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Full Length Article

Discrete harmonic analysis associated with Jacobi expansions III: The Littlewood–Paley–Stein g_k-functions and the Laplace type multipliers ☆

Alberto Arenas, Óscar Ciauri*, Edgar Labarga

Departamento de Matemáticas y Computación, Universidad de La Rioja, Complejo Científico-Tecnológico, Calle Madre de Dios 53, 26006 Logroño, Spain

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Abstract

The research about harmonic analysis associated with Jacobi expansions carried out in Arenas et al. (2020) and Arenas et al. (2022) is continued in this paper. Given the operator $\mathcal{J}^{(\alpha,\beta)} = J^{(\alpha,\beta)} - I$, where $J^{(\alpha,\beta)}$ is the three-term recurrence relation for the normalized Jacobi polynomials and I is the identity operator, we define the corresponding Littlewood–Paley–Stein $g_k^{(\alpha,\beta)}$-functions associated with it and we prove an equivalence of norms with weights for them. As a consequence, we deduce a result for Laplace type multipliers.

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* Corresponding author.
E-mail addresses: alberto.arenas@unirioja.es (A. Arenas), oscar.ciaurri@unirioja.es (Ó. Ciaurri), edgar.labarga@unirioja.es (E. Labarga).

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1. Introduction

We begin by setting some aspects of our context as in the previous papers [2,3]. For $\alpha, \beta > -1$, we take the sequences $\{a_n^{(\alpha,\beta)}\}_{n\in\mathbb{N}}$ and $\{b_n^{(\alpha,\beta)}\}_{n\in\mathbb{N}}$ given by

$$a_n^{(\alpha,\beta)} = \frac{2}{2n+\alpha+\beta+2}\sqrt{\frac{(n+1)(n+\alpha+1)(n+\beta+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+3)}}, \quad n \geq 1,$$

$$a_0^{(\alpha,\beta)} = \frac{2}{\alpha+\beta+2}\sqrt{\frac{(\alpha+1)(\beta+1)}{\alpha+\beta+3}},$$

$$b_n^{(\alpha,\beta)} = \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}, \quad n \geq 1,$$

and

$$b_0^{(\alpha,\beta)} = \frac{\beta-\alpha}{\alpha+\beta+2}.$$

For each sequence $\{f(n)\}_{n\geq 0}$, we define the operator $\{J^{(\alpha,\beta)}f(n)\}_{n\geq 0}$ by the relation

$$J^{(\alpha,\beta)}f(n) = a_{n-1}^{(\alpha,\beta)}f(n-1) + b_n^{(\alpha,\beta)}f(n) + a_n^{(\alpha,\beta)}f(n+1), \qquad n \geq 1,$$

and $J^{(\alpha,\beta)}f(0) = b_0^{(\alpha,\beta)}f(0) + a_0^{(\alpha,\beta)}f(1)$.

Defining the Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}_{n\geq 0}$ through Rodrigues' formula (see [19, p. 67, eq. (4.3.1)])

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n\, n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left\{(1-x)^{\alpha+n}(1+x)^{\beta+n}\right\},$$

it is well known that they are orthogonal on the interval $[-1,1]$ with respect to the measure

$$d\mu_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta\, dx.$$

Moreover, the sequence $\{p_n^{(\alpha,\beta)}(x)\}_{n\geq 0}$, given by $p_n^{(\alpha,\beta)}(x) = w_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(x)$ where

$$w_n^{(\alpha,\beta)} = \frac{1}{\|P_n^{(\alpha,\beta)}\|_{L^2([-1,1],d\mu_{\alpha,\beta})}} = \sqrt{\frac{(2n+\alpha+\beta+1)\,n!\,\Gamma(n+\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}}, \quad n \geq 1,$$

and

$$w_0^{(\alpha,\beta)} = \frac{1}{\|P_0^{(\alpha,\beta)}\|_{L^2([-1,1],d\mu_{\alpha,\beta})}} = \sqrt{\frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\,\Gamma(\beta+1)}},$$

is an orthonormal and complete system in $L^2([-1,1], d\mu_{\alpha,\beta})$, and it satisfies that

$$J^{(\alpha,\beta)}p_n^{(\alpha,\beta)}(x) = xp_n^{(\alpha,\beta)}(x), \qquad x \in [-1,1].$$

Along this paper we will work with the operator

$$\mathcal{J}^{(\alpha,\beta)}f(n) = (J^{(\alpha,\beta)} - I)f(n),$$

where I denotes the identity operator, instead of $J^{(\alpha,\beta)}$ since the translated operator $-\mathcal{J}^{(\alpha,\beta)}$ is non-negative. In fact, the spectrum of $J^{(\alpha,\beta)}$ is the interval $[-1,1]$, so that the spectrum of $-\mathcal{J}^{(\alpha,\beta)}$ is $[0,2]$.

This paper continues in a natural way the study of harmonic analysis associated with $\mathcal{J}^{(\alpha,\beta)}$ of [2] and [3]. In [2] we carried out an exhaustive analysis of the heat semigroup for $\mathcal{J}^{(\alpha,\beta)}$ and in [3] we investigated the Riesz transform. The main aim of this paper is to study another classical operator in harmonic analysis, the Littlewood–Paley–Stein g_k-function.

For an appropriate sequence $\{f(n)\}_{n\in\mathbb{N}}$ and $t>0$, the heat semigroup associated with $\mathcal{J}^{(\alpha,\beta)}$ is defined by the identity

$$W_t^{(\alpha,\beta)} f(n) = \sum_{m=0}^{\infty} f(m) K_t^{(\alpha,\beta)}(m,n),$$

where

$$K_t^{(\alpha,\beta)}(m,n) = \int_{-1}^{1} e^{-t(1-x)} p_m^{(\alpha,\beta)}(x) p_n^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x).$$

Then, the Littlewood–Paley–Stein $g_k^{(\alpha,\beta)}$-functions in this context are given by

$$g_k^{(\alpha,\beta)}(f)(n) = \left(\int_0^{\infty} t^{2k-1} \left| \frac{\partial^k}{\partial t^k} W_t^{(\alpha,\beta)} f(n) \right|^2 dt \right)^{1/2}, \qquad k \geq 1. \tag{1}$$

The history of g-functions goes back to the seminal paper by J. E. Littlewood and R. E. A. C. Paley [12], published in 1937, where they introduced the g-function for the trigonometric Fourier series. The extension to the Fourier transform on $\mathbb{R}^n$ was given by E. M. Stein in [16] more than twenty years later. He himself treated the question in a very abstract setting in [17]. In the last few years, there has been a deep research of these operators in different contexts and considering weights. For example, for the Hankel transform they were studied in [6], for Jacobi expansions in [13], for Laguerre expansions in [14], for Hermite expansions in [18], and for Fourier–Bessel expansions in [8].

Our work on discrete harmonic analysis related to Jacobi polynomials pretends to be a generalization of the work in [7] for the discrete Laplacian

$$\Delta_d f(n) = f(n-1) - 2f(n) + f(n+1), \qquad n \in \mathbb{Z}, \tag{2}$$

and in [5] in the case of ultraspherical expansions, which corresponds to the case $\alpha = \beta = \lambda - 1/2$ of $J^{(\alpha,\beta)}$. In both cases the corresponding g_k-functions were analysed (in [7] only for $k=1$).

To present our main result we need to introduce some notation. A weight on $\mathbb{N}$ will be a strictly positive sequence $w = \{w(n)\}_{n\geq 0}$. We consider the weighted ℓ^p-spaces

$$\ell^p(\mathbb{N}, w) = \left\{ f = \{f(n)\}_{n\geq 0} : \|f\|_{\ell^p(\mathbb{N},w)} := \left(\sum_{m=0}^{\infty} |f(m)|^p w(m) \right)^{1/p} < \infty \right\},$$

$1 \leq p < \infty$, and we simply write $\ell^p(\mathbb{N})$ when $w(n) = 1$ for all $n \in \mathbb{N}$.

Furthermore, we say that a weight w belongs to the discrete Muckenhoupt $A_p(\mathbb{N})$ class, $1 < p < \infty$, provided that

$$\sup_{\substack{0\leq n\leq m \\ n,m\in\mathbb{N}}} \frac{1}{(m-n+1)^p} \left(\sum_{k=n}^{m} w(k) \right) \left(\sum_{k=n}^{m} w(k)^{-1/(p-1)} \right)^{p-1} < \infty,$$

holds.

The main result of this paper is the following one.

Theorem 1.1. *Let $\alpha, \beta \geq -1/2$, $1 < p < \infty$, $k \in \mathbb{N}$, $k \geq 1$, and $w \in A_p(\mathbb{N})$. Then,*

$$C_1\|f\|_{\ell^p(\mathbb{N},w)} \leq \|g_k^{(\alpha,\beta)}(f)\|_{\ell^p(\mathbb{N},w)} \leq C_2\|f\|_{\ell^p(\mathbb{N},w)}, \qquad f \in \ell^2(\mathbb{N}) \cap \ell^p(\mathbb{N}, w), \tag{3}$$

where C_1 and C_2 are constants independent of f.

To prove this theorem we will start by showing that the second inequality in (3) implies the first one. After two appropriate reductions, the former will be deduced from the case $(\alpha, \beta) = (-1/2, -1/2)$ and $k = 1$ that we will obtain from discrete Calderón–Zygmund theory.

It is very common to define g_k-functions in terms of the Poisson semigroup instead of the heat semigroup. In our case the Poisson semigroup can be defined by subordination through the identity

$$P_t^{(\alpha,\beta)} f(n) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} W_{t^2/(4u)}^{(\alpha,\beta)} f(n)\, du, \quad t \geq 0, \tag{4}$$

and then we have the $\mathfrak{g}_k^{(\alpha,\beta)}$-function

$$\mathfrak{g}_k^{(\alpha,\beta)}(f)(n) = \left(\int_0^\infty t^{2k-1} \left| \frac{\partial^k}{\partial t^k} P_t^{(\alpha,\beta)} f(n) \right|^2 dt \right)^{1/2}, \qquad k \geq 1.$$

The following result will be a consequence of Theorem 1.1.

Corollary 1.2. *Let $\alpha, \beta \geq -1/2$, $1 < p < \infty$, $k \in \mathbb{N}$, $k \geq 1$, and $w \in A_p(\mathbb{N})$. Then,*

$$C_1\|f\|_{\ell^p(\mathbb{N},w)} \leq \|\mathfrak{g}_k^{(\alpha,\beta)}(f)\|_{\ell^p(\mathbb{N},w)} \leq C_2\|f\|_{\ell^p(\mathbb{N},w)}, \qquad f \in \ell^2(\mathbb{N}) \cap \ell^p(\mathbb{N}, w),$$

where C_1 and C_2 are constants independent of f.

We will prove this corollary by controlling the $\mathfrak{g}_k^{(\alpha,\beta)}$-function by a finite sum of $g_k^{(\alpha,\beta)}$-functions (see Lemma 4.2). This fact will follow from the subordination identity (4).

As an application of Theorem 1.1, we will prove the boundedness of some multipliers of Laplace type for the discrete Fourier–Jacobi series. As it is well known, for each function $F \in L^2([-1, 1], d\mu_{\alpha,\beta})$ its Fourier–Jacobi coefficients are given by

$$c_m^{(\alpha,\beta)}(F) = \int_{-1}^1 F(x) p_m^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x)$$

and

$$F(x) = \sum_{m=0}^\infty c_m^{(\alpha,\beta)}(F) p_m^{(\alpha,\beta)}(x),$$

where the equality holds in $L^2([-1, 1], d\mu_{\alpha,\beta})$. Moreover, $\{c_m^{(\alpha,\beta)}(F)\}_{m\geq 0}$ is a sequence in $\ell^2(\mathbb{N})$. Conversely, for each sequence $f \in \ell^2(\mathbb{N})$, the function

$$F_{\alpha,\beta}(x) = \sum_{m=0}^\infty f(m) p_m^{(\alpha,\beta)}(x) \tag{5}$$

belongs to $L^2([-1, 1], d\mu_{\alpha,\beta})$ and Parseval's identity

$$\|f\|_{\ell^2(\mathbb{N})} = \|F_{\alpha,\beta}\|_{L^2([-1,1],d\mu_{\alpha,\beta})} \tag{6}$$

holds. Moreover, $c_m^{(\alpha,\beta)}(F_{\alpha,\beta}) = f(m)$. An obvious consequence of (6) is the relation

$$\sum_{m=0}^{\infty} f(m)\overline{g}(m) = \int_{-1}^{1} F_{\alpha,\beta}(x)\overline{G_{\alpha,\beta}}(x)\,d\mu_{\alpha,\beta}(x), \qquad f, g \in \ell^2(\mathbb{N}), \tag{7}$$

where $F_{\alpha,\beta}$ is given by (5) and $G_{\alpha,\beta}$ is defined in a similar way.

Given a bounded function M defined on $[0,2]$, the multiplier associated with M is the operator defined, initially on $\ell^2(\mathbb{N})$, by the identity

$$T_M f(n) = c_n^{(\alpha,\beta)}(M(1-\cdot)F_{\alpha,\beta}).$$

We say that T_M is a Laplace type multiplier when

$$M(x) = x\int_0^\infty e^{-xt}a(t)\,dt, \qquad x \in [0,2],$$

with a being a bounded function.

The Laplace type multipliers were introduced by Stein in [17, Ch. 2]. There, it is observed that they verify $|x^k M^{(k)}(x)| \le C_k$ for $k = 0, 1, \dots$ and $x \in [0,2]$, and then form a subclass of Marcinkiewicz multipliers. For the operators T_M we have the following result.

Theorem 1.3. *Let $\alpha, \beta \ge -1/2$, $1 < p < \infty$, and $w \in A_p(\mathbb{N})$. Then,*

$$\|T_M f\|_{\ell^p(\mathbb{N},w)} \le C\|f\|_{\ell^p(\mathbb{N},w)}, \qquad f \in \ell^2(\mathbb{N}) \cap \ell^p(\mathbb{N}, w),$$

where C is a constant independent of f.

From the identity,

$$x^{i\gamma} = \frac{x}{\Gamma(1-i\gamma)}\int_0^\infty e^{-xt}t^{-i\gamma}\,dt, \qquad x > 0, \quad \gamma \in \mathbb{R},$$

we deduce the following corollary.

Corollary 1.4. *Let $\alpha, \beta \ge -1/2$, $1 < p < \infty$, and $w \in A_p(\mathbb{N})$. Then,*

$$\|(-\mathcal{J}^{(\alpha,\beta)})^{i\gamma} f\|_{\ell^p(\mathbb{N},w)} \le C\|f\|_{\ell^p(\mathbb{N},w)}, \qquad f \in \ell^2(\mathbb{N}) \cap \ell^p(\mathbb{N}, w),$$

where C is a constant independent of f.

The rest of the paper is organized as follows. Section 2 contains the proof of Theorem 1.1, which relies on a transplantation theorem and the Calderón–Zygmund theory. The proofs of two propositions that are necessary to apply the Calderón–Zygmund theory are provided in Section 3. Section 4 and Section 5 contain the proofs of Corollary 1.2 and Theorem 1.3, respectively.

2. Proof of Theorem 1.1

For $k \in \mathbb{N}$ and $k \ge 1$, we consider the Banach space

$$\mathbb{B}_k = \{f : (0,\infty) \to \mathbb{C} \text{ measurable} : \|f\|_{\mathbb{B}_k} < \infty\},$$

where

$$\|f\|_{\mathbb{B}_k} = \left(\int_0^\infty |f(t)|^2 t^{2k-1}\,dt\right)^{1/2};$$

i.e., $\mathbb{B}_k = L^2\left((0,\infty), t^{2k-1}\,dt\right)$. Moreover, we take the operator

$$G_{t,k}^{(\alpha,\beta)} f(n) = \sum_{m=0}^{\infty} f(m) G_{t,k}^{(\alpha,\beta)}(m,n),$$

with

$$\begin{aligned} G_{t,k}^{(\alpha,\beta)}(m,n) &= \frac{\partial^k}{\partial t^k} K_t^{(\alpha,\beta)}(m,n) \\ &= (-1)^k \int_{-1}^{1} (1-x)^k e^{-t(1-x)} p_m^{(\alpha,\beta)}(x) p_n^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x). \end{aligned}$$

Then, it is clear that

$$g_k^{(\alpha,\beta)}(f)(n) = \left\| G_{t,k}^{(\alpha,\beta)} f(n) \right\|_{\mathbb{B}_k}.$$

A first tool to prove Theorem 1.1 is the following result about the ℓ^2-boundedness of the $g_k^{(\alpha,\beta)}$-functions.

Lemma 2.1. *Let $\alpha, \beta \geq -1/2$, $k \in \mathbb{N}$, and $k \geq 1$. Then, for every $f \in \ell^2(\mathbb{N})$,*

$$\|g_k^{(\alpha,\beta)}(f)\|_{\ell^2(\mathbb{N})}^2 = \frac{\Gamma(2k)}{2^{2k}} \|f\|_{\ell^2(\mathbb{N})}^2. \tag{8}$$

Proof. For a sequence $f \in \ell^2(\mathbb{N})$, it is satisfied that

$$\begin{aligned} G_{t,k}^{(\alpha,\beta)} f(n) &= (-1)^k \int_{-1}^{1} (1-x)^k e^{-t(1-x)} F_{\alpha,\beta}(x) p_n^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x) \\ &= (-1)^k c_n^{(\alpha,\beta)}((1-\cdot)^k e^{-t(1-\cdot)} F_{\alpha,\beta}). \end{aligned}$$

Then, by using (6), we have

$$\begin{aligned} \|g_k^{(\alpha,\beta)}(f)\|_{\ell^2(\mathbb{N})}^2 &= \sum_{n=0}^{\infty} \int_0^{\infty} t^{2k-1} \left| c_n^{(\alpha,\beta)}((1-\cdot)^k e^{-t(1-\cdot)} F_{\alpha,\beta}) \right|^2 dt \\ &= \int_0^{\infty} t^{2k-1} \sum_{n=0}^{\infty} \left| c_n^{(\alpha,\beta)}((1-\cdot)^k e^{-t(1-\cdot)} F_{\alpha,\beta}) \right|^2 dt \\ &= \int_0^{\infty} t^{2k-1} \int_{-1}^{1} (1-x)^{2k} e^{-2t(1-x)} \left| F_{\alpha,\beta}(x) \right|^2 d\mu_{\alpha,\beta}(x)\, dt \\ &= \int_{-1}^{1} (1-x)^{2k} \left| F_{\alpha,\beta}(x) \right|^2 \int_0^{\infty} t^{2k-1} e^{-2t(1-x)}\, dt\, d\mu_{\alpha,\beta}(x) \\ &= \frac{\Gamma(2k)}{2^{2k}} \int_{-1}^{1} \left| F_{\alpha,\beta}(x) \right|^2 d\mu_{\alpha,\beta}(x) = \frac{\Gamma(2k)}{2^{2k}} \|f\|_{\ell^2(\mathbb{N})}^2 \end{aligned}$$

and the proof is completed. □

Now, let us see that it is enough to prove the second inequality in (3).

Lemma 2.2. *Let $\alpha, \beta \geq -1/2$, $1 < p < \infty$, $k \in \mathbb{N}$, $k \geq 1$, and $w \in A_p(\mathbb{N})$. If*

$$\|g_k^{(\alpha,\beta)}(f)\|_{\ell^p(\mathbb{N},w)} \leq C \|f\|_{\ell^p(\mathbb{N},w)}, \qquad f \in \ell^2(\mathbb{N}) \cap \ell^p(\mathbb{N}, w), \tag{9}$$

then the reverse inequality

$$\|f\|_{\ell^p(\mathbb{N},w)} \le C_1 \|g_k^{(\alpha,\beta)}(f)\|_{\ell^p(\mathbb{N},w)}, \qquad f \in \ell^2(\mathbb{N}) \cap \ell^p(\mathbb{N}, w), \tag{10}$$

holds.

Proof. First, we have to observe that (9) implies

$$\|g_k^{(\alpha,\beta)}(f)\|_{\ell^{p'}(\mathbb{N},w')} \le C\|f\|_{\ell^{p'}(\mathbb{N},w')}, \qquad f \in \ell^2(\mathbb{N}) \cap \ell^{p'}(\mathbb{N}, w'), \tag{11}$$

where $w' = w^{-1/(p-1)}$ and p' is the conjugate exponent of p; i.e., $1/p + 1/p' = 1$. Note that $w' \in A_{p'}(\mathbb{N})$ if $w \in A_p(\mathbb{N})$. Now, polarizing the identity (8), for $f, h \in \ell^2(\mathbb{N})$ we have

$$\sum_{n=0}^{\infty} f(n)\overline{h}(n) = \frac{2^{2k}}{\Gamma(2k)} \sum_{n=0}^{\infty} \int_0^{\infty} t^{2k-1} \left(\frac{\partial^k}{\partial t^k} W_t^{(\alpha,\beta)} f(n) \right) \overline{\left(\frac{\partial^k}{\partial t^k} W_t^{(\alpha,\beta)} h(n) \right)} \, dt$$

and, obviously,

$$\left| \sum_{n=0}^{\infty} f(n)\overline{h}(n) \right| \le C \sum_{n=0}^{\infty} g_k^{(\alpha,\beta)}(f)(n) g_k^{(\alpha,\beta)}(h)(n).$$

Taking $h(n) = w^{1/p}(n) f_1(n)$ with $f_1 \in c_{00}$, the space of sequences having a finite number of non-null terms, we deduce that

$$\begin{aligned}
\left| \sum_{n=0}^{\infty} f(n) w^{1/p}(n) \overline{f_1}(n) \right| &\le C \sum_{n=0}^{\infty} g_k^{(\alpha,\beta)}(f)(n) g_k^{(\alpha,\beta)}(w^{1/p} f_1)(n) \\
&= C \sum_{n=0}^{\infty} g_k^{(\alpha,\beta)}(f)(n) w^{1/p}(n) w^{-1/p}(n) g_k^{(\alpha,\beta)}(w^{1/p} f_1)(n) \\
&\le C \|g_k^{(\alpha,\beta)}(f)\|_{\ell^p(\mathbb{N},w)} \|g_k^{(\alpha,\beta)}(w^{1/p} f_1)\|_{\ell^{p'}(\mathbb{N},w')},
\end{aligned}$$

and, by (11),

$$\|g_k^{(\alpha,\beta)}(w^{1/p} f_1)\|_{\ell^{p'}(\mathbb{N},w')} \le C \|w^{1/p} f_1\|_{\ell^{p'}(\mathbb{N},w')} = \|f_1\|_{\ell^{p'}(\mathbb{N})}.$$

So, we obtain the estimate

$$\left| \sum_{n=0}^{\infty} f(n) w^{1/p}(n) \overline{f_1}(n) \right| \le C \|g_k^{(\alpha,\beta)}(f)\|_{\ell^p(\mathbb{N},w)} \|f_1\|_{\ell^{p'}(\mathbb{N})} < \infty$$

and taking the supremum over all $f_1 \in c_{00}$ such that $\|f_1\|_{\ell^{p'}(\mathbb{N})} \le 1$ the inequality (10) is proved. □

In this way, we have reduced the proof of Theorem 1.1 to prove (9). Now, we proceed with two new reductions. First, we are going to use a proper transplantation operator to deduce (9) from the case $(\alpha, \beta) = (-1/2, -1/2)$ for $k \ge 1$. Finally, we will see how to obtain (9) for $g_k^{(-1/2,-1/2)}$ with $k > 1$ from the case $k = 1$. These reductions in the proof are inspired by the work in [9].

For $f \in \ell^2(\mathbb{N})$ and $\alpha, \beta, \gamma, \delta \ge -1/2$ with $(\alpha, \beta) \ne (\gamma, \delta)$, we define the transplantation operator

$$T_{\alpha,\beta}^{\gamma,\delta} f(n) = \sum_{m=0}^{\infty} f(m) K_{\alpha,\beta}^{\gamma,\delta}(n, m)$$

where

$$K_{\alpha,\beta}^{\gamma,\delta}(n,m)=\int_{-1}^{1} p_n^{(\gamma,\delta)}(x)p_m^{(\alpha,\beta)}(x)\,d\mu_{\gamma/2+\alpha/2,\delta/2+\beta/2}(x).$$

This operator was analysed in [1], where an extension of a classical result from R. Askey [4] was given. In fact, for $\alpha,\beta,\gamma,\delta\geq -1/2$ with $(\alpha,\beta)\neq(\gamma,\delta)$ and $1<p<\infty$, it was proved that

$$\|T_{\alpha,\beta}^{\gamma,\delta}f\|_{\ell^p(\mathbb{N},w)}\leq C\|f\|_{\ell^p(\mathbb{N},w)},\qquad f\in\ell^2(\mathbb{N})\cap\ell^p(\mathbb{N},w),$$

with weights $w\in A_p(\mathbb{N})$, and the analogous weak inequality from $\ell^1(\mathbb{N},w)$ into $\ell^{1,\infty}(\mathbb{N},w)$ for weights in the $A_1(\mathbb{N})$ class. It is clear that the transplantation operator can be extended to the whole space $\ell^p(\mathbb{N},w)$ and, by a result due to Krivine (see [11, Theorem 1.f.14]), it is possible to give, in an obvious way, a vector-valued extension of it to the space $\mathbb{B}_k$, denoted by $\overline{T}_{\alpha,\beta}^{\gamma,\delta}$, satisfying

$$\|\overline{T}_{\alpha,\beta}^{\gamma,\delta}f\|_{\ell^p_{\mathbb{B}_k}(\mathbb{N},w)}\leq C\|f\|_{\ell^p_{\mathbb{B}_k}(\mathbb{N},w)},\qquad f\in\ell^p_{\mathbb{B}_k}(\mathbb{N},w),$$

for weights in $A_p(\mathbb{N})$, $\alpha,\beta,\gamma,\delta\geq -1/2$ with $(\alpha,\beta)\neq(\gamma,\delta)$, and $1<p<\infty$.

In this way, for $f\in\ell^2(\mathbb{N})$ and $\alpha,\beta\geq-1/2$ with $(\alpha,\beta)\neq(-1/2,-1/2)$, we have

$$G_{t,k}^{(\alpha,\beta)}f=\overline{T}_{-1/2,-1/2}^{\alpha,\beta}G_{t,k}^{(-1/2,-1/2)}T_{\alpha,\beta}^{-1/2,-1/2}f. \tag{12}$$

Indeed, by a continuity argument we can consider sequences f in c_{00}. Then,

$$\begin{aligned} G_{t,k}^{(-1/2,-1/2)}T_{\alpha,\beta}^{-1/2,-1/2}f(n) &= \frac{\partial^k}{\partial t^k}W_t^{(-1/2,-1/2)}T_{\alpha,\beta}^{-1/2,-1/2}f(n) \\ &= \sum_{m=0}^{\infty}f(m)\sum_{j=0}^{\infty}G_{t,k}^{(-1/2,-1/2)}(j,n)K_{\alpha,\beta}^{-1/2,-1/2}(j,m) \end{aligned}\tag{13}$$

and, by using (7) and the identities

$$G_{t,k}^{(-1/2,-1/2)}(j,n)=(-1)^k c_j^{(-1/2,-1/2)}((1-\cdot)^k e^{-t(1-\cdot)}p_n^{(-1/2,-1/2)})$$

and

$$K_{\alpha,\beta}^{-1/2,-1/2}(j,m)=c_j^{(-1/2,-1/2)}((1-\cdot)^{\alpha/2+1/4}(1+\cdot)^{\beta/2+1/4}p_m^{(\alpha,\beta)}),$$

we deduce that

$$\begin{aligned} &\sum_{j=0}^{\infty}G_{t,k}^{(-1/2,-1/2)}(j,n)K_{\alpha,\beta}^{-1/2,-1/2}(j,m) \\ &\quad=(-1)^k\int_{-1}^{1}(1-x)^k e^{-t(1-x)}p_n^{(-1/2,-1/2)}(x)p_m^{(\alpha,\beta)}(x)\,d\mu_{\alpha/2-1/4,\beta/2-1/4}(x). \end{aligned}$$

To justify the use of Fubini's theorem in (13) we can use the bounds (see, respectively, [1, Proposition 2.3] and [2, Lemma 4.4])

$$|K_{\alpha,\beta}^{-1/2,-1/2}(j,m)|\leq\frac{C}{1+|j-m|}$$

and

$$K_t^{(-1/2,1/2)}(j,n)\leq\frac{C}{1+|j-n|}.$$

The interchange of summation with derivative is a consequence of the estimate

$$\left| \int_{-1}^{1} (1-x)^k e^{-t(1-x)} p_n^{(-1/2,-1/2)}(x) p_m^{(\alpha,\beta)}(x)\, d\mu_{\alpha/2-1/4,\beta/2-1/4}(x) \right| \le C.$$

Applying a similar argument to the other composition, the proof of (12) follows.

Now, let us see that it is enough to analyse the $g_1^{(-1/2,-1/2)}$-function. In fact, using induction we can deduce the boundedness of the $g_k^{(-1/2,-1/2)}$-functions for $k > 1$. Let us suppose that the operator $G_{t,k}^{(-1/2,-1/2)}$ is bounded from $\ell^p(\mathbb{N}, w)$ into $\ell^p_{\mathbb{B}_k}(\mathbb{N}, w)$. Taking $k = 1$ and applying Krivine's theorem again, we deduce that the operator $\tilde{G}_{t,1}^{(-1/2,-1/2)} : \ell^p_{\mathbb{B}_k}(\mathbb{N}, w) \longrightarrow \ell^p_{\mathbb{B}_k \times \mathbb{B}_1}(\mathbb{N}, w)$, given by

$$\{f_s(n)\}_{s\ge 0} \longmapsto \{G_{t,1}^{(-1/2,-1/2)} f_s\}_{t,s\ge 0},$$

is bounded. Moreover, $\tilde{G}_{t,1}^{(-1/2,-1/2)} \circ G_{s,k}^{(-1/2,-1/2)}$ is a bounded operator from $\ell^p(\mathbb{N}, w)$ into $\ell^p_{\mathbb{B}_k \times \mathbb{B}_1}(\mathbb{N}, w)$. Now, from the Kolmogorov–Chapman type identity [2, Remark 3]

$$\sum_{n=0}^{\infty} K_t^{(\alpha,\beta)}(m,n) K_s^{(\alpha,\beta)}(n,k) = K_{t+s}^{(\alpha,\beta)}(m,k),$$

we can deduce the relation

$$\frac{\partial}{\partial t}\left(W_t^{(-1/2,-1/2)} \left(\frac{\partial^k}{\partial s^k} W_s^{(-1/2,-1/2)} f \right)\right) = \left. \frac{\partial^{k+1}}{\partial u^{k+1}} W_u^{(-1/2,-1/2)} f \right|_{u=s+t},$$

to obtain that

$$\begin{aligned}
&\left\| \tilde{G}_{t,1}^{(-1/2,-1/2)} \circ G_{s,k}^{(-1/2,-1/2)} f \right\|^2_{\mathbb{B}_k \times \mathbb{B}_1} \\
&= \int_0^\infty \int_0^\infty t s^{2k-1} \left| \left. \frac{\partial^{k+1}}{\partial u^{k+1}} W_u^{(-1/2,-1/2)} f \right|_{u=s+t} \right|^2 ds\, dt \\
&= \int_0^\infty \int_t^\infty t (r-t)^{2k-1} \left| \left. \frac{\partial^{k+1}}{\partial u^{k+1}} W_u^{(-1/2,-1/2)} f \right|_{u=r} \right|^2 dr\, dt \\
&= \int_0^\infty \left| \frac{\partial^{k+1}}{\partial r^{k+1}} W_r^{(-1/2,-1/2)} f \right|^2 \int_0^r t (r-t)^{2k-1}\, dt\, dr \\
&= \frac{1}{(2k+1)(2k)} \int_0^\infty r^{2k+1} \left| \frac{\partial^{k+1}}{\partial r^{k+1}} W_r^{(-1/2,-1/2)} f \right|^2 dr \\
&= \frac{\left(g_{k+1}^{(-1/2,-1/2)}(f) \right)^2}{(2k+1)(2k)}.
\end{aligned}$$

Finally, to complete the proof of Theorem 1.1 we have to prove (9) for $(\alpha, \beta) = (-1/2, -1/2)$ and $k = 1$. This fact will be a consequence of the following propositions.

Proposition 2.3. *Let $n, m \in \mathbb{N}$ with $n \neq m$. Then,*

$$\|G_{t,1}^{(-1/2,-1/2)}(m,n)\|_{\mathbb{B}_1} \le C|n-m|^{-1}. \tag{14}$$

Moreover

$$\|G_{t,1}^{(-1/2,-1/2)}(n,n)\|_{\mathbb{B}_1} \le C. \tag{15}$$

Proposition 2.4. *Let $n, m \in \mathbb{N}$ with $n \neq m$. Then,*

$$\|G_{t,1}^{(-1/2,-1/2)}(m+1,n) - G_{t,1}^{(-1/2,-1/2)}(m,n)\|_{\mathbb{B}_1} \leq C|n-m|^{-2}$$

and

$$\|G_{t,1}^{(-1/2,-1/2)}(m,n+1) - G_{t,1}^{(-1/2,-1/2)}(m,n)\|_{\mathbb{B}_1} \leq C|n-m|^{-2}.$$

The proof of these propositions is the most delicate part of the proof of Theorem 1.1, so it is postponed to the next section.

Now, using the decomposition

$$\begin{aligned} &|g_1^{(-1/2,-1/2)} f(n)| \\ &\leq \left\| \sum_{\substack{m=0 \\ m\neq n}}^{\infty} f(m) G_{t,1}^{(-1/2,-1/2)}(m,n) \right\|_{\mathbb{B}_1} + \left\| f(n) G_{t,1}^{(-1/2,-1/2)}(n,n) \right\|_{\mathbb{B}_1} \\ &:= T_1 f(n) + T_2 f(n), \end{aligned}$$

we can apply (14) of Propositions 2.3, 2.4, and Lemma 2.1 to deduce from the Calderón–Zygmund theory the inequality

$$\|T_1 f\|_{\ell^p(\mathbb{N},w)} \leq C\|f\|_{\ell^p(\mathbb{N},w)},$$

and (15) to obtain that

$$\|T_2 f\|_{\ell^p(\mathbb{N},w)} \leq C\|f\|_{\ell^p(\mathbb{N},w)},$$

finishing the proof of Theorem 1.1.

3. Proof of Propositions 2.3 and 2.4

Denoting by T_n the Chebyshev polynomials, we have

$$p_n^{(-1/2,-1/2)}(x) = \sqrt{\frac{2}{\pi}} T_n(x) = \sqrt{\frac{2}{\pi}} \cos(n\theta),$$

for $n \neq 0$ and where $x = \cos\theta$, and $p_0^{(-1/2,-1/2)}(x) = \sqrt{\frac{1}{\pi}} T_0(x) = \sqrt{\frac{1}{\pi}}$. Then, the identity (see [15, p. 456])

$$\frac{1}{\pi} \int_0^{\pi} e^{z\cos\theta} \cos(m\theta)\, d\theta = I_m(z), \qquad |\arg(z)| < \pi,$$

where I_m denotes the Bessel function of imaginary argument and order m, implies

$$K_t^{(-1/2,-1/2)}(m,n) = e^{-t}(I_{m+n}(t) + I_{n-m}(t)), \qquad n, m \neq 0, \tag{16}$$

$$K_t^{(-1/2,-1/2)}(m,0) = \sqrt{2} e^{-t} I_m(t), \qquad \text{and} \qquad K_t^{(-1/2,-1/2)}(0,n) = \sqrt{2} e^{-t} I_n(t). \tag{17}$$

To simplify notation, we set $R_t(n) = e^{-t} I_n(t)$.

The proofs of Propositions 2.3 and 2.4 follow the ideas in [7] and [5]. In particular, being

$$I_k^{\gamma,\alpha,\beta}(t) = \frac{t^{k+\gamma}}{2^{k+\gamma}} \int_{-1}^{1} e^{-t(1+s)} (1-s)^{k+\alpha} (1+s)^{k+\beta}\, ds,$$

we will apply the estimate

$$\|I_k^{\gamma,\alpha,\beta}\|_{\mathbb{B}_1} \le C\frac{\Gamma(k+1)}{(k+1)^{\beta-2\gamma}}, \tag{18}$$

with C a constant independent of k and where $k \in \mathbb{N}$ and $\alpha, \beta, \gamma \in \mathbb{R}$ verify the restrictions $k+2\gamma+1 > \beta > \gamma$, $k+\alpha+1 > 0$, and $k+\beta+1 > 0$. This result is [5, Lemma 4.1] in our context.

Proof of Proposition 2.3. The identities [10, eq. (5.7.9)]

$$2I_n'(t) = I_{n+1}(t) + I_{n-1}(t), \qquad n \ge 1,$$

and $I_0'(t) = I_1(t)$ yield

$$\frac{\partial R_t(n)}{\partial t} = \frac{1}{2}(R_t(n+1) - 2R_t(n) + R_t(n-1)), \qquad n \ge 1, \tag{19}$$

and

$$\frac{\partial R_t(0)}{\partial t} = R_t(1) - R_t(0). \tag{20}$$

The next identity is known as Schläfli's integral representation of Poisson type for modified Bessel functions (see [10, eq. (5.10.22)]):

$$I_\nu(z) = \frac{z^\nu}{\sqrt{\pi}\,2^\nu\Gamma(\nu+1/2)}\int_{-1}^{1} e^{-zs}(1-s^2)^{\nu-1/2}\,ds, \quad |\arg z| < \pi, \quad \nu > -\frac{1}{2}. \tag{21}$$

Integrating by parts once and twice in (21), we have, respectively, the identities

$$I_\nu(z) = -\frac{z^{\nu-1}}{\sqrt{\pi}\,2^{\nu-1}\Gamma(\nu-1/2)}\int_{-1}^{1} e^{-zs}s(1-s^2)^{\nu-3/2}\,ds, \quad \nu > \frac{1}{2}, \tag{22}$$

and

$$I_\nu(z) = \frac{z^{\nu-2}}{\sqrt{\pi}\,2^{\nu-2}\Gamma(\nu-3/2)}\int_{-1}^{1} e^{-zs}\frac{1+zs}{z}s(1-s^2)^{\nu-5/2}\,ds, \quad \nu > \frac{3}{2}. \tag{23}$$

Then, from (19), using (21), (22), and (23) with $\nu = n-1$, $\nu = n$, and $\nu = n+1$, respectively, we deduce that for $n \ge 1$

$$\frac{\partial R_t(n)}{\partial t} = \frac{1}{2\sqrt{\pi}\Gamma(n-1/2)}\left(I_{n-1}^{0,-1/2,3/2}(t) + \frac{1}{2}I_{n-2}^{0,1/2,3/2}(t) - \frac{1}{2}I_{n-2}^{0,1/2,1/2}(t)\right).$$

Applying (18) for $n \ge 3$ we have

$$\left\|\frac{\partial R_t(n)}{\partial t}\right\|_{\mathbb{B}_1} \le \frac{C}{\Gamma(n-1/2)}\left(\frac{\Gamma(n)}{n^{3/2}} + \frac{\Gamma(n-1)}{n^{3/2}} + \frac{\Gamma(n-1)}{n^{1/2}}\right) \le \frac{C}{n}, \tag{24}$$

where we have used that $\Gamma(n+a)/\Gamma(n+b) \sim n^{a-b}$ in the last step.

Now, we prove that

$$\left\|\frac{\partial R_t(0)}{\partial t}\right\|_{\mathbb{B}_1} + \left\|\frac{\partial R_t(1)}{\partial t}\right\|_{\mathbb{B}_1} + \left\|\frac{\partial R_t(2)}{\partial t}\right\|_{\mathbb{B}_1} \le C. \tag{25}$$

By Minkowski's integral inequality, it is clear that

$$\left\|\frac{\partial R_t(0)}{\partial t}\right\|_{\mathbb{B}_1} = \frac{1}{\pi}\left\|\frac{\partial}{\partial t}\int_0^\pi e^{-t(1-\cos\theta)}\,d\theta\right\|_{\mathbb{B}_1}$$

$$= \frac{1}{\pi}\left\|\int_0^\pi e^{-t(1-\cos\theta)}(1-\cos\theta)\,d\theta\right\|_{\mathbb{B}_1}$$
$$\leq \frac{1}{\pi}\int_0^\pi (1-\cos\theta)\left\|e^{-t(1-\cos\theta)}\right\|_{\mathbb{B}_1} d\theta \leq C.$$

Similarly, we obtain the bounds for $\frac{\partial R_t(1)}{\partial t}$ and $\frac{\partial R_t(2)}{\partial t}$ and the proof of (25) is finished.
Finally, using (16), (17), (24), (25), and the identity

$$I_{-n}(t) = I_n(t), \tag{26}$$

we conclude the proof of the proposition. □

Proof of Proposition 2.4. By using (16), (17), and (26), the proof will follow from the estimate

$$\left\|\frac{\partial}{\partial t}(R_t(n+1)-R_t(n))\right\|_{\mathbb{B}_1} \leq \frac{C}{n^2}, \qquad \text{for } n \neq 0. \tag{27}$$

Using (19), we have

$$\frac{\partial}{\partial t}(R_t(n+1)-R_t(n)) = \frac{1}{2}(R_t(n+2)-3R_t(n+1)+3R_t(n)-R_t(n-1)). \tag{28}$$

Integrating by parts three times in (21) gives

$$I_\nu(z) = -\frac{z^{\nu-3}}{\sqrt{\pi}\,2^{\nu-3}\Gamma(\nu-5/2)} \times \int_{-1}^1 e^{-zs}\,\frac{s(s^2z^2+3sz+3)}{z^2}(1-s^2)^{\nu-7/2}\,ds, \qquad \nu > \frac{5}{2}. \tag{29}$$

Then, using (21), (22), (23), and (29) with $\nu = n-1$, $\nu = n$, $\nu = n+1$, and $\nu = n+2$, respectively, for $n \geq 5$ (28) becomes

$$\frac{\partial}{\partial t}(R_t(n+1)-R_t(n)) = \frac{-1}{2\sqrt{\pi}\,\Gamma(n-1/2)} \times \left(I_{n-1}^{0,-1/2,5/2}(t) + \frac{3}{2}I_{n-2}^{0,1/2,5/2}(t) - \frac{3}{2}I_{n-2}^{0,1/2,3/2}(t) + \frac{3}{4}I_{n-3}^{0,3/2,5/2}(t) - \frac{3}{4}I_{n-3}^{0,3/2,3/2}(t)\right).$$

In this way, by using (18), we deduce (27) for $n \geq 5$. The remainder cases can be analysed as (25) in the previous proposition to obtain that

$$\left\|\frac{\partial}{\partial t}(R_t(n+1)-R_t(n))\right\|_{\mathbb{B}_1} \leq C, \qquad n = 1, 2, 3, 4,$$

and the proof is finished. □

4. Proof of Corollary 1.2

First, it is easy to check that

$$P_t^{(\alpha,\beta)} f(n) = \sum_{m=0}^\infty f(m)\mathcal{K}_t^{(\alpha,\beta)}(m,n),$$

with

$$\mathcal{K}_t^{(\alpha,\beta)}(m,n)=\int_{-1}^{1} e^{-t\sqrt{1-x}}\, p_m^{(\alpha,\beta)}(x)\, p_n^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x).$$

Then, we have the following result for the $\mathfrak{g}_k^{(\alpha,\beta)}$-function which is the analogue of Lemma 2.1.

Lemma 4.1. *Let $\alpha,\beta\ge -1/2$, $k\in\mathbb{N}$, and $k\ge 1$. Then,*

$$\|\mathfrak{g}_k^{(\alpha,\beta)}(f)\|^2_{\ell^2(\mathbb{N})}=\frac{\Gamma(2k)}{2^{2k}}\|f\|^2_{\ell^2(\mathbb{N})}. \tag{30}$$

This lemma can be proved following step by step the proof of Lemma 2.1, so we omit the details. Now, using polarization, for $f,h\in\ell^2(\mathbb{N})$ we deduce the identity

$$\sum_{n=0}^{\infty} f(n)\overline{h}(n)=\frac{2^{2k}}{\Gamma(2k)}\sum_{n=0}^{\infty}\int_0^{\infty} t^{2k-1}\left(\frac{\partial^k}{\partial t^k}P_t^{(\alpha,\beta)}f(n)\right)\overline{\left(\frac{\partial^k}{\partial t^k}P_t^{(\alpha,\beta)}h(n)\right)}\,dt.$$

From this fact, we obtain the inequality

$$\|f\|_{\ell^p(\mathbb{N},w)}\le C\|\mathfrak{g}_k^{(\alpha,\beta)}(f)\|_{\ell^p(\mathbb{N},w)}$$

from the direct inequality

$$\|\mathfrak{g}_k^{(\alpha,\beta)}(f)\|_{\ell^p(\mathbb{N},w)}\le C\|f\|_{\ell^p(\mathbb{N},w)} \tag{31}$$

as we did in the proof of Theorem 1.1. Finally, inequality (31) is an immediate consequence of the following lemma.

Lemma 4.2. *Let $\alpha,\beta>-1$. Then*

$$\mathfrak{g}_k^{(\alpha,\beta)}(f)\le\sum_{j=0}^{[k/2]}A_{k,j}\,g_{k-j}^{(\alpha,\beta)}(f),$$

where $A_{k,j}$ are some constants and $[\cdot]$ denotes the floor function.

Proof. First, we observe that

$$\frac{\partial^k}{\partial t^k}h\left(\frac{t^2}{4u}\right)=\sum_{j=0}^{[k/2]}B_{k,j}\,\frac{\partial^{k-j}}{\partial s^{k-j}}h(s)\bigg|_{s=\frac{t^2}{4u}}\frac{t^{k-2j}}{(4u)^{k-j}},$$

for some constants $B_{k,j}$. Then, from (4), we have

$$\frac{\partial^k}{\partial t^k}P_t^{(\alpha,\beta)}f(n)=\frac{1}{\sqrt{\pi}}\sum_{j=0}^{[k/2]}B_{k,j}\int_0^{\infty}\frac{e^{-u}}{\sqrt{u}}\left(\frac{\partial^{k-j}}{\partial s^{k-j}}W_s^{(\alpha,\beta)}f(n)\bigg|_{s=\frac{t^2}{4u}}\right)\frac{t^{k-2j}}{(4u)^{k-j}}\,du$$

and, by Minkowski's integral inequality,

$$\mathfrak{g}_k^{(\alpha,\beta)}(f)(n)\le\sum_{j=0}^{[k/2]}B_{k,j}P_j(n)$$

where

$$P_j(n)$$

$$= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}(4u)^{k-j}} \left(\int_0^\infty t^{4k-4j-1} \left| \frac{\partial^{k-j}}{\partial s^{k-j}} W_s^{(\alpha,\beta)} f(n) \Big|_{s=\frac{t^2}{4u}} \right|^2 dt \right)^{1/2} du.$$

Now, by using an appropriate change of variables, we have

$$\begin{aligned} P_j(n) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \left(\int_0^\infty s^{2k-2j-1} \left| \frac{\partial^{k-j}}{\partial s^{k-j}} W_s^{(\alpha,\beta)} f(n) \right|^2 ds \right)^{1/2} du \\ &= \frac{1}{\sqrt{2}} g_{k-j}^{(\alpha,\beta)}(f)(n) \end{aligned}$$

and the result follows. □

5. Proof of Theorem 1.3

We need only prove that

$$g_1^{(\alpha,\beta)}(T_M f)(n) \le C g_2^{(\alpha,\beta)}(f)(n), \tag{32}$$

since by Theorem 1.1 we get that

$$\|T_M f\|_{\ell^p(\mathbb{N},w)} \le C \|g_1^{(\alpha,\beta)}(T_M f)\|_{\ell^p(\mathbb{N},w)} \le C \|g_2^{(\alpha,\beta)}(f)\|_{\ell^p(\mathbb{N},w)} \le C \|f\|_{\ell^p(\mathbb{N},w)}.$$

Moreover, it is enough to prove (32) for sequences $f \in c_{00}$. First, we have

$$T_M f(n) = -\int_0^\infty a(s) \frac{\partial}{\partial s} W_s^{(\alpha,\beta)} f(n)\, ds,$$

which is an elementary consequence of the relation

$$\begin{aligned} &\int_{-1}^1 M(1-x) p_m^{(\alpha,\beta)}(x) p_n^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x) \\ &\quad = \int_0^\infty a(s) \int_{-1}^1 (1-x) e^{-s(1-x)} p_m^{(\alpha,\beta)}(x) p_n^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x)\, ds \\ &\quad = -\int_0^\infty a(s) \frac{\partial}{\partial s} \int_{-1}^1 e^{-s(1-x)} p_m^{(\alpha,\beta)}(x) p_n^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x)\, ds. \end{aligned}$$

Then, applying the semigroup property of $W_t^{(\alpha,\beta)}$ we obtain

$$W_t^{(\alpha,\beta)}(T_M f)(n) = -\int_0^\infty a(s) \frac{\partial}{\partial s} W_{s+t}^{(\alpha,\beta)} f(n)\, ds$$

and hence,

$$\begin{aligned} \frac{\partial}{\partial t} W_t^{(\alpha,\beta)}(T_M f)(n) &= -\int_0^\infty a(s) \frac{\partial}{\partial t} \frac{\partial}{\partial s} W_{s+t}^{(\alpha,\beta)} f(n)\, ds \\ &= -\int_0^\infty a(s) \frac{\partial^2}{\partial s^2} W_{s+t}^{(\alpha,\beta)} f(n)\, ds. \end{aligned}$$

In this way,

$$\begin{aligned} \left| \frac{\partial}{\partial t} W_t^{(\alpha,\beta)}(T_M f)(n) \right| &\le C \int_t^\infty s \left| \frac{\partial^2}{\partial s^2} W_s^{(\alpha,\beta)} f(n) \right| \frac{ds}{s} \\ &\le C t^{-1/2} \left(\int_t^\infty s^2 \left| \frac{\partial^2}{\partial s^2} W_s^{(\alpha,\beta)} f(n) \right|^2 ds \right)^{1/2}. \end{aligned}$$

Finally,

$$(g_1^{(\alpha,\beta)}(T_M f)(n))^2 = \int_0^\infty t \left| \frac{\partial}{\partial t} W_t^{(\alpha,\beta)}(T_M f)(n) \right|^2 dt$$
$$\leq C \int_0^\infty \int_t^\infty s^2 \left| \frac{\partial^2}{\partial s^2} W_s^{(\alpha,\beta)} f(n) \right|^2 ds\, dt$$
$$= C \int_0^\infty s^3 \left| \frac{\partial^2}{\partial s^2} W_s^{(\alpha,\beta)} f(n) \right|^2 ds = C(g_2^{(\alpha,\beta)} f(n))^2$$

and the proof of (32) is completed.

Data availability

No data was used for the research described in the article.

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