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Abstract. This paper is a sequel to [10], where we mainly consider semi-simple Leibniz algebras. It turns out that the analogue of the Hochschild-Serre spectral sequence for Leibniz cohomology cannot be applied to many ideals, and therefore this spectral sequence seems not to be applicable for computing the cohomology of non-semi-simple Leibniz algebras. The main idea of the present paper is to use similar tools as developed by Farnsteiner for Hochschild cohomology (see [7] and [8]) to work around this. Unfortunately, it does not seem to be possible to relate the cohomology of a Leibniz algebra directly to Hochschild cohomology as is the case for Lie algebras, but all the desired results can be obtained in a similar way. In particular, this enables us to generalize the vanishing theorems of Dixmier and Barnes for nilpotent and (super)solvable Lie algebras to Leibniz algebras. Moreover, we compute the cohomology of the one-dimensional Lie algebra with values in an arbitrary Leibniz bimodule and show that it is periodic with period two. As a consequence, we prove the Leibniz analogue of a non-vanishing theorem of Dixmier. Although not needed in full for the aforementioned results, we prove a Fitting lemma for Leibniz bimodules that might be useful elsewhere.

Introduction

In a previous paper [10] we started to study the cohomology of (left) Leibniz algebras. One of our main results is the second Whitehead lemma for finite-dimensional semi-simple Leibniz algebras in characteristic zero (see [10, Theorem 4.3]). More generally, we systematically adapted Pirashvili's spectral sequences (see [17]) to cohomology and general Leibniz bimodules. One of these spectral sequences (see [10, Theorem 3.4 or Corollary 3.5]) is the Leibniz analogue of the Hochschild-Serre spectral sequence. It is clear from the E_2-term that this spectral sequence is not useful for the computation of the cohomology of Leibniz algebras with abelian ideals different from the Leibniz kernel.

In this paper we consider methods developed in order to prove vanishing theorems for Hochschild cohomology. This enables us to extend several well-known vanishing theorems from the cohomology of solvable Lie algebras to Leibniz algebras. In fact, in [7] and [8], Farnsteiner's aim is (among other results) to unify the proofs of Dixmier's vanishing theorem for nilpotent Lie algebras (see [6, Théorème 1]) and Barnes' vanishing theorems for (super)solvable Lie algebras (see [1, Theorems 2 and 3]). The crucial idea to prove the vanishing of cohomology is to employ

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a generalization of Casimir elements. The latter were used by Whitehead to prove the vanishing of the cohomology of semi-simple Lie algebras in characteristic zero. Farnsteiner showed that similar ideas can be employed to prove cohomological vanishing theorems for Lie algebras with non-zero abelian ideals and Lie algebras in prime characteristic. More precisely, one needs the existence of certain elements that act invertibly on the cohomology, while the algebra acts trivially on its cohomology due to certain Cartan type relations. The conclusion is then that the cohomology must vanish. Since it does not seem to be possible to express Leibniz cohomology in terms of Hochschild cohomology, Farnsteiner's results cannot be applied directly, but they have to be adapted to Leibniz algebras. It turns out that this is possible (see Section 3). In the first section we recall some definitions and prove several basic results that will be useful later in the paper. Section 2 is devoted to the Fitting decomposition for Leibniz bimodules. Note that in the proof of one of the main results in Section 3, namely, Theorem 3.3, we only need that the Fitting-0-component of a Leibniz bimodule is a sub-bimodule. Since this might be useful elsewhere, we also prove the analogue of Fitting's lemma for Leibniz algebras.

The main results of the present paper are contained in Section 4. In 1955, Dixmier [6] proved (non-)vanishing theorems for the Chevalley-Eilenberg cohomology of finite-dimensional nilpotent Lie algebras. The vanishing behavior depends on the coefficients having or not having a trivial composition factor. Later, Barnes [1] gave a different proof of Dixmier's vanishing theorem (see the proof of [1, Lemma 3]) using the Hochschild-Serre spectral sequence and induction on the dimension of the Lie algebra. On the other hand, Dixmier's proof of his non-vanishing theorem (see [6, Théorème 2]) relies on a long exact sequence related to the kernel of the restriction map from the cohomology of the nilpotent Lie algebra to the cohomology of an ideal of codimension one. Our overall goal in the present paper is to generalize these results to (left) Leibniz algebras. We prove analogues of Dixmier's vanishing theorem for nilpotent Lie algebras (see Theorem 4.2) and Barnes' vanishing theorems for (super)solvable Lie algebras (see Theorems 4.9 and 4.10). Another part of Section 4 is then devoted to establish a Leibniz analogue of Dixmier's non-vanishing theorem for nilpotent Lie algebras. We proceed as close as possible to Dixmier's proof, but there are several obstacles. The base step follows from the computation of the cohomology of the one-dimensional Leibniz algebra with values in an arbitrary Leibniz bimodule. Similar to the cohomology of finite cyclic groups, in positive degrees this cohomology is periodic with period 2 (see Theorem 4.3). This allows us then to proceed by induction on the dimension of the nilpotent Leibniz algebra, but our cohomological non-vanishing theorem (see Theorem 4.4) is weaker than what one would expect from Dixmier's result (see Proposition 1.5 and Examples A and C). Nevertheless, a consequence of Theorem 4.4 is that the cohomology of a nilpotent Leibniz algebra with trivial coefficients does not vanish in any degree (see Corollary 4.5). Moreover, the sufficient condition for the non-vanishing of the adjoint cohomology of a nilpotent Leibniz algebra in every degree (see Corollary 4.6), which one obtains as a special case of Theorem 4.4, is easy to verify, and it is always satisfied for a nilpotent Lie algebra (see Corollary 4.7).

As an application of the vanishing theorems in Section 4 we extend some structure theorems for (super)solvable Lie algebras (see [1, Section 3]) to Leibniz algebras.

In this paper we will follow the notation used in [9] and [10]. An algebra without any specification will be a vector space with a bilinear multiplication that not necessarily satisfies any other identity. Ideals will always be two-sided ideals if not explicitly stated otherwise. All vector spaces and algebras are defined over an arbitrary field which is only explicitly mentioned when some additional assumptions on the ground field are made or this enhances the understanding of the reader. In particular, all tensor products are over the relevant ground field and will be denoted by $\otimes$. For a subset X of a vector space V over a field $\mathbb{F}$ we let $\langle X\rangle_{\mathbb{F}}$ be the subspace of V spanned by X. We will denote the space of linear transformations from an $\mathbb{F}$-vector space V to an $\mathbb{F}$-vector space W by $\mathrm{Hom}_{\mathbb{F}}(V,W)$. In particular, $\mathrm{End}_{\mathbb{F}}(V) := \mathrm{Hom}_{\mathbb{F}}(V,V)$ is the space of linear operators on V, and $V^* := \mathrm{Hom}_{\mathbb{F}}(V,\mathbb{F})$ is the space of linear forms on V. Finally, the identity function on a set X will be denoted by id_X, the set $\{1,2,\dots\}$ of positive integers will be denoted by $\mathbb{N}$, and the set $\mathbb{N}\cup\{0\}$ of non-negative integers will be denoted by $\mathbb{N}_0$.

1. Preliminaries

In this section we recall some definitions and collect several results that will be useful in the remainder of the paper.

A *left Leibniz algebra* is an algebra $\mathfrak{L}$ such that every left multiplication operator $L_x : \mathfrak{L} \to \mathfrak{L}$, $y \mapsto xy$ is a derivation. This is equivalent to the identity

$$x(yz) = (xy)z + y(xz)$$

for all $x, y, z \in \mathfrak{L}$, which in turn is equivalent to the identity

$$(xy)z = x(yz) - y(xz)$$

for all $x, y, z \in \mathfrak{L}$. There is a similar definition of a *right Leibniz algebra*, but in this paper we will only consider left Leibniz algebras which often will just be called Leibniz algebras unless this might make matters easier to understand for the reader.

Note that every Lie algebra is a left and a right Leibniz algebra. On the other hand, every Leibniz algebra has an important ideal, its Leibniz kernel, that measures how much the Leibniz algebra deviates from being a Lie algebra. Namely, let $\mathfrak{L}$ be a Leibniz algebra over a field $\mathbb{F}$. Then

$$\mathrm{Leib}(\mathfrak{L}) := \langle x^2 \mid x \in \mathfrak{L}\rangle_{\mathbb{F}}$$

is called the *Leibniz kernel* of $\mathfrak{L}$. The Leibniz kernel $\mathrm{Leib}(\mathfrak{L})$ is an abelian ideal of $\mathfrak{L}$, and $\mathrm{Leib}(\mathfrak{L}) \neq \mathfrak{L}$ whenever $\mathfrak{L} \neq 0$ (see [9, Proposition 2.20]). Moreover, $\mathfrak{L}$ is a Lie algebra if, and only if, $\mathrm{Leib}(\mathfrak{L}) = 0$.

By definition of the Leibniz kernel, $\mathfrak{L}_{\mathrm{Lie}} := \mathfrak{L}/\mathrm{Leib}(\mathfrak{L})$ is a Lie algebra which we call the *canonical Lie algebra* associated to $\mathfrak{L}$. In fact, the Leibniz kernel is the smallest ideal such that the corresponding factor algebra is a Lie algebra (see [9, Proposition 2.22]).

Next, we will briefly discuss left modules and bimodules of left Leibniz algebras. Let $\mathfrak{L}$ be a left Leibniz algebra over a field $\mathbb{F}$. A *left $\mathfrak{L}$-module* is a vector space M over $\mathbb{F}$ with an $\mathbb{F}$-bilinear left $\mathfrak{L}$-action $\mathfrak{L} \times M \to M$, $(x, m) \mapsto x \cdot m$ such that

$$(xy) \cdot m = x \cdot (y \cdot m) - y \cdot (x \cdot m)$$

is satisfied for every $m \in M$ and all $x, y \in \mathfrak{L}$.

Moreover, every left $\mathfrak{L}$-module M gives rise to a homomorphism $\lambda : \mathfrak{L} \to \mathfrak{gl}(M)$ of left Leibniz algebras, defined by $\lambda_x(m) := x \cdot m$, and vice versa. We call λ the *left representation* of $\mathfrak{L}$ associated to M.

The correct concept of a module for a left Leibniz algebra $\mathfrak{L}$ is the notion of a Leibniz bimodule (see [9, Section 3] for the motivation behind this definition of a bimodule for a left Leibniz algebra). An $\mathfrak{L}$-*bimodule* is a vector space M with an $\mathbb{F}$-bilinear left $\mathfrak{L}$-action and an $\mathbb{F}$-bilinear right $\mathfrak{L}$-action that satisfy the following compatibility conditions:

(LLM) $$(xy) \cdot m = x \cdot (y \cdot m) - y \cdot (x \cdot m)$$
(LML) $$(x \cdot m) \cdot y = x \cdot (m \cdot y) - m \cdot (xy)$$
(MLL) $$(m \cdot x) \cdot y = m \cdot (xy) - x \cdot (m \cdot y)$$

for every $m \in M$ and all $x, y \in \mathfrak{L}$.

It is an immediate consequence of (LLM) that every Leibniz bimodule is a left Leibniz module.

On the other hand, a pair (λ, ρ) of linear transformations $\lambda : \mathfrak{L} \to \mathrm{End}_{\mathbb{F}}(V)$ and $\rho : \mathfrak{L} \to \mathrm{End}_{\mathbb{F}}(V)$ is called a *representation* of $\mathfrak{L}$ on the $\mathbb{F}$-vector space V if the following conditions are satisfied:

$$\lambda_{xy} = \lambda_x \circ \lambda_y - \lambda_y \circ \lambda_x \tag{1.1}$$

$$\rho_{xy} = \lambda_x \circ \rho_y - \rho_y \circ \lambda_x \tag{1.2}$$

$$\rho_y \circ \rho_x = -\rho_y \circ \lambda_x \tag{1.3}$$

for any elements $x, y \in \mathfrak{L}$. Note that (LML) and (MLL) are equivalent to (1.2) and (1.3).

Then every $\mathfrak{L}$-bimodule M gives rise to a representation (λ, ρ) of $\mathfrak{L}$ on M via $\lambda_x(m) := x \cdot m$ and $\rho_x(m) := m \cdot x$. Conversely, every representation (λ, ρ) of $\mathfrak{L}$ on the vector space M defines an $\mathfrak{L}$-bimodule structure on M via $x \cdot m := \lambda_x(m)$ and $m \cdot x := \rho_x(m)$.

By virtue of [9, Lemma 3.3], every left $\mathfrak{L}$-module is an $\mathfrak{L}_{\mathrm{Lie}}$-module in a natural way, and vice versa. Consequently, many properties of left Leibniz modules follow from the corresponding properties of modules for the canonical Lie algebra.

The usual definitions of the notions of *sub(bi)module*, *irreducibility*, *complete reducibility*, *composition series*, *homomorphism*, *isomorphism*, etc., hold for left Leibniz modules and Leibniz bimodules. (Note that by definition an irreducible Leibniz (bi)module is always non-zero.)

Let $\mathfrak{L}$ be a left Leibniz algebra over a field $\mathbb{F}$, and let M be an $\mathfrak{L}$-bimodule. Then M is said to be *symmetric* if $m \cdot x = -x \cdot m$ for every $x \in \mathfrak{L}$ and every $m \in M$, and M is said to be *anti-symmetric* if $m \cdot x = 0$ for every $x \in \mathfrak{L}$ and every $m \in M$. Moreover, an $\mathfrak{L}$-bimodule M is called *trivial* if $x \cdot m = 0 = m \cdot x$ for every $x \in \mathfrak{L}$ and every $m \in M$. Note that an $\mathfrak{L}$-bimodule M is trivial if, and only if, M is symmetric and anti-symmetric. We call

$$M_0 := \langle x \cdot m + m \cdot x \mid x \in \mathfrak{L}, m \in M \rangle_{\mathbb{F}}$$

the *anti-symmetric kernel* of M. It is well known that M_0 is an anti-symmetric $\mathfrak{L}$-subbimodule of M such that $M_{\mathrm{sym}} := M/M_0$ is symmetric (see [9, Proposition 3.12 and Proposition 3.13]).

Recall that every left $\mathfrak{L}$-module M of a left Leibniz algebra $\mathfrak{L}$ determines a unique symmetric $\mathfrak{L}$-bimodule structure on M by defining $m \cdot x := -x \cdot m$ for every element

$m \in M$ and every element $x \in \mathfrak{L}$ (see [9, Proposition 3.15 (b)]). We will denote this symmetric $\mathfrak{L}$-bimodule by M_s. Similarly, every left $\mathfrak{L}$-module M with trivial right action is an anti-symmetric $\mathfrak{L}$-bimodule (see [9, Proposition 3.15 (a)]) which will be denoted by M_a.

Recall that for a subset S of a left Leibniz algebra $\mathfrak{L}$ the *space of right S-invariants* of an $\mathfrak{L}$-bimodule M is

$$M^S := \{m \in M \mid \forall\, s \in S : m \cdot s = 0\}\,.$$

In particular, we have that $M^\emptyset := M$.

Our first result is an obvious generalization of [10, Lemma 1.2].

Lemma 1.1. *Let $\mathfrak{L}$ be a left Leibniz algebra, let $\mathfrak{I}$ be a left ideal of $\mathfrak{L}$, and let M be an $\mathfrak{L}$-bimodule. Then $M^\mathfrak{I}$ is an $\mathfrak{L}$-subbimodule of M.*

Proof. It follows from (LML) that $M^\mathfrak{I}$ is invariant under the left $\mathfrak{L}$-action on M, and it follows from (MLL) that $M^\mathfrak{I}$ is invariant under the right $\mathfrak{L}$-action on M. □

Let $\mathfrak{L}$ be a left Leibniz algebra over a field $\mathbb{F}$, and let M be an $\mathfrak{L}$-bimodule with associated representation (λ, ρ). We say that $\mathrm{Ann}^\ell_\mathfrak{L}(M) := \mathrm{Ker}(\lambda)$ is the *left annihilator* of M. Similarly, $\mathrm{Ann}^r_\mathfrak{L}(M) := \mathrm{Ker}(\rho)$ is the *right annihilator* of M, and $\mathrm{Ann}_\mathfrak{L}(M) := \mathrm{Ann}^\ell_\mathfrak{L}(M) \cap \mathrm{Ann}^r_\mathfrak{L}(M)$ is called the *annihilator* of M.

It is clear from the definition of $M^\mathfrak{L}$ that an $\mathfrak{L}$-bimodule M is anti-symmetric if, and only if, $M^\mathfrak{L} = M$. The following generalization of [10, Lemma 1.1] will be useful later in this paper:

Lemma 1.2. *Let $\mathfrak{L}$ be a left Leibniz algebra, let S be a subset of $\mathfrak{L}$, and let M be an $\mathfrak{L}$-bimodule such that $M^S = 0$. Then M is symmetric. In particular,* $\mathrm{Leib}(\mathfrak{L}) \subseteq \mathrm{Ann}_\mathfrak{L}(M)$.

Proof. Since M_0 is anti-symmetric, it follows from the hypothesis that

$$M_0 = M_0^S \subseteq M^S = 0\,.$$

Hence we obtain from the definition of M_0 that M is symmetric. The second part is then an immediate consequence of [9, Lemma 3.10]. □

We continue with two results that will be needed at the end of this section.

Lemma 1.3. *Let $\mathfrak{L}$ be a left Leibniz algebra, and let M be an $\mathfrak{L}$-bimodule. Then $M^\mathfrak{L} = 0$ if, and only if, M is symmetric, and M does not contain a non-zero trivial $\mathfrak{L}$-subbimodule.*

Proof. One implication is an immediate consequence of Lemma 1.2. Conversely, suppose that $M^\mathfrak{L} \neq 0$. If M is not symmetric, then there is nothing to prove. On the other hand, if M is symmetric, then it follows from Lemma 1.1 that $M^\mathfrak{L}$ is a non-zero trivial $\mathfrak{L}$-subbimodule of M. □

Note that Lemma 1.3 generalizes [10, Corollary 1.3] from irreducible to arbitrary bimodules. Moreover, for $S = \mathfrak{L}$ one implication of Lemma 1.3 is the converse of Lemma 1.2.

Lemma 1.4 generalizes Lemma 1.3 further, but needs a stronger hypothesis on the ground field and seems to hold only for finite-dimensional bimodules. Recall that $\mathfrak{L}\mathfrak{L} := \langle xy \mid x, y \in \mathfrak{L}\rangle_\mathbb{F}$ is the *derived subalgebra* of a left Leibniz algebra $\mathfrak{L}$ (see [18]).

Lemma 1.4. *Let $\mathfrak{L}$ be a left Leibniz algebra over an algebraically closed field $\mathbb{F}$, and let M be a finite-dimensional $\mathfrak{L}$-bimodule. Then $M^{\mathfrak{L}\mathfrak{L}} = 0$ if, and only if, M is symmetric, and M does not contain a one-dimensional $\mathfrak{L}$-subbimodule.*

Proof. Assume first that $M^{\mathfrak{L}\mathfrak{L}} = 0$. By virtue of Lemma 1.2, we have that M is symmetric. Suppose now that M contains a one-dimensional $\mathfrak{L}$-subbimodule $\mathbb{F}m_0$. Then there exists a linear form $\mu \in \mathfrak{L}^*$ on $\mathfrak{L}$ such that $x \cdot m_0 = \mu(x)m_0$ for every element $x \in \mathfrak{L}$, and thus we obtain from (LLM) that

$$(xy) \cdot m_0 = x \cdot (y \cdot m_0) - y \cdot (x \cdot m_0) = \mu(x)\mu(y)m_0 - \mu(x)\mu(y)m_0 = 0$$

for any elements $x, y \in \mathfrak{L}$. Hence $0 \neq m_0 \in M^{\mathfrak{L}\mathfrak{L}}$ which is a contradiction.

On the other hand, assume that M is symmetric and does not contain a one-dimensional $\mathfrak{L}$-subbimodule. Furthemore, suppose that $N := M^{\mathfrak{L}\mathfrak{L}} \neq 0$. Then N is a finite-dimensional $(\mathfrak{L}/\mathfrak{L}\mathfrak{L})$-bimodule. Since $\dim_{\mathbb{F}} N < \infty$, we conclude that N contains an irreducible left $(\mathfrak{L}/\mathfrak{L}\mathfrak{L})$-submodule U. As $\mathfrak{L}/\mathfrak{L}\mathfrak{L}$ is an abelian Lie algebra and the ground field $\mathbb{F}$ is algebraically closed, we obtain that $\dim_{\mathbb{F}} U = 1$. But then U_s is also a one-dimensional $\mathfrak{L}$-subbimodule of M which contradicts the hypothesis. □

The last two results of this section will be important in Section 4.

Proposition 1.5. *Let $\mathfrak{L}$ be a left Leibniz algebra, and let M be a finite-dimensional $\mathfrak{L}$-bimodule such that every composition factor of M is non-trivial. Then $M^{\mathfrak{L}} = M_0$, and if in addition M is symmetric, then $M^{\mathfrak{L}} = 0$.*

Proof. Firstly, for symmetric $\mathfrak{L}$-bimodules the assertion is an immediate consequence of Lemma 1.3.

Next, if M is arbitrary, then in the short exact sequence

$$0 \to M_0 \to M \to M_{\mathrm{sym}} \to 0$$

the first term is anti-symmetric and the third term is symmetric (see [9, Propositions 3.12 and 3.13]). Consequently, an application of the long exact cohomology sequence in conjunction with the statement for the symmetric case yields the exact sequence

$$0 \to M_0 = M_0^{\mathfrak{L}} \hookrightarrow M^{\mathfrak{L}} \to M_{\mathrm{sym}}^{\mathfrak{L}} = 0\,,$$

which then implies that $M^{\mathfrak{L}} = M_0$. □

The next example shows that the converse of Proposition 1.5 is not true.

Example A. Let $\mathfrak{L} := \mathbb{F}e$ be the one-dimensional Lie algebra. Consider the matrices

$$A := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$B := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then the vector space $M := \mathbb{F}^3$ is a Leibniz $\mathfrak{L}$-bimodule via $\lambda_e(m) := Am$ and $\rho_e(m) := Bm$ for any column vector $m \in M$ because the matrices A and B satisfy the identities $AB = BA$ and $B^2 = -BA$. Note that we have

$$M^{\mathfrak{L}} = \mathrm{Ker}(\rho_e) = \langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rangle_{\mathbb{F}}$$

and

$$M_0 = \mathrm{Im}(\lambda_e + \rho_e) = \langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \rangle_{\mathbb{F}},$$

which imply that $M^{\mathfrak{L}} = M_0$, but M contains the trivial $\mathfrak{L}$-subbimodule

$$\mathrm{Ker}(\lambda_e) \cap \mathrm{Ker}(\rho_e) = \langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rangle_{\mathbb{F}}.$$

Finally, a proof very similar to the one for Proposition 1.5, in which one uses Lemma 1.4 instead of Lemma 1.3, shows:

Proposition 1.6. *Let $\mathfrak{L}$ be a left Leibniz algebra over an algebraically closed field $\mathbb{F}$, and let M be a finite-dimensional $\mathfrak{L}$-bimodule such that no composition factor of M is one-dimensional. Then $M^{\mathfrak{L}\mathfrak{L}} = M_0$, and if in addition M is symmetric, then $M^{\mathfrak{L}\mathfrak{L}} = 0$.*

2. Fitting decomposition of a Leibniz bimodule

Let V be a vector space over a field $\mathbb{F}$, and let $T \in \mathrm{End}_{\mathbb{F}}(V)$ be a linear operator on V. Then T is called *locally nilpotent* if for every vector $v \in V$ there exists a positive integer $n(v)$ such that $T^{n(v)}(v) = 0$.

Moreover, we recall *Fitting's lemma for linear operators* which asserts that for every linear operator T on a finite-dimensional vector space V one has

$$V = V_0(T) \oplus V_1(T)$$

such that $T_{|V_0(T)}$ is nilpotent and $T_{|V_1(T)}$ is invertible, where

$$V_0(T) := \bigcup_{n \in \mathbb{N}} \mathrm{Ker}(T^n)$$

and

$$V_1(T) := \bigcap_{n \in \mathbb{N}} \mathrm{Im}(T^n)$$

are T-invariant subspaces of V (see [13, Section 4 of Chapter II]). In [13, Theorem 4 of Chapter II] Fitting's lemma is generalized to nilpotent Lie algebras of linear operators. For our purposes we will need an even slightly more general version of Fitting's lemma which we will prove at the end of this section. In order to be able do so, we show the following two results (see [13, Lemma 1 of Chapter II] and [8, Lemma 4.6] for Lie algebra versions of Lemma 2.1). The only new insight is that the two identities (1.1) and (1.2) are sufficient to carry out the proofs.

Lemma 2.1. *Let $\mathfrak{L}$ be a left Leibniz algebra, and let M be a left $\mathfrak{L}$-module with associated left representation $\lambda : \mathfrak{L} \to \mathfrak{gl}(M)$. If $a \in \mathfrak{L}$ is an element such that the left multiplication operator $L_a : \mathfrak{L} \to \mathfrak{L}$, $x \mapsto ax$ is locally nilpotent, then $M_0(\lambda_a)$ is an $\mathfrak{L}$-subbimodule of M.*

Proof. Let x and y be arbitrary elements of $\mathfrak{L}$. Using (1.1) and (1.2) one can prove by induction on n that

$$\lambda_x^n \circ \lambda_y = \sum_{k=0}^{n} \binom{n}{k} \lambda_{L_x^k(y)} \circ \lambda_x^{n-k} \tag{2.1}$$

and

$$\lambda_x^n \circ \rho_y = \sum_{k=0}^{n} \binom{n}{k} \rho_{L_x^k(y)} \circ \lambda_x^{n-k} \tag{2.2}$$

for every positive integer n.

Now let $m \in M_0(\lambda_a)$ and $x \in \mathfrak{L}$ be arbitrary. Since $m \in M_0(\lambda_a)$ and L_a is locally nilpotent, there exists a positive integer n such that $\lambda_a^n(m) = 0$ and $L_a^n(x) = 0$.

Firstly, we obtain from (2.1) that

$$\begin{aligned}
\lambda_a^{2n}(x \cdot m) &= (\lambda_a^{2n} \circ \lambda_x)(m) \\
&= \sum_{k=0}^{2n} \binom{2n}{k} \left(\lambda_{L_a^k(x)} \circ \lambda_a^{2n-k}\right)(m) \\
&= \sum_{k=0}^{n-1} \binom{2n}{k} \left(\lambda_{L_a^k(x)} \circ \lambda_a^{2n-k}\right)(m) \\
&= 0\,,
\end{aligned}$$

which shows that $x \cdot m \in M_0(\lambda_a)$.

Next, we obtain from (2.2) that

$$\begin{aligned}
\lambda_a^{2n}(m \cdot x) &= (\lambda_a^{2n} \circ \rho_x)(m) \\
&= \sum_{k=0}^{2n} \binom{2n}{k} \left(\rho_{L_a^k(x)} \circ \lambda_a^{2n-k}\right)(m) \\
&= \sum_{k=0}^{n-1} \binom{2n}{k} \left(\rho_{L_a^k(x)} \circ \lambda_a^{2n-k}\right)(m) \\
&= 0\,,
\end{aligned}$$

which shows that $m \cdot x \in M_0(\lambda_a)$. □

It should be mentioned that we will only need Lemma 2.1 in the remainder of the paper. But Theorem 2.3 below might be useful when one studies other aspects of Leibniz bimodules.

Lemma 2.2. *Let $\mathfrak{L}$ be a left Leibniz algebra, and let M be a finite-dimensional left $\mathfrak{L}$-module with associated left representation $\lambda : \mathfrak{L} \to \mathfrak{gl}(M)$. If $a \in \mathfrak{L}$ is an element such that the left multiplication operator $L_a : \mathfrak{L} \to \mathfrak{L}$, $x \mapsto ax$ is locally nilpotent, then $M_1(\lambda_a)$ is an $\mathfrak{L}$-subbimodule of M.*

Proof. Let x and y be arbitrary elements of $\mathfrak{L}$. Using (1.1) and (1.2) one can prove by induction on n that

$$\lambda_y \circ \lambda_x^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \lambda_x^k \circ \lambda_{L_x^{n-k}(y)} \tag{2.3}$$

and

$$\rho_y \circ \lambda_x^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \lambda_x^k \circ \rho_{L_x^{n-k}(y)} \tag{2.4}$$

for every positive integer n.

Let r be the smallest positive integer such that

$$\mathrm{Im}(\lambda_a^r) = \mathrm{Im}(\lambda_a^{r+1}) = \cdots = M_1(\lambda_a)$$

and let s be the smallest positive integer such that

$$\mathrm{Ker}(\lambda_a^s) = \mathrm{Ker}(\lambda_a^{s+1}) = \cdots = M_0(\lambda_a)\,,$$

and set $t := \max\{r, s\}$ (see [13, Section 4 of Chapter II]).

Now let $m \in M_1(\lambda_a)$ and $x \in \mathfrak{L}$ be arbitrary. Since L_a is locally nilpotent, there exists a positive integer n such that $L_a^n(x) = 0$. Moreover, there exists an element $m_0 \in M$ such that $m = \lambda_a^{t+n-1}(m_0)$. Note that the hypothesis $\dim_{\mathbb{F}} M < \infty$ implies that the integers r and s always exist.

Firstly, we obtain from (2.3) that

$$\begin{aligned} x \cdot m &= x \cdot \lambda_a^{t+n-1}(m_0) \\ &= (\lambda_x \circ \lambda_a^{t+n-1})(m_0) \\ &= \sum_{k=0}^{t+n-1} (-1)^{t+n-1-k} \binom{t+n-1}{k} \left(\lambda_a^k \circ \lambda_{L_a^{t+n-1-k}(x)}\right)(m_0) \\ &= \sum_{k=t}^{t+n-1} (-1)^{t+n-1-k} \binom{t+n-1}{k} \lambda_a^k \left(\lambda_{L_a^{t+n-1-k}(x)}(m_0)\right) \in \mathrm{Im}(\lambda_a^t) = M_1(\lambda_a)\,. \end{aligned}$$

Next, we obtain from (2.4) that

$$\begin{aligned} m \cdot x &= \lambda_a^{t+n-1}(m_0) \cdot x \\ &= (\rho_x \circ \lambda_a^{t+n-1})(m_0) \\ &= \sum_{k=0}^{t+n-1} (-1)^{t+n-1-k} \binom{t+n-1}{k} \left(\lambda_a^k \circ \rho_{L_a^{t+n-1-k}(x)}\right)(m_0) \\ &= \sum_{k=t}^{t+n-1} (-1)^{t+n-1-k} \binom{t+n-1}{k} \lambda_a^k \left(\rho_{L_a^{t+n-1-k}(x)}(m_0)\right) \in \mathrm{Im}(\lambda_a^t) = M_1(\lambda_a)\,, \end{aligned}$$

which completes the proof. □

By abuse of language we write

$$M_0(S) := \bigcap_{s \in S} M_0(\lambda_s)$$

and

$$M_1(S) := \sum_{s \in S} M_1(\lambda_s)\,.$$

The main result of this section is the following *Fitting lemma for Leibniz algebras*. Note that our proof of part (d) will reduce the statement to the corresponding statement of [13, Theorem 4 of Chapter II].

Theorem 2.3. *Let $\mathfrak{L}$ be a left Leibniz algebra over a field $\mathbb{F}$, and let M be a $\mathfrak{L}$-bimodule with associated representation (λ, ρ). If S is a subset of $\mathfrak{L}$ such that the left multiplication operator $L_s : \mathfrak{L} \to \mathfrak{L}$, $x \mapsto sx$ is locally nilpotent for every element $s \in S$, then the following statements hold:*

(a) *$M_0(S)$ is an $\mathfrak{L}$-subbimodule of M.*
(b) *Every element of S acts locally nilpotently on $M_0(S)$ from the left and from the right.*

Moreover, if $\dim_{\mathbb{F}} M < \infty$, then

(c) *$M_1(S)$ is an $\mathfrak{L}$-subbimodule of M.*
(d) *$M = M_0(S) \oplus M_1(S)$.*

Proof. (a) is an immediate consequence of the definition of $M_0(S)$ and Lemma 2.1.

(b): For the left action of S the assertion is an immediate consequence of the definition of $M_0(S)$ and for the right action the claim then follows from [15, Lemma 6].

(c): Since a sum of $\mathfrak{L}$-subbimodules is again an $\mathfrak{L}$-subbimodule, the assertion is an immediate consequence of the definition of $M_1(S)$ and Lemma 2.2.

(d): It follows from [9, Lemma 3.3] that M is a left $\mathfrak{L}_{\mathrm{Lie}}$-module via $\overline{x} \cdot m := x \cdot m$ for any elements $x \in \mathfrak{L}$ and $m \in M$. Let $\overline{\lambda} : \mathfrak{L}_{\mathrm{Lie}} \to \mathfrak{gl}(M)$ denote the corresponding representation of $\mathfrak{L}_{\mathrm{Lie}}$ on M. Then we have that $\overline{\lambda}_{\overline{x}} = \lambda_x$ for every element $x \in \mathfrak{L}$. In particular, we obtain that

$$M_0(S) = \bigcap_{s \in S} M_0(\lambda_s) = \bigcap_{s \in S} M_0(\overline{\lambda}_{\overline{s}})$$

and

$$M_1(S) = \sum_{s \in S} M_1(\lambda_s) = \sum_{s \in S} M_1(\overline{\lambda}_{\overline{s}}) \,,$$

and therefore the assertion follows from [13, Theorem 4 in Chapter II] applied to the Lie subalgebra $\mathfrak{g} := \langle \overline{\lambda}_{\overline{s}} \mid s \in S \rangle_{\mathbb{F}}$ of $\mathfrak{gl}(M)$. Namely, it follows from identity (1.1) by induction on n that

$$\left(\mathrm{ad}_{\mathfrak{g}} \overline{\lambda}_{\overline{s}}\right)^n \left(\overline{\lambda}_{\overline{t}}\right) = \lambda_{L_s^n(t)}$$

for any elements $s, t \in S$. Then this in conjunction with Engel's theorem (see [12, Theorem 3.2]) shows that $\mathfrak{g}$ is nilpotent, and thus Jacobson's result can be applied. □

The following example shows that in Theorem 2.3 the elements of S do not necessarily act invertibly on the Fitting-1-component $M_1(S)$ of a left Leibniz module or a Leibniz bimodule M.

Example B. Let $\mathfrak{g}$ be the abelian Lie subalgebra of the general linear Lie algebra $\mathfrak{gl}_2(\mathbb{F})$ generated by

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} .$$

Then $M := \mathbb{F}^2$ is a left $\mathfrak{g}$-module via $\lambda_X(m) := Xm$ for any matrix $X \in \mathfrak{g}$ and any column vector $m \in M$. If we set $S := \{E, I\}$, then $M_0(S) = \{0\}$ and $M_1(S) = M$, but clearly E does not act invertibly on M. (Note that M can be made into a

Leibniz $\mathfrak{g}$-bimodule by considering its symmetrization M_s or its anti-symmetrization M_a.)

3. Cohomology of non-semi-simple Leibniz algebras

Similar to the coboundary operator in [14, Section 1.8] for the cohomology of a right Leibniz algebra with coefficients in a Leibniz bimodule one can also introduce a coboundary operator $\mathrm{d}^\bullet$ for the cohomology of a left Leibniz algebra with coefficients in a Leibniz bimodule as follows. Let $\mathfrak{L}$ be a left Leibniz algebra over a field $\mathbb{F}$, and let M be an $\mathfrak{L}$-bimodule. For every non-negative integer n set $\mathrm{CL}^n(\mathfrak{L},M) := \mathrm{Hom}_\mathbb{F}(\mathfrak{L}^{\otimes n},M)$ and consider the linear transformation $\mathrm{d}^n : \mathrm{CL}^n(\mathfrak{L},M) \to \mathrm{CL}^{n+1}(\mathfrak{L},M)$ defined by

$$\begin{aligned}(\mathrm{d}^n f)(x_1 \otimes \cdots \otimes x_{n+1}) \quad := \quad & \sum_{j=1}^{n}(-1)^{j+1}x_j \cdot f(x_1 \otimes \cdots \otimes \hat{x}_j \otimes \cdots \otimes x_{n+1}) \\ + \quad & (-1)^{n+1} f(x_1 \otimes \cdots \otimes x_n)\cdot x_{n+1} \\ + \quad & \sum_{1\le i<j\le n+1} (-1)^i f(x_1 \otimes \cdots \otimes \hat{x}_i \otimes \cdots \otimes x_{j-1} \otimes x_i x_j \otimes x_{j+1} \otimes \cdots \otimes x_{n+1})\end{aligned}$$

for any $f \in \mathrm{CL}^n(\mathfrak{L},M)$ and any elements $x_1,\dots,x_{n+1} \in \mathfrak{L}$.

It is proved in [5, Lemma 1.3.1] that $\mathrm{CL}^\bullet(\mathfrak{L},M) := (\mathrm{CL}^n(\mathfrak{L},M),\mathrm{d}^n)_{n\in\mathbb{N}_0}$is a cochain complex, i.e., $\mathrm{d}^{n+1}\circ \mathrm{d}^n = 0$ for every non-negative integer n. Hence, one can define the *cohomology of $\mathfrak{L}$ with coefficients in an $\mathfrak{L}$-bimodule M* by

$$\mathrm{HL}^n(\mathfrak{L},M) := \mathrm{H}^n(\mathrm{CL}^\bullet(\mathfrak{L},M)) := \mathrm{ZL}^n(\mathfrak{L},M)/\mathrm{BL}^n(\mathfrak{L},M)$$

for every non-negative integer n, where

$$\mathrm{ZL}^n(\mathfrak{L},M) := \mathrm{Ker}(\mathrm{d}^n) \text{ and } \mathrm{BL}^n(\mathfrak{L},M) := \mathrm{Im}(\mathrm{d}^{n-1})\,.$$

(Note that $\mathrm{d}^{-1} := 0$.)

Moreover, we will also need the linear operator $\theta_a^n : \mathrm{CL}^n(\mathfrak{L},M) \to \mathrm{CL}^n(\mathfrak{L},M)$ defined by

$$\theta_a^n(f)(x_1 \otimes \cdots \otimes x_n) \quad := \quad a\cdot f(x_1 \otimes \cdots \otimes x_n) - \sum_{j=1}^{n} f(x_1 \otimes \cdots \otimes x_{j-1} \otimes a x_j \otimes x_{j+1} \otimes \cdots \otimes x_n)$$

for any $f \in \mathrm{CL}^n(\mathfrak{L},M)$ and any elements $a,x_1,\dots,x_n \in \mathfrak{L}$ as well as the linear transformation $\iota_a^n : \mathrm{CL}^n(\mathfrak{L},M) \to \mathrm{CL}^{n-1}(\mathfrak{L},M)$ defined by

$$\iota_a^n(f)(x_1 \otimes \cdots \otimes x_{n-1}) := f(a \otimes x_1 \otimes \cdots \otimes x_{n-1})$$

for any $f \in \mathrm{CL}^n(\mathfrak{L},M)$ and any elements $a,x_1,\dots,x_{n-1} \in \mathfrak{L}$.

Then the following identities hold for every element $a \in \mathfrak{L}$ (see [5, Proposition 1.3.2 (1) & (4)]):

$$\mathrm{d}^{n-1}\circ \iota_a^n + \iota_a^{n+1}\circ \mathrm{d}^n = \theta_a^n \tag{3.1}$$

for every positive integer n, and

$$\theta_a^{n+1}\circ \mathrm{d}^n = \mathrm{d}^n \circ \theta_a^n \tag{3.2}$$

for every non-negative integer n.

Our first result in this section is the Leibniz analogue of [8, Corollary 4.3].

Lemma 3.1. *Let V and W be left modules over a left Leibniz algebra $\mathfrak{L}$. If x is an element of $\mathfrak{L}$ such that*

(i) *x acts locally nilpotently on V, and*
(ii) *x acts invertibly on W,*

then x acts invertibly on $\mathrm{Hom}_{\mathbb{F}}(V, W)$.

Proof. Similarly to the proof in [10, Lemma 1.4 (b)], one can show that $\mathrm{Hom}_{\mathbb{F}}(V, W)$ is a left $\mathfrak{L}$-module via

$$(x \cdot f)(y) := x \cdot f(y) - f(xy)$$

for every $f \in \mathrm{Hom}_{\mathbb{F}}(V, W)$ and any elements $x, y \in \mathfrak{L}$. Then the assertion is a special case of [8, Lemma 4.2]. □

The next two results are the Leibniz analogues of results that Farnsteiner obtained for Hochschild cohomology (see [8, Theorem 4.4] and [8, Theorem 4.7]). The proofs follow those in [8] very closely, but for the convenience of the reader we include the details.

Theorem 3.2. *Let $\mathfrak{L}$ be a left Leibniz algebra, and let M be an $\mathfrak{L}$-bimodule with associated representation (λ, ρ). If a is an element of $\mathfrak{L}$ such that*

(i) *$L_a : \mathfrak{L} \to \mathfrak{L}$, $x \mapsto ax$ is locally nilpotent, and*
(ii) *$\lambda_a : M \to M$ is invertible,*

then $\mathrm{HL}^n(\mathfrak{L}, M) = 0$ for every positive integer n. Moreover, if M is symmetric, then $\mathrm{HL}^n(\mathfrak{L}, M) = 0$ for every non-negative integer n.

Proof. Let n be an arbitrary non-negative integer. Then it follows similarly as in the proof of [5, Proposition 1.3.2 (2)] that $\mathfrak{L}^{\otimes n}$ is a left $\mathfrak{L}$-module via

$$\tau_a^n(x_1 \otimes \cdots \otimes x_n) := \sum_{j=1}^{n} x_1 \otimes \cdots \otimes x_{j-1} \otimes ax_j \otimes x_{j+1} \otimes \cdots \otimes x_n \,.$$

Since by hypothesis L_a is locally nilpotent on $\mathfrak{L}$, we conclude that τ_a^n is locally nilpotent on $\mathfrak{L}^{\otimes n}$.

Recall that θ^n denotes the representation of $\mathfrak{L}$ on $\mathrm{CL}^n(\mathfrak{L}, M) := \mathrm{Hom}_{\mathbb{F}}(\mathfrak{L}^{\otimes n}, M)$ obtained from τ^n and λ. It follows from Lemma 3.1 that θ_a^n is invertible. By virtue of identity (3.2), θ^n induces a representation $\overline{\theta}^n$ of $\mathfrak{L}$ on $\mathrm{HL}^n(\mathfrak{L}, M)$ via

$$\overline{\theta}_x^n(f + \mathrm{BL}^n(\mathfrak{L}, M)) := \theta_x^n(f) + \mathrm{BL}^n(\mathfrak{L}, M) \,.$$

As a consequence, we deduce that $\overline{\theta}_a^n$ is invertible on $\mathrm{HL}^n(\mathfrak{L}, M)$.

Let n be any positive integer. Then we obtain from the identity (3.1) that $\theta_x^n(f) = \mathrm{d}^{n-1}(\iota_x^n(f)) \in \mathrm{BL}^n(\mathfrak{L}, M)$ for every $x \in \mathfrak{L}$ and every $f \in \mathrm{ZL}^n(\mathfrak{L}, M)$ which implies that $\overline{\theta}^n$ is the trivial representation of $\mathfrak{L}$ on $\mathrm{HL}^n(\mathfrak{L}, M)$. Consequently, we have that $\overline{\theta}_a^n = 0$ is an invertible linear operator on $\mathrm{HL}^n(\mathfrak{L}, M)$, and therefore $\mathrm{HL}^n(\mathfrak{L}, M) = 0$.

On the other hand, if M is symmetric, then a also acts invertibly and trivially on $M^{\mathfrak{L}}$ which yields in addition that $\mathrm{HL}^0(\mathfrak{L}, M) = M^{\mathfrak{L}} = 0$. □

Theorem 3.3. *Let $\mathfrak{L}$ be a left Leibniz algebra over a field $\mathbb{F}$, and let M be an $\mathfrak{L}$-bimodule. If S is a subset of $\mathfrak{L}$ such that*

(i) *$L_s : \mathfrak{L} \to \mathfrak{L}$, $x \mapsto sx$ is locally nilpotent for every element $s \in S$, and*

(ii) $\dim_{\mathbb{F}} M/M_0(S) < \infty$,

then $\mathrm{HL}^n(\mathfrak{L}, M) \cong \mathrm{HL}^n(\mathfrak{L}, M_0(S))$ *(as* $\mathbb{F}$*-vector spaces) for every integer* $n \geq 2$*. Moreover, if* M *is symmetric, then* $\mathrm{HL}^n(\mathfrak{L}, M) \cong \mathrm{HL}^n(\mathfrak{L}, M_0(S))$ *for every non-negative integer* n*.*

Proof. We proceed by induction on $d := \dim_{\mathbb{F}} M/M_0(S)$. The base step $d = 0$ is trivially true. So let $d > 0$. Then there exists an element $s_0 \in S$ such that $N := M_0(\lambda_{s_0}) \neq M$. Note that it follows from Lemma 2.1 that N is an $\mathfrak{L}$-subbimodule of M.

Next, we show that $N_0(S) = M_0(S)$. As $N \subseteq M$, we have that $N_0(S) \subseteq M_0(S)$. In order to prove the reverse inclusion, let $m \in M_0(S)$ be an arbitrary element. As $s_0 \in S$, we have that $m \in M_0(S) \subseteq M_0(\lambda_{s_0}) = N$, and thus $m \in N_0(S)$.

Because of $N_0(S) = M_0(S)$, we have $\dim_{\mathbb{F}} N/N_0(S) < d$, and therefore the induction hypothesis yields that

$$\mathrm{HL}^n(\mathfrak{L}, N) \cong \mathrm{HL}^n(\mathfrak{L}, N_0(S)) = \mathrm{HL}^n(\mathfrak{L}, M_0(S))$$

for any integer $n \geq 2$. Now from the short exact sequence

$$0 \to N \to M \to M/N \to 0$$

of $\mathfrak{L}$-bimodules we obtain the long exact cohomology sequence

$$\cdots \to \mathrm{HL}^{n-1}(\mathfrak{L}, M/N) \to \mathrm{HL}^n(\mathfrak{L}, N) \to \mathrm{HL}^n(\mathfrak{L}, M) \to \mathrm{HL}^n(\mathfrak{L}, M/N) \to \cdots .$$

It follows from the definition of N that the linear operator on the finite-dimensional vector space M/N induced by λ_{s_0} is injective, and therefore invertible. As a consequence, we deduce from Theorem 3.2 that $\mathrm{HL}^n(\mathfrak{L}, M/N) = 0$ for every integer $n \geq 1$, and thus we conclude

$$\mathrm{HL}^n(\mathfrak{L}, M) \cong \mathrm{HL}^n(\mathfrak{L}, N) \cong \mathrm{HL}^n(\mathfrak{L}, M_0(S))$$

for every integer $n \geq 2$.

On the other hand, if M is symmetric, then we obtain the same conclusion for every integer $n \geq 0$. □

Remark 1. The example in [8, Section 6, p. 663] in conjunction with [10, Theorem 2.6] shows that hypothesis (ii) in Theorem 3.3 is necessary. Moreover, if $\dim_{\mathbb{F}} M < \infty$, then it follows from the Fitting decomposition for the linear operator λ_{s_0} in conjunction with Theorem 3.2 that Theorem 3.3 remains true for $n = 1$. We believe that Theorem 3.3 is always true for $n = 1$, but at the moment we do not know how to prove this.

Recall that an algebra $\mathfrak{R}$ is called *nilpotent* if there exists a positive integer n such that any product of n elements in $\mathfrak{R}$, no matter how associated, is zero (see [18]). Let $\mathfrak{L}$ be a left Leibniz algebra. Then the *left descending central series*

$$^1\mathfrak{L} \supseteq {}^2\mathfrak{L} \supseteq {}^3\mathfrak{L} \supseteq \cdots$$

of $\mathfrak{L}$ is defined recursively by $^1\mathfrak{L} := \mathfrak{L}$ and $^{r+1}\mathfrak{L} := \mathfrak{L}\,(^r\mathfrak{L})$ for every positive integer r. It follows from [9, Proposition 5.3 and Lemma 5.6] that a left Lie algebra $\mathfrak{L}$ is nilpotent exactly when there exists a positive integer m such that $^m\mathfrak{L} = 0$.

The following immediate consequence of Theorem 3.3 is the analogue of [8, Corollary 6.3] for Leibniz cohomology:

Corollary 3.4. *Let $\mathfrak{L}$ be a left Leibniz algebra over a field $\mathbb{F}$, and let $\mathfrak{N}$ be a nilpotent right ideal of $\mathfrak{L}$. If M is an $\mathfrak{L}$-bimodule such that $\dim_{\mathbb{F}} M/M_0(\mathfrak{N}) < \infty$, then $\mathrm{HL}^n(\mathfrak{L}, M) \cong \mathrm{HL}^n(\mathfrak{L}, M_0(\mathfrak{N}))$ (as $\mathbb{F}$-vector spaces) for every integer $n \geq 2$. Moreover, if M is symmetric, then $\mathrm{HL}^n(\mathfrak{L}, M) \cong \mathrm{HL}^n(\mathfrak{L}, M_0(\mathfrak{N}))$ for every non-negative integer n.*

Proof. According to Theorem 3.3, we need only to show that $L_a : \mathfrak{L} \to \mathfrak{L}$ is locally nilpotent for every element $a \in \mathfrak{N}$. Since $\mathfrak{N}$ is nilpotent, this follows from $L_a^r(x) \in {}^r\mathfrak{N}$ for any element $x \in \mathfrak{L}$ and any positive integer r. □

Remark 2. The proof of Corollary 3.4 shows that L_a is nilpotent for every element $a \in \mathfrak{N}$. In fact, the same exponent can be chosen for any such element. According to Remark 1, Corollary 3.4 also holds for $n = 1$ provided $\dim_{\mathbb{F}} M < \infty$.

As a consequence of the previous result we obtain the following vanishing theorem for Leibniz cohomology which will be useful in the next section.

Corollary 3.5. *Let $\mathfrak{L}$ be a left Leibniz algebra, and let $\mathfrak{N}$ be a nilpotent right ideal of $\mathfrak{L}$. If M is a finite-dimensional $\mathfrak{L}$-bimodule such that $M^{\mathfrak{N}} = 0$, then $\mathrm{HL}^n(\mathfrak{L}, M) = 0$ for every non-negative integer n.*

Proof. It follows from Lemma 1.2 in conjunction with the hypothesis $M^{\mathfrak{N}} = 0$ that M is symmetric. Suppose that $M_0(\mathfrak{N}) \neq 0$. Since by definition the elements of $\mathfrak{N}$ act nilpotently from the left on the $\mathfrak{L}$-subbimodule $M_0(\mathfrak{N})$ of M, the Leibniz analogue of Engel's theorem for Lie algebras of linear transformations (see [15, Theorem 7] or [9, Theorem 5.17]) implies that $M_0(\mathfrak{N})^{\mathfrak{N}} \neq 0$. Consequently, we have that $0 \neq M_0(\mathfrak{N})^{\mathfrak{N}} \subseteq M^{\mathfrak{N}}$ which contradicts the hypothesis. We conclude that $M_0(\mathfrak{N}) = 0$, and thus Corollary 3.4 yields the assertion. □

As a first application of the previous results we conclude this section by proving the following analogues of [7, Theorem 2.4] for Leibniz cohomology.

We say that an $\mathfrak{L}$-bimodule M with associated representation (λ, ρ) is *right faithful* if its right annihilator $\mathrm{Ann}_{\mathfrak{L}}^r(M) := \mathrm{Ker}(\rho)$ is zero.

Corollary 3.6. *Let $\mathfrak{L}$ be a left Leibniz algebra, and let M be a finite-dimensional right faithful irreducible $\mathfrak{L}$-bimodule. If $\mathfrak{L}$ is either a non-semi-simple Lie algebra or any non-Lie Leibniz algebra, then $\mathrm{HL}^n(\mathfrak{L}, M) = 0$ for every non-negative integer n.*

Proof. Either $\mathfrak{L}$ is a Lie algebra or $\mathrm{Leib}(\mathfrak{L}) \neq 0$. In the first case there exists a non-zero abelian ideal of $\mathfrak{L}$ and in the other case $\mathrm{Leib}(\mathfrak{L})$ is such an ideal which we both call $\mathfrak{A}$. If $M^{\mathfrak{A}} = 0$, the assertion follows from Corollary 3.5. Otherwise we obtain from Lemma 1.1 that $M^{\mathfrak{A}}$ is a non-zero $\mathfrak{L}$-subbimodule of M. As M is irreducible, this implies that $M^{\mathfrak{A}} = M$, and thus $0 \neq \mathfrak{A} \subseteq \mathrm{Ann}_{\mathfrak{L}}^r(M)$. But by hypothesis the latter is zero which is a contradiction. □

Remark 3. Note that for non-semi-simple Lie algebras Corollary 3.6 can also be obtained from [7, Theorem 2.4] in conjunction with [10, Theorem 2.6].

Corollary 3.7. *Let $\mathfrak{L}$ be a left Leibniz algebra over a field of characteristic zero. If M is a finite-dimensional right faithful irreducible $\mathfrak{L}$-bimodule, then $\mathrm{HL}^n(\mathfrak{L}, M) = 0$ for every non-negative integer n.*

Proof. If $\mathfrak{L}$ is semisimple, the assertion follows from [10, Theorem 4.2], and if $\mathfrak{L}$ is not semisimple, then the assertion is a consequence of Corollary 3.6. □

Finally, as a consequence of Corollary 3.7 and Corollary 3.6 we obtain the Leibniz analogue of [7, Theorem 2.4]:

Corollary 3.8. *Let $\mathfrak{L}$ be a left Leibniz algebra over a field $\mathbb{F}$, and let M be a finite-dimensional right faithful irreducible $\mathfrak{L}$-bimodule such that $\mathrm{HL}^n(\mathfrak{L}, M) \neq 0$ for some non-negative integer n. Then $\mathrm{char}(\mathbb{F}) > 0$ and $\mathfrak{L}$ is a semi-simple Lie algebra.*

4. Cohomology of solvable Leibniz algebras

As a special case of Corollary 3.5 we obtain the following generalization of the analogue of Barnes' vanishing theorem for finite-dimensional nilpotent Lie algebras [1, Lemma 3 or Theorem 1] to Leibniz algebras of arbitrary dimension:

Theorem 4.1. *Let $\mathfrak{L}$ be a nilpotent left Leibniz algebra. If M is a finite-dimensional $\mathfrak{L}$-bimodule such that $M^{\mathfrak{L}} = 0$, then $\mathrm{HL}^n(\mathfrak{L}, M) = 0$ for every non-negative integer n.*

Theorem 4.1 and Proposition 1.5 enable us now to prove a Leibniz analogue of Dixmier's vanishing theorem for nilpotent Lie algebras [6, Théorème 1] (see also [10, Proposition 2.8] for our preliminary attempt to obtain such a result).

Theorem 4.2. *Let $\mathfrak{L}$ be a finite-dimensional nilpotent left Leibniz algebra, and let M be a finite-dimensional $\mathfrak{L}$-bimodule. If every composition factor of M is non-trivial, then*

$$\mathrm{HL}^n(\mathfrak{L}, M) \cong \begin{cases} M_0 & \text{if } n = 0 \\ 0 & \text{if } n \geq 1\,. \end{cases}$$

Moreover, if M is symmetric, then $\mathrm{HL}^n(\mathfrak{L}, M) = 0$ for every non-negative integer n.

Proof. For $n = 0$ the assertion is just Proposition 1.5. The proof for $n > 0$ is divided into three steps. Firstly, for symmetric $\mathfrak{L}$-bimodules the statement follows from Proposition 1.5 and Theorem 4.1.

Now suppose that M is anti-symmetric. It is clear that subbimodules and homomorphic images of anti-symmetric bimodules are again anti-symmetric. By using the long exact cohomology sequence, it is therefore enough to prove the first part of the theorem for irreducible anti-symmetric $\mathfrak{L}$-bimodules. In this case we obtain from [10, Lemma 1.4 (b)] that

$$\mathrm{HL}^n(\mathfrak{L}, M) \cong \mathrm{HL}^{n-1}(\mathfrak{L}, \mathrm{Hom}_{\mathbb{F}}(\mathfrak{L}, M)_s) \cong \mathrm{HL}^{n-1}(\mathfrak{L}, (\mathfrak{L}^* \otimes M)_s)$$

for every positive integer n. By refining the left descending central series of $\mathfrak{L}$ (see [9, Section 5]), one can construct a composition series

$$\mathfrak{L}_{\mathrm{ad},\ell} = \mathfrak{L}_k \supset \mathfrak{L}_{k-1} \supset \cdots \supset \mathfrak{L}_1 \supset \mathfrak{L}_0 = 0$$

of the left adjoint $\mathfrak{L}$-module such that $\mathfrak{L}_j/\mathfrak{L}_{j-1}$ is the trivial one-dimensional $\mathfrak{L}$-module $\mathbb{F}$ for every integer $1 \leq j \leq k$. From the short exact sequences $0 \to \mathfrak{L}_{j-1} \to \mathfrak{L}_j \to \mathbb{F} \to 0$, we obtain by dualizing, tensoring each term with M, and symmetrizing the short exact sequences:

$$0 \to M_s \to (\mathfrak{L}_j^* \otimes M)_s \to (\mathfrak{L}_{j-1}^* \otimes M)_s \to 0$$

for every integer $1 \leq j \leq k$. Since M is a non-trivial irreducible left $\mathfrak{L}$-module, we conclude that M_s is a non-trivial irreducible symmetric $\mathfrak{L}$-bimodule. Hence, we obtain inductively from the long exact cohomology sequence that $\mathrm{HL}^n(\mathfrak{L}, M) \cong \mathrm{HL}^{n-1}(\mathfrak{L}, (\mathfrak{L}^* \otimes M)_s) = 0$ for every positive integer n.

Finally, if M is arbitrary, then in the short exact sequence

$$0 \to M_0 \to M \to M_{\mathrm{sym}} \to 0$$

the first term is anti-symmetric and the third term is symmetric. Hence, another application of the long exact cohomology sequence in conjunction with the statements for the symmetric and the anti-symmetric case yields that $\mathrm{HL}^n(\mathfrak{L}, M) = 0$ for every positive integer. □

Let $\mathfrak{L} := \mathbb{F}e$ be the one-dimensional Lie algebra. It is immediate from the definition of Leibniz cohomology that $\dim_{\mathbb{F}} \mathrm{HL}^n(\mathfrak{L}, \mathbb{F}) = 1$ for every non-negative integer n. In the following we will generalize this to arbitrary $\mathfrak{L}$-bimodules M. We will see that in positive degrees $\mathrm{HL}^n(\mathfrak{L}, M)$ is periodic with period 2. This is very similar to the cohomology of a finite cyclic group.

Theorem 4.3. *Let $\mathfrak{L} := \mathbb{F}e$ be the one-dimensional Lie algebra, and let M be a Leibniz $\mathfrak{L}$-bimodule. Then*

$$\mathrm{HL}^n(\mathfrak{L}, M) \cong \begin{cases} M^{\mathfrak{L}} & \text{if } n = 0 \\ M^0/M\mathfrak{L} & \text{if } n \text{ is odd} \\ M^{\mathfrak{L}}/M_0 & \text{if } n \text{ is even and } n \neq 0 \end{cases}$$

(as $\mathbb{F}$-vector spaces) for every non-negative integer n, where

$$M^0 := \{m \in M \mid e \cdot m + m \cdot e = 0\}\,.$$

Moreover, if M is finite dimensional, then

$$M^0/M\mathfrak{L} \cong M^{\mathfrak{L}}/M_0$$

(as $\mathbb{F}$-vector spaces).

Proof. As $\dim_{\mathbb{F}} \mathfrak{L} = 1$, we have that $\mathrm{CL}^n(\mathfrak{L}, M) := \mathrm{Hom}_{\mathbb{F}}(\mathfrak{L}^{\otimes n}, M) \cong M$ for every integer $n \geq 0$. Moreover, we obtain that

$$\mathrm{d}^n(m) = \sum_{j=1}^{n} (-1)^{j+1} e \cdot m + (-1)^{n+1} m \cdot e$$

for any $f \in \mathrm{CL}^n(\mathfrak{L}, M)$ from which it follows that

$$\mathrm{d}^n(m) = \begin{cases} -m \cdot e & \text{if } n \text{ is even} \\ e \cdot m + m \cdot e & \text{if } n \text{ is odd} \end{cases}$$

for every non-negative integer n, and therefore

$$\mathrm{Ker}(\mathrm{d}^n) = \begin{cases} M^{\mathfrak{L}} & \text{if } n \text{ is even} \\ M^0 & \text{if } n \text{ is odd} \end{cases}$$

and

$$\mathrm{Im}(\mathrm{d}^{n-1}) = \begin{cases} M_0 & \text{if } n \text{ is even} \\ M\mathfrak{L} & \text{if } n \text{ is odd}\,. \end{cases}$$

This immediately implies the first part of the theorem.

Now let us assume that $\dim_{\mathbb{F}} M < \infty$. If (λ, ρ) denotes the associated representation of M, then we have that $M^{\mathfrak{L}} = \mathrm{Ker}(\rho_e)$, $M\mathfrak{L} = \mathrm{Im}(\rho_e)$, $M^0 = \mathrm{Ker}(\lambda_e + \rho_e)$, and $M_0 = \mathrm{Im}(\lambda_e + \rho_e)$. In this case we deduce from the dimension formula for linear transformations that

$$\dim_{\mathbb{F}} M^{\mathfrak{L}} + \dim_{\mathbb{F}} M\mathfrak{L} = \dim_{\mathbb{F}} M = \dim_{\mathbb{F}} M^0 + \dim_{\mathbb{F}} M_0\,,$$

or equivalently,

$$\dim_{\mathbb{F}} M^0/M\mathfrak{L} = \dim_{\mathbb{F}} M^{\mathfrak{L}}/M_0\,,$$

which finishes the proof of the second part of the theorem. □

Note that in the special case of the one-dimensional Lie algebra $\mathfrak{L} = \mathbb{F}e$ the inclusions $M_0 \subseteq M^{\mathfrak{L}}$ and $M\mathfrak{L} \subseteq M^0$ can be obtained directly without the coboundary property. Namely, it follows from identity (1.3) that $\rho_e \circ (\lambda_e + \rho_e) = 0$, and thus $M_0 \subseteq M^{\mathfrak{L}}$. Similarly, we conclude from (1.2) and $e^2 = 0$ that $\lambda_e \circ \rho_e = \rho_e \circ \lambda_e$, and therefore

$$(\lambda_e + \rho_e) \circ \rho_e = \lambda_e \circ \rho_e + \rho_e^2 = \rho_e \circ \lambda_e + \rho_e^2 = \rho_e \circ (\lambda_e + \rho_e) = 0\,,$$

which yields $M\mathfrak{L} \subseteq M^0$.

It would be very interesting to find other Leibniz algebras with periodic cohomology or even characterize all such Leibniz algebras. We hope to come back to these questions on another occasion.

Next, we use Theorem 4.3 to prove a Leibniz analogue of Dixmier's non-vanishing theorem for nilpotent Lie algebras [6, Théorème 2]. According to Proposition 1.5, $M^{\mathfrak{L}} \neq M_0$ implies that the $\mathfrak{L}$-bimodule M has a trivial composition factor. In the next result we prove that the stronger of these two conditions is sufficient for the non-vanishing of $\mathrm{HL}^n(\mathfrak{L}, M)$ in every positive degree. But note that it follows from Theorem 4.3 that for the three-dimensional Leibniz bimodule M over the one-dimensional Lie algebra in Example A of Section 1 the cohomology vanishes in every positive degree, but every composition factor of M is trivial. This shows that the obvious analogue of the hypothesis in [6, Théorème 2] is not strong enough to guarantee the non-vanishing of Leibniz cohomology in positive degrees.

Theorem 4.4. *Let $\mathfrak{L}$ be a non-zero finite-dimensional nilpotent left Leibniz algebra, and let M be a finite-dimensional $\mathfrak{L}$-bimodule. If $M^{\mathfrak{L}} \neq M_0$, then $\mathrm{HL}^n(\mathfrak{L}, M) \neq 0$ for every non-negative integer n.*

Proof. For the proof we will use Dixmier's exact sequence (see [6, Proposition 1]), adapted to Leibniz algebras. Since $\mathfrak{L}$ is nilpotent, it has an ideal $\mathfrak{I}$ of codimension one, and therefore $\mathfrak{L} = \mathfrak{I} \oplus \mathbb{F}x$ for some element $x \in \mathfrak{L} \setminus \mathfrak{I}$. The restriction of a Leibniz cochain of $\mathfrak{L}$ to the ideal $\mathfrak{I}$ induces a short exact sequence of cochain complexes

$$0 \to \mathrm{DL}^\bullet(\mathfrak{L}, M) \to \mathrm{CL}^\bullet(\mathfrak{L}, M) \stackrel{\mathrm{res}^\bullet}{\to} \mathrm{CL}^\bullet(\mathfrak{I}, M) \to 0\,,$$

but the kernel cochain complex $\mathrm{DL}^\bullet(\mathfrak{L}, M) := \mathrm{Ker}(\mathrm{res}^\bullet)$ is much more complicated than in the Lie algebra case because of

$$\mathrm{DL}^n(\mathfrak{L}, M) = \bigoplus_{j=1}^{n} \mathrm{Hom}_{\mathbb{F}}(\mathfrak{I}^{\otimes(j-1)} \otimes \mathbb{F}x \otimes \mathfrak{I}^{\otimes(n-j)}, M) \oplus \ldots \oplus \mathrm{Hom}_{\mathbb{F}}([\mathbb{F}x]^{\otimes n}, M)\,,$$

where the dots indicate Hom spaces with two, three, ..., $n-1$ occurrences of factors $\mathbb{F}x$ (at arbitrary places). By using Dixmier's method and the Cartan relations for

Leibniz cohomology, we can identify one summand in the cohomology of $\mathrm{DL}^\bullet(\mathfrak{L}, M)$, namely, the summand corresponding to

$$\mathrm{DL}_1^n(\mathfrak{L}, M) := \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}x \otimes \mathfrak{I}^{\otimes(n-1)}, M) \subseteq \mathrm{CL}^n(\mathfrak{L}, M)\,.$$

Let us show that the isomorphism of vector spaces

$$(-1)^n\varphi^n : \mathrm{DL}_1^n(\mathfrak{L}, M) \to \mathrm{CL}^{n-1}(\mathfrak{I}, M), \quad \varphi^n(f) := \mathrm{res}^{n-1}(\iota_x^n(f))$$

(where $\iota_x^\bullet$ is the insertion operator into the first component) is compatible with the Leibniz coboundary operator. Indeed, by identity (3.1), we have

$$(\varphi^{n+1} \circ \mathrm{d}^n)(f) = \mathrm{res}^n(\iota_x^{n+1}(\mathrm{d}^n f)) = -\mathrm{res}^n(\mathrm{d}^{n-1}(i_x^n(f))) = -(\mathrm{d}^{n-1} \circ \varphi^n)(f)$$

as $\mathrm{res}^n(\theta_x^n(f)) = 0$ since the first component of the cochain vanishes on $\mathfrak{I}$. Note that this reasoning works for any integer $n \geq 1$ because of the validity of the Cartan identity (3.1) in this range.

Furthermore, the composition of the connecting homomorphism

$$\partial^n : \mathrm{HL}^n(\mathfrak{I}, M) \to \mathrm{H}^{n+1}(\mathrm{DL}^\bullet(\mathfrak{L}, M), \mathrm{d}^\bullet)$$

with φ^{n+1} is given by $\mathrm{res}^n \circ \theta_x^n$. Indeed, ∂^n is defined by lifting a cocycle $c \in \mathrm{ZL}^n(\mathfrak{I}, M)$ to $\tilde{c}$ in $\mathrm{CL}^n(\mathfrak{L}, M)$, then taking $\mathrm{d}^n(\tilde{c})$ and observing that the result lies in $\mathrm{DL}^{n+1}(\mathfrak{L}, M)$. The composition with φ^{n+1} gives thus a cocycle $c \in \mathrm{ZL}^n(\mathfrak{I}, M)$

$$(\varphi^{n+1} \circ \partial^n)(c) = \mathrm{res}^n(\iota_x^{n+1}(\mathrm{d}^n(\tilde{c}))) = (\mathrm{res}^n \circ \theta_x^n)(\tilde{c})$$

again because of the Cartan formula (3.1) saying $\theta_x^n(\tilde{c}) = \iota_x^{n+1}(\mathrm{d}^n(\tilde{c})) + \mathrm{d}^{n-1}(\iota_x^n(\tilde{c}))$, where the last term is a coboundary.

Now let us prove the theorem. We proceed by induction on the dimension of $\mathfrak{L}$. The base step follows from Theorem 4.3 in conjunction with the hypothesis. Suppose therefore that for all nilpotent Leibniz algebras of dimension less than the dimension of $\mathfrak{L}$ the cohomology with values in a finite-dimensional bimodule M for which $M^{\mathfrak{L}} \neq M_0$ is non-zero. We apply the induction hypothesis to an ideal $\mathfrak{I}$ in $\mathfrak{L}$ of codimension one. Note that $M^{\mathfrak{K}} \neq M_0(\mathfrak{K})$ is satisfied for every subalgebra $\mathfrak{K}$ of $\mathfrak{L}$, where $M_0(\mathfrak{K}) := \langle y \cdot m + m \cdot y \mid y \in \mathfrak{K}, m \in M\rangle_{\mathbb{F}}$. Namely one obtains from $M^{\mathfrak{K}} = M_0(\mathfrak{K})$, that $M^{\mathfrak{L}} = M_0(\mathfrak{L})$ as $M^{\mathfrak{L}} \subseteq M^{\mathfrak{K}} = M_0(\mathfrak{K}) \subseteq M_0(\mathfrak{L})$, and the other inclusion is always true as one can see from [9, Lemma 3.7]. Moreover, the long exact sequence which we constructed in the beginning of the proof reads

$$\cdots \to \mathrm{HL}^n(\mathfrak{L}, M) \stackrel{\mathrm{res}^n}{\to} \mathrm{HL}^n(\mathfrak{I}, M) \stackrel{\partial^n}{\to} \mathrm{H}^{n+1}(\mathrm{DL}^\bullet(\mathfrak{L}, M), \mathrm{d}^\bullet) \stackrel{\sigma^{n+1}}{\to} \mathrm{HL}^{n+1}(\mathfrak{L}, M) \to \cdots\,,$$

where the linear transformation $\sigma^n : \mathrm{H}^n(\mathrm{DL}^\bullet(\mathfrak{L}, M), \mathrm{d}^\bullet) \to \mathrm{HL}^n(\mathfrak{L}, M)$ is induced by the inclusion $\mathrm{DL}^n(\mathfrak{L}, M) \hookrightarrow \mathrm{CL}^n(\mathfrak{L}, M)$. By the induction hypothesis and the preceding constructions, the connecting homomorphism factors over

$$0 \neq \mathrm{HL}^n(\mathfrak{I}, M) \stackrel{(\theta_x^n)_{|\mathfrak{I}}}{\to} \mathrm{HL}^n(\mathfrak{I}, M) \subseteq \mathrm{H}^{n+1}(\mathrm{DL}^\bullet(\mathfrak{L}, M), \mathrm{d}^\bullet)\,.$$

By virtue of Theorem 3.3 and Remark 1, we may assume that $M = M_0(\mathfrak{L})$. Hence, $\mathfrak{L}$ acts locally nilpotently on M, and thus x acts nilpotently on M. From this one deduces that θ_x^n is nilpotent, and thus ∂^n cannot be an isomorphism. More precisely, ∂^n cannot be surjective onto the factor $\mathrm{DL}_1^{n+1}(\mathfrak{L}, M)$. This entails that $\sigma^{n+1} \neq 0$, and therefore $\mathrm{HL}^{n+1}(\mathfrak{L}, M) \neq 0$. □

As an immediate consequence of Theorem 4.4 we obtain the following result:

Corollary 4.5. *Let $\mathfrak{L}$ be a non-zero finite-dimensional nilpotent left Leibniz algebra. Then $\mathrm{HL}^n(\mathfrak{L}, \mathbb{F}) \neq 0$ for every non-negative integer n.*

Remark 4. Note that $\dim_{\mathbb{F}} \mathrm{HL}^n(\mathbb{F}e, \mathbb{F}) = 1$ for every non-negative integer n shows that Corollary 4.5 (and thus Theorem 4.4) is best possible (see also Example C below for a non-Lie Leibniz algebra with one-dimensional trivial cohomology in every non-negative degree).

Next, we apply Theorem 4.4 to prove that the cohomology of a finite-dimensional nilpotent left Leibniz algebra with coefficients in the adjoint bimodule does not vanish in any degree provided its left center is not contained in its Leibniz kernel. Recall that

$$C^\ell(\mathfrak{L}) := \{c \in \mathfrak{L} \mid \forall\, x \in \mathfrak{L} : cx = 0\}$$

is the *left center* of a left Leibniz algebra $\mathfrak{L}$. It is well-known that $\mathrm{Leib}(\mathfrak{L}) \subseteq C^\ell(\mathfrak{L}) = (\mathfrak{L}_{\mathrm{ad}})^{\mathfrak{L}}$ (see [9, Proposition 2.13]), but not necessarily conversely.

Corollary 4.6. *Let $\mathfrak{L}$ be a finite-dimensional nilpotent left Leibniz algebra such that $C^\ell(\mathfrak{L}) \neq \mathrm{Leib}(\mathfrak{L})$. Then $\mathrm{HL}^n(\mathfrak{L}, \mathfrak{L}_{\mathrm{ad}}) \neq 0$ for every non-negative integer n.*

Proof. Note that $(\mathfrak{L}_{\mathrm{ad}})^{\mathfrak{L}} = C^\ell(\mathfrak{L})$, and by virtue of [9, Example 3.11], we have $(\mathfrak{L}_{\mathrm{ad}})_0 \subseteq \mathrm{Leib}(\mathfrak{L})$. Consequently, we obtain from $C^\ell(\mathfrak{L}) \neq \mathrm{Leib}(\mathfrak{L})$ that $(\mathfrak{L}_{\mathrm{ad}})^{\mathfrak{L}} \neq (\mathfrak{L}_{\mathrm{ad}})_0$, and thus we can apply Theorem 4.4 to obtain the assertion. □

Since the center of a non-zero nilpotent Lie algebra is always non-zero, we deduce from Corollary 4.6:

Corollary 4.7. *If $\mathfrak{g}$ is a non-zero finite-dimensional nilpotent Lie algebra, then $\mathrm{HL}^n(\mathfrak{g}, \mathfrak{g}_{\mathrm{ad}}) \neq 0$ for every non-negative integer n.*

Let us illustrate the above results by some cohomology computations for the two-dimensional nilpotent non-Lie Leibniz algebra $\mathfrak{N} := \mathbb{F}e \oplus \mathbb{F}f$ with multiplication $ee = ef = fe = 0$ and $ff = e$ (see [9, Example 2.4]).

Example C: In the following we consider the cohomology of the two-dimensional nilpotent non-Lie Leibniz algebra $\mathfrak{N}$ with trivial or adjoint coefficients.

In Example C of [10], we stated incorrectly that the higher differentials d_r for $r \geq 2$ in Pirashvili's spectral sequence (see [10, Corollary 3.5]) are zero. But this is not the case. Indeed, the differential $\mathrm{d}_2 : E_2^{0,1} \to E_2^{2,0}$ sends $e^* \in E_2^{0,1} = \mathrm{HL}^0(\mathfrak{N}_{\mathrm{Lie}}, \mathrm{Leib}(\mathfrak{N})_s^*) \otimes \mathrm{HL}^1(\mathfrak{N}, \mathbb{F})$ to $\mathrm{d}_2 e^* \in E_2^{2,0} = \mathrm{HL}^2(\mathfrak{N}_{\mathrm{Lie}}, \mathrm{Leib}(\mathfrak{N})_s^*) \otimes \mathrm{HL}^0(\mathfrak{N}, \mathbb{F})$ with $(\mathrm{d}_2 e^*)(f \otimes f) = -e^*(ff) = -e^*(e) = -1$.

By computing explicitly cocycles and coboundaries, one can show

$$\dim_{\mathbb{F}} \mathrm{HL}^0(\mathfrak{N}, \mathbb{F}) = \dim_{\mathbb{F}} \mathrm{HL}^1(\mathfrak{N}, \mathbb{F}) = \dim_{\mathbb{F}} \mathrm{HL}^2(\mathfrak{N}, \mathbb{F}) = \dim_{\mathbb{F}} \mathrm{HL}^3(\mathfrak{N}, \mathbb{F}) = 1\,.$$

More precisely, we have

$$\mathrm{HL}^0(\mathfrak{N}, \mathbb{F}) = \langle e \rangle_{\mathbb{F}}\,,$$
$$\mathrm{HL}^1(\mathfrak{N}, \mathbb{F}) = \langle f^* \rangle_{\mathbb{F}}\,,$$
$$\mathrm{HL}^2(\mathfrak{N}, \mathbb{F})) = \langle f^* \otimes e^* \rangle_{\mathbb{F}}\,,$$
$$\mathrm{HL}^3(\mathfrak{N}, \mathbb{F}) = \langle f^* \otimes e^* \otimes f^* \rangle_{\mathbb{F}}\,.$$

In fact, Gnedbaye has proven in [11, (4.2), p. 22] that $\dim_{\mathbb{F}} \mathrm{HL}^n(\mathfrak{N}, \mathbb{F}) = 1$ for every non-negative integer n.

Let us now discuss the adjoint cohomology of $\mathfrak{N}$. We have that

$$\mathrm{HL}^0(\mathfrak{N}, \mathfrak{N}_{\mathrm{ad}}) = C^\ell(\mathfrak{N}) = \mathbb{F}e\,.$$

Moreover, by computing explicitly cocycles and coboundaries, one can show

$$\dim_{\mathbb{F}} \mathrm{HL}^1(\mathfrak{N}, \mathfrak{N}_{\mathrm{ad}}) = \dim_{\mathbb{F}} \mathrm{HL}^2(\mathfrak{N}, \mathfrak{N}_{\mathrm{ad}}) = 1\,.$$

We believe that this pattern continues in higher degrees, i.e., we conjecture that $\dim_{\mathbb{F}} \mathrm{HL}^n(\mathfrak{N}, \mathfrak{N}_{\mathrm{ad}}) = 1$ for every non-negative integer n.

Note that in the case $\mathrm{char}(\mathbb{F}) \neq 2$ we have that $\mathfrak{N}_{\mathrm{ad}}^{\mathfrak{N}} = \mathbb{F}e = (\mathfrak{N}_{\mathrm{ad}})_0$. Our conjecture would imply that the condition in Theorem 4.4 is not necessary for the non-vanishing of $\mathrm{HL}^n(\mathfrak{N}, \mathfrak{N}_{\mathrm{ad}})$ in any positive degree n. This example also shows that for Leibniz algebras the condition in Theorem 4.2 cannot be replaced by $M^{\mathfrak{L}} = M_0$, and thus the dichotomy in the (non-)vanishing of their cohomology as present in Dixmier's theorems does not seem to hold for Leibniz cohomology.

We say that a Leibniz algebra $\mathfrak{L}$ is *supersolvable* if there exists a chain

$$\mathfrak{L} = \mathfrak{L}_k \supset \mathfrak{L}_{k-1} \supset \cdots \supset \mathfrak{L}_1 \supset \mathfrak{L}_0 = 0$$

of ideals of $\mathfrak{L}$ such that $\dim_{\mathbb{F}} \mathfrak{L}_j/\mathfrak{L}_{j-1} = 1$ for every integer $1 \leq j \leq k$. Note that finite-dimensional nilpotent Leibniz algebras are supersolvable and supersolvable Leibniz algebras are solvable. Moreover, over algebraically closed fields of characteristic zero, every finite-dimensional solvable Leibniz algebra is supersolvable (see [16, Corollary 2] or [9, Corollary 6.7]). Finally, as for Lie algebras, it is not difficult to see that subalgebras and homomorphic images of supersolvable Leibniz algebras are again supersolvable.

Since $\mathfrak{L}$ is supersolvable, we have that the derived subalgebra $\mathfrak{L}\mathfrak{L}$ of $\mathfrak{L}$ is a nilpotent ideal of $\mathfrak{L}$ (see the proof of [16, Corollary 3]). Hence, the following result is a special case of Corollary 3.5:

Theorem 4.8. *Let $\mathfrak{L}$ be a supersolvable left Leibniz algebra. If M is a finite-dimensional $\mathfrak{L}$-bimodule such that $M^{\mathfrak{L}\mathfrak{L}} = 0$, then $\mathrm{HL}^n(\mathfrak{L}, M) = 0$ for every non-negative integer n.*

Similarly to the nilpotent case, Theorem 4.8 and Proposition 1.6 enable us to prove a Leibniz analogue of Barnes' vanishing theorem for supersolvable Lie algebras [1, Theorem 3] (see also [10, Proposition 2.7]). More general than in Barnes' result, we allow Leibniz bimodules that are not necessarily irreducible. Note also that the proof of Theorem 4.9 is very similar to the proof of Theorem 4.2.

Theorem 4.9. *Let $\mathfrak{L}$ be a supersolvable left Leibniz algebra over an algebraically closed field $\mathbb{F}$, and let M be a finite-dimensional $\mathfrak{L}$-bimodule. If no composition factor of M is one-dimensional, then*

$$\mathrm{HL}^n(\mathfrak{L}, M) \cong \begin{cases} M_0 & \text{if } n = 0 \\ 0 & \text{if } n \geq 1\,. \end{cases}$$

Moreover, if M is symmetric, then $\mathrm{HL}^n(\mathfrak{L}, M) = 0$ for every non-negative integer n.

Proof. For $n = 0$ the assertion is just Proposition 1.5. The proof for $n > 0$ is divided into three steps. Firstly, for symmetric $\mathfrak{L}$-bimodules the statement follows from Proposition 1.6 and Theorem 4.8.

Now suppose that M is anti-symmetric. It is clear that subbimodules and homomorphic images of anti-symmetric bimodules are again anti-symmetric. By using the long exact cohomology sequence, it is therefore enough to prove the first part of the theorem for irreducible anti-symmetric $\mathfrak{L}$-bimodules. In this case we obtain from [10, Lemma 1.4 (b)] that

$$\mathrm{HL}^n(\mathfrak{L}, M) \cong \mathrm{HL}^{n-1}(\mathfrak{L}, \mathrm{Hom}_{\mathbb{F}}(\mathfrak{L}, M)_s) \cong \mathrm{HL}^{n-1}(\mathfrak{L}, (\mathfrak{L}^* \otimes M)_s)$$

for every positive integer n. By definition of supersolvability, the left adjoint $\mathfrak{L}$-module has a composition series

$$\mathfrak{L}_{\mathrm{ad},\ell} = \mathfrak{L}_k \supset \mathfrak{L}_{k-1} \supset \cdots \supset \mathfrak{L}_1 \supset \mathfrak{L}_0 = 0$$

such that $\dim_{\mathbb{F}} \mathfrak{L}_j/\mathfrak{L}_{j-1} = 1$ for every integer $1 \le j \le k$. From the short exact sequences $0 \to \mathfrak{L}_{j-1} \to \mathfrak{L}_j \to \mathfrak{L}_j/\mathfrak{L}_{j-1} \to 0$, we obtain by dualizing, tensoring each term with M, and symmetrizing the short exact sequences:

$$0 \to [(\mathfrak{L}_j/\mathfrak{L}_{j-1})^* \otimes M]_s \to (\mathfrak{L}_j^* \otimes M)_s \to (\mathfrak{L}_{j-1}^* \otimes M)_s \to 0$$

for every integer $1 \le j \le k$. Since M is irreducible and $\dim_{\mathbb{F}} \mathfrak{L}_j/\mathfrak{L}_{j-1} = 1$, we conclude that $[(\mathfrak{L}_j/\mathfrak{L}_{j-1})^* \otimes M]_s$ is an irreducible symmetric $\mathfrak{L}$-bimodule. Moreover, we have that $\dim_{\mathbb{F}}[(\mathfrak{L}_j/\mathfrak{L}_{j-1})^* \otimes M]_s \neq 1$ as $\dim_{\mathbb{F}} M \neq 1$. Hence we obtain inductively from the long exact cohomology sequence that $\mathrm{HL}^n(\mathfrak{L}, M) \cong \mathrm{HL}^{n-1}(\mathfrak{L}, (\mathfrak{L}^* \otimes M)_s) = 0$ for every positive integer n.

The remainder of the proof is exactly the same as in the proof of Theorem 4.2. □

Remark 5. According to Lie's theorem for Leibniz algebras, every finite-dimensional irreducible Leibniz bimodule of a finite-dimensional solvable Leibniz algebra over an algebraically closed field of characteristic zero is one-dimensional (see [16, Theorem 2] or [9, Corollary 6.5 (a)]). Consequently, in this case the hypothesis of Theorem 4.9 is never satisfied, and thus this result is only applicable over fields of prime characteristic.

Note that in addition to not knowing of a general condition that guarentees the non-vanishing of the cohomology of a supersolvable Leibniz algebra in all or at least in certain degrees, in many situations Theorem 4.9 cannot be applied. Let us illustrate this by some computations for the cohomology of the two-dimensional supersolvable non-Lie Leibniz algebra $\mathfrak{A} = \mathbb{F}h \ltimes_\ell \mathbb{F}e$.

Example D: Let $\mathfrak{A} = \mathbb{F}h \oplus \mathbb{F}e$ denote the two-dimensional supersolvable non-Lie Leibniz algebra with multiplication $he = e$ and $hh = eh = ee = 0$ (see [9, Example 2.3]).

In [10, Example D] we computed the cohomology of $\mathfrak{A}$ with trivial coefficients. Namely, we obtained that

$$\dim_{\mathbb{F}} \mathrm{HL}^n(\mathfrak{A}, \mathbb{F}) = 1$$

for every non-negative integer n.

For the adjoint cohomology of $\mathfrak{A}$ we have that

$$\mathrm{HL}^0(\mathfrak{A}, \mathfrak{A}_{\mathrm{ad}}) = C^\ell(\mathfrak{A}) = \mathbb{F}e\,.$$

Note that $\mathfrak{A}_{\mathrm{ad}}$ has only one-dimensional composition factors, but by computing explicitly cocycles and coboundaries, one can show that

$$\mathrm{HL}^1(\mathfrak{A}, \mathfrak{A}_{\mathrm{ad}}) = \mathrm{HL}^2(\mathfrak{A}, \mathfrak{A}_{\mathrm{ad}}) = 0\,.$$

In particular, this shows that $\mathfrak{A}$ has only inner derivations. Moreover, it follows from [4, Théorème 3] that over an algebraically closed field of characteristic zero $\mathfrak{A}$ is a rigid Leibniz algebra, i.e., in this case $\mathfrak{A}$ has no non-trivial infinitesimal deformations.

We conjecture that the adjoint cohomology of $\mathfrak{A}$ also vanishes in higher degrees, namely, $\mathrm{HL}^n(\mathfrak{A}, \mathfrak{A}_{\mathrm{ad}}) = 0$ for every positive integer n.

Let us remark that this vanishing behavior would be analogous to [10, Proposition 2.11]. As $\mathfrak{A} = \mathbb{F}h \ltimes_\ell \mathbb{F}e$ is a hemi-semidirect product and $\mathbb{F}h$ acts semisimply on $\mathbb{F}e$, one might speculate whether the adjoint cohomology of any hemi-semidirect product of an abelian Lie algebra that acts semisimply on a nilpotent Lie algebra vanishes in every positive degree.

We conclude this section by generalizing Barnes' vanishing theorem for solvable Lie algebras [1, Theorem 2] to Leibniz algebras.

Theorem 4.10. *Let $\mathfrak{L}$ be a solvable left Leibniz algebra. If M is a finite-dimensional right faithful irreducible $\mathfrak{L}$-bimodule, then $\mathrm{HL}^n(\mathfrak{L}, M) = 0$ for every non-negative integer n.*

Proof. Suppose that $\mathfrak{L} \neq 0$ is semi-simple. Since by hypothesis $\mathfrak{L}$ is solvable, this implies that $0 \neq \mathfrak{L} = \mathrm{Leib}(\mathfrak{L})$ which contradicts [9, Proposition 2.20]. As a consequence, we have that either $\mathfrak{L} = 0$ or $\mathfrak{L}$ is not semisimple. The first case contradicts the original assumption, and in the second case the assertion follows from Corollary 3.6. □

5. Applications

In [1] Barnes uses the cohomological vanishing theorems proved in his paper to derive several structure theorems for finite-dimensional nilpotent and (super)solvable Lie algebras. The result for nilpotent Lie algebras [1, Theorem 5] has already been generalized and extended to Leibniz algebras by Barnes himself (see [2, Theorem 5.5]). In order to show the usefulness of Theorem 4.9, we prove the analogue of [1, Theorem 6] for Leibniz algebras and extend two characterizations of supersolvable Lie algebras (Theorems 7 and 8 in [1]) to Leibniz algebras. Recall that the *Frattini subalgebra* $F(\mathfrak{L})$ of a Leibniz algebra $\mathfrak{L}$ is the intersection of the maximal subalgebras of $\mathfrak{L}$ (see [20, Section 2] or [2, Definition 5.4]). For the convenience of the reader we include the details of the proofs.

Contrary to solvable Leibniz algebras, extensions of supersolvable Leibniz algebras by supersolvable Leibniz algebras are not always supersolvable. In certain situations the following result can be used as a substitute.

Theorem 5.1. *Let $\mathfrak{L}$ be a finite-dimensional left Leibniz algebra over an algebraically closed field $\mathbb{F}$. If $\mathfrak{I}$ is an ideal of $\mathfrak{L}$ such that $\mathfrak{I} \subseteq F(\mathfrak{L})$ and $\mathfrak{L}/\mathfrak{I}$ is supersolvable, then $\mathfrak{L}$ is supersolvable.*

Proof. If $\mathfrak{I} = 0$, then the assertion is trivial. So suppose that $\mathfrak{I} \neq 0$ and proceed by induction on the dimension of $\mathfrak{L}$. For the base step there is nothing to prove. Now choose a non-zero ideal $\mathfrak{A}$ of $\mathfrak{L}$ of minimal dimension that is contained in $\mathfrak{I}$. It follows from $\mathfrak{A} \subseteq \mathfrak{I} \subseteq F(\mathfrak{L})$ and [2, Corollary 5.6] that $\mathfrak{A}$ is nilpotent. But since $\mathfrak{A}$ has minimal dimension, we then obtain that $\mathfrak{A}$ is abelian, and therefore $\mathfrak{A}$ is an irreducible $\mathfrak{L}/\mathfrak{A}$-module. From $\mathfrak{A} \subseteq \mathfrak{I} \subseteq F(\mathfrak{L})$ and [20, Proposition 4.3 (ii)] we deduce that $\mathfrak{I}/\mathfrak{A} \subseteq F(\mathfrak{L})/\mathfrak{A} = F(\mathfrak{L}/\mathfrak{A})$. On the other hand, we have that $(\mathfrak{L}/\mathfrak{A})/(\mathfrak{I}/\mathfrak{A}) \cong \mathfrak{L}/\mathfrak{I}$ is supersolvable, and thus the induction hypothesis yields that $\mathfrak{L}/\mathfrak{A}$ is also supersolvable.

Suppose now that $\dim_{\mathbb{F}} \mathfrak{A} > 1$. In this case we conclude from Theorem 4.9 that the extension of $\mathfrak{A}$ by $\mathfrak{L}/\mathfrak{A}$ splits (see [14, Section 1.7] or [5, Theorem 1.3.13]), and thus there exists a subalgebra $\mathfrak{H}$ of $\mathfrak{L}$ such that $\mathfrak{L} = \mathfrak{A} \oplus \mathfrak{H}$. But then it follows from $\mathfrak{A} \subseteq \mathfrak{I} \subseteq F(\mathfrak{L})$ that $\mathfrak{L} = F(\mathfrak{L}) + \mathfrak{H}$, and we conclude from [20, Lemma 2.1] that $\mathfrak{L} = \mathfrak{H}$. Consequently, we have that $\mathfrak{A} = \mathfrak{A} \cap \mathfrak{L} = \mathfrak{A} \cap \mathfrak{H} = 0$, which is a contradiction. Hence, we obtain that $\mathfrak{A}$ is one-dimensional.

Since $\mathfrak{L}/\mathfrak{A}$ is supersolvable, there exists a chain of ideals

$$\mathfrak{L}/\mathfrak{A} = L_k \supset L_{k-1} \supset \cdots \supset L_1 \supset L_0 = 0$$

such that $\dim_{\mathbb{F}} L_j/L_{j-1} = 1$ for every integer $1 \leq j \leq k$. Then there exists ideals $\mathfrak{L}_j$ of $\mathfrak{L}$ such that $\mathfrak{A} \subseteq \mathfrak{L}_j$ and $L_j = \mathfrak{L}_{j+1}/\mathfrak{A}$ for every integer $0 \leq j \leq k$. Finally, the chain of ideals

$$\mathfrak{L} = \mathfrak{L}_{k+1} \supset \mathfrak{L}_k \supset \cdots \supset \mathfrak{L}_2 \supset \mathfrak{L}_1 = \mathfrak{A} \supset \mathfrak{L}_0 = 0$$

with $\dim_{\mathbb{F}} \mathfrak{L}_{j+1}/\mathfrak{L}_j = \dim_{\mathbb{F}} L_j/L_{j-1} = 1$ for every integer $1 \leq j \leq k$ shows that $\mathfrak{L}$ is supersolvable. □

As an application of Theorem 5.1 we obtain the following characterization of supersolvable Leibniz algebras in terms of their maximal subalgebras:

Corollary 5.2. *Let $\mathfrak{L}$ be a finite-dimensional left Leibniz algebra over an algebraically closed field $\mathbb{F}$. Then $\mathfrak{L}$ is supersolvable if, and only if, $\mathfrak{L}$ is solvable and every maximal subalgebra of $\mathfrak{L}$ has codimension one.*

Proof. Suppose that $\mathfrak{L}$ is supersolvable. We proceed by induction on $\dim_{\mathbb{F}} \mathfrak{L}$. Let $\mathfrak{M}$ be a maximal subalgebra. Choose a minimal ideal $\mathfrak{A}$ of $\mathfrak{L}$. If $\mathfrak{M} \supseteq \mathfrak{A}$, then $\mathfrak{M}/\mathfrak{A}$ is a maximal subalgebra of $\mathfrak{L}/\mathfrak{A}$, and therefore it follows from the induction hypothesis that $\dim_{\mathbb{F}}(\mathfrak{L}/\mathfrak{A})/(\mathfrak{M}/\mathfrak{A}) = 1$. Hence, we obtain that $\mathfrak{L}/\mathfrak{M} \cong (\mathfrak{L}/\mathfrak{A})/(\mathfrak{M}/\mathfrak{A})$ is one-dimensional. On the other hand, if $\mathfrak{M} \not\supseteq \mathfrak{A}$, then $\mathfrak{L} = \mathfrak{M} + \mathfrak{A}$. Since $\mathfrak{L}$ is supersolvable, $\mathfrak{A}$ is one-dimensional, and thus $\mathfrak{M} \cap \mathfrak{A} = 0$. Consequently, we obtain that $\mathfrak{L} = \mathfrak{M} \oplus \mathfrak{A}$, which implies that $\dim_{\mathbb{F}} \mathfrak{L}/\mathfrak{M} = \dim_{\mathbb{F}} \mathfrak{A} = 1$.

Conversely, suppose that $\mathfrak{L}$ is solvable and every maximal subalgebra of $\mathfrak{L}$ has codimension one. We again proceed by induction on $\dim_{\mathbb{F}} \mathfrak{L}$. Let $\mathfrak{A}$ be a minimal ideal of $\mathfrak{L}$. Then it follows from the induction hypothesis that $\mathfrak{L}/\mathfrak{A}$ is supersolvable. If $\mathfrak{A} \subseteq F(\mathfrak{L})$, we conclude from Theorem 5.1 that $\mathfrak{L}$ is supersolvable. Otherwise, if $\mathfrak{A} \not\subseteq F(\mathfrak{L})$, then there exists a maximal subalgebra $\mathfrak{M}$ that does not contain $\mathfrak{A}$, and therefore $\mathfrak{L} = \mathfrak{M} + \mathfrak{A}$. Since by hypothesis $\mathfrak{L}$ is solvable, $\mathfrak{A}$ is abelian. Hence, $\mathfrak{M} \cap \mathfrak{A}$ is an ideal of $\mathfrak{L}$, and thus $\mathfrak{A} \cap \mathfrak{M} = 0$. Consequently, we obtain that $\mathfrak{L} = \mathfrak{M} \oplus \mathfrak{A}$, which yields $\dim_{\mathbb{F}} \mathfrak{A} = \dim_{\mathbb{F}} \mathfrak{L}/\mathfrak{M} = 1$. The rest of the proof is then exactly the same as in the proof of Theorem 5.1. □

Remark 6: After finishing our paper we became aware of the paper [3]. In Corollary 3.10 of this paper Barnes proves Corollary 5.2 for arbitrary fields by using the theory of formations and projectors.

As a consequence of Corollary 5.2, we can deduce the following lattice-theoretic characterization of supersolvable Leibniz algebras:

Corollary 5.3. *Let $\mathfrak{L}$ be a finite-dimensional left Leibniz algebra over an algebraically closed field $\mathbb{F}$. Then $\mathfrak{L}$ is supersolvable if, and only if, $\mathfrak{L}$ is solvable and all maximal chains of subalgebras of $\mathfrak{L}$ have the same length.*

Proof. Suppose that $\mathfrak{L}$ is supersolvable. According to Corollary 5.2, every subalgebra in a maximal chain of subalgebras of $\mathfrak{L}$ has codimension one, and therefore the length of such a chain is $\dim_{\mathbb{F}} \mathfrak{L}$.

Conversely, suppose that $\mathfrak{L}$ is solvable and all maximal chains of subalgebras of $\mathfrak{L}$ have the same length. Then there exists a chain

$$\mathfrak{L} = \mathfrak{H}_d \supset \mathfrak{H}_{d-1} \supset \cdots \supset \mathfrak{H}_1 \supset \mathfrak{H}_0 = 0$$

of subalgebras of $\mathfrak{L}$ such that $\mathfrak{H}_{i-1}$ is a maximal ideal of $\mathfrak{H}_i$ (but not necessarily an ideal of $\mathfrak{L}$). Since $\mathfrak{L}$ is solvable, $\dim_{\mathbb{F}} \mathfrak{H}_i/\mathfrak{H}_{i-1} = 1$ for every integer $1 \leq i \leq d$. In particular, this chain of subalgebras has length $\dim_{\mathbb{F}} \mathfrak{L}$, and therefore by hypothesis every maximal chain of subalgebras of $\mathfrak{L}$ has length $\dim_{\mathbb{F}} \mathfrak{L}$.

Now let $\mathfrak{M}$ be a maximal subalgebra of $\mathfrak{L}$. By successively choosing maximal subalgebras we obtain a chain

$$\mathfrak{L} = \mathfrak{M}_r \supset \mathfrak{M} = \mathfrak{M}_{r-1} \supset \cdots \supset \mathfrak{M}_1 \supset \mathfrak{M}_0 = 0$$

of subalgebras of $\mathfrak{L}$ such that $\mathfrak{M}_{j-1}$ is maximal in $\mathfrak{M}_j$ for every integer $1 \leq j \leq r$. This chain is clearly maximal, and thus we have that $r = \dim_{\mathbb{F}} \mathfrak{L}$, or equivalently, $\dim_{\mathbb{F}} \mathfrak{M}_j/\mathfrak{M}_{j-1} = 1$ for every integer $1 \leq j \leq r$. In particular, we obtain that $\dim_{\mathbb{F}} \mathfrak{L}/\mathfrak{M} = 1$, and then the assertion follows from Corollary 5.2. □

Remark 7: After finishing our paper we became aware of the paper [19]. The equivalence of the statements (ii) and (iii) in Proposition 5.1 of this paper is closely related to our Corollary 5.3.

Let $\mathfrak{L}$ be a Leibniz algebra, and let S be any subset of $\mathfrak{L}$. Then

$$C_{\mathfrak{L}}^r(S) := \{x \in \mathfrak{L} \mid \forall\, s \in S : sx = 0\}$$

denotes the *right centralizer* of S in $\mathfrak{L}$. We conclude this section by extending [1, Theorem 4] from Lie algebras to Leibniz algebras. If D is a derivation of an algebra $\mathfrak{R}$ over the real or complex numbers, then

$$\exp(D) := \mathrm{id}_{\mathfrak{R}} + D + \frac{1}{2}D^2 + \frac{1}{3!}D^3 + \cdots$$

is an automorphism of $\mathfrak{R}$. The same is true for a nilpotent derivation D of an algebra $\mathfrak{R}$ over an arbitrary field of characteristic zero (see [12, Section 2.3]). Note that we do not need to assume that the characteristic of the ground field of $\mathfrak{R}$ is zero if $D^2 = 0$. We say that two subalgebras $\mathfrak{K}$ and $\mathfrak{H}$ of a left Leibniz algebra $\mathfrak{L}$ are *conjugate* if there exists an element $x \in \mathfrak{L}$ such that $\exp(L_x)(\mathfrak{K}) = \mathfrak{H}$, where L_x denotes the left multiplication operator of x on $\mathfrak{L}$ satisfying the appropriate nilpotency condition depending on the ground field of $\mathfrak{L}$.

Note that the Leibniz algebra in the next result does not have to be finite dimensional as in Barnes' result [1, Theorem 4].

Theorem 5.4. *Let $\mathfrak{L}$ be a solvable left Leibniz algebra. If $\mathfrak{A}$ is a finite-dimensional minimal ideal of $\mathfrak{L}$ such that $C_{\mathfrak{L}}^r(\mathfrak{A}) = \mathfrak{A}$, then every extension of $\mathfrak{A}$ by $\mathfrak{L}/\mathfrak{A}$ splits and all complements of $\mathfrak{A}$ in $\mathfrak{L}/\mathfrak{A}$ are conjugate.*

Proof. Since every minimal ideal of a solvable Leibniz algebra is abelian, $\mathfrak{A}$ is abelian, and therefore $\mathfrak{A}$ is an $\mathfrak{L}/\mathfrak{A}$-bimodule via the action induced by left and right multiplication on $\mathfrak{L}$. The hypothesis that $\mathfrak{A}$ is right self-centralizing implies that $\mathfrak{A}$ is a right faithful $\mathfrak{L}$-bimodule. Finally, we obtain from the minimality of $\mathfrak{A}$ that $\mathfrak{A}$ is an irreducible $\mathfrak{L}/\mathfrak{A}$-bimodule. Hence, it follows from Theorem 4.10 that $\mathrm{HL}^2(\mathfrak{L}/\mathfrak{A}, \mathfrak{A}) = 0$, and therefore every extension of $\mathfrak{A}$ by $\mathfrak{L}/\mathfrak{A}$ splits (see [14, Section 1.7] and [5, Theorem 1.3.13]).

Now let $\mathfrak{K}$ and $\mathfrak{K}'$ be two complements of $\mathfrak{A}$ in $\mathfrak{L}/\mathfrak{A}$, i.e., $\mathfrak{K}$ and $\mathfrak{K}'$ are subalgebras of $\mathfrak{L}$ such that $\mathfrak{L} = \mathfrak{A} \oplus \mathfrak{K}$ and $\mathfrak{L} = \mathfrak{A} \oplus \mathfrak{K}'$, respectively. Let $x \in \mathfrak{L}$ be arbitrary. Then $x = a + k$ for some uniquely determined elements $a \in \mathfrak{A}$ and $k \in \mathfrak{K}$. Similarly, $x = a' + k'$ for some uniquely determined elements $a' \in \mathfrak{A}$ and $k' \in \mathfrak{K}'$. Then the linear transformation $D : \mathfrak{L}/\mathfrak{A} \to \mathfrak{A}$, $x + \mathfrak{A} \mapsto k - k'$ is well-defined. Namely, note that k and k' do not change if x is replaced by $x + a_0$ for some $a_0 \in \mathfrak{A}$. Moreover, let $\pi : \mathfrak{L} \to \mathfrak{L}/\mathfrak{A}$ denote the natural epimorphism of Leibniz algebras. The computation $\pi(k - k') = \pi(k) - \pi(k') = \pi(x) - \pi(x) = 0$ shows that $k - k' \in \mathrm{Ker}(\pi) = \mathfrak{A}$.

Next, we prove that D is a derivation. For any two elements $x, y \in \mathfrak{L}$ there exist unique elements $k_x, k_y \in \mathfrak{K}$ and $a_x, a_y \in \mathfrak{A}$ such that $x = a_x + k_x$ and $y = a_y + k_y$. Since $\mathfrak{A}$ is abelian, we have that

$$xy = (a_x + k_x)(a_y + k_y) = a_x k_y + k_x a_y + k_x k_y \,.$$

Similarly, there exist unique elements $k'_x, k'_y \in \mathfrak{K}'$ and $a'_x, a'_y \in \mathfrak{A}$ such that $x = a'_x + k'_x$ and $y = a'_y + k'_y$, and we obtain that

$$xy = (a'_x + k'_x)(a'_y + k'_y) = a'_x k'_y + k'_x a'_y + k'_x k'_y \,.$$

From this we conclude that $D(xy + \mathfrak{A}) = k_x k_y - k'_x k'_y$, and we compute that

$$\begin{aligned} D[(x + \mathfrak{A})(y + \mathfrak{A})] &= D(xy + \mathfrak{A}) = k_x k_y - k'_x k'_y \\ &= (k_x - k'_x) k_y + k'_x (k_y - k'_y) \\ &= (k_x - k'_x) \cdot (k_y + \mathfrak{A}) + (k'_x + \mathfrak{A}) \cdot (k_y - k'_y) \\ &= (k_x - k'_x) \cdot (y + \mathfrak{A}) + (x + \mathfrak{A}) \cdot (k_y - k'_y) \\ &= D(x + \mathfrak{A}) \cdot (y + \mathfrak{A}) + (x + \mathfrak{A}) \cdot D(y + \mathfrak{A}) \,. \end{aligned}$$

It follows from Theorem 4.10 that $\mathrm{HL}^1(\mathfrak{L}/\mathfrak{A}, \mathfrak{A}) = 0$, and thus we obtain from [9, Proposition 4.3] that there exists an element $a \in \mathfrak{A}$ such that $D(x + \mathfrak{A}) = -ax$ for any element $x \in \mathfrak{L}$. Since $\mathfrak{A}$ is abelian, we have that $L_a^2 = 0$, and therefore $\sigma := \mathrm{id}_K + L_a$ defines an automorphism of $\mathfrak{L}$ (see the argument in [12, Section 2.3]).

For any element $k \in \mathfrak{K}$ we have that $k = 0 + k = a' + k'$ for some elements $a' \in \mathfrak{A}$ and $k' \in \mathfrak{K}'$. In particular, we obtain that $D(k + \mathfrak{A}) = k - k'$, and thus

$$\sigma(k) = k + ak = k - D(k + \mathfrak{A}) = k - (k - k') = k' \in \mathfrak{K}' \,,$$

which shows that $\sigma(\mathfrak{K}) \subseteq \mathfrak{K}'$. But as an automorphism σ is injective. Hence, the restriction $\sigma_{|\mathfrak{K}}$ of σ to $\mathfrak{K}$ is also injective. Now it follows from $\dim_{\mathbb{F}} \mathfrak{K} = \dim_{\mathbb{F}} \mathfrak{L}/\mathfrak{A} = \dim_{\mathbb{F}} \mathfrak{K}'$ that $\sigma_{|\mathfrak{K}} : \mathfrak{K} \to \mathfrak{K}'$ is surjective, i.e., $\sigma(\mathfrak{K}) = \mathfrak{K}'$. □

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