follows from Theorem 3.4. □

Remark. Condition (c) in Theorem 3.11 just says that there exists a nontrivial one-dimensional restricted L-module (cf. Corollary 3.5). By means of Theorem 3.11 it is possible to obtain a better upper bound than in (4) which is attained for nilpotent restricted Lie algebras (see [Fa5, Theorem 3.5(2)]).

Let us finish this section by pointing out the following generalization of Corollary 3.5 to *arbitrary* restricted Lie algebras, which follows from Theorem 6.5 in conjunction with Theorem 2.5:

Theorem 3.12. *Let L be a finite dimensional restricted Lie algebra and suppose L is not a torus. Then there are infinitely many positive and infinitely many negative integers n such that $\hat{H}^n_*(L, \mathbb{F}) \neq 0$.* □

Note that for (restricted) *simple Lie algebras of classical type* or their *Borel subalgebras* the following stronger statement is true (cf. [AJ, FP1, Jan2]):

Conjecture. [6] *Let L be a finite dimensional restricted Lie algebra and suppose L is not a torus. Then $\hat{H}^n_*(L, \mathbb{F}) \neq 0$ for every even integer n.*

§4. Block Structure

Let L denote a finite dimensional restricted Lie algebra over an arbitrary field $\mathbb{F}$. In the following we are interested in the category $\operatorname{mod}(L, \chi)$ of finite dimensional L-modules for an arbitrary character $\chi \in L^*$. In §2, we mentioned that a complete classification of simple objects in $\operatorname{mod}(L, \chi)$ is known only in a few cases. Moreover, this supplies enough information to describe $\operatorname{mod}(L, \chi)$ only if $u(L, \chi)$ is a semisimple algebra. But $u(L, \chi)$ does not always possess this property (cf. Theorem 2.5 and §7). So in the other cases one should try to get (at least) some information on the indecomposable $u(L, \chi)$-modules. Since this is a very hard problem in general (cf. §7), we consider in this section a decomposition of $u(L, \chi)$ into smaller subalgebras such that $\operatorname{mod}(L, \chi)$ decomposes into the corresponding smaller module categories (*block decomposition*):

$$u(L, \chi) = \bigoplus_{j=1}^{b} B_j,$$

[6] Recently this conjecture has been proved by R. Farnsteiner in [Fa10, Corollary 2.3].

where each B_j is an indecomposable two-sided ideal of $u(L,\chi)$. The B_j are called *block ideals* of $u(L,\chi)$. This decomposition is in one-to-one correspondence with a *primitive central idempotent decomposition* of the identity element 1 of $u(L,\chi)$:

$$1=\sum_{j=1}^{b}c_j,$$

where $B_j=u(L,\chi)c_j$ is a finite dimensional associative $\mathbb{F}$-algebra with identity element c_j. The c_j's are called *block idempotents* of $u(L,\chi)$ (cf. [HB, Theorem VII.12.1]). Every indecomposable $u(L,\chi)$-module is a (unitary left) B_j-module for some uniquely determined j, i.e.,

$$\operatorname{mod}(L,\chi)=\bigoplus_{j=1}^{b}\operatorname{mod}B_j.$$

In particular, this induces an equivalence relation "*belonging to a block*" or "*linked*" on the finite set $\operatorname{Irr}(L,\chi)$ of all isomorphism classes of (irreducible or) simple $u(L,\chi)$-modules

$$\operatorname{Irr}(L,\chi)=\bigcup_{j=1}^{b}\mathbb{B}_j,$$

such that the equivalence classes

$$\mathbb{B}_j=\{[S]\in\operatorname{Irr}(L,\chi)\mid c_j\cdot S=S\},$$

the so-called *blocks* or *linkage classes* of $\operatorname{Irr}(L,\chi)$, are in one-to-one correspondence with the set of isomorphism classes of simple B_j-modules. For the convenience of the reader we state some cohomological features of the linkage relation which will be useful in the sequel:

Lemma 4.1. *Let L be a finite dimensional restricted Lie algebra and $\chi\in L^*$. Then the following statements hold:*

(a) *If two $u(L,\chi)$-modules M and N belong to different blocks, then $\operatorname{Ext}^n_{u(L,\chi)}(M,N)$ vanishes for every integer $n\geq 0$.*

(b) *Two simple $u(L,\chi)$-modules M and N belong to the same block if and only if there exists a finite sequence $S_1,...,S_n$ of simple $u(L,\chi)$-modules such that $M=S_1$, $N=S_n$ and $\operatorname{Ext}^1_{u(L,\chi)}(S_j,S_{j+1})\neq 0$ or $\operatorname{Ext}^1_{u(L,\chi)}(S_{j+1},S_j)\neq 0$ for every $1\leq j\leq n-1$.*

Proof. (a) is an immediate consequence of [Fa2, Corollary 4.10] applied to the block idempotents corresponding to M resp. N and (b) is well-known in the literature (see e.g. [Sta, Corollary 1]. □

Remark. One can also show that two simple $u(L,\chi)$-modules M and N belong to the same block if and only if there exists a finite sequence $P_1, ..., P_n$ of projective indecomposable $u(L,\chi)$-modules such that P_1 resp. P_n is the projective cover (see §5) of M resp. N and P_j and P_{j+1} have a common composition factor for every $1 \leq j \leq n-1$ (see e.g. [Fe1]).

There exists a nice combinatorial description of the linkage relation, the so-called *Gabriel quiver* $\vec{Q}(L,\chi)$ of $u(L,\chi)$, i.e., the finite directed graph with the set $\mathrm{Irr}(L,\chi)$ as vertices and $\dim_{\mathbb{F}} \mathrm{Ext}^1_{u(L,\chi)}(M,N)$ arrows from $[M]$ to $[N]$. By virtue of Lemma 4.1(b), it is obvious that the $\mathbb{B}_j$ are in one-to-one correspondence with the *connected components* of the underlying (undirected) graph $Q(L,\chi)$ of $\vec{Q}(L,\chi)$.

Next, we briefly describe how $\vec{Q}(L,\chi)$ determines the so-called *basic algebra* of $u(L,\chi)$ and thus $\mathrm{mod}(L,\chi)$ (up to equivalence). For any finite dimensional associative $\mathbb{F}$-algebra A there exists a (finite dimensional) Morita equivalent algebra A_{basic} such that every simple A_{basic}-module S is one-dimensional over its centralizer $\mathrm{End}_{A_{\mathrm{basic}}}(S)$. It turns out that A_{basic} is uniquely determined up to isomorphism which justifies to call it *the* basic algebra of A (cf. [Ben1, p. 23]). In order to explain the connection between $\vec{Q}(L,\chi)$ and $u(L,\chi)_{\mathrm{basic}}$, we recall that the vector space $\mathbb{F}[\vec{Q}(L,\chi)]$, which is freely generated by all (directed) paths in $\vec{Q}(L,\chi)$, becomes a (not necessarily finite dimensional) associative $\mathbb{F}$-algebra with identity element by the usual concatenation of paths. According to a result of P. Gabriel, $u(L,\chi)_{\mathrm{basic}}$ is a certain factor algebra of $\mathbb{F}[\vec{Q}(L,\chi)]^{\mathrm{op}}$, where $?^{\mathrm{op}}$ denotes the opposite ring structure (cf. [Ben1, Proposition 4.1.7]). Therefore the knowledge of the Gabriel quiver $\vec{Q}(L,\chi)$ of $u(L,\chi)$ is a first step in computing $u(L,\chi)_{\mathrm{basic}}$. Nevertheless, in general, the explicit determination of $\vec{Q}(L,\chi)$ already turns out to be very difficult (e.g. for the p-dimensional Witt algebra ($p > 5$) only parts of the "restricted" quiver $\vec{Q}(W_1, 0)$ are known). It should also be mentioned that the relations of $\mathbb{F}[\vec{Q}(L,\chi)]^{\mathrm{op}}$ are determined by the structure of the projective indecomposable $u(L,\chi)$-modules about whose structure not much is known in general (but see §5). In the sequel, we will consider the following

Problems.

I. Describe the Gabriel quiver $\vec{Q}(L,\chi)$ of $u(L,\chi)$, in particular,

(1) determine the number $b(L,\chi)$ of blocks of $u(L,\chi)$ (i.e., determine the number of connected components of $Q(L,\chi)$),
(2) determine the number $|\mathbb{B}_j|$ of isomorphism classes of simple modules in every block $\mathbb{B}_j$ (i.e., determine the number of vertices of the connected components of $Q(L,\chi)$ corresponding to $\mathbb{B}_j$),
(3) investigate which special properties (e.g. no loops, etc.) of $\vec{Q}(L,\chi)$ (resp. of the connected components $\mathbb{B}_j$) can occur.

II. Determine the algebra structure resp. the representation theory of the block ideals B_j $(1 \le j \le b(L,\chi))$.

The second problem was (at least partially) motivated by the desire to determine all characters χ of L for which the indecomposable $u(L,\chi)$-modules can be classified (up to isomorphism), i.e., to decide for which characters $u(L,\chi)$ is *tame* (cf. §7 for the definition). By the above remarks on the Gabriel quiver, Problem I can be considered as a first step in solving Problem II. It is also of independent interest because the solution of Problem I.2 for all blocks with a fixed character χ would give the number of isomorphism classes of the simple $u(L,\chi)$-modules which is still unknown in many cases (cf. e.g. [Str3] for (non-nilpotent) solvable restricted Lie algebras). In the following, we will describe partial solutions to both problems for *supersolvable* resp. *simple* restricted Lie algebras.

Since a closed formula for the *block invariants* mentioned in Problems I.1 and I.2 seems to be difficult to obtain, we restrict ourself first to the simplest possible cases, i.e.,

- $|\mathbb{B}| = 1$ for every block $\mathbb{B}$ of $u(L,\chi)$ ($\Longrightarrow b(L,\chi) = |\operatorname{Irr}(L,\chi)|$),
- $b(L,\chi) = 1$ ($\Longrightarrow |\mathbb{B}| = |\operatorname{Irr}(L,\chi)|$) (“*block degeneracy*”)

and attempt to discover classes of restricted Lie algebras resp. characters for which these conditions hold. It is well-known from classical ring theory (cf. [Pie, Proposition 6.5a]) that in the first case every block ideal B is *primary*, i.e., $B/\operatorname{Jac}(B)$ is a simple algebra. This condition can be generalized as follows:

(∗) $\dim_{\mathbb{F}} S$ is constant for every simple module in $\mathbb{B}$,

(∗∗) $|\mathbb{B}|$ is constant for every block $\mathbb{B}$ of $u(L,\chi)$.

It turns out that $(*)$ holds for *supersolvable* restricted Lie algebras (see Corollary 4.3), and even in this case it does not seem to be obvious under which conditions $(**)$ will be satisfied (see [Fe5, Example 1]). Recall that by virtue of Lie's Theorem (which fails in the modular situation) every non-modular solvable Lie algebra (over an algebraically closed field) is supersolvable and thus Theorem 4.2 resp. Corollary 4.3 and 4.4 should be considered as a modular analogue of Lie's Theorem.

The block of $u(L,0)$ which contains the one-dimensional trivial L-module is called the *principal block* of L and will be denoted by $\mathbb{B}_0$. Then the following result provides a class of restricted Lie algebras which satisfies $(*)$, and it reduces in this case the problem of finding an upper bound for the number of simple modules in an arbitrary block to the principal block.

Theorem 4.2. [Fe5] *Let L be a finite dimensional supersolvable restricted Lie algebra and $\chi \in L^*$. Then the following statements hold:*

(a) *Every simple module in the principal block of L is one-dimensional.*
(b) *If two simple $u(L,\chi)$-modules M and N belong to the same block, then there exists a simple module S in the principal block of L such that $N \cong S \otimes_{\mathbb{F}} M$.*
(c) *$|\mathbb{B}| \le |\mathbb{B}_0|$ holds for any block $\mathbb{B}$ of $u(L,\chi)$.*

Proof. (a): Let X and Y be simple restricted L-modules such that $\dim_{\mathbb{F}} X \neq 1$ and $\dim_{\mathbb{F}} Y = 1$. Then $\mathrm{Hom}_{\mathbb{F}}(X,Y) \cong X^* \otimes_{\mathbb{F}} Y$ and $\mathrm{Hom}_{\mathbb{F}}(Y,X) \cong Y^* \otimes_{\mathbb{F}} X$ are also simple restricted L-modules such that $\dim_{\mathbb{F}} \mathrm{Hom}_{\mathbb{F}}(X,Y) \neq 1 \neq \dim_{\mathbb{F}} \mathrm{Hom}_{\mathbb{F}}(Y,X)$. From Theorem 3.7 we derive

$$\mathrm{Ext}^n_{u(L,0)}(X,Y) \cong H^n_*(L, \mathrm{Hom}_{\mathbb{F}}(X,Y)) = 0, \text{ and}$$
$$\mathrm{Ext}^n_{u(L,0)}(Y,X) \cong H^n_*(L, \mathrm{Hom}_{\mathbb{F}}(Y,X)) = 0.$$

If we apply this to a simple module S in the principal block of L and to $\mathbb{F}$, we obtain $\dim_{\mathbb{F}} S = 1$ by the transitivity of the linkage relation.

In order to prove (b) and (c), we set $H := \mathrm{Hom}_{\mathbb{F}}(M,N)$. Without loss of generality we may assume $H^n_*(L,H) \cong \mathrm{Ext}^1_{u(L,0)}(M,N) \neq 0$ and by virtue of a slight generalization of Lemma 4.1(a) (see [Fe5, Lemma 2]), H possesses a simple submodule S belonging to the principal block of L. From the adjointness of Hom and $\otimes$ we obtain $\mathrm{Hom}_L(S \otimes_{\mathbb{F}} M, N) \cong \mathrm{Hom}_L(S,H) \neq 0$. According to (a), S is one-dimensional and thus $S \otimes_{\mathbb{F}} M$ is simple. Hence (b) follows from Schur's Lemma and finally, (c) is an immediate consequence of (b). $\square$

Since all composition factors of an indecomposable module belong to the same block, we can immediately derive from Theorem 4.2(a):

Corollary 4.3. [Voigt1, Fe5] *Let L be a finite dimensional supersolvable restricted Lie algebra and $\chi \in L^*$. Then all simple modules in the same block of $u(L,\chi)$ have the same dimension. In particular, all composition factors of a finite dimensional indecomposable $u(L,\chi)$-module have the same dimension.* □

A finite dimensional associative $\mathbb{F}$-algebra A is called *basic* if $A = A_{\mathrm{basic}}$, i.e., $A/\operatorname{Jac}(A)$ is a direct product of division algebras. If $\mathbb{F}$ is algebraically closed, A is basic if and only if every simple A-module is one-dimensional.

Corollary 4.4. [Voigt1, Fe5] *Every block ideal of a reduced universal enveloping algebra of a finite dimensional supersolvable restricted Lie algebra is a full matrix algebra over a basic algebra.*

Proof. Let B be an arbitrary block ideal of a reduced universal enveloping algebra of a finite dimensional supersolvable restricted Lie algebra. According to Corollary 4.3, all the simple B-modules have the same dimension d. Let $P_1, ..., P_s$ be a representative set of the isomorphism classes of the projective indecomposable B-modules. Then [Ben1, Lemma 1.7.5] implies $B = \bigoplus_{i=1}^{s} dP_i$ (see the proof of Theorem 5.2) and therefore we obtain (cf. [Pie, Corollary 3.4a]):

$$B \cong \operatorname{End}_B(B)^{\mathrm{op}} \cong \operatorname{End}_B(dP)^{\mathrm{op}} \cong \operatorname{Mat}_d(\operatorname{End}_B(P)^{\mathrm{op}}),$$

where $P := \bigoplus_{i=1}^{s} P_i$ and $?^{\mathrm{op}}$ denotes the opposite ring structure. Since P is multiplicity-free, $\operatorname{End}_B(P)^{\mathrm{op}}$ is a basic algebra (cf. [Pie, Lemma 6.6a]). □

As an immediate consequence of Corollary 4.3, we obtain the following (partial) generalization of Theorem 3.7:

Corollary 4.5. [Fe5] *Let L be a finite dimensional supersolvable restricted Lie algebra and $\chi \in L^*$. If M and N are simple $u(L,\chi)$-modules such that $\dim_{\mathbb{F}} M \neq \dim_{\mathbb{F}} N$, then $\operatorname{Ext}^n_{u(L,\chi)}(M,N) = 0$ for every integer $n \geq 0$.* □

Remark. From [Fa2, Corollary 6.4] and Theorem 3.1(b) it follows that for arbitrary simple L-modules M, N such that $\dim_{\mathbb{F}} M \neq \dim_{\mathbb{F}} N$, we have $\operatorname{Ext}^n_{U(L)}(M,N) = 0$ for every $n \geq 0$. Is this also true in zero characteristic?

The following module-theoretic characterization of supersolvable restricted Lie algebras over algebraically closed fields was given by D. Voigt [Voigt1, Satz 2.40] in the more general context of infinitesimal algebraic group schemes (cf. also [Sta, Corollary 2] for the case of finite groups).

Theorem 4.6. [Voigt1, Fe5] *A finite dimensional restricted Lie algebra L is supersolvable if and only if the principal block ideal of L is a basic algebra.*

Proof. One implication is an immediate consequence of Corollary 4.4 (resp. its proof) and the other implication follows from Lemma 4.1(a) in conjunction with Theorem 3.9. $\square$

In order to solve Problem I.2, we consider for any finite dimensional restricted Lie algebra L the finite abelian p-subgroup G^L of the (additive) group of L^* (cf. §3). For any $\chi \in L^*$, G^L acts on $\mathrm{Irr}(L, \chi)$ via

$$\gamma \cdot [S] := [F_\gamma \otimes S] \qquad \forall\, \gamma \in G^L, [S] \in \mathrm{Irr}(L, \chi).$$

Question 1. Does there exist a (sufficiently large) subgroup G_0^L of G^L such that $\mathbb{B}_j$ is G_0^L-invariant for every $1 \le j \le b(L, \chi)$?

Consider $G_0^L := \{\gamma \in G^L \mid [F_\gamma] \in \mathbb{B}_0\}$ which indeed is a subgroup of G^L. (For the proof apply Lemma 4.1(b)!) According to Theorem 4.6, G_0^L is as large as possible (i.e., $|G_0^L| = |\mathbb{B}_0|$) if and only if L is supersolvable.

Question 2. Let L be a finite dimensional supersolvable restricted Lie algebra, $\chi \in L^*$ and $\mathbb{B}$ be a block of $u(L, \chi)$. For which simple modules S in $\mathbb{B}_0$ and for which simple modules M in $\mathbb{B}$ does $S \otimes_{\mathbb{F}} M$ again belong to $\mathbb{B}$? Under which conditions on L resp. χ is this satisfied for *every* simple module in $\mathbb{B}_0$ and *every* block $\mathbb{B}$?

In the case of an affirmative answer to the second part of Question 2, Theorem 4.2(b) would imply that G_0^L acts transitively on every block $\mathbb{B}$ of $u(L, \chi)$, i.e., $|\mathbb{B}|$ would be always a p-power (which?) $\le |\mathbb{B}_0|$.

In order to show how powerful this approach is, we use the simplest possible case for which the second question has an affirmative answer to characterize block degenerate supersolvable restricted Lie algebras (see also [Fe5, Theorem 4] for a more general result in this direction). The proof was inspired by a result of R. Farnsteiner [Fa8] (see also [Fe5, Proposition 5, Proposition 6 and Theorem 4] for a different approach avoiding projective covers).

Theorem 4.7. [Fe5] *A finite dimensional supersolvable restricted Lie algebra over an algebraically closed field has exactly one block if and only if it has no non-zero toral ideals.*

Proof. Suppose that L has no non-zero toral ideals. Since L is supersolvable, there is a torus T such that $L = T \oplus \mathrm{Nil}(L)$. But $\mathrm{Tor}_p(\mathrm{Nil}(L))$ is an ideal in L and thus by hypothesis zero. Hence $\mathrm{Nil}(L)$ is p-nilpotent and therefore every simple restricted L-module is one-dimensional. By virtue of the universal property of projective covers (see §5) and Lemma 3.3, we obtain that the projective cover $P(S)$ of any simple restricted L-module is isomorphic to $\mathrm{Ind}_T^L(S,0)$ (see e.g. [Hum1, Proposition 4.3] or [Fe1, Satz I.2.3]). Let $\{\alpha_1,...,\alpha_n\}$ be a basis of the roots of $\mathrm{Nil}(L)$ with respect to T and $\mathcal{G}_0^L := \sum_{j=1}^n \mathbb{F}_p\alpha_j$. Applying the Cartan-Weyl formula n times to the eigenvectors of $P(S)$, we see that the composition factors of $P(S)$ form the orbit of S under $\mathcal{G}_0^L$, i.e., $G_0^L \cong \mathcal{G}_0^L$ (see the Remark after Lemma 4.1) and thus $|\mathbb{B}_0| = |G_0^L| = p^{\dim_{\mathbb{F}_p}\mathcal{G}_0^L}$. But $\dim_{\mathbb{F}_p}\mathcal{G}_0^L$ is just $\dim_{\mathbb{F}} T$ since the canonical pairing between $\mathcal{G}_0^L$ and the $\mathbb{F}_p$-form of T is by hypothesis non-degenerate. Hence we finally obtain

$$|\mathbb{B}_0| = p^{\dim_{\mathbb{F}} T} = |\,\mathrm{Irr}(L,0)|,$$

i.e., $u(L,0)$ has precisely one block.

In order to prove the converse implication let us assume that L has a non-zero toral ideal T_0. Then $0 \neq \varepsilon_0(s_0)^{-1}\cdot s_0 \neq 1$ is a central idempotent of $u(L,0)$, where ε_0 denotes the *augmentation mapping* of $u(T_0,0) \hookrightarrow u(L,0)$ and s_0 denotes its *trace element* (see §3). □

As a consequence of Theorem 4.7 we obtain the following sufficient condition for simple modules to belong to the principal block:

Corollary 4.8. [Fe5] *Let L be a finite dimensional supersolvable restricted Lie algebra and S be a simple restricted L-module. If $\mathrm{Tor}_p(L) \subseteq \mathrm{Ann}_L(S)$, then S belongs to the principal block of L.*

Proof. The hypothesis implies that S is a simple restricted $L/\mathrm{Tor}_p(L)$-module. Since factor algebras of supersolvable Lie algebras are also supersolvable, Theorem 4.7 in conjunction with Proposition 1.2(b) implies that $L/\mathrm{Tor}_p(L)$ has precisely one block and therefore S necessarily belongs to the principal block of $L/\mathrm{Tor}_p(L)$. Finally, the five-term exact sequence for restricted cohomology in conjunction with Lemma 4.1(b) shows that then S also belongs to the principal block of L. □

As an illustration of the foregoing result, we consider again the three-dimensional supersolvable restricted Lie algebra from Example (ii) in §2:

Example. Let $L := \mathbb{F}t \oplus \mathbb{F}e \oplus \mathbb{F}z$, where $[t, e] = e + z$, $t^{[p]} = t$, $e^{[p]} = 0$ and $z^{[p]} = z$. Then we have $\mathrm{Tor}_p(L) = \mathbb{F}z$ and the restricted universal enveloping algebra $u(L, 0)$ has the following Gabriel quiver (see [FeS1, Proposition 3.2]):

If L is supersolvable, the principal block of L forms an abelian group under $[S_1]+[S_2] := [S_1 \otimes_{\mathbb{F}} S_2]$ which is isomorphic to G_0^L via $\gamma \mapsto [F_\gamma]$. In the above example, $\mathbb{B}_0$ is generated by the only non-trivial *chief factor* (i.e., composition factor of the adjoint module) F_1 and therefore $\mathbb{B}_0$ is cyclic. In general, it follows from [Ba2, Theorem 1] that chief factors of *solvable* restricted Lie algebras always belong to the principal block (cf. also [Fe5, Proposition 2]). Recall that a Lie algebra L is called *unimodular* if $\mathrm{tr}(\mathrm{ad}_L(x)) = 0$ for every element $x \in L$. Then we have the following

Lemma 4.9. *Let L be a finite dimensional unimodular supersolvable restricted Lie algebra. Then the principal block of L is generated by the (non-trivial) chief factors of L.*

Proof. Let S be a simple module belonging to $\mathbb{B}_0$. According to the main result of [Schue1], our hypothesis implies that the restricted universal enveloping algebra of L is symmetric. (In fact, by [Hum2, Theorem 4] our assumption is also necessary for $u(L, 0)$ to be symmetric.) Then [La, Lemma 3] yields a finite sequence $\mathbb{F} =: M_0, \ldots, M_n := S$ of restricted simple L-modules such that $\mathrm{Ext}^1_{u(L,0)}(M_{j-1}, M_j) \neq 0$ for every integer $1 \leq j \leq n$. In particular, all the M_j's belong to $\mathbb{B}_0$. Since by Theorem 4.2(a) every simple module in $\mathbb{B}_0$ is one-dimensional, for any $1 \leq j \leq n$ there exists a simple restricted L-module

S_j (namely, $M_{j-1}^* \otimes_{\mathbb{F}} M_j$) such that $M_j \cong M_{j-1} \otimes_{\mathbb{F}} S_j$ and $H_*^1(L, S_j) \neq 0$. Hence [Ba2, Theorem 1] shows that the S_j's are chief factors of L and

$$S \cong S_1 \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} S_n, \text{ i.e., } [S] = [S_1] + \cdots + [S_n]. \quad \square$$

As a first consequence of the results we proved so far, we obtain the following characterization of simple modules in the principal block of a *unimodular* supersolvable restricted Lie algebra. The example after Corollary 4.8 indicates that this might be true without assuming the symmetry of the restricted universal enveloping algebra which in fact is the case (see [Fe5, Theorem 5]).

Corollary 4.10. *Let L be a finite dimensional unimodular supersolvable restricted Lie algebra. Then S belongs to the principal block of L if and only if* $\operatorname{Tor}_p(L) \subseteq \operatorname{Ann}_L(S)$.

Proof. Assume that S belongs to the principal block of L. Then (the proof of) Lemma 4.9 yields the existence of chief factors S_j $(1 \leq j \leq n)$ such that $S \cong S_1 \otimes_{\mathbb{F}} ... \otimes_{\mathbb{F}} S_n$. Since $\operatorname{Tor}_p(L)$ is central, we obtain

$$\operatorname{Tor}_p(L) \subseteq \bigcap_{j=1}^{n} \operatorname{Ann}_L(S_j) \subseteq \operatorname{Ann}_L(\bigotimes_{j=1}^{n} S_j) \subseteq \operatorname{Ann}_L(S).$$

The other implication is just Corollary 4.8. $\square$

Now we are ready to determine the number of simple modules in the principal block, namely

Theorem 4.11. *Let L be a finite dimensional unimodular supersolvable restricted Lie algebra over an algebraically closed field $\mathbb{F}$. Then*

$$|\mathbb{B}_0| = p^{\dim_{\mathbb{F}} L/\operatorname{Nil}(L)}.$$

Proof. By hypothesis, L is unimodular supersolvable and thus Corollary 4.10 applies and yields a bijection between the principal block of L and the set of isomorphism classes of simple restricted $L/\operatorname{Tor}_p(L)$-modules. As we have already used in the proof of Theorem 3.7, the latter algebra is strongly solvable with maximal torus T if $L = T \oplus \operatorname{Nil}(L)$. Hence

$$|\mathbb{B}_0| = |\operatorname{Irr}(L/\operatorname{Tor}_p(L), 0)| = p^{\dim_{\mathbb{F}} T} = p^{\dim_{\mathbb{F}} L/\operatorname{Nil}(L)}. \quad \square$$

Remark. According to the remark before Corollary 4.10, Theorem 4.11 is also true in general (see [Fe5, Theorem 6]). Since $T \cong (T \oplus \mathrm{Tor}_p(L))/\mathrm{Tor}_p(L)$ is a maximal torus of $T/\mathrm{Tor}_p(L)$, [SF, Theorem II.4.5(2)] shows that $T \oplus \mathrm{Tor}_p(L)$ is a maximal torus of L and therefore the above formula can be interpreted as

$$|\mathbb{B}_0| = p^{\dim_{\mathbb{F}} T_{\max}/\mathrm{Tor}_p(L)},$$

where $T_{\max}$ is a(ny) maximal torus of L (see again [Fe5, Theorem 6]).

Let us now specialize to the nilpotent case. Let $S \not\cong \mathbb{F}$ be a simple restricted L-module over a finite dimensional restricted nilpotent Lie algebra. Then because of $S^L = 0 = (S^*)^L$ an application of Theorem 3.4 or the classical cohomological vanishing theorem of J. Dixmier (cf. [Dix, Ba1]) shows

$$\mathrm{Ext}^1_{u(L,0)}(\mathbb{F}, S) \cong H^1_*(L, S) \hookrightarrow H^1(L, S) = 0,$$

$$\mathrm{Ext}^1_{u(L,0)}(S, \mathbb{F}) \cong H^1_*(L, S^*) \hookrightarrow H^1(L, S^*) = 0,$$

and therefore the block structure of L is determined by the following result:

Corollary 4.12. [Voigt1, Fe5] *Let L be a finite dimensional nilpotent restricted Lie algebra and $\chi \in L^*$. Then every block of $u(L,\chi)$ contains only one isomorphism class of simple modules. In particular, every finite dimensional indecomposable $u(L,\chi)$-module has only one composition factor (up to isomorphism).* □

Remark. The proof of the second part of Corollary 4.12 gives a new conceptual approach to a special case of an old result of C.W. Curtis [Cu2]. In fact, it can be shown that the same argument works even more generally for *locally finite* indecomposable L-modules [Fe6].

Corollary 4.13. [Voigt1, Fe5] *Every block ideal of a reduced universal enveloping algebra of a finite dimensional nilpotent restricted Lie algebra is a full matrix algebra over a local algebra.*

Proof. This is quite analogous to the proof of Corollary 4.4 using Corollary 4.12 instead of Corollary 4.3 (cf. [Pie, Proposition 6.5a]). □

Lemma 4.1(a), Corollary 4.12 and Theorem 6.5 (or Proposition 6.4) in conjunction with Theorem 3.4 and the Reduction Theorem yield:

Corollary 4.14. *Let L be a finite dimensional nilpotent restricted Lie algebra, $\chi \in L^*$ and M, N be finite dimensional non-projective $u(L,\chi)$-modules. Then the following statements are equivalent:*

(a) $\hat{\operatorname{Ext}}^n_{u(L,\chi)}(M,N) \neq 0$ *for every integer n.*
(b) *There exists an integer n such that* $\hat{\operatorname{Ext}}^n_{u(L,\chi)}(M,N) \neq 0$.
(c) *M and N belong to the same block of $u(L,\chi)$.*
(d) $M \cong N$. $\square$[7]

Example. Consider the three-dimensional Heisenberg algebra

$$L = \mathcal{H}_1(\mathbb{F}) = \mathbb{F}e_+ \oplus \mathbb{F}z \oplus \mathbb{F}e_-, \quad [e_+, e_-] = z,\ e_\pm^{[p]} = 0,\ z^{[p]} = z.$$

Put $I := \mathbb{F}z \oplus \mathbb{F}e_-$ and assume $\chi(z) = 0$ (see Example (i) in §2). By virtue of the Reduction Theorem, it is easy to compute that $\operatorname{Ext}^1_{u(L,\chi)}(F_\chi, F_\chi) \cong H^1_*(L,\mathbb{F})$ is two-dimensional. Let $V(\omega)$ denote the restricted I-module $F_{-\omega} \otimes_{\mathbb{F}} \operatorname{Ind}_I^L(F_\omega,\chi)_{|I}$. Using Frobenius Reciprocity and the five-term exact sequence for restricted cohomology in conjunction with Lemma 3.2, we obtain

$$\begin{aligned}\operatorname{Ext}^1_{u(L,\chi)}(\operatorname{Ind}_I^L(F_\omega,\chi), \operatorname{Ind}_I^L(F_\omega,\chi)) &\cong \operatorname{Ext}^1_{u(I,\chi_{|I})}(F_\omega, \operatorname{Ind}_I^L(F_\omega,\chi)_{|I}),\\ &\cong H^1_*(I, V(\omega)) \cong H^1_*(\mathbb{F}e_-, V(\omega)_{|\mathbb{F}e_-})^{\mathbb{F}z}.\end{aligned}$$

According to Theorem 3.4, it is enough to show that the 0-th complete cohomology space $\hat{H}^0_*(\mathbb{F}e_-, V(\omega)_{|\mathbb{F}e_-})$ vanishes. But this is a consequence of the description of the latter space in §3 because the trace element of $u(\mathbb{F}e_-, 0)$ is e_-^{p-1} and therefore $\dim_{\mathbb{F}} V(\omega)^{\mathbb{F}e_-} = 1 = \dim_{\mathbb{F}} e_-^{p-1} \cdot V(\omega)$. Hence we have the following Gabriel quiver $\vec{Q}(L,\chi)$ of $u(L,\chi)$

where the vertices corresponding to the simple $u(L,\chi)$-modules are labelled by the respective eigenvalues $\omega(z)$ of z. In particular, it follows from Corollary 4.14 that $\operatorname{Ind}_I^L(F_\omega,\chi)$ is projective for any ω with $\omega(z) \neq 0$.

[7] If M and N are simple, then there exists a very short direct argument (see [Fe5, Proposition 4]).

Analogously to the supersolvable case we obtain the following module-theoretic characterization of nilpotent restricted Lie algebras (cf. [Voigt1, Satz 2.41] for an algebraically closed ground field).

Theorem 4.15. [Voigt1, Fe5] *A finite dimensional restricted Lie algebra L is nilpotent if and only if the principal block ideal of L is a local algebra.*

Proof. One implication is a special case of Corollary 4.12 and the other implication follows immediately from Lemma 4.1(a) in conjunction with Theorem 3.6. □

In the non-solvable case there is at least a first step for restricted *simple* Lie algebras (In fact, the first part of the next result is a special case of the *linkage principle* for Lie algebras of classical type.):

Theorem 4.16. *Let L be a restricted simple Lie algebra over an algebraically closed field $\mathbb{F}$. Then the following statements hold:*

(a) [KW] *If L is of classical type and M, N are simple restricted L-modules of highest weights μ, ν, respectively, then M and N belong to the same block of $u(L, 0)$ if and only if there exists an reflection σ in the Weyl group of L such that $\mu + \delta = \sigma(\nu + \delta)$, where δ denotes the halfsum of the positive roots of L.*

(b) [HN2] *If L is of Cartan type over a field of characteristic > 3, then $u(L, 0)$ has exactly one block.* □

Examples.

(i) Consider the three-dimensional (restricted) simple Lie algebra

$$\mathfrak{sl}_2(\mathbb{F}) = \mathbb{F}e_+ \oplus \mathbb{F}h \oplus \mathbb{F}e_-,$$

$$[h, e_\pm] = \pm 2 \cdot e_\pm, \ [e_+, e_-] = h, \ h^{[p]} = h, \ e_\pm^{[p]} = 0.$$

over an algebraically closed field $\mathbb{F}$ of characteristic $p > 2$. Then we have the following Gabriel quivers for the different orbits of characters:

Figure 1. Restricted orbit [Po1, Po2, Fi, FP5]

Figure 2. Regular nilpotent orbit [FP5, Proposition 2.3]

Figure 3. Regular semisimple orbit [FP5, Corollary 2.2]

(ii) Consider the eight-dimensional simple Jacobson-Witt algebra $W_2(\mathbb{F})$ over an algebraically closed field $\mathbb{F}$ of characteristic 2. Then $W_2(\mathbb{F})$ has four (isomorphism classes of) simple restricted modules, namely the one-dimensional trivial module, two three-dimensional modules resulting from the defining representation as derivations on the truncated polynomial algebra $P_2(\mathbb{F})$ in two variables resp. its dual $P_2(\mathbb{F})^*$ and the eight-dimensional adjoint module. The latter is a projective $u(W_2(\mathbb{F}), 0)$-module and the other three simples belong to the principal block of $W_2(\mathbb{F})$. Hence Theorem 4.16(b) is *not* true for $\mathrm{char}(\mathbb{F}) = 2$. Very similarly, the three-dimensional Witt algebra $W_1(\mathbb{F})$ over an algebraically closed field of characteristic 3 (being isomorphic to $\mathfrak{sl}_2(\mathbb{F})$) has two blocks. This shows that the assumption $\mathrm{char}(\mathbb{F}) > 3$ in Theorem 4.16(b) is really necessary.

Moreover, in both cases there is some information on the dimensions of the blocks resp. indecomposable projective modules and the Cartan invariants via generalized Reciprocity Theorems (cf. [Hum1, Na1, HN1, HN2, Chiu3, Hol2]).

A Lie algebra is called *strongly degenerate* if there exists $0 \neq x \in L$ such that $(\mathrm{ad}_L\, x)^2 = 0$. Then Theorem 4.16 leads to the following conjecture which is closely related to Theorem 4.7:

Conjecture. [HN2] *The restricted universal enveloping algebra of a strongly degenerate restricted Lie algebra L without classical or non-zero toral ideals has precisely one block.*

§5. Projective Modules and Block Stability

In this section, we investigate the structure of projective indecomposable $u(L, \chi)$-modules for an arbitrary character χ of a finite dimensional nilpotent restricted Lie algebra L.

A projective module $P(M)$ is called a *projective cover* of a module M if there exists a module epimorphism π_M from $P(M)$ onto M such that the kernel of π_M is *small* in $P(M)$, i.e., there are no proper submodules X of $P(M)$ with $P(M) = X + \mathrm{Ker}(\pi_M)$ or equivalently, $\mathrm{Ker}(\pi_M)$ is contained in the radical of $P(M)$ (see [DK, p. 53]). It is well-known that projective covers of finite dimensional modules over finite dimensional algebras always exist (cf. [DK, Theorem 3.3.7.1)]) and are again finite dimensional. Moreover, the *projective indecomposable modules* of a finite dimensional algebra are isomorphic to the projective covers of the simple modules (cf. [DK, Theorem 3.3.7.2) and Corollary 3.2.5]).

According to a classical result of H. Zassenhaus (cf. [SF, Corollary I.4.4 and Proposition I.4.6]), for every simple module S of a finite dimensional nilpotent Lie algebra L there exists a function $\omega : L \to \mathbb{F}$ such that

$$\forall\, x \in L\ \exists\, n = n(x) \in \mathbb{N} : \quad [(x)_S - \omega(x) \cdot \mathrm{id}_S]^n = 0.$$

ω is called (*weight function* or) *eigenvalue function* of S. Note that ω is *not* necessarily *linear* on L. It is well-known (and originally also due to

Zassenhaus) that the simple L-modules are uniquely determined (up to isomorphism) by its eigenvalue function (see [Str3]). More precisely, if we fix a toral basis $\mathcal{T} := \{t_1, \dots t_b\}$ of $\mathrm{Tor}_p(L)$ (cf. [SF, Theorem II.3.6(1)]), we have for any $\chi \in L^*$ a bijective mapping (depending on $\mathcal{T}$.)

$$\mathrm{Irr}(L,\chi) \longrightarrow \Pi_{\mathcal{T}}(L,\chi),\ [S] \longmapsto (\omega(t_1), \dots, \omega(t_b)),$$

where ω is the eigenvalue function of S and the set of parameters is defined by

$$\Pi_{\mathcal{T}}(L,\chi) := \{(z_1, \dots, z_b) \in \mathbb{F}^b \mid z_i^p - z_i = \chi(z_i)^p \ \forall\ 1 \le i \le b\}$$

(see [Str3, p. 31]). In particular, we obtain (see [Str3, Satz 6]):

$$|\,\mathrm{Irr}(L,\chi)| = |\Pi_{\mathcal{T}}(L,\chi)| = p^{\dim_{\mathbb{F}} T_p(L)}. \tag{6}$$

We will need the following slight generalization of [SF, Theorem V.8.7(2)]:

Lemma 5.1. *Let L be a finite dimensional nilpotent restricted Lie algebra and $\chi \in L^*$. If X and Y are simple $u(L,\chi)$-modules, then*

$$X \cong Y \quad \text{if and only if} \quad X_{|\mathrm{Tor}_p(L)} \cong Y_{|\mathrm{Tor}_p(L)}.$$

Proof. Note that the action of the p-nilpotent radical of the center of L on any simple $u(L,\chi)$-module is uniquely determined by χ (cf. [Str3, p. 31]). Hence the assertion follows from Theorem 1.1 and [SF, Theorem V.8.7(2)]. □

Consider a simple $u(L,\chi)$-module S with eigenvalue function ω as $\mathrm{Tor}_p(L)$-module which we denote by $S_{|\mathrm{Tor}_p(L)}$. Because $\mathrm{Tor}_p(L)$ is central, Schur's Lemma implies that $(t)_S = \omega(t) \cdot \mathrm{id}_S$ for all $t \in \mathrm{Tor}_p(L)$. Hence the semisimplicity of $u(\mathrm{Tor}_p(L), \chi_{|\mathrm{Tor}_p(L)})$ (see Lemma 3.2) yields $S_{|\mathrm{Tor}_p(L)} \cong \bigoplus^{\dim_{\mathbb{F}} S} F_\omega$, where F_ω is the one-dimensional $u(\mathrm{Tor}_p(L), \chi_{|\mathrm{Tor}_p(L)})$-module with action $t \cdot f := \omega(t) \cdot f$ for $t \in \mathrm{Tor}_p(L)$, $f \in F_\omega$. In particular, $S_{|\mathrm{Tor}_p(L)}$ has a unique (one-dimensional) composition factor (up to isomorphism) which we denote by $F(S)$.

Theorem 5.2. [Fe4] *Let S be a simple module of a finite dimensional nilpotent restricted Lie algebra L with character χ. If $F(S)$ is the (up to isomorphism) unique composition factor of $S_{|\mathrm{Tor}_p(L)}$, then there is an L-module isomorphism $\mathrm{Ind}_{\mathrm{Tor}_p(L)}^L(F(S),\chi) \cong \bigoplus^{\dim_{\mathbb{F}} S} P(S)$.*

Proof. Put $T_0 := \mathrm{Tor}_p(L)$. According to Lemma 3.2, $F(S)$ is a projective $u(T_0, \chi_{|T_0})$-module. By the additivity of induction, $P := \mathrm{Ind}_{T_0}^L(F(S),\chi)$ is thus a projective $u(L,\chi)$-module. Since $\{P(X) \mid X \in \operatorname{Irr}(L,\chi)\}$ is a full set of representatives of isomorphism classes of indecomposable projective $u(L,\chi)$-modules, there exist non-negative integers m_X such that

$$P \cong \bigoplus_{X \in \operatorname{Irr}(L,\chi)} m_X \cdot P(X).$$

Hence Frobenius Reciprocity in conjunction with Schur's Lemma and the additivity of the Hom-functor in conjunction with [Ben1, Lemma 1.7.5] yield for any simple $u(L,\chi)$-module Y:

$$\dim_{\mathbb{F}} \operatorname{Hom}_L(P,Y) = \dim_{\mathbb{F}} \operatorname{Hom}_{T_0}(F(S), Y_{|T_0}) = \begin{cases} \dim_{\mathbb{F}} S \text{ if } Y_{|T_0} \cong S_{|T_0} \\ 0 \text{ otherwise} \end{cases},$$

resp.

$$\dim_{\mathbb{F}} \operatorname{Hom}_L(P,Y) = \sum_{X \in \operatorname{Irr}(L,\chi)} m_X \cdot \dim_{\mathbb{F}} \operatorname{Hom}_L(P(X),Y) = m_Y.$$

Finally, an application of Lemma 5.1 shows that $Y_{|T_0} \cong S_{|T_0}$ if and only if $Y \cong S$. Hence

$$m_X = \begin{cases} \dim_{\mathbb{F}} S \text{ if } X \cong S \\ 0 \text{ otherwise} \end{cases}. \quad \square$$

Corollary 5.3. [Fe4] *Let L be a finite dimensional nilpotent Lie algebra. If S is a one-dimensional $u(L,\chi)$-module, then there is an L-module isomorphism $P(S) \cong \operatorname{Ind}_{\operatorname{Tor}_p(L)}^L(S_{|\operatorname{Tor}_p(L)}, \chi)$.* $\square$

In particular, this applies to the projective cover of the trivial simple module and moreover, Theorem 5.2 can be used to show the following Reciprocity Theorem for projective indecomposable modules (cf. [Fe4, Theorem 2]):

Corollary 5.4. [Fe4] *For any finite dimensional nilpotent restricted Lie algebra L the following statements hold:*

(a) $S^* \otimes_{\mathbb{F}} P(S) \cong P(\mathbb{F})$.

(b) $P(\mathbb{F}) \cong \operatorname{Ind}_{\operatorname{Tor}_p(L)}^L(\mathbb{F}, 0)$. $\square$

It should be remarked that Corollary 5.4 yields the projectivity of the p-dimensional simple $\mathcal{H}_1(\mathbb{F})$-modules much faster than in the corresponding example in §3 and, moreover, immediately generalizes to $\mathcal{H}_n(\mathbb{F})$ for any positive integer n (cf. [Fe4]).

Definition. A block ideal B of a finite dimensional associative $\mathbb{F}$-algebra A is called *stable* if $\dim_{\mathbb{F}} B : |\operatorname{Irr}(B)| = \dim_{\mathbb{F}} A : |\operatorname{Irr}(A)|$. A is called *block stable* if every block ideal of A is stable.

Brauer-Humphreys Reciprocity in conjunction with Theorem 4.16(a) shows that restricted universal enveloping algebras of simple Lie algebras of classical type are block stable (cf. [Hum1, Theorem 4.5]) and since block degenerate algebras are trivially block stable, the same is true for restricted simple Lie algebras of Cartan type (see Theorem 4.16(b)). But note that already for a(ny) regular nilpotent character of $\mathfrak{sl}_2$ the corresponding reduced universal enveloping algebra is *not* block stable (see Example (iii) in §2 and Example (i) after Theorem 4.16). Nevertheless, Corollary 5.4 in conjunction with Corollary 4.12 and (6) implies (cf. also [Pet]):

Corollary 5.5. *Every reduced universal enveloping algebra of a finite dimensional nilpotent restricted Lie algebra L is block stable, i.e., every block ideal B satisfies* $\dim_{\mathbb{F}} B = p^{\dim_{\mathbb{F}} L/\operatorname{Tor}_p(L)}$. $\square$

Remark. More generally, it can be shown that for every simple $u(L,\chi)$-module S the induced module $\operatorname{Ind}_{\operatorname{Tor}_p(L)}^{L}(F(S),\chi)$ has a ring structure and that it is isomorphic to $B(S)$ as an $\mathbb{F}$-algebra (cf. [Fe4, Theorem 1]).

§6. Complexity and Support Varieties

Throughout this section, L will denote a finite dimensional restricted Lie algebra over a field $\mathbb{F}$. Then for any $u(L,\chi)$-module M there exists a projective $u(L,\chi)$-module P and an epimorphism π of P onto M (see §5). By virtue of the Krull-Remak-Schmidt Theorem and since $u(L,\chi)$ is a Frobenius algebra, the kernel of π is isomorphic to $\Omega(M) \oplus Q$, where Q is a projective $u(L,\chi)$-module and $\Omega(M)$ has no projective submodules. Schanuel's Lemma implies that the isomorphism class of $\Omega(M)$ is independent of the choice of P and π. $\Omega(?)$ is called the *loop-space functor* (cf. [He]). Define $\Omega^n(?)$ recursively by $\Omega^n(M) := \Omega(\Omega^{n-1}(M))\ \forall\ n \geq 1$ and $\Omega^0(M) := M$ for every $u(L,\chi)$-module M.

By the definition of $\Omega(M)$ we obtain $\Omega(M) = \operatorname{Ker}(\pi_M)$ if $\pi_M : P(M) \to M$ is a *projective cover* of M (see §5). A projective resolution

$$P_\bullet : \cdots \to P_n \to \cdots \to P_0 \to M \to 0$$

is called *minimal* if P_n is isomorphic to the projective cover $P(\Omega^n(M))$ of $\Omega^n(M)$ for every integer $n \geq 0$. By using *injective hulls* instead of projective covers this definition can be extended to all integers and then $\Omega \circ \Omega^{-1}$ resp.

$\Omega^{-1} \circ \Omega$ are identity functors on the subcategory of modules with no direct projective (= injective) summands.

Recall that the *rate of growth* $\mathrm{gr}(V_\bullet)$ of a family of finite dimensional vector spaces $V_\bullet = (V_n)_{n\geq 0}$ is the smallest non-negative integer c such that there is a constant b with $\dim_{\mathbb{F}} V_n \leq b \cdot n^{c-1}$ $\forall\, n \geq 1$. If such c exists, then $V_\bullet$ is said to have *polynomial growth*; otherwise we set $\mathrm{gr}(V_\bullet) := \infty$. We will see in the following that every minimal projective resolution of a finite dimensional $u(L,\chi)$-module has polynomial growth. From the short exact sequence

$$(7) \qquad 0 \longrightarrow \Omega^{n+1}(M) \longrightarrow P_n \longrightarrow \Omega^n(M) \longrightarrow 0 \qquad \forall\, n \geq 0$$

it is clear that $\mathrm{gr}(P_\bullet) = \mathrm{gr}(\Omega^\bullet(M))$. These coinciding integers are called the *complexity* $\mathrm{cx}_L(M)$ of M. Since $u(L,\chi)$ is a Frobenius algebra, M is projective if and only if $\mathrm{cx}_L(M) = 0$.

Lemma 6.1. *Let M be an arbitrary $u(L,\chi)$-module and S be a simple $u(L,\chi)$-module. Then for every non-negative integer n there is an isomorphism*

$$\mathrm{Ext}^n_{u(L,\chi)}(M,S) \cong \mathrm{Hom}_L(\Omega^n(M),S).$$

Proof. Let $n \geq 2$. Then an application of the contravariant $\mathrm{Hom}_L(?,S)$-functor to the short exact sequence $0 \to \Omega(M) \to P(M) \to M \to 0$ in conjunction with the projectivity of $P(M)$ yields the following exact sequence:

$$0 = \mathrm{Ext}^{n-1}_{u(L,\chi)}(P(M),S) \longrightarrow \mathrm{Ext}^{n-1}_{u(L,\chi)}(\Omega(M),S) \longrightarrow \mathrm{Ext}^{n}_{u(L,\chi)}(M,S) \longrightarrow \mathrm{Ext}^{n}_{u(L,\chi)}(P(M),S) = 0.$$

By induction on n we therefore obtain:

$$\mathrm{Ext}^n_{u(L,\chi)}(M,S) \cong \mathrm{Ext}^1_{u(L,\chi)}(\Omega^{n-1}(M),S) \qquad \forall\, n \geq 2.$$

Since the case $n = 0$ is clear, it is enough to show

$$\mathrm{Ext}^1_{u(L,\chi)}(M,S) \cong \mathrm{Hom}_L(\Omega(M),S).$$

Looking at the beginning of the long exact cohomology sequence used above we have the following short exact sequence:

$$0 \longrightarrow \mathrm{Hom}_L(M,S) \longrightarrow \mathrm{Hom}_L(P(M),S) \longrightarrow \mathrm{Hom}_L(\Omega(M),S) \longrightarrow \mathrm{Ext}^1_{u(L,\chi)}(M,S) \longrightarrow 0.$$

By the exactness of the sequence it is enough to show that the first (non-zero) mapping is surjective. Let π_M denote the epimorphism from $P(M)$ onto M and let $0 \neq \varphi \in \mathrm{Hom}_L(P(M), S)$. Because S is simple, Schur's Lemma implies that φ is surjective and therefore $\mathrm{Ker}(\varphi)$ is a maximal submodule of $P(M)$. By definition $\Omega(M) = \mathrm{Ker}(\pi_M)$ is small in $P(M)$. Hence $\varphi \neq 0$ and the maximality of $\mathrm{Ker}(\varphi)$ imply the inclusion $\Omega(M) \subseteq \mathrm{Ker}(\varphi)$, i.e., there exists an L-module homomorphism ϕ from M into S such that $\phi \circ \pi_M = \varphi$. □

Let M, N be finite dimensional $u(L,\chi)$-modules. Then we define

$$\mathrm{cx}_L(M,N) := \mathrm{gr}(\mathrm{Ext}^{\bullet}_{u(L,\chi)}(M,N)).$$

Proposition 6.2. *For every finite dimensional $u(L,\chi)$-module M the following equalities hold:*

$$\mathrm{cx}_L(M) = \max\{\mathrm{cx}_L(M,S) \mid S \text{ simple}\} = \max\{\mathrm{cx}_L(M,N) \mid \dim_{\mathbb{F}} N < \infty\}.$$

Proof. Let S be a simple $u(L,\chi)$-module and put $d_S := \dim_{\mathbb{F}} \mathrm{End}_L(S)$. An application of the contravariant $\mathrm{Hom}_L(?, S)$-functor to (7) in conjunction with the projectivity of P_n yields the following exact sequence:

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_L(\Omega^n(M), S) \longrightarrow \mathrm{Hom}_L(P_n, S) \longrightarrow \mathrm{Hom}_L(\Omega^{n+1}(M), S) \\ \longrightarrow \mathrm{Ext}^1_{u(L,\chi)}(\Omega^n(M), S) \longrightarrow 0. \end{aligned}$$

By virtue of (the proof of) Lemma 6.1, the third (non-zero) map is bijective. According to the exactness of the sequence, then the first (non-zero) map must also be bijective. Therefore another application of Lemma 6.1 implies

$$\dim_{\mathbb{F}} \mathrm{Ext}^n_{u(L,\chi)}(M,S) = \dim_{\mathbb{F}} \mathrm{Hom}_L(P_n, S) = d_S \cdot [P_n : P(S)],$$

where $[P_n : P(S)]$ denotes the multiplicity of $P(S)$ as a direct summand of P_n (cf. [Ben1, Lemma 1.7.5]). Hence we obtain for every non-negative integer n:

$$\dim_{\mathbb{F}} P_n = \sum_{S \in \mathrm{Irr}(L,\chi)} \frac{\dim_{\mathbb{F}} \mathrm{Ext}^n_{u(L,\chi)}(M,S)}{d_S} \cdot \dim_{\mathbb{F}} P_L(S), \text{ i.e.,}$$

$$\mathrm{cx}_L(M) \leq \max\{\mathrm{cx}_L(M,S)) \mid S \text{ simple }\} \leq \max\{\mathrm{cx}_L(M,N) \mid \dim_{\mathbb{F}} N < \infty\}.$$

The remaining inequality $\max\{\mathrm{cx}_L(M,N) \mid \dim_{\mathbb{F}} N < \infty\} \leq \mathrm{cx}_L(M)$ is an immediate consequence of

$$\dim_{\mathbb{F}} \mathrm{Ext}^n_{u(L,\chi)}(M,N) \leq \dim_{\mathbb{F}} \mathrm{Hom}_{\mathbb{F}}(P_n, N) = \dim_{\mathbb{F}} N \cdot \dim_{\mathbb{F}} P_n \ \forall\, n \geq 1. \quad \square$$

In particular, $\mathrm{cx}_L(\mathbb{F})$ is the maximal growth of the restricted cohomology of L with coefficients in a finite dimensional restricted L-module:

Corollary 6.3. *Let L be a finite dimensional restricted Lie algebra. Then*

$$\begin{aligned}\mathrm{cx}_L(\mathbb{F}) &= \max\{\mathrm{gr}(H^\bullet_*(L,S)) \mid S \text{ simple}\}\\ &= \max\{\mathrm{gr}(H^\bullet_*(L,M)) \mid \dim_{\mathbb{F}} M < \infty\}. \quad \square\end{aligned}$$

Fundamental to the rest of this section will be the fact that (via the cup product) $H^\bullet_*(L,M)$ is a *noetherian* $H^\bullet_*(L,\mathbb{F})$-module for every *finite dimensional* restricted L-module M (cf. [FP3, Proposition 1.3(c)]). Let us immediately apply this in order to obtain the following useful modification of Proposition 6.2:

Proposition 6.4. *For every finite dimensional $u(L,\chi)$-module M we have $\mathrm{cx}_L(M) = \mathrm{cx}_L(M,M)$.*

Proof. By an argument similar to the proof of [Ben2, Corollary 4.2.4], one can show that (via the Yoneda product) $\mathrm{Ext}^\bullet_{u(L,\chi)}(M,S)$ is a finitely generated $\mathrm{Ext}^\bullet_{u(L,\chi)}(M,M)$-module for every simple $u(L,\chi)$-module S. Hence we obtain

$$\mathrm{cx}_L(M,S) = \mathrm{gr}(\mathrm{Ext}^\bullet_{u(L,\chi)}(M,S)) \le \mathrm{gr}(\mathrm{Ext}^\bullet_{u(L,\chi)}(M,M)) = \mathrm{cx}_L(M,M)$$

for every simple $u(L,\chi)$-module S. From this we conclude by applying Proposition 6.2 twice:

$$\mathrm{cx}_L(M) = \max\{\mathrm{cx}_L(M,S) \mid S \text{ simple}\} \le \mathrm{cx}_L(M,M) \le \mathrm{cx}_L(M). \quad \square$$

An immediate consequence of Proposition 6.4 is the following *projectivity criterion* (cf. also [Ben2, Lemma 5.2.3] for a slightly more direct proof or [Schulz, Theorem 2.5] for a ring-theoretical approach):

Theorem 6.5. *Let L be a finite dimensional restricted Lie algebra, $\chi \in L^*$ and M be a finite dimensional $u(L,\chi)$-module. If there exists an integer n_0 such that $\hat{\mathrm{Ext}}^n_{u(L,\chi)}(M,M) = 0$ for every $n \ge n_0$ (or $n \le n_0$), then M is projective.* $\square$

Remark. If L is nilpotent, then Theorem 3.4 in conjunction with the Reduction Theorem shows that the vanishing of $\hat{\mathrm{Ext}}^{n_0}_{u(L,\chi)}(M,M)$ for only one integer n_0 implies the projectivity of M (cf. Corollary 4.14 and also [Fa10, Proposition 2.2(2)]).

The following result summarizes some of the main properties of complexity.

Theorem 6.6. *Let M,N be finite dimensional $u(L,\chi)$-modules, M' be a finite dimensional $u(L,\chi')$-module and K be a p-subalgebra of L. Then the following statements hold:*

(a) $\mathrm{cx}_K(M_{|K}) \leq \mathrm{cx}_L(M)$.
(b) $\mathrm{cx}_L(M \oplus N) = \max\{\mathrm{cx}_L(M), \mathrm{cx}_L(N)\}$.
(c) $\mathrm{cx}_L(M \otimes_{\mathbb{F}} M') \leq \min\{\mathrm{cx}_L(M), \mathrm{cx}_L(M')\}$.
(d) $\mathrm{cx}_L(M^*) = \mathrm{cx}_L(M) = \mathrm{cx}_L(\Omega^{\pm n}(M)) \quad \forall\, n \in \mathbb{Z}$.
(e) $\mathrm{cx}_L(M) \leq \mathrm{cx}_L(\mathbb{F})$.

Proof. (a): Since $u(L,\chi)$ is a free $u(K,\chi_{|K})$-module, every (minimal) projective resolution over $u(L,\chi)$ is also a projective resolution over $u(K,\chi_{|K})$.

Consider a short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ of $u(L,\chi)$-modules and let S be any simple $u(L,\chi)$-module. By applying Proposition 6.2 we can read off from the long exact cohomology sequence associated to $\mathrm{Hom}_L(?,S)$ that $\mathrm{cx}_L(M_i) \leq \max\{\mathrm{cx}_L(M_j), \mathrm{cx}_L(M_k)\}$ for any $i,j,k \in \{1,2,3\}$. This immediately implies (b) and, in view of (7), the second equality in (d).

(c): Let Y be a finite dimensional restricted L-module. Then the adjointness of $\otimes$ and Hom yields the isomorphism

$$\mathrm{Ext}^n_{u(L,0)}(M \otimes_{\mathbb{F}} M', Y) \cong \mathrm{Ext}^n_{u(L,\chi)}(M, \mathrm{Hom}_{\mathbb{F}}(M', Y)) \qquad \forall\, n \geq 0,$$

i.e., according to Proposition 6.2, we have $\mathrm{cx}_L(M \otimes_{\mathbb{F}} M') \leq \mathrm{cx}_L(M)$ and thus the assertion follows by symmetry.

Finally, for an algebraically closed field $\mathbb{F}$, (d) is a consequence of Theorem 6.8(a), whereas the general case follows from $\mathrm{cx}_{L\otimes_{\mathbb{F}}\overline{\mathbb{F}}}(M \otimes_{\mathbb{F}} \overline{\mathbb{F}}) = \mathrm{cx}_L(M)$ if $\overline{\mathbb{F}}$ denotes the algebraic closure of $\mathbb{F}$, and (e) is the special case $M' := \mathbb{F}$ in (c). $\square$

Remark. Contrary to the modular representation theory of finite groups, the analogue of (a) for induced modules, namely

$$\mathrm{cx}_L(\mathrm{Ind}_K^L(V,\chi)) = \mathrm{cx}_K(V),$$

is *not* true in general. Look at the example of $\mathfrak{sl}_2$ and its (standard) Borel subalgebra B, where $\mathrm{cx}_B(F_{p-1}) = 1$, but $\mathrm{cx}_{\mathfrak{sl}_2}(V(p-1,0)) = 0$ as the Steinberg module $V(p-1,0)$ is projective (cf. Example (iii) in §3).

Corollary 6.7. *If L is a finite dimensional p-nilpotent restricted Lie algebra and M is a finite dimensional restricted L-module, then*

$$\mathrm{cx}_L(M) = \mathrm{gr}(H_*^\bullet(L, M)).$$

Proof. According to Theorem 6.6(d) and Proposition 6.2 in conjunction with $\mathrm{Ext}^n_{u(L,0)}(M^*, \mathbb{F}) \cong \mathrm{Ext}^n_{u(L,0)}(\mathbb{F}, M)\ \forall\ n \geq 0$, we obtain

$$\begin{aligned}\mathrm{cx}_L(M) &= \mathrm{cx}_L(M^*) = \max\{\mathrm{cx}_L(M^*, S) \mid S \text{ simple}\}\\ &= \mathrm{cx}_L(M^*, \mathbb{F}) = \mathrm{cx}_L(\mathbb{F}, M) = \mathrm{gr}(H_*^\bullet(L, M)),\end{aligned}$$

where the third equality follows from the fact that p-nilpotent restricted Lie algebras have only one simple restricted module (up to isomorphism), namely the one-dimensional trivial module $\mathbb{F}$. □

For the remainder of this section we assume that the ground field $\mathbb{F}$ is algebraically closed. We refer the reader to [Ben0, Ben2, Ca1, Ca2, Ca3, Ev] for the origin of module varieties in the cohomology theory of modular group algebras, to [FP3, FP4, FP5, Jan1, Jan2, Jan3] for more details and to [Na6, Section 3] for a slightly different presentation and examples. The *cohomology variety* X_L of a restricted Lie algebra L is the maximal spectrum of the noetherian commutative graded $\mathbb{F}$-algebra $H_*^{\mathrm{ev}}(L, \mathbb{F})$ of cohomology classes of even degree. Let M be a finite dimensional $u(L, \chi)$-module. Since $M^* \otimes_{\mathbb{F}} M$ is restricted (cf. [SF, Theorem V.2.7(3)]), $H_*^\bullet(L, M^* \otimes_{\mathbb{F}} M)$ is a $H_*^{\mathrm{ev}}(L, \mathbb{F})$-module via the cup product. The corresponding annihilator $\mathcal{A}_M$ of $H_*^\bullet(L, M^* \otimes_{\mathbb{F}} M)$ in $H_*^{\mathrm{ev}}(L, \mathbb{F})$ defines a closed homogeneous subvariety $X_L(M) := \mathrm{Supp}(H_*^\bullet(L, M^* \otimes_{\mathbb{F}} M)) := \{\mathcal{M} \in X_L \mid \mathcal{M} \supset \mathcal{A}_M\}$ of X_L, the so-called *support variety* of M. Note that $X_L = X_L(\mathbb{F})$.

The natural $\mathbb{F}$-linear mapping Φ^2 from L^* into $H_*^2(L, \mathbb{F})^{(-1)}$ induces a homomorphism $\Phi^\bullet$ of commutative graded $\mathbb{F}$-algebras from the symmetric algebra $S^\bullet(L^*)$ into $H_*^{\mathrm{ev}}(L, \mathbb{F})^{(-1)}$, where $?^{(-1)}$ means that the scalars α act through multiplication by α^p (cf. [Ho1]). Via $\Phi^\bullet$ any $H_*^\bullet(L, M^* \otimes_{\mathbb{F}} M)^{(-1)}$ can be viewed as an $S^\bullet(L^*)$-module. Then $V_L(M)$ is defined as the support of $H_*^\bullet(L, M^* \otimes_{\mathbb{F}} M)^{(-1)}$ in $S^\bullet(L^*)$. In more geometric terms, $V_L(M)$ is the image of $X_L(M)$ under the morphism Φ from X_L to $L \cong \mathbb{A}^{\dim_{\mathbb{F}} L}(\mathbb{F})$, which is induced by $\Phi^\bullet$.

Theorem 6.8. [Jan1, FP3, FP4, FP5] *For every finite dimensional $u(L,\chi)$-module M the following statements hold:*

(a) $\mathrm{cx}_L(M) = \mathrm{cx}_L(M^* \otimes_{\mathbb{F}} M) = \dim X_L(M) = \dim V_L(M)$.
(b) $V_L(M) = \{x \in L \mid x^{[p]} = 0 \text{ and } M_{|\langle x\rangle_p} \text{ is not projective }\} \cup \{0\}$.

Sketch of Proof. (a): Set $\mathcal{H} := H^{\mathrm{ev}}_*(L,\mathbb{F})$ and $\mathcal{E}_M := H^\bullet_*(L, M^* \otimes_{\mathbb{F}} M)$. Since $\mathcal{E}_M$ is a finitely generated faithful $\mathcal{H}/\operatorname{Ann}_{\mathcal{H}}(\mathcal{E}_M)$-module, we obtain in view of Proposition 6.4:

$$\begin{aligned}\mathrm{cx}_L(M) &= \mathrm{cx}_L(M,M) = \mathrm{gr}(\mathcal{E}_M) = \mathrm{gr}(\mathcal{H}/\operatorname{Ann}_{\mathcal{H}}(\mathcal{E}_M)) \\ &= \text{Krull-dim}(\mathcal{H}/\operatorname{Ann}_{\mathcal{H}}(\mathcal{E}_M)) = \dim \operatorname{Supp}_{\mathcal{H}}(\mathcal{E}_M) = \dim X_L(M).\end{aligned}$$

According to $X_L(M) = \bigcup_{S\in \mathrm{Irr}(L,\chi)} \operatorname{Supp}(\operatorname{Ext}^\bullet_{u(L,\chi)}(M,S))$, we have

$$X_L(M) \subseteq X_L(M^* \otimes_{\mathbb{F}} M).$$

This fact and an application of Theorem 6.6(c) in conjunction with the already established equality $\mathrm{cx}_L(M) = \dim X_L(M)$ imply that

$$\mathrm{cx}_L(M) = \dim X_L(M) \leq \dim X_L(M^* \otimes_{\mathbb{F}} M) = \mathrm{cx}_L(M^* \otimes_{\mathbb{F}} M) \leq \mathrm{cx}_L(M).$$

Because $\mathcal{H}$ is a finitely generated $S^\bullet(L^*)$-module, the morphism Φ from X_L into L is finite, from which we finally deduce that $\dim X_L(M) = \dim V_L(M)$.

In order to prove (b) we consider the so-called "rank variety"

$$R_L(N) := \{x \in L \mid x^{[p]} = 0 \text{ and } N_{|\langle x\rangle_p} \text{ is not projective}\} \cup \{0\}$$

of N in L for any finite dimensional $u(L,\chi)$-module N. [Fe2, Lemma 2.3] and the Reduction Theorem $\operatorname{Ext}^n_{u(L,\chi}(M,M) \cong H^n_*(L, M^* \otimes_{\mathbb{F}} M) \forall\ n \geq\ 1$ (see §3) in conjunction with Proposition 6.4 show that $R_L(M) = R_L(M^* \otimes_{\mathbb{F}} M)$. Since $M^* \otimes_{\mathbb{F}} M$ is restricted, the proof of $V_L(M) = R_L(M)$ can be reduced to the main result of [Jan1], namely the case of the one-dimensional trivial module $M = \mathbb{F}$ (see [FP4]). □

As a consequence of Theorem 6.8(b), we have

Corollary 6.9. [FP3, FP4, FP5] *Let M, N be finite dimensional $u(L,\chi)$-modules, M' be a finite dimensional $u(L,\chi')$-module and K be a p-subalgebra of L. Then the following statements hold:*

(a) $V_K(M_{|K}) = V_L(M) \cap K$.
(b) $V_L(M \oplus N) = V_L(M) \cup V_L(N)$.
(c) $V_L(M \otimes_{\mathbb{F}} M') = V_L(M) \cap V_L(M')$.
(d) $V_L(M^*) = V_L(M) = V_L(\Omega^{\pm n}(M)) \quad \forall\ n \in \mathbb{Z}$.
(e) $V_L(M) \subseteq V_L(\mathbb{F})$. □

Corollary 6.10. *Let M and N be finite dimensional $u(L,\chi)$-modules such that $V_L(M) \cap V_L(N) = 0$. Then $\operatorname{Ext}^n_{u(L,\chi)}(M,N) = 0$ for every integer $n \geq 1$.*

Proof. According to Corollary 6.9(c) and Theorem 6.8(a), we obtain that $M^* \otimes_{\mathbb{F}} N$ is projective, i.e., the assertion follows from the Reduction Theorem $\operatorname{Ext}^n_{u(L,\chi)}(M,N) \cong H^n_*(L, M^* \otimes_{\mathbb{F}} N)\ \forall\ n \geq 1$ (see §3). □

If we consider the projective completion $\operatorname{Proj}(V_L(M)) \subseteq \mathbb{P}^{\dim_{\mathbb{F}} L-1}(\mathbb{F})$ of the support variety $V_L(M)$ of a finite dimensional module M, then the connectedness of $\operatorname{Proj}(V_L(M))$ reflects the indecomposability of M:

Theorem 6.11. [FP5] *For any finite dimensional indecomposable $u(L,\chi)$-module M the projective variety $\operatorname{Proj}(V_L(M))$ is connected.* □

Finally, we present an application of the algebro-geometric methods developed in this section to a purely module-theoretic question (for another application see [Na3]).

Theorem 6.12. *Let L be a finite dimensional restricted Lie algebra and M be a finite dimensional restricted L-module. Then M is cohomologically trivial if and only if M is projective.*

Proof. Since both properties are independent of the ground field, we can assume that it is algebraically closed. Suppose that M is cohomologically trivial. In particular, we have for every element $x \in L$ that $\hat{H}^\bullet_*(\langle x\rangle_p, M_{|\langle x\rangle_p}) = 0$. If $x^{[p]} = 0$, then $\langle x\rangle_p$ is p-nilpotent and Lemma 3.3 implies that $M_{|\langle x\rangle_p}$ is projective. Hence we can conclude from Theorem 6.8 that $\operatorname{cx}_L(M) = 0$, i.e., M is projective. The other implication is trivial. □

§7. Module Type

As in §4 we are interested in the category $\operatorname{mod}(L,\chi)$ of finite dimensional L-modules with a fixed character χ over a finite dimensional restricted Lie algebra L.

Problem. Determine the characters $\chi \in L^*$ for which every $u(L,\chi)$-module is semisimple.

As far as I know, there are only two classes of restricted Lie algebras for which a complete answer to this question is known:

- For a torus L, every $u(L,\chi)$-module is semisimple (see Lemma 3.2).
- Let L be a (restricted) simple Lie algebra of classical type over an algebraically closed field $\mathbb{F}$ of characteristic > 3. Then E.M. Friedlander and B.J. Parshall have shown in [FP5] that $u(L,\chi)$ is semisimple

if and only if χ is *regular semisimple*, i.e., χ belongs to the same orbit under $\mathrm{Aut}_p(L)$ as χ_s which satisfies $\chi_s(L_{\pm\alpha}) = 0 \neq \chi_s([L_\alpha, L_{-\alpha}])$ for every root α of L relative to a classical Cartan subalgebra of L). We should mention that in this case

$$X_0(L) := \{\chi \in L^* \mid \chi \text{ is regular semisimple}\}$$

is an open and therefore dense subset of L^* (cf. [KW]). Hence $u(L,\chi)$ is semisimple for (in a geometrical sense) "most" of the characters of L.

The second example makes it reasonable to consider the set

$$X_0(L) := \{\chi \in L^* \mid u(L,\chi) \text{ is semisimple}\}$$

as a subvariety of the affine space $L^* = \mathbb{A}^{\dim_{\mathbb{F}} L}(\mathbb{F})$ (with the Zariski topology) and ask whether $X_0(L)$ is *non-empty* resp. *open*? (It is not difficult to find examples for which $X_0(L)$ is empty, e.g. consider the three-dimensional Heisenberg algebra with $z^{[p]} = 0$.)

It is well-known that any non-zero finite dimensional restricted Lie algebra has indecomposable modules of arbitrarily high dimensions [Zas2, Zas3]. Nevertheless, the dimensions of indecomposable L-modules with a given character χ might well be bounded. Following the terminology of [Po1, Po2], we say that the reduced universal enveloping algebra $u(L,\chi)$ is of *bounded module type* if there is an integer $d(L,\chi)$ such that $\dim_{\mathbb{F}} M \leq d(L,\chi)$ for every finite dimensional indecomposable $u(L,\chi)$-module M. $u(L,\chi)$ is said to be of *finite module type* if there are only finitely many isomorphism classes of finite dimensional indecomposable $u(L,\chi)$-modules. Algebras of finite module type are obviously of bounded module type, while the converse statement is a consequence of Roiter's famous solution of the first Brauer-Thrall Conjecture [Roi].

Problem. Determine the characters $\chi \in L^*$ for which $u(L,\chi)$ is of finite module type. Moreover, prove[8] that

$$X_1(L) := \{\chi \in L^* \mid u(L,\chi) \text{ is of finite module type}\}$$

is open in L^*.

[8] This can be shown by using the methods of [Ga] (see also [Kraft]) which was recently done by R. Farnsteiner [Fa10, Theorem 4.5(2)]

A module is called *uniserial* if it has a unique composition series. In this case, the finite number of members of the composition series exhaust all submodules, and therefore every submodule and every factor module of a uniserial module is also uniserial. $u(L,\chi)$ is called *serial* if every projective indecomposable $u(L,\chi)$-module is uniserial (cf. [ARS, EG1, EG2, Hup]).

We say that L has *bounded cohomology* if for every finite dimensional restricted L-module M there is an integer $b(L,M)$ such that $\dim_{\mathbb{F}} H_*^n(L,M) \leq b(L,M)$ for every $n \geq 0$. It is easy to see that L has bounded cohomology if $u(L,0)$ is of bounded module type. L has *periodic cohomology* if there exists an integer q such that for every positive integer n and all (not necessarily finite dimensional) restricted L-modules there are natural isomorphisms $H_*^n(L,M) \cong H_*^{n+q}(L,M)$. Restricted Lie algebras with periodic cohomology clearly have bounded cohomology (with upper bound $b(L,M) := \max\{\dim H_*^n(L,M) \mid 0 \leq n \leq q\}$). The reverse implication follows by using the methods of §6 (see also [Fe2, Theorem 4.1]). Moreover, [He, Proposition 2] implies that L has periodic cohomology if $u(L,0)$ is of finite module type.

Since the restricted cohomology of $\mathfrak{sl}_2(\mathbb{F})$ is *not* bounded (see e.g. [Fi, Chapter 3], it is an immediate consequence of an old classification result of J.R. Schue [Schue1] that restricted Lie algebras with bounded cohomology (over fields of characteristic > 3) need to be solvable (see also [Fe2, Proposition 4.3]). H. Strade and the author have shown the somewhat surprising fact that the five properties mentioned above are indeed equivalent. Moreover we classified the corresponding restricted Lie algebras completely.

A restricted Lie algebra L is called *cyclic* if there is an element $x \in L$ such that $L = \langle x \rangle_p$ and a p-nilpotent cyclic restricted Lie algebra is called *nilcyclic*.

Theorem 7.1. [FeS1] *Let L be a finite dimensional restricted Lie algebra over a perfect field $\mathbb{F}$ of prime characteristic p. Then the following statements are equivalent:*

(a) *$u(L,0)$ is serial.*
(b) *$u(L,0)$ is of finite module type.*
(c) *$u(L,0)$ is of bounded module type.*
(d) *L has periodic cohomology.*
(e) *L has bounded cohomology.*
(f) *L has a unique maximal abelian p-ideal I, $I/\operatorname{Tor}_p(L)$ is nilcyclic and L/I is an at most one-dimensional torus.* $\square$

Remark. W. Pfautsch and D. Voigt have announced in [PV] the classification of the restricted Lie algebras of finite module type over algebraically closed fields. They approach this problem in the context of algebraic group schemes. Our original proof of Theorem 7.1 in [FeS1] works for *perfect* fields but without some modifications not in general since we use [Fe2, Remark after Corollary 3.6] which is not valid in the generality as it was stated.[9]

From [FeS1, §3] we can see that the three-dimensional supersolvable restricted Lie algebra from Example (ii) in §2 is the "generic" case of a restricted Lie algebra of finite module type. Since its reduced universal enveloping algebras are either isomorphic to the restricted universal enveloping algebra or semisimple, the following consequence of Theorem 7.1 connecting the restricted case with the general case is not surprising (compare this with Lemma 3.2):[10]

Corollary 7.2. *Let L be a finite dimensional restricted Lie algebra over a perfect field. If $u(L, 0)$ is of finite module type, then $u(L, \chi)$ is of finite module type for every $\chi \in L^*$.* □

Remark. Corollary 7.2 shows that the restricted representations of L are the most complicated ones similar to the relationship between the principal block and the other blocks.

It is possible to derive from the complete classification of indecomposable restricted modules in [FeS1] the following generalization of an old result of B. Pareigis [Par, Korollar IV.2.4]:

Corollary 7.3. [FeS1] *If a finite dimensional restricted Lie algebra over an algebraically closed field has periodic cohomology, then the period is at most* 2. □

Corollary 7.3 has been generalized further by R. Farnsteiner [Fa10, Theorem 2.5(4)] from the trivial simple module to any finite dimensional *periodic* $u(L, \chi)$-module over an arbitrary finite dimensional restricted Lie algebra L. Recently, K. Erdmann [Erd2] has used Farnsteiner's result to give necessary conditions for the *graph structure* of the *connected components* of the *stable Auslander-Reiten quiver* of $u(L, 0)$ (see also [Fa10]).

A finite dimensional algebra A over an (at least infinite) field $\mathbb{F}$ is called *tame* if for any positive integer d almost all indecomposable A-modules

[9] This can be shown by a counterexample due to A.D. Bell which was communicated to the author by R. Farnsteiner.

[10] It is not at all clear (to the author) how one could prove Corollary 7.2 without using our classification result.

of dimension d belong to a finite number of one-parameter families (e.g. parametrized by the simple $\mathbb{F}[x]$-modules resp. the affine line $\mathbb{A}^1(\mathbb{F})$) (see [Ben1, Section 4.4]). Using a recent result of J. Rickard [Ric], H. Strade and the author [FeS2] were able to classify the restricted Lie algebras of tame module type over an algebraically closed field of characteristic $p > 2$ (cf. also [Voigt2] for a necessary condition if $p > 3$). The analogue of Corollary 7.2 for "tame" instead of "finite module type" should still be valid, and we can also introduce a subvariety $X_2(L)$ analogous to $X_1(L)$. Moreover, the classification of the (isomorphism classes of) finite dimensional indecomposable restricted $\mathfrak{sl}_2(\mathbb{F})$-modules over an algebraically closed field of characteristic > 2 was carried out (independently) by Yu.A.Drozd [Dro], A.N. Rudakov [Ru3] and G. Fischer [Fi] (with parts also due to R.D. Pollack [Po1]) by reduction to the *Kronecker quiver* (see [Ben1, Section 4.3]), similar to the classification for dihedral 2-groups in characteristic 2. The analogue problem for $\mathfrak{sl}_2(\mathbb{F})$ in characteristic 2 was solved earlier by D. Voigt [Voigt1].

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