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Research Paper

Location, separation and approximation of solutions of nonlinear Hammerstein-type integral equations ☆

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ABSTRACT

From Newton's method, we construct a Newton-type iterative method that allows studying a class of nonlinear Hammerstein-type integral equations. This method is reduced to Newton's method if the kernel of the integral equation is separable and, unlike Newton's method, can be applied to approximate a solution if the kernel is nonseparable. In addition, from an analysis of the global convergence of the method, we can locate and separate solutions of the nonlinear Hammerstein-type integral equations involved. For this study of the global convergence, we use auxiliary functions and obtain restricted global convergence domains that are usually balls.

1. Introduction

In this work, we consider nonlinear Hammerstein-type integral equations of the form [2,10,12]

$$\phi(x) = f(x) + \lambda \int_a^b \mathcal{K}(x,t)\mathcal{H}(\phi)(t)\,dt, \quad x \in [a,b], \quad \lambda \in \mathbb{R}, \tag{1}$$

where $f \in C[a,b]$, the kernel $\mathcal{K}(x,t)$ is a known function in $[a,b]\times[a,b]$, $\mathcal{H}$ is the Nemytskii operator $\mathcal{H} : \Omega \subseteq C[a,b] \to C[a,b]$ such that Ω is a nonempty open convex subset and $\mathcal{H}(\phi)(x) = H(\phi(x))$, where $H : \mathbb{R} \to \mathbb{R}$, and $\phi(x)$ is the unknown function to be determined.

For the integral equation (1), we consider the operator $\mathcal{G} : \Omega \subseteq C[a,b] \to C[a,b]$ with

$$[\mathcal{G}(\phi)](x) = \phi(x) - f(x) - \lambda \int_a^b \mathcal{K}(x,t)\mathcal{H}(\phi)(t)\,dt, \quad x \in [a,b], \quad \lambda \in \mathbb{R}, \tag{2}$$

where Ω is a nonempty open convex subset, so that a solution of the operator equation (2) is a solution of the integral equation (1).

An iterative method commonly used to approximate such solution is Newton's method [1,3], which is defined as follows

$$\phi_{n+1} = \phi_n - [\mathcal{G}'(\phi_n)]^{-1}\mathcal{G}(\phi_n), \quad n \geq 0, \quad \text{with } \phi_0 \text{ given in } C[a,b]. \tag{3}$$

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In [9], for solving the equation (1), the authors present the following adapted modification of Newton's method

$$y_{n+1}(x) = f(x) - \lambda \int_a^b \mathcal{K}(s,t)\mathcal{H}(y_n)(t)\,dt + \lambda \int_a^b \mathcal{K}(x,t)\mathcal{H}'(y_n)(y_n(t) - y_{n-1}(t))\,dt, \quad n \in \mathbb{N},$$

where $y_0(x) = f(x)$ and

$$y_1(x) = f(x) - \lambda \int_a^b \mathcal{K}(x,t)\mathcal{H}(f)(t)\,dt + \lambda \int_a^b \mathcal{K}(s,t)\mathcal{H}'(f)f(t)\,dt,$$

which is conceived from (3) when it is written as

$$\mathcal{G}(\phi_{n+1}) = \mathcal{G}(\phi_n) + \mathcal{G}'(\phi_n)(\phi_{n+1} - \phi_n), \quad n \geq 0.$$

In [11], Newton's method and quadrature methods are combined to develop a new method for solving the equation (1), so that Newton's method is applied to solve (1) as follows

$$\begin{cases} \xi_n(x) = \xi_{n-1}(x) + \nu_{n-1}(x), \\ \nu_{n-1}(x) = \chi_{n-1}(x) + \lambda \displaystyle\int_a^b \mathcal{K}(x,t)\mathcal{H}'(\xi_{n-1})\nu_{n-1}(t)\,dt, \\ \chi_{n-1}(x) = f(s) - \xi_{n-1}(x) + \lambda \displaystyle\int_a^b \mathcal{K}(x,t)\mathcal{H}(y_{n-1})(t)\,dt, \end{cases}$$

with $\xi_0(s) = f(s)$. Then, the two integrals appearing in the last scheme are approximated by numerical quadrature, so that

$$\xi_n(x_i) = f(x_i) + \lambda \sum_{j=0}^{n} w_j \mathcal{K}(x_i, t_j)\mathcal{H}(\xi_{n-1})(t_j)$$
$$+ \lambda \sum_{j=0}^{n} w_j \mathcal{K}(x_i, t_j)\mathcal{H}'(\xi_{n-1})(\xi_n(x_j) - \xi_{n-1}(x_j)), \quad n \in \mathbb{N},$$

where $\xi_0(x_i) = f(x_i)$, for $i = 0, 1, \ldots, n$, and the nodes x_j, t_j and the weights w_j are known.

In [7], we apply Newton's method (3) directly in a continuous form and justify numerically that in this way we significantly improve the results obtained in [9,11]. However, we have an important problem: we can apply the study if the kernel of the integral equation involved is separable. Remember that, the kernel $\mathcal{K}(x,t)$ is separable if $\mathcal{K}(x,t) = \sum_{i=1}^{m} \ell_i(x)\varpi_i(t)$, where $\ell_i, \varpi_i \in \mathcal{C}[a,b]$, for $i = 1, 2, \ldots, m$.

It is known that we can calculate explicitly $[\mathcal{G}'(\phi_n)]^{-1}$ if the kernel $\mathcal{K}$ is separable (see [4]), but this is not possible if the kernel is non-separable. In this case, from the ideas given in [8], we construct a Newton-type iterative method that can be applied in a continuous form. So, we can consider the operator $\mathcal{J} : \Omega \subseteq \mathcal{C}[a,b] \to \mathcal{C}[a,b]$ with

$$[\mathcal{J}(\phi)](x) = \phi(x) - f(x) - \lambda \int_a^b \mathcal{N}(x,t)\mathcal{H}(\phi)(t)\,dt, \quad x \in [a,b], \quad \lambda \in \mathbb{R},$$

where $\mathcal{N}$ is a separable approximation of the kernel $\mathcal{K}$. In this case, it is known that we can obtain explicitly $[\mathcal{J}'(\phi_n)]^{-1}$ (see [8]), since

$$[\mathcal{J}'(\phi)\varphi](x) = \varphi(x) - \lambda \int_a^b \mathcal{N}(x,t)[\mathcal{H}'(\phi)\varphi](t)\,dt = \varphi(x) - \lambda \int_a^b \mathcal{N}(x,t)H'(\phi(t))\varphi(t)\,dt,$$

choose $[\mathcal{J}'(\phi_n)]^{-1}$ as an approximation of $[\mathcal{G}'(\phi_n)]^{-1}$ and consider the Newton-type iterative method given by

$$\phi_{n+1} = \phi_n - [\mathcal{J}'(\phi_n)]^{-1}\mathcal{G}(\phi_n), \quad n \geq 0, \quad \text{with } \phi_0 \text{ given in } \mathcal{C}[a,b]. \tag{4}$$

Note that, if the kernel $\mathcal{K}$ is sufficiently differentiable in either of its two variables, we can consider $\mathcal{K}(x,t) = \mathcal{N}(x,t) + \mathcal{R}(\theta, x, t)$, where $\mathcal{N}(x,t)$ is a Taylor series, so that $\mathcal{N}(x,t) = \sum_{i=1}^{m} c_i(x)d_i(t)$ with $c_i, d_i \in \mathcal{C}[a,b]$, for $i = 1, 2, \ldots, m$, and then it is separable. Observe that $\mathcal{R}(\theta, x, t) = 0$ if $\mathcal{K}(x,t)$ is separable and the iterative method (4) is then reduced to Newton's method (3).

One of the aims of this paper is to show that the Newton-type iterative method given in (4) is a good approximation of Newton's method (3) in the sense of the approximations given by both methods are similar. For this, we prove that the operators $[\mathcal{G}'(\phi_n)]^{-1}$ and $[\mathcal{J}'(\phi_n)]^{-1}$ are as close as we want by adjusting the value of m in the separable approximation $\mathcal{N}$ of the kernel $\mathcal{K}$.

Other of the aims is the qualitative study that we can obtain from the Newton-type iterative method (4). For this, we do an analysis of the restricted global convergence for (4), so that we can obtain the existence of a solution of (1) in a certain domain in which the global convergence of (4) is guaranteed. Moreover, we obtain a result on the uniqueness of solution that allows separating solutions of (1). So, we use the theoretical results obtained from the convergence of the method (4) to draw conclusions about the existence and separation of solutions of (1). In addition, we obtain global convergence in restricted domains that are usually balls. For this, we analyse the convergence of the method by a technique based on recurrence relations that use an auxiliary function [5,6]. Finally, we are also interested in solving the operator equation (1) by means of the Newton-type iterative method (4).

The paper is organized as follows. In Section 2, we analyse the restricted global convergence of the method (4) by using a system of recurrence relations that base on an auxiliary function. This analysis allows us to locate and separate solutions of equations involved. In Section 3, we establish a result on the uniqueness of solutions that allows us to improve the separation of solutions obtained previously. In Section 4, we present some practical remarks that lead us to a practical application of the method (4). All results are illustrated with examples of integral equations of the form (1).

Throughout the paper, we denote $\overline{B(\phi,\varrho)} = \{\alpha \in C[a,b];\, \|\alpha-\phi\| \le \varrho\}$, $B(\phi,\varrho) = \{\alpha \in C[a,b];\, \|\alpha-\phi\| < \varrho\}$ and the set of bounded linear operators from $C[a,b]$ to $C[a,b]$ by $\mathcal{L}(C[a,b],C[a,b])$ and use the infinity norm in $C[a,b]$.

2. Restricted global convergence

We study the convergence of the iterative method (4) under the following conditions:

(A1) Let Ω be a nonempty open convex subset of $C[a,b]$ such that there exists $[\mathcal{J}'(\widetilde{\phi})]^{-1} \in \mathcal{L}(C[a,b],C[a,b])$, for some $\widetilde{\phi} \in \Omega$, with $\|[\mathcal{J}'(\widetilde{\phi})]^{-1}\| \le \beta$ and $\|[\mathcal{J}'(\widetilde{\phi})]^{-1}\mathcal{J}(\widetilde{\phi})\| \le \eta$.
(A2) There exists a constant $L_0 \ge 0$ such that $\|\mathcal{H}(\phi)-\mathcal{H}(\widetilde{\phi})\| \le L_0\|\phi-\widetilde{\phi}\|$, for $\phi \in \Omega$.
(A3) There exists a constant $L \ge 0$ such that $\|\mathcal{H}'(\phi)-\mathcal{H}'(\varphi)\| \le L\|\phi-\varphi\|$, for $\phi,\varphi \in \Omega$.

Next, we give a technical lemma which is used in the following.

Lemma 1. *Let $\phi,\varphi \in \Omega$ and suppose the conditions (A1)-(A2)-(A3). Then,*

(a) $\|\mathcal{J}'(\phi)-\mathcal{J}'(\varphi)\| \le M\|\phi-\varphi\|$ *with* $M = |\lambda|PL$ *and* $P = max_{x\in[a,b]}\int_a^b |\mathcal{N}(x,t)|\,dt$.
(b) $\|\mathcal{J}(\phi)-\mathcal{G}(\phi)\| \le \mu(\|\phi-\widetilde{\phi}\|)$ *with* $\mu:\mathbb{R}\to\mathbb{R}$ *and* $\mu(t) = |\lambda|Q(\|\mathcal{H}(\widetilde{\phi})\| + L_0 t)$ *where* $Q = max_{x\in[a,b]}\int_a^b |\mathcal{R}(\theta,x,t)|\,dt$.

Proof. Item (a). From $\psi \in \Omega$ and

$$[\mathcal{J}'(\phi)\psi](x) = \psi(x) - \lambda\int_a^b \mathcal{N}(x,t)[\mathcal{H}'(\phi)\psi](t)\,dt,$$

it follows

$$\left(\mathcal{J}'(\phi)-\mathcal{J}'(\varphi)\right)\psi(x) = -\lambda\int_a^b \mathcal{N}(x,t)[(\mathcal{H}'(\phi)-\mathcal{H}'(\varphi))\psi](t)\,dt,$$

$$\|\mathcal{J}'(\phi)-\mathcal{J}'(\varphi)\| \le |\lambda|PL\|\phi-\varphi\| = M\|\phi-\varphi\|.$$

Item (b). From

$$\mathcal{J}(\phi)(x)-\mathcal{G}(\phi)(x) = \lambda\int_a^b \left(\mathcal{K}(x,t)-\mathcal{N}(x,t)\right)\mathcal{H}(\phi)(t)\,dt,$$

and $\|\mathcal{H}(\phi)\| \le \|\mathcal{H}(\widetilde{\phi})\| + L_0\|\phi-\widetilde{\phi}\|$, it follows

$$\|\mathcal{J}(\phi)-\mathcal{G}(\phi)\| \le |\lambda|Q\left(\|\mathcal{H}(\widetilde{\phi})\| + L_0\|\phi-\widetilde{\phi}\|\right) = \mu(\|\phi-\widetilde{\phi}\|).$$

The proof is complete. □

Now, the first aim is to prove the existence of $[\mathcal{J}'(\phi)]^{-1}$ in Ω.

Lemma 2. *Suppose that the conditions (A1)-(A2)-(A3) are satisfied and $\phi \in \Omega$. If $\|\phi - \widetilde{\phi}\| < \dfrac{1}{M\beta}$, then there exists $[\mathcal{J}'(\phi)]^{-1}$. Moreover, $\|[\mathcal{J}'(\phi)]^{-1}\mathcal{J}'(\widetilde{\phi})\| \leq h(M\beta\|\phi - \widetilde{\phi}\|)$ and $\|[\mathcal{J}'(\phi)]^{-1}\| \leq \beta h(M\beta\|\phi - \widetilde{\phi}\|)$ with $h : (0,1) \to \mathbb{R}$ and $h(t) = \dfrac{1}{1-t}$.*

Proof. From $\|\phi - \widetilde{\phi}\| < \frac{1}{M\beta}$, we have

$$\|I - [\mathcal{J}'(\widetilde{\phi})]^{-1}\mathcal{J}'(\phi)\| \leq \|[\mathcal{J}'(\widetilde{\phi})]^{-1}\|\|\mathcal{J}'(\widetilde{\phi}) - \mathcal{J}'(\phi)\| \leq M\beta\|\widetilde{\phi} - \phi\| < 1$$

and, by the Banach lemma on invertible operators, it follows that the operator $[\mathcal{J}'(\phi)]^{-1}\mathcal{J}'(\widetilde{\phi})$ exists with

$$\left\|[\mathcal{J}'(\phi)]^{-1}\mathcal{J}'(\widetilde{\phi})\right\| \leq \frac{1}{1 - M\beta\|\widetilde{\phi} - \phi\|} = h(M\beta\|\phi - \widetilde{\phi}\|).$$

Moreover,

$$\left\|[\mathcal{J}'(\phi)]^{-1}\right\| \leq \left\|[\mathcal{J}'(\phi)]^{-1}\mathcal{J}'(\widetilde{\phi})\right\| \left\|[\mathcal{J}'(\widetilde{\phi})]^{-1}\right\| \leq \beta h(M\beta\|\phi - \widetilde{\phi}\|)$$

and the proof is complete. □

After that, we provide a system of recurrence relations that are used to prove the convergence of the iterative method (4).

Lemma 3. *Suppose that the conditions (A1)-(A2)-(A3) are satisfied. If $\phi_{n-1}, \phi_n \in \Omega$, then*

(a) $\|\mathcal{G}(\phi_n)\| \leq \dfrac{M}{2}\|\phi_n - \phi_{n-1}\|^2 + \delta(\|\phi_{n-1} - \widetilde{\phi}\| + \|\phi_n - \widetilde{\phi}\|)\|\phi_n - \phi_{n-1}\|$ *with* $\delta : \mathbb{R} \to \mathbb{R}$ *and* $\delta(t) = |\lambda| Q\left(\|\mathcal{H}'(\widetilde{\phi})\| + \frac{L}{2}t\right)$, *for* $n \geq 1$.

(b) $\|\phi_{n+1} - \phi_n\| \leq \Delta_n\|\phi_n - \phi_{n-1}\|$, *where*

$$\Delta_n = \beta h(M\beta\|\phi_n - \widetilde{\phi}\|)\left(\frac{M}{2}\|\phi_n - \phi_{n-1}\| + \delta(\|\phi_{n-1} - \widetilde{\phi}\| + \|\phi_n - \widetilde{\phi}\|)\right),$$

for $n \geq 1$.

(c) $\|\phi_{n+1} - \widetilde{\phi}\| \leq h(M\beta\|\phi_n - \widetilde{\phi}\|)\left(\beta\mu(\|\phi_n - \widetilde{\phi}\|) + \eta + \frac{1}{2}M\beta\|\phi_n - \widetilde{\phi}\|^2\right)$, *for* $n \geq 0$.

Proof. Item (a). From

$$[\mathcal{G}'(\varphi)\psi](x) = \psi(x) - \lambda\int_a^b \mathcal{K}(x,t)[\mathcal{H}'(\varphi)\psi](t)\,dt$$

and, since $\mathcal{K}(x,t) = \mathcal{N}(x,t) + \mathcal{R}(\theta, x, t)$, it follows

$$[\left(\mathcal{G}'(\varphi) - \mathcal{J}'(\phi)\right)\psi](x) = \lambda\int_a^b \mathcal{N}(x,t)\left(\mathcal{H}'(\phi) - \mathcal{H}'(\varphi)\right)\psi(t)\,dt - \lambda\int_a^b \mathcal{R}(\theta,x,t)[\mathcal{H}'(\varphi)\psi](t)\,dt,$$

with $\phi, \varphi, \psi \in \Omega$. Besides, taking into account (4), we have

$$\mathcal{J}'(\phi_{n-1})(\phi_n - \phi_{n-1}) = -\mathcal{G}(\phi_{n-1}).$$

Thus,

$$\begin{aligned}
\mathcal{G}(\phi_n) &= \mathcal{G}(\phi_n) - \mathcal{J}'(\phi_{n-1})(\phi_n - \phi_{n-1}) - \mathcal{G}(\phi_{n-1}) \\
&= \int_{\phi_{n-1}}^{\phi_n} (\mathcal{G}'(z) - \mathcal{J}'(\phi_{n-1}))\,dz \\
&= \int_0^1 \left(\mathcal{G}'(\phi_{n-1} + \tau(\phi_n - \phi_{n-1})) - \mathcal{J}'(\phi_{n-1})\right)(\phi_n - \phi_{n-1})d\tau
\end{aligned}$$

and

$$\|\mathcal{G}(\phi_n)\| \leq \int_0^1 \left\|\mathcal{G}'(\phi_{n-1} + \tau(\phi_n - \phi_{n-1})) - \mathcal{J}'(\phi_{n-1})\right\| d\tau \|\phi_n - \phi_{n-1}\|$$

$$\leq \int_0^1 \left(|\lambda| P \|\mathcal{H}'(\phi_{n-1}+\tau(\phi_n-\phi_{n-1}))-\mathcal{H}'(\phi_{n-1})\| \right.$$
$$\left. +|\lambda| Q \|\mathcal{H}'(\phi_{n-1}+\tau(\phi_n-\phi_{n-1}))\|\right) d\tau \|\phi_n-\phi_{n-1}\|$$
$$\leq \frac{M}{2}\|\phi_n-\phi_{n-1}\|^2+\delta(\|\phi_{n-1}-\widetilde{\phi}\|+\|\phi_n-\widetilde{\phi}\|)\|\phi_n-\phi_{n-1}\|,$$

since $\|\mathcal{H}'(\phi_{n-1}+\tau(\phi_n-\phi_{n-1}))\| \leq \|\mathcal{H}'(\widetilde{\phi})\|+L((1-\tau)\|\phi_{n-1}-\widetilde{\phi}\|+\tau\|\phi_n-\widetilde{\phi}\|)$, $\tau \in [0,1]$.

Item (b). We have

$$\begin{aligned}\|\phi_{n+1}-\phi_n\| &\leq \|[\mathcal{J}'(\phi_n)]^{-1}\| \ \|\mathcal{G}(\phi_n)\| \\ &\leq \beta h(M\beta\|\phi_n-\widetilde{\phi}\|)\left(\frac{M}{2}\|\phi_n-\phi_{n-1}\|^2+\delta(\|\phi_{n-1}-\widetilde{\phi}\|+\|\phi_n-\widetilde{\phi}\|)\|\phi_n-\phi_{n-1}\|\right) \\ &= \Delta_n\|\phi_n-\phi_{n-1}\|.\end{aligned}$$

Item (c). Observe that

$$\begin{aligned}\|\phi_{n+1}-\widetilde{\phi}\| &= \|\phi_n-[\mathcal{J}'(\phi_n)]^{-1}\mathcal{G}(\phi_n)-\widetilde{\phi}\| \\ &= \left\|[\mathcal{J}'(\phi_n)]^{-1}\left(-\mathcal{G}(\phi_n)-\mathcal{J}'(\phi_n)(\widetilde{\phi}-\phi_n)\right)\right\| \\ &\leq \|[\mathcal{J}'(\phi_n)]^{-1}\mathcal{J}'(\widetilde{\phi})\| \left\|[\mathcal{J}'(\widetilde{\phi})]^{-1}\left(-\mathcal{G}(\phi_n)-\mathcal{J}'(\phi_n)(\widetilde{\phi}-\phi_n)\right)\right\|.\end{aligned}$$

As

$$\begin{aligned}-\mathcal{G}(\phi_n)-\mathcal{J}'(\phi_n)(\widetilde{\phi}-\phi_n) &= \mathcal{J}(\phi_n)-\mathcal{G}(\phi_n)-\mathcal{J}(\phi_n)-\mathcal{J}'(\phi_n)(\widetilde{\phi}-\phi_n) \\ &= \mathcal{J}(\phi_n)-\mathcal{G}(\phi_n)+\mathcal{J}(\widetilde{\phi})-\mathcal{J}(\phi_n)-\mathcal{J}(\widetilde{\phi})-\mathcal{J}'(\phi_n)(\widetilde{\phi}-\phi_n) \\ &= \mathcal{J}(\phi_n)-\mathcal{G}(\phi_n)-\mathcal{J}(\widetilde{\phi})+\int_{\phi_n}^{\widetilde{\phi}}(\mathcal{J}'(z)-\mathcal{J}'(\phi_n))\,dz,\end{aligned}$$

we have from item (b) of Lemma 1 that

$$\begin{aligned}\|\phi_{n+1}-\widetilde{\phi}\| &= \|[\mathcal{J}'(\phi_n)]^{-1}\mathcal{J}'(\widetilde{\phi})\| \left\|[\mathcal{J}'(\widetilde{\phi})]^{-1}(\mathcal{J}(\phi_n)-\mathcal{G}(\phi_n))-[\mathcal{J}'(\widetilde{\phi})]^{-1}\mathcal{J}(\widetilde{\phi})\right. \\ &\quad \left. +\ [\mathcal{J}'(\widetilde{\phi})]^{-1}\int_0^1(\mathcal{J}'(\phi_n+\tau(\widetilde{\phi}-\phi_n))-\mathcal{J}'(\phi_n))\,d\tau(\widetilde{\phi}-\phi_n)\right\| \\ &\leq h(M\beta\|\phi_n-\widetilde{\phi}\|)\left(\beta\mu(\|\phi_n-\widetilde{\phi}\|)+\eta+\frac{1}{2}M\beta\|\phi_n-\widetilde{\phi}\|^2\right).\end{aligned}$$

The proof is complete. □

Then, we are ready to prove the convergence of the iterative method (4) and draw conclusions about the existence of solution of the integral equation (1).

Theorem 4. *Suppose that the conditions (A1)-(A2)-(A3) are satisfied and consider $R>0$ such that $R<\frac{1}{M\beta}$, $\Delta = \beta h(M\beta R)\left(\frac{M}{2}\|\phi_1-\phi_0\|+\delta(2R)\right)<1$ and $h(M\beta R)\left(\beta\mu(R)+\eta+\frac{1}{2}M\beta R^2\right)\leq R$. If $\overline{B(\widetilde{\phi},R)}\subset\Omega$, then the iterative method (4) converges starting at any $\phi_0\in\overline{B(\widetilde{\phi},R)}$ to a solution ϕ^* of $\mathcal{G}(\phi)=0$ in $\overline{B(\widetilde{\phi},R)}$. Moreover, $\phi_n,\phi^*\in\overline{B(\widetilde{\phi},R)}$, for $n\geq 0$.*

Proof. From item (c) of Lemma 3 and $\phi_0\in\overline{B(\widetilde{\phi},R)}$, we have

$$\|\phi_1-\widetilde{\phi}\|\leq h(M\beta R)\left(\beta\mu(R))+\eta+\frac{1}{2}M\beta R^2\right)\leq R$$

and $\phi_1\in\overline{B(\widetilde{\phi},R)}\subset\Omega$.

Now, as $R<\frac{1}{M\beta}$, it follows from Lemma 2 that there exists $[\mathcal{J}'(\phi_1)]^{-1}$, so that ϕ_2 is well-defined. In addition, from item (b) of Lemma 3, we obtain

$$\|\phi_2-\phi_1\|\leq\Delta_1\|\phi_1-\phi_0\|<\Delta\|\phi_1-\phi_0\|<\|\phi_1-\phi_0\|. \tag{5}$$

Taking next into account item (c) of Lemma 3, we have

$$\begin{aligned}\|\phi_2-\widetilde{\phi}\| &\le h(M\beta\|\phi_1-\widetilde{\phi}\|)\left(\beta\mu(\|\phi_1-\widetilde{\phi}\|)+\eta+\frac{1}{2}M\beta\|\phi_1-\widetilde{\phi}\|^2\right)\\ &\le h(M\beta R)\left(\beta\mu(R)+\eta+\frac{1}{2}M\beta R^2\right)\\ &\le R\end{aligned}$$

and $\phi_2 \in \overline{B(\widetilde{\phi}, R)} \subset \Omega$.

Moreover, by (5),

$$\begin{aligned}\Delta_2 &= \beta h(M\beta\|\phi_2-\widetilde{\phi}\|)\left(\frac{M}{2}\|\phi_2-\phi_1\|+\delta(\|\phi_1-\widetilde{\phi}\|+\|\phi_2-\widetilde{\phi}\|)\right)\\ &< \beta h(M\beta R)\left(\frac{M}{2}\|\phi_1-\phi_0\|+\delta(2R)\right)\\ &= \Delta.\end{aligned}$$

From mathematical induction on $n\in\mathbb{N}$ is now easy to prove that

$$\phi_n \in \overline{B(\widetilde{\phi}, R)} \subset \Omega \qquad \text{and} \qquad \Delta_n < \Delta.$$

Furthermore, by item (b) of Lemma 3, we obtain

$$\|\phi_{n+j}-\phi_n\| \le \sum_{i=1}^{j}\|\phi_{n+i}-\phi_{n+i-1}\| \le \sum_{i=1}^{j}\Delta^i\|\phi_n-\phi_{n-1}\| = \frac{\Delta-\Delta^{j+1}}{1-\Delta}\Delta^n\|\phi_1-\phi_0\|,$$

so that $\{\phi_n\}$ is a Cauchy sequence and therefore convergent. So, $\phi^* = \lim_{n\to\infty}\phi_n$.

Finally, from item (a) of Lemma 3, we obtain $\mathcal{G}(\phi^*)=0$ without more than taking into account the continuity of $\mathcal{G}$ and letting $n\to\infty$. □

Remark 5. Condition (A2) can be deduced from (A3). Indeed, from the Mean Value Theorem, we have

$$\mathcal{H}(\phi)-\mathcal{H}(\widetilde{\phi}) = \mathcal{H}'(\zeta)(\phi-\widetilde{\phi}), \quad \phi\in C[a,b],$$

where $\zeta=\tau\phi+(1-\tau)\widetilde{\phi}$ for some $\tau\in[0,1]$. By applying now condition (A3), we obtain

$$\begin{aligned}\|\mathcal{H}(\phi)-\mathcal{H}(\widetilde{\phi})\| &\le \|\mathcal{H}'(\zeta)\|\|\phi-\widetilde{\phi}\|\\ &\le \left(\|\mathcal{H}'(\widetilde{\phi})\|+\|\mathcal{H}'(\zeta)-\mathcal{H}'(\widetilde{\phi})\|\right)\|\phi-\widetilde{\phi}\|\\ &\le \left(\|\mathcal{H}'(\widetilde{\phi})\|+L\|\zeta-\widetilde{\phi}\|\right)\|\phi-\widetilde{\phi}\|\\ &= L_0\|\phi-\widetilde{\phi}\|.\end{aligned}$$

As a consequence, the condition (A2) is not necessary.

Next, we illustrate the previous study with an example, where Theorem 4 is applied to draw conclusions about the existence and separation of solutions.

Example 6. Consider the integral equation

$$\phi(x) = \frac{1}{2}\sin x + \frac{1}{4}\int_0^1 \cos(xt)\phi(t)^2\,dt, \quad x\in[0,1]. \tag{6}$$

Solving equation (6) is equivalent to solving the equation $\mathcal{G}(\phi)=0$, where $\mathcal{G}: C[a,b]\longrightarrow C[a,b]$ and

$$[\mathcal{G}(\phi)](x) = \phi(x) - \frac{1}{2}\sin x - \frac{1}{4}\int_0^1 \cos(xt)\phi(t)^2\,dt. \tag{7}$$

As the kernel $\mathcal{K}(x,t)=\cos(xt)$ is non-separable, we consider the Taylor series $\mathcal{N}(x,t)$ for $\mathcal{K}(x,t)$ when it is developed for the second derivative in $x=t=0$, so that

$$\begin{aligned}&\mathcal{K}(x,t)=\cos(xt)=\mathcal{N}(x,t)+\mathcal{R}(\theta,x,t),\\ &\mathcal{N}(x,t)=1-\frac{1}{2}x^2t^2 \qquad \text{and} \qquad \mathcal{R}(\theta,x,t)=\frac{1}{3}\left(4\theta^3x^3t^3\sin(\theta^2xt)-6\theta x^2t^2\cos(\theta^2xt)\right),\end{aligned}$$

with $\theta\in(0,1)$.

On the other hand, as $\mathcal{H}(\phi)(t) = \phi(t)^2$, we have

$$\|\mathcal{H}(\phi) - \mathcal{H}(\widetilde{\phi})\| \le \|\phi^2 - \widetilde{\phi}^2\| \le (2\|\widetilde{\phi}\| + R)\|\phi - \varphi\| = L_0\|\phi - \varphi\|,$$
$$\|\mathcal{H}'(\phi) - \mathcal{H}'(\varphi)\| \le 2\|\phi - \varphi\| = L\|\phi - \varphi\|.$$

From the last, we obtain $P = 1$, $Q = 0.3924\ldots$ and $M = \frac{1}{2}$.

Now, if we choose $\widetilde{\phi}(x) = \frac{1}{2}\sin x$, then $\beta = 1.2664\ldots$ and $\|\mathcal{H}'(\widetilde{\phi})\| \le 0.8414\ldots$ and, if $\phi_0(x) = \frac{1}{2}\sin x$, then $\|[\mathcal{J}'(\phi_0)]^{-1}\| \le 1.2664\ldots$, $\|\mathcal{G}(\phi_0)\| \le 0.0442\ldots$ and $\|\phi_1 - \phi_0\| \le 0.0560\ldots$ Therefore, the three main conditions of Theorem 4 are $R < \frac{1}{M\beta} = 1.5792\ldots$, $\Delta < 1$, which is satisfied if $R < 0.9954\ldots$, and $h(M\beta R)\left(\beta\mu(R) + \eta + \frac{1}{2}M\beta R^2\right) \le R$, which is satisfied provided that $R \in [0.0988\ldots, 0.7347\ldots]$. Since $R = 0.0988\ldots$ is the smallest value that R can take and $R = 0.7347\ldots$ the largest one, we can consider $\overline{B(\widetilde{\phi}, 0.0988\ldots)}$ as the best ball of location of solution of the integral equation (6) and $B(\widetilde{\phi}, 0.7347\ldots)$ as the best ball of separation of solution from other possible solutions. We also observe that global convergence is obtained in the ball $\overline{B(\widetilde{\phi}, R)}$ with $R \in [0.0988\ldots, 0.7347\ldots]$.

3. Uniqueness of solution

In this section, we present a result about the uniqueness of the solution ϕ^* of the equation $\mathcal{G}(\phi) = 0$.

Theorem 7. *Under the conditions of Theorem 4, the solution ϕ^* of the equation $\mathcal{G}(\phi) = 0$ is unique in $B(\widetilde{\phi}, r) \cap \Omega$, where $r = \frac{2}{L}\left(\frac{1}{|\lambda| S} - \|\mathcal{H}'(\widetilde{\phi})\|\right) - R$, where $S = max_{x\in[a,b]} \int_a^b |\mathcal{K}(x,t)|\,dt$.*

Proof. To prove the uniqueness of the solution ϕ^*, we suppose that φ^* is another solution of the equation $\mathcal{G}(\phi) = 0$ in $B(\widetilde{\phi}, r) \cap \Omega$. So, from

$$0 = \mathcal{G}(\varphi^*) - \mathcal{G}(\phi^*) = \left(\int_0^1 \mathcal{G}'(\phi^* + \tau(\varphi^* - \phi^*))\,d\tau\right)(\varphi^* - \phi^*) = \Phi(\varphi^* - \phi^*),$$

we have to prove that the operator $\Phi = \int_0^1 \mathcal{G}'(\phi^* + \tau(\varphi^* - \phi^*))\,d\tau$ is invertible and, as a consequence, we have $\varphi^* = \phi^*$. For this, we only have to prove, by the Banach lemma on invertible operators, that $\|I - \Phi\| < 1$. So, from

$$\begin{aligned}[(I - \Phi)\psi](x) &= \left(\left(\int_0^1 (I - \mathcal{G}'(\phi^* + \tau(\varphi^* - \phi^*)))d\tau\right)\psi\right)(x)\\ &= \lambda \int_0^1 \left(\int_a^b \mathcal{N}(x,t)[\mathcal{H}'(\phi^* + \tau(\varphi^* - \phi^*))\psi](t)\,dt\right)d\tau,\end{aligned}$$

it follows

$$\begin{aligned}\|I - \Phi\| &\le |\lambda| S \int_0^1 \|\mathcal{H}'(\phi^* + \tau(\varphi^* - \phi^*))\|\,d\tau\\ &< |\lambda| S \int_0^1 \left(\|\mathcal{H}'(\widetilde{\phi})\| + L(R + \tau(r - R))\right)d\tau\\ &= |\lambda| S\left(\|\mathcal{H}'(\widetilde{\phi})\| + \frac{L}{2}(r + R)\right)\\ &= 1.\end{aligned}$$

The proof is complete. □

In addition, it follows that the uniqueness of solution is given in $\overline{B(\widetilde{\phi}, R)}$ provided that $|\lambda| S\left(\|\mathcal{H}'(\widetilde{\phi})\| + LR\right) < 1$.

After that, we apply the previous theorem to the integral equation (6) of the last example to draw conclusions about the separation of solution.

Example 8. Consider again the integral equation (6). Then, from Theorem 7, we obtain that the ball of uniqueness is $B(\widetilde{\phi}, 3.0596\ldots)$, so that this new domain of separation of solution greatly improves that obtained from Theorem 4.

4. Practical remarks

First, we notice that, from item (a) of Lemma 3, it is clear that the convergence of the iterative method (4) approximates quadratic convergence when the number of terms m of the separable approximation $\mathcal{N}(x,t)$ of the kernel $\mathcal{K}(x,t)$ tends to infinity, since Q tends to zero. Obviously, if the kernel $\mathcal{K}(x,t)$ is separable, then $\mathcal{R}(\theta,x,t)=0$, the convergence is quadratic and (4) is Newton's method.

Second, we study the quality of the inverse approximation. Let $\phi \in \overline{B(\widetilde{\phi}, R)}$ be such that there exists $[\mathcal{J}'(\phi)]^{-1}$ with $\|[\mathcal{J}'(\phi)]^{-1}\| \le \beta_*$. Then,

$$\begin{aligned}\|I-[\mathcal{J}'(\phi)]^{-1}\mathcal{G}'(\phi)\| &\le \|[\mathcal{J}'(\phi)]^{-1}\|\|\mathcal{J}'(\phi)\|-\mathcal{G}'(\phi)\| \\ &\le \|[\mathcal{J}'(\phi)]^{-1}\||\lambda|\left\|\int_a^b \left(\mathcal{N}(x,t)-\mathcal{K}(x,t)\right) H'(\phi(t))\,dt\right\| \\ &\le |\lambda|\beta_* Q\left(\|H'(\widetilde{\phi})\|+LR\right).\end{aligned}$$

Observe that

$$|\lambda|\beta_* Q\left(\|H'(\widetilde{\phi})\|+LR\right)<1,$$

since, according to the value of m, the value of Q is as small as we want. Then, by the Banach lemma on invertible operators, there exists $[\mathcal{G}'(\phi)]^{-1}$ and

$$\|[\mathcal{G}'(\phi)]^{-1}\| \le \frac{\beta_*}{1-|\lambda|\beta_* Q\left(\|H'(\widetilde{\phi})\|+LR\right)}.$$

So,

$$\|[\mathcal{J}'(\phi)]^{-1}-[\mathcal{G}'(\phi)]^{-1}\| \le \|[\mathcal{G}'(\phi)]^{-1}\|\|[\mathcal{J}'(\phi)]^{-1}\mathcal{G}'(\phi)-I\| \le \frac{|\lambda|\beta_*^2 Q\left(\|H'(\widetilde{\phi})\|+LR\right)}{1-|\lambda|\beta_* Q\left(\|H'(\widetilde{\phi})\|+LR\right)}.$$

Thus, given $\epsilon>0$, we have $\|[\mathcal{J}'(\phi)]^{-1}-[\mathcal{G}'(\phi)]^{-1}\|<\epsilon$ if

$$Q<\frac{\epsilon}{(\beta_*+\epsilon)|\lambda|\beta_*\left(\|H'(\widetilde{\phi})\|+LR\right)}. \tag{8}$$

Now, as $\lim_{m\to\infty}\mathcal{R}(\theta,x,t)=0$, then there exists $m\in\mathbb{N}$ such that (8) holds. Therefore, $[\mathcal{J}'(\phi)]^{-1}$ and $[\mathcal{G}'(\phi)]^{-1}$ are as close as we want and, for this, simply set the value of m.

Third, we study the initial conditions on the auxiliary function $\widetilde{\phi}$. In the study developed throughout this work, the existence of the operator $[\mathcal{J}'(\widetilde{\phi})]^{-1}$ has been required. We see below if we can impose any condition for the existence of such operator.

Observe that

$$\begin{aligned}[\mathcal{J}'(\widetilde{\phi})\psi](x) &= \psi(x)-\lambda\int_a^b \mathcal{N}(x,t)[\mathcal{H}'(\widetilde{\phi})\psi](t)\,dt \\ &= \psi(x)-\lambda\int_a^b \mathcal{N}(x,t)H'(\widetilde{\phi}(t))\psi(t)\,dt.\end{aligned}$$

If $[\mathcal{J}'(\widetilde{\phi})\psi](x)=w(x)$, then $\psi(x)=\left([\mathcal{J}'(\widetilde{\phi})]^{-1}w\right)(x)$ with

$$\begin{aligned}\psi(x) &= w(x)+\lambda\int_a^b \mathcal{N}(x,t)H'(\widetilde{\phi}(t))\psi(t)\,dt \\ &= w(x)+\lambda\sum_{i=1}^m c_i(x)\int_a^b d_i(t)H'(\widetilde{\phi}(t))\psi(t)\,dt.\end{aligned}$$

Now, if we denote $\int_a^b d_i(t)H'(\widetilde{\phi}(t))\psi(t)\,dt=\mathcal{I}_i$, for $i=1,2,\dots,m$, then $\mathcal{I}_i\in\mathbb{R}$ and

$$\int_a^b d_j(x)H'(\widetilde{\phi}(x))\psi(x)\,dx=\int_a^b d_j(x)H'(\widetilde{\phi}(x))w(x)\,dx+\lambda\sum_{i=1}^m\int_a^b c_i(x)d_j(x)H'(\widetilde{\phi}(x))\,dx\,\mathcal{I}_j.$$

Thus, $\mathcal{I}_j = \gamma_j + \lambda \sum_{i=1}^{m} \delta_{ij}\, \mathcal{I}_i$, for $j = 1, 2, \dots, m$, where

$$\gamma_j = \int_a^b d_j(x) H'(\widetilde{\phi}(x)) w(x)\, dx \qquad \text{and} \qquad \delta_{ij} = \int_a^b c_i(x) d_j(x) H'(\widetilde{\phi}(x))\, dx,$$

so that we have the linear system

$$\begin{pmatrix} 1-\lambda\delta_{11} & -\lambda\delta_{12} & -\lambda\delta_{13} & \cdots & -\lambda\delta_{1m} \\ -\lambda\delta_{21} & 1-\lambda\delta_{22} & -\lambda\delta_{23} & \cdots & -\lambda\delta_{2m} \\ -\lambda\delta_{31} & \lambda\delta_{32} & 1-\lambda\delta_{33} & \cdots & -\lambda\delta_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\lambda\delta_{m1} & \lambda\delta_{m2} & -\lambda\delta_{m3} & \cdots & 1-\lambda\delta_{mm} \end{pmatrix} \begin{pmatrix} \mathcal{I}_1 \\ \mathcal{I}_2 \\ \mathcal{I}_3 \\ \vdots \\ \mathcal{I}_m \end{pmatrix} = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \vdots \\ \gamma_m \end{pmatrix}.$$

This system has a unique solution if

$$(-\lambda^m) \begin{vmatrix} \delta_{11}-\frac{1}{\lambda} & \delta_{12} & \delta_{13} & \cdots & \delta_{1m} \\ \delta_{21} & \delta_{22}-\frac{1}{\lambda} & \delta_{23} & \cdots & \delta_{2m} \\ \delta_{31} & \delta_{32} & \delta_{33}-\frac{1}{\lambda} & \cdots & \delta_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \delta_{m1} & \delta_{m2} & \delta_{m3} & \cdots & \delta_{mm}-\frac{1}{\lambda} \end{vmatrix} \neq 0,$$

or equivalently, if $\frac{1}{\lambda}$ is not an eigenvalue of the matrix $\left(\delta_{ij}\right)_{i,j=1}^{m}$, and then there exists $[\mathcal{J}'(\widetilde{\phi})]^{-1}$. Moreover, if $\left\{\mathcal{I}_i\right\}_{i=1}^{m}$ is the solution of the last linear system, we obtain

$$[\mathcal{J}'(\widetilde{\phi})]^{-1}\mathcal{G}(\widetilde{\phi})(x) = \mathcal{G}(\widetilde{\phi})(x) + \lambda \sum_{i=1}^{m} c_i(x)\, \mathcal{I}_i.$$

Notice that, if $\widetilde{\phi}$ is a solution of the equation $\mathcal{G}(\phi) = 0$, then $\gamma_i = 0$, for $i = 0, 1, \dots, m$, and the last linear system is homogeneous, so that the solution $\left\{\mathcal{I}_i\right\}_{i=1}^{m}$ is the zero solution.

From the above-mentioned, the application of the Newton-type iterative method (4) is given by the following algorithm:

$$\begin{cases} \phi_0 \ \text{ given in } \mathcal{C}[a,b], \\ \phi_{n+1} = \phi_n - [\mathcal{J}'(\phi_n)]^{-1}\mathcal{G}(\phi_n)(x) = \phi_n - \mathcal{G}(\phi_n)(x) - \lambda \displaystyle\sum_{i=1}^{m} c_i(x)\, \mathcal{I}_i. \end{cases}$$

We finish the work with a practical application of what was seen above.

Example 9. Consider the integral equation

$$\phi(x) = \frac{1}{7}\left(1 + (7 - \mathrm{e}^2)\mathrm{e}^x\right) + \frac{1}{7}\int_0^1 (x+2)\,\mathrm{e}^{xt}\phi(t)^2\, dt, \quad x \in [0,1]. \tag{9}$$

It is easy to check that $\phi^*(x) = \mathrm{e}^x$ is a solution of (9).

Observe that the kernel is $\mathcal{K}(x,t) = (x+2)\,\mathrm{e}^{xt}$. To be able to apply method (4) with guarantees, we approximate $\mathcal{K}(x,t)$ by a separable kernel. Thus, we use Taylor's series to approximate e^{xt},

$$\mathrm{e}^{xt} = \sum_{i=0}^{m-1} \frac{x^i\, t^i}{i!} + \frac{\mathrm{e}^{x\theta}}{m!} x^m\, t^m, \qquad \theta \in (0,1).$$

In this case,

$$\mathcal{K}(x,t) = \mathcal{N}(x,t) + \mathcal{R}(\theta, x, t),$$

where

$$\mathcal{N}(x,t) = \sum_{i=0}^{m-1}\left(\frac{x^{i+1}\, t^i}{i!} + 2\frac{x^i\, t^i}{i!}\right) \qquad \text{and} \qquad \mathcal{R}(\theta,x,t) = (x+2)\frac{\mathrm{e}^{x\theta}}{m!} x^m\, t^m.$$

Table 1
Absolute errors $\|\phi^*(x) - \phi_n(x)\|$ from Example 9.

n	$m = 2$	$m = 4$	$m = 9$	$m = 19$
0	$7.1828\ldots \times 10^{-1}$	$7.1828\ldots \times 10^{-1}$	$7.1828\ldots \times 10^{-1}$	$7.1828\ldots \times 10^{-1}$
1	$1.4860\ldots \times 10^{-1}$	$1.2965\ldots \times 10^{-2}$	$1.3032\ldots \times 10^{-1}$	$1.3032\ldots \times 10^{-1}$
2	$4.7300\ldots \times 10^{-2}$	$5.1100\ldots \times 10^{-3}$	$6.2108\ldots \times 10^{-3}$	$6.2108\ldots \times 10^{-3}$
3	$4.4413\ldots \times 10^{-2}$	$1.1487\ldots \times 10^{-3}$	$1.3884\ldots \times 10^{-5}$	$1.3926\ldots \times 10^{-5}$
4	$4.4404\ldots \times 10^{-2}$	$1.1440\ldots \times 10^{-3}$	$4.1981\ldots \times 10^{-8}$	$7.1284\ldots \times 10^{-11}$

If we choose $\phi_0(x) = 1 + x$, then we can conclude that the approximation of the solution obtained by method (4) is better the greater the number of addends that are chosen in $\mathcal{N}(x,t)$. Table 1 shows the absolute errors obtained for different values of m that corroborate what is said above.

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