CARPATHIAN J. MATH.
Volume **40** (2024), No. 1,
Pages 47 - 64

Online version at `https://www.carpathian.cunbm.utcluj.ro/`
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2024.01.05

Influence of symmetric first-order divided differences on Secant-like methods

M. A. HERNÁNDEZ-VERÓN[1], JOSÉ. L. HUESO[2] and EULALIA MARTÍNEZ[2]

ABSTRACT. In this paper, by using symmetric first-order divided differences, we introduce a new family of Secant-like iterative methods with quadratic convergence. Afterthought, we analyze its semilocal and local behavior when the nonlinear operator F is not differentiable by imposing appropriate bounding conditions in each case. Theoretical results have also been tested by solving a problem which shows the applicability of our work.

1. INTRODUCTION

Solving nonlinear equations is a fundamental issue of numerical analysis because a great variety of applied problems in engineering, physics, chemistry, biology, and statistics, involve such kind of equations as a part of its solving process. We often use iterative processes to approximate a simple solution x^* of a nonlinear equation

$$F(x) = 0. \tag{1.1}$$

We are interested in approximating a solution x^* of a nonlinear system of equations (1.1), where $F : D \subseteq \mathbb{R}^m \to \mathbb{R}^m$ is a continuous but non-differentiable nonlinear operator, and D is a non-empty open convex domain in the space $\mathbb{R}^m$, with values in the same space $\mathbb{R}^m$.

Newton's method

$$x_0 \in D, \qquad x_n = x_{n-1} - [F'(x_{n-1})]^{-1} F(x_{n-1}), \quad n \in \mathbb{N},$$

is the one of the most used iterative methods to approximate the solution x^* of $F(x) = 0$. The quadratic convergence and the low operational cost of Newton's method ensure that it has a good computational efficiency. As Newton's method needs the existence of F', it cannot be applied when the operator F is not differentiable. So, if the operator F is not differentiable, we have to choose the iterative processes carefully. The iterative processes that do not use derivatives are generally less studied in the literature. This type of methods generally includes divided differences [7] instead of derivatives. We shall use the standard definition for the first order divided differences of an operator. Let us denote by $L(\mathbb{R}^m, \mathbb{R}^m)$ the space of bounded linear operators from $\mathbb{R}^m$ to $\mathbb{R}^m$, see [2, 13]. An operator $[x, y; H] \in L(\mathbb{R}^m, \mathbb{R}^m)$ is called a first order divided difference for the operator $H : D \subseteq \mathbb{R}^m \to \mathbb{R}^m$ on the points x and y ($x \neq y$) if the following equality holds

$$[x, y; H](x - y) = H(x) - H(y). \tag{1.2}$$

Received: 15.04.2021. In revised form: 06.10.2023. Accepted: 13.10.2023
2020 *Mathematics Subject Classification.* 45G10, 47H17, 65J15.
Key words and phrases. *Iterative processes, semilocal convergence, divided differences, derivative-free iterative processes, Secant-like iterative methods.*
Corresponding author: Eulalia Martínez; eumarti@mat.upv.es

The best known iterative method that does not use derivatives in its algorithm is the Secant method [5, 3],

$$\left\{\begin{array}{l} x_0, x_{-1} \text{ given in } D, \\ x_{n+1} = x_n - [x_{n-1}, x_n; F]^{-1} F(x_n), \quad n \geq 0, \end{array}\right. \tag{1.3}$$

The use of the Secant method is interesting since the calculation of the first derivative F' is not required and the speed of convergence of the method of successive substitutions is improved, although it is slower than Newton's method. It is well known that the Secant method is superlineal convergent with order of convergence $(1+\sqrt{5})/2$ (see [15]).

From the geometrical interpretation of the Secant method in the real case, it is clear that if we consider a point of the segment that joins x_{n-1} and x_n, $y_n = \lambda x_n + (1-\lambda)x_{n-1}$ with $\lambda \in [0,1]$, the closer x_n and y_n are, the higher the speed of the convergence is. Accordingly, in [16], an uniparametric family of Secant-like methods is considered, given by the following algorithm:

$$\left\{\begin{array}{l} x_{-1},\ x_0 \text{ given in } D, \\ y_n = \lambda x_n + (1-\lambda)x_{n-1}, \quad \lambda \in [0,1], \\ x_{n+1} = x_n - [y_n, x_n; F]^{-1} F(x_n), \end{array}\right. \tag{1.4}$$

This uniparametric family of iterative processes can be considered as a combination of the Secant method ($\lambda = 0$) and, in the differentiable case, Newton's method ($\lambda = 1$). Moreover, its speed of convergence is close to that of Newton's iteration, when λ is near 1(see [18]). However, the Secant-like methods (1.4) are superlineal convergent with order of convergence $(1+\sqrt{5})/2$(see [18]).

The main objective of our work is to carry out a modification of the family of iterative processes (1.4) that, maintaining a single parameter, allows us to define an uniparametric family of iterative processes that, unlike (1.4), has quadratic convergence for whatever value the parameter takes. It is well known that the symmetric divided differences approximate better the derivative than the one sided divided differences. We can see that the Center-Steffensen and Kurchatov methods [14, 22] maintain the quadratic convergence of Newton's method by approximating the derivative through symmetric divided differences. Following this idea, from the uniparametric family (1.4), in this paper we consider $[x_n - (y_n - x_{n-1}), x_n + (y_n - x_{n-1}); F]$ for approximating $F'(x_n)$ in Newton's method. So, we obtain the uniparametric family of iterative processes given by the following algorithm

$$\left\{\begin{array}{l} x_{-1},\ x_0 \text{ given in } D,\ \lambda \in [0,1], \\ y_n = (1-\lambda)x_n + \lambda x_{n-1}, \\ z_n = (1+\lambda)x_n - \lambda x_{n-1}, \\ x_{n+1} = x_n - [y_n, z_n; F]^{-1} F(x_n). \end{array}\right. \tag{1.5}$$

This new uniparametric family of iterative processes can be considered as a combination of the Kurchatov's method ($\lambda = 1$) and, for differentiable case, Newton's method ($\lambda = 0$), both iterative processes with quadratic convergence.

Another objective of this work is to analyze the semilocal and the local convergence of the new uniparametric family of iterative process (1.5). First, the semilocal study of the convergence is based on demanding conditions to the initial approximations x_{-1} and x_0, from certain conditions on the operator F, and provide the so-called domain of parameters corresponding to the conditions required to the initial approximations that guarantee the convergence of sequence (1.5) to the solution x^*. Second, the local study of the convergence is based on demanding conditions to the solution x^*, from certain conditions on the operator F, and provide the so-called ball of convergence of (1.5), that shows the accessibility to x^* from the initial approximation x_0 belonging to the ball.

The rest of the paper is structured as follows: Section 2 is devoted to obtain the local order of convergence of (1.5) assuming differentiable operators. Then, in the next sections we set the semilocal and local convergence study of (1.5) for both differentiable and non-differentiable operators. In Section 5, we perform some numerical tests and finally, we give the conclusions.

Throughout the paper we consider $F : D \subseteq \mathbb{R}^m \to \mathbb{R}^m$ a continuous nonlinear operator and D is a non-empty open convex domain in the Banach space $\mathbb{R}^m$. Moreover, as it is known [7], there exists a divided difference of order one $[z, w; F] \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$ for each pair of different points $z, w \in \Omega$ and we denote $\overline{B(x, \varrho)} = \{y \in \mathbb{R}^m; \|y - x\| \leq \varrho\}$ and $B(x, \varrho) = \{y \in \mathbb{R}^m; \|y - x\| < \varrho\}$, respectively for the closed and open balls with center in x and of radius $\varrho > 0$.

2. Local order of convergence

The speed of convergence of an iterative method is usually measured by the order of convergence of the method. An excellent study about the speed of convergence of a sequence can be seen in [8]. The first definition of order of convergence was given in 1870 by Schröder [27], but a very commonly measure of speed of convergence in Banach spaces is the R-order of convergence [24], which is defined as follows:

Let $\{x_n\}$ a sequence of points of a Banach space X converging to a point $x^* \in X$ and let $\sigma \geq 1$ and

$$e_n(\sigma) = \begin{cases} n & \text{if} \quad \sigma = 1, \\ \sigma^n & \text{if} \quad \sigma > 1, \end{cases} \qquad n \geq 0.$$

(a) We say that σ is an R-order of convergence of the sequence $\{x_n\}$ if there are two constants $b \in (0, 1)$ and $B \in (0, +\infty)$ such that

$$\|x_n - x^*\| \leq B b^{e_n(\sigma)}.$$

(b) We say that σ is the exact R-order of convergence of the sequence $\{x_n\}$ if there are four constants $a, b \in (0, 1)$ and $A, B \in (0, +\infty)$ such that

$$A a^{e_n(\sigma)} \leq \|x_n - x^*\| \leq B b^{e_n(\sigma)}, \quad n \geq 0.$$

In general, check double inequalities of (b) is complicated, so that normally only seek upper inequalities as (a). Therefore, if we find an R-order of convergence σ of sequence $\{x_n\}$, we then say that sequence $\{x_n\}$ has order of convergence at least σ.

However, in the scalar case, $x_{n+1} = \Phi(x_n)$ with $\{x_n\} \in \mathbb{R}$, a simple procedure to obtain this lower bound for the R-order of convergence, as long as the iteration function Φ is sufficiently differentiable with these derivatives satisfying certain conditions, it consists in applying Taylor expansions. Thus, if for example their local error difference equation is $e_{n+1} = e_n^\sigma + O(e_n^{\sigma+1})$, where $e_j = x_j - x^*$ with $x^* = \Phi(x^*)$, then $\{x_n\}$ has R-order of convergence at least σ. Taking into account the expression of the local error difference equation, we extend this situation in the m-dimensional case, $\mathbb{R}^m$, and we consider, without using norms, the definition of the local order of convergence [11].

Definition 2.1. Given a one-step iterative method without memory, $x_{n+1} = \Phi(x_n)$, the local order of convergence is $\rho \in \mathbb{N}$ if there is an operator $C \in L_\rho(\mathbb{R}^m, \mathbb{R}^m)$, where $L_\rho(\mathbb{R}^m, \mathbb{R}^m)$ the set of bounded ρ-linear operators, such that

$$e_{n+1} = C e_n^\rho + O(e_n^{\rho+1}),$$

where $e_n^\rho \in \mathbb{R}^m \times \ldots(\rho \ldots \times \mathbb{R}^m$.

Given a one-step iterative method with memory, $x_{n+1} = \Phi(x_n, x_{n-1}, ..., x_{n-j+1})$, the local order of convergence is $\rho \in \mathbb{N}$ if there are an operator $C \in L_{\alpha_1+\alpha_2+...+\alpha_j}(\mathbb{R}^m, \mathbb{R}^m)$ and α_k, nonnegative integers for $1 \le k \le j$, such that

$$e_{n+1} = C e_n^{\alpha_1} e_{n-1}^{\alpha_2} ... e_{n-j+1}^{\alpha_j} + o(e_n^{\alpha_1} e_{n-1}^{\alpha_2} ... e_{n-j+1}^{\alpha_j}), \tag{2.6}$$

and ρ is the unique real positive root of the polynomial equation $p_j(t) = t^j - \alpha_1 t^{j-1} + ... + \alpha_{j-1} t - \alpha_j = 0$ associated with (2.6).

Notice that if we apply Descartes's rule to the previous polynomial, there is a unique real positive root ρ that coincides with the local order of convergence (see [23, 29]).

On the other hand, note that a priori, when considering the m-dimensional space $\mathbb{R}^m$, this concept that we have just defined cannot be related to the R-order of convergence as it happened in the scalar case $\mathbb{R}$.

In this section we study the local order of convergence for the new uniparametric family of iterative processes given by (1.5), assuming that the nonlinear operator F is differentiable and we can obtain Taylor's expansion in a neighborhood of the solution x^*. For this purpose, we consider the characterization of divided difference operator introduced in [23], given by

$$[x, x+h; F] = \int_0^1 F'(x+th)\,\mathrm{d}t, \qquad (x,h) \in \mathbb{R}^m \times \mathbb{R}^m,$$

and integrating the Taylor's expansion of $F'(x+th)$ around x we have:

$$[x, x+h; F] = F'(x) + \frac{1}{2}F''(x)h + \frac{1}{6}F'''(x)h^2 + O(h^3).$$

Then, we establish the following result:

Theorem 2.1. *Let $F : D \subseteq \mathbb{R}^m \longrightarrow \mathbb{R}^m$ a sufficiently Fréchet differentiable function in a non-empty open set $D \subseteq \mathbb{R}^m$ containing the solution x^* of $F(x) = 0$. Suppose that $F'(x)$ is continuous and non-singular at x^*, the initial approximations x_{-1} and x_0 are chosen sufficiently close to x^*. Then, the family of iterative processes* (1.5) *converges to x^* and has local order of convergence* 2 *for any value of the parameter $\lambda \in [0,1]$.*

Proof. Let $e_n = x_n - x^*$ the error in the n-th approximation to the solution x^* of the system $F(x) = 0$. As the initial approximations x_{-1} and x_0 are chosen sufficiently close to x^*, we can consider that the sequence $\{x_n\}$ converges to x^*.

We are going to develop the operator $[y_n, z_n; F]$, that approximates the derivative, considering in this case $x = y_n$, $x + h = z_n$ and, therefore, $h_n = z_n - y_n = 2\lambda(e_n - e_{n-1})$. Thus, we obtain

$$[y_n, z_n; F] = F'(y_n) + \frac{1}{2}F''(y_n)(z_n - y_n) + O((z_n - y_n)^2), \tag{A}$$

and, by using

$$y_n = x_n - \lambda(e_n - e_{n-1}) = x_n - \frac{h_n}{2},$$
$$z_n = x_n + \lambda(e_n - e_{n-1}) = x_n + \frac{h_n}{2},$$

we obtain the following Taylor's expansions around x_n

$$F'(y_n) = F'(x_n) - F''(x_n)\frac{h_n}{2} + \frac{1}{2}F'''(x_n)\Big(\frac{h_n}{2}\Big)^2 + \dots, \tag{2.7}$$

$$F''(y_n) = F''(x_n) - F'''(x_n)\frac{h_n}{2} + \frac{1}{2}F^{(iv)}(x_n)\Big(\frac{h_n}{2}\Big)^2 + \dots. \tag{2.8}$$

Now, we use the developments of function $F(x_n)$ and its successive derivatives in a neighborhood of x^* which take the form:

$$\begin{aligned} F(x_n) &= F'(x^*)(e_n + A_2 e_n^2 + A_3 e_n^3 + A_4 e_n^4 + A_5 e_n^5) + O(e_n^6), \\ F'(x_n) &= F'(x^*)(I + 2A_2 e_n + 3A_3 e_n^2 + 4A_4 e_n^3 + 5A_5 e_n^4) + O(e_n^5), \\ F''(x_n) &= F'(x^*)(2A_2 + 6A_3 e_n + 12A_4 e_n^2 + 20A_5 e_n^3) + O(e_n^4). \end{aligned}$$

where I is the identity operator in $\mathbb{R}^m$ and being $A_j = \frac{1}{j!}F'(x^*)^{-1}F^{(j)}(x^*) \in L_j(\mathbb{R}^m, \mathbb{R}^m)$ with $L_j(\mathbb{R}^m, \mathbb{R}^m)$ the set of bounded j-linear operators, $j = 1, 2, 3, \ldots$.

Then, we substitute this Taylor series in (2.7)-(2.8) and, taking into account the value of h_n in (A), we obtain the error equation for the divided difference operator $[y_n, z_n; F]$:

$$[y_n, z_n; F] = F'(x^*)(I - 3\lambda^2 A_3 e_{n-1}^2 + 2A_2 e_n + 6\lambda^2 A_3 e_{n-1} e_n) + O\left(e_n, e_{n-1}\right), \tag{2.9}$$

where $O\left(e_n, e_{n-1}\right)$ denotes high order terms of e_n and e_{n-1}.

Now, we consider

$$[y_n, z_n; F]^{-1} = (I + B_1 e_{n-1}^2 + B_2 e_n + B_3 e_{n-1} e_n) F'(x^*)^{-1} + O\left(e_n, e_{n-1}\right),$$

and therefore, from $[y_n, z_n; F]^{-1}[y_n, z_n; F] = I$, we deduce by product of series, (see [10, 12]), the following

$$B_1 = 3\lambda^2 A_3, \quad B_2 = -2A_2, \text{ and } B_3 = -6\lambda^2 A_3.$$

Then, we obtain for the inverse operator

$$[y_n, z_n; F]^{-1} = (I + 3\lambda^2 A_3 e_{n-1}^2 - 2A_2 e_n - 6\lambda^2 A_3 e_{n-1} e_n) F'(x^*)^{-1} + O\left(e_n, e_{n-1}\right).$$

So, we get the error equation of the method:

$$e_{n+1} = x_n - x^* - [y_n, z_n; F]^{-1} F(x_n) = -3\lambda^2 A_3 e_{n-1}^2 e_n - A_2 e_n^2 + O\left(e_n, e_{n-1}\right). \tag{2.10}$$

Therefore, if we assume that the family of iterative processes (1.5) has local order of convergence r, we have that $e_{n+1} \approx D_n e_n^r$ and also $e_n \approx D_{n-1} e_{n-1}^r$. With these equivalences the error equation (2.10) turns up in

$$e_{n+1} \approx D_n D_{n-1}^r e_{n-1}^{r^2} \approx -3\lambda^2 A_3 D_{n-1} e_{n-1}^{r+2} - A_2 D_{n-1}^2 e_{n-1}^{2r},$$

whose associated equations are $r^2 - 2r = 0$, if we consider $r + 2 \leq 2r$, or $r^2 - r - 2 = 0$, if we consider $r + 2 \geqslant 2r$. Both of them with positive solution equal 2, which proves that the proposed method has at least quadratic local order of convergence. $\square$

In the Remarks section, at point 2, we study the computational order of convergence (5.27) for the iterative processes (1.5), seeing that their computational order of convergence tends to be quadratic. While, in the case of iterative processes (1.4), their computational order of convergence tends to $(1 + \sqrt{5})/2$.

Note that the difference in operational cost between families of iterative processes (4) and (5) is 2m more products per iteration in family (5). Taking into account the quadratic convergence that we have just tested for (1.5), and that (1.4) has superlinear convergence $(1 + \sqrt{5})/2$, it is evident that the new family of iterative processes (1.5) has a computational efficiency greater that of (1.4).

3. SEMILOCAL CONVERGENCE

To analyze the semilocal convergence of iterative processes that do not use derivatives in their algorithms, the conditions usually required for the divided difference operator are the Lipschitz and Hölder continuous conditions (see [19]). Notice that in these conditions the operator F must be differentiable [16]. To generalize the above conditions and even to consider situations in which operator F is non-differentiable, we will consider the ω-continuous condition (3.11) in D for the operator first order divided difference of operator F. Therefore, we will consider $\lambda \in (0,1]$, eliminating the case corresponding to Newton's method, which requires the differentiability of the operator F.

Now, we provide a semilocal convergence result for the uniparametric family of iterative processes given in (1.5). For this purpose, we denote $A_n = [y_n, z_n, F]$ and assume the following conditions (*SL*):

(SL1): There exist $x_{-1}, x_0 \in D$, with $\|x_{-1} - x_0\| = \alpha$, such that there exists ${A_0}^{-1}$ with $\|{A_0}^{-1}\| \leq \beta$ and $\|{A_0}^{-1} F(x_0)\| \leq \eta$.

(SL2): There exists $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, a non decreasing continuous function in its two arguments, such that

$$\|[x, y; F] - [v, w; F]\| \leq \omega(\|x - v\|, \|y - w\|); \;\; x, y, v, w \in D, \tag{3.11}$$

with $x \neq y$ and $u \neq v$.

(SL3): The equation

$$(1 - Q(t))t - \eta = 0, \tag{3.12}$$

has at least one positive zero, and we denote by R the smallest positive zero. Where $Q(t) = \dfrac{q}{1 - \beta\omega(t + \lambda\alpha, t + \lambda(\eta + \alpha))}$, with $q = max\{\omega(\lambda\alpha, \eta + \lambda\alpha), \omega(\lambda\eta, (1+\lambda)\eta)\}$.

(SL4): $B(x_0, R + \eta) \subseteq D$ and $q + \beta w(R + \lambda\alpha, t + \lambda(\eta + \alpha)) < 1$.

Theorem 3.2. *Assume that conditions (SL) are verified. Then, by taking two different starting points $x_{-1}, x_0 \in B(x_0, R + \eta)$, the sequence $\{x_n\}$ given by the family of iterative processes* (1.5) *is well defined, belongs to $B(x_0, R)$ and converges to $x^* \in \overline{B(x_0, R)}$ a solution of the equation $F(x) = 0$.*

Proof:
First of all we notice that, by taking two different points $x_{-1}, x_0 \in B(x_0, R + \eta)$, with $\|x_{-1} - x_0\| = \alpha$, from (1.5), we obtain

$$\begin{aligned} \|y_0 - x_0\| &\leq \lambda\|x_{-1} - x_0\| = \lambda\alpha < R + \eta \\ \|z_0 - x_0\| &\leq \lambda\|x_0 - x_{-1}\| = \lambda\alpha < R + \eta. \end{aligned} \tag{3.13}$$

So, $y_0, z_0 \in B(x_0, R + \eta)$. Moreover, as $\lambda \in (0,1]$ and $x_{-1} \neq x_0$, it follows that

$$z_0 - y_0 = 2\lambda(x_0 - x_{-1}) \neq 0,$$

therefore, y_0 and z_0 are a pair of different points in $B(x_0, R + \eta)$. On the other hand, from (3.12), we have that $\|x_1 - x_0\| \leq \|{A_0}^{-1}\| \|F(x_0)\| \leq \eta < R$. Then $x_1 \in B(x_0, R)$.

By a reasoning similar to that considered in the case $n = 0$, from (1.5), we have that

$$\begin{aligned} \|y_1 - x_0\| &\leq (1 - \lambda)\|x_1 - x_0\| = \|x_1 - x_0\| - \lambda\|x_1 - x_0\| < R + \eta, \\ \|z_1 - x_0\| &\leq (1 + \lambda)\|x_1 - x_0\| \leq \|x_1 - x_0\| + \lambda\|x_1 - x_0\| < R + \lambda\eta < R + \eta. \end{aligned} \tag{3.14}$$

Moreover, if x_0 is not a solution of $F(x) = 0$ then, as $\lambda \in (0,1]$ and $x_1 \neq x_0$, we have $z_1 - y_1 = 2\lambda(x_1 - x_0) \neq 0$. Therefore, y_1 and z_1 are a pair of different points in $B(x_0, R + \eta)$.

Notice that, if x_0 were a solution of $F(x) = 0$ then $x_n = x_1 = x_0$, for all $n \geqslant 2$, and the family of iterative processes would end with $x^* = x_0$. Then, converges to x^*.

As y_1 and z_1 are a pair of different points in $B(x_0, R + \eta)$, $[y_1, z_1; F]$ is well defined. Moreover, we are now in conditions to prove that A_1^{-1} exists. So, we have that

$$
\begin{aligned}
\|I - {A_0}^{-1}A_1\| &\leq \|A_0^{-1}\|\|A_0 - A_1\| \leq \beta\omega(\|y_1 - y_0\|, \|z_1 - z_0\|) \\
&\leq \beta\omega(\|(1-\lambda)(x_1 - x_0) + \lambda(x_0 - x_{-1})\|, \|(1+\lambda)(x_1 - x_0) - \lambda(x_0 - x_{-1})\|) \\
&\leq \beta\omega((1-\lambda)\eta + \lambda\alpha, (1+\lambda)\eta + \lambda\alpha) \\
&\leq \beta\omega((1-\lambda)\eta + \lambda\alpha, R + \lambda(\eta + \alpha)) < 1.
\end{aligned}
$$

Notice that the previous bound follows from the hypothesis and using the non decreasing character of $\omega-$function, for being $\lambda \in (0, 1]$ and $\eta < R$, so it follows that $\omega((1-\lambda)\eta + \lambda\alpha, (1+\lambda)\eta + \lambda\alpha) \leq \omega(R + \lambda\alpha, (1+\lambda)R + \lambda\alpha) < 1$.

Now, by applying Banach Lemma [20], we have that A_1^{-1} exists and

$$
\|A_1^{-1}\| \leq \frac{\beta}{1 - \beta\omega((1-\lambda)\eta + \lambda\alpha, (1+\lambda)\eta + \lambda\alpha)}.
$$

Therefore, x_2 is well defined.

Now, we use the characterization of divided differences,

$$
[x_0, x_1, F](x_0 - x_1) = F(x_0) - F(x_1),
$$

and, by the definition of function iteration (1.5) for obtaining x_1, it follows: $F(x_0) = A_0(x_0 - x_1)$. Then

$$
F(x_1) = F(x_0) - [x_0, x_1, F](x_0 - x_1) = (A_0 - [x_0, x_1, F])(x_0 - x_1).
$$

This allows to have

$$
\|F(x_1)\| \leq \omega(\|y_0 - x_0\|, \|z_0 - x_1\|)\|x_0 - x_1\| \leq \omega(\lambda\alpha, \eta + \lambda\alpha)\|x_0 - x_1\|
$$

and then, by (1.5) for obtaining x_2 and as $\eta < R$, it is verified

$$
\begin{aligned}
\|x_2 - x_1\| &\leq \|A_1^{-1}\|\|F(x_1)\| \leq \frac{\beta\omega(\lambda\alpha, \eta + \lambda\alpha)}{1 - \beta\omega((1-\lambda)R + \lambda\alpha, R + \lambda(\eta + \alpha))}\|x_0 - x_1\| \\
&\leq Q(R)\|x_1 - x_0\| < \|x_1 - x_0\| \leq \eta.
\end{aligned}
$$

Moreover, we also get

$$
\|x_2 - x_0\| \leq \|x_2 - x_1\| + \|x_1 - x_0\| \leq (Q(R) + 1)\eta < \frac{\eta}{1 - Q(R)} = R.
$$

Then, $x_2 \in B(x_0, R)$ and $\|x_2 - x_1\| < \|x_1 - x_0\|$.

Next, by a mathematical induction procedure, we assume that for $k = 1, 2, ..., n-1$, the following assertions are verified:

$(i)_k$ $y_k, z_k \in B(x_0, R + \eta)$ with $y_k \neq z_k$.

$(ii)_k$ $\|A_k^{-1}\| \leq \dfrac{\beta}{1 - \beta\omega(R + \lambda\alpha, R + \lambda(\eta + \alpha))}$.

$(iii)_k$ $\|x_{k+1} - x_k\| \leq Q(R)\|x_k - x_{k-1}\| \leq ... \leq Q(R)^k\|x_1 - x_0\| < \|x_1 - x_0\| \leq \eta$.

$(iv)_k$ $\|x_{k+1} - x_0\| \leq (Q(R)^k + Q(R)^{k-1} + ... + Q(R) + 1)\eta < \dfrac{\eta}{1 - Q(R)} = R$.

Under these hypothesis we conclude the process studding the bounds for $k = n$.

As before, we suppose that x_{n-1} is not solution of $F(x) = 0$. Otherwise, $x_{n-1} = x_n = x_k$, for all $k \geqslant n$, and then the result is proved with $x^* = x_{n-1}$. Next, by using the introduced process and by (1.5), we have that

$$\begin{aligned}
&\|y_n - x_0\| \le \|x_n - x_0\| + \lambda\|x_{n-1} - x_n\| < R + \lambda\|x_1 - x_0\| \le R + \eta,\\
&\|z_n - x_0\| \le \|x_n - x_0\| + \lambda\|x_n - x_{n-1}\| < R + \lambda\|x_1 - x_0\| \le R + \lambda\eta,\\
&\|y_n - x_n\| \le \lambda\|x_n - x_{n-1}\| < \lambda\|x_1 - x_0\| \le \lambda\eta,\\
&\|z_n - x_n\| \le \lambda\|x_{n-1} - x_n\| < \lambda\|x_1 - x_0\| \le \lambda\eta,\\
&\quad y_n - z_n = 2\lambda(x_{n-1} - x_n) \neq 0.
\end{aligned}$$

So, we have that y_n and z_n are a pair of different points such that $y_n, z_n \in B(x_0, R + \eta)$, what proves item $(i)_n$. Moreover, $[y_n, z_n; F]$ is well defined and we obtain

$$\begin{aligned}
\|I - A_0{}^{-1}A_n\| &\le \|A_0^{-1}\|\|A_0 - A_n\|\\
&\le \beta\omega(\|y_n - y_0\|, \|z_n - z_0\|)\\
&\le \beta\omega(R + \lambda\alpha, R + \lambda(\eta + \alpha)) < 1,
\end{aligned}$$

where we have used the induction hypotheses and

$$\|y_n - y_0\| \le \|(1 - \lambda)(x_n - x_0) + \lambda(x_{n-1} - x_0) + \lambda(x_0 - x_{-1})\| < R + \lambda\alpha,$$

$$\|z_n - z_0\| \le \|(x_n - x_0) + \lambda(x_n - x_{n-1}) - \lambda(x_0 - x_{-1})\| < R + \lambda(\eta + \alpha).$$

So, by Banach Lemma [20], we have the existence of A_n^{-1} and $(ii)_n$ is proved.

Now, we use the definition of first order divided difference and our iterative process (1.5) to obtain:

$$F(x_n) = F(x_{n-1}) - [x_{n-1}, x_n, F](x_{n-1} - x_n) = (A_{n-1} - [x_{n-1}, x_n, F])(x_{n-1} - x_n).$$

This allows to have

$$\|F(x_n)\| \le \omega(\|y_{n-1} - x_{n-1}\|, \|z_{n-1} - x_n\|)\|x_{n-1} - x_n\| \le \omega(\lambda\eta, (1 + \lambda)\eta)\|(x_n - x_{n-1})\|, \tag{3.15}$$

where we have used $\|y_{n-1} - x_{n-1}\| \le \lambda\eta$, and

$$\|z_{n-1} - x_n\| \le \|(x_{n-1} - x_n) + \lambda(x_{n-1} - x_{n-2})\| \le (1 + \lambda)\eta.$$

Then, to prove $(iii)_n$, we consider

$$\begin{aligned}
\|x_{n+1} - x_n\| &\le \|A_n^{-1}\|\|F(x_n)\|\\
&\le \frac{w(\lambda\eta, (1 + \lambda)\eta)}{1 - \beta w(R + \lambda\alpha, R + \lambda(\eta + \alpha))}\|(x_{n-1} - x_n)\|\\
&\le Q(R)\|x_{n-1} - x_n\|\\
&\le Q(R)^n\|x_1 - x_0\| < \|x_1 - x_0\|.
\end{aligned}$$

On the other hand, by the induction hypothesis $(iv)_{n-1}$, we have that the iterates $\{x_n\}$ defined by (1.5) remain in the ball $B(x_0, R)$ since

$$\|x_{n+1} - x_0\| \le \|x_{n+1} - x_n\| + \|x_n - x_0\| \le (Q(R)^n + Q(R)^{n-1} + ... + Q(R) + 1)\eta < R.$$

Therefore the mathematical induction procedure is finished.

Next, to conclude the proof, we prove that $\{x_n\}$ is a Cauchy sequence. For this, we consider

$$\begin{aligned}\|x_{n+k} - x_n\| &\leq \sum_{j=1}^{k} \|x_{n+j} - x_{n-(j-1)}\| \\ &\leq (Q(R)^{k-1} + Q(R)^{k-2} + ... + Q(R) + 1)\|x_{n+1} - x_n\| \\ &\leq \frac{1 - Q(R)^k}{1 - Q(R)}\|x_{n+1} - x_n\| \leq \frac{1}{1 - Q(R)} Q(R)^n \eta.\end{aligned}$$

So, $\{x_n\}$ is a Cauchy sequence that converges to $x^* \in \overline{B(x_0, R)}$. Moreover, in order to have that the limit x^* is a solution of $F(x) = 0$, we notice that by using (3.15) we get

$$\|F(x_n)\| \leq \omega(\lambda\eta, (1 + \lambda)\eta)\|(x_n - x_{n-1})\| \tag{3.16}$$

and taking limits when $n \to +\infty$, by the continuity of the operator F, we have that $F(x^*) = 0$. □

Concerning the uniqueness of the solution x^*, we have the following result.

Theorem 3.3. *Under the conditions (SL) suppose that there exists $\widetilde{R} \geq R$ such that*

$$\beta\omega(R, \widetilde{R} + \lambda\alpha) < 1. \tag{3.17}$$

Then, x^* is the unique solution of equation (1.1) in $\overline{B(x_0, \widetilde{R})} \cap D$.

Proof. If we assume the existence of $y^* \in \overline{B(x_0, \widetilde{R})} \cap D$, be such that $F(y^*) = 0$, we can write

$$[x^*, y^*; F](x^* - y^*) = F(x^*) - F(y^*) = 0. \tag{3.18}$$

Then, using $(SL1)$ and $(SL2)$, we get in turn that

$$\begin{aligned}\|I - [y_0, z_0; F]^{-1}[x^*, y^*; F]\| &\leq \|[y_0, z_0; F]^{-1}\|\|[y_0, z_0; F] - [x^*, y^*; F]\| \\ &\leq \beta\omega(\|y_0 - x^*\|, \|z_0 - y^*\|) \\ &\leq \beta\omega(\|y_0 - x^*\|, \|x_0 - y^*\| + \lambda\|x_0 - x_{-1}\|) \\ &\leq \beta\omega(R, \widetilde{R} + \lambda\alpha) < 1.\end{aligned}$$

Hence, there exists $[x^*, y^*; F]^{-1}$. Then, from (3.18), we deduce that $x^* = y^*$. □

4. Local convergence from auxiliary points

In this Section, we focus our attention on the analysis of the local convergence of sequence (1.5). We present a local convergence analysis for the uniparametric family of iterative processes (1.5) in order to approximate a locally unique solution of the equation (1.1), both in the differentiable case and in the non-differentiable case for the operator F. Therefore, we will consider $\lambda \in (0, 1]$ throughout this section to contemplate both situations.

On the other hand, it is common for the study of local convergence of derivative-free iterative processes to show a small contradiction. Usually, in many known results of local convergence (see [6], [9], [19], [21], [25], [26], and references therein given) the existence of the operator $[F'(x^*)]^{-1}$ is required, forcing the operator F to be Fréchet differentiable. These results therefore study the accessibility of the iterative process for Fréchet differentiable operators. However, in [17], by modifying the hypothesis about the solution x^* and using an auxiliary point, a result of local convergence for (1.4) is obtained, where the operator F is non-differentiable. This is the procedure that we are going to follow to obtain our local convergence result for (1.5).

We shall show the local convergence of method (1.4) based on the following conditions (L):

(L1) There exists $x^* \in D$ with $F(x^*) = 0$, $\delta, \beta > 0$ and $\tilde{x} \in D$, with $\|\tilde{x} - x^*\| = \delta$, such that there exists $[x^*, \tilde{x}; F]^{-1} \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$ with $\|[x^*, \tilde{x}; F]^{-1}\| \leq \beta$.

(L2) There exists $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ a non-decreasing continuous function in its two variables, such that

$$\|[x, y; F] - [u, v; F]\| \leq \omega(\|x - u\|, \|y - v\|)$$

holds for each $x, y, u, v \in D$, with $x \neq y$ and $u \neq v$.

(L3) The equation

$$d(1 + \lambda)\beta\omega(2\lambda t, (1 + 2\lambda)t) + (d - t)(1 - \beta\omega_0(t, \delta + t) = 0 \tag{4.19}$$

has at least one positive zero, and we denote by R the smallest positive zero. Where $d = \|x_0 - x^*\|$ and ω_0 is given by the relation

$$\|[x, y; F] - [x^*, \tilde{x}; F]\| \leq \omega_0(\|x - x^*\|, \|y - \tilde{x}\|) \tag{4.20}$$

for each $x, y \in D$, with $x \neq y$, being $\omega_0 : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ a non-decreasing continuous function in its two variables.

(L4) $B(x^*, R) \subseteq D$ and $\beta\left(\omega(2\lambda R, (1 + 2\lambda)R) + \omega_0(R, \delta + R)\right) < 1$.

Notice that condition (4.20) is not additional to (3.11), because obviously, if an operator divided difference is ω-continuous in D, condition (3.11), the operator divided difference is ω_0-center-continuous in $x^*, \tilde{x} \in D$ for each pair of different points $x, y \in D$. In practice the computation of function ω involves the computation of function ω_0 as a special case. However, we have $\omega_0(s, t) \leq \omega(s, t)$ for each $\mathrm{s}, \mathrm{t} \in \mathbb{R}_+$ and the function $\dfrac{\omega}{\omega_0}$ can be arbitrarily large ([6],[25]). Taking into account a result given in [6], by combining ω-continuous and ω_0-center-continuous conditions, we can to obtain a wider choice of initial points to increase the accessibility of iterative process, tighter error distances and a more precise uniqueness ball.

Next, we present an auxiliary perturbation result on the divided difference of order one for the operator F.

Lemma 4.1. *Suppose that (L) conditions hold. If $y, z \in B(x^*, R)$, with $y \neq z$, then there exists $[y, z; F]^{-1} \in \mathcal{L}(Y, X)$ and*

$$\|[y, z; F]^{-1}\| \leq \frac{\beta}{1 - \beta\omega_0(\|y - x^*\|, \|z - \tilde{x}\|)} \leq \frac{\beta}{1 - \beta\omega_0(R, \delta + R)}. \tag{4.21}$$

Proof. Using (4.20), we obtain in turn

$$\begin{aligned}
\|I - [x^*, \tilde{x}; F]^{-1}[y, z; F]\| &\leq \|[x^*, \tilde{x}; F]^{-1}\|\|[x^*, \tilde{x}; F] - [y, z; F]\| \\
&\leq \beta\omega_0(\|x^* - y\|, \|\tilde{x} - x^*\| + \|x^* - z\|) \\
&< \beta\omega_0(R, \delta + R).
\end{aligned}$$

Now, from $(L4)$, we have that $\beta\omega_0(R, \delta + R) < 1$. Then, by the Banach Lemma on invertible operators [20], there exists the operator $[y, z; F]^{-1}$ so that (4.21) is satified. □

Lemma 4.2. *Suppose that (L) conditions hold and consider* $n \geqslant 1$. *If* $x_{n-1}, x_{n-2} \in B(x^*, R)$, *with* $x_{n-1} \neq x_{n-2}$, *and* $y_{n-1}, z_{n-1} \in D$, *then, sequence* $\{x_n\}$ *generated by the family of iterative processes (1.5) is well defined and*

$$\|x_n - x^*\| < M(R)\|x_{n-1} - x^*\|, \quad \text{where} \quad M(R) = \frac{\beta\omega(2\lambda R, (1+2\lambda)R)}{1 - \omega_0(R, \delta + R)}.$$

Proof. Firstly, note that there exists $[y_{n-1}, z_{n-1}; F]$, since that $y_{n-1} - z_{n-1} = 2\lambda(x_{n-2} - x_{n-1}) \neq 0$, and therefore $y_{n-1} \neq z_{n-1}$. Secondly, from the hypotheses given and applying Lemma 4.1, it follows that A_{n-1} is invertible and

$$\|A_{n-1}^{-1}\| = \|\,[y_{n-1}, z_{n-1}; F]^{-1}\,\| \leq \frac{\beta}{1 - \beta\omega_0(R, \delta + R)}.$$

So that x_n is well defined. Now, from (1.5), it follows

$$\begin{aligned} x_n - x^* &= x_{n-1} - A_{n-1}^{-1}F(x_{n-1}) - x^* \\ &= .A_{n-1}^{-1}(A_{n-1}(x_{n-1} - x^*) - F(x_{n-1})) \\ &= A_{n-1}^{-1}([y_{n-1}, z_{n-1}; F]\,(x_{n-1} - x^*) - F(x_{n-1}) - F(x^*)) \\ &= A_{n-1}^{-1}([y_{n-1}, z_{n-1}; F] - [x_{n-1}, x^*; F])(x_{n-1} - x^*). \end{aligned}$$

Taking norms in the previous expression and applying conditions (3.11) and (4.20), we obtain

$$\|x_n - x^*\| \leq \|A_{n-1}^{-1}\|\omega(\|y_{n-1} - x_{n-1}\|, \|z_{n-1} - x^*\|)\|x_{n-1} - x^*\|$$

$$\leq \frac{\beta}{1 - \beta\omega_0(R, \delta + R)}\omega(\lambda\|x_{n-1} - x_{n-2}\|, (1+\lambda)\|x_{n-1} - x^*\| + \lambda\|x_{n-2} - x^*\|)\|x_{n-1} - x^*\|$$

$$< \frac{\beta\omega(2\lambda R, (1+\lambda)R + \lambda R)}{1 - \beta\omega_0(R, \delta + R)}\|x_{n-1} - x^*\| = M(R)\|x_{n-1} - x^*\|.$$

So, the proof is complete. □

Taking into account the previous results, we need to require that $y_{n-1}, z_{n-1} \in B(x^*, R)$ is verified when the hypotheses $x_{n-1}, x_{n-2} \in B(x^*, R)$ are true. Our immediate aim is to analyze this condition. If we denote $d = \|x_0 - x^*\|$ and assume that $x_1, x_2, \ldots, x_{n-1} \in B(x^*, R)$ and the conditions of the previous results are satisfied, it follows that

$$\|y_{n-1} - x^*\| \leq (1-\lambda)\|x_{n-1} - x^*\| + \lambda\|x^* - x_{n-2}\| < R,$$

and

$$\|z_{n-1} - x^*\| \leq (1+\lambda)\|x_{n-1} - x^*\| + \lambda\|x_{n-2} - x^*\| \leq ((1+\lambda)M(R) + \lambda)\,\|x_{n-2} - x^*\|.$$

However, from the Lemma 4.2, we have

$$\|x_{n-2} - x^*\| \leq M(R)^{n-2}\|x_0 - x^*\|.$$

Therefore, as $M(R) < 1$ from condition $(L4)$, we obtain

$$\|x_{n-2} - x^*\| < \|x_0 - x^*\| = d$$

and then, we can consider

$$\|z_{n-1} - x^*\| < ((1+\lambda)M(R) + 1)d = R$$

Obviously, we get $d < R$.

Then, R must be a positive root of the equation:

$$((1+\lambda)M(R) + 1)d = R, \tag{4.22}$$

that is to say, the equation (4.19). Obviously, we get $R > d$.

We can now show the main result of local convergence for method (1.5) using the (*L*) conditions.

Theorem 4.4. *Suppose that (L) conditions hold. If $x_{-1}, x_0 \in B(x^*, R)$, with $x_{-1} \neq x_0$, $\|x_0 - x^*\| = d$ and $\|x_{-1} - x^*\| < \frac{1}{\lambda}(R-d)$, then, sequence $\{x_n\}$ generated for $x_{-1}, x_0 \in B(x^*, R)$ by the family of iterative processes (1.5) is well defined, remains in $B(x^*, R)$, for each $n = 0, 1, 2, \ldots$, and converges to x^*.*

Proof. Taking into account that $x_{-1}, x_0 \in B(x^*, R)$, from (1.5) and $\lambda \in (0, 1]$, we have

$$\|y_0 - x^*\| \leq (1-\lambda)\|x_0 - x^*\| + \lambda\|x_{-1} - x^*\| < R$$

and

$$\|z_0 - x^*\| \leq \|x_0 - x^*\| + \lambda\|x_{-1} - x^*\| < R.$$

So, $y_0, z_0 \in B(x^*, R) \subseteq D$. Moreover, as $\lambda \in (0, 1]$, it follows that

$$z_0 - y_0 = 2\lambda(x_0 - x_{-1}) \neq 0,$$

therefore, y_0 and z_0 are a pair of different points in $B(x_0, R)$ and, from Lemma 4.1, there exists $A_0 = [y_0, x_0; F]^{-1}$. So, x_1 is well defined. Then, from Lemma 4.2, we have $\|x_1 - x^*\| < M(R)\|x_0 - x^*\| < \|x_0 - x^*\| = d < R$. So, if $x_1 = x_0$ then $x_n = x^*$, for all $n \geqslant 0$, and the result is proved. In other case, $x_1 \neq x_0$, we have $x_1, x_0 \in B(x^*, R)$ and, moreover, we obtain

$$\|y_1 - x^*\| \leq (1-\lambda)\|x_1 - x^*\| + \lambda\|x_0 - x^*\| < R$$

and, from (4.22), we have

$$\|z_1 - x^*\| \leq (1+\lambda)\|x_1 - x^*\| + \lambda\|x_0 - x^*\| < [(1+\lambda)M(R) + \lambda]d \leq R.$$

So, $y_1, z_1 \in B(x^*, R) \subseteq D$. Moreover, as $\lambda \in (0, 1]$, it follows that

$$z_1 - y_1 = 2\lambda(x_1 - x_0) \neq 0,$$

therefore, y_1 and z_1 are a pair of different points in $B(x^*, R)$ and, from Lemma 4.1, there exists $A_1 = [y_1, x_1; F]^{-1}$. Therefore, x_2 is well defined.

Then, by a mathematical inductive procedure, it is easy to check that for all $n \geqslant 1$ the following items are verified:

(i) $x_n \in B(x^*, R)$,
(ii) $y_n, z_n \in B(x^*, R) \subseteq D$ are a pair of different points,
(iii) $\|x_n - x^*\| < M(R)\|x_{n-1} - x^*\| < M(R)^{n-1}\|x_1 - x_0\| < \|x_1 - x_0\|$.

Therefore, the sequence $\{x_n\}$ given by (1.5) remains in $B(x^*, R)$. Moreover, as $\{\|x_n - x^*\|\}$ is a strictly decreasing sequence of positive real numbers, it follows that $\{x_n\}$ converges to x^*. □

Concerning the uniqueness of the solution x^*, we have the following result.

Theorem 4.5. *Under the conditions (L) suppose that there exists $\widetilde{R} \geq R$ such that*

$$\beta\omega_0(0, \delta + \widetilde{R}) < 1. \tag{4.23}$$

Then, x^* is the unique solution of equation (1.1) in $\overline{B(x^*, \widetilde{R})} \cap D$.

Proof. Let $y^* \in \overline{B(x^*, \widetilde{R})} \cap D$ be such that $F(y^*) = 0$. Then, using (4.20), we get in turn that

$$\begin{aligned}\|I - [x^*, \tilde{x}; F]^{-1}[x^*, y^*; F]\| &\leq \|[x^*, \tilde{x}; F]^{-1}\|\|[x^*, \tilde{x}; F] - [x^*, y^*; F]\| \\ &\leq \beta\omega_0(\|x^* - x^*\|, \|\tilde{x} - y^*\|) \\ &\leq \beta\omega_0(0, \delta + \widetilde{R}) < 1.\end{aligned}$$

Hence, there exists $[x^*, y^*; F]^{-1}$. Then, from $0 = F(x^*) - F(y^*) = [x^*, y^*; F](x^* - y^*)$, we deduce that $x^* = y^*$. □

5. Numerical experiments

An important class of problems for ordinary differential equations consist of what are called boundary value problems [1, 28]. A typical example is to find a solution of the second-order differential equation

$$\frac{d^2x(t)}{dt^2} = \phi(t, x(t)), \tag{5.24}$$

where $\phi : \Omega \subseteq [a, b] \times \mathbb{R} \to \mathbb{R}$, $a, b \in \mathbb{R}$ which satisfies the boundary conditions

$$x(a) = A \text{ and } x(b) = B.$$

Extensive discussions of (5.24), with applications to a variety of physical problems, can be found in above mentioned references.

Next, we show the application of the previous study to the following boundary value problem:

$$\begin{aligned}&\frac{d^2x(t)}{dt^2} + x(t)^2 + |x(t)| + P = 0, \\ &x(0) = 0 = x(1).\end{aligned} \tag{5.25}$$

with $P \in \mathbb{R}$.

To solve this problem by finite differences, we start by drawing the usual grid line with grid points $t_i = ih$, where $h = 1/n$ and n is an appropriate integer. Note that x_0 and x_n are given by the boundary conditions, then $x_0 = 0 = x_n$, and our work is to find the other x_i ($i = 1, 2, \ldots, n-1$). To do this, we begin by replacing the second derivative $x''(t)$ in the differential equation with its approximation

$$\begin{aligned}x''(t) &\approx [x(t+h) - 2x(t) + x(t-h)]/h^2, \\ x''(t_i) &= (x_{i+1} - 2x_i + x_{i-1})/h^2, \quad i = 1, 2, \ldots, n-1.\end{aligned}$$

So, we have the following system of non-linear equations

$$\begin{cases} 2x_1 - h^2x_1^2 - h^2|x_1| - x_2 - h^2P = 0, \\ -x_{i-1} + 2x_i - h^2x_i^2 - h^2|x_i| - x_{i+1} - h^2P = 0, \quad i = 2, 3, \ldots, n-2, \\ -x_{n-2} + 2x_{n-1} - h^2x_{n-1}^2 - h^2|x_{n-1}| - h^2P = 0. \end{cases} \tag{5.26}$$

We therefore have an operator $F : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ such that $F(\mathbf{x}) = M\mathbf{x} - h^2\varphi(\mathbf{x})$, where

$$M = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{pmatrix}, \quad \varphi(\mathbf{x}) = \begin{pmatrix} x_1^2 + |x_1| + P \\ x_2^2 + |x_2| + P \\ \vdots \\ x_{n-1}^2 + |x_n| + P \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{pmatrix}$$

Let $\mathbf{x} \in \mathbb{R}^{n-1}$ and and choose the norm $\|\mathbf{x}\| = \max_{1\le i\le n-1} |x_i|$. The corresponding norm on $A \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ is

$$\|A\| = \max_{1\le i\le n-1} \sum_{j=1}^{n-1} |a_{ij}|.$$

Moreover, we then use the divided difference of first order given by $[\mathbf{u},\mathbf{v};F] = ([\mathbf{u},\mathbf{v};F]_{ij})_{i,j=1}^{n-1} \in \mathcal{L}(\mathbb{R}^{n-1},\mathbb{R}^{n-1})$, where

$$[\mathbf{u},\mathbf{v};F]_{ij} = \frac{1}{u_j - v_j}\left(F_i(u_1,\dots,u_j,v_{j+1},\dots,v_{n-1}) - F_i(u_1,\dots,u_{j-1},v_j,\dots,v_{n-1})\right),$$

$\mathbf{u} = (u_1,u_2,\dots,u_{n-1})^T$ and $\mathbf{v} = (v_1,v_2,\dots,v_{n-1})^T$. Then, in this case we have

$$[\mathbf{x},\mathbf{y};F] = M - h^2\cdot\ \mathrm{Diag}\left(\begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ \cdots\cdots\cdots \\ x_{n-1}+y_{n-1}\end{pmatrix} + \begin{pmatrix} \frac{|x_1|-|y_1|}{x_1-y_1} \\ \frac{|x_2|-|y_2|}{x_2-y_2} \\ \cdots\cdots\cdots \\ \frac{|x_{n-1}|-|y_{n-1}|}{x_{n-1}-y_{n-1}}\end{pmatrix}\right)$$

and, therefore, we obtain

$$\|[\mathbf{x},\mathbf{y};F] - [\mathbf{u},\mathbf{v};F]\| \le h^2(\|\mathbf{x}-\mathbf{u}\| + \|\mathbf{y}-\mathbf{v}\|) + 2h^2.$$

So, $\omega(s,t) = h^2(s+t)$.

5.1. **The semilocal case.** In this problem we consider $n = 10$, $P = 2$, $D = B(\mathbf{x_0}, 2)$ and we refer to stage 1 when we take starting guesses $\mathbf{x_{-1}} = (1/2, ..., 1/2)$, and $\mathbf{x_0} = (1/3, ..., 1/3)$. Then, by obtaining the bounds used in our theoretical study, one has $\alpha = 0.1667$, $\beta = 15.138021$, $\eta = 0.637074$ and the existence ball radius can be seen in Table 1, where we observe that as λ approximates to 0 the radius are better, which corresponds with Newton's method ($\lambda = 0$). Moreover in all cases the solution is unique in the whole domain $B(\mathbf{x_0}, 2)$.

λ	q	R
0.2	0.028919	0.6794
0.4	0.031467	0.6888
0.6	0.034015	0.6956
0.8	0.036564	0.7077

TABLE 1. Radii of existence ball for different values of λ and stage 1.

λ	q	R
0.1	0.022273	0.1969
0.3	0.023030	0.1974
0.5	0.023788	0.1979
0.7	0.024545	0.1984
0.9	0.025303	0.1989

TABLE 2. Radii of existence ball for different values of λ and stage 2.

Now, we refer to stage 2 when we take starting points

$$\mathbf{x}_{-1} = (0.2, 0.4, 0.6, 0.8, 0.9, 0.8, 0.6, 0.4, 0.2),$$
$$\mathbf{x}_0 = (0.3, 0.5, 0.7, 0.9, 0.1, 0.9, 0.7, 0.5, 0.3).$$

In this case, we obtain the parameters $\alpha = 0.1$, $\beta = 17.166258$, and $\eta = 0.1894$. So, Table 2 shows the corresponding values of the existence ball radii for stage 2. Note that the results are similar to that of Table 1.

Moreover, we run the family of iterative processes (1.5) with Matlab 2018 by using as stopping criteria $||\mathbf{x}_{n+1} - \mathbf{x}_n|| \leq 10^{-23}$ and we can check in Tables 3 and 4 the number of iterations needed k, the distance between the last two iterates and the value of the nonlinear operator F at the approximated solution.

	$\lambda = 0.2$	$\lambda = 0.4$	$\lambda = 0.6$	$\lambda = 0.8$
k	5	5	6	6
$\|\|\mathbf{x}_{n+1} - \mathbf{x}_n\|\|$	6.2242e-29	6.2242e-29	2.5150e-40	2.3414e-40
$\|\|F(\mathbf{x}_{n+1})\|\|$	2.9284e-40	1.6978e-40	9.7575e-41	2.0877e-40

TABLE 3. Numerical results for stage 1.

	$\lambda = 0.1$	$\lambda = 0.3$	$\lambda = 0.5$	$\lambda = 0.7$	$\lambda = 0.9$
k	6	6	6	6	6
$\|\|\mathbf{x}_{n+1} - \mathbf{x}_n\|\|$	3.0122e-36	3.0115e-36	1.983e-29	1.06895e-27	4.91376e-27
$\|\|F(\mathbf{x}_{n+1})\|\|$	1.24831e-40	8.79882e-41	1.04425e-40	1.0474e-40	6.76703e-41

TABLE 4. Numerical results for stage 2.

In Table 5 we can observe the approximated solution obtained for problem (5.25) with different starting guesses, in both stages 1 and 2 the iterates converge to the same solution.

t	$x(t)$	t	$x(t)$
0	0	0.6	0.275864
0.1	0.101628	0.7	0.240492
0.2	0.182137	0.8	0.182137
0.3	0.240492	0.9	0.101628
0.4	0.275864	1.0	0
0.5	0.287717		

TABLE 5. Approximated solution for problem (5.25)

Now, we compare the results obtained in Table 3 and Table 4 with the ones obtained for the already known uniparametric family of Secant-like methods whose iterative function is given by (1.4), the corresponding results in Table 6 and 7 indicate that the new methods introduced in this study are more competitive since with less iterations we reach same accuracy.

	$\lambda = 0$	$\lambda = 0.4$	$\lambda = 0.6$	$\lambda = 0.8$
k	7	7	7	6
$\|\|\mathbf{x}_{n+1} - \mathbf{x}_n\|\|$	8.5853e-36	2.0877e-37	1.1679e-25	3.0302e-28
$\|\|F(\mathbf{x}_{n+1})\|\|$	2.2799e-59	4.8636e-62	3.3351e-43	1.3818e-47

TABLE 6. Numerical results for stage 1 and family (1.4).

	$\lambda = 0$	$\lambda = 0.4$	$\lambda = 0.6$	$\lambda = 0.8$
k	7	7	7	7
$\lVert \mathbf{x}_{n+1} - \mathbf{x}_n \rVert$	4.8228e-27	2.6166e-29	2.3952e-31	2.8383e-35
$\lVert F(\mathbf{x}_{n+1}) \rVert$	3.1482e-45	5.1081e-49	2.0493e-52	6.2289e-59

TABLE 7. Numerical results for stage 2 and family (1.4).

5.1.1. *Remarks.* 1.- Notice that if we take different starting guess making the non-differentiable case more evident when some components of the initial points are negative and others positive, that is,

$$\mathbf{x}_{-1} = (-0.2, -0.4, -0.6, -0.8, 0.9, 0.8, 0.6, 0.4, 0.2),$$

$$\mathbf{x}_0 = (-0.3, -0.5, -0.7, -0.9, 0.1, 0.9, 0.7, 0.5, 0.3),$$

the iterative method converges to the same solution and we obtain similar results that with previous initial guesses, the results can be checked in Table 8.

	$\lambda = 0.2$	$\lambda = 0.4$	$\lambda = 0.6$	$\lambda = 0.8$
k	6	6	6	6
$\lVert \mathbf{x}_{n+1} - \mathbf{x}_n \rVert$	1.2281e-30	1.43003e-26	1.99746e-25	5.58169e-25
$\lVert F(\mathbf{x}_{n+1}) \rVert$	1.17495e-58	8.05584e-55	1.57166e-52	1.22725e-51

TABLE 8. Numerical results for starting guess given in Remark with (1.5).

2.- Moreover, in this case we also check the approximated computational order of convergence by obtaining in each iteration the following value:

$$p_n \approx \frac{log(||x_{n+1} - x_n)||/||x_n - x_{n-1})||}{log(||x_n - x_{n-1})||/||x_{n-1} - x_{n-2})||} \quad n = 2, 3, ... \tag{5.27}$$

In Table (9) we can check the behavior of the method studied in this work in the first 6 iterations, with $\lambda = 0.4$ and for starting guess given in previous remark. We notice that in the last column the sequence of iterates converges to 2, that is the value of the computational order of convergence. We compare the results with the ones obtained for secant method (1.4) with same value of λ, see Table (10), one can check the improvement obtained by the new iterative method.

n	$\lVert \mathbf{x}_{n+1} - \mathbf{x}_n \rVert$	$\lVert F(\mathbf{x}_{n+1}) \rVert$	p_n
1	1.83974	0.02076	
2	0.18714	0.00088	
3	0.00871	2.98893e-7	1.34161
4	0.000003	4.7367e-14	2.55328
5	5.56006e-13	1.21735e-27	1.99856
6	1.43003e-26	8.05584e-55	1.99987

TABLE 9. Numerical results for the 6 first iterations by iterative method (1.5).

n	$\|\|\mathbf{x}_{n+1} - \mathbf{x}_n\|\|$	$\|\|F(\mathbf{x}_{n+1})\|\|$	p_n
1	1.84328	0.02004	
2	0.16339	0.00234	
3	0.01681	0.000007	0.93858
4	0.00008	3.49104e-9	2.31441
5	3.97475e-8	8.18631e-15	1.46115
6	9.57931e-14	8.99367e-24	1.68193

TABLE 10. Numerical results for the 6 first iterations by iterative method (1.4).

5.2. **The local case.** In this case we consider $P = 0$. So, the solution of $F(\mathbf{x}) = 0$ is $\mathbf{x}^* = (0, ..., 0) \in \mathbb{R}^{n-1}$. In order to obtain the local convergence ball for this problem we take $n = 10$, $D = B(\mathbf{x_0}, 2)$ and we refer to stage 3 when we take starting guesses $\mathbf{x}_{-1} = (1/3, ..., 1/3)$, $\mathbf{x}_0 = (1/5, ..., 1/5)$, and $\tilde{\mathbf{x}} = (1/2, ..., 1/2)$. Then, the conditions established in section 4 follow with $\delta = 0.1667$, $\beta = 14.2392$ and the equation given in $(L3)$ gives us the local convergence radii that can be seen in Table 11, for different values of parameter λ. We observe that as λ increases, so do the radius. Moreover in all cases the solution is unique in the whole domain, being $\tilde{R} = 2$ the uniqueness radius. Now, we refer to stage 4 when we take starting points $\mathbf{x}_{-1} = (0.2, 0.4, 0.6, 0.8, 0.9, 0.8, 0.6, 0.4, 0.2)/3$, $\mathbf{x}_0 = \mathbf{x}_{-1}/2$, and $\tilde{\mathbf{x}} = (\mathbf{x}_{-1} + \mathbf{x}_0)/2$. Then, the corresponding values for stage 4 can also be seen in Table 12, being the parameters $\delta = 0.225$ and $\beta = 14.2553$.

λ	0.2	0.4	0.6	0.8
R	0.3523	0.4138	0.5127	0.7612

TABLE 11. Radii of convergence ball for different values of λ and stage 3.

λ	0.1	0.3	0.5	0.7
R	0.2394	0.2688	0.3075	0.3616

TABLE 12. Radii of convergence ball for different values of λ and stage 4.

6. CONCLUSIONS

In this work, by using symmetric first-order divided differences, we establish a modification of a known family of secant-like iterative processes. From this modification, we obtain a new family of secant-like iterative processes more efficient than the previous one, since we get quadratic convergence with a similar operational cost. The local and semilocal convergence study shows an improved behavior of secant-like methods analyzed in previous papers. The theoretical results established are based in weaker assumptions for the nonlinear operator that the ones used before. Moreover, numerical examples have been developed to corroborate the theoretical results under new conditions for the operators. The obtained results show the advantages of using this new family of iterative methods.

REFERENCES

[1] Andronow, A. A.; Chaikin, C. E. *Theory of Oscillations* (Princenton University Press, Princenton, N. J., 1949).
[2] Argyros, I. K. On the Secant method. *Publ. Math. Debrecen* **43** (1993), 223–238.
[3] Argyros, I. K.; Cho, Y. J.; Hilout, S. *Numerical Methods for Equations and Its Applications.* CRC Press/Taylor and Francis, Boca Raton, Florida, USA, 2012.
[4] Argyros, I. K.; Hilout, S. *Numerical Methods in Nonlinear Analysis.* World Scientific Publ. Comp., New Jersey, 2013.
[5] Argyros, I. K. The Secant method and fixed points of nonlinear operators. *Mh. Math.* **106** (1988), 85–94.
[6] Argyros, I. K.; Ren, H. On an improved local convergence analysis for the Secant method. *Numer. Algor.* **52** (2009), 257–271.
[7] Balazs, M.; Goldner, G. On existence of divided differences in linear spaces. *Rev. Anal. Numer. Theor. Approx.* **2** (1973), 3–6.
[8] Catinas, E. *How many steps still left to x^*?* SIAM Review **63** (2021), no. 3, 585–624, doi: 10.1137/19M1244858.
[9] Chen, J.; Shen, Z. Convergence analysis of the secant type methods *Appl. Math. Comput.* **188** (2007), 514–524.
[10] Cordero, Alicia; Hueso, José, L.; Martínez, Eulalia; Torregrosa, Juan, R. A modified Newton-Jarrat's composition. *Numer. Algor.* **55** (2010), 87–99.
[11] Grau-Sánchez, M.; Noguera, M. *On Convergence and Efficiency in the Resolution of Systems of Nonlinear Equations from a Local Analysis* Chapter 10, Springer International Publishing Switzerland 2016. S. Amat, S. Busquier (eds.), Advances in Iterative Methods for Nonlinear Equations, SEMA SIMAI Springer Series 10.
[12] Grau-Sánchez, M.; Grau, A.; Noguera, M. Frozen divided differences scheme for solving systems of nonlinear equations. *J. Comput. Appl. Math.* **235** (2011), 1739–1743.
[13] Grau-Sánchez, M.; Noguera, M.; Amat, S. On the approximation of derivatives using divided difference operators preserving the local convergence order of iterative methods. *J. Comput. Appl. Math.* **237** (2013), 363–372.
[14] Hernández-Verón, M. A.; Magreñán, Á. A.; Rubio, M. J. Dynamics and local convergence of a family of derivative-free iterative processes *J. Comput. Appl. Math.* **354** (2019), 414–430.
[15] Hernández, M. A.; Rubio, M. J. A new type of recurrence relations for the Secant method.*Int. J. Comput. Math.* **72** (1999), 477–490.
[16] Hernández, M. A.; Rubio, M. J. A uniparametric family of iterative processes for solving nondifferentiable equations, *J. Math. Anal. Appl.* **275** (2002), 821–834.
[17] Hernández-Verón, M. A.; Rubio, M. J. On the ball of convergence of Secant-like methods for non- differentiable operators. *Applied Mathematics and Computatio*n (2015), http://dx.doi.org/10.1016/j.amc.2015.10.007.
[18] Hernández, M. A.; Rubio, M. J.; Ezquerro, J. A. Solving a special case of conservative problems by secant-like methods, *Appl. Math. Comput.* **169**, **2**2 (2005), 926–942.
[19] Hongmin, R.; Qingiao, W. The convergence ball of the Secant method under Hölder continuous divided differences, *J. Comput. Appl. Math.* **194** (2006), 284–293.
[20] Kantorovich, L. V.; Akilov, G. P. *Functional analysis*. Pergamon Press, Oxford, 1982.
[21] Kewei, L. Homocentric convergence ball of the Secant method *Appl. Math. J. Chinese Univ. Ser. B.* **22** (2007), no. 3, 353–365.
[22] Kurchatov, V. A. On a method of linear interpolation for the solution of functional equations. (Russian) *Dokl. Akad. Nauk SSSR.* **198** (1971), no. 3, 524–526; translation in Soviet Math. Dokl. **12** (1971), 835–838.
[23] Ortega, J. M.; Rheinboldt, W. C. *Iterative Solution of Nonlinear Equations in Several Variables.* Academic Press, New York, 1970.
[24] Potra, F. A.; Pták, V. *Nondiscrete induction and iterative methods.* Pitman Publishing Limited, London, 1984.
[25] Ren, H.; Argyros, I. K. Local convergence of efficient Secant-type methods for solving nonlinear equations. *Appl. Math. Comput.* **218** (2012), 7655–7664.
[26] Shakhno, S. On the Secant method under generalized Lipschitz conditions for the divided difference operator. *PAMM-Proc. Appl. Math. Mech.* **7** (2007), 2060083–2060084.
[27] Schröeder, E. Über unendlich viele algotithmen zur auflösung der Gleichugen. *Math. Ann.* **2** (1870), 317–365.
[28] Stoker, J. J. *Nonlinear Vibrations* (Interscience-Wiley, New York, 1950).
[29] Tornheim, L. Convergence of multipoint iterative methods. *J. ACM* **11** (1964), 210–220.

[1] DEPT. OF MATHEMATICS AND COMPUTATION
UNIVERSITY OF LA RIOJA. LOGROÑO, SPAIN
Email address: `mahernan@unirioja.es`

[2] INSTITUTO UNIVERSITARIO DE MATEMÁTICA MULTIDISCIPLINAR
UNIVERSITAT POLITÈCNICA DE VALÈNCIA. VALÈNCIA, SPAIN
Email address: `jlhueso@mat.upv.es; eumarti@mat.upv.es`