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Families of graphs with maximum nullity equal to zero forcing number

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Abstract: The maximum nullity of a simple graph G, denoted $\mathrm{M}(G)$, is the largest possible nullity over all symmetric real matrices whose ijth entry is nonzero exactly when $\{i, j\}$ is an edge in G for $i \neq j$, and the iith entry is any real number. The zero forcing number of a simple graph G, denoted $\mathrm{Z}(G)$, is the minimum number of blue vertices needed to force all vertices of the graph blue by applying the color change rule. This research is motivated by the longstanding question of characterizing graphs G for which $\mathrm{M}(G) = \mathrm{Z}(G)$. The following conjecture was proposed at the 2017 AIM workshop *Zero forcing and its applications*: If G is a bipartite 3-semiregular graph, then $\mathrm{M}(G) = \mathrm{Z}(G)$. A counterexample was found by J. C.-H. Lin but questions remained as to which bipartite 3-semiregular graphs have $\mathrm{M}(G) = \mathrm{Z}(G)$. We use various tools to find bipartite families of graphs with regularity properties for which the maximum nullity is equal to the zero forcing number; most are bipartite 3-semiregular. In particular, we use the techniques of twinning and vertex sums to form new families of graphs for which $\mathrm{M}(G) = \mathrm{Z}(G)$ and we additionally establish $\mathrm{M}(G) = \mathrm{Z}(G)$ for certain Generalized Petersen graphs.

Keywords: zero forcing number; maximum nullity; semiregular bipartite graph; Generalized Petersen graph

MSC: 05C50, 15A03, 15A18

1 Introduction

Let V be a finite nonempty set. A *graph* $G = (V, E)$ is a pair of sets such that E is a set of two element subsets of V. The elements of V are called *vertices* and the elements of E are called *edges*. The vertex set of a graph G is often denoted by $V(G)$ and the edge set by $E(G)$. The *order* of G is the cardinality of $V(G)$ and the *size* of G is the cardinality of $E(G)$. An edge $\{u, v\}$ is usually written as uv. Vertices u and v are *adjacent* in G if $uv \in E(G)$. A vertex u is a *neighbor* of v if $uv \in E(G)$. The *neighborhood* of v, denoted by $N(v)$, is the set of neighbors of v.

The *adjacency matrix* of G is $A(G) = [a_{ij}]$ where $a_{ij} = 1$ if $ij \in E(G)$ and $a_{ij} = 0$ otherwise. For a graph G, the *set of symmetric matrices* of G over $\mathbb{R}$, denoted by $\mathcal{S}(G)$, is the set of real symmetric matrices $A = [a_{ij}]$ such that a_{ij} is non-zero if $ij \in E(G)$, a_{ij} is any real number if $i = j$, and a_{ij} is 0 otherwise. The *minimum rank* of G is

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$\mathrm{mr}(G) = \min\{\mathrm{rank}(A) \mid A \in \mathcal{S}(G)\}$. The *maximum nullity* of G is defined as $\mathrm{M}(G) = \max\{\mathrm{null}(A) \mid A \in \mathcal{S}(G)\}$. Observe that $\mathrm{mr}(G) + \mathrm{M}(G) = |V(G)|$ where $|\cdot|$ denotes cardinality.

In order to introduce zero forcing we will define the *color change rule* as follows: Suppose a graph G has every vertex colored either blue or white, and b is a blue vertex. If b has exactly one white neighbor, w, then we change the color of w to blue. We say that b *forces* w, and this can be denoted by $b \to w$. Let $S \subseteq V(G)$. The *final coloring of* S is the result of initially coloring every vertex in S blue and every vertex in $V(G)\setminus S$ white, and then applying the color-change rule until no more color changes can be made. Note that the order in which forces occur does not affect the final coloring of G. The set S is called a *zero forcing set* if the final coloring of S is all blue. The *zero forcing number* of a graph G is $\mathrm{Z}(G) = \min\{|S| \mid S \text{ is a zero forcing set of } G\}$. It is well known from [1] that $\mathrm{M}(G) \leq \mathrm{Z}(G)$. This paper addresses the longstanding question of determining graphs G for which $\mathrm{M}(G) = \mathrm{Z}(G)$ (see [1, Question 1]).

The *degree* of v, $\deg(v)$, is the number of edges incident to v. Note that $\deg(v) = |N(v)|$. A vertex with degree equal to 1 is called a *leaf*. The *minimum degree* of G is $\delta(G) = \min\{\deg(v) \mid v \in V(G)\}$ and the *maximum degree* of G is $\Delta(G) = \max\{\deg(v) \mid v \in V(G)\}$. A graph G is *regular* if $\Delta(G) = \delta(G)$. We call G *cubic* if $\Delta(G) = 3 = \delta(G)$. A graph G is *bipartite* if $V(G)$ can be partitioned into two sets X and Y such that $N(x) \subseteq Y$ and $N(y) \subseteq X$ for $x \in X$ and $y \in Y$; the partition of vertices can be denoted by $G(X, Y)$. Let $G = G(X, Y)$ be a bipartite graph. We say G is *k-semiregular* if the degree of x is k for all $x \in X$.

A graph H is a *subgraph* of graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Let $S \subseteq V(G)$ and $E[S] = \{uv \in E(G) \mid u, v \in S\}$. The subgraph of G induced by S is $G[S] = (S, E[S])$. That is, an *induced graph* is one obtained by deleting vertices and incident edges.

A graph G is a *path* on n vertices, denoted by P_n, if $V(G) = \{v_1, v_2, \dots, v_n\}$ and $E(G) = \{v_1v_2, v_2v_3, \dots, v_{n-2}v_{n-1}, v_{n-1}v_n\}$. A graph G is a *cycle* on n vertices, denoted by C_n, if $V(G) = \{v_1, v_2, \dots, v_n\}$ and $E(G) = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_nv_1\}$. A *complete graph* on n vertices, denoted by K_n, is a graph of order n with all possible edges between its vertices. A *complete bipartite* graph is a bipartite graph with all possible edges between the two parts and is denoted by $K_{n,m}$ where n and m are the orders of the two parts.

The *union* of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $(V_1 \cup V_2, E_1 \cup E_2)$, denoted $G_1 \cup G_2$. If $V_1 \cap V_2 \neq \emptyset$, then the *intersection* of G_1 and G_2 is the graph $G_1 \cap G_2 = (V_1 \cap V_1, E_1 \cap E_2)$. Suppose that G_1 and G_2 are graphs such that $V(G_1) \cap V(G_2) = \{v\}$ (and $V(G_i) \neq \{v\}$ for $i = 1, 2$). Then $G_1 \cup G_2$ is called the *vertex sum of* G_1 *and* G_2 *at* v and is denoted $G_1 \underset{v}{\oplus} G_2$. A graph $G = (V, E)$ is *connected* if for each pair of vertices $u, v \in V$ there exists a path from u to v. A (*connected*) *component* of G is a maximal connected subgraph of G. A vertex v is a *cut vertex* of G if the number of components of G is less than the number of components of $G - v$, where $G - v = G[V \setminus \{v\}]$. Observe that v is a cut vertex of $G_1 \underset{v}{\oplus} G_2$, and if v is a cut vertex of G, then G can be expressed as a vertex sum at v.

At the 2017 American Institute of Mathematics workshop *Zero forcing and its applications*, it was conjectured that $\mathrm{M}(G) = \mathrm{Z}(G)$ if G is a bipartite 3-semiregular graph [2]. It is known that all bipartite 1-semiregular and bipartite 2-semiregular graphs satisfy $\mathrm{M}(G) = \mathrm{Z}(G)$, but not all bipartite graphs have $\mathrm{M}(G) = \mathrm{Z}(G)$. A counterexample to the conjecture was found by J.C.-H. Lin [8] (see Example 1.1), but questions remain as to which bipartite 3-semiregular graphs have $\mathrm{M}(G) = \mathrm{Z}(G)$. In this paper, we construct bipartite families of graphs with regularity properties for which the maximum nullity is equal to the zero forcing number. In Section 2 we establish that $\mathrm{M}(G) = \mathrm{Z}(G)$ for many Generalized Petersen graphs (these graphs are all cubic and some are bipartite). In Section 3 we develop expansion techniques that preserve $\mathrm{M}(G) = \mathrm{Z}(G)$ and apply them to families of graphs in Section 4. In particular, we use twinning and vertex sums to form new families of bipartite graphs for which $\mathrm{M}(G) = \mathrm{Z}(G)$; some of these are 3-semiregular. In Section 5 we show that $\mathrm{M}(G) = \mathrm{Z}(G)$ for two well-known cubic graphs.

Example 1.1. Let L be the bipartite 3-semiregular graph shown in Figure 1. The *path cover number* of a graph G, denoted by $P(G)$, is the minimum number of induced paths needed to cover all vertices of G. A graph is outerplanar if there is a drawing of the graph with no crossings and all vertices on the infinite face. Sinkovic

[10] showed that $\mathrm{M}(G) \leq P(G)$ for an outerplanar graph G. Since L is outerplanar and $P(L) = 3$, $\mathrm{M}(L) \leq 3$. By use of software [6] it is straightforward to verify that $\mathrm{Z}(L) = 4$.

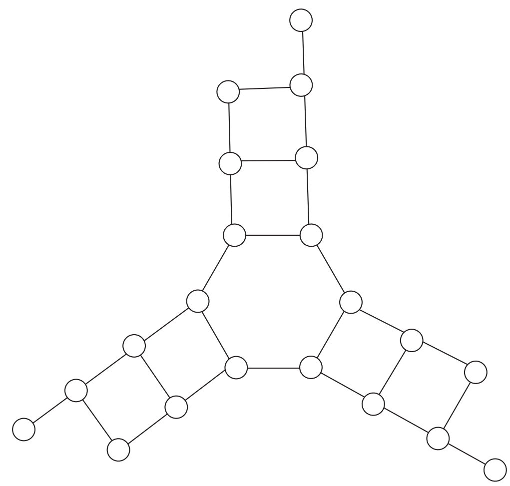

Figure 1: Lin's example L of a bipartite 3-semiregular graph with $\mathrm{M}(L) < \mathrm{Z}(L)$

2 Generalized Petersen Graphs

In this section, we present results on the zero forcing number and maximum nullity of the Generalized Petersen (abbreviated GP) graphs, including exact values for a few subfamilies of GP graphs. We first define the aforementioned family using notation similar to that in [11]. For $n \geq 3$ and $0 < k \leq \left\lfloor \frac{n-1}{2} \right\rfloor$, the *Generalized Petersen* (GP) graph $P(n, k)$ is the graph with vertex set $V = \{u_0, u_1, u_2, \ldots, u_{n-1}, v_0, v_1, v_2, \ldots, v_{n-1}\}$ and edge set $E = \{u_i u_{i+1}$ for all $0 \leq i \leq n-1$, $u_i v_i$ for all $0 \leq i \leq n-1$, $v_j v_{j+k}$ for all $0 \leq j \leq n-1\}$ where subscripts are taken modulo n.

The graph $P(n, k)$ is a cubic graph consisting of an outside cycle on vertices $u_0, \ldots u_{n-1}$ with a perfect matching connecting this cycle to one or more inner cycles on vertices $v_0, \ldots, v_{n-1}$ where the vertices on the inner and outer cycles are ordered counterclockwise (see Figure 2). In the literature, the term *Generalized Petersen* graph often requires that n and k be relatively prime. However, for the purposes of this paper we do not require this condition. Note that in the above definition k is restricted as $0 < k \leq \left\lfloor \frac{n-1}{2} \right\rfloor$. This is because the symmetry of the GP graphs gives us that for $\left\lceil \frac{n-1}{2} \right\rceil < \ell < n$, $P(n, \ell)$ is isomorphic to $P(n, n-\ell)$, and $0 < n - \ell < \left\lfloor \frac{n-1}{2} \right\rfloor$. Some GP graphs are known by different names such as the n-Prism, which is $P(n, 1)$, and the well-known Petersen graph $P(5, 2)$. It is known that $\mathrm{M}(P(n, 1)) = \mathrm{Z}(P(n, 1)) = 4$ for $n \geq 4$, $\mathrm{M}(P(3, 1)) = \mathrm{Z}(P(3, 1)) = 3$, and $\mathrm{M}(P(5, 2)) = \mathrm{Z}(P(5, 2)) = 5$ [1].

Remark 2.1. A GP graph is bipartite if and only if n is even and k is odd. This is because $P(n, k)$ contains an odd cycle if n is odd or if k is even, and a graph is bipartite if and only if it does not contain an odd cycle.

Theorem 2.2. *For any $n \geq 3$, $\mathrm{Z}(P(n, k)) \leq 2k + 2$.*

Proof. Let $S_0 = \{v_0, u_0, u_1, u_2, \ldots, u_{2k-1}, v_{2k-1}\}$, $S_1 = S_0 \cup \{v_1, v_2, \ldots, v_{2k-2}\}$, and $S_i = S_{i-1} \cup \{v_{n-i+1}, u_{n-i+1}\}$ for $2 \leq i \leq n - 2k$. Initially, the vertices in S_0 can force the vertices in S_1 because u_i forces v_i for $i = 1, \ldots, 2k-2$. Then u_{n-i+2} forces u_{n-i+1} and v_{k-i+1} forces v_{n-i+1}, for $2 \leq i \leq n+1$. Since all vertices are eventually forced, $\mathrm{Z}(P(n, k)) \leq |S_0| = 2k + 2$. □

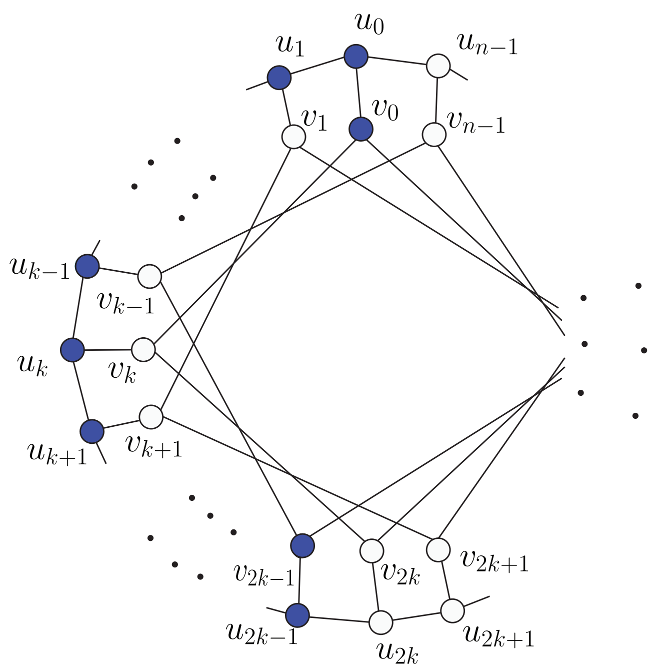

Figure 2: A Generalized Petersen graph (shown with a zero forcing set)

Since some but not all of our arithmetic is modular, we define notation for the residue mod n of an integer ℓ by

$$(\ell)_n = \ell - \left\lfloor \frac{\ell}{n} \right\rfloor n.$$

In this section we index the rows and columns of matrices and vectors starting with zero. To form the adjacency matrix of a GP graph, we order the vertices as $u_0, u_1, u_2, \dots, u_i, \dots, u_{n-1}$ followed by the vertices $v_0, v_1, v_2, \dots, v_i, \dots, v_{n-1}$. In the GP graph $P(n, k)$, the neighbors of an outer cycle vertex u_i are v_i, $u_{(i-1)_n}$, and $u_{(i+1)_n}$ for $0 \le i \le n-1$. The neighbors of an inner cycle(s) vertex v_i are u_i, $v_{(i+k)_n}$, and $v_{(i-k)_n}$ for $0 \le i \le n-1$. Thus, the adjacency matrix is a block matrix of the following form

$$\begin{bmatrix} A(C_n) & I_n \\ I_n & A'(C_n) \end{bmatrix}$$

where $A(C_n)$ is the adjacency matrix of the cycle on n vertices and the matrix $A'(C_n)$ is a matrix with 1's on the kth and $(n-k)$th super and subdiagonals and zeros elsewhere (i.e., $A'(C_n)$ is the adjacency matrix of the inner cycle(s)).

Next we establish a technical lemma about eigenvalue multiplicities of GP graphs and then use it to show certain subfamilies of GP graphs have maximum nullity equal to zero forcing number.

Lemma 2.3. *Let $n \ge 3$, $0 < k \le \left\lfloor \frac{n-1}{2} \right\rfloor$, $n' = nr$ for $r \ge 1$, and k' satisfies $\left(k'\right)_n = k$ and $0 < k' \le \left\lfloor \frac{n'-1}{2} \right\rfloor$. Suppose λ is an eigenvalue of $A(P(n, k))$ with multiplicity m. Then, λ is an eigenvalue of $A(P(n', k'))$ with multiplicity at least m.*

Proof. For each eigenvector of $A(P(n, k))$ for λ, we construct an eigenvector of $A(P(n', k'))$ for λ. Since by construction independent eigenvectors of $A(P(n, k))$ yield independent eigenvectors of $A(P(n', k'))$, this shows λ is an eigenvalue of multiplicity at least m for $A(P(n', k'))$.

Suppose $\mathbf{x} = [x_j]$ is an eigenvector of $A(P(n, k))$ for eigenvalue λ. By considering row j for $0 \le j \le n-1$ and using that $j = (j)_n$ we have

$$x_{(j-1)_n} + x_{(j+1)_n} + x_{n+(j)_n} = \left(A(P(n, k))\,\mathbf{x}\right)_j = (\lambda \mathbf{x})_j = \lambda x_{(j)_n}. \tag{1}$$

Considering row j for $n \le j \le 2n-1$ and using that $j - n = (j)_n$ and $(j-n)_n = (j)_n$ yields

$$x_{(j)_n} + x_{n+(j-k)_n} + x_{n+(j+k)_n} = \left(A(P(n,k))\,\mathbf{x}\right)_j = (\lambda \mathbf{x})_j = \lambda x_{n+(j)_n}. \tag{2}$$

We define

$$\hat{\mathbf{x}} = [\underbrace{\underbrace{x_0, x_1, \dots, x_{n-1}}, \dots, \underbrace{x_0, x_1, \dots, x_{n-1}}}_{r \text{ times}}, \underbrace{\underbrace{x_n, x_{n+1}, \dots, x_{2n-1}}, \dots, \underbrace{x_n, x_{n+1}, \dots, x_{2n-1}}}_{r \text{ times}}]^T.$$

Observe that independence of a set of n-vectors $\mathbf{x}, \mathbf{y}, \dots$ guarantees the independence of the set of n'-vectors $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \dots$ just constructed.

We show that $\hat{\mathbf{x}}$ is an eigenvector of $A(P(n', k'))$ for eigenvalue λ. Note that $\hat{x}_j = x_{(j)_n}$ for $j = 0, \dots, n'-1$, and $\hat{x}_j = x_{n+(j-n')_n} = x_{n+(j)_n}$ for $j = n', \dots, 2n'-1$. Observe that $\left((c)_{n'}\right)_n = (c)_n$ for all integers c. Thus for $j = 0, \dots, n'-1$,

$$\begin{aligned}(A(P(n',k'))\hat{\mathbf{x}})_j &= \hat{x}_{(j-1)_{n'}} + \hat{x}_{(j+1)_{n'}} + \hat{x}_{n'+(j)_{n'}} \\ &= x_{(j-1)_n} + x_{(j+1)_n} + x_{n+(j)_n} \\ &= \lambda x_{(j)_n} && \text{by (1)} \\ &= \lambda \hat{x}_j,\end{aligned}$$

For $j = n', \dots, 2n'-1$

$$\begin{aligned}(A(P(n',k'))\hat{\mathbf{x}})_j &= \hat{x}_{(j)_{n'}} + \hat{x}_{n'+(j-k')_{n'}} + \hat{x}_{n'+(j+k')_{n'}} \\ &= x_{(j)_n} + x_{n+(j-k')_n} + x_{n+(j+k')_n} \\ &= x_{(j)_n} + x_{n+(j-k)_n} + x_{n+(j+k)_n} && \text{because } \left(k'\right)_n = k \\ &= \lambda x_{n+(j)_n} && \text{by (2)} \\ &= \lambda \hat{x}_j,\end{aligned}$$

Thus, we conclude $A(P(n', k'))\hat{\mathbf{x}} = \lambda \hat{\mathbf{x}}$. □

Theorem 2.4. *Let $n \ge 3$ and $0 < k \le \left\lfloor \frac{n-1}{2} \right\rfloor$. Suppose there is an eigenvalue λ of $A(P(n,k))$ with multiplicity m. Then λ is an eigenvalue of $A(P(nr,k))$ with multiplicity at least m. If $m = 2k+2$, then $\mathrm{M}(P(nr,k)) = \mathrm{Z}(P(nr,k)) = 2k+2$ for all integers $r \ge 1$. In particular, the following Generalized Petersen graphs have maximum nullity equal to zero forcing number for $r \ge 1$:*

(a) $\mathrm{M}(P(15r, 2)) = \mathrm{Z}(P(15r, 2)) = 6$.
(b) $\mathrm{M}(P(24r, 5)) = \mathrm{Z}(P(24r, 5)) = 12$.

Proof. By Lemma 2.3, λ is an eigenvalue of $A(P(nr,k))$ with multiplicity at least m. When $m = 2k+2$,

$$2k+2 = \operatorname{null}(A(P(nr,k)) - \lambda I) \le \mathrm{M}(P(nr,k)) \le \mathrm{Z}(P(nr,k)) \le 2k+2,$$

by Theorem 2.2. For (a), -2 is an eigenvalue of $A(P(15,2))$ with multiplicity 6. For (b), 0 is an eigenvalue of $A(P(24,5))$ with multiplicity 12. □

Theorem 2.5. *Suppose $n \ge 3$ and λ is an eigenvalue of $A\left(P\left(n, \left\lfloor \frac{n-1}{2} \right\rfloor\right)\right)$ with multiplicity m. Then for any odd positive integer t, λ is an eigenvalue of $A\left(P\left(nt, \left\lfloor \frac{nt-1}{2} \right\rfloor\right)\right)$ with multiplicity at least m.*

Proof. The result follows from Lemma 2.3 when it is established that $\left(\left\lfloor \frac{nt-1}{2} \right\rfloor\right)_n = \frac{n-1}{2}$ for odd t. Let $t = 2s+1$ where s is a nonnegative integer. Then

$$\left(\left\lfloor \frac{nt-1}{2} \right\rfloor\right)_n = \left(\left\lfloor \frac{n(2s+1)-1}{2} \right\rfloor\right)_n = \left(ns + \left\lfloor \frac{n-1}{2} \right\rfloor\right)_n = \left\lfloor \frac{n-1}{2} \right\rfloor.$$ □

Theorem 2.6. *For any $k \geq 3$, $\mathrm{Z}(P(2k+1,k)) \leq 6$.*

Proof. Define $S_0 = \{u_0, u_1, v_1, u_k, u_{k+1}, v_{k+1}\}$

$$S_i = \begin{cases} S_{i-1} \cup \left\{u_{k+1+\frac{i+1}{2}}, v_{k+1+\frac{i+1}{2}}\right\}, & \text{for } i \text{ odd} \\ S_{i-1} \cup \left\{u_{1+\frac{i}{2}}, v_{1+\frac{i}{2}}\right\}, & \text{for } i \text{ even} \end{cases}$$

for $1 \leq i \leq 2k-3$, and $S_{2k-2} = S_{2k-3} \cup \{v_0, v_k\}$. We force in order of increasing i. For i odd, $u_{k+1+\frac{i-1}{2}} \to u_{k+1+\frac{i+1}{2}}$ and $v_{1+\frac{i-1}{2}} \to v_{k+1+\frac{i+1}{2}}$. For i even, $u_{\frac{i}{2}} \to u_{1+\frac{i}{2}}$ and $v_{k+1+\frac{i}{2}} \to v_{1+\frac{i}{2}}$. Thus $\mathrm{Z}(P(2k+1,k)) \leq |S_0| = 6$. □

Corollary 2.7. *For all odd $t \geq 1$, $\mathrm{M}(P(15t, \frac{15t-1}{2})) = 6 = \mathrm{Z}(P(15t, \frac{15t-1}{2}))$.*

Proof. By Theorem 2.6, $\mathrm{Z}(P(15t, \frac{15t-1}{2})) \leq 6$. It is straightforward to verify that -2 is an eigenvalue of $A(P(15,7))$ with multiplicity 6. So -2 is an eigenvalue of $A(P(15t, \frac{15t-1}{2}))$ with multiplicity at least 6 by Theorem 2.5. Thus

$$6 \leq \mathrm{M}(P(15t, \tfrac{15t-1}{2})) \leq \mathrm{Z}(P(15t, \tfrac{15t-1}{2})) \leq 6.$$ □

Question 2.8. *Does there exist a Generalized Peterson graph $P(n,k)$ having $\mathrm{M}(P(n,k)) < \mathrm{Z}(P(n,k))$?*

3 Expansion Procedures

In this section, we introduce expansion procedures that determine the maximum nullity and minimum rank of graphs with special characteristics. In a graph G, vertices v and w that have the same set of neighbors (except possibly v and w) are called *twins*. Let v and w be twins; if v and w are not adjacent, then v and w are *independent twins*, whereas if v and w are adjacent, then v and w are *adjacent twins*. Define *twinning* of $v \in V(G)$ as a graph operation in which we add a new vertex x such that $N(v) = N(x)$ (so x is an independent twin of v). Define $\mathrm{twin}(G, v, k)$ to be the graph resulting from performing the twinning operation k times on vertex v in the graph G.

Proposition 3.1. [7] *Let G be a graph with independent twin vertices v and w. If there exists $A \in \mathcal{S}(G-w)$ such that* $\mathrm{rank}(A) = \mathrm{mr}(G)$ *and the diagonal entry $a_{vv} = 0$, then* $\mathrm{mr}(G) = \mathrm{mr}(G-w)$. *Furthermore, there exists $A' \in \mathcal{S}(G)$ with* $\mathrm{rank}(A') = \mathrm{mr}(G)$ *and $a'_{vv} = 0$.*

Corollary 3.2. (Twinning) *Suppose* $\mathrm{M}(G) = \mathrm{Z}(G)$, *$v \in V(G)$, and there exists $A = [a_{ij}] \in \mathcal{S}(G)$ such that* $\mathrm{null}(A) = \mathrm{M}(G)$ *and $a_{vv} = 0$. Then* $\mathrm{M}(\mathrm{twin}(G,v,k)) = \mathrm{Z}(\mathrm{twin}(G,v,k)) = \mathrm{M}(G) + k$ *and there exists $A' \in \mathcal{S}(\mathrm{twin}(G,v,k))$ with $a'_{vv} = 0$.*

Proof. Since $\mathrm{null}(A) = \mathrm{M}(G)$, we have $\mathrm{rank}(A) = \mathrm{mr}(G)$. By Proposition 3.1 there is a matrix $A' \in \mathcal{S}(\mathrm{twin}(G,v,k))$ with $\mathrm{rank}(A') = \mathrm{mr}(\mathrm{twin}(G,v,k)) = \mathrm{mr}(G)$ and $a'_{vv} = 0$. Since $\mathrm{twin}(G,v,k)$ has k more vertices than G, $\mathrm{M}(\mathrm{twin}(G,v,k)) = \mathrm{M}(G) + k$. Since we can simply color an independent twin blue and add it to a given zero forcing set, we know that $\mathrm{Z}(\mathrm{twin}(G,v,k)) \leq \mathrm{Z}(G) + k$. Thus, $\mathrm{M}(G) + k = \mathrm{M}(\mathrm{twin}(G,v,k)) \leq \mathrm{Z}(\mathrm{twin}(G,v,k)) \leq \mathrm{Z}(G) + k = \mathrm{M}(G) + k$. □

Theorem 3.3. [4](Cut vertex reduction) *Let G be a graph with a cut vertex v. Let $W_1, \ldots, W_k$ be the vertex sets for the connected components of $G - v$, and for $1 \leq i \leq k$, let $G_i = G[W_i \cup \{v\}]$. Then* $\mathrm{mr}(G) = \min\left\{\sum_{i=1}^{k} \mathrm{mr}(G_i), \sum_{i=1}^{k} \mathrm{mr}(G_i - v) + 2\right\}$.

Corollary 3.4. *Suppose that* $G = G_1 \underset{v}{\oplus} G_2$ *and* $\mathrm{mr}(G_i - v) = \mathrm{mr}(G_i)$ *for* $i \in \{1, 2\}$. *Then* $\mathrm{mr}(G) = \mathrm{mr}(G_1) + \mathrm{mr}(G_2)$ *and* $\mathrm{M}(G) = \mathrm{M}(G_1) + \mathrm{M}(G_2) - 1$.

Let S be a zero forcing set of a graph G. A *chronological list of forces* is a list of the forcing operations used to color every vertex of G in the order in which they are performed. A *forcing chain* for such a list is a sequence of vertices $(v_1, v_2, \ldots, v_k)$ such that for $i = 1, 2, \ldots, k-1$, v_i forces v_{i+1}. A forcing chain is called *maximal* if it is not a proper subsequence of any other forcing chain. A *reversal* of S is the set of last vertices of the maximal zero forcing chains of a chronological list of forces.

Theorem 3.5. [3] *If S is a zero forcing set of G then so is any reversal of S.*

Lemma 3.6. [9] *Let G be a graph with a cut vertex v. Let $W_1, \ldots, W_k$ be the vertex sets for the connected components of $G - v$, and for $1 \le i \le k$, let $G_i = G[W_i \cup \{v\}]$. Then* $\mathrm{Z}(G) \ge \sum_{i=1}^{k} \mathrm{Z}(G_i) - k + 1$.

Corollary 3.7. *Suppose* $G = G_1 \underset{v}{\oplus} G_2$ *and for* $i \in \{1, 2\}$ *there exist minimum zero forcing sets S_i for G_i such that $v \in S_i$. Then* $\mathrm{Z}(G) = \mathrm{Z}(G_1) + \mathrm{Z}(G_2) - 1$.

Proof. By Lemma 3.6 with $k = 2$ we have $\mathrm{Z}(G) \ge \mathrm{Z}(G_1) + \mathrm{Z}(G_2) - 1$. To see that $\mathrm{Z}(G) \le \mathrm{Z}(G_1) + \mathrm{Z}(G_2) - 1$, note that v is non-forcing in G_1 with respect to a reversal of S_1. In G, color blue the vertices of the reversal ($|S_1|$ of them) and proceed to force the rest of the vertices of G_1 in G by the chronological list of forces of the reversal. This leaves all of G_1's vertices in G colored blue, including v, and now by coloring the $|S_2| - 1$ other vertices of S_2 in G we may force the rest of the graph. □

Corollary 3.8. (Cut vertex expansion) *Suppose* $G = G_1 \underset{v}{\oplus} G_2$. *If* $\mathrm{M}(G_i) = \mathrm{Z}(G_i)$, $\mathrm{mr}(G_i - v) = \mathrm{mr}(G_i)$ *and there exists a minimum zero forcing set S_i for G_i such that $v \in S_i$ for $i = 1, 2$, then* $\mathrm{M}(G) = \mathrm{Z}(G)$.

4 Families of bipartite graphs constructed by expansion

In this section we apply results from the previous section to construct families of bipartite graphs, some of which are 3-semiregular.

4.1 Vertex sums of bat graphs

Define the family of *bat graphs* as follows: The *basic bat graph*, which is denoted by B_0, is the graph given by a C_4 with a leaf appended to each of two non-adjacent vertices. See Figure 3. This graph has $\mathrm{M}(B_0) = \mathrm{Z}(B_0) = 2$ with its adjacency matrix being of minimum rank. The set consisting of one leaf and one degree two vertex is a minimum zero forcing set. In the forcing process, the other leaf does not force. Call the degree 3 vertices x_1 and x_2, the degree 2 vertices y_1 and y_2, and the leaves y_3 and y_4. Any graph constructed by a sequence of twinning operations applied to x_1 or x_2 is a *bat graph*.

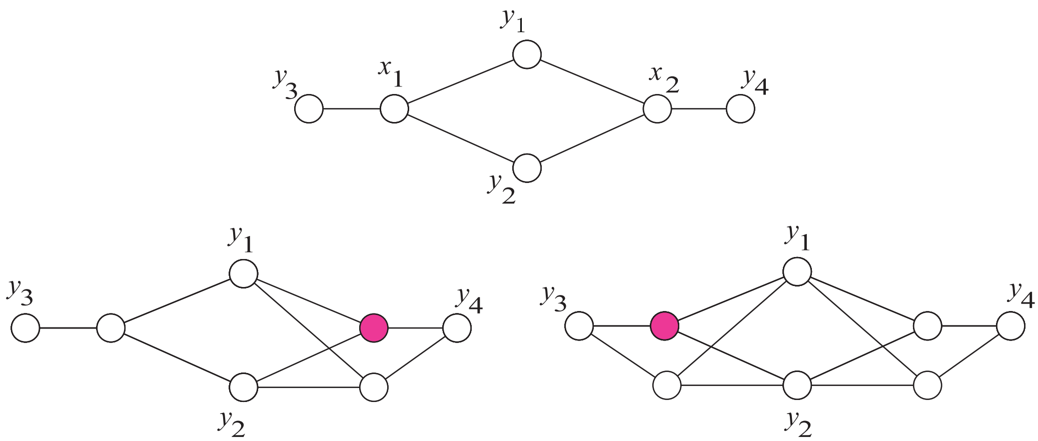

Figure 3: The basic bat graph and two expansions by twinning (each pink vertex has a newly created twin)

Theorem 4.1. *For every bat graph* B*,* $\mathrm{M}(B) = \mathrm{Z}(B)$.

Proof. For the adjacency matrix of B_0, $\mathrm{rank}(A(B_0)) = 4 = \mathrm{mr}(B_0)$ and all diagonal elements are zero, so we may apply the twinning operation described in Corollary 3.2. □

Note that any bat graph is a 3-semiregular bipartite graph. Once at least two independent twins have been added to the X set of $V(B_0)$, the resulting graph has $|X| \geq |Y|$.

Remark 4.2. If we take G_1 and G_2 to be bat graphs and let $G = G_1 \underset{v}{\oplus} G_2$ with v taken to be either y_3 or y_4 in each of the G_i, all hypotheses of the cut vertex expansion procedure are satisfied and as a result $\mathrm{M}(G) = \mathrm{Z}(G)$. In particular, y_3 or y_4 may be in a minimum zero forcing set of any bat graph, and a reversal of that set will include whichever of the two was not in the original set. Similarly, if G is a vertex sum of bat graphs as described above, additional bat graphs may be appended to the y_3 or y_4 vertices of the "bat subgraphs" not already involved in a vertex sum while preserving the property that maximum nullity equals zero forcing number, allowing the construction of chains of bats, called *bat chains*, of arbitrary length. See Figure 4.

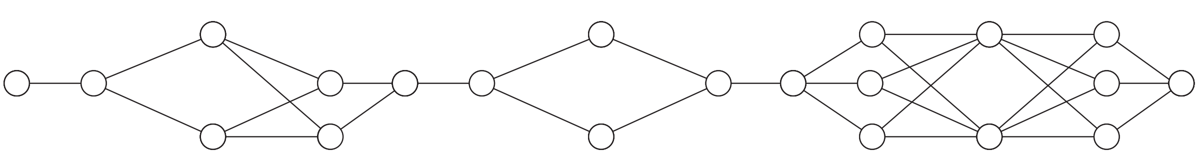

Figure 4: A 3-semiregular bipartite graph G with $\mathrm{M}(G) = \mathrm{Z}(G)$

4.2 Jewel necklace graphs

In this section, we introduce a family of graphs called jewel necklaces and show that maximum nullity equals the zero forcing number for these graphs. Define an *r-jewel* to be a $K_{r,r}$ where one edge is deleted. Define an s, *r-jewel necklace*, $J_{s,r}$, for $s \geq 2$ and $r \geq 3$ to be a cycle of jewel graphs connected appropriately by edges between vertices v that have $\deg(v) = r - 1$. We require that $r \geq 3$ and $s \geq 2$ because otherwise we would be examining a cycle ($r = 2$), s copies of P_2 ($r = 1$), or a $K_{r,r}$ ($s = 1$), for which maximum nullity and zero forcing number are known. In this process of connecting s copies of r-jewels, exactly one connecting edge is incident with each vertex of degree $r - 1$, so $J_{s,r}$ is r-regular. Figure 5 shows the general form of $J_{s,r}$. Note that the order of $J_{s,r}$ is $2sr$ and this graph is bipartite.

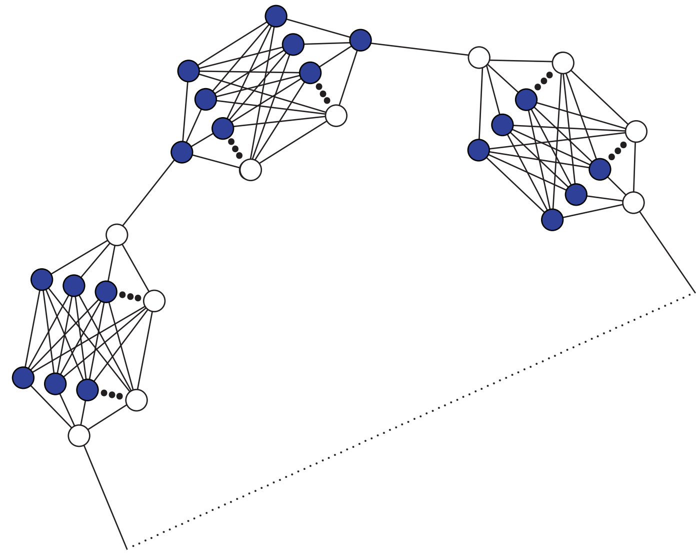

Figure 5: The jewel necklace $J_{s,r}$ with a zero forcing set

Theorem 4.3. *The* s, r*-jewel necklace* $J_{s,r}$ *has* $\mathrm{M}(J_{s,r}) = \mathrm{Z}(J_{s,r}) = 2sr - 4s + 2$ *for* $s \geq 2$ *and* $r \geq 3$.

Proof. First we show that s, r-jewel necklace $J_{s,r}$ has $\mathrm{Z}(J_{s,r}) \leq 2sr - 4s + 2$. We number the vertices of the kth jewel as (k, i), $i = 0, \ldots, 2r-1$. A zero forcing set for the case $s = 2$ consists of vertices $\{(0, 0), (0, 1), \ldots, (0, r-2), (0, r), \ldots, (0, 2r-3), (0, 2r-1), (1, 1), \ldots, (1, r-2), (1, r), \ldots, (1, 2r-3)\}$ and is shown in Figure 6. We see that vertices $(0, 1)$ and $(0, r)$ force vertices $(0, 2r-2)$ and $(0, r-1)$, respectively. Once these vertices are forced, vertices $(0, 0)$ and $(0, 2r-1)$ force vertices $(1, 0)$ and $(1, 2r-1)$, respectively. Now $(1, 1)$ and $(1, r)$ can force the final vertices. This process can be generalized for $J_{s,r}$, with the zero forcing set shown Figure 5, with one jewel having its end vertices colored and the others not.

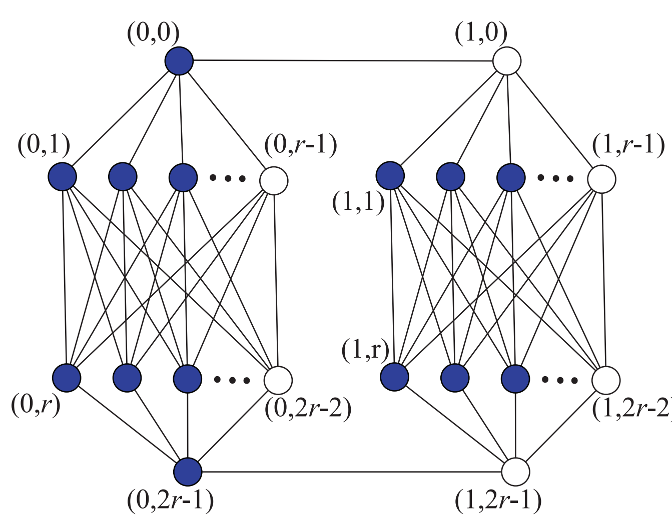

Figure 6: The double jewel, $J_{2,r}$

Twinning on a cycle can be used to construct $J_{s,r}$ and prove that $\mathrm{M}(J_{s,r}) = \mathrm{Z}(J_{s,r})$. We begin with the $4s$-cycle C_{4s} and observe that the adjacency matrix $A(C_{4s})$ has $\operatorname{rank}(A(C_{4s})) = 4s - 2$, so $2 = \mathrm{M}(A(C_{4s})) = \mathrm{Z}(C_{4s})$ (and all diagonal entries of $A(C_{4s})$ are zero). We add an (independent) twin vertex of $v_i \in V(C_{4s})$ for all $i \equiv 3$

mod 4 and for all $i \equiv 0 \mod 4$ (see Figure 7). Perform this twinning of $2s$ vertices $r-2$ times to construct $J_{s,r}$. Then $\mathrm{mr}(J_{s,r}) = 4s-2$ by Proposition 3.1, which implies $\mathrm{M}(J_{s,r}) = 2sr-4s+2$. Thus, $\mathrm{M}(J_{s,r}) = \mathrm{Z}(J_{s,r}) = 2sr-4s+2$. □

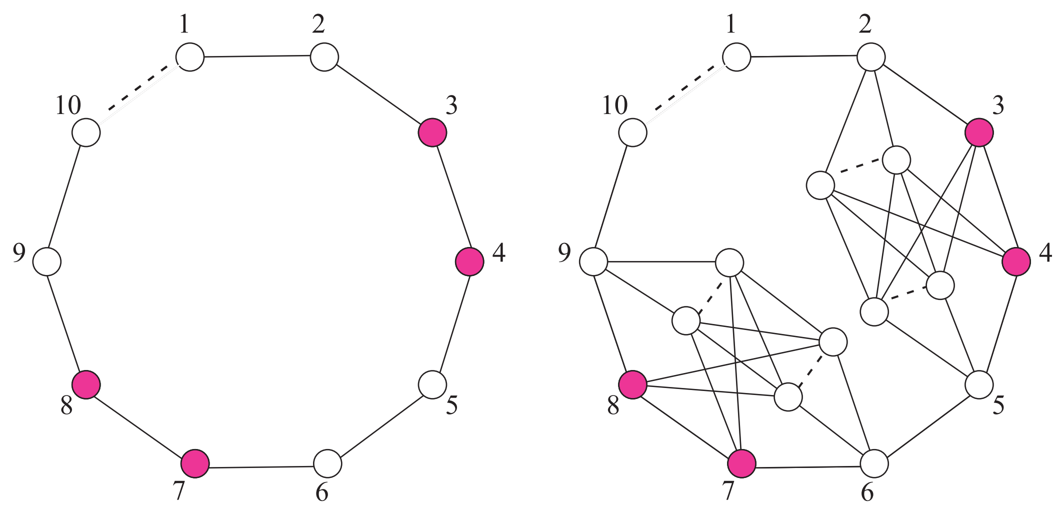

Figure 7: C_{4s} before and after twinning (pink vertices selected for twinning)

5 Other cubic graphs with M(G) = Z(G)

In this section we show that $\mathrm{M}(G) = \mathrm{Z}(G)$ for two other well-known cubic graphs. The *Bidiakis cube*, BC, is a 12-vertex graph consisting of a cube in which two opposite faces (which we call top and bottom) have edges drawn across them which connect the centers of opposite sides of the faces in such a way that the orientation of the edges added on top and bottom are perpendicular to each other. Note that the Bidiakis cube is also isomorphic to a C_{12} with edges added between three pairs of vertices on opposite sides of the cycle. This graph is shown below in Figure 8.

Proposition 5.1. *The Bidiakis cube has* $\mathrm{M}(BC) = \mathrm{Z}(BC) = 4$.

Proof. It is straightforward to verify that the rank of the adjacency matrix of the Bidiakis cube is eight, so we know that $\mathrm{M}(BC) \geq 4$. As a zero forcing set consider $S = \{1, 2, 3, 5\}$ with the vertex labeling in Figure 8. Consequently, $4 \leq \mathrm{M}(BC) \leq \mathrm{Z}(BC) \leq 4$ and so $\mathrm{M}(BC) = \mathrm{Z}(BC)$. □

The *Tutte-Coxeter graph TC* is defined to have a set of 30 vertices denoted by

$$a_0, a_1, \ldots, a_9, b_0, b_1, \ldots, b_9, c_0, c_1, \ldots, c_9,$$

and edges

$$\{a_i, a_{i+1}\}, \{a_i, b_i\}, \{b_i, b_{i+5}\}, \{b_i, c_i\}, \text{ and } \{c_i, c_{i+3}\},$$

where i ranges over $\{0, 1, \ldots, 9\}$ and arithmetic is taken modulo 10. This graph is shown below in Figure 9.

Proposition 5.2. *The Tutte-Coxeter graph has* $\mathrm{M}(TC) = \mathrm{Z}(TC) = 10$.

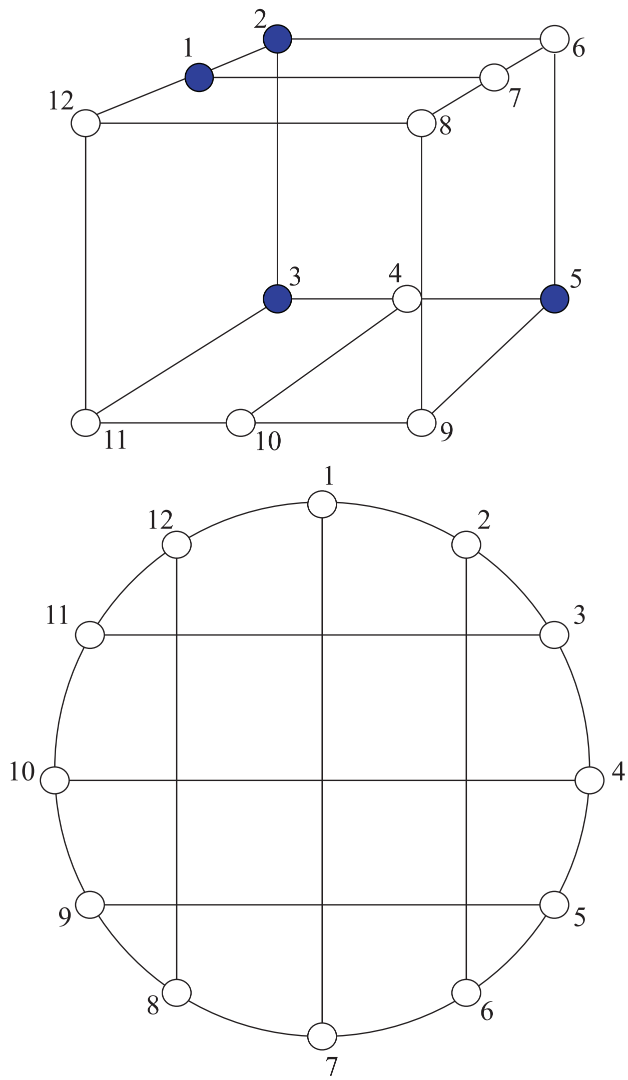

Figure 8: A zero forcing set on the Bidiakis Cube and an alternate drawing

Proof. Note that $\mathrm{rank}(A(TC)) = 20$, so $\mathrm{mr}(TC) \le 20$ and $\mathrm{M}(TC) \ge 10$. The set $S = \{a_1, a_0, b_0, b_5, c_5, c_8, b_8, b_3, c_3, b_7\}$ is a zero forcing set for the Tutte-Coxeter graph with the vertex labeling shown in Figure 9. Consequently, $10 \le \mathrm{M}(TC) \le \mathrm{Z}(TC) \le 10$, implying $\mathrm{M}(TC) = \mathrm{Z}(TC) = 10$. □

6 Conclusion

We establish $\mathrm{M}(G) = \mathrm{Z}(G)$ for the following graphs:

- The following Generalized Petersen graphs (cubic):
  - $P(24r, 5)$ for $r \ge 1$ (bipartite).
  - $P(15r, 2)$ for $r \ge 1$.
  - $P(15t, \frac{15t-1}{2})$ for odd $t \ge 1$.

- Bat graphs and bat chains (bipartite 3-semiregular).
- Jewel necklaces (bipartite and regular).
- The Bidiakis Cube (cubic).

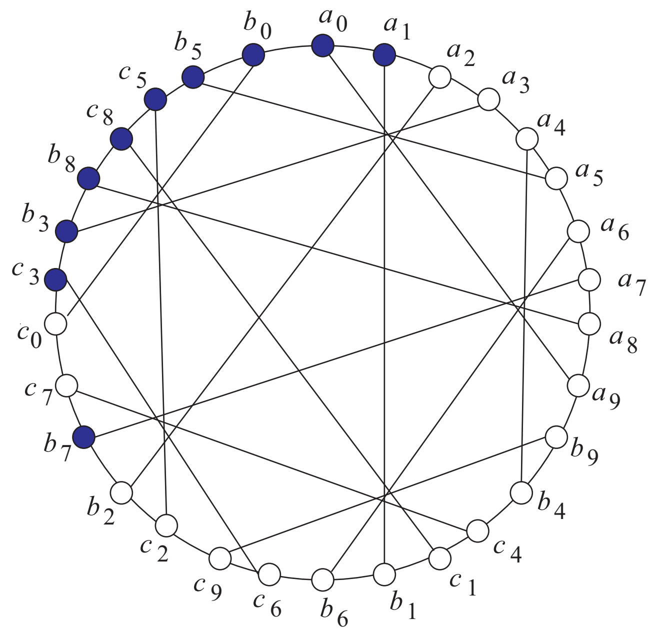

Figure 9: Tutte-Coxeter graph with a zero forcing set

– The Tutte-Coxeter graph (bipartite and cubic).

We also provide expansion techniques for constructing graphs with $\mathrm{M}(G) = \mathrm{Z}(G)$ from smaller graphs with $\mathrm{M}(G) = \mathrm{Z}(G)$ in Section 3.

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