A RIGOROUS MODEL FOR INVERTING EDDY-CURRENT DATA

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INTRODUCTION

Inverse scattering models, of the type that are often used to invert eddy-current data, are inherently nonlinear, because they involve the product of two unknowns, the flaw conductivity, and the true electric field within the flaw. Computational inverse models, therefore, often linearize the problem by assuming that the electric field within the flaw is known a priori. In this paper we describe how conjugate gradients might be applied to solve the nonlinear problem. The model is developed for an anisotropic material such as graphite epoxy, and is based on a method-of-moment discretization of two coupled integral equations.

THE NONLINEAR COUPLED INTEGRAL EQUATIONS

Let region 1, where the excitation source and sensors are located, be above the slab, and region 2 be the slab, which contains the anomaly. Assume that we measure the EMF's; then the appropriate pair of coupled integral equations is

$$\bar{E}(\bar{r}) = \bar{E}^{(0)}(\bar{r}) + \int_{flaw} \bar{\bar{G}}_{22}^{(ee)}(\bar{r}|\bar{r}') \cdot \bar{J}_e^{(a)}(\bar{r}')d\bar{r}'$$

$$= \bar{E}^{(0)}(\bar{r}) + \int_{flaw} \bar{\bar{G}}_{22}^{(ee)}(\bar{r}|\bar{r}') \cdot \bar{\bar{\sigma}}^{(a)}(\bar{r}') \cdot \bar{E}(\bar{r}')d\bar{r}' \qquad (1)(a)$$

$$EMF(\bar{r}) = \int_{flaw} \bar{G}_{12}^{(Ee)}(\bar{r}|\bar{r}') \cdot \bar{\bar{\sigma}}^{(a)}(\bar{r}') \cdot \bar{E}(\bar{r}')d\bar{r}'. \qquad (1)(b)$$

The superscripts on the Green's functions denote their type, the first denoting the type of field (electric or magnetic), and the second the type of current source (electric or magnetic), whereas the subscripts denote the regions which are coupled by the Green's function; the first subscript denotes the region which contains the field point, and the second the region that contains the source point. $\bar{E}^{(0)}$ is the incident field that exists in the presence of the slab without the flaw. Finally, $\bar{G}_{12}^{(Ee)}$ is a row-vector that takes the anomalous current density vector, $\bar{J}_e^{(a)}$, into the scalar EMF. From here on we drop all subscripts that are related to regions 1 and 2.

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This system is nonlinear (or, more precisely, bilinear) because of the presence of the product $\bar{\bar{\sigma}}^{(a)}(\bar{r}') \cdot \bar{E}(\bar{r}')$. We usually linearize the problem by replacing $\bar{E}(\bar{r}')$ in (1)(b) by $\bar{E}^{(0)}(\bar{r}')$, and then ignoring (1)(a).

To solve this system, we first discretize it in the usual way by means of the method of moments, and then apply an iterative technique to the resulting algebraic equations. One possible technique would start by replacing $\bar{E}$ in (1)(b) by $\bar{E}^{(0)}$, and then solving for $\bar{\bar{\sigma}}^{(a)}$. This step uses measured data, and is the 'inverse' phase of the problem.

Once we have an acceptable approximation to $\bar{\bar{\sigma}}^{(a)}$, we substitute it into (1)(a) and solve the resulting 'direct problem' for an improved version of $\bar{E}$, the electric field within the flawed region.

The result of the direct phase is then substituted into (1)(b) and the second-level inverse problem is solved. The process is continued until the error in the solution is of the order of the error in the measured data.

The process that we have just suggested may not be the most efficient way to solve system (1). We will look at conjugate gradients, and see how to handle the nonlinearity there.

DISCRETIZATION OF THE COUPLED SYSTEM: METHOD OF MOMENTS

The discretization of the coupled system of integral equations is done by subdividing the region of space occupied by the anomaly into a regular grid of $(N_x+1) \times (N_y+1) \times (N_z+1)$ cells, each of size $\delta_x \times \delta_y \times \delta_z$, and then expanding the electric field and anomalous conductivity tensor using pulse functions defined over the grid:

$$\bar{E}(\bar{r}) = \sum_{l=0}^{N_x} \sum_{m=0}^{N_y} \sum_{j=0}^{N_z} \bar{E}_{lmj} P_l(\frac{x}{\delta_x}) P_m(\frac{y}{\delta_y}) P_j(\frac{z}{\delta_z}), \tag{2}$$

and

$$\bar{\bar{\sigma}}^{(a)}(\bar{r}) = \sum_{l=0}^{N_x} \sum_{m=0}^{N_y} \sum_{j=0}^{N_z} \bar{\bar{\sigma}}_{lmj} P_l(\frac{x}{\delta_x}) P_m(\frac{y}{\delta_y}) P_j(\frac{z}{\delta_z}). \tag{3}$$

The pulse function, $P_j(s)$, satisfies

$$P_j(s) = \begin{cases} 1, & \text{if } j \le s < j+1; \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Note that the anomalous current density, which is given by the product of (2) and (3), has exactly the same expansion in pulse functions as either (2) or (3), except, of course, that the expansion coefficients are given by the term-by-term product $\bar{\bar{\sigma}}_{lmj} \cdot \bar{E}_{lmj}$.

We are going to use Galerkin's variant of the method of moments to complete the discretization. In Galerkin's method, we 'test' the integral equations (1) with the same pulse functions that we used to expand the unknowns in (2). For example, we form moments of (1)(a) by multiplying (1)(a) by $P_l(x/\delta_x)P_m(y/\delta_y)P_j(z/\delta_z))/\delta_x\delta_y\delta_z$, and then integrating over each cell. This yields an algebraic system for $\bar{E}_{lmj}$:

$$\bar{E}_{lmj}^{(0)} = \bar{E}_{lmj} - \sum_{L=0}^{N_x} \sum_{M=0}^{N_y} \sum_{J=0}^{N_z} \bar{G}_{jJ}^{(ee)}(l-L, m-M) \cdot \bar{\bar{\sigma}}_{LMJ} \cdot \bar{E}_{LMJ}. \tag{5}$$

In taking moments of $(1)(b)$ we must keep in mind that the sensors lie in a fixed z-plane above the workpiece. Hence, we multiply by only $P_l(x/\delta x)P_m(y/\delta y)$

and then integrate to get the algebraic system:

$$EMF_{lm} = \sum_{L=0}^{N_x} \sum_{M=0}^{N_y} \sum_{J=0}^{N_z} \bar{G}_J^{(Ee)}(l-L, m-M) \cdot \bar{\bar{\sigma}}_{LMJ} \cdot \bar{E}_{LMJ}. \tag{6}$$

Equations (5) and (6) are the coupled system of discrete equations that define the rigorous inverse problem. The system is nonlinear (bilinear) in the unknowns, $\bar{\sigma}_{lmj}$, $\bar{E}_{lmj}$. Note that in two of the three dimensions, the indices appear in a Toeplitz (or convolution) form.

MULTIFREQUENCY OR MULTIVIEW RECONSTRUCTION METHODS

Equation (6) indicates that if we measure the EMF's at a single z-level, then we don't have enough data to reconstruct a three-dimensional flaw. The additional data to reconstruct the third dimension can be obtained several ways. For example, we can excite the system from a single coil at a number, N_f, of frequencies, where $N_f \geq N_z + 1$. In that case, the Green's functions and electric field in (5) and (6), as well as the measured EMF's, vary with frequency (but the unknown conductivities are assumed to be independent of frequency).

If we let the integer, n, index the frequency parameter, then (5) and (6) become

$$\bar{E}_{lmj}^{(0)}(n) = \bar{E}_{lmj}(n) - \sum_{L=0}^{N_x} \sum_{M=0}^{N_y} \sum_{J=0}^{N_z} \bar{G}_{jJ}^{(ee)}(l-L, m-M; n) \cdot \bar{\bar{\sigma}}_{LMJ} \cdot \bar{E}_{LMJ}(n) \tag{7(a)}$$

$$EMF_{lm}(n) = \sum_{L=0}^{N_x} \sum_{M=0}^{N_y} \sum_{J=0}^{N_z} \bar{G}_J^{(Ee)}(l-L, m-M; n) \cdot \bar{\bar{\sigma}}_{LMJ} \cdot \bar{E}_{LMJ}(n), \tag{7(b)}$$

where $\bar{G}_J^{(Ee)}$ is a vector, not a dyad.

This is the ''multifrequency reconstruction algorithm''.

Another method is to excite the system at a single frequency, but with a number of different exciting coils. In this case, only the electric fields and measured EMF's vary with n, which now indexes the location of the exciting coil. Hence, the system of equations becomes

$$\bar{E}_{lmj}^{(0)}(n) = \bar{E}_{lmj}(n) - \sum_{L=0}^{N_x} \sum_{M=0}^{N_y} \sum_{J=0}^{N_z} \bar{G}_{jJ}^{(ee)}(l-L, m-M) \cdot \bar{\bar{\sigma}}_{LMJ} \cdot \bar{E}_{LMJ}(n) \tag{8(a)}$$

$$EMF_{lm}(n) = \sum_{L=0}^{N_x} \sum_{M=0}^{N_y} \sum_{J=0}^{N_z} \bar{G}_J^{(Ee)}(l-L, m-M) \cdot \bar{\bar{\sigma}}_{LMJ} \cdot \bar{E}_{LMJ}(n). \tag{8(b)}$$

This is the ''multiview reconstruction algorithm''.

THE CONJUGATE GRADIENT ALGORITHM FOR THE COUPLED SYSTEM

We are going to apply the conjugate algorithm to the nonlinear coupled system (8), and work in the coordinate system in which $\bar{\bar{\sigma}}$ is diagonal. Then we can rewrite (8)(b) as

$$0 = R_{lmn} = -EMF_{lm}(n) + \sum_{J=0}^{N_z} \left[G_{Jx}^{(Ee)} \otimes \sigma_{Jx} E_{Jx}(n) + G_{Jy}^{(Ee)} \otimes \sigma_{Jy} E_{Jy}(n) + G_{Jz}^{(Ee)} \otimes \sigma_{Jz} E_{Jz}(n) \right], \tag{9}$$

where R_{lmn} is the lmnth component of the residual vector, and $\otimes$ denotes a two-dimensional discrete convolution.

We form the squared-norm of the residuals

$$\Phi(\sigma_{LMJx}, \sigma_{LMJy}, \sigma_{LMJz}) = \sum_{l=0}^{N_x} \sum_{m=0}^{N_y} \sum_{n=0}^{N_v} R_{lmn} R_{lmn}^*, \tag{10}$$

where N_v is the number of "views" (i.e., the number of source locations), and then differentiate with respect to $\bar{\sigma}_{LMJ} = \{\sigma_{LMJx}, \sigma_{LMJy}, \sigma_{LMJz}\}$:

$$\frac{\partial \Phi}{\partial \bar{\bar{\sigma}}_{LMJ}} = 2 \mathrm{Re} \sum_{l=0}^{N_x} \sum_{m=0}^{N_y} \sum_{n=0}^{N_v} R_{lmn} \left[\mathrm{Diag}\left(\bar{G}_J^{(Ee)*}(l-L, m-M) \bar{E}_{LMJ}^*(n) \right) + \right.$$

$$\left. \bar{G}_J^{(Ee)*}(l-L, m-M) \cdot \bar{\bar{\sigma}}_{LMJ} \cdot \frac{\partial \bar{E}_{LMJ}^*(n)}{\partial \bar{\bar{\sigma}}_{LMJ}} \right] \tag{11}$$

$$= 2 \overline{\overline{Grad}}_{LMJ}.$$

In (10) and (11) the asterisk denotes complex conjugation. We are using dyadic notation in (11). For example, the first term within the square brackets is the diagonal part of the dyadic product of the vectors $\bar{G}_J^{(Ee)*}$ and $\bar{E}_{LMJ}^*(n)$.

The Fletcher-Reeves Conjugate Gradient Algorithm for minimizing the objective function $\Phi(x)$ is [1]:

Step 1. Given $\bar{x}_0$, compute $\bar{g}_0 = \nabla\Phi(\bar{x}_0)'$ and set $\bar{d}_0 = -\bar{g}_0$.

Step 2. For $k = 0, 1, \ldots, n-1$

    (a) Set $\bar{x}_{k+1} = \bar{x}_k + \alpha_k \bar{d}_k$, where α_k minimizes $\Phi(\bar{x}_k + \alpha \bar{d}_k)$
    (b) Compute $\bar{g}_{k+1} = \nabla\Phi(\bar{x}_{k+1})'$
    (c) Unless $k = n-1$, set $\bar{d}_{k+1} = -\bar{g}_{k+1} + \beta_k \bar{d}_k$ where

$$\beta_k = \frac{\bar{g}_{k+1}' \bar{g}_{k+1}}{\bar{g}_k' \bar{g}_k}.$$

Step 3. Replace $\bar{x}_0$ by $\bar{x}_n$ and go back to Step 1.

The prime denotes the conjugate-transpose of a complex vector.

This algorithm requires the exact gradient of Φ, as well as a line-search. The gradient, (11), is obviously quite complicated, and it will be to our advantage to have a simpler formulation of the conjugate gradient algorithm. For this purpose we will turn to some ideas of Stephen Norton, [2], who has developed a quasi-linearized version of the conjugate gradient algorithm.

The quasi-linear conjugate gradient algorithm starts with the usual iterative step $\bar{\bar{\sigma}}_{LMJ}^{(k)} = \bar{\bar{\sigma}}_{LMJ}^{(k-1)} + a_k \bar{\bar{f}}_{LMJ}^{(k)}$. We choose a_k to minimize $\Phi(\bar{\bar{\sigma}}_{LMJ}^{(k-1)} + a_k \bar{\bar{f}}_{LMJ}^{(k)})$ with respect to a_k, for a given $\bar{\bar{f}}_{LMJ}^{(k)}$. We will shortly determine the optimum $\bar{\bar{f}}_{LMJ}^{(k)}$.

Let us first derive an expression for the kth residual, (9), using a linearized expression, in which the electric field is replaced by its $(k-1)$th approximation:

$$R_{lmn}^{(k)} = -EMF_{lm}(n) + \sum_{L=0}^{N_x} \sum_{M=0}^{N_y} \sum_{J=0}^{N_z} \bar{G}_J^{(Ee)}(l-L, m-M) \cdot (\bar{\bar{\sigma}}_{LMJ}^{(k-1)} + a_k \bar{\bar{f}}_{LMJ}^{(k)}) \cdot \bar{E}_{LMJ}^{(k-1)}(n)$$

$$= R_{lmn}^{(k-1)} + a_k F_{lmn}^{(k)}. \tag{12}$$

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Upon substituting this result into (10), we get

$$\Phi^{(k)} = \Phi^{(k-1)} + a_k 2\text{Re}\big[F^{(k)} \cdot R^{(k-1)}\big] + a_k^2 \|F^{(k)}\|^2, \tag{13}$$

where we are using vector-matrix inner-product notation. Hence, $\Phi^{(k)}$ is minimized when

$$a_k = -\frac{\text{Re}\big[R^{(k-1)} \cdot F^{(k)}\big]}{\|F^{(k)}\|^2}, \tag{14}$$

and when this is substituted into (13) we find the minimum value to be

$$\Phi^{(k)} = \Phi^{(k-1)} - \frac{\big[\text{Re}\big(F^{(k)} \cdot R^{(k-1)}\big)\big]^2}{\|F^{(k)}\|^2}. \tag{15}$$

The greatest decrease occurs when $F^{(k)} = R^{(k-1)}$ in (15). What this means in terms of the direction, $\vec{f}_{LMJ}^{(k)}$, of the change in $\bar{\sigma}_{LMJ}$ can be determined by returning to (12):

$$\text{Re}\bigg[R^{(k-1)} \cdot F^{(k)}\bigg]$$

$$= \text{Re}\sum_{l=0}^{N_x}\sum_{m=0}^{N_y}\sum_{n=0}^{N_v} R_{lmn}^{(k-1)}\bigg[\sum_{L=0}^{N_x}\sum_{M=0}^{N_y}\sum_{J=0}^{N_z} \bar{G}_J^{(Ee)*}(l-L, m-M) \cdot \vec{f}_{LMJ}^{(k)} \cdot \bar{E}_{LMJ}^{(k-1)*}(n)\bigg]$$

$$= \sum_{L=0}^{N_x}\sum_{M=0}^{N_y}\sum_{J=0}^{N_z} \vec{f}_{LMJ}^{(k)} \cdot \bigg[\text{Re}\sum_{l=0}^{N_x}\sum_{m=0}^{N_y}\sum_{n=0}^{N_v} R_{lmn}^{(k-1)}\text{Diag}\Big(\bar{G}_J^{(Ee)*}(l-L, m-M)\bar{E}_{LMJ}^{(k-1)*}(n)\Big)\bigg]. \tag{16}$$

Upon comparing the summation term within the large square brackets of (16) with the expression for the gradient, (11), we see that the summation term is the linearized gradient, which is (11) with $\partial\bar{E}_{LMJ}^*(n)/\partial\bar{\sigma}_{LMJ} = 0$. Hence, (16) shows that the maximum decrease in the norm of the residuals occurs when

$$\vec{f}_{LMJ}^{(k)} = \overline{\overline{Grad}}_{LMJ}^{(k-1)}. \tag{17}$$

This is the steepest-descent direction.

There is an important orthogonality relation that holds:

$$\sum_{L=0}^{N_x}\sum_{M=0}^{N_y}\sum_{J=0}^{N_z} \vec{f}_{LMJ}^{(k)} \cdot \overline{\overline{Grad}}_{LMJ}^{(k)} = 0. \tag{18}$$

In order to derive this result, substitute (12) into the expression, (16), for the linearized gradient and get the following recursion relation:

$$\overline{\overline{Grad}}_{LMJ}^{(k)} = \overline{\overline{Grad}}_{LMJ}^{(k-1)} + a_k\text{Re}\sum_{l=0}^{N_x}\sum_{m=0}^{N_y}\sum_{n=0}^{N_v} F_{lmn}^{(k)}\text{Diag}\Big(\bar{G}_J^{(Ee)*}(l-L, m-M)\bar{E}_{LMJ}^{(k-1)*}(n)\Big). \tag{19}$$

When the definition of $F_{lmn}^{(k)}$ from (12) is substituted into (19), and the resulting expression is multiplied by $\vec{f}_{LMJ}^{(k)}$, and then summed, we get

$$\sum_{L=0}^{N_x}\sum_{M=0}^{N_y}\sum_{J=0}^{N_z} \vec{f}_{LMJ}^{(k)} \cdot \overline{\overline{Grad}}_{LMJ}^{(k)} = \sum_{L=0}^{N_x}\sum_{M=0}^{N_y}\sum_{J=0}^{N_z} \vec{f}_{LMJ}^{(k)} \cdot \overline{\overline{Grad}}_{LMJ}^{(k-1)} + a_k\text{Re}\|F^{(k)}\|^2$$
$$= \text{Re}\big[R^{(k-1)} \cdot F^{(k)}\big] - \text{Re}\big[R^{(k-1)} \cdot F^{(k)}\big]$$
$$= 0. \tag{20}$$

We have used (14) and (16) in arriving at the final result.

The conjugate gradient algorithm starts with a pure gradient step, (17), for $k = 1$, and then continues with

$$\bar{f}_{LMJ}^{(k)} = \bar{\bar{G}}rad_{LMJ}^{(k-1)} + b_k \bar{f}_{LMJ}^{(k-1)}. \tag{21}$$

b_k will be chosen to minimize the denominator in (15), thereby guaranteeing an improved convergence rate [3].

Before stating an expression for b_k, we will state another expression for the numerator of a_k, which appears in (14). Starting with (16), and using (18) and (21), we get:

$$\text{Numerator of } a_k = \|\bar{\bar{G}}rad^{(k-1)}\|^2. \tag{22}$$

The derivation of b_k is lengthy, and will be omitted. If we assume that the change in the model (i.e., conductivity) is sufficiently small to permit us to say that the electric field within the anomalous region does not not change too much from iteration to iteration, then the result is

$$b_k = -\frac{\|\bar{\bar{G}}rad^{(k-1)}\|^2}{\|\bar{\bar{G}}rad^{(k-2)}\|^2}. \tag{23}$$

Let Y be the array of complex scalars, $\{EMF_{lm}(n)\}$, where lmn index the array elements, and define the operator

$$\mathcal{A}^{(k)}(\bar{\bar{f}}) = \sum_{L=0}^{N_x} \sum_{M=0}^{N_y} \sum_{J=0}^{N_z} \text{Diag}\left(\bar{G}_J^{(Ee)}(l-L, m-M)\bar{E}_{LMJ}^{(k-1)}(n)\right) \cdot \bar{\bar{f}}_{LMJ}, \tag{24}$$

together with its adjoint

$$\mathcal{A}^{*(k)}(R) = \text{Re} \sum_{l=0}^{N_x} \sum_{m=0}^{N_y} \sum_{n=0}^{N_v} R_{lmn}\text{Diag}\left(\bar{G}_J^{(Ee)*}(l-L, m-M)\bar{E}_{LMJ}^{(k-1)*}(n)\right). \tag{25}$$

Note that $\mathcal{A}$ produces an array of complex scalars from an array of real vectors, whereas $\mathcal{A}^*$ produces an array of real vectors from an array of complex scalars. The electric field at the kth step is computed from (8)(a), using the kth approximation of $\bar{\bar{\sigma}}_{LMJ}$.

We summarize these results by saying that the nonlinear (quasi-linear) conjugate gradient algorithm is similar to the linear one, except that the operators must be updated at each iteration, because the electric field is updated. In addition, we may have to ensure that the increments in the solution vector are not too large.

The algorithm starts with an initial guess, X_0, from which we compute $R_0 = Y - \mathcal{A}^{(1)}X_0$, $P_1 = Q_0 = \mathcal{A}^{*(1)}R_0$. In addition, we have a convergence parameter, ϵ. Then for $k = 1, ..,$ if $Test = \|R_k\|/\|Y\| < \epsilon$, stop; X_k is the optimal solution of (8)(b). Otherwise, update X_k by the following steps:

$$S_k = \mathcal{A}^{(k)}P_k$$

$$a_k = \frac{\|Q_{k-1}\|^2}{\|S_k\|^2}$$

$$X_k = X_{k-1} + a_k P_k$$

$$R_k = R_{k-1} - a_k S_k \tag{26}$$

$$Q_k = \mathcal{A}^{*(k)}R_k$$

$$b_k = \frac{\|Q_k\|^2}{\|Q_{k-1}\|^2}$$

$$P_{k+1} = Q_k + b_k P_k.$$

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The convolution and correlation operations that are a part of $\mathcal{A}$ and $\mathcal{A}^*$ are evaluated by using the FFT. This, together with the fact that the storage requirements are reasonably modest, are the reasons why the conjugate gradient algorithm becomes attractive for large problems.

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