TOMOGRAPHIC RECONSTRUCTION OF INTERNAL TEMPERATURE

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INTRODUCTION

The development of a sensor to measure internal temperature distributions of hot bodies would significantly improve the productivity and quality of materials processing. The American Iron and Steel Institute (AISI) and the National Bureau of Standards have initiated a joint research program to develop such a sensor for steels capable of generating internal temperature maps with ±10°C accuracy and 20 mm spatial resolution. Numerous applications exist for such a sensor during the processing of steel (and other metals); two of particular importance are in the control of continuous casting and slab reheating[1].

The approach under development is based upon the strong, almost linear, dependence of ultrasonic velocity upon temperature[2]. This velocity-temperature dependence was measured in our laboratory for 304 stainless steel and the results are plotted in figure 1. In this case, a 1°C temperature shift changes the longitudinal velocity by $\sim 0.68\ \mathrm{ms}^{-1}$. This is a large change given the great precision of modern ultrasonic techiques. A single velocity value, when used with a calibration curve of velocity versus temperature, would yield the average temperature along the ultrasonic ray path. To produce two-dimensional temperature maps, we exploit the tomographic approach originally developed as a medical imaging modality with x-rays and ultrasound.

There are a number of important differences, however, between medical tomography and the proposed application. In medical ultrasonic tomography, diagnostically significant changes in the elastic modulus of tissue are typically larger than the changes in elastic modulus of steel arising due to temperature. Also, the speed of sound in water, the relevant velocity in medical ultrasonic tomography, is less than one third the velocity in steel. Thus, the time resolution for the temperature problem must be at least three times better than in previous techniques. Moreover, due to time constraints and the complexity of making parallel measurements, the number of ultrasonic measurements is severely limited compared to medical tomography.

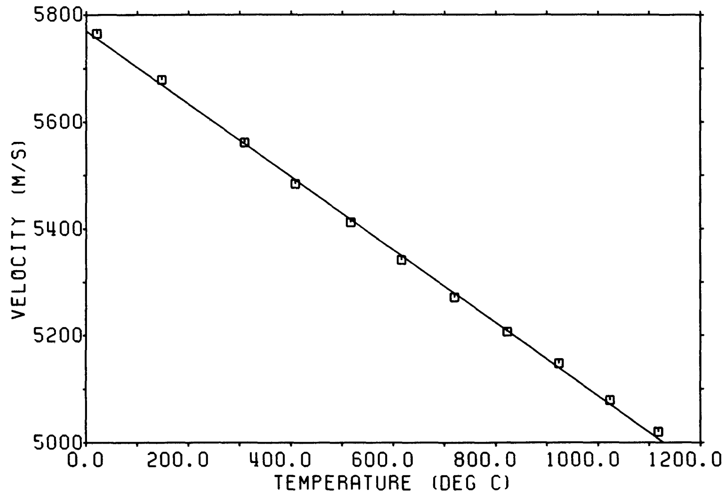

Fig. 1. Temperature dependence of longitudinal velocity in 304 stainless steel.

We see that the application of tomography to temperature distribution measurement is in some ways a more difficult problem than that of medical tomography. However, the temperature measurement problem has an important factor in its favor--the availability of a priori heat-flow information. The well understood thermal diffusion process and independent surface temperature values constitute a priori information that serves to limit the range of admissible solutions to the inverse reconstruction problem. This paper shows how a priori information can be exploited in formulating reconstruction algorithms for the imaging of internal temperature distributions.

TIME-OF-FLIGHT TOMOGRAPHY

The time-of-flight (TOF) of an ultrasonic pulse along a path through an object is the line integral of the reciprocal sound velocity along that ray path. The path length divided by the TOF is the average velocity along the path, and in conjunction with calibration curves such as that of figure 1, this could be used to compute the average temperature along that path. For some applications, average temperature alone may constitute sufficient information. If many TOF measurements are acquired over multiple paths, a tomographic approach can be employed to reconstruct a cross-sectional image of the sound velocity (and hence temperature) within the sample.

In principle, at least as many TOF measurements as pixels are required to compute a tomogram. In practice, errors in the TOF and path length measurement combine with inherent ill-conditioning in the inversion process to require considerable measurement redundancy (factors of 3-10 or more), in which case least squares techniques may be exploited to best estimate the temperature field. A priori information can be used to reduce this dependency on redundant information.

The most important a priori constraint is the assumption of symmetrical heat flow, which is often reasonable in bodies of simple geometric shape (e.g., circular or rectangular cross-section). Knowledge that the temperature field is symmetrical drastically reduces the number of unknowns characterizing the temperature field, and thus reduces the number of required measurements by a comparable amount. Furthermore, heat flow is well modeled by the thermal conduction equation (a diffusion equation). Because temperature is a solution to this equation, it is, in effect, being subjected to a low pass spatial filter whose spatial-frequency cutoff decreases in proportion to the square root of the cooling time. Stated another way, rapid spatial temperature fluctuations disappear with time due to thermal diffusion. This limit on the spatial frequency bandwidth (smoothness) of the temperature field implies the existence of a limit on the density of data sampling (number of TOF measurements) needed to recover the temperature distribution. Compared to medical tomography, far coarser spatial resolution is adequate to reconstruct the internal temperature field.

CYLINDRICAL BAR GEOMETRIES

Consider the problem of reconstructing a circular cross-sectional map of the temperature through a cylinder. If heat diffuses uniformly through the surface, the temperature is radially symmetric, i.e. the unknown temperature may be represented by a one-dimensional function of radius, $T(r)$.

Between the temperatures of phase transformations, the velocity-temperature relation of most materials is approximately linear and may be expressed:

$$T(r) = T_0 + b\,(v(r) - v_0) \quad , \qquad (1)$$

where T_0, v_0 and b are experimentally determined constants. Our objective is to use TOF data to reconstruct the radial velocity profile, $v(r)$, from which we then compute the temperature profile, $T(r)$.

The simplest set of TOF measurements to obtain is from a single omnidirectional ultrasonic source at one point on the cylinder and multiple receivers around the circumference (this would be referred to as a single "fan beam" measurement in the terminology of medical x-ray tomography), as illustrated in figure 2. Let τ_m denote the measured TOF over a path L_m where M is the number of paths. Then we can write:

$$\tau_m = \int_{L_m} \frac{dl}{v(r)} \quad , \qquad m = 1, 2 \ldots M. \qquad (2)$$

In practice, M will be small because of the limited time available and the experimental complexity of making many simultaneous measurements.

Numerous techniques exist for tomographic reconstruction. However, to reconstruct the radial velocity profile we use a "series expansion" algorithm[3], since we find it both a natural approach in terms of imposing constraints and effective in reducing the number of unknowns (pixels) to an absolute minimum. This, as we have already emphasized, reduces the number of measurements and improves the numerical conditioning of the problem. This approach is also amenable to the utilization of a priori information.

The series expansion technique consists of expanding the unknown temperature profile (i.e. the reciprocal velocity) in a suitable set of N basis functions $\phi_n(r)$:

$$\frac{1}{v(r)} = \sum_{n=1}^{N} a_n \phi_n(r), \tag{3}$$

where $\{\phi_n(r)\}$ is a basis set orthogonal on the interior of a circle of radius R (the radius of the cylinder). The number of basis functions, N, must be kept small, and is determined by the permitted error between the actual and reconstructed profile and by the choice of basis set.

Inserting (3) into (2) and interchanging orders of summation and integration gives:

$$\tau_m = \sum_{n=1}^{N} a_n \Phi_{mn} \quad , \quad m = 1, 2, \ldots, M \quad , \tag{4}$$

where

$$\Phi_{mn} = \int_{L_m} \phi_n(r) d\ell \quad . \tag{5}$$

Once the basis functions $\phi_n(r)$ are chosen, the matrix elements Φ_{mn} can be numerically computed and stored. Our problem then reduces to solving the linear system (4) for the unknown coefficients a_n, where the τ_m are measured. Upon solving (4) for a_n, (3) gives the reconstruction of the reciprocal velocity $1/v(r)$, and thus $T(r)$ using (1). To mitigate the effect of measurements errors, many more TOF measurements

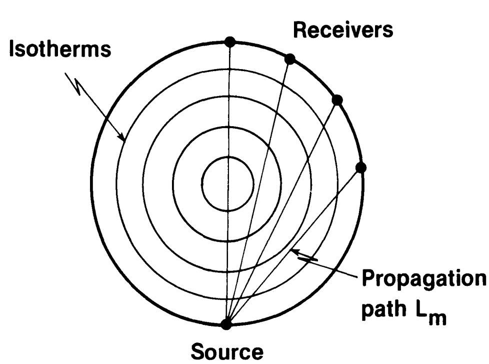

Fig. 2. Cylindrical cross-section with ray paths diverging from a single ultrasonic source. The temperature distribution is assumed to have cylindrical symmetry.

(M) than unknowns (N) are desirable. In this case, (4) will be overdetermined, and it is natural to compute the pseudoinverse (the minimum-norm least mean-square-error solution) of (4). This most simply can be achieved by writing (4) in matrix form:

$$\underline{\tau} = \Phi\underline{a} \quad , \tag{6}$$

where Φ is an M by N matrix ($M \geq N$), $\underline{a}$ is the N-component coefficient vector and $\underline{\tau}$ is the M-component measurement vector. Minimizing the mean-square-error $E = \underline{e}^T\underline{e}$, where $\underline{e} = \Phi\underline{a} - \underline{\tau}$, results in the pseudo-inverse of (6), given by:

$$\underline{a} = (\Phi^T\Phi)^{-1}\Phi^T\underline{\tau} \quad , \tag{7}$$

where ^{T} denotes transpose.

In our work, two candidate basis sets $\{\phi_n\}$, Bessel functions and "ring functions," were studied. Both are orthogonal in the sense that

$$\int_0^R \phi_n(r)\phi_m(r)r\,dr = N_n\delta_{nm} \quad ,$$

where N_n is a normalization constant. They are defined as follows:

i) Bessel function basis:

$$\phi_n(r) = J_0(k_n r) \quad , \tag{8a}$$

where $J_0(\cdot)$ is the zero-order Bessel function and k_n is the n-th root of $J_0(kR) = 0$.

ii) Ring function basis:

$$\phi_n(r) = \text{ring}_n(r) \quad , \tag{8b}$$

where

$$\text{ring}_n(r) = \begin{cases} 1 \text{ for } r_{n-1} \leq r \leq r_n \\ 0 \text{ otherwise} \end{cases} \quad ,$$

and $r_n = Rn/N$, $n = 0, 1, \ldots, N$.

The Bessel basis is a particularly interesting choice because Bessel functions are smooth over the circular domain, and we recall that the solution to the thermal conductivity equation in a cylindrical geometry is also given by a Bessel function series[4]. This suggests that the Bessel basis is a natural choice for the temperature reconstruction problem with circular symmetry and that the approximation (3) may even provide a good fit when truncated after the first few terms. This is because the higher-order terms in the series solution to the conductivity equation are exponentially damped with time. As a result, after a relatively short cooling time, the temperature profile increasingly resembles a single Bessel function, in which case only one term in (3) may be sufficient to approximate the profile.

The ring basis on the other hand provides a discrete or "staircase" approximation to the temperature profile, and thus does not provide the characteristic smooth temperature profile expected.

Using actual TOF measurements, the above tomographic scheme has been used to reconstruct temperature profiles through a 6 in. diameter stainless steel cylinder initially heated to 400°C. The results, and the experimental techniques employed, are reported in [3].

RECTANGULAR BAR GEOMETRIES

When the temperature distribution is bounded by a rectangle, the tomographic reconstruction problem can sometimes be greatly simplified. To illustrate this, consider, for simplicity, a temperature distribution on a square cross-section (defined by $-a < x < a$ and $-a < y < a$). A major simplification arises when the two-dimensional temperature field, $T(x,y,t)$, factors into the product of two one-dimensional solutions to the heat flow equation:

$$T(x,y,t) = f(x,t) \cdot f(y,t) \quad . \tag{9}$$

In general, the function $T(x,y,t)$ can be shown to be factorable in this way under the following conditions:

(i) The initial temperature distribution is factorable (the simplest example is a uniform initial temperature):

$$T(x,y,0) = f(x,0) \cdot f(y,0) \quad ,$$

(ii) The boundary conditions are homogeneous and constant over each face (length 2a) of the square boundary. That is, the boundary conditions must be expressible as:

$$\alpha_x \frac{\partial T}{\partial x} \pm \beta_x T = 0, \; x = \pm a, \; t \geq 0$$

$$\alpha_y \frac{\partial T}{\partial y} \pm \beta_y T = 0, \; y = \pm a, \; t \geq 0 \; ,$$

where α_x, α_y, β_x, β_y are constants.

When equation (9) holds, the dimensionality of the problem is reduced from two to one. This represents a significant simplification of the reconstruction problem, because it can be shown that only a single set of TOF measurements (a single projection) along parallel paths (in, say, the x-direction; see Fig. 3) is sufficient to recover the function $f(x,t)$, and hence $T(x,y,t)$. This should be contrasted to the general problem encountered, for example, in medical tomography in which hundreds of projections are required at small angular increments over 180°.

There are many obvious practical cases in which the above criteria for factorability are satisfied. However, less obvious is the fact that even initially unfactorable distributions often, with time, evolve to a form where only small approximations are involved to achieve factorability. This is because the lowest order term in the series-expansion solution to the heat flow equation is of the form $\phi(x) \cdot \phi(y)$ and this term rapidly comes to dominate because of the exponential damping (with time) of the higher order terms.

Under the assumption that the two-dimensional temperature is factorable, our approach is to solve the equation:

$$\tau(y_m) = \int_{-a}^{a} \frac{dx}{v(x,y_m)} , \qquad m=1, 2, \ldots M,$$

where the velocity is of the form

$$v(x,y) = \alpha T(x,y) + \beta$$

$$= \alpha f(x) \cdot f(y) + \beta$$

and α and β are known constants. Thus we have

$$\tau(y_m) \cong \int_{-a}^{a} \frac{dx}{\alpha f(x) f(y_m) + \beta} \equiv \hat{\tau}(y_m) , \tag{10}$$

where $\tau(y_m)$ denotes the TOF measurements and $\hat{\tau}(y_m)$ symbolizes the integral quantity in (10).

From the solution to the heat flow equation, f(x) has the following general form (where, for brevity, we have suppressed the t-dependence in f)

$$f(x) = \sum_{n=1}^{\infty} c_n e^{-\alpha_n t} \phi_n(x) , \tag{11}$$

where the coefficients c_n are determined by the initial conditions, α_n are parameters depending on the thermal conductivity and the boundary conditions (e.g., heat transfer coefficient), and $\phi_n(x)$ are the eigenfunctions of the heat flow equation and depend on the problem geometry and boundary conditions.

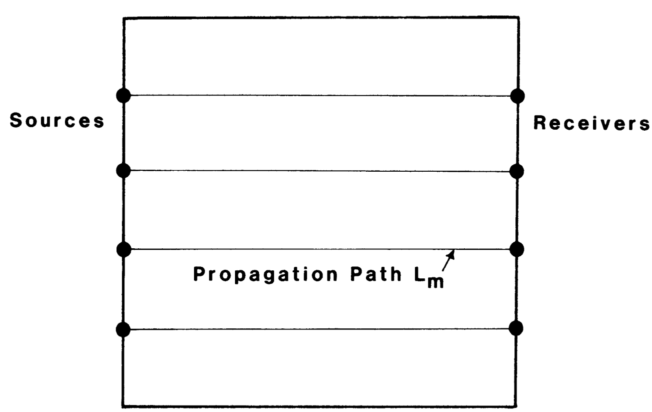

Fig. 3. Square cross-section with parallel ray paths between opposite pairs of sources and receivers.

Our approach to solving (10) is first to approximate f(x) with a truncated expansion of the form (11), where N terms are retained, i.e.,

$$f(x) \cong \hat{f}(x) \equiv \sum_{n=1}^{N} a_n \phi_n(x) \quad . \tag{12}$$

Our objective is to determine the N unknown coefficients a_n from the M measurements $\tau(y_m)$. Inserting (12) into (10), the mean square error E, defined below, is obtained as a function of the coefficients a_n:

$$E(a_1,\ldots,a_n) = \sum_{m=1}^{M} [\tau(y_m) - \hat{\tau}(y_m)]^2 \quad , \tag{13}$$

where τ are measurements and $\hat{\tau}$ is defined in (10). The least-mean-square solution is obtained by minimizing E with respect to the a_n and substituting these values into (12). The reconstructed temperature is then $\hat{T}(x,y) = \hat{f}(x)\hat{f}(y)$. Iterative techniques can be used in the minimization of (13).

The above results were derived assuming a square cross section. A similar approach can be applied to the case of a rectangular cross section for which the temperature factors as follows: $T(x,y) = f(x)\cdot g(y)$. This case requires, in general, two perpendicular projections, e.g., TOF measured over paths parallel to both the x and y axes. If, however, the heat transfer coefficient is known on the boundary, it is not difficult to show that the rectangular cross section can be transformed by a rescaling into a square cross section, in which case the foregoing algorithm (using one projection) is directly applicable.

SIMULATED RECONSTRUCTIONS

The above algorithm was tested on computer-simulated TOF data with randomly distributed errors added. The solution to the two-dimensional thermal conduction equation was first computed for an object with square cross section (6 in. on a side) assuming an initially uniform temperature of 400°C and cooling into a constant ambient temperature of 25°C. The thermal conductivity of 304 stainless steel was used and a heat transfer coefficient of 300 watt/°C-m^2 was assumed. The 2-D temperature distributions were computed at cooling times of 1, 2 and 5 minutes. At each of these times, 15 simulated TOF values were calculated along parallel paths through the existing temperature distribution assuming a known, linear velocity-temperature relation (with slope -0.68 m-sec^{-1}/°C). Time-of-flight errors uniformly distributed between ±0.05 microsec were added to the 15 simulated TOF values. (This error is of the order of the precision of actual laboratory measurements made previously.) These corrupted values were then substituted into (13) and a minimization algorithm applied to obtain the coefficients a_n.

For the case of 1 and 2 minute cooling times, we found that an optimum number (N) of coefficients was about 4. This is the number used in the reconstructions shown in figures 4a and 4b, where the dotted lines indicate the true temperature profile and the solid line is the reconstruction. This small number reflects the fact that the temperature after 1 and 2 minutes is already quite smooth, i.e., only the lowest few Fourier components dominate in defining the distribution. For N larger

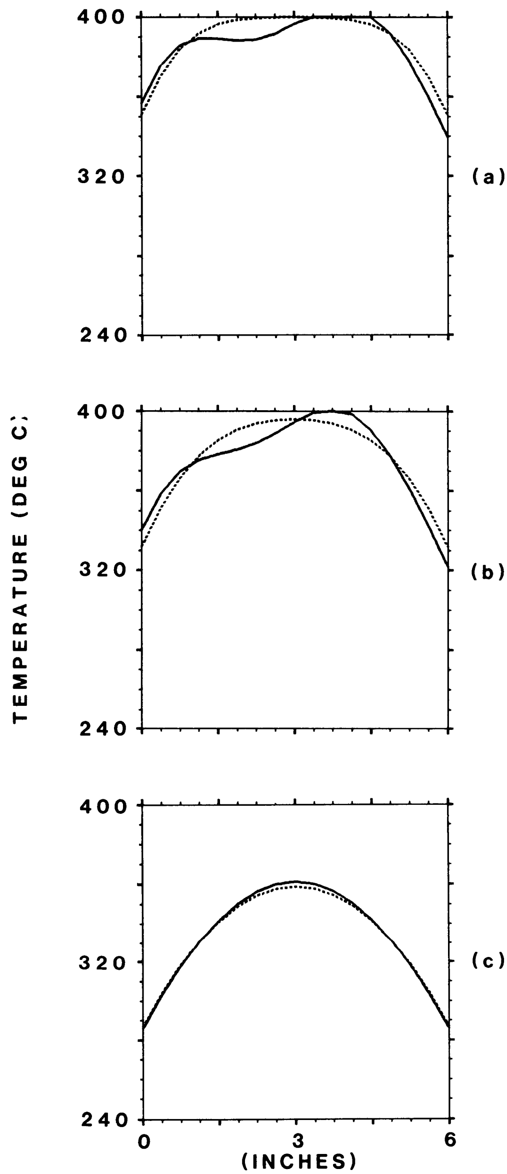

Fig. 4. Profiles of the true (dashed line) and reconstructed (solid line) temperature through the center of a block of 6 inch cross section. Assuming an initially uniform temperature of 400°C, reconstructions were computed from simulated noisy TOF data measured at cooling times of (a) 1 minute, (b) 2 minutes, and (c) 5 minutes. In (a) and (b), the asymmetry in the solid line is due to the random errors added to the simulated TOF data.

than 4, no noticeable improvement in the reconstructed distribution was observed. Although the true distribution is symmetrical, the asymmetry of the reconstructed temperature in figures 4a and 4b (solid line) is a result of the random errors added to the simulated TOF measurements. In the final simulation (Fig. 4c), in which the cooling time is 5 minutes, only a single coefficient was used in the reconstruction (N=1); no improvement was noted for larger N. Thus, after 5 minutes of cooling, the lowest order Fourier coefficient in the temperature distribution clearly dominates.

The above simulations demonstrate that quite accurate tomographic reconstructions are feasible in a rectangular geometry, with a relatively small number of measurements, when factorability of the temperature distribution is assumed. The assumption of factorability represents a powerful constraint upon the solution, and is a good illustration of the use of a priori heat flow information to simplify the tomographic reconstruction problem. The factorability constraint has at least two important consequences: first, as noted earlier, it significantly reduces the number of measurements and, second, by greatly reducing the number of degrees of freedom that characterize the temperature distribution, it improves the numerical conditioning of the inversion process. This last point is also important because numerical ill-conditioning gives rise to the amplification of measurement errors in the inversion process, and thus can degrade, sometimes severely, the accuracy of the reconstructed temperature.

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